Wave equations estimates and the nonlinear stability of slowly rotating Kerr black holes

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Abstract. *This is the last part of our proof of the nonlinear stability of the Kerr family for small angular momentum, i.e* $|a|/m \ll 1$*, in which we deal with the nonlinear wave type estimates needed to complete the project. More precisely we provide complete proofs for Theorems M1 and M2 as well the curvature estimates of Theorem M8, which were stated without proof in sections 3.7.1 and 9.4.7 of [53]. Our procedure is based on a new general interest formalism (detailed in Part I of this work), which extends the one used in the stability of Minkowski space. Together with [53] and the GCM papers [51], [52], [66], this work completes proof of the Main Theorem stated in Section 3.4 of [53].*

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Chapter 1

Introduction

This is the last part of our proof of the nonlinear stability of the Kerr family for small angular momentum, i.e $|a|/m \ll 1$, in which we deal with the nonlinear wave type estimates needed to complete the project. More precisely we provide complete proofs for Theorems M1 and M2 as well the curvature estimates of Theorem M8, which were stated without proof in sections 3.7.1 and 9.4.7 of [53]. Our procedure is based on a new general interest formalism (detailed in Part I of this work), which extends the one used in the stability of Minkowski space. Together with [53] and the GCM papers [51], [52], [66], this work completes proof of the Main Theorem stated in Section 3.4 of [53].

1.1 Black hole stability problem

We give below a quick introduction to the black hole stability problem.

1.1.1 Einstein Vacuum equations

We restrict our attention to the Einstein vacuum equations (EVE), i.e. spacetimes $(\mathcal{M}, \mathbf{g})$ with vanishing Ricci curvature, i.e.

$$\mathbf{R}_{\alpha\beta} = 0. \tag{1.1.1}$$

Note that a solution of (1.1.1) is in fact a class of equivalence of solutions with respect to diffeomorphisms Φ of $\mathcal{M}$, i.e. $\mathbf{g}$ and $\Phi^*\mathbf{g}$ are indistinguishable as solutions of EVE. This is precisely what is meant by the general covariance of the Einstein equations.

(1.1.1) corresponds to an evolution problem and an associated initial data set $(\Sigma_0, g_{(0)}, k_{(0)})$ for EVE consists of 3 dimensional manifold Σ together with a complete Riemannian metric $g_{(0)}$ and a symmetric 2-tensor $k_{(0)}$ which verify compatibility conditions known as the constraint equations. A Cauchy development of an initial data set is a globally hyperbolic space-time $(\mathcal{M}, g)$, verifying EVE and an embedding $i : \Sigma \to \mathcal{M}$ such that $i_*(g_{(0)}), i_*(k_{(0)})$ are the first and second fundamental forms of $i(\Sigma_{(0)})$ in $\mathcal{M}$. A well known foundational result in GR associates a unique maximal, global hyperbolic, future development to all sufficiently regular initial data sets, see [18] [19]. We further restrict the discussion to asymptotically flat initial data sets, i.e. we assume that outside a sufficiently large compact set K, $\Sigma_{(0)} \setminus K$ is diffeomorphic to the complement of the unit ball in $\mathbb{R}^3$ and admits a system of coordinates in which $g_{(0)}$ is asymptotically euclidean and $k_{(0)}$ vanishes at appropriate order.

1.1.2 Kerr family

(EVE) admits a remarkable two parameter family of explicit solutions, the Kerr spacetimes $\mathcal{K}(a, m)$, $0 \leq |a| \leq m$, which are stationary and axisymmetric. In the usual Boyer-Lindquist coordinates they take the form

$$\mathbf{g} = -\frac{q^2\Delta}{\Sigma^2}(dt)^2 + \frac{\Sigma^2(\sin\theta)^2}{|q|^2}\left(d\phi - \frac{2amr}{\Sigma^2}dt\right)^2 + \frac{|q|^2}{\Delta}(dr)^2 + |q|^2(d\theta)^2, \tag{1.1.2}$$

where

$$\begin{cases} \Delta = r^2 + a^2 - 2mr, \qquad q = r + ia\cos\theta, \\ \Sigma^2 = (r^2 + a^2)|q|^2 + 2mra^2(\sin\theta)^2 = (r^2 + a^2)^2 - a^2(\sin\theta)^2\Delta. \end{cases} \tag{1.1.3}$$

Among them one distinguishes the Schwarzschild family of spherically symmetric solutions, of mass $m > 0$,

$$\mathbf{g} = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1}dr^2 + r^2 d\sigma_{\mathbb{S}^2}. \tag{1.1.4}$$

Though the metric seems singular at $r = 2m$ ($r = r_+$, the largest root of $\Delta(r) = 0$, in the case of Kerr) it turns out that one can glue together two regions $r > 2m$ and two regions $r < 2m$ of the Schwarzschild metric to obtain a metric which is smooth along the null hypersurface $\mathcal{E} = \{r = 2m\}$ called the Schwarzschild event horizon. The portion of $r < 2m$ to the future of the hypersurface $t = 0$ is a black hole whose future boundary $r = 0$ is singular. The region $r > 2m$, free of singularities, is called the domain of outer communication. The more general family of Kerr solutions, which are both stationary and axially symmetric, possesses (in addition to well defined event horizons, black holes and domains of outer communication) Cauchy horizons ($r = r_-$, the smallest root of $\Delta(r) = 0$) inside the black hole region across which predictability fails[1]. Once more, one can easily check, from the precise nature of the Kerr metric, that the region outside the event horizon, i.e. outside the Kerr black hole, is free of singularities[2].

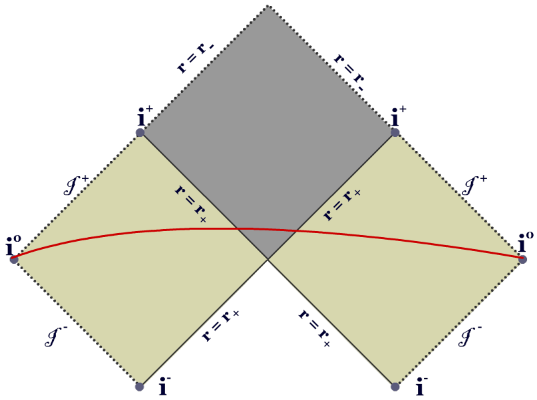

Figure 1.1: The Penrose diagram of Kerr for $0 < |a| < m$.

Finally we note that the Kerr spacetimes $\mathcal{K}(a, m)$ possess two Killing vectorfields: the stationary vectorfield $\mathbf{T} = \partial_t$, which is time-like in the asymptotic region, away from the horizon, and the axial symmetric Killing field $\mathbf{Z} = \partial_\varphi$.

1.1.3 Stability of Kerr conjecture

The issue of the stability[3] of the Kerr family has been at the center of attention of GR physics and mathematical relativity for more than half a century, ever since their discovery by Kerr in [43]. Roughly the problem here is to show that all spacetime developments of initial data sets, sufficiently close to that of a Kerr spacetime, behave asymptotically like a Kerr solution. Here is a more precise formulation of the conjecture.

Kerr stability conjecture. *Vacuum initial data sets sufficiently close to Kerr initial data have a maximal development with complete future null infinity*[4] *and with a domain of outer communication which approaches (globally) a nearby Kerr solution.*

[1] Infinitely many smooth extensions are possible beyond the boundary.

[2] Consistent with the *weak cosmic censorship conjecture (WCC)* of Penrose.

[3] This is not only a deep mathematical question but one with serious astrophysical implications. Indeed, if the Kerr family would be unstable, black holes would be nothing more than mathematical artifacts.

[4] Thus observers which are far away from the black hole will never experience its effects.

Until very recently the only space-time for which full nonlinear stability had been established was the Minkowski space. The proof is based on some important PDE advances of late last century:

(i) Robust vectorfield approach to derive quantitative decay based on generalized energy estimates and commutation with (approximate) Killing and conformal killing vectorfields.

(ii) The *null condition* identifying the deep mechanism for nonlinear stability, i.e. the specific structure of the nonlinear terms which enables stability despite the low decay of the perturbations.

(iii) Elaborate bootstrap argument according to which one makes educated assumptions about the behavior of solutions to nonlinear wave equations and then proceeds, by a long sequence of a-priori estimates, to show that they are in fact satisfied. This amounts to a *conceptual linearization*, i.e. a method by which the equations become, essentially, linear[5] without actually linearizing them.

There are three, related, major obstacles in passing from the stability of Minkowski to that of Kerr:

1. The first can be understood in the general framework of nonlinear evolution equations. Given a nonlinear evolution equation $\mathcal{N}[\phi] = 0$ and a stationary solution ϕ_0, we have two notions of stability, *orbital stability*, according to which small perturbations of ϕ_0 lead to solutions ϕ which remain close for all time, and *asymptotic stability (AS)* according to which the perturbed solutions converge as $t \to \infty$ to ϕ_0. In the case where ϕ_0 is non trivial, there is a third notion of stability, which we call *asymptotic orbital stability (AOS)*, to describe the fact that the perturbed solutions may converge to a different stationary solution[6]. For quasilinear equations[7], such as EVE, a proof of stability means necessarily AS or AOS stability. Both require a detailed understanding of the decay properties of the linearized equation. One is thus led to study the linearized equation $\mathcal{L}[\phi_0]\psi = 0$, with $\mathcal{L}[\phi_0]$ the Fréchet derivative $\mathcal{N}'[\phi_0]$, which is, essentially, a linear hyperbolic system with variable coefficients and, typically, presents instabilities. In the exceptional situation, when stability can

[5]Note that in the context of EVE, and other quasilinear hyperbolic systems, this differs substantially from the usual notion of linearization around a fixed background.

[6]This happens if ϕ_0 belongs to a multi parameter smooth family of stationary solutions, or by applying a gauge transform to ϕ_0 which keeps the equation invariant. In the case of Kerr, both cases are present as we shall see below.

[7]Orbital stability can be established directly (i.e. without establishing the stronger version) only for Hamiltonian equations with weak nonlinearities.

ultimately be established, one can tie all the instability modes to either the gauge invariance of the equation or the presence of a continuum of other distinct[8] stationary solutions[9] near ϕ_0. These instabilities at the linearized level are responsible for the fact that a small perturbation of the fixed stationary solution ϕ_0 may not converge to ϕ_0 but to another nearby stationary solution, this is the case of AOS. The methodology of tracking this asymptotic final state, in general different from ϕ_0, is usually referred to as modulation. In the case of the Einstein equations, this problem is compounded by the presence of infinitely many instabilities related to full group of diffeomorphism, i.e. to the general covariance of the Einstein equations[10].

2. A fundamental insight in the stability of the Minkowski space was that the Bianchi identities decouple at first order from the null structure equations which allows one to control curvature first, as a Maxwell type system (see [22]), and then proceed with the rest of the solution. This cannot work for perturbations of Kerr due to the fact that some of the null components[11] of the curvature tensor are non-trivial in Kerr.

3. Even if one succeeds in tackling the above mentioned issues, there are still major obstacles in understanding the decay properties of the solution. Indeed, when one considers the simplest, relevant, linear equation on a fixed Kerr background, i.e. the scalar wave equation $\square_{\mathbf{g}}\psi = 0$, one encounters serious difficulties to prove decay. Below is a very short description of these:

   - *The problem of trapped null geodesics.* This concerns the existence of null geodesics[12] neither crossing the event horizon nor escaping to null infinity, along which solutions can concentrate for arbitrary long times. This leads to degenerate energy-Morawetz estimates which require a very delicate analysis.
   - *The trapping properties of the horizon.* The horizon itself is ruled by null geodesics, which do not communicate with null infinity and can thus concentrate energy. This problem was solved by understanding the so called red-shift effect associated to the event horizon, which counteracts this type of trapping.
   - *The problem of superradiance.* This is the failure of the stationary Killing field $\mathbf{T} = \partial_t$ to be everywhere timelike in the domain of outer communication[13], and

[8] I.e. solutions not tied to ϕ_0 by a gauge transformation.

[9] In the case of the stability of Kerr, there exists a 2 parameter family of solutions $\mathcal{K}(a, m)$.

[10] Note that in the stability of Minkowski, even though the linearized system does not contain instabilities, one must still take general covariance into account in the far r region of the perturbed space-time due the presence of a non trivial mass. On the other hand, in perturbations of Kerr, the general covariance affects the entire construction of the spacetime.

[11] With respect to the *principal null directions of Kerr,* i.e a distinguished null pair which diagonalizes the full curvature tensor, the middle component $P = \rho + i\ {}^*\rho$ is nontrivial.

[12] In the Schwarzschild case, these geodesics are associated with the celebrated photon sphere $r = 3m$.

[13] $\mathbf{T}$ is timelike only outside of the so-called ergoregion.

thus, of the associated conserved energy to be positive. Note that this problem is absent in Schwarzschild and, in general, for axially symmetric solutions of EVE.

- *Superposition problem.* This is the problem of combining the estimates in the near region, close to the horizon, (including the ergoregion and trapping) with estimates in the asymptotic region, where the spacetime looks Minkowskian.

4. The full linearized system, whatever its formulation, presents many additional difficulties due to the huge gauge covariance of the equations[14]. In particular, the full linearized system is not conservative and we thus lack, unlike in the case of the scalar wave equation $\square_{\mathbf{g}}\psi = 0$, the most basic ingredient in controlling the solutions of the equation, i.e. energy estimates.

1.2 Linear stability

1.2.1 Linearized gravity system

Historically, two versions of linearization for EVE have been considered:

(a) At the level of the metric itself, i.e. with $\mathbf{G}_{\alpha\beta} := \mathbf{R}_{\alpha\beta} - \frac{1}{2}\mathbf{R}\mathbf{g}_{\alpha\beta}$,

$$\mathbf{G}'(\mathbf{g}_0)\,\delta\mathbf{g} = 0. \tag{1.2.1}$$

(b) At the level of curvature via the Newman-Penrose (NP) formalism, based on null frames.

In our work, we rely on a geometric variant of the second approach, see the comparison between the NP formalism, the Geroch-Held-Penrose formalism and our approach in section 2.2.4. In what follows, we describe the main known results concerning solutions to linearized equations in a Kerr background.

[14]Note that rates of decay are heavily dependent on a proper choice of gauge, thus affecting the issue of convergence.

1.2.2 Formal mode analysis

The first important results concerning both items (a) and (b) above were obtained by mathematical physicists based on the classical method of separation of variables and formal mode analysis. In the particular case when $\mathbf{g}_0$ is the Schwarzschild metric, the LGE equations (1.2.1) can be formally decomposed into modes, by using Fourier transform in time and spherical harmonics. A similar decomposition, using oblate spheroidal harmonics, can be done in Kerr. The formal study of fixed modes from the point of view of *metric perturbations* as in (1.2.1) was initiated by Regge-Wheeler [64], in perturbations of Schwarzschild, who discovered the master Regge-Wheeler equation for axial or odd-parity perturbations. This study was completed by Vishveshwara [73] and Zerilli [77]. A gauge-invariant formulation of *metric perturbations* was then given by Moncrief [59]. An alternative approach via the Newman-Penrose (NP) formalism was first undertaken by Bardeen-Press [6]. This latter type of analysis was later extended to the Kerr family by Teukolsky [72] who made the important discovery that the extreme curvature components, relative to a principal null frame, satisfy decoupled, separable, wave equations. These extreme curvature components also turn out to be gauge invariant in the sense discussed above. The full extent of what could be done by mode analysis, in both approaches, can be found in Chandrasekhar's book [16]. Chandrasekhar also introduced (see [15]) a transformation theory relating the two approaches. More precisely, he exhibited a transformation which connects the Teukolsky equations to the Regge-Wheeler one. This transformation was further elucidated and extended by R. Wald [74]. The full mode stability, i.e. lack of exponentially growing modes, for the Teukolsky equation in Kerr is due to Whiting [76].

1.2.3 Classical vectorfield method

Mode stability is far from establishing even boundedness of solutions. To achieve that and, in addition, to derive realistic decay estimates one needs an entirely different approach based on the vectorfield method[15], used in the proof of the nonlinear stability of Minkowski [23].

The vectorfield method was first developed in connection with the wave equation in Minkowski space. As well known, solutions of the wave equation $\square\psi = 0$ in the Minkowski space $\mathbb{R}^{n+1}$ both conserve energy and decay uniformly in time like $t^{-\frac{n-1}{2}}$. While conservation of energy can be established by a simple integration by parts, and is thus robust to perturbations of the Minkowski metric, decay was first derived either using the Kirchhoff

[15]Robust method based on the symmetries of Minkowski space to derive decay for nonlinear wave equations, see [44] and [46].

formula or by Fourier methods, which are manifestly not robust. An integrated version of local energy decay, based on an inspired integration by parts argument, was first derived by C. Morawetz [60], [61]. The first derivation of decay based on the commutations properties of $\square$ with Killing and conformal Killing vectorfields of Minkowski space together with energy conservation appear in [44] and [47]. That method also provides precise information about the decay properties of derivatives of solutions with respect to the standard null frame of Minkowski space, an important motivating factor in the discovery of the null condition [45], [20] and [46]. The methodology initiated with these papers, to which we refer as the classical vectorfield method, has had numerous applications to nonlinear wave equations and has played an important role in the proof of the nonlinear stability of Minkowski space [23]. The vectorfield method has also been applied to later versions of the stability of Minkowski result in [48], [55], [7], [39], and extensions of it to Einstein equation coupled with various matter fields in [8], [31], [9], [75], [54], [56], [42].

1.2.4 Scalar wave equation in Kerr and new vectorfield method

The new vectorfield method is an extension of the classical vectorfield method which compensates for the lack of enough Killing and conformal Killing vectorfields on a Schwarzschild or Kerr background by introducing new vectorfields whose deformation tensors have coercive properties in different regions of spacetime, not necessarily causal. The new method has emerged in the last fifteen years in connection to the study of boundedness and decay for the scalar wave equation in $\mathcal{K}(a,m)$, $\square_{\mathbf{g}_{a,m}}\phi = 0$. The starting and most demanding part of the new method, originating in [10], is the derivation of a global, simultaneous, *Energy-Morawetz* estimate which degenerates in the trapping region. Once an Energy-Morawetz estimate is established one can commute with the Killing vectorfields of the background, and the so called red shift vectorfield introduced in [24], to derive uniform bounds for solutions. The most efficient way to also get decay, and solve the *superposition problem*, originating in [25], is based on the presence of family of r^p*-weighted*, quasi-conformal vectorfields defined in the far r region of spacetime[16].

The first Energy-Morawetz type results for scalar wave equations in Schwarzschild are due to Soffer-Blue [10], [11] and Blue-Sterbenz [12], based on a modified version of the classical Morawetz integral energy decay estimate. Further developments appear in the works of Dafermos-Rodnianski[17], see [24], [25], and Marzuola-Metcalfe-Tataru-Tohaneanu,

[16]These replace the scaling and inverted time translation vectorfields used in [44] or their corresponding deformations used in [23]. A recent improvement of the method allowing one to derive higher order decay can be found in [5].

[17]We note in particular the red shift vectorfield, introduced in [24] to deal with the degeneracy of the Morawetz-energy estimates along the horizon and the r^p weighted estimates introduced in [25] as an

[58]. The vectorfield method can also be extended to derive decay for axially symmetric solutions in Kerr, see [41] and [68], but it is known to fail for general solutions in Kerr, see [1].

In the absence of axial symmetry the derivation of an Energy-Morawetz estimate in $\mathcal{K}(a, m)$ for $|a/m| \ll 1$ requires a more refined analysis involving both the vectorfield method and Fourier or mode decompositions, see Tataru-Tohaneanu [70] for the first full quantitative decay result (see also Dafermos-Rodnianski [26] for boundedness of solutions). The derivation of such an estimate in the full sub-extremal case $|a| < m$ is even more subtle and was achieved by Dafermos-Rodnianski-Shlapentokh-Rothman [27]. A purely physical space proof the Energy-Morawetz estimate for small $|a/m|$, which extends the classical vectorfield method to include second order operators (in this case the Carter operator [13]) was pioneered by Andersson-Blue in [4]. Their approach has the usual advantages of the classical vectorfield method, i.e it is robust with respect to perturbations, which is the very reason we pursue it in this paper.

1.2.5 Linear stability of Schwarzschild

A first quantitative proof of the linear stability of Schwarzschild spacetime was established by Dafermos-Holzegel-Rodnianski in [28]. Notable in their analysis is the treatment of the Teukolsky equation (TE) in a fixed Schwarzschild background. While TE is separable, and amenable to mode analysis, it is not Lagrangian and thus cannot be treated by energy type estimates. It is for this reason that [28] relies on a physical space version of the Chandrasekhar transformation, which takes solutions of TE to solutions of Regge-Wheeler (RW), an equation which is manifestly Lagrangian, and in addition has a positive potential. Once decay estimates for the RW equation have been established[18], the authors recover the expected decay for solutions to the original TE. The remaining work in [28] is to derive decay for the other curvature components and most of the linearized Ricci coefficients associated to the double null foliation. This last step requires carefully chosen gauge conditions, which the authors make within the framework of a double null foliation, initialized by the given Cauchy data[19] of the fixed Schwarzschild background.

effective method to derive decay estimates in the asymptotic region of black holes, as mentioned above.

[18]Based on the technology developed earlier for the scalar wave equation in Schwarzschild.

[19]Note that there is also a scalar condition for the linearized lapse along the event horizon (part of what the authors call future normalized gauge), itself initialized from initial data, see (212) and (214) in [28]. This gauge fixing from initial data leads to sub-optimal decay estimates for some metric coefficients (see (250)–(252) and (254) in [28]) and potentially for $\underline{\omega}$, and is thus inapplicable to the nonlinear case. This deficiency was fixed in [33], by relying on a linearized version of the GCM construction in [50].

1.2.6 Linear stability of Kerr for small angular momentum

The first breakthrough result on the linear stability of Kerr, for $|a|/m \ll 1$, is due to Ma [57], see also [29]. Both results are based on a generalization of the Chandrasekhar transformation to Kerr which takes the Teukolsky equations to a generalized version of the Regge-Wheeler (gRW) equation. Also, both methods depend on a combination of mode decomposition and vectorfield techniques similar to those developed for the scalar wave equation in slowly rotating Kerr. These results were recently partially[20] extended to the full subextremal range in [67].

The first stability results for the full linearized Einstein vacuum equations near $Kerr(a, m)$, for $|a|/m \ll 1$, appeared in [2] and [37]. The first paper, based on the GHP formalism[21], see [32], builds on the results of [57] while the second paper is based on an adapted version of the metric formalism and builds on the seminal work of the authors on Kerr-de-sitter [38]. Though the ultimate relevance of these papers to nonlinear stability remains open, they are both remarkable results in so far as they deal with difficulties that looked insurmountable even ten years ago.

1.3 Nonlinear stability

1.3.1 Nonlinear stability of Schwarzschild

The first nonlinear stability result of the Schwarzschild space was established in [50]. In its simplest version, the result states the following.

Theorem 1.3.1 (Klainerman-Szeftel [50])**.** *The future globally hyperbolic development of an* axially symmetric, polarized*, asymptotically flat initial data set, sufficiently close (in a specified topology) to a Schwarzschild initial data set of mass $m_0 > 0$, has a complete future null infinity $\mathcal{I}^+$ and converges in its causal past $\mathcal{J}^{-1}(\mathcal{I}^+)$ to another nearby Schwarzschild solution of mass m_∞ close to m_0.*

The restriction to axial polarized perturbations is the simplest assumption which insures that the final state is itself Schwarzschild and thus avoids the additional complications of the Kerr stability problem which we discuss below. We refer the reader to the introduction in [50] for a full discussion of the result.

[20]The analysis of [67] is still limited to modes. The authors have however announced a full proof of the result.

[21]An adapted spinorial version of the NP formalism.

Recently Dafermos-Holzegel-Rodnianski-Taylor [30] have extended the result of [50] by properly preparing a co-dimension 3 subset of the initial data such that the final state is still Schwarzschild. Like in [50], the starting point of [30] is to anchor the entire construction on a far away[22] GCM type sphere, in the sense of [51] [52], with no direct reference to the initial data. It also uses the same definition of the angular momentum as in (7.19) of [52]. Finally, the spacetime in [30] is separated in an exterior region ${}^{(ext)}\mathcal{M}$ and an interior region ${}^{(int)}\mathcal{M}$, with the ingoing foliation of ${}^{(int)}\mathcal{M}$ initialized on a timelike hypersurface, as in [50]. We note, however, that [30] does not use the geodesic foliation of [50], but instead both ${}^{(int)}\mathcal{M}$ and ${}^{(ext)}\mathcal{M}$ are foliated by double null foliations, and thus, the process of estimating the gauge dependent variables is somewhat different.

1.3.2 Nonlinear stability of slowly rotating Kerr black holes

In [53], we have stated the following theorem on the resolution of the Kerr stability conjecture for small angular momentum whose complete proof relies on [53], [51], [52], [66] and the present paper.

Theorem 1.3.2 (Kerr stability for $|a|/m \ll 1$)**.** *The future globally hyperbolic development of a general, asymptotically flat, initial data set, sufficiently close (in a suitable topology) to a Kerr(a_0, m_0) initial data set, for sufficiently small $|a_0|/m_0$, has a complete future null infinity $\mathcal{I}^+$ and converges in its causal past $\mathcal{J}^{-1}(\mathcal{I}^+)$ to another nearby Kerr spacetime Kerr(a_∞, m_∞) with parameters (a_∞, m_∞) close to the initial ones (a_0, m_0).*

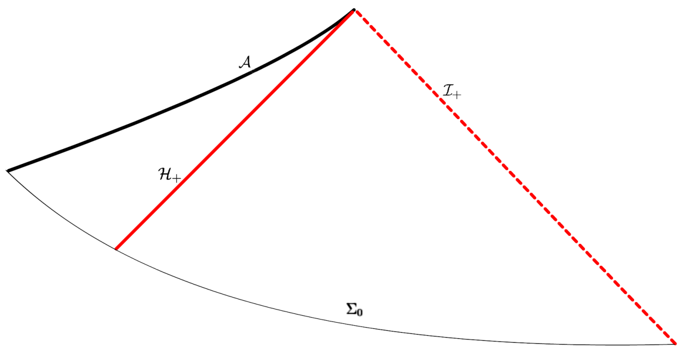

Figure 1.2: The Penrose diagram of the final space-time in the Main Theorem 1.3.2.

Remark 1.3.3. *The analog of our Main Theorem 1.3.2 in the context of the Einstein vacuum equation with a strictly positive cosmological constant, i.e. the nonlinear stability of the stationary part of Kerr-de Sitter with small angular momentum, was proved in the seminal paper [38] of Hintz and Vasy, see our discussion in section 1.1.4 in [53].*

[22] I.e. $r \gg u$, similar to the dominant in r condition (3.3.4) of [50].

The proof of the Main Theorem 1.3.2 rests on the following major ingredients:

1. A formalism to derive tensorial versions of the Teukolsky and generalized Regge-Wheeler type (gRW) equations in the full nonlinear setting. The formalism, first introduced in [36], is self-contained and vastly expanded in Part I of this paper.

2. An analytic mechanism to derive combined Morawetz-energy estimates for solutions of these gRW equations, based on an extension of the Andersson-Blue method, introduced in [4], to spin-2 wave equations in suitable perturbations of Kerr. This is developed in Part II of this paper.

3. A dynamical mechanism for finding the right gauge conditions, based on GCM (generally covariant modulated) spheres and hypersurfaces, in which convergence to the final state takes place. GCM spheres are codimension 2 compact surfaces, unrelated to the initial conditions, on which specific geometric quantities take Schwarzschildian values (made possible by taking into account the full general covariance of the Einstein vacuum equations), see the discussion in the introductions to [51], [52]. It is hard to overstate the importance of admissible GCM spheres, they are literally the keystone of our entire approach to the proof of the nonlinear stability of Kerr. The related concepts of GCM admissible spheres and hypersurfaces (these are codimension-1 spacelike hypersurfaces foliated by GCM spheres, where additional conditions are verified) have appeared first in [50] in the context of polarized symmetry. The construction of GCM spheres, without any symmetries, in realistic perturbations of Kerr, is treated in [51], [52][23], and the case of spacelike GCM hypersurfaces is treated in [66].

