GEOMETRY OF THE KERR BLACK HOLES

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Abstract. In this paper, we will explore the geometry of the Kerr spacetime, a solution to the Einstein Equation in general relativity. We will first give physical and mathematical motivations, and sketch the constructions for the theory of general relativity using objects and ideas from differential geometry such as Lorentz vector spaces, geodesics, and Ricci curvature tensor. A short treatment of several solutions to the Einstein Equation, including the Minkowski, Schwarzschild, and Kerr metric tensors, will be followed by a more detailed analysis of the Kerr solution, which leads to the mathematical model of rotating black holes and interesting possibilities such as time machine.

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1. Introduction

The theory of general relativity, formulated by Albert Einstein in 1915, is among the greatest mathematical and physical feats of the twentieth century. The model provides a general and complete description of the structure of spacetime with gravitation as the main element. The idea is simple yet groundbreaking: instead of the separate Newtonian time and space, these concepts are fused into a single connected four-dimensional manifold called "spacetime", whose particular characteristics depend on the physical content of interest. Very few solutions to the Einstein Equation have been found; most notable results were produced by Minkowski (1908 - solution for the case of special relativity), Schwarzschild (1916), and Kerr (1966). Physical predictions posed by these models, such as the precession of Mercury, black holes, and gravitational waves, have been confirmed over the course of the past hundred of years. With so much potential yet to be uncovered, general relativity is guaranteed to receive a significant amount of attention in the scientific community.

In this paper, we will introduce the theory of general relativity and discuss in detail one solution to the Einstein Equation—the Kerr metric, which physically

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manifests in the form of rotating black holes. Prerequisite knowledge assumed by the author of this paper is general manifold theory, basic tensor algebra, and tensor analysis on manifolds (brief summaries of these subjects can be found in [1, 2]; detailed treatments are contained in [3, 4]). Sections 2 and 3 will include an introduction to Lorentz vector spaces and semi-Riemannian geometry—necessary mathematical tools for building the model of general relativity, which will be motivated and constructed in section 4. In addition, the Einstein Equation, the central model of relativity theory, and its solutions will be discussed in sections 4 and 5. The metrics for the Minkowski, Schwarzschild, and Kerr spacetimes will be briefly analyzed in the section 5. The final section of this paper focuses on the geometry of the Kerr spacetime and explores causality and time machine possibility in the model.

2. Lorentz Vector Spaces

Let V be a real vector space of dimension n.

Definition 2.1. A *scalar product* g on V is a symmetric, nondegenerate bilinear form on V.

From this definition, we define the notion of *orthogonality*: given two vectors $v, w \in V$, v and w are orthogonal (write $v \perp w$) if $g(v, w) = 0$. The nondegerate property of the scalar product implies that if $v \perp w$ for all $w \in V$ then $v = 0$. A characteristic that identifies nondegeneracy is that there exists a basis $v_1, ..., v_n$ for V such that the matrix $g_{ij} = g(v_i, v_j)$ has a nonzero determinant. A common notation for $g(v, w)$ is $\langle v, w\rangle$; these notations will be used interchangeably throughout the paper. A vector space equipped with a scalar product is called a *scalar product space*.

Definition 2.2. The *norm* $|v|$ of v is defined to be $|v| = |\langle v, v\rangle|^{1/2}$.

If $|u| = 1$, then we call u a *unit vector*. A set of mutually orthogonal unit vectors is called *orthonormal*.

Proposition 2.3. *Every scalar product space $V \neq 0$ has an orthonormal basis. Moreover, any orthonormal set $e_1, ..., e_k$ in V can be enlarged to an orthonormal basis for V.*

Given an orthonormal basis $e_1, ..., e_n$ for V, there exist n numbers $\epsilon_i = \pm 1$ (called *signs*) such that $\langle e_i, e_j\rangle = \delta_{ij}\epsilon_i$.

Proposition 2.4. *Every orthonormal basis for V has the same signs $\epsilon_1, ..., \epsilon_n$ (up to permutation).*

The proofs for these propositions are analogous to standard arguments in linear algebra. The n-tuple $(\epsilon_1, ..., \epsilon_n)$ is called the *signature* of V (by convention, negative signs are listed first). The number of negative signs in the signature of V is called the *index* of V, denoted by ν. An important application of the signs is the *orthonormal expansion*: for every $v \in V$, we can write $v = \sum \epsilon_i \langle v, e_i\rangle e_i$.

Definition 2.5. A *Lorentz vector space* V is a scalar product space of dimension $n \geq 2$ and index $\nu = 1$.

The signature of a Lorentz vector space, called the *Lorentz signature*, is hence $(-1, +1, ..., +1)$.

Let V be a Lorentz vector space of dimension n.

Definition 2.6. Given $v \in V$, we say that the vector v is

- *spacelike* if $\langle v, v\rangle > 0$ or $v = 0$,
- *null* (or *lightlike*) if $\langle v, v\rangle = 0$ and $v \neq 0$,
- *timelike* if $\langle v, v\rangle < 0$.

The type to which v belongs is called its *causal character*.

Regarding this definition, the following lemma is perhaps the most important fact about a Lorentz vector space for the construction of the model of general relativity.

Lemma 2.7. *A vector orthogonal to a timelike vector z in V is spacelike.*

Proof. Define $u = z/|z|$. Then u is a unit vector. By Proposition 2.3, choose an orthonormal basis $u, e_1, ..., e_{n-1}$ for V. If $v \perp z$, then $v \perp u$. Thus the orthonormal expansion for v is $v = \sum_{i=1}^{n-1} v_i e_i$ for components v_i. Hence $\langle v, v\rangle = \sum_{i=1}^{n-1} {v_i}^2 \geq 0$; the equality implies that $v = 0$. We conclude that v is spacelike. □

Corollary 2.8. *The space $z^\perp = \{v \in V : v \perp z\}$ is an inner product space (a vector space equipped with a positive-definite scalar product). Moreover, $V = \mathbb{R}z \oplus z^\perp$.*

Proof. Given $u = z/|z|$, $u^\perp$ is a subspace of V spanned by $e_1, ..., e_{n-1}$, as defined in the previous argument. The scalar product in $u^\perp$ is positive-definite, hence it is an inner product space. Given $v \in V$, orthonormal expansion gives $v = au + \sum_{i=1}^{n-1} v_i e_i$. So $v \in \mathbb{R}u \oplus u^\perp$. Thus $V \subset \mathbb{R}u \oplus u^\perp$. Since the converse inclusion is true by definition, $V = \mathbb{R}u \oplus u^\perp$. Because $u^\perp = z^\perp$, the claim is proved. □

The upshot of this corollary is that it allows us to express an arbitrary vector $v \in V$ as a sum $v = au + x$, where u is a timelike unit vector and $x \in u^\perp$. Then we have a valuable identity: $\langle v, v\rangle = -a^2 + |x|^2$.

