Lecture XXVI: Kerr black holes: I. Metric structure and regularity of particle orbits

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I. OVERVIEW

We now wish to consider the case of a neutral black hole with angular momentum. This is an astrophysically relevant case – collapsing stars may be rotating, and accreting black holes (such as the ones we see!) collect material with angular momentum. We will see that the problem is very similar to that for the charged black hole, with the corrections to the radial equation from angular momentum leading to a similar structure – two event horizons, etc. However the deviation from spherical symmetry leads to some interesting additional properties outside the outer horizon, i.e. in the part of the hole we can actually see.

Reading:

- MTW Ch. 33.
- If you're interested in learning more about the global structure, see Hawking & Ellis, *The Large-Scale Structure of Spacetime*.
- The great details of perturbations of the Reissner-Nordstrøm metric, and what goes wrong at the inner horizon, can be found in Chandrasekhar, *The Mathematical Theory of Black Holes*.

II. THE KERR METRIC

The *Kerr metric* is the only stationary, axisymmetric vacuum black hole solution. (For a long proof, see Chandrasekhar §§52–55.) Its form is

$$ds^2 = -\left(1-\frac{2Mr}{\Sigma}\right)dt^2 - \frac{4aMr}{\Sigma}\sin^2\theta\,dt\,d\phi + \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\Sigma}\sin^2\theta\,d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma\,d\theta^2, \tag{1}$$

where M and a are constants, and

$$\Delta = r^2 - 2Mr + a^2 \quad\text{and}\quad \Sigma = r^2 + a^2\cos^2\theta. \tag{2}$$

There are many algebraically equivalent forms in which this can be written with the same set of coordinates $\{t,r,\theta,\phi\}$ called *Boyer-Lindquist coordinates*.

The Kerr metric tensor components are independent of t and ϕ and therefore there are two Killing fields $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ with $\xi^\alpha = (1,0,0,0)$ and $\zeta = (0,0,0,1)$. They commute, $[\boldsymbol{\xi},\boldsymbol{\zeta}]=0$, and correspond respectively to the conservation of energy $E = -\boldsymbol{p}\cdot\boldsymbol{\xi}$ and z-angular momentum $L = \boldsymbol{p}\cdot\boldsymbol{\zeta}$.

It is clear that for $a=0$, the Kerr metric becomes the Schwarzschild metric with mass M. The significance of a can be obtained by taking the large-r limit – i.e. expanding to order M/r or a/r^2:

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 - \frac{4aM}{r^2}\sin^2\theta\,dt\,d\phi + \left(1+\frac{2M}{r}\right)dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2), \tag{3}$$

or – switching to Cartesian-like coordinates where $x^1 = r\sin\theta\cos\phi$, $x^2 = r\sin\theta\sin\phi$, and $x^3 = r\cos\theta$ –

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 - \frac{4aM}{r^3}\epsilon_{3ij}x^i\,dt\,dx^j + [(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + \frac{2M}{r}dr^2. \tag{4}$$

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The a term thus corresponds to an angular momentum of the black hole (as measured by gyroscope experiments from the outside) of magnitude $J \equiv aM$ and directed in the $+3$ direction. If we flip the sign of a, this is equivalent to a black hole rotating in the opposite direction ($\phi \to -\phi$): thus in astrophysical applications involving accretion disks, one often refers to a retrograde system (disk orbiting opposite to the black hole spin) as $a < 0$.

The angular momentum has units of mass squared, so in addition to the parameter a it is common to define a dimensionless angular momentum for a black hole:

$$\chi \equiv a_\star \equiv \frac{a}{M} = \frac{J}{M^2}. \tag{5}$$

III. THE GLOBAL STRUCTURE

We wish to understand the global behavior of the Kerr metric, just as we did for Schwarzschild. It turns out the most convenient way to do this is to complete the square in the metric. In general, we have for the $t\phi$ sector of the metric:

$$A\,dt^2 + 2B\,dt\,d\phi + C\,d\phi^2 = C\left(d\phi - \frac{B}{C}dt\right)^2 + \frac{AC - B^2}{C}dt^2. \tag{6}$$

