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GOLDEN OLDIE EDITORIAL

Editorial note to: R. P. Kerr and A. Schild, A new class of vacuum solutions of the Einstein field equations

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Part 1: Explanation of some details of derivation

By Andrzej Krasiński

The Kerr solution is today textbook material and a basic element of education in relativity. However, derivations of it are not easy to find in the literature—most textbooks and monographs simply quote it as a given thing. There exist several uniqueness theorems (see part 2 of this note) that refer to physical and geometrical properties of the Kerr solution, but none of them gives a hint on how to derive it starting from the Einstein

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equations. Given the great physical and astrophysical significance of this solution,[1] this is a rather frustrating situation. Therefore, we (the editors) decided to go back to the historical origins. In Part 3 of this note, Roy Kerr gives a first-hand account on how he arrived at his famous result. Here, some comments on the Kerr–Schild paper.

The original publication [3] reported only the final result and some of its basic properties, but gave no hint on how to derive it. A derivation was published 2 years later, in a volume of conference proceedings that has never been easily accessible, and was not frequently referred to. This is the text reprinted here. The idea of the derivation is mostly self-explanatory, but not the details. The authors warn the reader at one point that the calculations are not simple, but even where they say that something follows by "simple calculation", the results are not necessarily easy to reproduce. This note is meant to be a guide for those readers who wish to verify all the details.[2] We will use the following notation for the tetrad vectors:

$$\{e_1, e_2, e_3, e_4\} = \{m, \overline{m}, \ell, k\}\,. \tag{1}$$

The following auxiliary results are helpful in verifying the "simple direct calculation" at the beginning of Sect. 2:

The combinations $\left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} l^\beta l^\gamma$, $\left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} l_\alpha l^\gamma$ and $\left\{ \begin{matrix} \alpha \\ \beta\alpha \end{matrix} \right\}$ have the same value, no matter whether $g_{\alpha\beta}$ or $\eta_{\alpha\beta}$ is used to calculate the Christoffel symbols.

Then:

$$\begin{aligned} \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} l^\gamma &= \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\}(\eta) l^\gamma - \frac{1}{2} l^\rho \left(l^\alpha l_\beta \right);_\rho\,, \\ \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} l_\alpha &= \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\}(\eta) l_\alpha + \frac{1}{2} l^\rho \left(l_\beta l_\gamma \right);_\rho\,, \end{aligned} \tag{2}$$

where $\left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\}$ are calculated from $g_{\alpha\beta}$, and $\left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\}(\eta)$ are calculated from $\eta_{\alpha\beta}$ (remember: $\eta_{\alpha\beta}$ is flat, but expressed in arbitrary coordinates, so in general $\left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\}(\eta) \neq 0$). Using these formulae in (2.1), after a lot of algebra, one is led to (2.2).

The other "simple calculation" mentioned before (2.8) is again not really simple. To carry it out one needs to note that, in consequence of (2.6), Eqs. (2) above simplify to:

$$\begin{aligned} \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} k^\gamma &= \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\}(\eta) k^\gamma - k^\rho H,_\rho k^\alpha k_\beta, \\ \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} k_\alpha &= \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\}(\eta) k_\alpha + k^\rho H,_\rho k_\beta k_\gamma \end{aligned} \tag{3}$$

[1] This is attested by the very large number of papers discussing its various astrophysical applications—see Refs. [1] and [2].

[2] According to the information from R. P. Kerr (see part 3 of this note), the Kerr–Schild paper is a text specially adapted to deriving the Kerr metric. The derivation by the original method was published still later, see Ref. [4].

while the general Christoffel symbols of the two metrics are related by

$$\left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} = \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\}(\eta) - k^\alpha k_\beta H,_\gamma - Hk^\alpha k_{\beta;\gamma} - Hk^\alpha;_\gamma k_\beta - H,_\beta k^\alpha k_\gamma$$
$$- Hk^\alpha k_{\gamma;\beta} - Hk^\alpha;_\beta k_\gamma + \eta^{\alpha\rho} H,_\rho k_\beta k_\gamma + \eta^{\alpha\rho} H k_{\beta;\rho} k_\gamma$$
$$+ \eta^{\alpha\rho} H k_\beta k_{\gamma;\rho} + 2Hk^\rho H,_\rho k^\alpha k_\beta k_\gamma. \tag{4}$$

To derive (3.9) one observes that $R_{12} = 0$ is equivalent to $k^\rho H,_\rho (z + \overline{z}) + H\left(z^2 + \overline{z}^2\right) = 0$, while z obeys $k^\rho z,_\rho = -z^2$. This last equation is not trivial either. To obtain it, one must take the Ricci identities for the vector field k^α:

$$k_{\alpha;\beta\gamma} - k_{\alpha;\gamma\beta} = R_{\rho\alpha\beta\gamma} k^\rho \tag{5}$$

and contract them first with $k^\gamma g^{\alpha\beta}$, obtaining an equation analogous to the Raychaudhuri equation:

$$k^\gamma \theta,_\gamma + \sigma^2 - \omega^2 + \theta^2 = -\frac{1}{2} R_{\rho\gamma} k^\rho k^\gamma, \tag{6}$$