4. A dynamical mechanism to identify the values of (a_∞, m_∞) and the axis of rotation of the final Kerr, see sections 3.2.4 and 8.5.2 in [53]. This is based on the fact that our GCM approach allows us to define the mass m, as well as the angular momentum a in terms of intrinsically defined (using effective uniformization, see [52]) $\ell = 1$ modes of $curl\, \beta$. This was introduced[24] in [52] and used in [53][25].

5. As mentioned above, the main novelty of the GCM approach is that it relies on gauge conditions initialized at a far away co-dimension 2 sphere S_* with no direct reference to the initial conditions. This gauge choice needs however to be connected, somehow,

[23]See also chapter 16 of [30] in the particular case of perturbations of Schwarzschild, where they appear instead under the name “teleological”.

[24]Note that our definition of angular momentum, see (7.19) in [52], is unnatural from a physical point of view (though very effective for our proof). A more realistic definition was introduced in [65], and another general definition can be found in [17], see also [69] for a comprehensive discussion of the subject.

[25]As well as in [30] in the particular case of perturbations of Schwarzschild.

to the initial conditions. This is achieved in both [50] and [53] by transporting[26] the sphere S_* to a sphere S_1 in the the initial layer and compare it, using the rigidity properties of the GCM conditions, to a sphere of the initial data layer. This leads to a new foliation of the initial layer which differs substantially from the original one, due to a shift of the center of mass frame of the final black hole, known in the physics literature as a gravitational wave recoil. We refer the reader to section 8.3 in [53] for the details.

6. A precisely formulated continuity argument, based on a grand bootstrap scheme, which assigns to all geometric quantities involved in the process specific decay rates, which can then be dynamically recovered from the initial conditions by a long series of estimates, and thus ensure convergence to a final Kerr state, see sections 3.5 and 3.7 in [53].

7. The continuity argument is based on the crucial concept of finite *GCM admissible* spacetimes $\mathcal{M} = {}^{(ext)}\mathcal{M} \cup {}^{(int)}\mathcal{M} \cup {}^{(top)}\mathcal{M}$, see Figure 1.3, whose defining characteristic is its spacelike *GCM boundary* Σ_*. Note that the boundaries ${}^{(ext)}\mathcal{M} \cap {}^{(top)}\mathcal{M}$ and ${}^{(int)}\mathcal{M} \cap {}^{(top)}\mathcal{M}$ are timelike[27] and that ${}^{(top)}\mathcal{M}$ is needed to have the entire space $\mathcal{M}$ causal. The regions ${}^{(ext)}\mathcal{M}$ and ${}^{(int)}\mathcal{M}$ are separated by the timelike hypersurface $\mathcal{T}$ and the spacelike boundary $\mathcal{A}$ is beyond the future horizon $\mathcal{H}_+$ of the limiting space. Finally the region $\mathcal{L}_0$, is the initial data layer in which $\mathcal{M}$ is prescribed as a solution of the Einstein vacuum equations. We direct the reader to section 3.2 of [53] for more details on the construction.

Here is a short summary of where these issues are dealt with in our work:

- Papers [51], [52] provide a framework for dealing with the issue (4), by constructing general covariant modulated (GCM) spheres, generalizing those used in the nonlinear stability of Schwarzschild in the polarized case in [50], in the asymptotic region of a general perturbation of Kerr. Paper [52] also contains a definition of the angular momentum for GCM spheres. Relying on these GCM spheres, spacelike GCM hypersurfaces are constructed in [66], generalizing the construction of GCM hypersurfaces in polarized symmetry of [50] to the case of general perturbations of Kerr. These results are applied in [53] to the construction of the crucial spacelike GCM boundary Σ_*, see section 8.4 and 8.5 in that paper.

- Part I of this paper deals with issue (1) by developing a geometric formalism of non-integrable horizontal structures, well adapted to perturbations of Kerr, and use it

[26] That is, we transport the $\ell = 1$ modes of some quantities from S_* to S_1, see section 8.3.1 in [53].
[27] Asymptotically null as we pass to the limit.

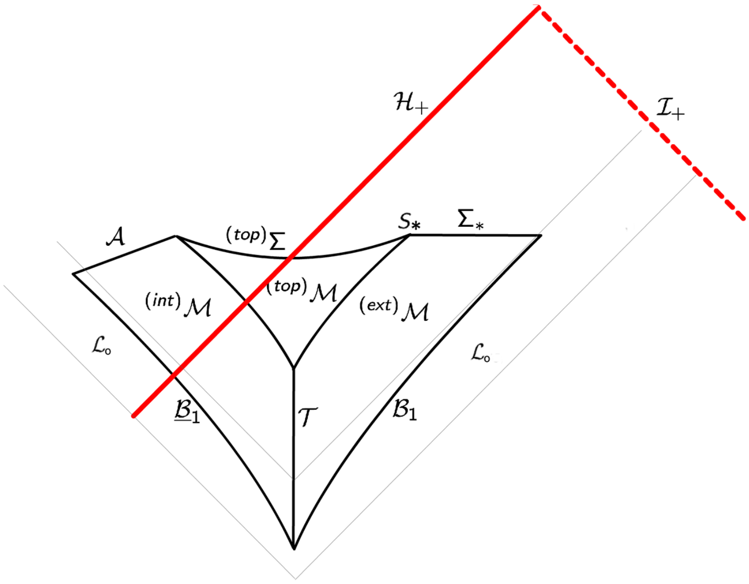

Figure 1.3: The GCM admissible space-time $\mathcal{M}$

to derive the generalized Regge-Wheeler (gRW) equations in the context of general perturbations of Kerr[28].

- Paper [53] contains a precise version of Theorem 1.3.2, definition of the main objects and a roadmap for the entire proof. The proof is split into 9 distinct steps, Theorems M0-M8, and full proofs are given to all but three of them. Theorems M1, M2 and half of M8 were carefully stated and their proofs were assumed as a blackbox and delayed to the present paper.

- Part II of the present paper deals with issue (2). It provides complete proofs of Theorems M1 and M2 mentioned above, by deriving estimates for gRW using an extension of the classical vectorfield method[29], based on commutation with a non-linear version of the Carter operator. In the context of the standard scalar wave equation in Kerr, such an approach was developed by Andersson and Blue in their important paper [4].

- The nonlinear terms present in the full version of the gRW equation derived in [36], as well as those generated by commutation with vectorfields and second order Carter

[28]In the linear case, recall that complex scalar versions of such equations were derived in [57], see also [29], based on an extension of the physical space Chandrasekhar type transformation introduced in [15].

[29]The results on decay in [57] and [29], on the other hand, depend heavily on mode decompositions for the linearized gRW equations in Kerr, an approach whose generalization to the full nonlinear setting seems to present substantial difficulties.

operator, are treated in a similar spirit[30] as the treatment of the nonlinear terms in [50], by showing that they verify a favorable null type structure.

- Part III of this paper gives a full derivation of the estimates for the top derivatives of the curvature components, i.e. a proof of the second part of Theorem M8 stated without proof in Theorem 9.4.15 in [53].

In the remaining part of this introduction, we give a short presentation of the main ingredients of this paper and refer to the introduction in [53] for a presentation of the other steps in the proof of Theorem 1.3.2, i.e Theorems M0, M3–M7, and the first part of M8.

1.4 Geometric set-up

1.4.1 Spacetime $\mathcal{M}$

The geometric setting of this paper consists of an Einstein vacuum Lorentzian manifold $(\mathcal{M}, \mathbf{g})$ with boundaries equipped with the following:

1. A regular horizontal structure defined by a null pair (e_3, e_4), and the space $\mathcal{H}$ orthogonal to it. Note that the horizontal structure considered here is not integrable[31]. The formalism of non-integrable horizontal structures, on which of our entire work is based, is developed in full in Chapter 2. Chapter 3 contains a description of the standard non-integrable horizontal structure of Kerr.

2. Two constants (a, m) with $|a| < m$, two scalar functions (r, θ) and a time function τ on $\mathcal{M}$. In addition, $\mathcal{M}$ possesses a horizontal complex 1-form[32] $\mathfrak{J}$, used to linearize all horizontal 1-forms in perturbations of Kerr.

3. Boundaries given by $\partial\mathcal{M} = \mathcal{A} \cup \Sigma(\tau_*) \cup \Sigma_* \cup \Sigma(1)$ where

   - $\mathcal{A}$ is the spacelike hypersurface given by

$$\mathcal{A} \;:=\; \mathcal{M} \cap \{r = r_+(1 - \delta_{\mathcal{H}})\}, \qquad r_+ := m + \sqrt{m^2 - a^2},$$

[30]We note however that Theorem M2 is treated different here, see discussion below.

[31]In other words, the space $\mathcal{H}$ forms a non integrable distribution. The formalism was originally mentioned in [40] and developed in [36].

[32]By this, we mean $\mathfrak{J} = j + i\,{}^*j$ where j is a real horizontal 1-form. In Kerr this quantity is specifically introduced in Definition 3.4.2.

where $\delta_{\mathcal{H}} > 0$ a sufficiently small constant.

- $\Sigma(1)$ and $\Sigma(\tau_*)$ denote the spacelike level hypersurfaces $\tau = 1$ and $\tau = \tau_*$, with $\tau_* > 1$ and $1 \le \tau \le \tau_*$ on $\mathcal{M}$.
- Σ_* is a uniformly spacelike hypersurface connecting $\Sigma(1)$ to $\Sigma(\tau_*)$.

4. Two spacetime regions ${}^{(int)}\mathcal{M}$ and ${}^{(ext)}\mathcal{M}$ such that

$$\mathcal{M} = {}^{(int)}\mathcal{M} \cup {}^{(ext)}\mathcal{M}, \qquad {}^{(ext)}\mathcal{M} = \mathcal{M}_{r \ge r_0}, \qquad {}^{(int)}\mathcal{M} = \mathcal{M}_{r \le r_0},$$

where $r_0 \gg m$ is a sufficiently large constant.

Remark 1.4.1. *Note that the spacetime $\mathcal{M}$ considered above does not require any specific gauge conditions. Indeed, in this paper, we only provide gauge independent curvature estimates. The control of Ricci coefficients is provided in [53] where specific gauge choices are made, see section 2.3 and 2.8 for the definitions of PG and PT structures in [53]. We also note that the scalar functions r, θ and τ are not aligned with the frame, i.e. unlike in the stability of Minkowski space, in [23], and all other subsequent works[33], our frames are in no way adapted to foliations.*

The function τ is used to define the regions of integrations $\mathcal{M}(\tau_1, \tau_2)$ where $\tau_1 \le \tau \le \tau_2$. We also define the following significant regions of $\mathcal{M}$, see Definition 9.1.2.

Definition 1.4.2. *We define the following regions of $\mathcal{M}$:*

1. *We define the trapping region of $\mathcal{M}$ to be the set*

$$\mathcal{M}_{trap} := \mathcal{M} \cap \left\{ \frac{|\mathcal{T}|}{r^3} \le \delta_{trap} \right\}, \qquad \delta_{trap} = \frac{1}{10},$$

where $\mathcal{T} = \mathcal{T} = r^3 - 3mr^2 + a^2 r + ma^2$. This is the region that contains all trapped null geodesics, for sufficiently small a/m.

2. *We denote $\mathcal{M}_{\not{trap}}$ the complement to the trapping region $\mathcal{M}_{trap}$.*

3. *We denote $\mathcal{M}_{red} := \mathcal{M} \cap \{ r \le r_+(1 + 2\delta_{red}) \}$, for a sufficiently small constant $\delta_{red} > 0$, the region where the red shift effect of the horizon is manifest.*

[33]We note however that in the treatment of the Regge Wheeler equation in Chapter 10 of [50] the foliations used are also not aligned with the frame.

1.4.2 Ricci and curvature coefficients

Definition of the Ricci and curvature coefficients

We can define, with respect to the horizontal structure associated to (e_3, e_4), connection and curvature coefficients similar to those in the integrable case, as in [23],

$$
\begin{aligned}
\underline{\chi}_{ab} &= \mathbf{g}(\mathbf{D}_a e_3, e_b), & \chi_{ab} &= \mathbf{g}(\mathbf{D}_a e_4, e_b), & \underline{\xi}_a &= \frac{1}{2}\mathbf{g}(\mathbf{D}_3 e_3, e_a), & \xi_a &= \frac{1}{2}\mathbf{g}(\mathbf{D}_4 e_4, e_a),\\
\underline{\omega} &= \frac{1}{4}\mathbf{g}(\mathbf{D}_3 e_3, e_4), & \omega &= \frac{1}{4}\mathbf{g}(\mathbf{D}_4 e_4, e_3), & \underline{\eta}_a &= \frac{1}{2}\mathbf{g}(\mathbf{D}_4 e_3, e_a), & \eta_a &= \frac{1}{2}\mathbf{g}(\mathbf{D}_3 e_4, e_a),\\
\zeta_a &= \frac{1}{2}\mathbf{g}(\mathbf{D}_a e_4, e_3),
\end{aligned}
$$

$$
\alpha_{ab} = \mathbf{R}_{a4b4}, \quad \beta_a = \frac{1}{2}\mathbf{R}_{a434}, \quad \underline{\beta}_a = \frac{1}{2}\mathbf{R}_{a334}, \quad \underline{\alpha}_{ab} = \mathbf{R}_{a3b3}, \quad \rho = \frac{1}{4}\mathbf{R}_{3434}, \quad {}^*\rho = \frac{1}{4}\,{}^*\mathbf{R}_{3434},
$$

and derive the corresponding null structure and null Bianchi equations, see Propositions 2.2.5 and 2.2.6. The non-symmetric 2 tensors $\chi, \underline{\chi}$ are decomposed as follows.

$$
\chi_{ab} = \widehat{\chi}_{ab} + \frac{1}{2}\delta_{ab}\mathrm{tr}\,\chi + \frac{1}{2}\in_{ab}\ {}^{(a)}\mathrm{tr}\chi, \qquad \underline{\chi}_{ab} = \widehat{\underline{\chi}}_{ab} + \frac{1}{2}\delta_{ab}\mathrm{tr}\,\underline{\chi} + \frac{1}{2}\in_{ab}\ {}^{(a)}\mathrm{tr}\underline{\chi},
$$

where the scalars $\mathrm{tr}\,\chi$, $\mathrm{tr}\,\underline{\chi}$ and ${}^{(a)}\mathrm{tr}\chi$, ${}^{(a)}\mathrm{tr}\underline{\chi}$ are given by

$$
\mathrm{tr}\,\chi := \delta^{ab}\chi_{ab}, \qquad \mathrm{tr}\,\underline{\chi} := \delta^{ab}\underline{\chi}_{ab}, \qquad {}^{(a)}\mathrm{tr}\chi := \in^{ab}\chi_{ab}, \qquad {}^{(a)}\mathrm{tr}\underline{\chi} := \in^{ab}\underline{\chi}_{ab}.
$$

Remark 1.4.3. *The non integrability of (e_3, e_4) corresponds to the non vanishing ${}^{(a)}tr\chi$ and ${}^{(a)}tr\underline{\chi}$. A well known example of a non integrable null frame, is the principal null frame of Kerr for which ${}^{(a)}tr\chi$ and ${}^{(a)}tr\underline{\chi}$ are indeed non trivial, see section 3.3.*

1.4.3 Frame transformations

To start with, given an arbitrary perturbation of Kerr, there is no reason to prefer an horizontal structure to any other one. It is thus essential that we consider all possible frame transformations from one horizontal structure $(e_4, e_3, \mathcal{H})$ to another one $(e_4', e_3', \mathcal{H}')$ together with the transformation formulas $\Gamma \to \Gamma'$, $R \to R'$ they generate for the Ricci and curvature coefficients. The most general transformation formulas between two null frames is given in Lemma 2.2.1 in [53]. It depends on two horizontal 1-forms $f, \underline{f}$ and a real scalar function λ and is given by

$$
\begin{aligned}
e_4' &= \lambda\left(e_4 + f^b e_b + \frac{1}{4}|f|^2 e_3\right),\\
e_a' &= \left(\delta_a^b + \frac{1}{2}\underline{f}_a f^b\right) e_b + \frac{1}{2}\underline{f}_a e_4 + \left(\frac{1}{2}f_a + \frac{1}{8}|f|^2\underline{f}_a\right) e_3, \qquad (1.4.1)\\
e_3' &= \lambda^{-1}\left(\left(1 + \frac{1}{2}f\cdot\underline{f} + \frac{1}{16}|f|^2|\underline{f}|^2\right) e_3 + \left(\underline{f}^b + \frac{1}{4}|\underline{f}|^2 f^b\right) e_b + \frac{1}{4}|\underline{f}|^2 e_4\right).
\end{aligned}
$$

The very important transformation formulas $\Gamma \to \Gamma'$, $R \to R'$ are given in Proposition 2.2.3 of [53].

1.4.4 Basic equations and complexification

The null structure and null Bianchi equations verified by the Ricci and curvature coefficients are derived in sections 2.2. These equations simplify considerably, see section 2.4, by introducing complex notations:

$$
\begin{aligned}
&A := \alpha + i\,{}^*\alpha, \quad B := \beta + i\,{}^*\beta, \quad P := \rho + i\,{}^*\rho, \quad \underline{B} := \underline{\beta} + i\,{}^*\underline{\beta}, \quad \underline{A} := \underline{\alpha} + i\,{}^*\underline{\alpha},\\
&X := \chi + i\,{}^*\chi, \quad \underline{X} := \underline{\chi} + i\,{}^*\underline{\chi}, \quad H := \eta + i\,{}^*\eta, \quad \underline{H} := \underline{\eta} + i\,{}^*\underline{\eta}, \quad Z := \zeta + i\,{}^*\zeta,\\
&\Xi := \xi + i\,{}^*\xi, \quad \underline{\Xi} := \underline{\xi} + i\,{}^*\underline{\xi},
\end{aligned}
$$

where * denotes the Hodge dual as defined in Definition 2.1.7. In particular, note that $\mathrm{tr}X = \mathrm{tr}\,\chi - i\,{}^{(a)}\mathrm{tr}\chi$, $\mathrm{tr}\underline{X} = \mathrm{tr}\,\underline{\chi} - i\,{}^{(a)}\mathrm{tr}\underline{\chi}$, while $\widehat{X}$ and $\widehat{\underline{X}}$ denote the symmetric traceless part of X and $\underline{X}$ respectively. Further useful simplifications of the equations can be obtained with the help of conformally invariant derivative operators introduced in section 2.2.9.

1.4.5 Kerr and $O(\epsilon)$-perturbations of Kerr

The preferred (principal) null pair of Kerr is given in Boyer Lindquist coordinates by[34]

$$
e_4 = \frac{r^2+a^2}{|q|^2}\partial_t + \frac{\Delta}{|q|^2}\partial_r + \frac{a}{|q|^2}\partial_\phi, \qquad e_3 = \frac{r^2+a^2}{\Delta}\partial_t - \partial_r + \frac{a}{\Delta}\partial_\phi.
$$

[34]There is an indeterminacy in the principal null pair as one may replace (e_3, e_4) with $(\lambda^{-1}e_3, \lambda e_4)$ for any $\lambda > 0$. The formulas provided here correspond to a choice of $\lambda > 0$ ensuring $\mathbf{D}_3 e_3 = 0$ and thus $\underline{\omega} = 0$, $\underline{\xi} = 0$, which is regular across the horizon towards the future. This is called the canonical incoming null frame of Kerr.

The horizontal structure associated to this null pair is spanned by the vectorfields

$$e_1 = \frac{1}{|q|}\partial_\theta, \qquad e_2 = \frac{a\sin\theta}{|q|}\partial_t + \frac{1}{|q|\sin\theta}\partial_\phi. \tag{1.4.2}$$

The complexified Ricci and curvature coefficients take a particularly simple form in Kerr, relative to the above principal null pair, see Section 3,

$$\widehat{X} = \underline{\widehat{X}} = \Xi = \underline{\Xi} = \underline{\omega} = 0, \qquad A = B = \underline{B} = \underline{A} = 0,$$

and

$$\begin{aligned} \mathrm{tr}X &= \frac{2\Delta\overline{q}}{|q|^4}, & \mathrm{tr}\underline{X} &= -\frac{2}{\overline{q}}, & P &= -\frac{2m}{q^3}, \\ \underline{H} &= -\frac{a\overline{q}}{|q|^2}\mathfrak{J}, & H &= \frac{aq}{|q|^2}\mathfrak{J}, & Z &= \frac{aq}{|q|^2}\mathfrak{J}, \end{aligned}$$

where $q = r + ia\cos\theta$ and $\Delta = r^2 + a^2 - 2mr$, relative to the Boyer-Lindquist (BL) coordinates (r, θ), and where the horizontal complex 1-form $\mathfrak{J}$ is given, relative to the horizontal basis (e_1, e_2), by the formula

$$\mathfrak{J}_1 = \frac{i\sin\theta}{|q|}, \qquad \mathfrak{J}_2 = \frac{\sin\theta}{|q|}. \tag{1.4.3}$$

Relative to the ingoing frame (e_3, e_4) and complex 1 form $\mathfrak{J}$ the Killing vectorfields $\mathbf{T}$ and $\mathbf{Z}$ take the form

$$\begin{aligned} \mathbf{T} &= \frac{1}{2}\left(e_4 + \frac{\Delta}{|q|^2}e_3 - 2a\Re(\mathfrak{J})^b e_b\right), \\ \mathbf{Z} &= \frac{1}{2}\left(2(r^2+a^2)\Re(\mathfrak{J})^b e_b - a(\sin\theta)^2 e_4 - \frac{a(\sin\theta)^2\Delta}{|q|^2}e_3\right). \end{aligned} \tag{1.4.4}$$

The same formula is used to define approximate Killing vectorfields in perturbations of Kerr, see Definition 4.3.1. Note that $\mathbf{T}$ becomes spacelike in the ergoregion $|q|^2 < 2mr$.

A spacetime $\mathcal{M}$, endowed with an horizontal structure $(e_3, e_4, \mathcal{H})$ is said to be an $O(\epsilon)$ perturbation of Kerr if all quantities which vanish in Kerr are $O(\epsilon)$, and if all other quantities stay bounded in an $O(\epsilon)$ neighborhood of their corresponding[35] Kerr values. The definition is, of course, ambiguous in the sense that any other horizontal structure $(e_3', e_4', \mathcal{H}')$ connected to $(e_3, e_4, \mathcal{H})$ by the frame transformation (1.4.1) with $f, \underline{f} = O(\epsilon)$ and $\lambda = 1 + O(\epsilon)$ is also an $O(\epsilon)$-perturbation of Kerr. Nevertheless the definition is useful

[35]To make this precise, we also need a definition of functions (r, θ) and of a complex 1-form $\mathfrak{J}$, see section 1.4.6.

in that it brings to light the remarkable fact that the extreme curvature components are in fact $O(\epsilon^2)$ invariant. This can be easily seen from the transformation formulas

$$\lambda^{-2}\alpha' = \alpha + \left(f\widehat{\otimes}\beta - {}^*f\widehat{\otimes}{}^*\beta\right) + \left(f\widehat{\otimes}f - \frac{1}{2}{}^*f\widehat{\otimes}{}^*f\right)\rho + \frac{3}{2}\left(f\widehat{\otimes}{}^*f\right){}^*\rho + O(\epsilon^3),$$
$$\lambda^{2}\underline{\alpha}' = \underline{\alpha} + \left(\underline{f}\widehat{\otimes}\underline{\beta} - {}^*\underline{f}\widehat{\otimes}{}^*\underline{\beta}\right) + \left(\underline{f}\widehat{\otimes}\underline{f} - \frac{1}{2}{}^*\underline{f}\widehat{\otimes}{}^*\underline{f}\right)\rho + \frac{3}{2}\left(\underline{f}\widehat{\otimes}{}^*\underline{f}\right){}^*\rho + O(\epsilon^3),$$

see Proposition 2.2.3 of [53].

Remark 1.4.4. *It is this fact that allows us to treat $\alpha, \underline{\alpha}$ differently from all other quantities, by choosing frames best adapted to their analysis.*

1.4.6 Linearization of the Ricci and curvature coefficients

Definition of linearized quantities

Since the quantities $\widehat{X}$, $\widehat{\underline{X}}$, Ξ, $\underline{\Xi}$, $\underline{\omega}$, A, B, $\underline{B}$, $\underline{A}$ all vanish in Kerr, it suffices to linearize the remaining quantities, i.e. $\mathrm{tr}X$, $\mathrm{tr}\underline{X}$, ω, H, $\underline{H}$, Z and P. The linearization is given by subtracting the Kerr values as follows, see section 4.1.1:

$$\widetilde{\mathrm{tr}X} := \mathrm{tr}X - \frac{2\overline{q}\Delta}{|q|^4}, \qquad \widetilde{\mathrm{tr}\underline{X}} := \mathrm{tr}\underline{X} + \frac{2}{\overline{q}}, \qquad \check{P} := P + \frac{2m}{q^3}, \qquad \check{\omega} := \omega + \frac{1}{2}\partial_r\left(\frac{\Delta}{|q|^2}\right),$$
$$\check{H} := H - \frac{aq}{|q|^2}\mathfrak{J}, \qquad \check{\underline{H}} := \underline{H} + \frac{a\overline{q}}{|q|^2}\mathfrak{J}, \qquad \check{Z} := Z - \frac{aq}{|q|^2}\mathfrak{J}.$$

Notation (Γ_g, Γ_b) for Ricci coefficients

We group the linearized Ricci coefficients in two subsets reflecting their expected decay properties, see section 4.1.2:

$$\Gamma_g := \left\{\widetilde{\mathrm{tr}X}, \quad \widehat{X}, \quad \widetilde{\mathrm{tr}\underline{X}}, \quad \check{\underline{H}}, \quad \check{Z}, \quad \check{\omega}, \quad \Xi\right\},$$
$$\Gamma_b := \left\{\widehat{\underline{X}}, \quad \check{H}, \quad \underline{\omega}, \quad \underline{\Xi}\right\}.$$

Remark 1.4.5. *In fact, (Γ_g, Γ_b) also include the linearization of the derivatives of the scalar functions $(r, \cos\theta)$, and of the complex horizontal 1-form $\mathfrak{J}$, see section 4.1.2.*

The justification for the above decompositions has to do with the expected decay properties of the linearized components in perturbations of Kerr, with respect to τ and r. More precisely, see (11.1.1),

$$\begin{aligned} \left|\mathfrak{d}^{\leq s}\Gamma_g\right| &\lesssim \epsilon \min\left\{r^{-2}\tau^{-1/2-\delta_{dec}},\ r^{-1}\tau^{-1-\delta_{dec}}\right\}, \\ \left|\mathfrak{d}^{\leq s}\Gamma_b\right| &\lesssim \epsilon r^{-1}\tau^{-1-\delta_{dec}}, \end{aligned} \tag{1.4.5}$$

for a small constant $\delta_{dec} > 0$, where $\mathfrak{d} = \{\nabla_3, r\nabla_4, r\nabla\}$ denotes weighted derivatives, and $\epsilon > 0$ is a sufficiently small bootstrap constant. We note also that the curvature components $\underline{A}, r\underline{B}$ behave in the same way as Γ_b, while $r(\check{P}, B, A)$ behave like Γ_g. Moreover A, B get the optimal decay in powers of r, i.e.

$$|A|, |B| \lesssim \epsilon r^{-7/2-\delta_{dec}}.$$

1.5 Main theorems

We refer to section 3.4 of [53] for a precise statement of our Main Theorem concerning the stability of Kerr and to section 3.7 of [53] the main steps in the proof. Here we concentrate on a simplified set of assumptions needed for the proof of Theorems M1, M2 and the curvature estimates for Theorem M8.

1.5.1 Smallness constants

The following constants are involved in the statement of Theorems M0-M8, see section 3.4. in [53]:

- The constants $m_0 > 0$ and $|a_0| \ll m_0$ are the mass and the angular momentum of the Kerr solution relative to which our initial perturbation is measured.
- The integer k_{large} which corresponds to the maximum number of derivatives of the solution.
- The size of the initial data perturbation is measured by $\epsilon_0 > 0$.
- The size of the bootstrap assumption norms are measured by $\epsilon > 0$.
- $r_0 > 0$ is tied to ${}^{(int)}\mathcal{M} \cap {}^{(ext)}\mathcal{M} = \{r = r_0\}$.