For example:

$$C = \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\Sigma}\sin^2\theta. \tag{7}$$

Then:

$$\begin{aligned}
\frac{AC-B^2}{C} &= \frac{-(\Sigma - 2Mr)[(r^2+a^2)^2 - a^2\Delta\sin^2\theta] - 4a^2M^2r^2\sin^2\theta}{[(r^2+a^2)^2 - a^2\Delta\sin^2\theta]\Sigma} \\
&= \frac{-\Sigma[(r^2+a^2)^2 - a^2\Delta\sin^2\theta] + 2Mr[(r^2+a^2)^2 - a^2\Delta\sin^2\theta - 2a^2Mr\sin^2\theta]}{[(r^2+a^2)^2 - a^2\Delta\sin^2\theta]\Sigma} \\
&= \frac{-\Sigma[(r^2+a^2)^2 - a^2\Delta\sin^2\theta] + 2Mr(r^2+a^2)\Sigma}{[(r^2+a^2)^2 - a^2\Delta\sin^2\theta]\Sigma} \\
&= \frac{-(r^2+a^2)(r^2+a^2-2Mr) + a^2\Delta\sin^2\theta}{(r^2+a^2)^2 - a^2\Delta\sin^2\theta} \\
&= \frac{-\Delta\Sigma}{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}.
\end{aligned} \tag{8}$$

We then find:

$$ds^2 = -\frac{\Delta\Sigma}{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}dt^2 + \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\Sigma}\sin^2\theta\left(d\phi - \frac{2aMr}{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}dt\right)^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma\,d\theta^2. \tag{9}$$

This is an algebraically equivalent form to Eq. (1).

What is interesting about Eq. (9) is that, like the usual form of Schwarzschild, it tells us about the range of allowed trajectories in the (t,r)-plane. This is because the normalization condition for the 4-velocity is now

$$-1 = -\frac{\Delta\Sigma(u^t)^2}{(r^2+a^2)^2 - a^2\Delta\sin^2\theta} + \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\Sigma}\sin^2\theta\left(u^\phi - \frac{2aMr}{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}u^t\right)^2 + \frac{\Sigma}{\Delta}(u^r)^2 + \Sigma(u^\theta)^2. \tag{10}$$

Thus we have the usual situation – the allowed regions in the (u^t, u^r)-plane are hyperbolae that allow one to move either inward or outward – as long as $\Delta > 0$. If $\Delta < 0$, then particles moving inward ($u^r < 0$) are forced to maintain this motion. The situation is exactly analogous to the Reissner-Nordstrøm case, with a replacing the charge: we have $\Delta = 0$ at

$$r = r_\pm = M \pm \sqrt{M^2 - a^2}, \tag{11}$$

a “normal region” at $r > r_+$, and an outer horizon at $r = r_+$. Particles falling through the outer horizon are doomed to have their r continue to decrease until they reach the inner horizon, $r = r_-$, at which point they see the end of the universe and are destroyed.

If particles could pass through $r = r_-$ without being destroyed, then the analytic continuation of Kerr, like Reissner-Nordstrøm, goes all the way down to the singularity at $r = 0$. In fact, this singularity is not a single point – if $\cos\theta \neq \pi/2$, then the metric, including spherical-like coordinates, are perfectly well-behaved at $r = 0$! There is thus a *ring singularity* in the analytic continuation of Kerr, and on the other side ($r < 0$) a whole universe with asymptotically Minkowski-like geometry. However, it seems that in the actual collapse of a rotating star, this structure does not form for the aforementioned reasons.

IV. EXTERIOR STRUCTURE OF KERR SPACETIME

Outside the outer horizon, at $r > r_+$, the Kerr spacetime allows particles to move either inward or outward in radius. However, their motion in longitude may be restricted. In particular, using Eq. (10), and that $d\phi/dt = u^\phi/u^t$, we see that

$$\left|\frac{d\phi}{dt} - \frac{2aMr}{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}\right| < \frac{\Sigma\sqrt{\Delta}}{[(r^2+a^2)^2 - a^2\Delta\sin^2\theta]\sin\theta}. \tag{12}$$