where

$$\omega^2 \stackrel{\text{def}}{=} \frac{1}{2} k_{[\alpha;\beta]} k^{\alpha;\beta} \tag{7}$$
$$\theta \stackrel{\text{def}}{=} k^\alpha;_\alpha \tag{8}$$
$$\sigma^2 \stackrel{\text{def}}{=} \frac{1}{2} k_{(\alpha;\beta)} k^{\alpha;\beta} - \theta^2 \tag{9}$$

are the rotation, expansion and shear of the null geodesic k-congruence. Then one contracts Eq. (5) with $k^\gamma \left(m^\alpha \overline{m}^\beta - m^\beta \overline{m}^\alpha\right)$ and uses the fact that $R_{\rho[\alpha\beta]\gamma} k^\rho k^\gamma \equiv 0$. The result is:

$$k^\gamma \left(\Gamma_{4[12]}\right),_\gamma + \frac{1}{2}\left[(\Gamma_{412})^2 - (\Gamma_{421})^2\right] = 0. \tag{10}$$

But θ and ω are the real and the imaginary part of the same Ricci rotation coefficient Γ_{412}, which is another exercise for the reader (Eqs. (2.17) in the paper seem to have misplaced indices). Knowing this, one can rewrite (10) as

$$k^\gamma \omega,_\gamma + 2\theta\omega = 0, \tag{11}$$

and then observe that (6) and (11) can be written as one complex equation:

$$k^\gamma z,_\gamma + z^2 + \sigma^2 = -\frac{1}{2} R_{\rho\gamma} k^\rho k^\gamma \equiv -\frac{1}{2} R_{44}. \tag{12}$$

In the shearfree vacuum case this reduces to $k^\rho z,_\rho = -z^2$, as stated above. Thus, $k^\rho H,_\rho (z+\overline{z}) + H\left(z^2+\overline{z}^2\right) = 0$ is equivalent to $k^\rho[H/(z+\overline{z})],_\rho = 0$, whose solution is (3.9).

Deriving (3.11) is again tricky. The set (3.10) implies the integrability condition $P_{|12} - P_{|21} = P_{|m}\,(\Gamma^m{}_{12} - \Gamma^m{}_{21})$ (from (2.21)), which reads explicitly

$$S \stackrel{\text{def}}{=} \frac{z}{\overline{z}^2} Y,_u\, m^\rho \overline{z},_\rho - \frac{z}{\overline{z}} m^\rho Y,_{u\rho} - \frac{\overline{z}}{z^2} \overline{Y},_u\, \overline{m}^\rho z,_\rho + \frac{\overline{z}}{z} \overline{m}^\rho \overline{Y},_{u\rho} + \left(\frac{z}{\overline{z}} - \frac{\overline{z}}{z}\right) Y,_u\, \overline{Y},_u - \ell^\rho P,_\rho (\overline{z} - z) = 0. \tag{13}$$

Other useful formulae are:

$$m^\rho z,_\rho = (\overline{z} - z)\overline{Y},_u\,, \tag{14}$$

$$\ell^\rho z,_\rho = Y,_u\, \overline{Y},_u - Hz^2 + \overline{m}^\rho \overline{Y},_{u\rho}\,, \tag{15}$$

and their complex conjugates. Also, it is important to note that $R_{33} = R^1{}_{313} + R^2{}_{323} \equiv 2R_{1323} = 2R^0{}_{323}$. Although R_{33} must be real, the equation $R_{1323} = R_{2313}$ does not hold identically.[3] Consequently, $2R_{2313}$ is not identically equal to $R_{1323} + R_{2313}$. In the equation $R^4{}_{132} = 0$ one must use the analogue of (2.23) for the full Riemann tensor and all the other equations derived so far. The result is:

$$\overline{z}\ell^\rho\left(\mathrm{e}^{3P}\right),_\rho (z+\overline{z}) + 3H\frac{z+\overline{z}}{z}\, Y,_u\, \overline{Y},_u + 3z\mathrm{e}^{3P} m^\rho Y,_{u\rho} - 3\frac{\overline{z}^2}{z}\mathrm{e}^{3P}\overline{m}^\rho \overline{Y},_{u\rho} + 3\frac{\overline{z}^2}{z^2}\mathrm{e}^{3P}\overline{Y},_u\, \overline{m}^\rho z,_\rho - 3\frac{z}{\overline{z}}\mathrm{e}^{3P} Y,_u\, m^\rho \overline{z},_\rho = 0. \tag{16}$$

Taking the imaginary part of the above we obtain $(z+\overline{z})S = 0$, where S is the expression defined in (13). Thus, the imaginary part of (16) vanishes by virtue of the integrability condition of (3.10). Using now (13) in (16) to eliminate the $m^\rho Y,_{u\rho}$ terms and recalling that $Y,_u = \ell^\rho Y,_\rho$, we obtain (3.11).