- The constant $\delta_{\mathcal{H}}$ tied to the definition of $\mathcal{A} = \{r = r_+(1-\delta_{\mathcal{H}})\}$.
- δ_{dec} is tied to decay estimates in τ for the linearized quantities of section 1.4.6.

These constants are chosen such that

$$\begin{aligned} &0 < \delta_{\mathcal{H}},\ \delta_{dec} \ \ll \min\{m_0 - |a_0|, 1\}, \\ &r_0 \gg \max\{m_0, 1\}, \qquad k_{large} \gg \frac{1}{\delta_{dec}}. \end{aligned} \tag{1.5.1}$$

Then, ϵ and ϵ_0 are chosen such that

$$0 < \epsilon_0, \epsilon \ll \min\left\{\delta_{dec}, \frac{1}{r_0}, \frac{1}{k_{large}}, m_0 - |a_0|, 1\right\}, \tag{1.5.2}$$

$$\epsilon_0, \epsilon \ll |a_0| \quad \text{in the case } a_0 \neq 0, \tag{1.5.3}$$

and

$$\epsilon = \epsilon_0^{\frac{2}{3}}. \tag{1.5.4}$$

Also, we introduce the integer k_{small} which corresponds to the number of derivatives for which the solution satisfies decay estimates. It is related to k_{large} by

$$k_{small} = \left\lfloor \frac{1}{2} k_{large} \right\rfloor + 1. \tag{1.5.5}$$

From now on, in the rest of the paper, $\lesssim$ means bounded by a constant depending only on geometric universal constants (such as Sobolev embeddings, elliptic estimates,...) as well as the constants

$$m_0,\ a_0,\ \delta_{\mathcal{H}},\ \delta_{dec},\ r_0,\ k_{large},$$

but not on ϵ and ϵ_0.

1.5.2 Initial data assumptions

The initial data norm denoted by $\mathfrak{I}_k$, measures the size of the perturbation from Kerr at $\tau = 1$, for the top k derivatives of the curvature tensor[36].

[36]The definition used here differs slightly from the one in Definition 9.4.9 in [53], but easily follows from it by a local existence argument.

Definition 1.5.1. *We define the following initial data norms on* Σ_1

$$\begin{aligned}\mathfrak{I}_k := \sup_{S\subset\Sigma_1} r^{\frac{5}{2}+\delta_B}\Big(\big\|\mathfrak{d}^k(A,B)\big\|_{L^2(S)}+\big\|\mathfrak{d}^k B\big\|_{L^2(S)}\Big)\\ +\sup_{S\subset\Sigma_1}\Big(r^2\big\|\mathfrak{d}^k\check{P}\big\|_{L^2(S)}+r\big\|\mathfrak{d}^k\underline{B}\big\|_{L^2(S)}+\big\|\mathfrak{d}^k\underline{A}\big\|_{L^2(S)}\Big).\end{aligned} \tag{1.5.6}$$

In this paper, we make the following assumption on the control of the initial data norm[37]

$$\mathfrak{I}_{k_{large}+7} \leq \epsilon_0. \tag{1.5.7}$$

The bound (1.5.7) will be used both in Part II and Part III as assumptions on the initial data.

1.5.3 Quantitative assumptions on the spacetime $\mathcal{M}$

The quantitative assumptions made in this article depend on a large positive integer k_L, representing the maximal number of derivatives for the linearized Ricci and curvature coefficients $(\check{\Gamma}, \check{R})$ which are required in the proof. There are in fact two types of assumptions:

1. For the proof of Theorem M1 and M2 of [53], we rely on the following pointwise quantitative assumptions on Γ_b and Γ_g, for $k \leq k_L$,

$$\begin{aligned}\Big(r^2\tau^{\frac{1}{2}+\delta_{dec}}+r\tau^{1+\delta_{dec}}\Big)|\mathfrak{d}^{\leq k}\Gamma_g| \leq \epsilon,\\ r\tau^{1+\delta_{dec}}|\mathfrak{d}^{\leq k}\Gamma_b| \leq \epsilon.\end{aligned} \tag{1.5.8}$$

2. For the proof of the curvature estimates of Theorem M8 of [53], we introduce weighted energy-Morawetz type norms for curvature and Ricci coefficients, denoted respectively by $\mathfrak{R}_k$ and $\mathfrak{G}_k$, see section 13.5 for the precise definition. We then rely on the following quantitative assumptions on $\mathfrak{R}_k$ and $\mathfrak{G}_k$

$$\mathfrak{R}_k+\mathfrak{G}_k \leq \epsilon, \qquad 0\leq k\leq k_L, \tag{1.5.9}$$

as well as the following pointwise quantitative assumptions on Γ_b and Γ_g

$$r^2|\mathfrak{d}^k\Gamma_g|+r|\mathfrak{d}^k\Gamma_b| \leq \frac{\epsilon}{\tau_{trap}^{1+\delta_{dec}}}, \qquad 0\leq k\leq \frac{k_L}{2}, \tag{1.5.10}$$

[37] The original assumption on initial data in [53] is stated for $k_{large}+10$ derivatives, see (3.4.7) in that paper, in a given frame of an initial data layer $\mathcal{L}(a_0, m_0)$. The control in the frames used in this paper are obtained in Theorem M0 of section 3.7.1 in [53], and in Theorem 9.4.12 in [53] for $k_{large}+7$ derivatives.

where the scalar function τ_{trap} defined by

$$\tau_{trap} := \begin{cases} 1+\tau & \text{on } \mathcal{M}_{trap}, \\ 1 & \text{on } \mathcal{M}_{\not{trap}}. \end{cases}$$

The integer k_L is chosen as follows:

- For the proof of Theorem M1 and M2 of [53] (restated in Theorem 1.5.2 and 1.5.3 below), we choose $k_L = k_{small} + 120$. Then, (1.5.8) follows by interpolation from the bootstrap assumptions (3.5.1) (3.5.2) in [53] together with the construction of the global frame in section 3.6.3 of [53], where (3.5.1) in [53] are bootstrap assumptions on boundedness for $k \leq k_{large}$ derivatives, and (3.5.2) in [53] are bootstrap assumptions on decay for $k \leq k_{small}$ derivatives.
- For the proof of the curvature estimates of Theorem M8 (see Theorem 1.5.4 below), we choose $k_L = k_{large} + 7$. Then, (1.5.9) follows from the bootstrap assumptions (9.4.20) of [53] together with the construction of the global frame in section 9.4 of [53]. Also, (1.5.10) is a non sharp consequence of the bootstrap assumptions (9.4.22) in [53] together with the construction of the global frame in section 9.4 of [53].

1.5.4 Statement of the main theorems

Recall that the nonlinear stability of the Kerr family for small angular momentum, i.e $|a|/m \ll 1$, is stated in the Main Theorem in section 3.4 of [53]. The proof is divided in a sequence of nine intermediary steps, called Theorem M0–M8, see section 3.7 in [53]. The goal of the present paper is to provide the proof of Theorems M1 and M2 as well the curvature estimates of Theorem M8, which were stated without proof in Theorem 9.4.15 of [53] and all involve curvature estimates of hyperbolic type.

Theorems M1 and M2

In what follows, we restate[38] Theorem M1 and M2, see section 3.7.1 in [53].

Theorem 1.5.2 (Theorem M1 in [53])**.** *Assume that the spacetime $\mathcal{M}$ as defined in section 1.4.1 verifies the quantitative assumptions* (1.5.8)*, and the assumption* (1.5.7) *on*

[38] A more precise statement is given in Theorems 11.7.1 and 12.4.4.

initial data. Then, if $\epsilon_0 > 0$ is sufficiently small, there exists $\delta_{extra} > \delta_{dec}$ such that we have the following estimates in $\mathcal{M}$, for all $k \leq k_L - 20$,

$$\sup_{\mathcal{M}} \left(\frac{r^2(2r+\tau)^{1+\delta_{extra}}}{\log(1+\tau)} + r^3(2r+\tau)^{\frac{1}{2}+\delta_{extra}} \right) \Big(|\mathfrak{d}^k A| + r|\mathfrak{d}^{k-1}\nabla_3 A| \Big) \lesssim \epsilon_0.$$

Also, the quantity $\mathfrak{q}$ introduced below, see section 1.6.1, satisfies, for all $k \leq k_L - 20$,

$$\int_{\Sigma_*(\geq\tau)} |\nabla_3 \mathfrak{d}^{k-1}\mathfrak{q}|^2 \lesssim \epsilon_0^2 \tau^{-2-2\delta_{extra}}.$$

Theorem 1.5.3 (Theorem M2 in [53])**.** *In addition to the assumptions of Theorem 1.5.2, we make the following assumption*[39] *on Σ_**

$$\min_{\Sigma_*} r \geq \delta_* \epsilon_0^{-1} \tau_*^{1+\delta_{dec}} \tag{1.5.11}$$

for some small universal constant $\delta_* > 0$. Then, we have the following decay estimates for $\underline{A}$ along Σ_*

$$\max_{0\leq k\leq k_L-40} \int_{\Sigma_*} \tau^{2+2\delta_{dec}} |\mathfrak{d}^k \underline{A}|^2 \lesssim \epsilon_0^2.$$

Both results are proved in Part II of this paper.

Curvature estimates in Theorem M8

Theorem M8 in [53] is proved through an iteration procedure described in section 9.4.7 of [53]. The control of the Ricci coefficients have been derived in Chapter 9 of [53]. In the present paper, we derive the remaining estimates for the proof of Theorem M8, i.e the estimates for curvature stated in Theorem 9.4.15 of [53]. To this end, we introduce weighed L^2 type norms $\mathfrak{R}_k$ and $\mathfrak{G}_k$ respectively for curvature and Ricci coefficients[40], and decompose $\mathfrak{R}_k$ and $\mathfrak{G}_k$ in their restrictions ${}^{(int)}\mathfrak{R}$, ${}^{(int)}\mathfrak{G}$ to ${}^{(int)}\mathcal{M}$ and ${}^{(ext)}\mathfrak{R}$, ${}^{(ext)}\mathfrak{G}$ to ${}^{(ext)}\mathcal{M}$, see section 13.5 for the precise definition of these norms. In view of the results in Chapter 9 of [53], the proof of Theorem 8 reduces to the following result on the control of the curvature norm $\mathfrak{R}_k$.

Theorem 1.5.4 (Theorem 9.4.15 of [53])**.** *Assume that the spacetime $\mathcal{M}$ as defined in section 1.4.1 verifies the quantitative assumptions* (1.5.9) (1.5.10) *for $k_L = k_{large} + 7$, and*

[39] This is the dominant condition of r on Σ_*, see (3.4.5) in [53].

[40] As well as derivatives of $(r, \cos\theta)$ and $\mathfrak{J}$.

the assumption (1.5.7) *on initial data. Let* $k_{small} - 1 \le J \le k_{large} + 6$*. Then, we have the following boundedness estimates for all components of curvature*

$$
\begin{aligned}
{}^{(int)}\mathfrak{R}^2_{J+1} \lesssim & r_0^{18}\Big(\epsilon_J(\mathfrak{G}_{J+1} + \mathfrak{R}_{J+1}) + \epsilon_J^2 + \epsilon_0^2\Big) + |a| r_0^3 \mathfrak{G}^2_{J+1} \\
& + r_0^{\frac{27}{4}} \mathfrak{G}^{\frac{3}{2}}_{J+1}\Big(\epsilon_0 + \sqrt{\epsilon_J}\sqrt{\mathfrak{G}_{J+1} + \mathfrak{R}_{J+1}}\Big)^{\frac{1}{2}}, \\
{}^{(ext)}\mathfrak{R}^2_{J+1} \lesssim & r_0^{3+\delta_B}\, {}^{(int)}\mathfrak{R}^2_{J+1} + r_0^{-\delta_B}\, {}^{(ext)}\mathfrak{G}^2_{J+1} + \epsilon_0^2,
\end{aligned}
$$

where the constant in $\lesssim$ *is independent of* r_0 *and* ϵ_J *is such that* $\mathfrak{G}_J + \mathfrak{R}_J \le \epsilon_J$*.*

Part III of this paper is entirely dedicated to the proof of Theorem 1.5.4.

1.6 Derivation and estimates for the gRW equations

1.6.1 Teukolsky and gRW equations in our approach

In section 1.4.1 we derive, using the formalism developed in the previous sections[41], the nonlinear version of the Teukolsky equations for A and $\underline{A}$ of the form

$$
\mathcal{L}[A] = \text{Err}[\mathcal{L}[A]], \qquad \underline{\mathcal{L}}[\underline{A}] = \text{Err}[[\underline{\mathcal{L}}[\underline{A}]], \tag{1.6.1}
$$

where $\mathcal{L}, \underline{\mathcal{L}}$ are second order tensorial wave operators on our spacetime $\mathcal{M}$, and where $\text{Err}[\mathcal{L}[A]]$, $\text{Err}[\,\not{\mathcal{L}}[\underline{A}]]$ are nonlinear errors depending on all linearized Ricci and curvature coefficients.

Just as in linear theory, to be able to control $A, \underline{A}$ we need to perform transformations $\mathfrak{q} = \mathfrak{q}[A]$, $\underline{\mathfrak{q}} = \underline{\mathfrak{q}}[\underline{A}]$, which take solutions $A, \underline{A}$ of the Teukolsky equation (1.6.1) into solutions of nonlinear, tensorial, versions of Regge-Wheeler equations, which we call gRW equations.

In the setting of polarized perturbations of Schwarzschild [50], the derivation of the RW equation for[42] $\mathfrak{q}$ was performed using null frames, which had the feature to be both adapted to an integrable foliation and diagonalize the curvature tensor up to error terms. One could thus rely on the geometric formalism developed in the context of the proof of the nonlinear stability of Minkowski space [23]. In the present paper, we rely on an extension of the formalism of [23] which allows for non integrable null frames and is presented in

[41]This follows from the complex form of the null Bianchi identities, see Proposition 2.4.11.
[42]Note that [50] did not rely on $\underline{\mathfrak{q}}$.

Chapter 2. Our results on the derivation of gRW in perturbations of Kerr are obtained in Chapter 5 and can be summarized as follows.

Theorem 1.6.1. *There exist complex* 2 *tensors* $\mathfrak{q}, \underline{\mathfrak{q}} \in \mathfrak{s}_2(\mathbb{C})$ *derived from* $A, \underline{A}$ *as follows,*

$$\begin{aligned} \mathfrak{q} = &\ q\overline{q}^3 \left({}^{(c)}\nabla_3 {}^{(c)}\nabla_3 A + C_1 {}^{(c)}\nabla_3 A + C_2 A \right), \\ \underline{\mathfrak{q}} = &\ \overline{q} q^3 \left({}^{(c)}\nabla_4 {}^{(c)}\nabla_4 A + \underline{C}_1 {}^{(c)}\nabla_3 A + \underline{C}_2 A \right), \end{aligned} \tag{1.6.2}$$

where $q = r + ia\cos\theta$ ${}^{(c)}\nabla_3$, ${}^{(c)}\nabla_4$ *are conformal derivatives, see section 2.2.9, and*

$$\begin{aligned} C_1 &= 2tr\,\underline{\chi} - 2\frac{{}^{(a)}tr\underline{\chi}^2}{tr\,\underline{\chi}} - 4i\,{}^{(a)}tr\underline{\chi}, \\ C_2 &= \frac{1}{2}tr\,\underline{\chi}^2 - 4\,{}^{(a)}tr\underline{\chi}^2 + \frac{3}{2}\frac{{}^{(a)}tr\underline{\chi}^4}{tr\,\underline{\chi}^2} + i\left(-2tr\,\underline{\chi}\,{}^{(a)}tr\underline{\chi} + 4\frac{{}^{(a)}tr\underline{\chi}^3}{tr\,\underline{\chi}}\right), \\ \underline{C}_1 &= 2tr\,\chi - 2\frac{{}^{(a)}tr\chi^2}{tr\,\chi} - 4i\,{}^{(a)}tr\chi, \\ \underline{C}_2 &= \frac{1}{2}tr\,\chi^2 - 4\,{}^{(a)}tr\chi^2 + \frac{3}{2}\frac{{}^{(a)}tr\chi^4}{tr\,\chi^2} + i\left(-2tr\,\chi\,{}^{(a)}tr\chi + 4\frac{{}^{(a)}tr\chi^3}{tr\,\chi}\right), \end{aligned} \tag{1.6.3}$$

which verify gRW equations of the form[43]

$$\begin{aligned} \dot{\square}_2\mathfrak{q} - i\frac{4a\cos\theta}{|q|^2}\nabla_{\mathbf{T}}\mathfrak{q} - V\mathfrak{q} &= L_{\mathfrak{q}}[A] + Err[\dot{\square}_2\mathfrak{q}], \\ \dot{\square}_2\underline{\mathfrak{q}} + i\frac{4a\cos\theta}{|q|^2}\nabla_{\mathbf{T}}\underline{\mathfrak{q}} - \underline{V}\underline{\mathfrak{q}} &= L_{\underline{\mathfrak{q}}}[\underline{A}] + Err[\dot{\square}_2\underline{\mathfrak{q}}], \end{aligned} \tag{1.6.4}$$

with $\mathbf{T}$ *defined as in* (1.4.4). *The potentials* $V, \underline{V}$ *are real and positive and the terms* $L_{\mathfrak{q}}[A], L_{\underline{\mathfrak{q}}}[\underline{A}]$ *are linear in* A*, resp* $\underline{A}$ *and have have important specific properties described in detail in subsections 5.2.3 and 5.3.3. Finally the error terms* $Err[\dot{\square}_2\mathfrak{q}]$, $Err[\dot{\square}_2\underline{\mathfrak{q}}]$ *depending on all linearized Ricci and curvature coefficients are acceptable error terms, i.e. they verify important structural properties, reminiscent to the null condition.*

Remark 1.6.2. *Due to the presence of the linear terms in* A*, resp.* $\underline{A}$*, on the right hand side of* (1.6.4)*, one has to view the wave equations in* (1.6.4) *as coupled with the defining equations for* $\mathfrak{q}$, $\underline{\mathfrak{q}}$ *given by* (1.6.2)*, that is coupled*[44] *with second order transport type equations in* A*, resp.* $\underline{A}$*.*

[43] Here $\dot{\square}_2$ is the covariant wave operator for horizontal 2-tensors, see section 2.3.
[44] This is different from the case of Schwarzschild, see [50], where these equations decouple.

Remark 1.6.3. *Note that, in the case of Kerr, the corresponding gRW type equations in [57] are complex scalars $\psi^{[\pm]}$ verifying the equations*[45]

$$\square_{a,m}\psi^{[\pm]} + ia\ c(r,\theta)\partial_t\psi^{[\pm]} + V(r,\theta)\psi^{[\pm]} = aL_\pm(\alpha^{[\pm 2]}). \tag{1.6.5}$$

These scalars are connected to our tensorial quantities $\mathfrak{q}, \underline{\mathfrak{q}}$ via the relations $\psi^{[+]} = \mathfrak{q}(e_1, e_1)$, $\psi^{[-]} = \underline{\mathfrak{q}}(e_1, e_1)$. The equations (1.6.5) *can be obtained by projecting our tensorial equations* (1.6.4)*. Note however that the projection modifies the equations by the appearance of Christoffel symbols*[46] *of the horizontal frame*[47]*.*

1.6.2 RW model equations

The most demanding part in the analysis of the gRW equations (1.6.4) is to derive global Energy-Morawetz type estimates for $(\mathfrak{q}, A)$ and respectively $(\underline{\mathfrak{q}}, \underline{A})$. To do this, it helps to analyze first the reduced equations in which the right hand side of both equations are treated as sources. Taking also $\psi = \Re(\mathfrak{q})$, $\underline{\psi} = \Re(\underline{\mathfrak{q}})$ we are led to the real RW model equations

$$\begin{aligned} \square_2\psi - V\psi &= -\frac{4a\cos\theta}{|q|^2}\ {}^*\nabla_T\psi + N, \qquad V = \frac{4\Delta}{(r^2+a^2)|q|^2}, \\ \dot{\square}_2\underline{\psi} - V\underline{\psi} &= \frac{4a\cos\theta}{|q|^2}\ {}^*\nabla_T\underline{\psi} + \underline{N}, \qquad V = \frac{4\Delta}{(r^2+a^2)|q|^2}. \end{aligned} \tag{1.6.6}$$

A significant part in the proof of Theorems 1.5.2-1.5.3 is to derive the following result for $\psi, \underline{\psi}$.

Theorem 1.6.4. *The following estimates hold true for solutions $\psi, \underline{\psi} \in \mathfrak{s}_2$ of the wave equations* (1.6.6) *on spacetime region $\mathcal{M}(\tau_1, \tau_2)$, for all $\delta \le p \le 2-\delta$ and $2 \le s \le k_L$,*

$$BEF^s_p[\psi](\tau_1,\tau_2) \lesssim E^s_p[\psi](\tau_1) + \mathcal{N}^s_p[\psi, N](\tau_1,\tau_2), \tag{1.6.7}$$

$$BEF^s_p[\underline{\psi}](\tau_1,\tau_2) \lesssim E^s_p[\underline{\psi}](\tau_1) + \mathcal{N}^s_p[\underline{\psi}, \underline{N}](\tau_1,\tau_2), \tag{1.6.8}$$

where

$$BEF^s_p[\psi](\tau_1,\tau_2) := \sup_{\tau\in[\tau_1,\tau_2]} E^s_p[\psi](\tau) + B^s_p[\psi](\tau_1,\tau_2) + F^s_p[\psi](\tau_1,\tau_2). \tag{1.6.9}$$

[45]With $\square_{a,m}$ the Kerr D'Alembertian, c, V are real function of r, θ and $L_\pm(\alpha^{[\pm 2]})$ lower order terms.
[46]Singular on the axis, i.e. at $\theta = 0, \pi$.
[47]See Section 5.2.2 for a discussion of the projection and the relation with equation (1.6.5).

The energy flux norms $E_p^s[\psi], F_p^s[\psi]$, bulk norms $B_p^s[\psi]$ and source norms $\mathcal{N}_p^s$, with p referring to r^p weights and s to the number of derivatives, are defined in section 6.1.5. For the sake of this introduction it suffices to take a closer look at the crucial bulk terms B_p^s, which degenerate at the trapped set $\mathcal{M}_{trap}$, see Definition 1.4.2.

Definition 1.6.5. *For $0 < p < 2$ we define, with $\mathfrak{d} = (r\nabla_4, r\nabla, \nabla_3)$, the bulk norms $B_p^s[\psi](\tau_1, \tau_2) := \sum_{k\le s} B_p[\mathfrak{d}^k\psi]$*

$$B_p[\psi](\tau_1,\tau_2) := Mor[\psi](\tau_1,\tau_2) + \int_{\mathcal{M}_{r\geq 4m}(\tau_1,\tau_2)} r^{-1-\delta}|\nabla_3\psi| + r^{p-3}\Big(|\mathfrak{d}\psi|^2 + |\psi|^2\Big),$$

$$Mor[\psi](\tau_1,\tau_2) := \int_{\mathcal{M}(\tau_1,\tau_2)} r^{-2}|\nabla_{\widehat{R}}\psi|^2 + r^{-3}|\psi|^2 + \int_{\mathcal{M}_{\not{trap}}(\tau_1,\tau_2)} \left(r^{-2}|\nabla_3\psi|^2 + r^{-1}|\nabla\psi|^2\right).$$

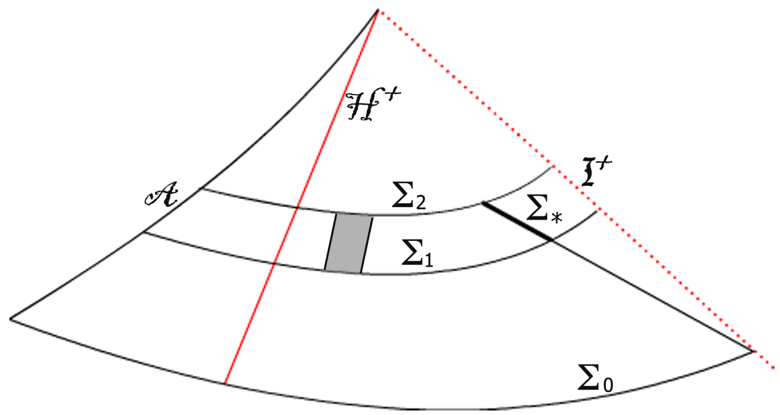

Figure 1.4: The spacetime region $\mathcal{M}(\tau_1,\tau_2) = \mathcal{M} \cap \{\tau_1 \le \tau \le \tau_2\}$ between the spacelike hypersurfaces $\Sigma_1 = \Sigma(\tau_1)$ and $\Sigma_2 = \Sigma(\tau_2)$, with the grey region denoting the trapped set.

The important thing in this definition is that $B_p[\psi]$ controls the spacetime integrals of $|\nabla_{\widehat{R}}\psi|^2$ and $|\psi|^2$ everywhere and all other derivatives away from the trapped set.

In addition, we also derive estimates for the quantity $\check{\psi} := r^2(e_4\psi + \frac{r}{|q|^2}\psi)$ for which one can prove stronger r^p estimates[48], see Theorem 6.2.2.

[48]These results are the analog in perturbations of Kerr, to Theorem 5.17 and Theorem 5.18 of [50] for perturbations of Schwarzschild. They are based on improved r^p weighted hierarchy first introduced in [5].

1.6.3 Main steps in the proof of Theorems M1 and M2

The proof of Theorem 1.6.4 is by far the most demanding part in the proof of Theorems 1.5.2-1.5.3. We provide an introduction to some of the main ideas of the proof in section 1.7.

To extend Theorem 1.6.4 to the full gRW equations (1.6.4), we have to control the terms $\mathcal{N}_p^s[\mathfrak{q}, N](\tau_1, \tau_2)$ with $N = L_{\mathfrak{q}}[A] + \mathrm{Err}[\dot{\square}_2\mathfrak{q}]$ and $\mathcal{N}_p^s[\underline{\mathfrak{q}}, \underline{N}](\tau_1, \tau_2)$ with $\underline{N} = L_{\underline{\mathfrak{q}}}[\underline{A}] + \mathrm{Err}[\dot{\square}_2\underline{\mathfrak{q}}]$ for the second equation. This is done in steps by first eliminating the linear error terms on the right hand side of our two gRW equations and then eliminating the remaining nonlinear quadratic terms. The procedure for doing this differs substantially for the two equations.

Estimates for $(\mathfrak{q}, A)$

The procedure of eliminating the error term $N = L_{\mathfrak{q}}[A] + \mathrm{Err}[\dot{\square}_2\mathfrak{q}]$ requires the use a global frame of $\mathcal{M}$ for which we have[49]

$$\check{H} \in \Gamma_g. \tag{1.6.10}$$

Step 1. To eliminate the contribution of the linear term $L_{\mathfrak{q}}[A]$ one has to first derive estimates for A using the second order transport equations which defines $\mathfrak{q}$. In the particular case of Kerr, we can write, see Proposition 5.2.4,

$$^{(c)}\nabla_3\, {}^{(c)}\nabla_3 \left(\frac{\overline{q}^4}{r^2} A\right) = \frac{\overline{q}}{q} r^{-2} \mathfrak{q}.$$

The above factorization[50] is used to derive appropriate BEF_p^s estimates for A in terms of $\mathfrak{q}$. These can then be combined with the estimates derived in Theorem 1.6.4 to obtain estimates for the norms $BEF_p^s[A, \mathfrak{q}]$ depending only on the the nonlinear error terms, see Theorem 11.2.3. We note that the smallness of a and the specific structure[51] of the top terms $L_{\mathfrak{q}}[A]$ is also essential in controlling[52] the terms in $\mathcal{N}_p^s[\mathfrak{q}, L_{\mathfrak{q}}[A]](\tau_1, \tau_2)$.