At large radii, this is not of particular note – it becomes $|d\phi/dt| < 1/r$. But near the hole, strange things start to happen. In particular, there is a region called the *ergosphere* where $d\phi/dt$ **must** have the same sign as a – any particle is forced to revolve in the direction of the hole's spin! This is where

$$2|a|Mr\sin\theta > \Sigma\sqrt{\Delta}. \tag{13}$$

Squaring this gives

$$4a^2M^2r^2\sin^2\theta > \Sigma^2\Delta. \tag{14}$$

Our work on Eq. (8) gave an alternative form for $\Sigma^2\Delta$:

$$\Sigma^2\Delta = (\Sigma - 2Mr)[(r^2+a^2)^2 - a^2\Delta\sin^2\theta] + 4a^2M^2r^2\sin^2\theta, \tag{15}$$

so the ergosphere is the region outside the horizon ($r > r_+$) and with $\Sigma < 2Mr$, or

$$r^2 - 2Mr + a^2\cos^2\theta < 0. \tag{16}$$

Therefore the region is

$$r_+ = M + \sqrt{M^2 - a^2} < r < r_{\rm sl}(\theta) = M + \sqrt{M^2 - a^2\cos^2\theta}. \tag{17}$$

The outer boundary of the ergosphere is the *static limit*. At the static limit, a photon can instantaneously have $d\phi/dt = 0$, but all matter particles are inexorably dragged along with the rotation of the hole. The static limit touches the horizon at the poles, and extends outward to $r = 2M$ at the equator. For the Schwarzschild case, it merges with the horizon (the ergosphere vanishes).

The behavior as one approaches the horizon itself is weirder. Recall that as we approach the horizon, $\Delta \to 0^+$. Then all objects near the horizon must orbit the hole at the rate

$$\frac{d\phi}{dt} = \Omega_{\rm H} = \frac{2aMr_+}{(r_+^2+a^2)^2} = \frac{a}{r_+^2+a^2}. \tag{18}$$

Thus (classically) the objects that fall onto the hole have their exponentially fading images appear to rotate at a constant velocity. The black hole thus appears to rotate as a solid body.

It is worth summarizing these results both in the case of small spin parameter, $\chi \ll 1$, and large spin parameter, $\chi = 1 - \epsilon$.

- For *small spins* $\chi \ll 1$, the outer horizon is located at $r_+ = (2 - \frac{1}{2}\chi^2)M$, and the rotation rate of the horizon is $\Omega_{\rm H} = \chi/(4M)$.
- For *large spins* $\chi = 1 - \epsilon$, the outer horizon is located at $r_+ = (1 + \sqrt{2\epsilon})M$, and the rotation rate of the horizon is $\Omega_{\rm H} = (1 - \sqrt{2\epsilon})/(2M)$.

V. PARTICLE ORBITS

We are now interested in test particle orbits in the Kerr spacetime. In order to study such orbits, we need to understand the conserved quantities. For a test particle of mass μ, the energy per unit mass $\mathcal{E} = -u_t$ and the z-angular momentum per unit mass $\mathcal{L} = u_\phi$ will be conserved because of the existence of the appropriate Killing fields. In order to understand the allowed motions, then, we must use the 4-velocity normalization,

$$g^{\alpha\beta} u_\alpha u_\beta = -1. \tag{19}$$

This requires us to derive the inverse-metric components by inversion of $g_{\alpha\beta}$. This is tedious but straightforward: the result is

$$\begin{aligned} g^{tt} &= -\frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\Delta\Sigma}, \\ g^{t\phi} &= -\frac{2aMr}{\Delta\Sigma}, \\ g^{\phi\phi} &= \frac{\Delta - a^2\sin^2\theta}{\Delta\Sigma\sin^2\theta}, \\ g^{rr} &= \frac{\Delta}{\Sigma}, \quad \text{and} \\ g^{\theta\theta} &= \frac{1}{\Sigma}. \end{aligned} \tag{20}$$

The normalization condition is then

$$-1 = \frac{-(r^2+a^2)^2\mathcal{E}^2 + a^2\Delta\sin^2\theta\,\mathcal{E}^2 + 4aMr\mathcal{L}\mathcal{E} + \Delta\mathcal{L}^2\csc^2\theta - a^2\mathcal{L}^2}{\Delta\Sigma} + \frac{\Delta(u_r)^2 + (u_\theta)^2}{\Sigma}. \tag{21}$$

In the Schwarzschild case, we found the allowed region by rotating to a coordinate system that made the orbit equatorial. No such luck for Kerr: the spacetime has a preferred axis. We could find an allowed region in the (r,θ)-plane by taking the last term in Eq. (21) to be non-negative. But we can do better.