From here on, until the end of Sect. 4, the derivation is general and covariant, and relatively easy to follow. The non-covariant arbitrary assumption is made at the beginning of Sect. 5. What the authors say here is not strictly correct: Eq. (5.1) does not result for any quadratic polynomial $\Phi(Y)$. One must assume $\Phi = \alpha Y^2 + \beta Y - \overline{\alpha}$, where α and β are arbitrary complex constants; only then can (5.1) be achieved by a translation of ξ and u or of ξ and v.

[3] The identities obeyed by the Riemann and Weyl tensors are deduced under the assumption that these tensors arise from commutators of second derivatives of tensors. However, if the basic objects in the theory are the Ricci rotation coefficients, like in the Newman–Penrose formalism, then the curvature tensors are present in first-order equations, and not all the 'identities' will automatically be fulfilled (in fact, only $R_{ijkl} = -R_{ijlk} = -R_{jikl}$ and $C^i{}_{jil} = 0$). Thus, some of those other 'identities' must be imposed as equations to fulfill. The equation $R_{1323} = R_{2313}$ holds by virtue of (13).

Also (5.5) requires a comment: the equation $F = 0$ has two solutions for Y, the other one follows from (5.5) by replacing $\rho \to -\rho$. However, what then results in (5.8) is a continuation of the Kerr metric to negative ρ, and is thus not a new solution.

Another instructive derivation of the Kerr metric can be found in the paper by Carter [5].[4] The Carter paper, and that of Boyer and Lindquist [6], are also good sources of information about geometrical properties of the Kerr metric.

The Kerr–Schild paper reprinted here is at the same time an important early source of knowledge on the, so-called today, Kerr–Schild metrics. That metric ansatz was quite popular in its time. A good overview of the exact solutions obtained on the basis of that ansatz is Chapter 32 in Ref. [7].[5]

Part 2: Later research stimulated by the Kerr solution

By Enric Verdaguer

The relevance of the Kerr solution in the development of general relativity during the 1970s cannot be overemphasized. On the one hand, it stimulated the search for solutions describing spinning isolated bodies, both in the exterior as well as in the interior, and also the development of new solution-generating techniques. On the other hand the development of black hole physics, and the so called uniqueness or "no hair" theorems, places the Kerr solution in a unique position as a key piece in our understanding of gravitational theory. Let us briefly comment on those points.

Solution-generating techniques

The Kerr–Schild ansatz and the Kerr solution played an inspiring role in the development of the powerful solution-generating techniques that were introduced into general relativity in the late 1970s in order to solve Einstein equations in the stationary and axisymmetric context. Thus Maison [8] proved that Einstein equations in vacuum for stationary and axisymmetric spacetimes are an integrable system. At about the same time Belinski and Zakharov [9,10] extended the inverse scattering method (ISM), which had been developed to solve some nonlinear wave equations in fluid dynamics, to general relativity under certain symmetry assumptions. They showed that the ISM could be applied to solve Einstein equations in spacetimes with two commuting Killing vector fields that admit the existence of 2-surfaces orthogonal to the group orbits, i.e. spacetimes that admit an orthogonally transitive two-parameter group of isometries. This includes the stationary axisymmetric spacetimes when one of the Killing fields is timelike (stationary) and the second Killing field generates closed curves and vanishes on the symmetry axis (axial symmetry).

[4] The Carter paper will also be reprinted in the Oldies series.

[5] J. Ehlers tells me that one remark should be added to the account in Ref. [7]: all the vacuum solutions obtained via the Kerr–Schild ansatz have nontrivial isometry groups that are subgroups of the Poincaré group.

In the stationary and axisymmetric context the ISM is closely related to other solution-generating techniques such as different Bäcklund transformations, like Harrison's [11,12] and Neugebauer's Bäcklund transformations [13,14], Kinnersley–Chitre transformations [15–17], or the Hauser–Ernst formalism [18,19]. All these techniques provide algorithms to obtain new solutions starting from some known solutions. Repeated applications of these transformations to a given solution leads to a large class of new solutions with an increasing number of parameters. Generally these solution-generating techniques can also be extended to the Einstein–Maxwell equations.

As an example, and to be specific, let us summarize briefly the soliton transformation of the ISM. In the stationary axisymmetric context it is convenient to write the metric in coordinates adapted to the Killing vector fields, ∂_t and ∂_φ, in the form:

$$ds^2 = f(\rho, z)(d\rho^2 + dz^2) + g_{ab}(\rho, z)dx^a dx^b, \tag{17}$$

where $a, b = 0, 1$ with $x^0 = t$, $x^1 = \varphi$. We can impose on the 2×2 matrix g the condition that

$$\det g = -\rho^2, \tag{18}$$

so that ρ and z are Weyl coordinates. The Einstein equations in vacuum for this metric separate in two sets of equations. The first set determines the matrix g,

$$\left(\rho g_{,\rho} g^{-1}\right)_{,\rho} + \left(\rho g_{,z} g^{-1}\right)_{,z} = 0, \tag{19}$$