Definition 2.9. A *cone* in V is a subset closed under multiplication by a positive scalar. In particular, the *nullcone* Λ of V is the set of all null vectors in V.

Strictly speaking, the nullcone of V is not a cone in V; however, it is the disjoint union of two opposite cones in V.

Lemma 2.10. *The nullcone Λ of V has two disjoint components Λ^+ and Λ^-, with $\Lambda^- = -\Lambda^+$. Each component is a cone in V diffeomorphic to $\mathbb{R}^+ \times S^{n-2}$.*

Proof. Let u be a timelike vector in V. By Corollary 2.8, for every $v \in \Lambda$, write $v = au + x$, where $x \in u^\perp$; notice that though the full orthonormal expansion is not unique (dependent on the choice of basis for $u^\perp$), the component $a = -\langle v, u\rangle$ is for each v since u is fixed. Define $\Lambda^+ = \{v : a > 0\}$ and $\Lambda^- = \{v : a < 0\}$. One can check that Λ, Λ^+, and Λ^- are all cones, and that $\Lambda^- = -\Lambda^+$.

Next, we want to show that Λ^+ is diffeomorphic to $\mathbb{R}^+ \times S^{n-2}$, where S^{n-2} is the unit sphere in the inner product space $u^\perp \cong \mathbb{R}^{n-1}$. Let $v \in \Lambda^+$. By Corollary 2.8, write $v = au + x$ where $x \in u^\perp$. Since v is null, $|x|^2 = a^2$; thus $|x| = a$. Define a map $f\colon \Lambda^+ \to \mathbb{R}^+ \times S^{n-2}$ by $v \mapsto (a, x/a)$. A map in the reverse direction is achieved by sending $(r, y) \in \mathbb{R}^+ \times S^{n-2}$ to $r(u + y)$, which is indeed a null vector. One can check that the two maps are inverses.

For $n \geq 3$, the sphere S^{n-2} is connected. Hence the disjoint sets Λ^+ and Λ^- are actually the connected components of the full nullcone.

For $n = 2$, since $S^0 = \{\pm 1\}$, Λ^+ and Λ^- are not connected; each consists of two rays. □

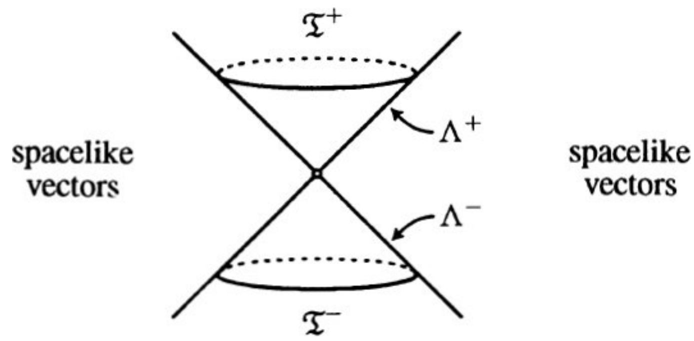

FIGURE 1. Nullcones and timecones in a Lorentz vector space [1].

Definition 2.11. The set $\mathfrak{T}$ of all timelike vectors in V is called its *timecone*.

Similar to the nullcone of V, its timecone is strictly speaking not a cone in V.

Lemma 2.12. *The timecone $\mathfrak{T}$ in V has two disjoint components $\mathfrak{T}^+$ and $\mathfrak{T}^-$, with $\mathfrak{T}^- = -\mathfrak{T}^+$. Each component is an open convex cone, and (given consistent choices of signs)*

$$\partial\mathfrak{T}^+ = \Lambda^+ \cup \{0\},$$
$$\partial\mathfrak{T}^- = \Lambda^- \cup \{0\}.$$

Proof. Let u be a unit timelike vector. For every vector v, write $v = au + x$, with $x \in u^\perp$. Hence we have v is timelike iff $\langle v, v\rangle = -a^2 + |x|^2 < 0$ or $a^2 > |x|^2$.

Define $\mathfrak{T}^+ = \{v \in \mathfrak{T} : a > 0\}$ and $\mathfrak{T}^- = \{v \in \mathfrak{T} : a < 0\}$. One can check that $\mathfrak{T}$, $\mathfrak{T}^+$, and $\mathfrak{T}^-$ are cones, with $\mathfrak{T}^- = -\mathfrak{T}^+$. Since they are defined by inequalities, all three cones are open sets of V.

We want to show that $\mathfrak{T}^+$ is convex (a similar argument can be used for $\mathfrak{T}^-$). Define $H_v = \{w \in V : \langle w, v\rangle < 0\}$ where $v \in V$. Note that H_v is convex for every v; that is, if $w_1, w_2 \in \mathfrak{T}^+$ then the vector $w_t = tw_1 + (1-t)w_2$ is also in $\mathfrak{T}$ for all $0 < t < 1$. Since an intersection of convex sets is convex, it suffices to show that $\mathfrak{T}^+$ is the intersection of the sets H_{u+e} for the unit vector u and all (spacelike) unit vectors $e \perp u$.

Suppose that $v \in \mathfrak{T}^+$. Write $v = au + x$ as before, then $a > |x|$. For any $e \perp u$, by the Schwarz inequality we have

$$\langle v, u+e\rangle = \langle au + x, u + e\rangle = -a + \langle x, e\rangle \leq -a + |x| < 0.$$

Hence $v \in \bigcap H_{u+e}$.

To show that $v \in \bigcap H_{u+e}$ implies that $v \in \mathfrak{T}^+$, first suppose that $v \neq \lambda u$. Then $v = au + x$, with $0 \neq x \perp u$. In particular, $v \in H_{u+x/|x|}$, hence

$$-a + |x| = \langle au + x, u + x/|x|\rangle < 0.$$

Thus $v \in \mathfrak{T}^+$.