[49] We remark that if $\check{H} \in \Gamma_b$, we cannot even derive a Morawetz estimate.

[50] In perturbations of Kerr, to avoid the presence of unacceptable nonlinear error terms, we use instead the following modified factorization, see Lemma 11.1.3,

$$^{(c)}\nabla_3 \left({}^{(c)}\nabla_3 \left(\frac{(\overline{\mathrm{tr}\underline{X}})^2}{(\Re(\mathrm{tr}\underline{X}))^2 (\mathrm{tr}\underline{X})^2} A \right) - \frac{r^2}{2} F A \right) = O(r^{-2})\mathfrak{q} + r\mathfrak{d}^{\leq 1}(\Gamma_b \cdot A),$$

for an appropriately defined scalar function F.

[51] This is needed to control the corresponding energy estimates.

[52] This requires several integration by parts.

Step 2. To eliminate the error terms $\mathrm{Err}[\dot{\square}_2\mathfrak{q}]$ in the $\mathfrak{q}$ equation we proceed as in [50]. As mentioned above, it is essential that the analysis is done in a global frame of $\mathcal{M}$ for which $\check{H} \in \Gamma_g$. One can show that $\mathrm{Err}[\dot{\square}_2\mathfrak{q}]$ can be written in the form, see equation (5.2.13),

$$\mathrm{Err}[\dot{\square}_2\mathfrak{q}] = r^2\mathfrak{d}^{\leq 3}(\Gamma_g \cdot (A, B)) + \nabla_3(r^2\mathfrak{d}^{\leq 2}(\Gamma_b \cdot (A, B))) + \mathfrak{d}^{\leq 1}(\Gamma_g \cdot \mathfrak{q}) + r^3\mathfrak{d}^{\leq 2}\big(\Gamma_b \cdot \Gamma_g \cdot \Gamma_g\big).$$

Step 3. Given the above form of $\mathrm{Err}[\dot{\square}_2\mathfrak{q}]$ we are able to derive estimates for the norms $BEF^s_p[\mathfrak{q}, A]$ for $\delta \leq p \leq 2-\delta$. Additional improved r^p estimates are then obtained for the quantity $\check{\mathfrak{q}} = r^2(\nabla_4\mathfrak{q} + \frac{r}{|q|^2}\mathfrak{q})$.

Estimates for $(\underline{\mathfrak{q}}, \underline{A})$

We rely on a different global frame of $\mathcal{M}$ for which we have, in ${}^{(ext)}\mathcal{M}$,

$$\Xi = 0, \qquad \underline{\check{H}} = 0. \tag{1.6.11}$$

Remark 1.6.6. *We note that the temporal frame of* ${}^{(ext)}\mathcal{M}$ *(see Definition 9.1.1 in [53]), in which the top derivative estimates in* ${}^{(ext)}\mathcal{M}$ *for the linearized Ricci coefficients were derived, verifies these properties. The conditions* (1.6.11) *can in fact be relaxed, i.e.* $\Xi \in r^{-2}\Gamma_g$, $\underline{\check{H}} \in r^{-1}\Gamma_g$ *suffice.*

The structure of the error terms $\mathrm{Err}[\dot{\square}_2\underline{\mathfrak{q}}]$, in that frame, turns out to be more subtle than that of $\mathrm{Err}[\dot{\square}_2\mathfrak{q}]$, as it depends in an essential way on the fact that the quantities $\underline{A}_4, \underline{B}_4, P_4$, introduced[53] in section 2.4.4, have improved decay properties in r. Similar improvements appear in the null structure equations, see Proposition 2.4.13. Thus, for example, the quantities ${}^{(c)}\nabla_4\mathrm{tr}\underline{X} + \frac{1}{2}\mathrm{tr}X\mathrm{tr}\underline{X}$ and ${}^{(c)}\nabla_4\underline{\widehat{X}} + \frac{1}{2}\mathrm{tr}X\,\underline{\widehat{X}}$ have better decay properties than respectively ${}^{(c)}\nabla_4\mathrm{tr}\underline{X}$ and ${}^{(c)}\nabla_4\underline{\widehat{X}}$. These improvements[54] carry over various quadratic and cubic error terms appearing in $\mathrm{Err}[\dot{\square}_2\underline{\mathfrak{q}}]$. Using these facts we can show that

$$\mathrm{Err}[\dot{\square}_2\underline{\mathfrak{q}}] \;=\; r^2\mathfrak{d}^{\leq 3}\big((A, B)\cdot\underline{A}\big) + \mathfrak{d}^{\leq 3}(\Gamma_b \cdot \Gamma_g). \tag{1.6.12}$$

Step 1. The simplest way to factorize $\underline{\mathfrak{q}}$ in Kerr is given by, see Proposition 5.3.4,

$$r\,{}^{(c)}\nabla_4\left(r^2\left({}^{(c)}\nabla_4(r\frac{q^4}{r^4}\underline{A})\right)\right) = \frac{q}{\overline{q}}\underline{\mathfrak{q}}.$$

[53] In the case of integrable S-foliations these notations were introduced in [23], Chapter 7.

[54] Similar improvements, in the particular case of perturbations of Schwarzschild, are used to treat the corresponding error terms in the estimates for the quantity corresponding to $\underline{\mathfrak{q}}$ in [30].

The above factorization[55] is used to derive appropriate BEF_p^s estimates for $\underline{A}$ in terms of $\underline{\mathfrak{q}}$. The contribution to $\mathcal{N}_p^s[\underline{\mathfrak{q}}, N](\tau_1, \tau_2)$ due to $L_{\underline{\mathfrak{q}}}[\underline{A}]$ can then be absorbed exactly as in the case of $\mathfrak{q}$.

Step 2. Given the form (1.6.12) for $\mathrm{Err}[\dot{\square}_2\underline{\mathfrak{q}}]$ we can only derive estimates for the norms $BEF_p^s[\underline{\mathfrak{q}}, \underline{A}]$ for $\delta \leq p \leq 1-\delta$.

Proof of Theorem 1.5.2

Once full r^p weighted estimates for $(\mathfrak{q}, A)$ are derived, the proof of the estimates of Theorem M1 follow steps similar to those used in Chapter 5 of [50], see section 11.7 for the details.

Proof of Theorem 1.5.3

Given that we only derive estimates for the norms $BEF_p^s[\underline{\mathfrak{q}}, \underline{A}]$ for $\delta \leq p \leq 1-\delta$, we obtain at first insufficient decay estimates in τ for $(\underline{\mathfrak{q}}, \underline{A})$. However, we can show that higher $\not{\mathcal{L}}_{\mathbf{T}}$ derivatives of $(\underline{\mathfrak{q}}, \underline{A})$ decay faster in powers of τ. Using this observation, we obtain suitable decay in τ for the flux of $(\not{\mathcal{L}}_{\mathbf{T}}^2\underline{\mathfrak{q}}, \not{\mathcal{L}}_{\mathbf{T}}^2\underline{A})$ along Σ_*. We then rely on this result, and on some version of Teukolsky-Starobinsky providing an identity between $\mathfrak{q}$ and $\underline{A}$ to recover the desired decay estimate for $\underline{A}$ on Σ_* stated in Theorem M2, see section 12.4 for the details.

1.7 Main ideas in the proof of Theorem 1.6.4

In the polarized situation of [50] where the wave equation[56] for $\mathfrak{q}$ decouples, linearly, from the transport equation for A, the proof of the analogue of Theorem 1.6.4 was done as follows:

[55] In perturbations of Kerr, to avoid the presence of unacceptable nonlinear error terms, we use instead the following modified factorization, see Lemma 12.1.2,

$$\underline{\mathfrak{q}} = \overline{q}q^3\left({}^{(c)}\nabla_4 + 2\mathrm{tr}X - \frac{|\mathrm{tr}X|^2}{2\mathrm{tr}\,\chi}\right)\left({}^{(c)}\nabla_4 + 2\mathrm{tr}X - \frac{3|\mathrm{tr}X|^2}{2\mathrm{tr}\,\chi}\right)\underline{A} + r^2\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b).$$

[56] We note that in [50] the quantity $\underline{\mathfrak{q}}$ was not actually used.

- An adapted *physical space* approach based on Morawetz-Energy, red shift and r^p-weighted estimates which extends the treatment of the linearized RW equation in Schwarzschild in [28] to the full nonlinear setting.
- Exploit the specific null structure of the nonlinear error terms.
- Once we control $\mathfrak{q}$, we then estimate A using the corresponding second order transport equation.

To pass to the case of perturbations of Kerr, one encounters the following additional problems:

1. *Complicated nature of the trapping region.* This is the region of the domain of outer communication $r > r_+$ which contains trapped null geodesics. For small $|a|/m$ one can show[57] that all such geodesics are included in the set $\mathcal{M}_{trap}$, see Definition 1.4.2.
2. *Presence of a non-trivial ergoregion.* This is the region of $r > r_+$ where $\mathbf{T}$ is spacelike.

As mentioned earlier, to treat these difficulties in linear theory, [57] and [29] rely on methods, first used in the context of the scalar wave equation in Kerr, based on mode decompositions and construction of vectorfields adapted to different modes. It is however not clear how to extend this method, without loss of derivatives, to general perturbations of Kerr.

In our work[58], we rely instead on a physical space method introduced by Blue and Andersson in [4] in the context of the scalar wave equation $\square_{a,m}\psi = 0$ in $Kerr(a, m)$ for small $|a|/m$. In the next section, we provide a very short review of this method in the simplest case of a scalar wave equation in Kerr.

1.7.1 Andersson-Blue method

The crucial new idea in [4] is to supplement the existing Killing vectorfields of $Kerr(a, m)$, i.e. $\mathbf{T} = \partial_t$ and $\mathbf{Z} = \partial_\phi$ in Boyer Lindquist (BL) coordinates, with a second order operator

[57] See a discussion of trapped null geodesics in Kerr in section 3.8.3. Note that the trapped set reduces to the hypersurface $r = 3m$ in the case of Schwarzschild.

[58] We need in fact to adapt the the method of [4] to the far more difficult case when the metric is a perturbation of the Kerr metric, ψ is a 2-horizontal tensor rather than a scalar and the gRW equation contains additional linear terms.

$\mathcal{K} = \mathbf{D}_\alpha(\mathbf{K}^{\alpha\beta}\mathbf{D}_\beta)$ which commutes with the scalar wave operator $\square_{a,m}$. Here $\mathbf{K}_{\alpha\beta}$ is a Killing tensor, i.e. symmetric and verifying $\mathbf{D}_{(\gamma}\mathbf{K}_{\alpha\beta)} = 0$. Note that if X, Y are Killing vectorfields, their symmetric tensor product $\frac{1}{2}(X \otimes Y + Y \otimes X)$ is automatically a Killing tensor but the remarkable thing about Kerr, discovered by Carter, is that it has an additional Killing tensor which cannot be reduced in this manner to Killing vectorfields. Relative to the basis (1.4.2), the Carter tensor takes the form

$$\mathbf{K} = -a^2\cos^2\theta\mathbf{g} + O, \qquad O = |q|^2\big(e_1 \otimes e_1 + e_2 \otimes e_2\big). \tag{1.7.1}$$

The associated second order operator $\mathcal{O} = \mathbf{D}_\alpha(O^{\alpha\beta}\mathbf{D}_\beta)$ verifies itself a commutation property with $\square_{a,m}$

$$[\mathcal{O}, |q|^2\square_{a,m}] = 0.$$

In their work, Andersson and Blue introduce the set of second order operators

$$\mathcal{S}_{\underline{a}} = \mathbf{D}_\alpha(S_{\underline{a}}^{\alpha\beta}\mathbf{D}_\beta), \qquad \underline{a} = 1, 2, 3, 4,$$

associated to the Killing tensors

$$\Big\{S_1 = \mathbf{T} \otimes \mathbf{T}, \quad S_2 = \frac{1}{2}a\big(\mathbf{T} \otimes \mathbf{Z} + \mathbf{Z} \otimes \mathbf{T}\big), \quad S_3 = a^2\mathbf{Z} \otimes \mathbf{Z}, \quad O\Big\}, \tag{1.7.2}$$

which commute with $|q|^2\square$. Thus if ψ is a solution to $\square_{a,m}\psi = 0$, so are $\psi_{\underline{a}} := \mathcal{S}_{\underline{a}}\psi$ for $\underline{a} = 1, 2, 3, 4$.

It is important to note that O appears naturally in the expression of the the inverse Kerr metric in BL coordinates, see Lemma 3.5.1,

$$|q|^2\mathbf{g}^{\alpha\beta} \;=\; \Delta\partial_r^\alpha\partial_r^\beta + \frac{1}{\Delta}\mathcal{R}^{\alpha\beta} \tag{1.7.3}$$

with the 2 tensor $\mathcal{R}^{\alpha\beta}$ a linear combination of the tensors $S_{\underline{a}}$ of the form

$$\mathcal{R}^{\alpha\beta} = \mathcal{R}^{\underline{a}}S_{\underline{a}}^{\alpha\beta},$$

with coefficients $\mathcal{R}^1 = -(r^2+a^2)^2$, $\mathcal{R}^2 = -2a(r^2+a^2)$, $\mathcal{R}^3 = -a^2$, $\mathcal{R}^4 = \Delta$.

The main idea in [4] is to derive a Morawetz-Energy integrated spacetime estimates for an appropriate combination the set of solutions $\{\psi, \psi_{\underline{a}} = \mathcal{S}_{\underline{a}}\psi \,|\, \underline{a} = 1, 2, 3, 4\}$. This is done, roughly, as follows.

Step 1. Integrate by parts an expression of the form

$$\int_{\mathcal{M}(\tau_1,\tau_2)} \sum_{\underline{a},\underline{b}=1}^{4} \left(X^{\underline{ab}} + \frac{1}{2}w^{\underline{ab}}\right)\psi_{\underline{a}} \cdot \square_{a,m}\psi_{\underline{b}}$$

for Morawetz type vectorfields $X^{\underline{ab}} = -zhf^{\underline{ab}}\partial_r$ and scalars $w^{\underline{ab}} = |q|^2\mathrm{Div}\big(|q|^{-2}X^{\underline{ab}}\big) + h\partial_r z f^{\underline{ab}}$ with a choice for the function $f^{\underline{ab}}$ of the form

$$f^{\underline{ab}} = \tilde{\mathcal{R}}'^{(\underline{a}}\mathcal{L}^{\underline{b})} = \frac{1}{2}\big(\tilde{\mathcal{R}}'^{\underline{a}}\mathcal{L}^{\underline{b}} + \tilde{\mathcal{R}}'^{\underline{b}}\mathcal{L}^{\underline{a}}\big)$$

where $\tilde{\mathcal{R}}'^{\underline{a}} = \partial_r\left(\frac{z}{\Delta}\mathcal{R}^{\underline{a}}\right)$, and $\mathcal{L}^{\underline{a}}$ are suitable constants. With appropriate choices[59] for z, h, after suitable integration by parts, summing and absorbing terms small in a/m one derives estimates of the form

$$\int_{\mathcal{M}(\tau_1,\tau_2)} P \lesssim \left|\int_{\partial\mathcal{M}(\tau_1,\tau_2)} B\right|, \tag{1.7.4}$$

where P, B are quadratic quantities involving combinations of $\psi_{\underline{a}}$ and their first derivatives with P everywhere positive. One can show that outside the trapped set, $\int_{\mathcal{M}(\tau_1,\tau_2)} P$ is coercive, that is, roughly, using the notation $|\psi|^2_{\mathcal{S}} := \sum_{\underline{a}=1}^{4} |\psi_{\underline{a}}|^2$, we can show:

- The integral $\int_{\mathcal{M}(\tau_1,\tau_2)} P$ controls, with appropriate r weights, the spacetime integrals of $|\nabla_r\psi|^2_{\mathcal{S}}$ and $|\psi|^2_{\mathcal{S}}$ in the entire domain of outer communication $r > r_+$.
- For $r > r_+$ and away from the trapping set $\int_{\mathcal{M}(\tau_1,\tau_2)} P$ also controls the corresponding spacetime integrals of all other derivatives, with appropriate r weights, $|\nabla_{\mathbf{T}}\psi|^2_{\mathcal{S}}$, $|\nabla_Z\psi|^2_{\mathcal{S}}$, $|\nabla\psi|^2_{\mathcal{S}}$.

Step 2. The boundary term $\int_{\partial\mathcal{M}(\tau_1,\tau_2)} B$ can be eliminated in favor of a positive energy-flux type integral[60] $\sup_{\tau\in[\tau_1,\tau_2]} E[\psi](\tau) + F[\psi](\tau_1,\tau_2)$, with the help of an energy identity induced by the Killing vectorfield $\mathbf{T}$. This is derived by integrating by parts in the integral

$$\int_{\mathcal{M}(\tau_1,\tau_2)} \mathbf{T}\psi \cdot \square_{a,m}\psi.$$

To avoid the fact that $\mathbf{T}$ becomes spacelike in the ergoregion one can replace it by a smooth, causal, vectorfield $\widehat{T}_\delta$ equal to $\mathbf{T}$ in a small neighborhood of size δ of the trapping region and equal to the future causal Hawking vectorfield $\widehat{T} = \partial_t + \frac{a}{r^2+a^2}\partial_\phi$, see (3.2.1), everywhere else. Combining such an energy inequality with the Morawetz estimate above we derive estimates of the form, roughly,

$$\int_{\mathcal{M}(\tau_1,\tau_2)} P + \sup_{\tau\in[\tau_1,\tau_2]} E[\psi](\tau) + F[\psi](\tau_1,\tau_2) \lesssim E[\psi](\tau_1)$$

[59]The scalar functions z and h are chosen by $z = z_0 - \delta_0 z_0^2$, $z_0 = \frac{\Delta}{(r^2+a^2)^2}$, $h = \frac{(r^2+a^2)^3}{r}$.

[60]Here $E[\psi](\tau)$ denote integrals on $\Sigma(\tau)$ while $F[\psi](\tau_1,\tau_2)$ are flux integrals on the spacelike boundaries $\mathcal{A}\cup\Sigma_*$.

where the integrands in $E[\psi]$ are positive definite.

Step 3. Both E and P degenerate at the horizon $r = r_+$. This problem can be easily fixed by using the standard red shift vectorfield technique, see [24], applied to the region $\mathcal{M}_{red}$.

Step 4. To control all second derivatives of ψ away from the trapping set one needs to make use of some type of coercivity for the operator $\mathcal{O}$. In that sense it is important to remark that in the integrable context of Schwarzschild, or Minkowski space $\mathcal{O} = r^2\triangle$ with $\triangle$ the standard Laplacian on the sphere and thus $\mathcal{O}\psi$ controls $\nabla^2\psi$ by standard elliptic estimates on spheres. Such estimates are not possible in the non-integrable case when there are no compact 2-surfaces on which $\mathcal{O}$ acts. We need in fact both $\mathcal{O}$ and $\mathbf{T}$ to control $\nabla^2\psi$, see sections 4.8 and 9.3.

Step 5. The method outlined above only provides estimates consistent with the value $s = 2$ in the estimates[61] of Theorem 1.6.4. To control the lower derivatives one has to rely on the conditional Morawetz estimates of Proposition 6.3.7, i.e. estimates which can only be closed when combined with the $s = 2$ estimates discussed above. These conditional estimates[62] are derived using the traditional vectorfield method, based on a vectorfield X of the form $\mathcal{F}(r)\partial_r$.

Step 6. Higher derivative estimates, i.e. $s > 2$, can also be derived by making use of the commutation properties of the symmetry operators $\mathbf{T}$, $\mathbf{Z}$ and $\mathcal{O}$.

Once a non-degenerate combined Energy-Morawetz estimate has been derived, one can also derive r^p-weighted type estimates on the large r region. This step does not differ much from the case of Schwarzschild since the corrections in a/r are small. Then, combining the Energy-Morawetz estimate with these r^p weighted estimates, one can derive the result of Theorem 1.6.4 for the particular case of the equation $\square_{m,a}\psi = 0$. We refer the reader to section 6.3.2 for a more extended outline of the main ideas in the proof. We also note that the case of the equation $\square_{m,a}\psi + V\psi$ with $V = \frac{4\Delta}{(r^2+a^2)|q|^2}$, which is more relevant to our situation, can be treated in the same manner[63].

[61] In reality, the method also generates lower order terms in derivatives of ψ so that the estimates cannot be closed without control of the lower derivatives.

[62] Note that these conditional estimates become unconditional in the axially symmetric case.

[63] The positive potential V does actually help in the estimates.

1.7.2 Proof of Theorem 1.6.4

Here are some of the necessary adjustments to the Andersson-Blue method to the case of our model gRW equations (1.6.6):

- Define approximate Killing vectorfields $\mathbf{T}, \mathbf{Z}$, using only the frame and the functions r, θ, according to the formula (1.4.4). Define also the vectorfields
$$\widehat{T} \;=\; \frac{1}{2}\left(\frac{|q|^2}{r^2+a^2}e_4 + \frac{\Delta}{r^2+a^2}e_3\right), \qquad \widehat{R} = \frac{1}{2}\left(\frac{|q|^2}{r^2+a^2}e_4 - \frac{\Delta}{r^2+a^2}e_3\right).$$

- Define an appropriate version of the O tensor, expressed only in terms of the horizontal structure and function $q = r + ia\cos\theta$, i.e.
$$O^{\alpha\beta} \;:=\; |q|^2\gamma^{ab}e_a^\alpha e_b^\beta,$$
where γ is the metric induced by $\mathbf{g}$ on the horizontal structure.

- Show that the error terms generated by commuting $\mathbf{T}, \mathbf{Z}$ and the operator $\mathcal{O} = \dot{\mathbf{D}}_\alpha(O^{\alpha\beta}\dot{\mathbf{D}}_\beta)$ with $|q|^2\dot{\square}_2$ are acceptable error terms. It is important to note that, due to the tensorial character of the gRW equations, even in the case of Kerr the commutators generate linear terms proportional to a, see section 3.7. These can ultimately be absorbed for sufficiently small a/m.

- Consider the approximate Killing tensors $S_{\underline{a}}$, $\underline{a} = 1, 2, 3, 4$,
$$\left\{S_1 = \mathbf{T}\otimes\mathbf{T}, \quad S_2 = \frac{1}{2}a\big(\mathbf{T}\otimes\mathbf{Z} + \mathbf{Z}\otimes\mathbf{T}\big), \quad S_3 = a^2\mathbf{Z}\otimes\mathbf{Z}, \quad S_4 = O\right\}$$
and the corresponding second order operators $\mathcal{S}_{\underline{a}} := \dot{\mathbf{D}}_a\big(S_{\underline{a}}^{\alpha\beta}\dot{\mathbf{D}}_\beta\big)$ such that the commutators with $|q|^2\dot{\square}_2$ produce only acceptable error terms.

The proof of Theorem 1.6.4 is the most technical part of the paper. It is carried out first in Kerr in Chapters 7 and 8, and is then extended to perturbations of Kerr in Chapter 9.

1.8 Main ideas in the proof of Theorem 1.5.4

To derive top derivative estimates for all curvature components we need to rely on the following facts:

- We work in the global frame of $\mathcal{M}$ for which the conditions (1.6.11) are verified in ${}^{(ext)}\mathcal{M}$, see also Remark 1.6.6.
- Control of the initial data, see 1.5.7, $\mathfrak{I}_{k_L} \lesssim \epsilon_0$.
- Improved[64] r^p-estimates for $\underline{\mathfrak{q}}$ in the case $p = \delta$ for all derivatives $s \leq k_L - 2$. This is done as in Part II, noticing that one can go to the maximum number of derivatives in the case $p = \delta$. These estimates rely heavily on the global frame of $\mathcal{M}$ in which the conditions (1.6.11) hold true.
- The fact that the desired estimates for $\mathfrak{R}_{J+1}$ are conditional on the constant ϵ_J where[65]

$$\mathfrak{G}_J + \mathfrak{R}_J \quad \leq \quad \epsilon_J, \tag{1.8.1}$$

  which allows to deal with lower order terms.
- We take into account the quantitative assumptions (1.5.9) on $\mathfrak{R}_k$ and $\mathfrak{G}_k$

$$\mathfrak{G}_k + \mathfrak{R}_k \quad \leq \quad \epsilon, \quad k \leq k_L, \tag{1.8.2}$$

  which allows to deal with nonlinear terms.

The proof of Theorem 1.5.4 can be divided in two main parts:

1. Global Energy-Morawetz estimates in $\mathcal{M}$. This step relies heavily on the following main ingredients:
   (a) Derive first BEF_δ^J estimates for $\check{P}$ by relying on a suitable linearization of the wave equation of P. The RHS of this scalar wave equation contains quadratic terms, which are dealt with our quantitative assumptions (1.8.2), and linear terms containing fewer derivatives which can be dealt by the iterations assumption (1.8.1).
   (b) Use these estimates for $\check{P}$ together with the Bianchi identities to derive BEF_δ^J estimates for all other curvature components $A, B, \underline{A}, \underline{B}$. The RHS of the Bianchi identities contain also quadratic terms, which are dealt with our quantitative assumptions (1.8.2), and linear terms containing fewer derivatives which

[64]By improved estimates, we mean in terms of ϵ_0 rather than ϵ.

[65]This is based on an iteration assumption allowing to recover the control of $J+1$ derivatives from the one of J derivatives. To initiate the iteration procedure, we start at $J = \frac{k_L}{2}$ for which we have full control of $\mathfrak{G}_J$ and $\mathfrak{R}_J$ thanks to the decay estimates established in Theorem M1 and M2 for A and $\underline{A}$, and in Theorems M3 to M7 of [53] for all other linearized curvature components and all linearized Ricci and metric coefficients, see Lemma 9.4.13 in [53] for the corresponding statement.

can be dealt by the iterations assumption (1.8.1). The most demanding part of the proof is to deal with the estimates in the trapping[66]. Given that $\check{P}$ is already under control, we rely on the triangular structure of the Bianchi identities which allow us to control first $(B, \underline{B})$ from $\check{P}$, and then A from B and $\underline{A}$ from $\underline{B}$.

2. Weighted estimates in the region of ${}^{(ext)}\mathcal{M}$, where $r \geq r_0$ for sufficiently large r_0, for all top derivatives of the curvature components. This is proved using r^p weighted estimates for Bianchi pairs in a similar fashion as the corresponding estimates in section 8.7 of [50].

1.9 Organization

The paper is organized in 3 Parts and 16 Chapters as follows:

- Part I:
  - In Chapter 2, we give a full account of our geometric framework based on non-integrable horizontal structures and derive the null structure and null Bianchi equations. We also rephrase these equations using complex notations.
  - In Chapter 3, we provide the main formulas in Kerr.
  - In Chapter 4, we discuss the linearized quantities, introduce our notations (Γ_b, Γ_g), as well as other important quantities in perturbation of Kerr. We also provide numerous useful commutators.
  - In Chapter 5, we derive, in perturbations of Kerr, the Teukolsky equations, the generalized Regge-Wheeler (gRW) equations, and the Teukolsky-Starobinsky identities.
- Part II:
  - In Chapter 6, we introduce a simplified RW model and provide precise statements for the corresponding Energy-Morawetz-r^p-weighted estimates.
  - In Chapters 7 and 8, we adapt the Blue Andersson method to derive basic Energy-Morawetz estimates for the RW model equations (1.6.6) in the particular case of Kerr.

[66]This requires in particular to differentiate between the good direction $\widehat{R}$ and the other directions, and make systematic use of the Bianchi identities to recover the remaining derivatives.

- In Chapter 9, we extend the results of Chapters 7 and 8 to perturbations of Kerr and also deal the case of higher order derivatives.
- In Chapter 10, we conclude the study of the RW model (1.6.6) by deriving r^p-weighted estimates for it, hence proving the statements of Chapter 6.
- In Chapter 11, we extend the results of Chapter 6 on Energy-Morawetz-r^p-weighted estimates for the RW model (1.6.6) to the full generalized Regge Wheeler equations for $(\mathfrak{q}, A)$. We then rely on this result to prove Theorem M1, restated here as Theorem 1.5.2.
- In Chapter 12, we extend the results of Chapter 6 on Energy-Morawetz-r^p-weighted estimates for the RW model (1.6.6) to the full generalized Regge Wheeler equations for $(\underline{\mathfrak{q}}, \underline{A})$. We then rely on this result to prove Theorem M2, restated here as Theorem 1.5.3.