A. The Carter constant

It turns out that the Kerr spacetime has a peculiar additional property that is non-obvious and whose reason for existence remains shrouded in mystery. We may write

$$\begin{aligned} \frac{du_\theta}{d\tau} &= \frac{d}{d\tau}(\Sigma u^\theta) \\ &= \Sigma\frac{du^\theta}{d\tau} + u^\theta\frac{d\Sigma}{d\tau} \\ &= -\Sigma\Gamma^\theta_{\mu\nu}u^\mu u^\nu + u^\theta\Sigma_{,\mu}u^\mu \\ &= \frac{1}{2}g_{\mu\nu,\theta}u^\mu u^\nu - g_{\mu\theta,\nu}u^\mu u^\nu + u^\theta\Sigma_{,\mu}u^\mu. \end{aligned} \tag{22}$$

In the $g_{\mu\theta,\nu}u^\mu u^\nu$, only the $\mu = 0$ terms can give a nonzero result; since $g_{\theta\theta} = \Sigma$, this cancels the last term. Therefore,

$$\frac{du_\theta}{d\tau} = \frac{1}{2}g_{\mu\nu,\theta}u^\mu u^\nu = \frac{1}{2}g_{\mu\nu,\theta}g^{\alpha\mu}g^{\beta\nu}u_\alpha u_\beta = -\frac{1}{2}g^{\alpha\beta}{}_{,\theta}u_\alpha u_\beta, \tag{23}$$

where in the last equality we have used the derivative of a matrix inverse. Given the prevalence of Σ in the denominator of the inverse metric, it is easier to write this as

$$\Sigma\frac{du_\theta}{d\tau} = -\frac{1}{2}(\Sigma g^{\alpha\beta})_{,\theta}u_\alpha u_\beta - \frac{1}{2}\Sigma_{,\theta}, \tag{24}$$

where we recall the normalization condition $g^{\alpha\beta}u_\alpha u_\beta = -1$. This relation simplifies if we recall that

$$\Sigma_{,\theta} = -2a^2\sin\theta\cos\theta \tag{25}$$

and

$$\Sigma \frac{du_\theta}{d\tau} = \frac{u_\theta}{d\theta/d\tau}\frac{du_\theta}{d\tau} = \frac{1}{2}\frac{d(u_\theta)^2}{d\theta}, \tag{26}$$

so that

$$\frac{d(u_\theta)^2}{d\theta} = -(\Sigma g^{\alpha\beta})_{,\theta} u_\alpha u_\beta + 2a^2 \sin\theta\cos\theta. \tag{27}$$

Now for the miracle: of the metric coefficients (rescaled by Σ), only Σg^{tt} and $\Sigma g^{\phi\phi}$ depend on θ, and the derivatives do not depend on r:

$$\frac{d(u_\theta)^2}{d\theta} = -a^2\mathcal{E}^2 \frac{d}{d\theta}\sin^2\theta - \mathcal{L}^2\frac{d}{d\theta}\csc^2\theta + 2a^2\sin\theta\cos\theta. \tag{28}$$

Therefore, one may integrate to get

$$(u_\theta)^2 = -a^2\mathcal{E}^2\sin^2\theta - \mathcal{L}^2\csc^2\theta + a^2\sin^2\theta + \text{constant}. \tag{29}$$

Using the Pythagorean identities $\sin^2\theta = 1-\cos^2\theta$ and $\csc^2\theta = 1+\cot^2\theta$, and noting that $\mathcal{E}$ and $\mathcal{L}$ are constant, we may write this as

$$(u_\theta)^2 = -a^2(1-\mathcal{E}^2)\cos^2\theta - \mathcal{L}^2\cot^2\theta + \mathcal{Q}, \tag{30}$$

where $\mathcal{Q}$ is the *Carter constant*. The Carter constant is an accidental conserved quantity of trajectories in the Kerr spacetime. It is obvious that for orbits with $|\mathcal{E}| \le 1$ (which will include all the bound orbits), we must have $\mathcal{Q} \ge 0$.