□

The corollary below follows nicely from the previous lemma:

Corollary 2.13. *Two timelike vectors z, z' are in the same timecone iff $\langle z, z'\rangle < 0$.*

Proof. Consider the function $f\colon \mathfrak{T} \to \mathbb{R}$ given by $f(v) = \langle v, z\rangle$. Now $f(z) < 0$, and the timecones $\mathfrak{T}^{\pm}$ are connected, hence $f < 0$ on the timecone that contains z. Assume, without lost of generality, that $f < 0$ on $\mathfrak{T}^+$. Thus if z' is also in $\mathfrak{T}^+$, $\langle z, z'\rangle = f(z') < 0$. But if $z' \in \mathfrak{T}^-$, then $-z' \in \mathfrak{T}^+$; so $\langle z, z'\rangle = -\langle z, -z'\rangle > 0$. □

A vector is said to be *nonspacelike* or *causal* if it is not spacelike. The *causal cones* in V are defined to be $\mathfrak{T}^+ \cup \Lambda^+$ and $\mathfrak{T}^- \cup \Lambda^-$ (with consistent signs). The previous corollary can be generalized to the following statement: a timelike vector z and a causal vector w are in the same causal cone iff $\langle z, w\rangle < 0$.

3. Semi-Riemannian Geometry

Given a smooth manifold M, a *metric tensor* g is a smooth covariant (2,0) tensor that assigns each point p of M a scalar product g_p on the tangent space $T_p(M)$.

Definition 3.1. A *semi-Riemannian manifold* is a smooth manifold M equipped with a metric tensor g.

If each g_p is a Lorentz scalar product, then g is a *Lorentz metric* and M is called a *Lorentz manifold*. This is the central mathematical subject in relativity theory.

On the domain $\mathcal{U}$ of a coordinate system $x^1, ..., x^n$ the components of the metric tensors are $g_{ij} = g(\partial_i, \partial_j) = \langle \partial_i, \partial_j\rangle$. Hence $g = \sum g_{ij} dx^i \otimes dx^j$. Equivalently, a metric tensor can be represented by its *line-element* q defined by $q(v) = \langle v, v\rangle$ for every tangent vector v to M. The line-element gives at each point p the associated quadratic form of g_p, hence we can reconstruct the metric tensor from its line-element by polarization. In terms of a coordinate system, $q = ds^2 = \sum g_{ij} dx^i dx^j$, where the juxtaposition of differentials is the ordinary multiplication of functions.

Definition 3.2. A mapping $\psi : M \to N$ of semi-Riemannian manifolds is an *isometry* if it is a diffeomorphism and preserves scalar products:

$$\langle d\psi(v), d\psi(w)\rangle = \langle v, w\rangle \text{ for all tangent vectors } v,\, w \text{ to } M.$$

Given vector fields V, W on a semi-Riemannian manifold M, we want to define a new vector field $D_V W$ whose value at each point p gives some notion of the vector rate of change of W in the direction V_p. The following theorem guarantees the existence and uniqueness of such an operation.

Theorem 3.3 (Fundamental Theorem of Semi-Riemannian Geometry)**.** *Given a semi-Riemannian manifold M, let $\mathfrak{X} = \mathfrak{X}(M)$ be the set of smooth vector fields on M and $\mathfrak{F} = \mathfrak{F}(M)$ be the set of smooth real-valued functions on M. Then there exists a unique function D: $\mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ such that*

(1) $D_V W$ is $\mathfrak{F}$-linear in V,
(2) $D_V W$ is $\mathbb{R}$-linear in W,
(3) $D_V(fW) = V[f]W + f D_V W$ for all $f \in \mathfrak{F}$,
(4) $[V, W] = D_V W - D_W V$,
(5) $X\langle V, W\rangle = \langle D_X V, W\rangle + \langle V, D_X W\rangle$.

This unique function D is called the *Levi-Civita connection* of M. Given two vector fields V, W, $D_V W$ is the *covariant derivative* of W with respect to V.

Given a coordinate system $x^1, ..., x^n$ of a neighborhood $\mathcal{U} \in M$, the Levi-Civita connection can be written as $D_{\partial_i}(\partial_j) = \sum \Gamma^m_{ij} \partial_m$ for all i, j, where the *Christoffel* *symbols* Γ^k_{ij} are given by

$$\Gamma^k_{ij} = \frac{1}{2}\sum g^{km}\Big[\frac{\partial g_{mi}}{\partial x^j} + \frac{\partial g_{mj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^m}\Big].$$

Let α be a curve in M and V be a vector field on α. If $\alpha' \neq 0$, we define the *covariant derivative of V along α* to be $V' = D_{\alpha'}V$. In terms of coordinates

$$V' = \sum\Big\{\frac{dV^k}{ds} + \sum_{ij}\Gamma^k_{ij}\frac{d(x^j\circ\alpha)}{ds}V^i\Big\}\partial_k.$$

The vector field V is said to be *parallel along* α given $V' = 0$. For parameter values s_1, s_2, we say $V(s_2)$ is obtained from $V(s_1)$ by a *parallel translation* along α. The tangent α' of α is a vector field on α, and the covariant derivative of α' is the *acceleration* $\alpha'' = D_{\alpha'}\alpha'$ of the curve.

In this framework, straight lines in Euclidean space are generalized to geodesics in a semi-Riemannian manifold.

Definition 3.4. A *geodesic* in a semi-Riemannian manifold M is a curve γ in M with zero acceleration: $\gamma'' = 0$.

In terms of coordinates, such a curve is described by the *geodesic equations*:

$$\frac{d^2x^k}{ds^2} + \sum\Gamma^k_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0 \text{ for } k = 1, ..., n$$

Here the coordinate functions $x^i\circ\alpha$ of the curve are abbreviated to x^i. By the existence and uniqueness theorem for solutions to differential equations, for every tangent vector v to M there is a unique geodesic $\gamma_v\colon I \to M$ with initial velocity $\gamma_v'(s_0) = v$. If every geodesic can be extended over the entire real line, M is said to be *geodesically complete*.

The most important invariant of a semi-Riemannian manifold is its curvature. Various notions of curvature offer different geometric interpretations of the manifold.

Definition 3.5. The *Riemann curvature tensor* of a semi-Riemannian manifold M is the function $R\colon \mathfrak{X}\times\mathfrak{X}\times\mathfrak{X}\to\mathfrak{X}$ given by

$$R_{XY}Z = D_X(D_YZ) - D_Y(D_XZ) - D_{[X,Y]}Z$$

where D is the Levi-Civita connection of M.

Proposition 3.6 (Symmetries of curvature)**.** *For $x, y, v, w \in T_p(M)$,*

(1) $R_{xy} = -R_{yx}$*;*
(2) $\langle R_{xy}v, w\rangle = -\langle R_{yx}w, v\rangle$*;*
(3) $R_{xy}z + R_{yz}x + R_{zx}y = 0$*;*
(4) $\langle R_{xy}v, w\rangle = \langle R_{vw}x, y\rangle$*.*

Given a frame field $\{E_i\}$, components of the curvature tensor are $R^i_{jkl} = \omega^i(R_{E_kE_l}E_j)$; hence by the duality formula, it is deduced that $R_{\partial_k\partial_l}(\partial_j) = \sum R^m_{jkl}\partial_m$. Due to the symmetries of curvature, there exists exactly one nonzero contraction of the curvature tensor; we define such quantity to be the *Ricci curvature tensor Ric*.