and the second set determines f, once g is known. The ISM focuses on the nonlinear equation (19), the integration of the second set to determine f is rather simple once g has been obtained. The idea is to associate to the nonlinear system (19) some "spectral equations" (a set of two linear equations involving g and its derivatives) for a "generating matrix" $\psi(\lambda, \rho, z)$ in such a way that the Einstein equations (19) are the integrability conditions of the spectral equations. The matrix g is then given in terms of the generating matrix when the "spectral" paramater λ is zero:

$$g(\rho, z) = \psi(0, \rho, z). \tag{20}$$

The procedure for the integration of the spectral equations is to assume a particular solution g_0 and then to find, by integration, the corresponding solution $\psi_0(\lambda, \rho, z)$. This is the only non algebraic step in the procedure, which is made easier by assuming a simple "background" solution g_0. A particularly simple and useful choice is the Minkowski metric, for which $f_0 = 1$ and $g_0 = \mathrm{diag}\,(-1, \rho^2)$, in the Weyl coordinates. The generating matrix is in this case

$$\psi_0 = \mathrm{diag}\,(-1, \rho^2 - 2z\lambda - \lambda^2). \tag{21}$$

Next one looks for solutions of ψ of the form $\psi = \chi\psi_0$, and the spectral equations become equations for the "dressing matrix" $\chi(\lambda, \rho, z)$. The fact that g must be symmetric and real imposes some restrictions on χ.

The "n-soliton transformation" of the ISM corresponds to the presence of n pole singularities of the dressing matrix in the complex plane of the spectral parameter λ. That is, when χ takes the form:

$$\chi = I + \sum_{k=1}^{n} \frac{R_k}{\lambda - \mu_k}, \tag{22}$$

where I is the unit matrix and R_k are 2×2 matrices independent of λ. The spectral equation for χ and the restriction that it must satisfy completely determine the so called "pole trajectories" $\mu_k(\rho, z)$ and the matrices $R_k(\rho, z)$. The pole trajectories are

$$\mu_k = w_k - z \pm \sqrt{(w_k - z)^2 + \rho^2}, \tag{23}$$

where w_k are arbitrary, generally complex, constants. The matrices R_k have the form

$$(R_k)_{ab} = n_a^{(k)} m_b^{(k)}, \tag{24}$$

where the two-component vectors $m_a^{(k)}$ are given by

$$m_a^{(k)} = m_{0b}^{(k)} \left[\psi_0^{-1}[\mu_k, \rho, z)\right]_{ba} \tag{25}$$

with arbitrary constants $m_{0b}^{(k)}$, and the vectors $n_a^{(k)}$ are solutions of the algebraic equations,

$$\sum_{l=1}^{n} \Gamma_{kl} n_a^{(l)} = \mu_k^{-1} m_c^{(k)} (g_0)_{ca}, \tag{26}$$

where the symmetric matrix Γ_{kl} is defined by

$$\Gamma_{kl} = m_c^{(k)} (g_0)_{cb} m_b^{(l)} (\rho^2 + \mu_k \mu_l)^{-1}. \tag{27}$$

All this leads to the following "n-soliton solution" of the nonlinear equations (19):

$$g = \psi(0) = \chi(0)\psi_0(0) = \left(I - \sum_{k=1}^{n} R_k \mu_k^{-1}\right) g_0. \tag{28}$$

This matrix g satisfies Eq. (19) but not the condition (18) and therefore is not a solution of the Einstein equations. However, a simple normalization procedure, which involves the use of the differential equation satisfied by $\det g$, shows that the matrix

$$g^{(ph)} = \rho^{-n} \left(\prod_{k=1}^{n} \mu_k \right) g, \tag{29}$$

where g is given by (28) satisfies Eqs. (18) and (19). It corresponds to the n-soliton solution of Einstein equations. An important property of the soliton transformation is that each pole trajectory changes the sign of $\det g$ so that if we start from a physical background the number of pole trajectories must be even. Furthermore, if a pole trajectory is complex its complex conjugate must also be included to ensure that the final solution is real.

So far we have ignored the metric coefficient $f(\rho, z)$ but the differential equation for f is easily integrated when the n-soliton solutions for g are known. The final coefficient, see [9,10] for the details, is given by the following algebraic expression:

$$f^{(ph)} = f_0 \rho^{-n^2/2} \left(\prod_{k=1}^{n} \mu_k \right)^{n+1} \left[\prod_{k>l=1}^{n} (\mu_k - \mu_l)^{-2} \right] \det \Gamma_{kl}, \tag{30}$$

where the second product above is equal to 1 for $n = 1$.

The Kerr metric and some generalizations

That the Kerr solution could describe the exterior gravitational field of a spinning mass was already noticed in Kerr original paper [3], where it was shown that the two parameters of the solution represented the mass and the angular momentum with no higher order multipole moments, and that it reduced to the Schwarzschild solution when the angular momentum vanished. This by itself was an important achievement since we had for the first time a solution describing the exterior field of stationary rotating isolated sources. The search for a matching interior solution soon began but, so far, this search has been unsuccessful.

The Kerr solution has some important generalizations. One of them is the Kerr-NUT solution and its extensions [7]. The Kerr-NUT solution is not asymptotically flat but it reduces to the Kerr solution when the NUT parameter vanishes.