• Part III:

- In Chapter 13, we provide the geometric set up and the precise statement of Theorem 1.5.4 on the control of high derivatives curvature estimates.
- In Chapter 14, we derive energy-Morawetz estimates for $\check{P}$.
- In Chapter 15, we derive energy-Morawetz estimates for $(B, \underline{B})$ and then $(A, \underline{A})$.
- In Chapter 16, we derive r^p weighted estimates for the Bianchi pairs to recover the correct r decay using the control of Energy-Morawetz for $(A, B, \check{P}, \underline{B}, \underline{A})$, hence concluding the proof of Theorem 1.5.4.

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Part I

Formalism and derivation of the main equations

Chapter 2

Non-integrable structures

2.1 A general formalism for non-integrable structures

We present here a general formalism extending the one used for perturbations of Minkowski space [23] to perturbations of Kerr spacetimes.

2.1.1 Null pairs and horizontal structures

Let $(\mathcal{M}, \mathbf{g})$ be a Lorentzian spacetime. Consider an arbitrary null pair $e_3 = \underline{L}$, $e_4 = L$, i.e.

$$\mathbf{g}(e_3, e_3) = \mathbf{g}(e_4, e_4) = 0, \qquad \mathbf{g}(e_3, e_4) = -2.$$

Definition 2.1.1. *A vectorfield X is $(L, \underline{L})$-horizontal, or simply horizontal, if*

$$\mathbf{g}(L, X) = \mathbf{g}(\underline{L}, X) = 0.$$

We denote by $\mathbf{O}(\mathcal{M})$ the set of horizontal vectorfields on $\mathcal{M}$. Given a fixed orientation on $\mathcal{M}$, with corresponding volume form $\in$, we define the induced volume form on $\mathbf{O}(\mathcal{M})$ by,

$$\in (X, Y) := \frac{1}{2} \in (X, Y, \underline{L}, L). \tag{2.1.1}$$

Given a null pair $(L, \underline{L})$, the horizontal vectorfields $\mathbf{O}(\mathcal{M})$ define a horizontal distribution, i.e. a sub-bundle of the tangent bundle $\mathbf{T}(\mathcal{M})$ of the manifold. In the standard terminology used in differential topology, a subbundle $E \subset \mathbf{T}(\mathcal{M})$ of the tangent bundle is said to be integrable if for any vectorfields X and Y taking values in E, the Lie bracket $[X, Y]$ takes values in E as well. According to the Frobenius theorem a subbundle E is integrable (or involutive) if and only if the subbundle E arises from a regular foliation of $\mathcal{M}$, i.e. if locally the subbundle E can be realized as the tangent space of a submanifold of $\mathcal{M}$. An useful example of integrable structures, which have played an important role in [23], is that provided by S-foliations, i.e. regular foliations whose leaves are topological spheres orthogonal, at every point, to the null pair $(L, \underline{L})$. In this work we consider general, not necessarily integrable, horizontal structures.

Given an arbitrary vectorfield X we denote by ${}^{(h)}X$ its horizontal projection,

$$ {}^{(h)}X \;=\; X + \frac{1}{2}\mathbf{g}(X, \underline{L})L + \frac{1}{2}\mathbf{g}(X, L)\,\underline{L}. $$

Definition 2.1.2. *A k-covariant tensor-field U is said to be horizontal, and denoted $U \in \mathbf{O}_k(\mathcal{M})$, if for any vectorfields $X_1, \ldots X_k$ we have,*

$$ U(X_1, \ldots X_k) = U({}^{(h)}X_1, \ldots {}^{(h)}X_k). $$

Define the projection operator

$$ \Pi^{\mu\nu} \;=\; \mathbf{g}^{\mu\nu} + \frac{1}{2}(\,\underline{L}^\mu L^\nu + L^\mu\,\underline{L}^\nu). $$

Clearly $\Pi_\alpha^\mu \Pi_\mu^\beta = \Pi_\alpha^\beta$. An arbitrary tensor $U_{\alpha_1 \ldots \alpha_m}$ is horizontal iff

$$ \Pi_{\alpha_1}^{\beta_1} \ldots \Pi_{\alpha_m}^{\beta_m} U_{\beta_1 \ldots \beta_m} = U_{\alpha_1 \ldots \alpha_m}. $$

Definition 2.1.3. *For any horizontal X, Y we define*[1]

$$ \gamma(X, Y) \;=\; \mathbf{g}(X, Y) \tag{2.1.2} $$

and

$$ \begin{cases} \chi(X, Y) = \mathbf{g}(\mathbf{D}_X L, Y), \\ \underline{\chi}(X, Y) = \mathbf{g}(\mathbf{D}_X\,\underline{L}, Y). \end{cases} \tag{2.1.3} $$

where $\mathbf{D}$ denotes the covariant derivative of $\mathbf{g}$.

Observe that χ and $\underline{\chi}$ are symmetric if and only if the horizontal structure is integrable. Indeed this follows easily from the formulas,

$$ \begin{aligned} \chi(X, Y) - \chi(Y, X) &= \mathbf{g}(\mathbf{D}_X L, Y) - \mathbf{g}(\mathbf{D}_Y L, X) = -\mathbf{g}(L, [X, Y]), \\ \underline{\chi}(X, Y) - \underline{\chi}(Y, X) &= \mathbf{g}(\mathbf{D}_X\,\underline{L}, Y) - \mathbf{g}(\mathbf{D}_Y\,\underline{L}, X) = -\mathbf{g}(\,\underline{L}, [X, Y]). \end{aligned} $$

[1]In the particular case where the horizontal structure is integrable, γ is the induced metric, and χ and $\underline{\chi}$ are the null second fundamental forms.

We can view γ, χ and $\underline{\chi}$ as horizontal 2-covariant tensor-fields by extending their definition to arbitrary vectorfields X, Y according to,

$$\gamma(X,Y) = \gamma({}^{(h)}X, {}^{(h)}Y)$$

and

$$\chi(X,Y) = \chi({}^{(h)}X, {}^{(h)}Y), \qquad \underline{\chi}(X,Y) = \underline{\chi}({}^{(h)}X, {}^{(h)}Y).$$

Given a general 2-covariant horizontal tensor U we decompose it in its symmetric and antisymmetric part as follows,

$$\begin{aligned} {}^{(s)}U(X,Y) &= \frac{1}{2}\big(U(X,Y) + U(Y,X)\big), \\ {}^{(a)}U(X,Y) &= \frac{1}{2}\big(U(X,Y) - U(Y,X)\big). \end{aligned}$$

Given a horizontal structure defined by $e_3 = \underline{L}$, $e_4 = L$ we associate a null frame by choosing orthonormal horizontal vectorfields e_1, e_2 such that $\gamma(e_a, e_b) = \delta_{ab}$. By convention, we say that (e_1, e_2) is positively oriented on $\mathbf{O}(\mathcal{M})$ if,

$$\in (e_1, e_2) = \frac{1}{2} \in (e_1, e_2, e_3, e_4) = 1. \tag{2.1.4}$$

Remark 2.1.4. *We note that the particular choice of an orthonormal basis is immaterial. All the quantities we work with are tensorial with respect to the horizontal structure.*

Given a covariant horizontal 2-tensor U and an arbitrary orthonormal horizontal frame $(e_a)_{a=1,2}$ we have,

$${}^{(s)}U_{ab} = \frac{1}{2}(U_{ab} + U_{ba}), \qquad {}^{(a)}U_{ab} = \frac{1}{2}(U_{ab} - U_{ba}).$$

Definition 2.1.5. *The trace of a horizontal 2-tensor U is defined by*

$$tr(U) := \delta^{ab}U_{ab} = \delta^{ab}\,{}^{(s)}U_{ab}. \tag{2.1.5}$$

We define the anti-trace of U to be

$${}^{(a)}tr(U) := \in^{ab} U_{ab} = \in^{ab}\ {}^{(a)}U_{ab}. \tag{2.1.6}$$

Observe that the first trace is independent of the particular choice of the frame e_1, e_2. On the other hand, for fixed e_3, e_4, ${}^{(a)}tr$ depends on the orientation of e_1, e_2. Also, by interchanging e_3, e_4, ${}^{(a)}tr$ changes sign.

A general horizontal 2-tensor U can be decomposed according to

$$U_{ab} = {}^{(s)}U_{ab} + {}^{(a)}U_{ab} = \widehat{U}_{ab} + \frac{1}{2}\delta_{ab}\,\mathrm{tr}(U) + \frac{1}{2}\in_{ab}\,{}^{(a)}\mathrm{tr}(U), \tag{2.1.7}$$

where $\widehat{U}$ denotes the symmetric traceless part of U.

Definition 2.1.6. *We introduce the notation*

$$tr\,\chi := tr(\chi), \quad {}^{(a)}tr\chi := {}^{(a)}tr(\chi), \quad tr\,\underline{\chi} := tr(\underline{\chi}), \quad {}^{(a)}tr\underline{\chi} := {}^{(a)}tr(\underline{\chi}). \tag{2.1.8}$$

The quantities $\widehat{\chi}, tr\,\chi$ and $\widehat{\underline{\chi}}, tr\,\underline{\chi}$ are called, respectively, the shear and expansion of the horizontal distribution $\mathbf{O}(\mathcal{M})$. The scalars ${}^{(a)}tr\chi$ and ${}^{(a)}tr\underline{\chi}$ measure the integrability defects of the distribution.

Accordingly, we decompose $\chi, \underline{\chi}$ as follows

$$\begin{aligned}\chi_{ab} &= \widehat{\chi}_{ab} + \frac{1}{2}\delta_{ab} tr\,\chi + \frac{1}{2}\in_{ab}\,{}^{(a)}tr\chi,\\ \underline{\chi}_{ab} &= \widehat{\underline{\chi}}_{ab} + \frac{1}{2}\delta_{ab} tr\,\underline{\chi} + \frac{1}{2}\in_{ab}\,{}^{(a)}tr\underline{\chi}.\end{aligned}$$

In what follows we fix a null pair (e_3, e_4) and an orientation on $\mathbf{O}(\mathcal{M})$.

Definition 2.1.7. *We define the left and right duals of a horizontal 1-form ξ and a 2-covariant tensor-field U,*

$$\begin{aligned} {}^*\xi_a &= \in_{ab}\xi_b, & \xi^*{}_a &= \xi_b \in_{ba},\\ ({}^*U)_{ab} &= \in_{ac} U_{cb}, & (U^*)_{ab} &= U_{ac}\in_{cb}. \end{aligned}$$

Lemma 2.1.8. *Given a horizontal 1-form ξ, we have*

$${}^*({}^*\xi) = -\xi, \qquad {}^*\xi = -\xi^*.$$

Lemma 2.1.9. *Given a covariant horizontal 2-tensor U, we have*

1. ${}^*({}^*U) = -U$.
2. *If U is symmetric, then* ${}^*U_{ab} = -U^*_{ba}$.
3. *If $U = \hat{U}$ is symmetric traceless, then* ${}^*\hat{U} = -\hat{U}^*$ *is also symmetric traceless.*

4. In general,

$$\begin{aligned} tr({}^*U) &= tr(U^*) = -{}^{(a)}tr(U), \\ {}^{(a)}tr({}^*U) &= {}^{(a)}tr(U^*) = tr(U), \\ \widehat{{}^*U} &= {}^*\hat{U}. \end{aligned}$$

Given a general horizontal 2-tensor U we have, according to (2.1.7),

$$\begin{aligned} U^*_{ab} &= \hat{U}^*_{ab} + \frac{1}{2}\in_{ab}\,\mathrm{tr}(U) - \frac{1}{2}\delta_{ab}\,{}^{(a)}\mathrm{tr}(U), \\ {}^*U_{ab} &= {}^*\hat{U}_{ab} + \frac{1}{2}\in_{ab}\,\mathrm{tr}(U) - \frac{1}{2}\delta_{ab}\,{}^{(a)}\mathrm{tr}(U). \end{aligned}$$

Hence,

$$U^*_{ab} = -{}^*U_{ab} + \in_{ab}\,\mathrm{tr}(U) - \delta_{ab}\,{}^{(a)}\mathrm{tr}(U).$$

We note the following lemma.

Lemma 2.1.10. *Given two 1-forms ξ, η we have,*

$${}^*\xi\cdot\eta = -\xi^{\,*}\cdot\eta = \xi\cdot\eta^*.$$

Given a 1-form ξ and 2-tensor U we have,

$$\begin{aligned} \xi_a U_{ab} &= \xi^a\hat{U}_{ab} + \frac{1}{2}\xi_b tr(U) - \frac{1}{2}\xi_b\,{}^{(a)}tr(U), \\ {}^*\xi^a U^*_{ab} &= \xi^a\hat{U}_{ab} + \frac{1}{2}\xi_b tr(U) - \frac{1}{2}\xi_b\,{}^{(a)}tr(U). \end{aligned}$$

Thus,

$${}^*\xi^a U^*_{ab} + \xi^a U_{ab} = \xi_b(trU) - {}^*\xi_b({}^{(a)}trU).$$

Also,

$${}^*\xi^a U^*_{ab} - \xi^a U_{ab} = -2\xi^a\hat{U}_{ab}.$$

Proof. We have

$$\begin{aligned} {}^*\xi^a U^*_{ab} &= {}^*\xi^a\left(\hat{U}^*_{ab} + \frac{1}{2}\in_{ab}\mathrm{tr}(U) - \frac{1}{2}\delta_{ab}\,{}^{(a)}\mathrm{tr}(U)\right) \\ &= -\xi^a\hat{U}_{ab} + \frac{1}{2}\xi_b\mathrm{tr}(U) - \frac{1}{2}\,{}^*\xi_b\,{}^{(a)}\mathrm{tr}(U), \\ \xi_a U_{ab} &= \xi^a\left(\hat{U}_{ab} + \frac{1}{2}\delta_{ab}\mathrm{tr}(U) + \frac{1}{2}\in_{ab}\,{}^{(a)}\mathrm{tr}(U)\right) \\ &= \xi^a\hat{U}_{ab} + \frac{1}{2}\xi_b\mathrm{tr}(U) - \frac{1}{2}\xi_b\,{}^{(a)}\mathrm{tr}(U), \end{aligned}$$

which implies the formulas involving U and ξ. □

Definition 2.1.11. *We denote[2] by $\mathbf{O}_k(\mathcal{M})$ the set of all horizontal tensor-fields of rank k on $\mathcal{M}$. We denote by $\mathfrak{s}_0 = \mathfrak{s}_0(\mathcal{M})$ the set of pairs of real scalar functions on $\mathcal{M}$, $\mathfrak{s}_1 = \mathfrak{s}_1(\mathcal{M})$ the set of real horizontal 1-forms on $\mathcal{M}$ and for, $k \geq 2$, $\mathfrak{s}_k(\mathcal{M})$ the set of fully symmetric traceless horizontal real tensors of rank k. In particular $\mathfrak{s}_2 = \mathfrak{s}_2(\mathcal{M})$ denotes the set of symmetric traceless horizontal real 2-tensors on $\mathcal{M}$.*

Definition 2.1.12. *Given real $\xi, \eta \in \mathfrak{s}_1$ we denote*

$$\xi \cdot \eta \ := \ \delta^{ab}\xi_a\eta_b, \qquad \xi \wedge \eta := \in^{ab} \xi_a\eta_b = \xi \cdot {}^*\eta, \qquad (\xi \widehat{\otimes} \eta)_{ab} := \xi_a\eta_b + \xi_b\eta_a - \delta_{ab}\xi \cdot \eta.$$

Given $\xi \in \mathfrak{s}_1$, $U \in \mathfrak{s}_2$ we denote

$$(\xi \cdot U)_a \ := \ \delta^{bc}\xi_b U_{ac}.$$

Given $U, V \in \mathfrak{s}_2$ we denote

$$(U \wedge V)_{ab} \ := \ \in^{ab} U_{ac}V_{cb}.$$

The following two lemmas are immediate.

Lemma 2.1.13. *Given $\xi \in \mathfrak{s}_1$, $U \in \mathfrak{s}_2$, we have*

$$U_{ab}\xi^b \ = \ \left(\hat{U} \cdot \xi + \frac{1}{2}tr(U)\xi + \frac{1}{2}{}^{(a)}tr(U)\ {}^*\xi\right)_a.$$

Lemma 2.1.14. *Given $\hat{U}, \hat{V} \in \mathfrak{s}_2$ we have, with respect to an arbitrary orthonormal basis,*

$$\hat{U}_{ac}\hat{V}_{cb} + \hat{V}_{ac}\hat{U}_{cb} = \delta_{ab}\hat{U} \cdot \hat{V}$$

where

$$\hat{U} \cdot \hat{V} = \delta^{ac}\delta^{bd}\hat{U}_{ab}\hat{V}_{cd}.$$

In particular

$$\hat{V}_{ac}\hat{V}_{cb} = \frac{1}{2}\delta_{ab}|\hat{V}|^2$$

with $|\hat{V}|^2 = \hat{V} \cdot \hat{V}$.

Remark 2.1.15. *The previous lemma implies in particular $\widehat{\hat{U}_{ac}\hat{V}_{cb}} = 0$.*

[2] Using the convention of raising and lowering indices we make no distinction here between covariant and contravariant tensors.

We generalize the lemma as follows.

Lemma 2.1.16. *Given U, V arbitrary 2-covariant horizontal tensor-fields, we have*

$$\begin{aligned} \delta^{ab} U_{ac} V_{cb} &= \hat{U} \cdot \hat{V} + \frac{1}{2}\big(tr(U) tr(V) - {}^{(a)}tr(U)\, {}^{(a)}tr(V)\big), \\ \in^{ab} U_{ac} V_{cb} &= \hat{U} \wedge \hat{V} + \frac{1}{2}\big({}^{(a)}tr(U) tr(V) + tr(U)\, {}^{(a)}tr(V)\big), \\ \widehat{U_{ac} V_{cb}} &= \frac{1}{2}\big(\widehat{U}_{ab} tr(V) + \widehat{V}_{ab} tr(U)\big) + \frac{1}{2}\big(-\ {}^*\widehat{U}_{ab}\, {}^{(a)}tr(V) +\ {}^*\widehat{V}_{ab}\, {}^{(a)}tr(U)\big), \end{aligned}$$

where

$$\begin{aligned} \hat{U} \cdot \hat{V} &= \delta^{ac} \delta^{bd} \hat{U}_{ab} \hat{V}_{cd}, \\ \hat{U} \wedge \hat{V} &= \hat{U} \cdot {}^*\hat{V} = \in^{ab} \hat{U}_{ac} \hat{V}_{cb}. \end{aligned}$$

Proof. In view of the decomposition (2.1.7), we have

$$\begin{aligned} U_{ac} V_{cb} &= \widehat{U}_{ac} \widehat{V}_{cb} + \frac{1}{2}\big(\mathrm{tr}(V) \widehat{U}_{ab} + \mathrm{tr}(U) \widehat{V}_{ab}\big) + \frac{1}{2}\big({}^{(a)}\mathrm{tr}(U)\ {}^*\widehat{V}_{ab} + {}^{(a)}\mathrm{tr}(V) \widehat{U}^*_{ab}\big) \\ &+ \frac{1}{4}\big(\mathrm{tr}(U)\mathrm{tr}(V) - {}^{(a)}\mathrm{tr}(U)\, {}^{(a)}\mathrm{tr}(V)\big)\delta_{ab} + \frac{1}{4}\big(\mathrm{tr}(U)\, {}^{(a)}\mathrm{tr}(V) + {}^{(a)}\mathrm{tr}(U)\mathrm{tr}(V)\big) \in_{ab} \end{aligned}$$

and the proof easily follows, using also the fact that $\widehat{\hat{U}_{ac} \hat{V}_{cb}} = 0$ according to Remark 2.1.15. □

The following is an immediate consequence of Lemma 2.1.16.

Corollary 2.1.17. *In the particular case when $U = V$, we have*

$$\begin{aligned} \delta^{ab} U_{ac} U_{cb} &= |\hat{U}|^2 + \frac{1}{2}\big((tr(U))^2 - ({}^{(a)}tr(U))^2\big), \\ \in^{ab} U_{ac} U_{cb} &= tr(U)\, {}^{(a)}tr(U), \\ \widehat{U_{ac} U_{cb}} &= tr(U) \widehat{U}_{ab}. \end{aligned}$$

As another corollary to Lemma 2.1.16 we have the following.

Lemma 2.1.18. *Let u be an arbitrary 2-horizontal tensor and $v \in \mathfrak{s}_2$. Then*

$$u_{ac} v_{cb} + u_{bc} v_{ca} = \delta_{ab} \hat{u} \cdot v + (tr u) v_{ab} + \frac{1}{2}\Big[\big(u_{ac} - u_{ca}\big) v_{cb} + \big(u_{bc} - u_{cb}\big) v_{ca}\Big].$$

Proof. We give below a direct proof based on Lemma 2.1.14 according to which, given $u, v \in \mathfrak{s}_2$, we have

$$u_{ac}v_{cb} + u_{bc}v_{ca} = \delta_{ab} u \cdot v.$$

If u is only symmetric and $v \in \mathfrak{s}_2$ we can write,

$$\begin{aligned} u_{ac}v_{cb} + u_{bc}v_{ca} &= \left(\hat{u}_{ac} + \frac{1}{2}\delta_{ac} tr(u)\right) v_{cb} + \left(\hat{u}_{bc} + \frac{1}{2}\delta_{bc} tr(u)\right) v_{ca} \\ &= \delta_{ab}\hat{u} \cdot v + (\mathrm{tr} u) v_{ab}. \end{aligned}$$

If u is an arbitrary 2-tensor and $v \in \mathfrak{s}_2$,

$$\begin{aligned} u_{ac}v_{cb} + u_{bc}v_{ca} &= \frac{1}{2}\Big(u_{ac} + u_{ca} + \big(u_{ac} - u_{ca}\big)\Big) v_{cb} + \frac{1}{2}\Big(u_{bc} + u_{cb} + \big(u_{bc} - u_{cb}\big)\Big) v_{ca} \\ &= \frac{1}{2}\delta_{ab}\Big(u_{ac} + u_{ca}\Big) v_{ac} + \frac{1}{2}\Big(\big(u_{ac} - u_{ca}\big) v_{cb} + \big(u_{bc} - u_{cb}\big) v_{ca}\Big) \\ &= \delta_{ab}\hat{u} \cdot v + (\mathrm{tr} u) v_{ab} + \frac{1}{2}\Big(\big(u_{ac} - u_{ca}\big) v_{cb} + \big(u_{bc} - u_{cb}\big) v_{ca}\Big). \end{aligned}$$

This concludes the proof of the lemma. □

Lemma 2.1.19. *The following formulas hold true:*

- *Given* $\xi, \eta \in \mathfrak{s}_1$, *we have*

$$\begin{gathered} {}^*\xi \cdot \eta = -\xi \cdot {}^*\eta, \qquad {}^*\xi \cdot {}^*\eta = \xi \cdot \eta, \qquad {}^*\xi \wedge \eta = -\xi \wedge {}^*\eta, \qquad {}^*\xi \wedge {}^*\eta = \xi \wedge \eta, \\ {}^*\xi \widehat{\otimes} \eta = \xi \widehat{\otimes} {}^*\eta, \qquad {}^*(\xi \widehat{\otimes} \eta) = {}^*\xi \widehat{\otimes} \eta, \qquad {}^*\xi \widehat{\otimes} {}^*\eta = -\xi \widehat{\otimes} \eta. \end{gathered}$$

- *Given* $\xi \in \mathfrak{s}_1, U \in \mathfrak{s}_2$, *we have*

$${}^*(\xi \cdot U) = \xi \cdot {}^*U, \qquad {}^*\xi \cdot U = -\xi \cdot {}^*U, \qquad {}^*\xi \cdot {}^*U = \xi \cdot U.$$

- *Given* $U, V \in \mathfrak{s}_2$, *we have,*

$${}^*U \cdot V = -U \cdot {}^*V, \quad {}^*U \cdot {}^*V = U \cdot V, \quad {}^*U \wedge V = -U \wedge {}^*V, \quad {}^*U \wedge {}^*V = U \wedge V.$$

Proof. The statements follow from the above results except the ones involving $\widehat{\otimes}$. To check those we write, for an arbitrary basis e_1, e_2,

$$\begin{aligned} {}^*(\xi \widehat{\otimes} \eta)_{11} &= (\xi \widehat{\otimes} \eta)_{21} = \xi_2\eta_1 + \xi_1\eta_2, \\ (\xi \widehat{\otimes} {}^*\eta)_{11} &= \xi_1({}^*\eta)_1 - \xi_2({}^*\eta)_2 = \xi_1\eta_2 + \xi_2\eta_1, \\ ({}^*\xi \widehat{\otimes} \eta)_{11} &= ({}^*\xi)_1\eta_1 - ({}^*\xi)_2\eta_2 = \xi_2\eta_1 + \xi_1\eta_2, \end{aligned}$$

and

$$\begin{aligned}
{}^*(\xi\widehat{\otimes}\eta)_{12} &= \xi_2\eta_2 - \xi_1\eta_1,\\
(\xi\widehat{\otimes}\ {}^*\eta)_{12} &= \xi_1\ {}^*\eta_2 + \xi_2\ {}^*\eta_1 = -\xi_1\eta_1 + \xi_2\eta_2,\\
(\ {}^*\xi\widehat{\otimes}\eta)_{12} &= \ {}^*\xi_1\eta_2 + \ {}^*\xi_2\eta_1 = -\xi_1\eta_1 + \xi_2\eta_2.
\end{aligned}$$

Hence,

$${}^*(\xi\widehat{\otimes}\eta) = \ {}^*\xi\widehat{\otimes}\eta = \xi\widehat{\otimes}\ {}^*\eta$$

as stated. □

Lemma 2.1.20. *Given $\xi, \eta \in \mathfrak{s}_1$, $u \in \mathfrak{s}_2$ we have*

$$\xi\widehat{\otimes}(\eta\cdot u) + \eta\widehat{\otimes}(\xi\cdot u) = 2(\xi\cdot\eta)u.$$

Proof. Straightforward verification. □

2.1.2 Horizontal covariant derivative

Given $X, Y \in \mathbf{O}(\mathcal{M})$, the covariant derivative $\mathbf{D}_X Y$ fails in general to be horizontal. We thus define the horizontal covariant operator ∇ as follows

$$\nabla_X Y :=^{(h)} (\mathbf{D}_X Y) = \mathbf{D}_X Y - \frac{1}{2}\underline{\chi}(X,Y)L - \frac{1}{2}\chi(X,Y)\,\underline{L}. \tag{2.1.9}$$

Proposition 2.1.21. *For all $X, Y \in \mathbf{O}(\mathcal{M})$,*

$$\begin{aligned}
\nabla_X Y - \nabla_Y X &= [X,Y] - {}^{(a)}\underline{\chi}(X,Y)L - {}^{(a)}\chi(X,Y)\,\underline{L}\\
&= [X,Y] - \frac{1}{2}\left({}^{(a)}tr\underline{\chi}\,L + {}^{(a)}tr\chi\ \underline{L}\right) \in (X,Y).
\end{aligned}$$

In particular,

$${}^{(h)}[X,Y] \;=\; \frac{1}{2}\left({}^{(a)}tr\underline{\chi}\,L + {}^{(a)}tr\chi\ \underline{L}\right) \in (X,Y). \tag{2.1.10}$$

For all $X, Y, Z \in \mathbf{O}(\mathcal{M})$,

$$Z\gamma(X,Y) = \gamma(\nabla_Z X, Y) + \gamma(X, \nabla_Z Y).$$

Remark 2.1.22. *In the integrable case, ∇ coincides with the Levi-Civita connection of the metric induced on the integral surfaces of $\mathbf{O}(\mathcal{M})$.*

Given a general covariant, horizontal tensor-field U we define its horizontal covariant derivative according to the formula,

$$\begin{aligned}\nabla_Z U(X_1, \dots X_k) = Z(U(X_1, \dots X_k)) &- U(\nabla_Z X_1, \dots X_k) - \\ \dots &- U(X_1, \dots \nabla_Z X_k).\end{aligned}$$

Given X horizontal, $\mathbf{D}_L X$ and $\mathbf{D}_{\underline{L}} X$ are in general not horizontal. We define $\nabla_L X$ and $\nabla_{\underline{L}} X$ to be the horizontal projections of the former. More precisely,

$$\begin{aligned}\nabla_L X &:= {}^{(h)}(\mathbf{D}_L X) = \mathbf{D}_L X - \mathbf{g}(X, \mathbf{D}_L \underline{L}) L - \mathbf{g}(X, \mathbf{D}_L L)\, \underline{L}, \\ \nabla_{\underline{L}} X &:= {}^{(h)}(\mathbf{D}_{\underline{L}} X) = \mathbf{D}_{\underline{L}} X - \mathbf{g}(X, \mathbf{D}_{\underline{L}} \underline{L}) L - \mathbf{g}(X, \mathbf{D}_{\underline{L}} L)\, \underline{L}.\end{aligned}$$

We can extend the operators ∇_L and $\nabla_{\underline{L}}$ to arbitrary k-covariant, horizontal tensor-fields U as follows,

$$\begin{aligned}\nabla_L U(X_1, \dots, X_k) = L(U(X_1, \dots, X_k)) &- U(\nabla_L X_1, \dots, X_k) - \\ \dots &- U(X_1, \dots, \nabla_L X_k), \\ \nabla_{\underline{L}} U(X_1, \dots, X_k) = \underline{L}(U(X_1, \dots, X_k)) &- U(\nabla_{\underline{L}} X_1, \dots, X_k) - \\ \dots &- U(X_1, \dots, \nabla_{\underline{L}} X_k).\end{aligned}$$

The following proposition follows easily from the definition.