Lemma 3.7. *Given an arbitrary frame field $\{E_i\}$, the components of the Ricci curvature tensor are $R_{ij} = Ric(E_i, E_j) = \sum R^m_{imj}$.*

In the theory of relativity, the Ricci curvature tensor is of utmost importance due to its direct appearance in the Einstein equation, the central model of the theory. Another geometric quantity that makes appearance in the Einstein equation is the *scalar curvature*, defined as the contraction of Ric, formally $S = \sum g^{ij} R_{ij}$.

The information contained in the curvature tensor R of a semi-Riemannian manifold M can also be extracted from the *sectional curvature* K, defined as follows:

Definition 3.8. For a semi-Riemannian manifold M, the sectional curvature K of M is a real-valued function on the set of all nondegenerate 2-planes Π tangent to M, such that for any basis $v, w \in \Pi$

$$K(\Pi) = \frac{-\langle R_{vw}v, w\rangle}{\langle v,v\rangle\langle w,w\rangle - \langle v,w\rangle^2}.$$

If S is a surface, then for $p \in S$ the sectional curvature K of T_pS is the Gaussian curvature of S at p, as classically defined in differential geometry of surfaces.

Another consequence of the symmetries of the curvature tensor R is the *Bianchi curvature identity*, which asserts a symmetry of its covariant differential DR:

$$(D_X R)_{YZ} + (D_Y R)_{ZX} + (D_Z R)_{XY} = 0.$$

The last concept in differential geometry to be discussed here is the *Killing vector field*. Let X be a vector field on a semi-Riemannian manifold M. Assume for simplicity that X is complete, so its flow $\{\psi_s\}$ is globally defined. Then X is a Killing vector field provided each stage $\psi_s\colon M \to M$ is an isometry. Thus X is also called an infinitesimal isometry. For example, on the circle S^1 any vector field of constant length is Killing since its flow consists of rotations of S^1.

If X is not complete, the definition above remains valid using flows defined only locally.

Proposition 3.9. *The following conditions on a vector field X are equivalent:*

(1) X is a Killing vector field.
(2) $X\langle V,W\rangle = \langle [X,V],W\rangle + \langle V,[X,W]\rangle$ for all V,W.
(3) $\langle D_V X, W\rangle = -\langle D_W X, V\rangle$ for all V,W.

As the definition above shows, Killing vector fields are very useful in finding isometries $M \to M$. They also have a vital role in the study of geodesics, because of the following conservation lemma.

Lemma 3.10. *If X is a Killing vector field on M and γ is a geodesic, then the scalar product $\langle X, \gamma'\rangle$ is constant along γ.*

Proof. Since $\gamma'' = 0$, $(d/ds)\langle X,\gamma'\rangle = \langle D_{\gamma'}(X), \gamma'\rangle$, which, by criterion (3) in the preceding proposition, is zero. □

If M is connected, then any isometry ψ of M is completely determined by its differential map $d\psi_p$ at a single point $p \in M$. The corresponding Killing fact is that a Killing vector field X is completely determined by its values on any arbitrarily small neighborhood of a single point.

4. General Relativity

Let M be a connected four-dimensional Lorentz manifold. Since each tangent space to M is a Lorentz vector space, for every $p \in M$, we may have two components of the timecone $\mathfrak{T}$ in T_pM. The act of selecting one such component for each $p \in M$

in a continuous manner to be the "future" is called *time-orienting* M. The selected components are called *future timecones*, the others *past timecones*.

Definition 4.1. A *spacetime* is a connected time-oriented four-dimensional Lorentz manifold.

A timelike tangent vector z is *future-pointing* if it is in a future timecone, otherwise *past-pointing*. The nullcone in the boundary of each future timecone is a future nullcone, and its vectors are future-pointing. A causal (nonspacelike) curve or vector field is future-pointing if all its tangent vectors are future-pointing.

Definition 4.2. A *material particle* α in a spacetime M is a future-pointing timelike curve $\alpha\colon I \to M$. The *proper time* τ of α is its arc length function, and its mass is $m = |\alpha'| > 0$.

Definition 4.3. A *lightlike particle* γ in a spacetime M is a future-pointing null geodesic.

It is geometrically true, for reasons explained later, that light "travels in straight lines." A lightlike particle γ is massless, i.e. $m = 0$, so it has no proper time.

Similarly, we can define basic physical notions such as velocity, momentum, and energy in this framework. Let α be a particle. The tangent vector $\alpha' = d\alpha/ds$ is called its *energy-momentum* 4-*vector*, denoted by $\boldsymbol{p}$. Physically, this vector is the relativistic fusion of the energy and momentum in Newtonian physics. It is possible to reparametrize α by its proper time τ; we denote the reparametrized curve by $\tilde{\alpha}$. Then $\boldsymbol{p} = d\alpha/ds = md\tilde{\alpha}/d\tau$ since $m = |d\alpha/ds| = d\tau/ds$. The unit vector $d\tilde{\alpha}/d\tau$ is called the 4-*velocity* of the particle; it is the fusion of the Newtonian energy per unit mass and Newtonian velocity. Note that for a lightlike particle γ, $\boldsymbol{p} = \gamma'$.

The most important idea in the theory of general relativity is that gravity curves spacetime. More generally, physical matter determines the curvature of spacetime, while the geometry of spacetime directs the laws of motion. For some physical motivations for this idea, consider the following Newtonian scenario: let there be two spaceships S_1 and S_2, and a star X in the spacetime M. Assume that (1) both spaceships have same initial nonzero velocities that do not align with the center of X, (2) S_1 is close to X while S_2 is infinitely far from X, and (3) the only force acting on the ships is the gravitational force by X. Newtonian physics tell us that gravity is attractive, and that its magnitude is roughly inversely proportional to the square of the distance. Hence S_1, under a strong gravitational effect, moves in an elliptic trajectory in M. On the other hand, the force acting on S_2 is negligible, so S_2 moves in a straight line. However, it is impossible for an astronaut on a ship to tell which ship he is on. This example was first noticed by Albert Einstein, who thought that physically imdiscriminate objects should not be mathematically different. In other words, the straight trajectory of S_2 and the curved trajectory of S_1 should be mathematically the same, since both ships are affected only by gravitation. This interesting observation results in the birth of the central idea in relativity: instead of the trajectory of S_1 being curved, it is the spacetime that is curved by the star's gravitation so that the trajectory is "straight" in this twisted spacetime.