In Eq. (30), for the bound orbits ($|\mathcal{E}| < 1$) the right-hand side achieves a maximum value of $\mathcal{Q}$ at $\theta = \pi/2$, and decreases to $-\infty$ at $\theta = 0$ or $\theta = \pi$. It crosses zero at $\theta = \pi/2 \pm I$, where I is the coordinate-inclination of the orbit. (Note: there are several other definitions of "inclination" in use in the Kerr spacetime.) Thus the particle is confined to the region within I of the equator, and it bounces back and forth between $\theta/2 - I$ and $\theta/2 + I$. The Carter constant is related to the inclination via

$$\mathcal{Q} = a^2(1-\mathcal{E}^2)\sin^2 I + \mathcal{L}^2\tan^2 I. \tag{31}$$

In the Newtonian limit, $1-\mathcal{E}^2$ is small and $\mathcal{L}$ is large, so we have $\tan I \approx \mathcal{Q}^{1/2}/\mathcal{L}$. Note that for retrograde orbits, $\mathcal{L} < 0$ and $\tan I < 0$. The only orbits that can reach the poles ($\theta = 0, \pi$) have $\mathcal{L} = 0$ and finite $\mathcal{Q}$.

We may now attempt to solve for the θ coordinate along the particle's trajectory. We see that

$$\frac{d\theta}{d\tau} = u^\theta = \Sigma^{-1}\sqrt{-a^2(1-\mathcal{E}^2)\cos^2\theta - \mathcal{L}^2\cot^2\theta + \mathcal{Q}}. \tag{32}$$

We unfortunately cannot integrate this to give $\theta(\tau)$ because Σ depends on r as well as θ. We can however define a reparameterization of the particle trajectory from the proper time τ to the rescaled coordinate

$$\lambda = \int \frac{d\tau}{\Sigma}. \tag{33}$$

[**Warning**: This is *not* an affine parameter!] Then we have

$$\int_{\pi/2-I}^{\theta} \frac{d\theta}{\sqrt{-a^2(1-\mathcal{E}^2)\cos^2\theta - \mathcal{L}^2\cot^2\theta + \mathcal{Q}}} = \lambda - c_\theta, \tag{34}$$

where c_θ is a constant. The integral on the left-hand side is an elliptic function, but we will not evaluate it here. In the Newtonian (or the Schwarzschild) limit where a can be dropped, it becomes $\mathcal{L}^{-1}\arccos(\cos\theta/\sin I)$ so that

$$\cos\theta \approx \sin I \cos[\mathcal{L}(\lambda - c_\theta)] \qquad (a \approx 0); \tag{35}$$

the more general case is qualitatively similar. In all cases, θ is a periodic function of λ with some period P_θ (which is $2\pi\mathcal{L}^{-1}$ in the Schwarzschild limit).

B. The radial motion

We are next interested in the radial motion of a particle. To find this, we take the normalization condition Eq. (21) and substitute Eq. (30):

$$-1 = \frac{-(r^2+a^2)^2\mathcal{E}^2 + a^2\Delta\mathcal{E}^2 + 4aMr\mathcal{L}\mathcal{E} + \Delta\mathcal{L}^2 - a^2\mathcal{L}^2 - a^2\Delta\cos^2\theta + \Delta\mathcal{Q}}{\Delta\Sigma} + \frac{\Delta(u_r)^2}{\Sigma}. \tag{36}$$

Multiplying through by $\Delta\Sigma$ gives $-\Delta\Sigma = -\Delta(r^2 + a^2\cos^2\theta)$ on the left-hand side; moving this to the right then gives

$$0 = \Delta r^2 - (r^2+a^2)^2\mathcal{E}^2 + a^2\Delta\mathcal{E}^2 + 4aMr\mathcal{L}\mathcal{E} + \Delta\mathcal{L}^2 - a^2\mathcal{L}^2 + \Delta\mathcal{Q} + \Delta^2(u_r)^2. \tag{37}$$