Proposition 2.1.23. *The operators ∇, ∇_L and $\nabla_{\underline{L}}$ take horizontal tensor-fields into horizontal tensor-fields. We have,*

$$\nabla \gamma = \nabla_L \gamma = \nabla_{\underline{L}} \gamma = 0. \tag{2.1.11}$$

We now extend the definition of horizontal covariant derivative to any $X \in \mathbf{T}(\mathcal{M})$ in the tangent space of $\mathcal{M}$ and $Y \in \mathbf{O}(\mathcal{M})$.

Definition 2.1.24. *Given $X \in \mathbf{T}(\mathcal{M})$ and $Y \in \mathbf{O}(\mathcal{M})$ we define,*

$$\dot{\mathbf{D}}_X Y := {}^{(h)}(\mathbf{D}_X Y).$$

Given an orthonormal frame $e_1, e_2 \in \mathbf{O}(\mathcal{M})$ we write

$$\dot{\mathbf{D}}_\mu e_a = \sum_{b=1,2} (\Lambda_\mu)_{ba}\, e_b, \qquad (\Lambda_\mu)_{\alpha\beta} := \mathbf{g}(\mathbf{D}_\mu e_\beta, e_\alpha).$$

Definition 2.1.25. *Given a general, covariant, horizontal tensor-field U we define its horizontal covariant derivative according to the formula*

$$\dot{\mathbf{D}}_X U(Y_1, \dots Y_k) = X(U(Y_1, \dots Y_k)) - U(\dot{\mathbf{D}}_X Y_1, \dots Y_k) - \dots - U(Y_1, \dots \dot{\mathbf{D}}_X Y_k),$$

where $X \in \mathbf{T}(\mathcal{M})$ and $Y_1, \dots Y_k \in \mathbf{O}(\mathcal{M})$.

Proposition 2.1.26. *For all $X \in \mathbf{T}(\mathcal{M})$ and $Y_1, Y_2 \in \mathbf{O}(\mathcal{M})$,*

$$X\gamma(Y_1, Y_2) = \gamma(\dot{\mathbf{D}}_X Y_1, Y_2) + \gamma(Y_1, \dot{\mathbf{D}}_X Y_2).$$

Proof. Indeed,

$$\begin{aligned} X\gamma(Y_1, Y_2) &= X\mathbf{g}(Y_1, Y_2) = \mathbf{g}(\mathbf{D}_X Y_1, Y_2) + \mathbf{g}(Y_1, \mathbf{D}_X Y_2) = \mathbf{g}(\dot{\mathbf{D}}_X Y_1, Y_2) + \mathbf{g}(Y_1, \dot{\mathbf{D}}_X Y_2) \\ &= \gamma(\dot{\mathbf{D}}_X Y_1, Y_2) + \gamma(Y_1, \dot{\mathbf{D}}_X Y_2) \end{aligned}$$

as desired. □

We consider tensors $\mathbf{T}_k(\mathcal{M}) \otimes \mathbf{O}_l(\mathcal{M})$, i.e. tensors of the form $U_{\nu_1\ldots\nu_k, a_1\ldots a_l}$ for which we define,

$$\begin{aligned} \dot{\mathbf{D}}_\mu U_{\nu_1\ldots\nu_k, a_1\ldots a_l} &= e_\mu U_{\nu_1\ldots\nu_k, a_1\ldots a_l} - U_{\mathbf{D}_\mu \nu_1\ldots\nu_k, a_1\ldots a_l} - \ldots - U_{\nu_1\ldots\mathbf{D}_\mu \nu_k, a_1\ldots a_l} \\ &- U_{\nu_1\ldots\nu_k, \dot{\mathbf{D}}_\mu a_1\ldots a_l} - U_{\nu_1\ldots\nu_k, a_1\ldots\dot{\mathbf{D}}_\mu a_l}. \end{aligned}$$

We are now ready to prove the following.

Proposition 2.1.27. *For a tensor $\Psi \in \mathbf{O}_1(\mathcal{M})$, we have the curvature formula*[3]

$$(\dot{\mathbf{D}}_\mu \dot{\mathbf{D}}_\nu - \dot{\mathbf{D}}_\nu \dot{\mathbf{D}}_\mu)\Psi_a = \dot{\mathbf{R}}_{ab\mu\nu}\Psi^b \tag{2.1.12}$$

where, with connection coefficients $(\Lambda_\alpha)_{\beta\gamma} = \mathbf{g}(\mathbf{D}_\alpha e_\gamma, e_\beta)$,

$$\begin{aligned} \dot{\mathbf{R}}_{ab\mu\nu} &:= \mathbf{R}_{ab\mu\nu} + \frac{1}{2}\mathbf{B}_{ab\mu\nu}, \\ \mathbf{B}_{ab\mu\nu} &:= (\Lambda_\mu)_{3a}(\Lambda_\nu)_{b4} + (\Lambda_\mu)_{4a}(\Lambda_\nu)_{b3} - (\Lambda_\nu)_{3a}(\Lambda_\mu)_{b4} - (\Lambda_\nu)_{4a}(\Lambda_\mu)_{b3}. \end{aligned} \tag{2.1.13}$$

More generally, for a mixed tensor $\Psi \in \mathbf{T}_1(\mathcal{M}) \otimes \mathbf{O}_1(\mathcal{M})$, we have

$$(\dot{\mathbf{D}}_\mu \dot{\mathbf{D}}_\nu - \dot{\mathbf{D}}_\nu \dot{\mathbf{D}}_\mu)\Psi_{\lambda a} = \mathbf{R}_\lambda{}^\sigma{}_{\mu\nu}\Psi_{\sigma a} + \dot{\mathbf{R}}_a{}^b{}_{\mu\nu}\Psi_{\lambda b}$$

with an immediate generalization to tensors $\Psi \in \mathbf{T}_k(\mathcal{M}) \otimes \mathbf{O}_l(\mathcal{M})$.

Proof. We have

$$\mathbf{D}_\mu e_a = \dot{\mathbf{D}}_\mu e_a - \frac{1}{2}\mathbf{g}(\mathbf{D}_\mu e_a, e_3)e_4 - \frac{1}{2}\mathbf{g}(\mathbf{D}_\mu e_a, e_4)e_3 = \dot{\mathbf{D}}_\mu e_a - \frac{1}{2}(\Lambda_\mu)_{3a}e_4 - \frac{1}{2}(\Lambda_\mu)_{4a}e_3.$$

[3] With an immediate generalization to tensors $\Psi \in \mathbf{O}_l(\mathcal{M})$.

We deduce

$$\mathbf{D}_\mu \Psi_a = \dot{\mathbf{D}}_\mu \Psi_a - \frac{1}{2}(\Lambda_\mu)_{3a}\Psi_4 - \frac{1}{2}(\Lambda_\mu)_{4a}\Psi_3 = \nabla_\mu \Psi_a.$$

Hence

$$\begin{aligned} \mathbf{D}_\mu \mathbf{D}_\nu \Psi_a &= e_\mu(\mathbf{D}_\nu \Psi_a) - \mathbf{D}_{\mathbf{D}_\mu e_\nu}\Psi_a - \mathbf{D}_\nu \Psi_{\mathbf{D}_\mu e_a} \\ &= \nabla_\mu \nabla_\nu \Psi_a - \dot{\mathbf{D}}_{\mathbf{D}_\mu e_\nu}\Psi_a + \frac{1}{2}(\Lambda_\mu)_{3a}\mathbf{D}_\nu \Psi_4 + \frac{1}{2}(\Lambda_\mu)_{4a}\mathbf{D}_\nu \Psi_3. \end{aligned}$$

On the other hand

$$\mathbf{D}_\nu \Psi_4 = -(\Lambda_\nu)_{b4}\Psi_b, \qquad \mathbf{D}_\nu \Psi_3 = -(\Lambda_\nu)_{b3}\Psi_b,$$

hence

$$\mathbf{D}_\mu \mathbf{D}_\nu \Psi_a = \nabla_\mu \nabla_\nu \Psi_a - \dot{\mathbf{D}}_{\mathbf{D}_\mu e_\nu}\Psi_a - \frac{1}{2}(\Lambda_\mu)_{3a}(\Lambda_\nu)_{b4}\Psi_b - \frac{1}{2}(\Lambda_\mu)_{4a}(\Lambda_\nu)_{b3}\Psi_b.$$

By symmetry

$$\mathbf{D}_\nu \mathbf{D}_\mu \Psi_a = \nabla_\nu \nabla_\mu \Psi_a - \dot{\mathbf{D}}_{\mathbf{D}_\nu e_\mu}\Psi_a - \frac{1}{2}(\Lambda_\nu)_{3a}(\Lambda_\mu)_{b4}\Psi_b - \frac{1}{2}(\Lambda_\nu)_{4a}(\Lambda_\mu)_{b3}\Psi_b.$$

Subtracting and using the Ricci formula and the Lemma above we deduce

$$\begin{aligned} \mathbf{R}_{ab\mu\nu}\Psi^b &= [\nabla_\mu, \nabla_\nu]\Psi_a - \nabla_{[e_\mu, e_\nu]}\Psi_a \\ &\quad -\frac{1}{2}(\Lambda_\mu)_{3a}(\Lambda_\nu)_{b4}\Psi_b - \frac{1}{2}(\Lambda_\mu)_{4a}(\Lambda_\nu)_{b3}\Psi_b + \frac{1}{2}(\Lambda_\nu)_{3a}(\Lambda_\mu)_{b4}\Psi_b + \frac{1}{2}(\Lambda_\nu)_{4a}(\Lambda_\mu)_{b3}\Psi_b \\ &= \dot{\mathbf{D}}_\mu \dot{\mathbf{D}}_\nu \Psi_a - \dot{\mathbf{D}}_\nu \dot{\mathbf{D}}_\mu \Psi_a - \frac{1}{2}\mathbf{B}_{ab\mu\nu}\Psi^b, \end{aligned}$$

from which the desired formula follows. □

Remark 2.1.28. *Note that the tensor $\mathbf{B}_{ab\mu\nu}$ is anti-symmetric in both $\mu\nu$ and ab.*

Corollary 2.1.29. *Let X, Y be arbitrary vectorfields on $\mathcal{M}$ and $U \in \mathbf{O}_1(\mathcal{M})$ an horizontal tensor. We have*[4]

$$\big(\nabla_X \nabla_Y - \nabla_Y \nabla_X\big)U = \nabla_{[X,Y]}U + \dot{\mathbf{R}}(X, Y)U$$

with an immediate generalization to $U \in \mathbf{O}_l(\mathcal{M})$.

[4] Here $(\dot{\mathbf{R}}(X, Y)U)_a := X^\mu Y^\nu \dot{\mathbf{R}}_{ab\mu\nu}U^b$.

Proof. We have

$$\begin{aligned}\nabla_Y\nabla_X U_a &= (Y^\lambda\dot{\mathbf{D}}_\lambda)(X^\mu\dot{\mathbf{D}}_\mu)U_a = Y^\lambda X^\mu\dot{\mathbf{D}}_\lambda\dot{\mathbf{D}}_\mu U_a + (Y^\lambda\dot{\mathbf{D}}_\lambda)(X^\mu)\dot{\mathbf{D}}_\mu U_a,\\ \nabla_X\nabla_Y U_a &= X^\mu Y^\lambda\dot{\mathbf{D}}_\mu\dot{\mathbf{D}}_\lambda U_a + (X^\mu\dot{\mathbf{D}}_\mu)(Y^\lambda)\dot{\mathbf{D}}_\lambda U_a.\end{aligned}$$

Hence,

$$\begin{aligned}\big(\nabla_X\nabla_Y - \nabla_Y\nabla_X\big)U_a &= Y^\lambda X^\mu\big(\dot{\mathbf{D}}_\lambda\dot{\mathbf{D}}_\mu - \dot{\mathbf{D}}_\mu\dot{\mathbf{D}}_\lambda\big)U_a + \big(\dot{\mathbf{D}}_X(Y^\mu) - \dot{\mathbf{D}}_Y(X^\mu)\big)\dot{\mathbf{D}}_\mu U_a\\ &= X^\mu Y^\nu\dot{\mathbf{R}}_{ab\mu\nu}U^b + \dot{\mathbf{D}}_{[X,Y]}U_a,\end{aligned}$$

as stated. □

2.1.3 Horizontal Hodge operators

In this section we recall the Hodge operators on 2-spheres as defined in [23] and extend their properties to the case of non-integrable horizontal structures.

We first define the following operators on horizontal tensors.

Definition 2.1.30. *For a given horizontal* 1*-form* ξ*, we define the frame independent operators*

$$div\,\xi = \delta^{ab}\nabla_b\xi_a, \qquad curl\,\xi = \in^{ab}\nabla_a\xi_b, \qquad (\nabla\widehat{\otimes}\xi)_{ba} = \nabla_b\xi_a + \nabla_a\xi_b - \delta_{ab}(div\,\xi).$$

We collect here some Leibniz rules regarding the horizontal Hodge operators.

Lemma 2.1.31. *We have for* $\xi,\eta\in\mathfrak{s}_1$, $u\in\mathfrak{s}_2$,

$$\begin{aligned}(div\,\eta)\xi - (curl\,\eta)\,{}^*\xi &= \xi\cdot\nabla\eta + \xi\cdot\,{}^*\nabla\,{}^*\eta,\\ \xi\widehat{\otimes}(div\,u) &= \xi\cdot\nabla u + \xi\cdot\,{}^*\nabla\,{}^*u,\\ \xi\cdot(\nabla\widehat{\otimes}\eta) &= \xi\cdot\nabla f - \xi\cdot\,{}^*\nabla\,{}^*\eta.\end{aligned}$$

Proof. Define the tensors on the left $Z_a = (\mathrm{div}\,\eta)\xi_a - (curl\,\eta)\,{}^*\xi_a$, $Y_{ab} = \xi\widehat{\otimes}(\mathrm{div}\,u)_{ab}$ and $W_a = \xi\cdot(\nabla\widehat{\otimes}\eta)_a = \xi_b(\nabla\widehat{\otimes}\eta)_{ab}$ and evaluate components. By simply manipulating the definitions we find

$$\begin{aligned}Z_1 &= \xi\cdot\nabla\eta_1 + (\xi\cdot\,{}^*\nabla)\,{}^*\eta_1,\\ Z_2 &= \xi\cdot\nabla\eta_2 + (\xi\cdot\,{}^*\nabla)\,{}^*\eta_2,\\ Y_{11} &= (\xi\cdot\nabla)u_{11} + (\xi\cdot\,{}^*\nabla)\,{}^*u_{11},\\ Y_{12} &= (\xi\cdot\nabla)u_{12} + (\xi\cdot\,{}^*\nabla)\,{}^*u_{12},\\ W_1 &= \xi\cdot\nabla\eta_1 - \xi\cdot\,{}^*\nabla\,{}^*\eta_1,\\ W_2 &= \xi\cdot\nabla\eta_2 - \xi\cdot\,{}^*\nabla\,{}^*\eta_2,\end{aligned}$$

which implies the stated identities. □

Definition 2.1.32. *Given an orthonormal basis of horizontal vectors e_1, e_2 we define the Hodge type operators (recall Definition 2.1.11), as introduced in [23].*

- $\mathcal{D}_1$ *takes* $\mathfrak{s}_1$ *into*[5] $\mathfrak{s}_0$*:*

$$\mathcal{D}_1\xi = (div\,\xi, curl\,\xi),$$

- $\mathcal{D}_2$ *takes* $\mathfrak{s}_2$ *into* $\mathfrak{s}_1$*:*

$$(\mathcal{D}_2\xi)_a = \nabla^b\xi_{ab},$$

- $\mathcal{D}_1^*$ *takes* $\mathfrak{s}_0$ *into* $\mathfrak{s}_1$*:*

$$\mathcal{D}_1^*(f, f_*) = -\nabla_a f + \in_{ab} \nabla_b f_*,$$

- $\mathcal{D}_2^*$ *takes* $\mathfrak{s}_1$ *into* $\mathfrak{s}_2$*:*

$$\mathcal{D}_2^*\xi = -\frac{1}{2}\nabla\widehat{\otimes}\xi.$$

Lemma 2.1.33. *Note the following pointwise identities:*

1. *Given* $(f, f_*) \in \mathfrak{s}_0$, $u \in \mathfrak{s}_1$, *we have*
$$\mathcal{D}_1^*(f, f_*) \cdot u = (f, f_*) \cdot \mathcal{D}_1 u - \nabla_a\big(fu^a + f_*({}^*u)^a\big). \tag{2.1.14}$$
2. *Given* $f \in \mathfrak{s}_1$, $u \in \mathfrak{s}_2$, *we have*
$$(\mathcal{D}_2^* f) \cdot u = f \cdot (\mathcal{D}_2 u) - \nabla_a\big(f_b u^{ab}\big). \tag{2.1.15}$$

Proof. To check (2.1.15) we write

$$(\nabla\widehat{\otimes}f)\cdot u = \big(\nabla_a f_b + \nabla_b f_a - \delta_{ab}\mathrm{d}iv\, f\big)u_{ab} = 2(\nabla_a f_b)u_{ab} = 2\nabla_a(u_{ab}f_b) - 2(\mathrm{d}iv\, u)\cdot f$$

which immediately yields the second identity. □

In the particular case when the horizontal structure is tangent to 2-spheres S these operators are elliptic on S and have remarkable properties discussed in Chapter 2 of [23] which we recall below.

[5] Recall that $\mathfrak{s}_0$ refers to pairs of scalar functions.

Hodge operators on spheres

The following results were derived in Chapter 2 of [23] in the context of general 2-dimensional compact surfaces S with strictly positive Gauss curvature K which we will refer from now on as a 2-sphere.

Lemma 2.1.34. *Given a 2-sphere S, we have the following:*

- *The kernels of both $\mathcal{D}_1$ and $\mathcal{D}_2$ in $L^2(S)$ are trivial while the kernel of $\mathcal{D}_1^*$ consists of pairs of constants in $\mathfrak{s}_0$.*
- *The operators $\mathcal{D}_1^*$, resp. $\mathcal{D}_2^*$ are the L^2 adjoints of $\mathcal{D}_1$, respectively $\mathcal{D}_2$.*
- *The kernel of $\mathcal{D}_2^*$ is the space of conformal Killing vectorfields on S.*

Moreover the following identities hold true[6]*, see [23]:*

$$\begin{aligned} \mathcal{D}_1^*\mathcal{D}_1 &= -\triangle_1 + K, \qquad & \mathcal{D}_1\mathcal{D}_1^* &= -\triangle_0, \\ \mathcal{D}_2^*\mathcal{D}_2 &= -\frac{1}{2}\triangle_2 + K, \qquad & \mathcal{D}_2\mathcal{D}_2^* &= -\frac{1}{2}(\triangle_1 + K). \end{aligned} \tag{2.1.16}$$

Proof. The statements about L^2-adjoints follow immediately by integrating formulas (2.1.14)-(2.1.15) on S. The formulas (2.1.16) follow easily by using the definitions and commuting derivatives. See also the more general Lemma 2.1.36. Note also that for $\xi \in \mathfrak{s}_1$

$$\mathcal{D}_2^*\xi = -\frac{1}{2}\mathcal{L}_\xi\gamma$$

where γ denotes the induced horizontal metric as in Definition 2.1.3. □

As a simple consequence of (2.1.16) one derives the following L^2 estimates.

Proposition 2.1.35. *Let (S,γ) be a compact manifold with Gauss curvature K.*

i. *The following identity holds for vectorfields f on S:*

$$\int_S \left(|\nabla f|^2 + K|f|^2\right) = \int_S \left(|div\, f|^2 + |curl\, f|^2\right) = \int_S |\mathcal{D}_1 f|^2. \tag{2.1.17}$$

[6] Here $\triangle_k : \mathfrak{s}_k \to \mathfrak{s}_k$, $k = 0, 1, 2$, is defined by $(\triangle_k U)_A = \nabla^a\nabla_a U_A$.

ii. *The following identity holds for symmetric, traceless, 2-tensorfields f on S:*

$$\int_S \left(|\nabla f|^2 + 2K|f|^2\right) = 2\int_S |div\, f|^2 = 2\int_S |\,{\mathcal D}\!\!\!/_2 f|^2. \tag{2.1.18}$$

iii. *The following identity holds for pairs of functions (f, f_*) on S:*

$$\int_S \left(|\nabla f|^2 + |\nabla f_*|^2\right) = \int_S |-\nabla f + (\,^*\nabla f_*)|^2 = \int_S |\,{\mathcal D}\!\!\!/_1^*(f, f_*)|^2. \tag{2.1.19}$$

iv. *The following identity holds for vectors f on S:*

$$\int_S \left(|\nabla f|^2 - K|f|^2\right) = 2\int_S |\,{\mathcal D}\!\!\!/_2^* f|^2. \tag{2.1.20}$$

Proof. See Chapter 2 in [23]. □

Bochner identities in the non-integrable case

We extend the identities above to the case of non-integrable horizontal structure.

Lemma 2.1.36. *Given a general possibly non-integrable horizontal structure, the Hodge operators and the Laplacians are related by the following relations for $\xi \in \mathfrak{s}_1$ and $u \in \mathfrak{s}_2$:*

$$\begin{aligned}
{\mathcal D}\!\!\!/_1^*{\mathcal D}\!\!\!/_1\xi &= -\triangle_1\xi - \frac{1}{2}\in_{ab}[\nabla_a, \nabla_b]\,^*\xi,\\
{\mathcal D}\!\!\!/_2\,{\mathcal D}\!\!\!/_2^*\xi &= -\frac{1}{2}\triangle_1\xi + \frac{1}{4}\in_{ab}[\nabla_a, \nabla_b]\,^*\xi,\\
{\mathcal D}\!\!\!/_2^*{\mathcal D}\!\!\!/_2 u &= -\frac{1}{2}\triangle_2 u - \frac{1}{4}\in_{ab}[\nabla_a, \nabla_b]\,^*u.
\end{aligned} \tag{2.1.21}$$

Proof. We check the last relation by evaluating the components of the tensor $Y_{ab} := (\,{\mathcal D}\!\!\!/_2^*(\,{\mathcal D}\!\!\!/_2 u))_{ab}$

$$\begin{aligned}
Y_{ab} &= -\frac{1}{2}(\nabla\widehat{\otimes}(\,{\mathcal D}\!\!\!/_2 u))_{ab} = -\frac{1}{2}(\nabla_a(\,{\mathcal D}\!\!\!/_2 u)_b + \nabla_b(\,{\mathcal D}\!\!\!/_2 u)_a - \delta_{ab}(\mathrm{d}iv\,(\,{\mathcal D}\!\!\!/_2 u)))\\
&= -\frac{1}{2}(\nabla_a\nabla^c u_{bc} + \nabla_b\nabla^c u_{ac} - \delta_{ab}\nabla^c\nabla^d u_{cd}).
\end{aligned}$$

For $a = b = 1$ we derive

$$\begin{aligned} Y_{11} &= -\frac{1}{2}(\nabla_1 \nabla^c u_{1c} + \nabla_1 \nabla^c u_{1c} - \delta_{11} \nabla^c \nabla^d u_{cd}) \\ &= -\nabla_1 \nabla_1 u_{11} - \nabla_1 \nabla_2 u_{12} + \frac{1}{2}(\nabla_1 \nabla_1 u_{11} + \nabla_2 \nabla_1 u_{21} + \nabla_1 \nabla_2 u_{12} + \nabla_2 \nabla_2 u_{22}) \\ &= -\frac{1}{2}(\nabla_1 \nabla_1 + \nabla_2 \nabla_2) u_{11} + \frac{1}{2}(\nabla_2 \nabla_1 u_{12} - \nabla_1 \nabla_2 u_{12}) \end{aligned}$$

which gives

$$Y_{11} = -\frac{1}{2}\triangle_2 u_{11} - \frac{1}{2}[\nabla_1, \nabla_2] u_{12} = -\frac{1}{2}\triangle_2 u_{11} - \frac{1}{2}[\nabla_1, \nabla_2]\ {}^*u_{11}.$$

For $a = 1, b = 2$ we derive

$$\begin{aligned} Y_{12} &= -\frac{1}{2}(\nabla_1 \nabla^c u_{2c} + \nabla_2 \nabla^c u_{1c} - \delta_{12} \nabla^c \nabla^d u_{cd}) \\ &= -\frac{1}{2}(\nabla_1 \nabla_1 u_{21} + \nabla_1 \nabla_2 u_{22} + \nabla_2 \nabla_1 u_{11} + \nabla_2 \nabla_2 u_{12}) \\ &= -\frac{1}{2}(\nabla_1 \nabla_1 u_{12} + \nabla_2 \nabla_2 u_{12}) + \frac{1}{2}(\nabla_1 \nabla_2 u_{11} - \nabla_2 \nabla_1 u_{11}) \end{aligned}$$

which gives

$$Y_{12} = -\frac{1}{2}\triangle_2 u_{12} + \frac{1}{2}[\nabla_1, \nabla_2] u_{11} = -\frac{1}{2}\triangle_2 u_{12} - \frac{1}{2}[\nabla_1, \nabla_2]\ {}^*u_{12}.$$

Hence

$$Y_{ab} = -\frac{1}{2}\triangle_2 u_{ab} - \frac{1}{2}[\nabla_1, \nabla_2]\ {}^*u_{ab}$$

as stated. The other relations can be checked in the same manner. □

Using the pointwise relations (2.1.14) and (2.1.15) and the above lemma, we can deduce the following pointwise version of the L^2 estimates of Proposition 2.1.35.