This conceptual picture is described mathematically by geodesics. Geometrically, geodesics are the generalization of the notion of "straight lines" in Euclidean spaces for arbitrary manifolds. Thus by definition, light does "travel in straight lines". A material particle may or may not be a geodesic. If it is, it is said to be *freely falling*, moving solely under the influence of gravity. In relativity, a particle is freely falling iff it is a geodesic. A freely falling particle has contant mass m, since $\langle \alpha', \alpha' \rangle = -m^2$ is constant for any geodesic α.

The precise model for the theory of general relativity is given by the Einstein Equation, which relates the curvature of spacetime with the matter content in our universe. Before stating the equation, we must first understand how matter is represented mathematically. Relativistically, gravitation results from the energy-momentum of matter, which is mathematically described by the *stress-energy tensor field* T on the spacetime M. In general, there is no definite formulation for T; however, there are general requirements for the stress-energy tensor: (1) T is a symmetric $(2,0)$ tensor, (2) for every instantaneous observer u at $p \in M$ (i.e. a future-pointing timelike unit vector u in T_pM), the spatial part of T generalizes the classical stress tensor as measured by u, (3) the energy density measured by u is $T(u,u)$, and most importantly (4) $div(T) = 0$ which expresses conservation of energy-momentum.

On the other hand, the geometry of spacetime appears in the Einstein Equation in the form of the *Einstein gravitational tensor* defined as follows:

Definition 4.4. The Einstein gravitational tensor of a spacetime M is

$$G = Ric - \frac{1}{2}Sg.$$

This odd combination of geometric notions is purely experimental with some mathematical motivations explained later. With this tensor in hand, we are finally able to state the precise form of general relativity, that is the *Einstein Equation*:

Construction 4.5 (Einstein Equation)**.** If M is a spacetime that contains matter described by a stress-energy tensor T, then

$$G = 8\pi T$$

where G is the Einstein gravitational tensor.

The Einstein Equation cannot be mathematically derived; it is rather Einstein's model for the laws of our universe. However, some observations may shed light on the credibility of the theory.

Lemma 4.6. *(1) G is a symmetric (2,0) tensor field; $div(G) = 0$.*
(2) $Ric = G - \frac{1}{2}C(G)g$, where $C(G)$ denotes the contraction of G.

Proof. Since both Ric and g are symmetric (2,0) tensors, so is G. One may compute that $div(Sg) = dS$. Then

$$div(G) = div(Ric - \frac{1}{2}Sg) = \frac{1}{2}(dS - dS) = 0.$$

Since $C(g) = 4$, $C(G) = C(Ric) - \frac{1}{2}SC(g) = -S$. Thus $Ric = G + \frac{1}{2}Sg = G - \frac{1}{2}C(G)g$. □

Thus the Einstein gravitational tensor is of the same nature as our desired stress-energy tensor. Moreover, recalling the definition of G, we observe that the symmetry states that G and Ric contain the same information; that is, the Einstein gravitational tensor contains all we need to know about the geometry of spacetime. The universal constant $k = 8\pi$ is experimentally determined based on comparisons with Newtonian physics at low speeds and weak gravitation, for example in the Mercury's precession problem.

5. Solutions to the Einstein Equation

The exact solutions to the Einstein Equation have been found to have significant physical implications. In this section, we will briefly discuss three such solutions: the *Minkowski solution* (1908), the *Schwarzschild solution* (1916), and the *Kerr solution* (1966).

Physically, the Minkowski solution results in a *vacuum* spacetime, i.e. there is no mass in the system. Such a spacetime is the realm of *special relativity.* Relativistic phenomena such as length contraction, time dilation, relativistic velocity addition, and the twin paradox are mathematically explained in this context. The Minkowski spacetime is flat, since there is no gravity; thus the geodesics in the Minkowski spacetime are straight lines.

The Schwarzschild solution describes a spacetime which contains a single stationary gravitational source of mass $M > 0$. Mathematically, the solution is required to be spherically symmetric to properly reflect the physical symmetry of the spacetime, as well as vacuum at infinity which entails that the grativitational effect is negligible at far distances from the source. The Schwarzschild spacetime is the simplest model for black holes; in fact, the Schwarzschild metric yields two solution sets, which are respectively defined to be the *Schwarzschild exterior spacetime* N and the *Schwarzschild black hole* B, joined by an *event horizon* H.

The last solution to be discussed is the Kerr spacetime, of which the stress-energy tensor contains information of an axially rotating mass. Thus the Kerr spacetime depends on two parameters: mass $M > 0$ and angular momentum per unit mass $a \neq 0$. Being a physical generalization of the Schwarzschild and Minkowski cases, the Kerr metric should reduce to the Schwarzschild metric for $a = 0$ (i.e. the gravitational source is stationary), and to the Minkowski metric for $M = 0$ (i.e. the spacetime is vacuum) or at infinity (i.e. the gravitational effect is negligible at infinity). The physical manifestation of the Kerr solution is the most interesting of the three metrics; one of the resulted Kerr spacetimes is an axially rotating black hole. A more detailed treatment of the Kerr geometry will be offered in the next section.

With this overview of the physical interpretations of three solutions to the Einstein Equation, we shall examine the metric components of these spacetimes, provided in Table 1. The simplest descriptions of these metric tensors are in terms of the coordinates (t, r, θ, ϕ) with a time coordinate t on $\mathbb{R}^1$ and spatial spherical coordinates on $\mathbb{R}^3$. When the Kerr metric is presented in these natural coordinates, they are called *Boyer-Lindquist coordinates* for historical reasons. Specifically for the Kerr spacetime, the following two functions pervade the mathematics of the subject:

$$\rho^2 = r^2 + a^2 \cos^2 \theta$$
$$\Delta = r^2 - 2Mr + a^2$$

A quick examination of the metric components shows that these metrics satisfy the relationships stated above: the Schwarzschild and Kerr solutions reduce to the Minkowski solution for $M = 0$ or $r \to \infty$, while the Kerr solution reduces to the Schwarzschild case for $a = 0$. On the other hand, the Schwarzschild metric fails on the z-axis (as spherical coordinates always do) and when $r = 0$ or $r = 2M$. It is

TABLE 1. The metric components of the Minkowski, Schwarzschild, and Kerr metrics.