We may now use the soliton transformation of the ISM sketched above to derive explicitly the Kerr-NUT solution as a 2-soliton transformation on the Minkowski background. Since the pole trajectories come in pairs, this is the simplest solution we can obtain from the Minkowski background. Thus, let us take μ_1 and μ_2 with the w_k parameters in Eq. (23) written as

$$w_1 = z_1 + \sigma, \quad w_2 = z_2 - \sigma, \tag{31}$$

where we assume that the new parameters z_1 and σ are both real, which means that the pole trajectories are real. Instead of Weyl's coordinates ρ and z it is convenient to introduce Boyer–Lindquist coordinates r and θ as

$$\rho = \sqrt{(r-m)^2 - \sigma^2} \sin\theta, \quad z - z_1 = (r-m)\cos\theta, \tag{32}$$

where m is a new parameter whose value will be specified below. The pole trajectories take now the form

$$\mu_1 = 2(r - m + \sigma)\sin^2(\theta/2), \quad \mu_2 = 2(r - m - \sigma)\sin^2(\theta/2), \tag{33}$$

where we have chosen the plus signs in front of the square roots of (23) for the two pole trajectories.

Since the generating matrix ψ_0 for the Minkowski background is given by (21) we get from (25) the following components for the vectors $m_a^{(k)}$:

$$m_0^{(k)} = C_0^{(k)}, \quad m_1^{(k)} = C_1^{(k)}\mu_k^{-1}, \tag{34}$$

where now $k = 1, 2$ and $C_0^{(k)}$ and $C_1^{(k)}$ are arbitrary constants. Without loss of generality we can impose the following two conditions on these constants

$$C_1^{(1)}C_0^{(2)} - C_0^{(1)}C_1^{(2)} = \sigma, \quad C_1^{(1)}C_0^{(2)} + C_0^{(1)}C_1^{(2)} = -m. \tag{35}$$

The first equation is possible because there is a normalization freedom on these constants: $C_a^{(k)} \to \zeta^{(k)}C_a^{(k)}$. The second condition is the definition of m. We can also introduce two new arbitrary constants a and b, defined by

$$C_1^{(1)}C_1^{(2)} - C_0^{(1)}C_0^{(2)} = -b, \quad C_1^{(1)}C_1^{(2)} + C_0^{(1)}C_0^{(2)} = a. \tag{36}$$

It follows from (35) and (36) that

$$\sigma^2 = m^2 - a^2 + b^2. \tag{37}$$

Finally, we can substitute the above expressions for the vectors $m_a^{(k)}$ and the pole trajectories μ_k into (27), solve (26) for $n_a^{(k)}$ and find the metric coefficients $g_{ab}^{(ph)}$ from (28) and (29). The coefficient f of the metric (17) is easily obtained from equation (30). The resulting expressions for the metric contain only those combinations of the constants $C_a^{(k)}$ which are expressible through the three independent arbitrary parameters m, a and b according to (35)–(37). One also needs to write the line element $d\rho^2 + dz^2$ in the new r and θ variables and the result is,

$$\begin{aligned} ds^2 = {} & \omega(\Delta^{-1}dr^2 + d\theta^2) - \omega^{-1}(\Delta - a^2\sin^2\theta)(dt + 2ad\varphi)^2 \\ & + \omega^{-1}[4\Delta b\cos\theta - 4a\sin^2\theta(mr + b^2)](dt + 2ad\varphi)d\varphi \\ & - \omega^{-1}[\Delta(a\sin^2\theta + 2b\cos\theta)^2 - \sin^2\theta(r^2 + b^2 + a^2)^2]d\varphi^2, \end{aligned} \tag{38}$$

where ω and Δ are defined as

$$\omega = r^2 + (b - a\cos\theta)^2, \quad \Delta = r^2 - 2mr + a^2 - b^2. \tag{39}$$

These formulas are the standard expression for the Kerr-NUT solution in the Boyer–Lindquist coordinates if we take $t + 2a\varphi$ as the new time variable.

Since σ is real $m^2 + b^2 > a^2$ which corresponds to the usual Kerr-NUT solution with an event horizon. Note that if one uses complex conjugate poles, $\mu_1 = \mu_2^\star$ this is equivalent to assume that σ in equation (31) is pure imaginary and one obtains $m^2 + b^2 < a^2$, which corresponds to the Kerr-NUT solution with naked singularities. Note also that if opposite signs in front of the square roots in (23) are taken for the two pole trajectories, the final solution is equivalent to that obtained here; provided appropriate relations are written between the constants $C_a^{(k)}$ and the parameters σ, m a and b. When $b = 0$ the metric (38) is the Kerr metric. The metric (38) is not asymptotically flat in the presence of the NUT parameter b, when $b \neq 0$. When $a = 0$ the metric (38) is the Taub-NUT metric. Of course, $b = a = 0$ corresponds to the Schwarzschild metric, and $b = 0$ and $a = m$ corresponds to the "extreme" Kerr solution.