Proposition 2.1.37. *Given a not necessarily integrable horizontal structure, the following pointwise relations hold:*

i. *The following identity holds for $f \in \mathfrak{s}_1$:*

$$\begin{aligned} |\nabla f|^2 - \frac{1}{2} \in_{ab} [\nabla_a, \nabla_b]\ {}^*f \cdot f = |\,{\not{\mathcal{D}}}_1 f|^2 \\ + \nabla_a \Big(\nabla^a f \cdot f - (div\, f) f^a - (curl\, f)(\ {}^*f)^a \Big). \end{aligned} \tag{2.1.22}$$

ii. *The following identity holds for $f \in \mathfrak{s}_2$:*

$$|\nabla f|^2 - \frac{1}{4} \in_{ab} [\nabla_a, \nabla_b] \, {}^*f \cdot f = 2|\not{\mathcal{D}}_2 f|^2 + \nabla_a\Big(\nabla^a f \cdot f - 2(div\, f)_b f^{ab}\Big). \tag{2.1.23}$$

iii. *The following identity holds for $f \in \mathfrak{s}_1$:*

$$|\nabla f|^2 + \frac{1}{4} \in_{ab} [\nabla_a, \nabla_b] \, {}^*f \cdot f = 2|\not{\mathcal{D}}_2^* f|^2 + \nabla_a\Big(\nabla^a f \cdot f + 2(\not{\mathcal{D}}_2^* f)^{ab} f_b\Big). \tag{2.1.24}$$

Proof. The above relations are obtained by multiplying relations (2.1.21) by f and integrating by parts in the horizontal directions. □

Remark 2.1.38. *In the integrable case the commutator $\in^{ab} [\nabla_a, \nabla_b]$ is given by the standard Gauss formula in terms of K. In the non-integrable case it can be computed by using the generalized Gauss equation, see Proposition 2.1.41.*

Observe that in the relations obtained in Proposition 2.1.37, the divergence terms cannot be discarded upon integration because of the absence of an integrable surface. There are various ways to deal with this difficulty, such as to integrate (2.1.22)-(2.1.24) on the entire spacetime manifold $\mathcal{M}$.

Remark 2.1.39. *Note that the divergence terms in Proposition 2.1.37 can be re-expressed in terms of spacetime divergences based on the following lemma.*

Lemma 2.1.40. *For $f \in \mathfrak{s}_1$, we have*[7]

$$\mathbf{D}^\alpha f_\alpha = \nabla^a f_a + (\eta + \underline{\eta}) \cdot f \tag{2.1.25}$$

where $\underline{\eta}_a := \frac{1}{2}\mathbf{g}(e_a, \mathbf{D}_L \underline{L})$ and $\eta_a := \frac{1}{2}\mathbf{g}(e_a, \mathbf{D}_{\underline{L}} L)$, see Definition 2.2.1.

Proof. We have, using (2.2.3),

$$\begin{aligned}\mathbf{D}^\alpha f_\alpha - \nabla^a f_a &= -\frac{1}{2}\big(\mathbf{D}_3 f_4 + \mathbf{D}_4 f_3\big) = -\frac{1}{2}\big(e_3(f_4) - f_{\mathbf{D}_3 4} + e_4(f_3) - f_{\mathbf{D}_4 3}\big)\\ &= \frac{1}{2}\big(2\eta_a f_a + 2\underline{\eta}_a f_a\big) = (\eta + \underline{\eta}) \cdot f\end{aligned}$$

as stated. □

[7] Here, we extend the horizontal 1-form f as a full 1-form on $\mathcal{M}$ by setting $f_3 = f_4 = 0$.

2.1.4 The Gauss equation

Note that in the case of a non-integrable structure, we are missing the traditional Gauss equation which connects the Gauss curvature of a sphere to a Riemann curvature component. In what follows we state a result which is its non-integrable analogue.

Proposition 2.1.41. *The following identity holds true.*

$$\begin{aligned}\nabla_a\nabla_b X_c - \nabla_b\nabla_a X_c = \mathbf{R}_{cdab}X^d + \frac{1}{2}\in_{ab}\left({}^{(a)}tr\chi\nabla_3 + {}^{(a)}tr\underline{\chi}\nabla_4\right)X_c \\ -\frac{1}{2}\Big(\chi_{ac}\underline{\chi}_{bd} + \underline{\chi}_{ac}\chi_{bd} - \chi_{bc}\underline{\chi}_{ad} - \underline{\chi}_{bc}\chi_{ad}\Big)X^d,\end{aligned} \tag{2.1.26}$$

where $\mathbf{R}_{cdab}$ denotes the Riemann curvature of $(\mathcal{M}, \mathbf{g})$.

Proof. Given the importance of the formula we give below a direct proof of it.

For $X \in \mathfrak{s}_1$ we have,

$$\begin{aligned}\mathbf{D}_b X_c &= \nabla_b X_c, \qquad \mathbf{D}_3 X_c = \nabla_3 X_c,\\ \mathbf{D}_4 X_c &= \nabla_4 X_c, \qquad \mathbf{D}_b X_3 = -\underline{\chi}_{bd}X_d, \qquad \mathbf{D}_b X_4 = -\chi_{bd}X_d.\end{aligned}$$

Also,

$$\begin{aligned}\mathbf{D}_a\mathbf{D}_b X_c &= \nabla_a\nabla_b X_c - \frac{1}{2}\chi_{ab}\mathbf{D}_3 X_c - \frac{1}{2}\underline{\chi}_{ab}\mathbf{D}_4 X_c - \frac{1}{2}\chi_{ac}\mathbf{D}_b X_3 - \frac{1}{2}\underline{\chi}_{ac}\mathbf{D}_b X_4\\ &= \nabla_a\nabla_b X_c - \frac{1}{2}\chi_{ab}\nabla_3 X_c - \frac{1}{2}\underline{\chi}_{ab}\nabla_4 X_c + \frac{1}{2}\chi_{ac}\underline{\chi}_{bd}X_d + \frac{1}{2}\underline{\chi}_{ac}\chi_{bd}X_d.\end{aligned}$$

Hence,

$$\begin{aligned}\mathbf{D}_a\mathbf{D}_b X_c &= \nabla_a\nabla_b X_c - \frac{1}{2}\chi_{ab}\nabla_3 X_c - \frac{1}{2}\underline{\chi}_{ab}\nabla_4 X_c + \frac{1}{2}\chi_{ac}\underline{\chi}_{bd}X_d + \frac{1}{2}\underline{\chi}_{ac}\chi_{bd}X_d,\\ \mathbf{D}_b\mathbf{D}_a X_c &= \nabla_b\nabla_a X_c - \frac{1}{2}\chi_{ba}\nabla_3 X_c - \frac{1}{2}\underline{\chi}_{ba}\nabla_4 X_c + \frac{1}{2}\chi_{bc}\underline{\chi}_{ad}X_d + \frac{1}{2}\underline{\chi}_{bc}\chi_{ad}X_d.\end{aligned}$$

Subtracting we derive

$$\begin{aligned}\mathbf{R}_{cdab}X^d &= \mathbf{D}_a\mathbf{D}_b X_c - \mathbf{D}_b\mathbf{D}_a X_c\\ &= \nabla_a\nabla_b X_c - \nabla_b\nabla_a X_c - \frac{1}{2}(\chi_{ab} - \chi_{ba})\nabla_3 X_c - \frac{1}{2}(\underline{\chi}_{ab} - \underline{\chi}_{ba})\nabla_4 X_c\\ &+ \frac{1}{2}\Big(\chi_{ac}\underline{\chi}_{bd} + \underline{\chi}_{ac}\chi_{bd} - \chi_{bc}\underline{\chi}_{ad} - \underline{\chi}_{bc}\chi_{ad}\Big)X^d.\end{aligned}$$

Thus,

$$\begin{aligned}\nabla_a\nabla_b X_c - \nabla_b\nabla_a X_c &= \frac{1}{2}(\chi_{ab}-\chi_{ba})\nabla_3 X_c + \frac{1}{2}(\underline{\chi}_{ab}-\underline{\chi}_{ba})\nabla_4 X_c\\ &- \frac{1}{2}\Big(\chi_{ac}\underline{\chi}_{bd} + \underline{\chi}_{ac}\chi_{bd} - \chi_{bc}\underline{\chi}_{ad} - \underline{\chi}_{bc}\chi_{ad}\Big)X^d + \mathbf{R}_{cdab}X^d.\end{aligned}$$

Since

$$\chi_{ab}-\chi_{ba} = \in_{ab}\ {}^{(a)}\mathrm{tr}\chi, \qquad \underline{\chi}_{ab}-\underline{\chi}_{ba} = \in_{ab}\ {}^{(a)}\mathrm{tr}\underline{\chi},$$

this concludes the proof of the proposition. □

Remark 2.1.42. *We note that* (2.1.26) *can be derived from Corollary 2.1.29 according to which, relative to an arbitrary frame* e_μ*,*

$$\big(\nabla_\mu\nabla_\nu - \nabla_\mu\nabla_\nu\big)X = \nabla_{[e_\mu,e_\nu]}X + \dot{\mathbf{R}}(e_\mu,e_\nu)X$$

with $\dot{\mathbf{R}} = \mathbf{R} + \frac{1}{2}\mathbf{B}$ *and* $\mathbf{B}$ *defined in* (2.1.13)*. The Gauss formula follows then easily by evaluating the components* $\mathbf{B}_{cdab}$ *of the tensor* $\mathbf{B}$ *and the term* $\nabla_{[e_a,e_b]}X$*.*

We now specialize the Gauss equation (2.1.26) to tensors.

Proposition 2.1.43. *The following identities hold true.*

1. *For a scalar* ψ*:*

$$[\nabla_a,\nabla_b]\psi = \left(\frac{1}{2}\big({}^{(a)}tr\chi\nabla_3 + {}^{(a)}tr\underline{\chi}\nabla_4\big)\psi\right)\in_{ab}. \tag{2.1.27}$$

2. *The only non-vanishing component of* $\mathbf{B}_{abcd}$ *is given by*

$$\mathbf{B}_{1212} = -\mathbf{B}_{1221} = \mathbf{B}_{2121} = -\frac{1}{2}tr\,\chi tr\underline{\chi} - \frac{1}{2}{}^{(a)}tr\chi\,{}^{(a)}tr\underline{\chi} + \widehat{\chi}\cdot\widehat{\underline{\chi}}. \tag{2.1.28}$$

3. *For* $\psi\in\mathfrak{s}_k$ *for* $k=1,2$*,*

$$[\nabla_a,\nabla_b]\psi = \left(\frac{1}{2}\big({}^{(a)}tr\chi\nabla_3 + {}^{(a)}tr\underline{\chi}\nabla_4\big)\psi + k\ {}^{(h)}K\ {}^*\psi\right)\in_{ab} \tag{2.1.29}$$

where

$${}^{(h)}K := -\frac{1}{4}tr\,\chi tr\underline{\chi} - \frac{1}{4}{}^{(a)}tr\chi\,{}^{(a)}tr\underline{\chi} + \frac{1}{2}\widehat{\chi}\cdot\widehat{\underline{\chi}} - \frac{1}{4}\mathbf{R}_{3434}. \tag{2.1.30}$$

Proof. The case of scalars can be easily checked directly.

We consider below the case $\psi \in \mathfrak{s}_2$. From Corollary 2.1.29 applied to $\psi \in \mathfrak{s}_2$, we have

$$\begin{aligned}\big(\nabla_a\nabla_b - \nabla_b\nabla_a\big)\psi_{st} &= \frac{1}{2}\in_{ab}\big({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4\big)\psi_{st} + \frac{1}{2}\mathbf{B}_{sdab}\psi_{dt} + \frac{1}{2}\mathbf{B}_{tdab}\psi_{sd}\\ &+ \mathbf{R}_{sdab}\psi_{dt} + \mathbf{R}_{tdab}\psi_{sd}\end{aligned}$$

where, by definition of $\mathbf{B}$ given in (2.1.13),

$$\mathbf{B}_{cdab} := \chi_{bc}\underline{\chi}_{ad} + \underline{\chi}_{bc}\chi_{ad} - \chi_{ac}\underline{\chi}_{bd} - \underline{\chi}_{ac}\chi_{bd}. \tag{2.1.31}$$

Note that by the symmetries of $\mathbf{B}$, all components of $\mathbf{B}_{abcd}$ vanish except for $\mathbf{B}_{1212}$. We have

$$\begin{aligned}\mathbf{B}_{1212} &= -\chi_{11}\underline{\chi}_{22} - \underline{\chi}_{11}\chi_{22} + \chi_{21}\underline{\chi}_{12} + \underline{\chi}_{21}\chi_{12}\\ &= -\left(\frac{1}{2}tr\,\chi + \widehat{\chi}_{11}\right)\left(\frac{1}{2}\mathrm{tr}\,\underline{\chi} + \widehat{\underline{\chi}}_{22}\right) - \left(\frac{1}{2}\mathrm{tr}\,\underline{\chi} + \widehat{\underline{\chi}}_{11}\right)\left(\frac{1}{2}tr\,\chi + \widehat{\chi}_{22}\right)\\ &+\left(-\frac{1}{2}\,{}^{(a)}\mathrm{tr}\chi + \widehat{\chi}_{21}\right)\left(\frac{1}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi} + \widehat{\underline{\chi}}_{12}\right) + \left(-\frac{1}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi} + \widehat{\underline{\chi}}_{21}\right)\left(\frac{1}{2}\,{}^{(a)}\mathrm{tr}\chi + \widehat{\chi}_{12}\right)\\ &= -\frac{1}{2}tr\,\chi\mathrm{tr}\,\underline{\chi} - \frac{1}{2}\,{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} - \widehat{\chi}_{11}\widehat{\underline{\chi}}_{22} - \widehat{\chi}_{22}\widehat{\underline{\chi}}_{11} + \widehat{\chi}_{21}\widehat{\underline{\chi}}_{12} + \widehat{\chi}_{12}\widehat{\underline{\chi}}_{21}\\ &= -\frac{1}{2}tr\,\chi\mathrm{tr}\,\underline{\chi} - \frac{1}{2}\,{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} + \widehat{\chi}\cdot\widehat{\underline{\chi}}.\end{aligned}$$

This implies for $\psi \in \mathfrak{s}_2$:

$$\begin{aligned}[\nabla_1, \nabla_2]\psi &= \frac{1}{2}\big({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4\big)\psi\\ &- \left(\frac{1}{2}tr\,\chi\mathrm{tr}\,\underline{\chi} + \frac{1}{2}\,{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} - \widehat{\chi}\cdot\widehat{\underline{\chi}} + \frac{1}{2}\mathbf{R}_{3434}\right)\,{}^*\psi\end{aligned}$$

as stated. The case $\psi \in \mathfrak{s}_1$ can be treated in the same manner. □

Remark 2.1.44. *The quantity* ${}^{(h)}K$ *defined by* (2.1.30) *becomes the standard Gauss curvature in the case of an integrable structure. We note also that the value of* ${}^{(h)}K$ *for the standard non-integrable structure (induced by the standard principal null directions, see Chapter 3) of Kerr is given by the formula*

$${}^{(h)}K = \frac{r^4 + a^2r^2\sin^2\theta - 4ma^2r\cos^2\theta - a^4\cos^2\theta}{|q|^6}.$$

Here is a more general version of Proposition 2.1.43.

Proposition 2.1.45. *The following identity holds true for any horizontal tensor $\psi \in \mathbf{O}_k$ and set of horizontal indices $I = i_1 \dots i_k$*

$$\begin{aligned}[\nabla_a, \nabla_b]\psi_I &= \left(\frac{1}{2}\big({}^{(a)}tr\chi\nabla_3 + {}^{(a)}tr\underline{\chi}\nabla_4\big)\psi_I\right) \in_{ab} \\ &\quad + {}^{(h)}K\Big[\big(g_{i_1a}g_{tb} - g_{i_1b}g_{ta}\big)\psi^t{}_{i_2\dots i_k} + \cdots \big(g_{i_ka}g_{tb} - g_{i_kb}g_{ta}\big)\psi_{i_1\dots}{}^t\Big]\end{aligned} \tag{2.1.32}$$

with ${}^{(h)}K$ given by (2.1.30).

Proof. The proof is a simple extension of the proof of Proposition 2.1.43, and is left to the reader. □

Remark 2.1.46. *Observe that in the case when the horizontal structure is tangent to a S-foliation, ${}^{(h)}K$ reduces to the Gauss curvature of S. In the integrable case, for $k = 1$, we can calculate directly*[8] *on any surface of integrability S with Gauss curvature K,*

$$[\nabla_a, \nabla_b]\psi_s = K\big(g_{sa}g_{tb} - g_{sb}g_{ta}\big)\psi^t = K\big(g_{sa}\psi_b - g_{sb}\psi_a\big) = K \in_{ab} {}^*\psi_s$$

which coincides with formula (2.1.29) *in this case. Also for $\psi \in \mathbf{O}_2$ (but not necessarily in $\mathfrak{s}_2$),*

$$\begin{aligned}[\nabla_a, \nabla_b]\psi_{s_1s_2} &= K\big(g_{s_1a}g_{tb} - g_{s_1b}g_{ta}\big)\psi^t{}_{s_2} + K\big(g_{s_2a}g_{tb} - g_{s_2b}g_{ta}\big)\psi_{s_1}{}^t \\ &= K\big(g_{s_1a}\psi_{bs_2} - g_{s_1b}\psi_{as_2}\big) + K\big(g_{s_2a}\psi_{s_1b} - g_{s_2b}\psi_{s_1a}\big).\end{aligned}$$

Using (2.1.29) we can rewrite Proposition 2.1.37 as follows.

Proposition 2.1.47. *Given a not necessarily integrable horizontal structure, the following pointwise relations hold*[9]*:*

i. *The following identity holds for $f \in \mathfrak{s}_1$:*

$$\begin{aligned}|\nabla f|^2 + {}^{(h)}K|f|^2 &= |\,\not{\mathcal{D}}_1 f|^2 + \frac{1}{2}\left(\Big({}^{(a)}tr\chi\nabla_3 + {}^{(a)}tr\underline{\chi}\nabla_4\Big)\,{}^*f\right)\cdot f + div\,[\,\not{\mathcal{D}}_1 f], \\ div\,[\,\not{\mathcal{D}}_1 f] &:= \nabla_a\Big(\nabla^a f\cdot f - (div\, f)f^a - (curl\, f)(\,{}^*f)^a\Big).\end{aligned} \tag{2.1.33}$$

ii. *The following identity holds for $f \in \mathfrak{s}_2$:*

$$\begin{aligned}|\nabla f|^2 + 2\,{}^{(h)}K|f|^2 &= 2|\,\not{\mathcal{D}}_2 f|^2 + \frac{1}{2}\left(\Big({}^{(a)}tr\chi\nabla_3 + {}^{(a)}tr\underline{\chi}\nabla_4\Big)\,{}^*f\right)\cdot f + div\,[\,\not{\mathcal{D}}_2 f], \\ div\,[\,\not{\mathcal{D}}_2 f] &:= \nabla_a\big(\nabla^a f\cdot f - 2(div\, f)_b f^{ab}\big).\end{aligned} \tag{2.1.34}$$

[8] One can check directly that $g_{sa}\psi_b - g_{sb}\psi_a = \in_{ab} {}^*\psi_s$.

[9] Note that according to Lemma 2.1.40, the divergence terms in the proposition can be re-expressed in terms of the spacetime divergences, see Remark 2.1.39.

iii. *The following identity holds for $f \in \mathfrak{s}_1$:*

$$
\begin{aligned}
|\nabla f|^2 - {}^{(h)}K|f|^2 &= 2|\,{\not{\mathcal{D}}}_2^* f|^2 - \frac{1}{2}\Big(\Big({}^{(a)}tr\chi\nabla_3 + {}^{(a)}tr\underline{\chi}\nabla_4\Big)\, {}^*f\Big)\cdot f + div\,[{\not{\mathcal{D}}}_2^* f],\\
div\,[{\not{\mathcal{D}}}_2^* f] &:= \nabla_a\Big(\nabla^a f\cdot f + 2({\not{\mathcal{D}}}_2^* f)^{ab} f_b\Big).
\end{aligned}
\tag{2.1.35}
$$

Proof. From (2.1.29), we have for $f \in \mathfrak{s}_1$ and $u \in \mathfrak{s}_2$:

$$
\begin{aligned}
\frac{1}{2}\in_{ab}[\nabla_a, \nabla_b]\, {}^*f &= \frac{1}{2}({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4)\, {}^*f - {}^{(h)}Kf,\\
\frac{1}{2}\in_{ab}[\nabla_a, \nabla_b]\, {}^*u &= \frac{1}{2}({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4)\, {}^*u - 2\,{}^{(h)}Ku,
\end{aligned}
$$

from which we obtain the stated identities. □

2.1.5 Bochner identities for the horizontal Laplacian

Proposition 2.1.48. *The following identities hold true.*

1. *Given a scalar function ψ we have*

$$
\big|\triangle\psi|^2 = \big|\nabla^2\psi\big|^2 + {}^{(h)}K|\nabla\psi|^2 + Err_0[\triangle\psi] + div\,_0[\triangle\psi],
$$

with

$$
\begin{aligned}
Err_0[\triangle\psi] &:= -\frac{1}{2}\nabla\psi\cdot\big({}^{(a)}tr\chi\nabla_3 + {}^{(a)}tr\underline{\chi}\nabla_4\big)\, {}^*\nabla\psi,\\
div\,_0[\triangle\psi] &:= \nabla_a\Big(\nabla^a\psi\cdot\triangle\psi - \frac{1}{2}\nabla^a|\nabla\psi|^2\Big).
\end{aligned}
$$

2. *For $\psi \in \mathfrak{s}_1$ we have*

$$
\big|\triangle\psi|^2 = |\nabla^2\psi|^2 + {}^{(h)}K\big(|\nabla\psi|^2 - 2\,{}^{(h)}K|\psi|^2\big) + Err_1[\triangle\psi] + div\,_1[\triangle\psi],
$$

with

$$
\begin{aligned}
Err_1[\triangle\psi] &:= -\frac{1}{2}\nabla\psi\cdot\big({}^{(a)}tr\chi\nabla_3 + {}^{(a)}tr\underline{\chi}\nabla_4\big)\, {}^*\nabla\psi + \frac{1}{2}\Big|\big({}^{(a)}tr\chi\nabla_3 + {}^{(a)}tr\underline{\chi}\nabla_4\big)\psi\Big|^2\\
&\quad -\frac{3}{2}\,{}^{(h)}K\big({}^{(a)}tr\chi\nabla_3 + {}^{(a)}tr\underline{\chi}\nabla_4\big)\psi\cdot\, {}^*\psi + {}^{(h)}K\,div\,[{\not{\mathcal{D}}}_1\psi],\\
div\,_1[\triangle\psi] &:= \nabla_a\Big(\nabla^a\psi\cdot\triangle\psi - \nabla_c\psi\cdot\nabla^c\nabla^a\psi\Big).
\end{aligned}
$$

3. *For $\psi \in \mathfrak{s}_2$ we have*

$$\big|\triangle\psi|^2 = |\nabla^2\psi|^2 + {}^{(h)}K\Big(|\nabla\psi|^2 - 6\,{}^{(h)}K|\psi|^2\Big) + Err_2[\triangle\psi] + div\,_2[\triangle\psi], \tag{2.1.36}$$

where

$$\begin{aligned} Err_2[\triangle\psi] &:= -\frac{1}{2}\nabla\psi\cdot\big({}^{(a)}tr\chi\nabla_3 + {}^{(a)}tr\underline{\chi}\nabla_4\big)\,{}^*\nabla\psi + \frac{1}{2}\Big|\big({}^{(a)}tr\chi\nabla_3 + {}^{(a)}tr\underline{\chi}\nabla_4\big)\psi\Big|^2 \\ &\quad -3\,{}^{(h)}K\big({}^{(a)}tr\chi\nabla_3 + {}^{(a)}tr\underline{\chi}\nabla_4\big)\psi\cdot{}^*\psi + 2\,{}^{(h)}K\,div\,[\not{\mathcal{P}}_2\psi], \\ div\,_2[\triangle\psi] &:= \nabla_a\Big(\nabla^a\psi\cdot\triangle\psi - \nabla_c\psi\cdot\nabla^c\nabla^a\psi\Big). \end{aligned}$$

Proof. See Appendix A.4. □

Remark 2.1.49. *In the integrable case, the horizontal structure is tangent to spheres S, and we derive the following by integration:*

1. *Given a scalar function ψ we have*

$$\int_S|\triangle\psi|^2 = \int_S\big|\nabla^2\psi\big|^2 + \int_S K|\nabla\psi|^2. \tag{2.1.37}$$

2. *For $\psi \in \mathfrak{s}_1$ we have*

$$\int_S|\triangle\psi|^2 = \int_S|\nabla^2\psi|^2 + K\big(|\nabla\psi|^2 - 2K|\psi|^2\big) + \int_S K\,div\,[\not{\mathcal{P}}_1\psi] \tag{2.1.38}$$

with

$$div\,[\not{\mathcal{P}}_1\psi] = \nabla_a\Big(\nabla^a\psi\cdot\psi - \big(div\,\psi\psi^a - curl\,\psi\,{}^*\psi^a\big)\Big).$$

3. *For $\psi \in \mathfrak{s}_2$ we have*

$$\int_S|\triangle\psi|^2 = \int_S|\nabla^2\psi|^2 + \int_S K\Big(|\nabla\psi|^2 - 6K|\psi|^2\Big) + \int_S 2K\,div\,[\not{\mathcal{P}}_2\psi] \tag{2.1.39}$$

with

$$div\,[\not{\mathcal{P}}_2\psi] = \nabla_a\Big(\nabla^a\psi\cdot\psi - 2(div\,\psi)_b\psi^{ab}\Big).$$

2.2 Horizontal structures and Einstein equations

We apply the general formalism for non-integrable structures to the case of a spacetime solution to the Einstein vacuum equation. For an application of the formalism to the non-vacuum case, such as the Einstein-Maxwell equation, see [34] [35].