	Minkowski	Schwarzschild	Boyer-Lindquist (Kerr)
g_{tt}	-1	$-1 + 2M/r$	$-1 + 2Mr/\rho^2$
g_{rr}	1	$r/(r - 2M)$	ρ^2/Δ
$g_{\theta\theta}$	r^2	r^2	$\rho^2 = r^2 + a^2 \cos^2 \theta$
$g_{\phi\phi}$	$r^2 \sin^2 \theta$	$r^2 \sin^2 \theta$	$[r^2 + a^2 + (2Mra^2 \sin^2 \theta)/\rho^2] \sin^2 \theta$
g_{ij} $(i \neq j)$	all vanish	all vanish	$g_{t\phi} = g_{\phi t} = -(2Mra \sin^2 \theta)/\rho^2$; all the others vanish

important to note that the points where $r = 2M$ give the *event horizon* H of the Schwarzschild spacetime, while the (single) point where $r = 0$ is the *point singularity* O of the spacetime. The regions where $r > 2M$ and $0 < r < 2M$ are respectively defined to be the *Schwarzschild exterior spacetime* and the *Schwarzschild black hole*; it is a very interesting mathematical result that the black hole's attraction is strong enough not to let any particle escape once it passes the event horizon.

Proposition 5.1. *No particle, whether material or lightlike, can escape from the black hole B. Furthermore, any particle in B moves inward, ending (on a finite parameter interval) at the central singularity $r = 0$, if not before.*

Hence the behavior of these mathematical black holes matches physical observations. The Schwarzschild spacetime provides with a simple model for black holes that works well in the limit of very slowly rotating (or stationary) stars. It is worth noticing that, however, most stars in our galaxy are axially rotating objects, hence the Schwarzschild model is experimentally fairly limited. For more details on the mathematics of the Schwarzschild black hole, consult [2].

The problem with the Schwarzschild case is subsequently solved with the Kerr solution. Similar to the Schwarzschild metric, the Kerr metric also fails on the z-axis; however, this metric does not fail when $r = 0$ but when $\rho^2 = 0$ or $\Delta = 0$. These points, as it turns out, will give the horizons and singularity of the Kerr spacetime.

A final remark when looking at the Kerr metric tensor is that since the coordinates t and ϕ do not appear in the preceding metric formulas, the coordinate vector fields ∂_t and ∂_ϕ are Killing vector fields. The flow ∂_t consists of the coordinate translations that send t to $t + \Delta t$, leaving the other coordinates fixed. Roughly speaking, these isometries express the time-invariance of the model. For ∂_ϕ the flow consists of coordinate rotations $\phi \to \phi + \Delta\phi$; these isometries express its axial symmetry.

6. Kerr Geometry

In this section, we will explore the geometry of the Kerr spacetime in more detail.

Lemma 6.1. *The Boyer-Lindquist metric components satisfy the following metric identities:*

(1) $g_{\phi\phi} + a(\sin^2\theta)g_{\phi t} = (r^2 + a^2)\sin^2\theta$,
(2) $g_{t\phi} + a(\sin^2\theta)g_{tt} = -a\sin^2\theta$,
(3) $ag_{\phi\phi} + (r^2 + a^2)g_{t\phi} = \Delta a\sin^2\theta$,
(4) $ag_{t\phi} + (r^2 + a^2)g_{tt} = -\Delta$.

Since the formulas for $g_{\phi\phi}$ and $g_{\phi t}$ are rather cumbersome, it is a good tactic to use the metric identities as early and often as possible. Because $g_{ij} = \langle \partial_i, \partial_j \rangle$ they can be interpreted as scalar products of vector fields as follows.

Definition 6.2. The *canonical Kerr vector fields* are

$$V = (r^2 + a^2)\partial_t + a\partial_\phi,$$
$$W = \partial_\phi + a\sin^2\theta\partial_t.$$

In terms of V and W the metric identities can be written as

$$\langle V, \partial_\phi \rangle = \Delta a\sin^2\theta, \qquad \langle W, \partial_\phi \rangle = (r^2 + a^2)\sin^2\theta,$$

$$\langle V, \partial_t \rangle = -\Delta, \qquad \langle W, \partial_t \rangle = -a\sin^2\theta.$$

Using these identities, it is straightforward to calculate the following inner products:

Lemma 6.3. $\langle V, V \rangle = -\Delta\rho^2$, $\quad \langle W, W \rangle = \rho^2\sin^2\theta$, $\quad \langle V, W \rangle = 0$.

Depending on the parameters M and a, Kerr spacetimes are divided into three categories: *slowly rotating Kerr spacetime* ("slow Kerr") for $0 < a^2 < M^2$, *extreme Kerr spacetime* for $a^2 = M^2$, and *rapidly rotating Kerr spacetime* ("fast Kerr") for $M^2 < a^2$. In this paper, we are concerned with slow Kerr spacetimes only, since most physically interesting phenomena manifest only for this type of spacetime.

Similar to the Schwarzschild case, the points where the Kerr metric fails provide important physical insights. Notice that the Kerr metric does not fail at $r = 0$, so we let r run over the whole real line. It is topologically convenient to think of the coordinates r and t as cartesian coordinates over $\mathbb{R}^2$ together with spherical coordinates θ and ϕ on S^2; hence the Kerr spacetime is modeled as the product manifold $\mathbb{R}^2 \times S^2$. There are three subsets of the space at which the metric fails:

(1) On the *horizons*: $\Delta = 0$. For slow Kerr, this horizon equation has two distinct solutions $r_\pm = M \pm \sqrt{M^2 - a^2}$. Define the outer and inner horizons H_+ and H_- to be the set of points where $r = r_+$ and $r = r_-$, respectively.
(2) On the *ring singularity* Σ: $\rho^2 = 0$. This terminology comes from the fact that $\rho^2 = 0$ if and only if both $r = 0$ and $\cos\theta = 0$. Thus, Σ is the cartesian product of a time axis $\mathbb{R}^1(t)$ and a circle S^1, namely, the equatorial circle $\theta = \pi/2$ in S^2 at radius $r = 0$. Informally, the circle itself is sometimes called the ring singularity, with $\Sigma = S^1 \times \mathbb{R}^1$ its history through time.
(3) On the *axis* A: $\sin\theta = 0$. In the sphere S^2, $\sin\theta = 0$ picks out the north pole p ($\theta = 0$) and the south pole $-p$ ($\theta = \pi$). Hence in the Kerr spacetime $\mathbb{R}^2 \times S^2$, the solution set to the axis equation is

$$[\mathbb{R}^2(r,t) \times \{p\}] \cup [\mathbb{R}^2(r,t) \times \{-p\}].$$

Each of these two components provides a complete "z-axis" enduring through time. For some intuition, consider a slice $t = const$ of $\mathbb{R}^2 \times S^2$ (Fig. 2). The