Expanding the Δs and algebraically simplifying gives

$$-\Delta^2(u_r)^2 = (1-\mathcal{E})^2 r^4 - 2Mr^3 + [a^2(1-\mathcal{E}^2) + \mathcal{L}^2 + \mathcal{Q}]r^2 - 2M[(a\mathcal{E}-\mathcal{L})^2 + \mathcal{Q}]r + a^2\mathcal{Q} \equiv V(r). \tag{38}$$

Here $V(r)$ is a fourth-order polynomial in r; the particle is trapped in the regions where it is negative. Clearly $V(r)$ becomes large for positive r, at least for bound ($|\mathcal{E}| < 1$) orbits, and it is positive for $r = 0$ (or zero for a nonspinning hole $a = 0$ or equatorial orbit $\mathcal{Q} = 0$). Descartes's rule of signs tells us that there can be no negative zeroes of $V(r)$ since all signs alternate, and finally by inspection of the intermediate step Eq. (37) we have $V(r_+) \le 0$. Thus there are two possibilities: there is either 1 zero of $V(r)$ between r_+ and ∞, or there are 3. In the former case, the allowed region in r spans from below the horizon to some $r_{\rm max}$, and all particles on such orbits fall into the hole. There are no stable orbits for these values of $(\mathcal{E},\mathcal{L},\mathcal{Q})$. In the latter case, there is a bounded trapped region where $V(r)$ is negative between two roots $r_{\rm min}$ and $r_{\rm max}$. It is in the latter case that stable orbits are possible.

We can construct the particle's trajectory by noting that

$$\frac{dr}{d\lambda} = \frac{dr}{d\tau}\frac{d\tau}{d\lambda} = \Sigma u^r = \Delta u_r = \sqrt{-V(r)}, \tag{39}$$

and so

$$\int_{r_{\rm min}}^{r} \frac{dr}{\sqrt{-V(r)}} = \lambda - c_r, \tag{40}$$

where c_r is a constant. Again the function $r(\lambda)$ is periodic, with period $P_r = 2\int_{r_{\rm min}}^{r_{\rm max}} dr/\sqrt{-V(r)}$. This is also an elliptic integral.

C. The longitude and time motions

The remaining two motions, $\phi(\lambda)$ and $t(\lambda)$, can be obtained from the inverse-metric:

$$\frac{dt}{d\lambda} = \Sigma u^t = \left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\mathcal{E} - \frac{2aMr\mathcal{L}}{\Delta} \tag{41}$$

and

$$\frac{d\phi}{d\lambda} = \Sigma u^t = \left[\csc^2\theta - \frac{a^2}{\Delta}\right]\mathcal{L} + \frac{2aMr\mathcal{E}}{\Delta}. \tag{42}$$

These functions, while complicated, are the sums of functions of θ and r (and constants of the motion). Therefore there are average values (over λ), $b_t = \langle dt/d\lambda\rangle_\lambda$ and $b_\phi = \langle d\phi/d\lambda\rangle_\lambda$. The coordinate time is then

$$t = c_t + b_t\lambda + D_{tr}(\lambda) + D_{t\theta}(\lambda), \tag{43}$$

where D_{tr} is periodic with period P_r and $D_{t\theta}$ is periodic with period P_θ. Similarly one may write

$$\phi = c_\phi + b_\phi\lambda + D_{\phi r}(\lambda) + D_{\phi\theta}(\lambda), \tag{44}$$

where $D_{\phi r}$ is periodic with period P_r and $D_{\phi\theta}$ is periodic with period P_θ.

We conclude that the motion is regular when measured with the parameter λ, and that the particle proceeds to exhibit oscillatory motions superposed on the mean march forward in time (at rate b_t) and longitude (at rate b_ϕ). The orbits are confined to a certain range in θ and r, and are integrable (in the sense of ordinary classical mechanics). In the next lecture, we will examine this motion in greater detail, and evaluate such quantities as precession rates, the energies and angular momenta of circular orbits, and stability.