Another important generalization is the Tomimatsu–Sato solution [20] and some extensions. The Tomimatsu–Sato family of solutions are asymptotically flat and include an arbitrary real "deformation parameter", δ, that may be related to the quadrupole moment of a massive isolated source. The solutions for integer deformation parameters may be obtained as the n-soliton transformation on the Minkowski background and a limiting procedure of pole fusion. When $n = 4$ this procedure leads to the extended Tomimatsu–Sato metric with five arbitrary parameters that was obtained by Kinnersley and Chitre [15,16]. It includes the original two parameter Tomimatsu–Sato solution with deformation parammeter $\delta = 2$: after two of the five parameters are taken to be zero and two other parameters are combined to impose asymptotic flatness of the metric. Therefore the Tomimatsu–Sato solution corresponds to an overlap of two identical Kerr solutions centered at the same point. The same limiting procedure can be made to obtain an overlap of $n/2$ identical Kerr solutions. The distorsion parameter is then $\delta = n/2$ and the solution can be made asymptotically flat, with a suitable choice of parameters.

Many of the generalizations of the Kerr metric obtained by the solution-generating techniques derived from a search for stationary asymmetric solutions with higher multipole moments, which could describe the exterior gravitational field of realistic rotating bodies. Unlike the case of spherical symmetry, where Birkhoff's theorem guarantees that the exterior solution is uniquely described by the Schwarzschild metric, one expects that a rotating axisymmetric body should contain an arbitrary number of multipole moments.

The Kerr solution and black holes

The Kerr solution, however, acquired a new and unique significance with black hole physics. In particular, from a new understanding of black holes as describing the spacetime geometry resulting from the gravitational collapse of isolated bodies. The great importance of the Kerr metric is not only that it describes the entire spacetime geometry of a spinning black hole but that it describes, in some sense, the most general final state of an uncharged black hole. This is the result of work by several authors such as Price, Israel, Carter, Robinson and Hawking among others, which lead to the formulation of the uniqueness theorems. In fact, one would expect that a black hole formed by gravitational collapse would settle down to a stationary final state. The uniqueness theorems then conclude that a stationary vacuum black hole is uniquely described by the Kerr metric.

The chain of theorems leading to this result may be summarized as follows. It begins with Hawking demonstration that the event horizon of a stationary black hole has S^2 topology, or more precisely, that the intersection of the horizon with a Cauchy surface has the topology of a two-sphere. This is followed by the result also proved by Hawking [21] that the event horizon of a stationary asymptotically flat spacetime is a Killing horizon, both in vacuum and in Einstein–Maxwell; see Hawking and Ellis book [22] for a complete exposition of these results. The corresponding Killing vector field may be normal to the horizon or not. When it is normal the spacetime is static and thus spherically symmetric and, consequently, it is described by the Schwarzschild solution. When it is not normal the spacetime is axisymmetric.

The chain is finally closed with theorems due to Carter [23] and Robinson [24]; see Chandrasekhar's book [25] for a detailed demonstration of Robinson's version. This theorem establishes that a stationary and axisymmetric vacuum solution of Einstein equations with a regular convex event horizon which is asymptotically flat and not singular outside the horizon is uniquely specified by two parameters which correspond to the mass and the angular momentum. Since the Kerr metric satisfies the conditions of this theorem it provides a unique specification of a stationary black hole.

It is interesting that, in recent years, research in string theory has stimulated the search for black hole solutions in higher dimensions, and that solution-generating techniques such as the ISM are playing and important role in this search. Thus, in five dimensions the analog of the Kerr solution was found by Myers and Perry [26]. It is a vacuum stationary axisymmetric and asymptotically flat metric representing a rotating five dimensional black hole with an event horizon of topology S^3. Remarkably the uniqueness of the four dimensional Kerr black hole is somewhat relaxed in higher dimensions, thus Emparan and Reall [27] discovered a black ring solution: a vacuum stationary axisymmetric and asymptotically flat solution in five dimensions that has an event horizon of topology $S^1 \times S^2$. The black ring rotates along the direction of the S^1. Both, the rotating five dimensional black hole and the rotating black ring solutions have been recently generated and generalized by the ISM [28,29].

Part 3: a brief history of the discovery of the Kerr solution and the Kerr–Schild metrics

By Roy Patrick Kerr

The motivation for my study of algebraically special (AS) was very simple. The only known *physically significant* solution of the Einstein equations was Schwarzschild. The Weyl solutions generalised this but were still static and added little to our understanding of stellar structures. Relativists had been searching for a stationary rotating solution for decades but had found nothing. The Schwarzschild metric is AS but *static*, so it was possible that there were interesting *stationary* AS metrics which were asymptotically flat (like Schwarzschild) but necessarily rotating.

When I realised that one previous attempt to find all rotating AS spaces had foundered and another seemed to have stopped at the static ones, I rushed headlong into the search for solutions that fit the conditions of the previous paragraph, found the now standard coordinates for algebraically special metrics in spring 1963, solved the easy field equations and then calculated the remaining hard ones. Since these were impossibly difficult, I then "solved Killing's equation", looking for all possible symmetries. There are two distinct types of Killing vectors with very simple canonical forms, $\mathrm{d}/\mathrm{d}\phi$ and $P\mathrm{d}/\mathrm{d}u$. I was fairly sure that if there were two commuting symmetries one of which was asymptotically timelike then canonical coordinates could be chosen for which each of the above vectors is a symmetry. Instead of wasting time proving this conjecture, I assumed it to be true and forged ahead. The complete solution with these conditions contains four parameters. These are m, a, the NUT parameter and a further garbage parameter. Since the metric was not asymptotically flat when either of the last two parameters is nonzero, I discarded them, kept m and a and the rest is history.