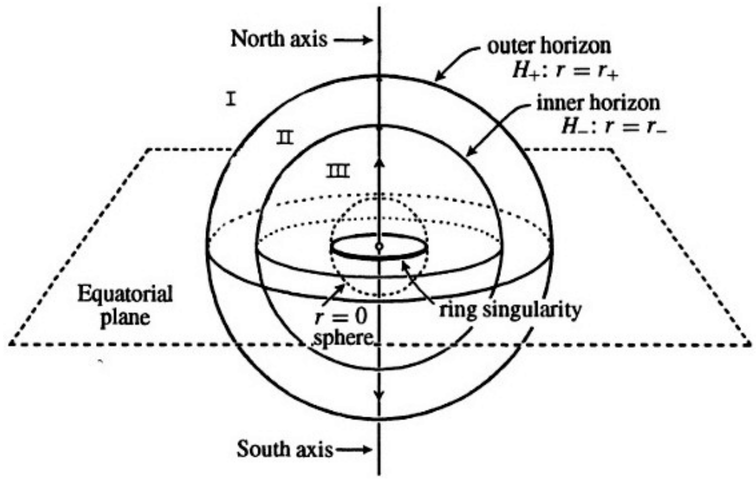

FIGURE 2. Schematics of the Kerr spacetime, presented by an exponential picture of a slice $t = const$ of $\mathbb{R}^2 \times S^2$ (the radius r is drawn as e^r, so $r = -\infty$ is at the center of the figure) [1].

trace of $\mathbb{R}^2(r,t) \times \{p\}$ on the slice is homeomorphic to $\mathbb{R}^1(r)$—a complete spatial "z-axis" at a specific moment in time; by convention, $\mathbb{R}^2(r,t) \times \{p\}$ is called the *north axis* of the spacetime. Similarly, the trace of $\mathbb{R}^2(r,t) \times \{-p\}$ on the slice gives another spatial axis for the spacetime at time t. The term *south axis* is used to denote $\mathbb{R}^2(r,t) \times \{-p\}$.

With some extra algebra, it can be shown that although the Boyer-Linquist coordinates are not well-defined on the rotational axis A, the Kerr metric extends its validity over A. However, the Boyer-Linquist form of the Kerr metric cannot be pushed any further. Hence the domain in which the Kerr metric is well-defined is $\mathbb{R}^2 \times S^2 - (H \cup \Sigma)$, which is subsequently divided into three connected components by the horizons.

Definition 6.4. For slow Kerr with horizons $r = r_\pm$, the Boyer-Lindquist blocks I, II, and III are the open subsets of $\mathbb{R}^2 \times S^2 - (H \cup \Sigma)$ which include points with $r > r_+$ (I), $r_- < r < r_+$ (II), and $r < r_-$ (III), respectively.

Each of these blocks is a connected Lorentz 4-manifold. However, recall that by definition a spacetime is a single connected time-oriented Lorentz 4-manifold. It turns out that by gluing the Boyer-Lindquist blocks along the horizons, we obtain the maximally extended Kerr spacetime that is the final Kerr solution. For the purpose of this paper, the discussion of such a construction is bypassed; readers may refer to [1] for more details.

Notice that $\Delta > 0$ on I $\cup$ III and $\Delta < 0$ on II; also $\rho^2 > 0$ everywhere. Hence it follows from Lemma 6.3 and the formulas for $g_{ij} = \langle \partial_i, \partial_j \rangle$ that

(1) ∂_θ and W are always spacelike;
(2) ∂_r is spacelike on I $\cup$ III and timelike on II;
(3) V is timelike on I $\cup$ III and spacelike on II;
(4) ∂_ϕ is spacelike when $r > 0$, hence on I $\cup$ II. On block III its causal character varies, though inspection of $g_{\phi\phi}$ shows that it is spacelike for $r \ll -1$;

(5) ∂_t is spacelike on block II since it is orthogonal to ∂_r (timelike on II), but its causal character varies on both I and III. However, it is timelike if $r > 2M$ or $r < 0$, as evidenced when g_{tt} is written as $p^{-2}[-r(r-2M) - a^2\cos^2\theta]$. In fact, ∂_t is still timelike for $r = 0$ since $\cos\theta = 0$ then gives the ring singularity.

Block I is called the *Kerr exterior*. In general, it bears a close resemblance to the usual Newtonian interpretation of the physical world; that is, provided that r is not too small, its gravitational force field is similar to the Newtonian central force field. The Boyer-Linquist coordinates likewise can be interpreted in the usual sense, consisting of a time coordinate t and a spatial spherical coordinate system (r, θ, ϕ) with the center of the black hole at its origin. For this regime, ∂_t is timelike, so it is natural to declare these vectors future-pointing. On the other hand, block II, the spacetime between the horizons, is utterly relativistic. Since the causal characters of the coordinate vector fields are reversed compared to blocks I and III, it is rather hard to determine a time-orientation for this region. Mathematically, however, it is not much different from block I. Finally, block III is the most interesting of all Boyer-Lindquist blocks. Its gravity is repulsive rather than attractive. Moreover, not only does it contain the ring singularity, but also the fact that it extends to the region with negative r-coordinate opens up lots of interesting mathematical possibilities, as we shall see an example in the next part of the paper.

Many interesting physical phenomena result from the mathematics of general relativity. Here, we discuss one of the most unintuitive yet fascinating possibilities: time travel! If a timelike curve α is closed, then α represents a material particle that starts from an event p, proceeds through an interval of its proper time then arrives back at exactly the same event p. Though physically unsettling, this idea is not mathematically impossible; in fact, as we shall show below, there is always such a violation of causality in the Kerr spacetime.

Definition 6.5. A spacetime M is *chronological* [*causal*] if there exist no closed timelike [nonspacelike] curves in M.

Proposition 6.6. *Boyer-Lindquist blocks I and II are causal.*

Proof. First we show that in block I, the hypersurfaces N: $t = t_0$ are spacelike. At each off-axis point $p \in N$, the vectors $\{\partial_r, \partial_\theta, \partial_\phi\}$ form a basis for $T_p(N)$. Since these vectors are spacelike and mutually orthogonal, the space is spacelike. If $p = ((t_0, r), q) \in N \subset \mathbb{R}^2 \times S^2$ is on the axis, then $q = (0, 0, \pm 1)$ in Cartesian coordinates (i.e. the north and south poles of S^2). It suffices to replace ∂_θ and ∂_ϕ by any basis for $T_q(S^2)$. Hence N is spacelike.

It follows that on any nonspacelike curve α, the coordinate t is strictly monotonic. Suppose, for contradiction, that $(d(t \circ \alpha)/ds)(s_0) = \alpha'(s_0)[t] = 0$. Then $\alpha'(s)$ is tangent to the hypersurface $t = t(\alpha(s_0))$, so $\alpha'(s_0)$ is spacelike. This contradicts the assumption that α is nonspacelike. Thus α cannot be a closed curve. We have shown that block I is causal.

A similiar argument can be applied for block II, with t and r reversed. Notice that the causal characters of ∂_r and ∂_t are reversed in this block, so the hypersurfaces N': $r = r_0$ are now spacelike. □

Although t does not measure time everywhere on block I, it is strictly increasing on any future-pointing particle. To verify this sign, note that *grad*(t) is timelike on the Kerr exterior since its orthogonal hypersurfaces are spacelike. Now

$\langle grad(t), \partial_t \rangle = dt/dt = 1 > 0$, and ∂_t is future-pointing (when timelike), so $grad(t)$ is past-pointing everywhere on block I. Thus, if a nonspacelike vector v is future-pointing, $v[t] = \langle v, grad(t) \rangle > 0$. In particular, if $v = \alpha'$ means $d(t \circ \alpha)/ds > 0$.

On block III, however, the argument above fails radically. Integral curves of the vector field ∂_ϕ are closed curves circling around the axis of rotation (except on the axis, where $\partial_\phi = 0$). Though ∂_ϕ is spacelike on blocks I and II, in block III there exists a region $\mathfrak{T} = \{g_{\phi\phi} < 0\}$ called the *Carter time machine*, where ∂_ϕ becomes timelike. This allows the possibility of "going back in time". For concreteness, the time-orientation on block III is given by the canonical vector field V (timelike on III).

Proposition 6.7. *Given any events $p,\ q \in III$, there exists a timelike future-pointing curve in III from p to q.*

As a result of this proposition, by first choosing a timelike future-pointing curve from p to q, then picking another future-pointing curve from q to p, we achieve a closed timelike future-pointing curve in block III. The proof of this proposition depends on the following observation: any point with $\theta = \pi/2$ and $r < 0$ sufficiently small is in $\mathfrak{T}$. This observation provides a "destination" for any future-pointing or past-pointing curve in block III to enter and exit the time machine, as expressed in the following lemma:

Lemma 6.8. *Given $p = (r_0, \theta_0, \phi_0, t_0) \in III$ there is a number $c > 0$ such that for any numbers $\Delta t \geq c$ and ϕ, there is*

(1) A future-pointing timelike curve α from p to $(\epsilon, \pi/2, \phi, t_0 + \Delta t) \in \mathfrak{T}$, and
(2) A past-pointing timelike curve β from p to $(\epsilon, \pi/2, \phi, t_0 - \Delta t) \in \mathfrak{T}$.

Proof. By the previous remark, choose ϵ small enough such that $(\epsilon, \pi/2, \phi, t) \in \mathfrak{T}$ for all ϕ and t. Let $\alpha_1(s) = (r(s), \theta(s), \phi_0, t_0)$ be a curve in block III from p to $(\epsilon, \pi/2, \phi_0, t_0)$. For a number $A > 0$ consider the curve

$$\alpha(s) = (r(s), \theta(s), \phi_0 + Aas, t_0 + At(s))$$

where $t(s)$ satisfies $dt/ds = r^2 + a^2$. Then

$$\alpha' = \alpha_1' + A[a\partial_\phi + (r^2 + a^2)\partial_t] = \alpha_1' + AV$$

where the canonical vector field V is restricted to α. Since $V \perp \alpha_1'$,

$$\langle \alpha', \alpha \rangle = \langle \alpha_1, \alpha_1' \rangle + A^2 \langle V, V \rangle = \langle \alpha_1', \alpha_1' \rangle - A^2 \Delta \rho^2.$$

The curves are defined on a closed interval, so $\langle \alpha_1', \alpha_1' \rangle$ is bounded above, and on α the positive functions Δ and r^2 are bounded away from 0. Thus for large A, α is timelike. Furthermore, α is future-pointing since V is future-pointing on III and $\langle \gamma', V \rangle = A \langle V, V \rangle < 0$.

Extending α by an integral curve of the future-pointing timelike vector fields V or $-\partial_\phi$ lets us freely increase the final t coordinate of α and adjust the ϕ coordinate to any value. This flexibility allows the existence of the required number c [1]. □

Proof of Proposition 6.7. The required curve will be constructed by joining three segments. The first is a curve α as in the preceding lemma. Then for a constant $B > 0$, consider the curve

$$\lambda(s) = (\epsilon, \pi/2, \overline{\phi} - Bs, \overline{t} - s)$$

for $s \geq 0$, that starts at the final point of α. Then λ is in the time machine and has velocity $\lambda' = -B\partial_\phi - \partial_t$. Thus, $\langle \lambda', \lambda' \rangle = B^2 g_{\phi\phi} - 2Bg_{\phi t} + g_{tt}$. Since the metric components depend only on r and θ, they are constant along λ. But $g_{\phi\phi} < 0$, hence for B large, λ is timelike. Furthermore, it is future-pointing since a short computation using the metric identities shows that $\langle \lambda', V \rangle = \Delta(1 - aB)$, which is negative for $B > 1/a$ since $\Delta > 0$ on III.

What makes the time machine work is that, on the curve λ,

$$dt/ds = \lambda'(s)[t] = -1.$$

Thus as λ proceeds into the future, its coordinate t is steadily decreasing. Let λ run until it reaches a point, say, $(\epsilon, \pi/2, \phi_1, t_1)$ that can be reached from event q by a past-pointing timelike curve β. Then reversing the parametrization of β gives a future pointing curve $\alpha \circ \lambda \circ \beta^{-1}$ from p to q as required [1].

$\square$

The segments of the curve going from p to $\mathfrak{T}$ and from $\mathfrak{T}$ to q require positive time period of travel; however, the arc within $\mathfrak{T}$ literally makes up for lost time.

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References

[1] Barrett O'Neill. *The Geometry of Kerr Black Holes.* A K Peters, Ltd., Wellesley, Massachusett, 1995.
[2] Barrett O'Neill. *Semi-Riemannian Geometry with Applications to Relativity.* Academic Press, San Diego, California, 1983.
[3] John M. Lee. *Introduction to Smooth Manifolds.* Springer, New York, 2003.
[4] Richard L. Bishop and Samuel I. Goldberg. *Tensor Analysis on Manifolds.* Dover Publication, Inc., New York, 1980.