This metric was originally found in a rather nasty coordinate system, but it did have a visible Kerr–Schild structure. I was able to transform the flat space part to $dx^2 + dy^2 + dz^2 - dt^2$, giving a final metric that was asymptotically flat. At this point I told Alfred that I was about to calculate its angular momentum. He came to my office and sat smoking his pipe while I compared its asymptotic expansion with the well-known linear approximation for any slow moving body (time independent in our case). When I found that the angular momentum is nonzero he was even more excited than myself.

When this solution was first found it was clear to everyone that it must have an event horizon for small a since it was a generalisation of Schwarzschild. For this the light cones at $r = m$ are tilted inwards. A small perturbation of the metric, $a = 10^{-3}$ m say, could not change this. Physicists knew that a spherically symmetric body that collapsed had to fall inside the event horizon and quickly become singular. However, it had not been known what would happen if the body had spin. Some thought that this would stop the formation of a horizon and the collapse to a singularity. After 1963 we knew that this did not happen for small angular momentum at least.

The first Texas Symposium was held in Dallas on 18 December 1963, a few months after the Kerr metric was found. Before I went I calculated the geodesics up and down the axis of rotation and found that there were two distinct horizons. I also attempted to calculate these off the axis, but I made a mistake and got the wrong surfaces, perhaps because I was in a rush and K–S coordinates are not the ones to use. This attempt was published in Kerr [30]. I stated there that the two horizons are the roots of the equation $r^4 + a^2z^2 = 2mr^3$, instead of the correct equation $r^4 + a^2r^2 = 2mr^3$.

Although I did not discuss the topology in Kerr [3], I did point out in Kerr [30] that the points on the axial ring $x^2 + y^2 = a^2,\ z = a$ are the branch points of the metric, and that the disc bounded by this ring is the corresponding branch cut. The two branch sheets are each asymptotically flat. The mass is positive on one branch and negative on the other, and the two event horizons are both in the first sheet.

One morning in autumn, 1963, I looked to see if there were more general situations where an AS metric splits in the same way as the Kerr metric does into two parts, $ds_o^2 + m_o k^2$, where ds_o^2 is itself an Einstein space (but not necessarily Lorentzian) and m_o is an arbitrary constant, i.e. where a first-order approximation is actually exact.

The final result appeared to be that ds_o^2 is necessarily Lorentzian and the second term contains an arbitrary function of a complex variable. I did not check this result, just put the calculations aside.

Sometime after Christmas, 1963, Jerzy Plebański visited Austin. Alfred held a party for him during which I heard him talking to Jerzy about metrics of the K–S type. I said "This may be rubbish because it was a fairly rough calculation, but I think I know of a generalisation of spinning Schwarzschild with an extra function of a complex variable".

Alfred was quite excited so he and I went into his office and calculated the connexion and the easy curvature components for a metric of K–S type. The simplest equation, $R_{ab}k^a k^b = 0$, reduced to "**k** is geodesic".[6] We then calculated those components of the Riemann tensor which determine whether the metrics were AS. Bingo! They were! This meant that even if I had made a mistake, we could soon find all (diverging) K–S metrics. We redid my previous calculations next day and found that they were correct.

Since the work on algebraically special spaces had not been published, we then worked everything out from first principles. This was a much better treatment anyway, one that could be extended to Einstein–Maxwell fields. The original derivation was published much later in Debney et al. [4].

Sometime in early 1964 I added an electromagnetic field to the Kerr–Schild ansatz. The analysis for this is given in Debney et al. [4]. The first stumbling block was that I could not prove that the special null vector was geodesic (and still cannot). I therefore took this as an additional assumption. The "easy" equations then showed that the Einstein–Maxwell field depends on three functions, M, A and γ, all constant along the rays and restricted by the "hard" field equations. I could not solve the latter unless $\gamma = 0$ so I temporarily took this as an additional assumption and continued. This led to the complete charged Kerr–Schild metrics including, of course, charged Kerr. The null congruence for the latter is the same as the Kerr congruence but the EM field depends on an arbitrary function of a complex variable. If this is a constant, $e + ib$, then e is the charge and b the magnetic charge. If it is a more complicated function then the EM field will probably be singular.[7]

At that point I turned the problem over to G.C. Debney, who had just completed his prelims for a Ph.D., to see if he could solve the problem when $\gamma \neq 0$. Naturally, it took some time for him to get up to speed, but in the end it became clear that the three of us could not solve the general problem. George then shifted to "Symmetries in Algebraically Special Spaces", extending the analysis that was used to find the Kerr metric to cover all possible symmetry groups [33]. This became the basis for his Ph.D. thesis.

Each year the American Mathematics Society holds a symposium on applied mathematics. The XVII one, held in New York on 20–23 April 1964, included two talks on relativity, one by A. Lichnerowicz and one by myself, but written by Alfred, on algebraically degenerate solutions of the Einstein equations [32]. This was to an audience of applied mathematicians and emphasised the K–S metrics. The manuscript had to

6 This does not work for charged K–S.

7 In Kerr and Wilson [34] it is proved that the only K–S metric that is nonsingular at infinity is Kerr. I suspect the same is true for charged K–S and that the proof is very simple, but I have not tried to prove it.

be written before the conference and so anything in it was discovered by early April at the latest. It included the following sentence:

Together with their graduate student, Mr. George Debney, the authors have examined solutions of the nonvacuum Einstein–Maxwell equations where the metric has the form (2.1).[8] Most of the results above apply to this more general case. This work is continuing.

This tells me that the charged metrics were known by the end of March since they were discovered before George joined the project, and that was before the Symposium paper was written. Later that year, Newman went ahead and published his own version, in which he considered the simplified problem where the background metric is Kerr and the electric field is generated by a real constant rather than an arbitrary complex function.

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Golden Oldie Editor's comment: Professor Kerr gave his permission to announce the following (this is a quotation from his email to A.K.):

"Fulvio Melia and I are writing a book 'Cracking the Einstein Code' which will probably be published by Farrar Straus and Giroux. It will give the history of Black holes from early last century to the present, with several chapters on the events of the 60's. It will be a popular book (we hope) for the general public, rather than for physicists."

Roy Patrick Kerr: a brief biography

By Andrzej Krasiński, compiled from Refs. [35] and [36].[9]

Roy Kerr was born on May 16, 1934, in Kurow, New Zealand.

He received his M.S. in 1954 from the New Zealand University (now dissolved) and his Ph.D. in 1960 at the Cambridge University.

Kerr's colleague (see Ref. [35]) describes his early years as follows: "His undergraduate career was not given wholly to mathematics and science; he admits to having played a lot of billiards, and in 1952 represented his College in boxing at the Easter Tournament, as a light-welterweight. I recall W.W. Sawyer, then a lecturer at Canterbury, expressing alarm and dismay over Roy's pugilism, on the ground that he didn't want the best brain he'd encountered in a student scrambled by a well-thrown punch; but history seems to confirm that Roy came to no lasting harm over it.

In 1955 he received a M.Sc with first class honours, and went to Cambridge with a Sir Arthur Sims Empire Scholarship. He was awarded a Ph.D. in 1960, for a thesis on the equations-of-motion problem in general relativity. This work appeared in a

[8] The usual Kerr–Schild form.

[9] Repeated attempts by the Editor of this series to get a first-hand autobiography of Roy Kerr were unsuccessful.

series of three . . . papers in Nuovo Cimento, and although later overshadowed by the Kerr metric, was extensively cited. He went on to a post-doctoral post at Syracuse University, and then to work with a US Air Force relativity group at Wright-Patterson Field, in Ohio. The USAF were interested in antigravity devices; one of the tasks of the relativity group was to assess and report on such devices proposed to it by inventors."

In 1962 Kerr joined the relativity group at the University of Texas in Austin. He found the famous Kerr solution in his first year there. The solution was publicly presented in 1963, during the First Texas Symposium on Relativistic Astrophysics in Austin, but did not seem to be properly appreciated by the public. Universal recognition came later. S. Chandrasekhar, in his Ryerson Lecture of 1978, said: "In my entire scientific life, extending over forty-five years, the most shattering experience has been the realization that an exact solution of Einstein's equations of general relativity, discovered by the New Zealand mathematician Roy Kerr, provides the absolutely exact representation of untold numbers of massive black holes that populate the universe." (quotation after Ref. [35]).

Kerr returned to New Zealand in 1971. In 1983 he became the head of the Department. His colleague describes his activities so: "Roy's style as HOD was at once uncompromising and dashing; in a series of moves which affronted some of our colleagues in other departments, who had grown comfortable with the traditional Canterbury view that Mathematics should be a low-cost department devoted to service teaching, he contrived to reduce student–staff ratios, encourage research, and equip the department with a computer system at the sort of cost hitherto associated with spectrographs. Morale rose markedly. In many respects Roy was an unusual figure in University administration; he had very little patience for the practice of wrapping self-interest up in politically correct pieties, and was perfectly willing to offend entrenched privilege. But he was successful, and we are the better for his efforts, and we love him for them." [35]

Roy Kerr retired from his position as Professor of Mathematics at the University of Canterbury in February 1993.

He received many awards, culminating in the Hughes Medal of the Royal Society of London in 1984.

Roy Kerr's extra-scientific activities are described in some detail in Ref. [37]. The most notable of them is his long career as "a national representative and champion bridge player".

A biography of Alfred Schild was published by Lawrence Shepley in *Gen. Relativ. Gravit* **8**, 955 (1977), shortly after his death. See also Amanda Oren, http://www.tsha.utexas.edu/handbook/online/articles/SS/fsc48.html.

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