2.2.1 Ricci coefficients

Definition 2.2.1. *We define the following horizontal* 1*-forms*

$$\begin{aligned}\underline{\eta}(X) &:= \frac{1}{2}\mathbf{g}(X, \mathbf{D}_L \underline{L}), \qquad \eta(X) := \frac{1}{2}\mathbf{g}(X, \mathbf{D}_{\underline{L}} L),\\ \underline{\xi}(X) &:= \frac{1}{2}\mathbf{g}(X, \mathbf{D}_{\underline{L}} \underline{L}), \qquad \xi(X) := \frac{1}{2}\mathbf{g}(X, \mathbf{D}_L L).\end{aligned}$$

With these definitions we have

$$\begin{aligned}\nabla_L X &:= {}^{(h)}(\mathbf{D}_L X) = \mathbf{D}_L X - \underline{\eta}(X) L - \xi(X)\underline{L},\\ \nabla_{\underline{L}} X &:= {}^{(h)}(\mathbf{D}_{\underline{L}} X) = \mathbf{D}_{\underline{L}} X - \underline{\xi}(X) L - \eta(X)\underline{L}.\end{aligned}$$

In addition to the horizontal tensor-fields $\chi, \underline{\chi}, \underline{\eta}, \eta, \xi, \underline{\xi}$ introduced above, we also define the scalars

$$\underline{\omega} := \frac{1}{4}\mathbf{g}(\mathbf{D}_{\underline{L}} \underline{L}, L), \qquad \omega := \frac{1}{4}\mathbf{g}(\mathbf{D}_L L, \underline{L}),$$

and the horizontal 1-form

$$\zeta(X) = \frac{1}{2}\mathbf{g}(\mathbf{D}_X L, \underline{L}).$$

We summarize below the definition of the the horizontal 1-forms $\xi, \underline{\xi}, \eta, \underline{\eta}, \zeta \in \mathbf{O}_1$:

$$\begin{cases}\xi(X) = \frac{1}{2}\mathbf{g}(\mathbf{D}_L L, X), \quad \underline{\xi}(X) = \frac{1}{2}\mathbf{g}(\mathbf{D}_{\underline{L}} \underline{L}, X),\\ \eta(X) = \frac{1}{2}\mathbf{g}(\mathbf{D}_{\underline{L}} L, X), \quad \underline{\eta}(X) = \frac{1}{2}\mathbf{g}(\mathbf{D}_L \underline{L}, X),\\ \zeta(X) = \frac{1}{2}\mathbf{g}(\mathbf{D}_X L, \underline{L}),\end{cases} \tag{2.2.1}$$

and the real scalars

$$\omega = \frac{1}{4}\mathbf{g}(\mathbf{D}_L L, \underline{L}), \qquad \underline{\omega} = \frac{1}{4}\mathbf{g}(\mathbf{D}_{\underline{L}} \underline{L}, L). \tag{2.2.2}$$

Definition 2.2.2. *The horizontal tensor-fields $\chi, \underline{\chi}, \eta, \underline{\eta}, \zeta, \xi, \underline{\xi}, \omega, \underline{\omega}$ are called the connection coefficients of the null pair $(L, \underline{L})$. Given an arbitrary basis of horizontal vectorfields e_1, e_2, we write using the short hand notation $\mathbf{D}_a = \mathbf{D}_{e_a}$, $a = 1, 2$,*

$$\begin{aligned}\underline{\chi}_{ab} &= \mathbf{g}(\mathbf{D}_a \underline{L}, e_b), \qquad \chi_{ab} = \mathbf{g}(\mathbf{D}_a L, e_b),\\ \underline{\xi}_a &= \frac{1}{2}\mathbf{g}(\mathbf{D}_{\underline{L}} \underline{L}, e_a), \qquad \xi_a = \frac{1}{2}\mathbf{g}(\mathbf{D}_L L, e_a),\\ \underline{\omega} &= \frac{1}{4}\mathbf{g}(\mathbf{D}_{\underline{L}} \underline{L}, L), \qquad \omega = \frac{1}{4}\mathbf{g}(\mathbf{D}_L L, \underline{L}),\\ \underline{\eta}_a &= \frac{1}{2}\mathbf{g}(\mathbf{D}_L \underline{L}, e_a), \qquad \eta_a = \frac{1}{2}\mathbf{g}(\mathbf{D}_{\underline{L}} L, e_a),\\ \zeta_a &= \frac{1}{2}\mathbf{g}(\mathbf{D}_a L, \underline{L}).\end{aligned}$$

We easily derive the Ricci formulae,

$$
\begin{aligned}
\mathbf{D}_a e_b &= \nabla_a e_b + \frac{1}{2}\chi_{ab} e_3 + \frac{1}{2}\underline{\chi}_{ab} e_4, \\
\mathbf{D}_a e_4 &= \chi_{ab} e_b - \zeta_a e_4, \\
\mathbf{D}_a e_3 &= \underline{\chi}_{ab} e_b + \zeta_a e_3, \\
\mathbf{D}_3 e_a &= \nabla_3 e_a + \eta_a e_3 + \underline{\xi}_a e_4, \\
\mathbf{D}_3 e_3 &= -2\underline{\omega} e_3 + 2\underline{\xi}_b e_b, \\
\mathbf{D}_3 e_4 &= 2\underline{\omega} e_4 + 2\eta_b e_b, \\
\mathbf{D}_4 e_a &= \nabla_4 e_a + \underline{\eta}_a e_4 + \xi_a e_3, \\
\mathbf{D}_4 e_4 &= -2\omega e_4 + 2\xi_b e_b, \\
\mathbf{D}_4 e_3 &= 2\omega e_3 + 2\underline{\eta}_b e_b.
\end{aligned}
\tag{2.2.3}
$$

2.2.2 Curvature and Weyl fields

Assume that $W \in \mathbf{T}_4^0(\mathcal{M})$ is a Weyl field, i.e.

$$
\begin{cases}
W_{\alpha\beta\mu\nu} = -W_{\beta\alpha\mu\nu} = -W_{\alpha\beta\nu\mu} = W_{\mu\nu\alpha\beta}, \\
W_{\alpha\beta\mu\nu} + W_{\alpha\mu\nu\beta} + W_{\alpha\nu\beta\mu} = 0, \\
\mathbf{g}^{\beta\nu} W_{\alpha\beta\mu\nu} = 0.
\end{cases}
\tag{2.2.4}
$$

We define the null components of the Weyl field W, $\alpha(W), \underline{\alpha}(W), \varrho(W) \in \mathbf{O}_2(\mathcal{M})$ and $\beta(W), \underline{\beta}(W) \in \mathbf{O}_1(\mathcal{M})$ by the formulas

$$
\begin{cases}
\alpha(W)(X,Y) = W(L,X,L,Y), \\
\underline{\alpha}(W)(X,Y) = W(\underline{L},X,\underline{L},Y), \\
\beta(W)(X) = \frac{1}{2}W(X,L,\underline{L},L), \\
\underline{\beta}(W)(X) = \frac{1}{2}W(X,\underline{L},\underline{L},L), \\
\varrho(W)(X,Y) = W(X,\underline{L},Y,L).
\end{cases}
\tag{2.2.5}
$$

Recall that if W is a Weyl field its Hodge dual ${}^*W$, defined by ${}^*W_{\alpha\beta\mu\nu} = \frac{1}{2}\in_{\mu\nu}{}^{\rho\sigma} W_{\alpha\beta\rho\sigma}$, is also a Weyl field. We easily check the formulas,

$$
\begin{cases}
\underline{\alpha}({}^*W) = {}^*\underline{\alpha}(W), \qquad \alpha({}^*W) = -{}^*\alpha(W), \\
\underline{\beta}({}^*W) = {}^*\underline{\beta}(W), \qquad \beta({}^*W) = -{}^*\beta(W), \\
\varrho({}^*W) = {}^*\varrho(W).
\end{cases}
\tag{2.2.6}
$$

It is easy to check that $\alpha, \underline{\alpha}$ are symmetric traceless horizontal tensor-fields. On the other hand the horizontal 2-tensorfield ϱ is neither symmetric nor traceless. It is convenient to express it in terms of the following two scalar quantities

$$\rho(W) = \frac{1}{4} W(L, \underline{L}, L, \underline{L}), \qquad {}^*\rho(W) = \frac{1}{4}\, {}^*W(L, \underline{L}, L, \underline{L}). \tag{2.2.7}$$

Observe also that,

$$\rho({}^*W) = {}^*\rho(W), \qquad {}^*\rho({}^*W) = -\rho.$$

Thus,

$$\varrho(X,Y) = \big(-\rho\,\gamma(X,Y) + {}^*\rho \in (X,Y)\big), \qquad \forall\, X,Y \in \mathbf{O}(\mathcal{M}). \tag{2.2.8}$$

We have

$$\begin{aligned}
W_{a3b4} &= \varrho_{ab} = (-\rho\delta_{ab} + {}^*\rho \in_{ab}),\\
W_{ab34} &= 2 \in_{ab} {}^*\rho,\\
W_{abcd} &= - \in_{ab}\in_{cd} \rho,\\
W_{abc3} &= \in_{ab} {}^*\underline{\beta}_c,\\
W_{abc4} &= - \in_{ab} {}^*\beta_c.
\end{aligned}$$

Remark 2.2.3. *In addition to the Hodge duality we will need to take into account the duality with respect to the interchange of L, $\underline{L}$, which we call a pairing transformation. Clearly, under this transformation, $\alpha \leftrightarrow \underline{\alpha}$, $\beta \leftrightarrow -\underline{\beta}$, $\rho \leftrightarrow \rho$, ${}^*\rho \leftrightarrow -{}^*\rho$, $\varrho \leftrightarrow \check{\varrho}$ with $\check{\varrho}_{ab} := \varrho_{ba}$. One has to be careful however when combining the Hodge dual and pairing transformations. In that case we have, ${}^*\underline{\alpha} \leftrightarrow -{}^*\alpha$, ${}^*\underline{\beta} \leftrightarrow {}^*\beta$. This is due to the fact that under the pairing transformation $\in_{ab} \to -\in_{ab}$ (since $\in_{ab}=\in_{ab34}$). Indeed, for example,*

$$\begin{aligned}
{}^*\underline{\alpha}_{ab} &= \underline{\alpha}({}^*W)_{ab} = {}^*W_{a3b3} = -\in_{a3c4} W_{c3b3} = \in_{ac34} W_{c3b3} = \in_{ac} \underline{\alpha}_{cb},\\
{}^*\alpha_{ab} &= \alpha({}^*W)_{ab} = {}^*W_{a4b4} = -\in_{a4c3} W_{c4b4} = -\in_{cb34} W_{c4b4} = -\in_{ac} \alpha_{cb}.
\end{aligned}$$

The decomposition above for Weyl fields applies in particular to the Riemann curvature tensor $\mathbf{R}$ of a vacuum spacetime.

In the case of a vacuum spacetime, the non-integrable Gauss curvature defined by (2.1.30) becomes

$$ {}^{(h)}K = -\frac{1}{4}\mathrm{tr}\,\chi\,\mathrm{tr}\,\underline{\chi} - \frac{1}{4}\,{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} + \frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}} - \rho. \tag{2.2.9}$$

2.2.3 Horizontal tensor B

We calculate below the components of the horizontal curvature tensor **B** defined by the formula, see (2.1.13),

$$\mathbf{B}_{ab\mu\nu} := (\Lambda_\mu)_{3a}(\Lambda_\nu)_{b4} + (\Lambda_\mu)_{4a}(\Lambda_\nu)_{b3} - (\Lambda_\nu)_{3a}(\Lambda_\mu)_{b4} - (\Lambda_\nu)_{4a}(\Lambda_\mu)_{b3}.$$

Proposition 2.2.4. *The components of* **B** *are given by the following formulas:*

$$\begin{aligned}\mathbf{B}_{abc3} &= 2\big(-\underline{\chi}_{ca}\eta_b + \underline{\chi}_{cb}\eta_a - \chi_{ca}\underline{\xi}_b + \chi_{cb}\underline{\xi}_a\big),\\ \mathbf{B}_{abc4} &= 2\big(-\chi_{ca}\underline{\eta}_b + \chi_{cb}\underline{\eta}_a - \underline{\chi}_{ca}\xi_b + \underline{\chi}_{cb}\xi_a\big),\\ \mathbf{B}_{ab34} &= 4\big(-\underline{\xi}_a\xi_b + \xi_a\underline{\xi}_b - \eta_a\underline{\eta}_b + \underline{\eta}_a\eta_b\big),\\ \mathbf{B}_{abcd} &= \chi_{bc}\underline{\chi}_{ad} + \underline{\chi}_{bc}\chi_{ad} - \chi_{ac}\underline{\chi}_{bd} - \underline{\chi}_{ac}\chi_{bd}.\end{aligned} \tag{2.2.10}$$

The above can also be written as

$$\begin{aligned}\mathbf{B}_{abc3} &= -tr\,\underline{\chi}\big(\delta_{ca}\eta_b - \delta_{cb}\eta_a\big) - {}^{(a)}tr\underline{\chi}\big(\in_{ca}\eta_b - \in_{cb}\eta_a\big)\\ &\quad + 2\big(-\widehat{\underline{\chi}}_{ca}\eta_b + \widehat{\underline{\chi}}_{cb}\eta_a - \chi_{ca}\underline{\xi}_b + \chi_{cb}\underline{\xi}_a\big),\\ \mathbf{B}_{abc4} &= -tr\,\chi\big(\delta_{ca}\underline{\eta}_b - \delta_{cb}\underline{\eta}_a\big) - {}^{(a)}tr\chi\big(\in_{ca}\underline{\eta}_b - \in_{cb}\underline{\eta}_a\big)\\ &\quad + 2\big(-\widehat{\chi}_{ca}\underline{\eta}_b + \widehat{\chi}_{cb}\underline{\eta}_a - \underline{\chi}_{ca}\xi_b + \underline{\chi}_{cb}\xi_a\big).\end{aligned} \tag{2.2.11}$$

Also, $\mathbf{B}_{abcd}$ *is given by*

$$\mathbf{B}_{abcd} = \left(-\frac{1}{2}tr\,\chi tr\underline{\chi} - \frac{1}{2}{}^{(a)}tr\chi\,{}^{(a)}tr\underline{\chi} + \widehat{\chi}\cdot\widehat{\underline{\chi}}\right)\in_{ab}\in_{cd}.$$

Proof. We write recalling the definition $(\Lambda_\mu)_{\alpha\beta} = \mathbf{g}(\mathbf{D}_\mu e_\beta, e_\alpha)$ and definition of Ricci coefficients, see Definition 2.2.2,

$$\begin{aligned}\mathbf{B}_{abc3} &= (\Lambda_c)_{3a}(\Lambda_3)_{b4} + (\Lambda_c)_{4a}(\Lambda_3)_{b3} - (\Lambda_3)_{3a}(\Lambda_c)_{b4} - (\Lambda_3)_{4a}(\Lambda_c)_{b3}\\ &= -2\underline{\chi}_{ca}\eta_b - 2\chi_{ca}\underline{\xi}_b + 2\underline{\xi}_a\chi_{cb} + 2\eta_a\underline{\chi}_{cb}\end{aligned}$$

and

$$\begin{aligned}\mathbf{B}_{ab34} &= (\Lambda_3)_{3a}(\Lambda_4)_{b4} + (\Lambda_3)_{4a}(\Lambda_4)_{b3} - (\Lambda_4)_{3a}(\Lambda_3)_{b4} - (\Lambda_4)_{4a}(\Lambda_3)_{b3}\\ &= 4(-\underline{\xi}_a)\xi_b + 4(-\eta_a)\underline{\eta}_b - 4(-\underline{\eta}_a)\eta_b - 4(\underline{\eta}_a)\eta_b - (-\xi_a)\underline{\xi}_b\\ &= 4\big(-\underline{\xi}_a\xi_b + \xi_a\underline{\xi}_b - \eta_a\underline{\eta}_b + \underline{\eta}_a\eta_b\big).\end{aligned}$$

For the remaining formulas see (2.1.31) and (2.1.28). □

2.2.4 Connection to the Newman-Penrose formalism

In the Newman-Penrose NP formalism, one chooses a specific orthonormal basis of horizontal vectors (e_1, e_2) and defines all connection coefficients relative to the complexified frame $(n, l, m, \overline{m})$ where $n = \frac{1}{2}e_3$, $l = e_4$, $m = e_1 + ie_2$, $\overline{m} = e_1 - ie_2$. Thus, all quantities of interest are complex scalars instead of our horizontal tensors such as $\mathfrak{s}_1, \mathfrak{s}_2$. The NP formalism works well for deriving the basic equations, but has the disadvantage of substantially increasing the number of variables. Moreover, the calculations become far more cumbersome when deriving equations involving higher derivatives of the main quantities, in perturbations of Kerr. Another advantage of the formalism used here is that all important equations look similar to the ones in [23]. We refer to [62] for the original form of the NP formalism.

The formalism used here is also related to the so-called Geroch-Held-Penrose formalism GHP formalism, which also introduced derivatives with boost weights, which are the scalar equivalent of the conformal derivatives used here, see Lemma 2.2.18. Nevertheless, the GHP formalism still involves complex scalars instead of horizontal tensors[10]. We refer to [32] for the original form of the GHP formalism.

2.2.5 Null structure equations

We state below the null structure equation in the general setting discussed above. We assume given a vacuum spacetime endowed with a general null frame (e_3, e_4, e_1, e_2) relative to which we define our connection and curvature coefficients.

Proposition 2.2.5 (Null structure equations)**.** *The connection coefficients verify the following equations:*

$$\begin{aligned}
\nabla_3 tr\underline{\chi} &= -|\widehat{\underline{\chi}}|^2 - \frac{1}{2}\big(tr\underline{\chi}^2 - {}^{(a)}tr\underline{\chi}^2\big) + 2div\,\underline{\xi} - 2\underline{\omega}tr\underline{\chi} + 2\underline{\xi}\cdot(\eta + \underline{\eta} - 2\zeta),\\
\nabla_3\, {}^{(a)}tr\underline{\chi} &= -tr\underline{\chi}\, {}^{(a)}tr\underline{\chi} + 2curl\,\underline{\xi} - 2\underline{\omega}\, {}^{(a)}tr\underline{\chi} + 2\underline{\xi}\wedge(-\eta + \underline{\eta} + 2\zeta),\\
\nabla_3\widehat{\underline{\chi}} &= -tr\underline{\chi}\,\widehat{\underline{\chi}} + \nabla\widehat{\otimes}\underline{\xi} - 2\underline{\omega}\widehat{\underline{\chi}} + \underline{\xi}\widehat{\otimes}(\eta + \underline{\eta} - 2\zeta) - \underline{\alpha},
\end{aligned}$$

[10]It also introduces spin weights which partially cure the dependance of these scalars on the choice of frames.

$$
\begin{aligned}
\nabla_3 tr\,\chi &= -\widehat{\underline{\chi}}\cdot\widehat{\chi} - \frac{1}{2}tr\underline{\chi}tr\,\chi + \frac{1}{2}{}^{(a)}tr\underline{\chi}\,{}^{(a)}tr\chi + 2div\,\eta + 2\underline{\omega}tr\,\chi + 2\big(\xi\cdot\underline{\xi} + |\eta|^2\big) + 2\rho,\\
\nabla_3\,{}^{(a)}tr\chi &= -\widehat{\underline{\chi}}\wedge\widehat{\chi} - \frac{1}{2}\big({}^{(a)}tr\underline{\chi}tr\,\chi + tr\,\underline{\chi}\,{}^{(a)}tr\chi\big) + 2curl\,\eta + 2\underline{\omega}\,{}^{(a)}tr\chi + 2\underline{\xi}\wedge\xi - 2\,{}^*\rho,\\
\nabla_3\widehat{\chi} &= -\frac{1}{2}\big(tr\,\chi\widehat{\underline{\chi}} + tr\underline{\chi}\widehat{\chi}\big) - \frac{1}{2}\big(-\,{}^*\widehat{\underline{\chi}}\,{}^{(a)}tr\chi + {}^*\widehat{\chi}\,{}^{(a)}tr\underline{\chi}\big) + \nabla\widehat{\otimes}\eta + 2\underline{\omega}\widehat{\chi}\\
&+ \underline{\xi}\widehat{\otimes}\xi + \eta\widehat{\otimes}\eta,
\end{aligned}
$$

$$
\begin{aligned}
\nabla_4 tr\,\underline{\chi} &= -\widehat{\chi}\cdot\widehat{\underline{\chi}} - \frac{1}{2}tr\,\chi tr\,\underline{\chi} + \frac{1}{2}{}^{(a)}tr\chi\,{}^{(a)}tr\underline{\chi} + 2div\,\underline{\eta} + 2\omega tr\,\underline{\chi} + 2\big(\xi\cdot\underline{\xi} + |\underline{\eta}|^2\big) + 2\rho,\\
\nabla_4\,{}^{(a)}tr\underline{\chi} &= -\widehat{\chi}\wedge\widehat{\underline{\chi}} - \frac{1}{2}\big({}^{(a)}tr\chi tr\,\underline{\chi} + tr\,\chi\,{}^{(a)}tr\underline{\chi}\big) + 2curl\,\underline{\eta} + 2\omega\,{}^{(a)}tr\underline{\chi} + 2\xi\wedge\underline{\xi} + 2\,{}^*\rho,\\
\nabla_4\widehat{\underline{\chi}} &= -\frac{1}{2}\big(tr\underline{\chi}\widehat{\chi} + tr\,\chi\widehat{\underline{\chi}}\big) - \frac{1}{2}\big(-\,{}^*\widehat{\chi}\,{}^{(a)}tr\underline{\chi} + {}^*\widehat{\underline{\chi}}\,{}^{(a)}tr\chi\big) + \nabla\widehat{\otimes}\underline{\eta} + 2\omega\widehat{\underline{\chi}}\\
&+ \xi\widehat{\otimes}\underline{\xi} + \underline{\eta}\widehat{\otimes}\underline{\eta},
\end{aligned}
$$

$$
\begin{aligned}
\nabla_4 tr\,\chi &= -|\widehat{\chi}|^2 - \frac{1}{2}\big(tr\,\chi^2 - {}^{(a)}tr\chi^2\big) + 2div\,\xi - 2\omega tr\,\chi + 2\xi\cdot(\underline{\eta} + \eta + 2\zeta),\\
\nabla_4\,{}^{(a)}tr\chi &= -tr\,\chi\,{}^{(a)}tr\chi + 2curl\,\xi - 2\omega\,{}^{(a)}tr\chi + 2\xi\wedge(-\underline{\eta} + \eta - 2\zeta),\\
\nabla_4\widehat{\chi} &= -tr\,\chi\,\widehat{\chi} + \nabla\widehat{\otimes}\xi - 2\omega\widehat{\chi} + \xi\widehat{\otimes}(\underline{\eta} + \eta + 2\zeta) - \alpha.
\end{aligned}
$$

Also,

$$
\begin{aligned}
\nabla_3\zeta + 2\nabla\underline{\omega} &= -\widehat{\underline{\chi}}\cdot(\zeta+\eta) - \frac{1}{2}tr\underline{\chi}(\zeta+\eta) - \frac{1}{2}{}^{(a)}tr\underline{\chi}({}^*\zeta + {}^*\eta) + 2\underline{\omega}(\zeta-\eta)\\
&\quad+\widehat{\chi}\cdot\underline{\xi} + \frac{1}{2}tr\,\chi\,\underline{\xi} + \frac{1}{2}{}^{(a)}tr\chi\,{}^*\underline{\xi} + 2\omega\underline{\xi} - \underline{\beta},\\
\nabla_4\zeta - 2\nabla\omega &= \widehat{\chi}\cdot(-\zeta+\underline{\eta}) + \frac{1}{2}tr\,\chi(-\zeta+\underline{\eta}) + \frac{1}{2}{}^{(a)}tr\chi(-\,{}^*\zeta + {}^*\underline{\eta}) + 2\omega(\zeta+\underline{\eta})\\
&\quad-\widehat{\underline{\chi}}\cdot\xi - \frac{1}{2}tr\underline{\chi}\,\xi - \frac{1}{2}{}^{(a)}tr\underline{\chi}\,{}^*\xi - 2\underline{\omega}\xi - \beta,\\
\nabla_3\underline{\eta} - \nabla_4\underline{\xi} &= -\widehat{\underline{\chi}}\cdot(\underline{\eta}-\eta) - \frac{1}{2}tr\underline{\chi}(\underline{\eta}-\eta) + \frac{1}{2}{}^{(a)}tr\underline{\chi}({}^*\underline{\eta} - {}^*\eta) - 4\omega\underline{\xi} + \underline{\beta},\\
\nabla_4\eta - \nabla_3\xi &= -\widehat{\chi}\cdot(\eta-\underline{\eta}) - \frac{1}{2}tr\,\chi(\eta-\underline{\eta}) + \frac{1}{2}{}^{(a)}tr\chi({}^*\eta - {}^*\underline{\eta}) - 4\underline{\omega}\xi - \beta,
\end{aligned}
$$

and

$$
\nabla_3\omega + \nabla_4\underline{\omega} - 4\omega\underline{\omega} - \xi\cdot\underline{\xi} - (\eta-\underline{\eta})\cdot\zeta + \eta\cdot\underline{\eta} \;=\; \rho.
$$

Also,

$$
\begin{aligned}
div\,\widehat{\chi}+\zeta\cdot\widehat{\chi} &= \frac{1}{2}\nabla tr\,\chi+\frac{1}{2}tr\,\chi\zeta-\frac{1}{2}\,{}^*\nabla\,{}^{(a)}tr\chi-\frac{1}{2}\,{}^{(a)}tr\chi\,{}^*\zeta-\,{}^{(a)}tr\chi\,{}^*\eta-\,{}^{(a)}tr\underline{\chi}\,{}^*\xi-\beta,\\
div\,\underline{\widehat{\chi}}-\zeta\cdot\underline{\widehat{\chi}} &= \frac{1}{2}\nabla tr\,\underline{\chi}-\frac{1}{2}tr\,\underline{\chi}\zeta-\frac{1}{2}\,{}^*\nabla\,{}^{(a)}tr\underline{\chi}+\frac{1}{2}\,{}^{(a)}tr\underline{\chi}\,{}^*\zeta-\,{}^{(a)}tr\underline{\chi}\,{}^*\underline{\eta}-\,{}^{(a)}tr\chi\,{}^*\underline{\xi}+\underline{\beta},
\end{aligned}
$$

and[11]

$$
curl\,\zeta = -\frac{1}{2}\widehat{\chi}\wedge\underline{\widehat{\chi}}+\frac{1}{4}\big(tr\,\chi\,{}^{(a)}tr\underline{\chi}-tr\,\underline{\chi}\,{}^{(a)}tr\chi\big)+\omega\,{}^{(a)}tr\underline{\chi}-\underline{\omega}\,{}^{(a)}tr\chi+\,{}^*\rho.
$$

Proof. Except for the fact that the order of indices in $\chi, \underline{\chi}$ is important, since they are no longer symmetric, the derivation is exactly as in section 7.4 of [23]. □

2.2.6 Null Bianchi identities

We state below the equations verified by the null curvature components of an Einstein vacuum space-time.

Proposition 2.2.6 (Null Bianchi identities)**.** *The curvature components verify the following equations:*

$$
\begin{aligned}
\nabla_3\alpha-\nabla\widehat{\otimes}\beta &= -\frac{1}{2}\big(tr\underline{\chi}\alpha+\,{}^{(a)}tr\underline{\chi}\,{}^*\alpha\big)+4\underline{\omega}\alpha+(\zeta+4\eta)\widehat{\otimes}\beta-3(\rho\widehat{\chi}+\,{}^*\rho\,{}^*\widehat{\chi}),\\
\nabla_4\beta-div\,\alpha &= -2(tr\,\chi\beta-\,{}^{(a)}tr\chi\,{}^*\beta)-2\omega\beta+\alpha\cdot(2\zeta+\underline{\eta})+3(\xi\rho+\,{}^*\xi\,{}^*\rho),\\
\nabla_3\beta+div\,\varrho &= -(tr\underline{\chi}\beta+\,{}^{(a)}tr\underline{\chi}\,{}^*\beta)+2\underline{\omega}\,\beta+2\underline{\beta}\cdot\widehat{\chi}+3(\rho\eta+\,{}^*\rho\,{}^*\eta)+\alpha\cdot\underline{\xi},\\
\nabla_4\rho-div\,\beta &= -\frac{3}{2}(tr\,\chi\rho+\,{}^{(a)}tr\chi\,{}^*\rho)+(2\underline{\eta}+\zeta)\cdot\beta-2\xi\cdot\underline{\beta}-\frac{1}{2}\underline{\widehat{\chi}}\cdot\alpha,\\
\nabla_4\,{}^*\rho+curl\,\beta &= -\frac{3}{2}(tr\,\chi\,{}^*\rho-\,{}^{(a)}tr\chi\rho)-(2\underline{\eta}+\zeta)\cdot\,{}^*\beta-2\xi\cdot\,{}^*\underline{\beta}+\frac{1}{2}\underline{\widehat{\chi}}\cdot\,{}^*\alpha,\\
\nabla_3\rho+div\,\underline{\beta} &= -\frac{3}{2}(tr\underline{\chi}\rho-\,{}^{(a)}tr\underline{\chi}\,{}^*\rho)-(2\eta-\zeta)\cdot\underline{\beta}+2\underline{\xi}\cdot\beta-\frac{1}{2}\widehat{\chi}\cdot\underline{\alpha},\\
\nabla_3\,{}^*\rho+curl\,\underline{\beta} &= -\frac{3}{2}(tr\underline{\chi}\,{}^*\rho+\,{}^{(a)}tr\underline{\chi}\rho)-(2\eta-\zeta)\cdot\,{}^*\underline{\beta}-2\underline{\xi}\cdot\,{}^*\beta-\frac{1}{2}\widehat{\chi}\cdot\,{}^*\underline{\alpha},\\
\nabla_4\underline{\beta}-div\,\check{\varrho} &= -(tr\,\chi\underline{\beta}+\,{}^{(a)}tr\chi\,{}^*\underline{\beta})+2\omega\,\underline{\beta}+2\beta\cdot\underline{\widehat{\chi}}-3(\rho\underline{\eta}-\,{}^*\rho\,{}^*\underline{\eta})-\underline{\alpha}\cdot\xi,\\
\nabla_3\underline{\beta}+div\,\underline{\alpha} &= -2(tr\,\underline{\chi}\,\underline{\beta}-\,{}^{(a)}tr\underline{\chi}\,{}^*\underline{\beta})-2\underline{\omega}\underline{\beta}-\underline{\alpha}\cdot(-2\zeta+\eta)-3(\underline{\xi}\rho-\,{}^*\underline{\xi}\,{}^*\rho),\\
\nabla_4\underline{\alpha}+\nabla\widehat{\otimes}\underline{\beta} &= -\frac{1}{2}\big(tr\,\chi\underline{\alpha}-\,{}^{(a)}tr\chi\,{}^*\underline{\alpha}\big)+4\omega\underline{\alpha}+(\zeta-4\underline{\eta})\widehat{\otimes}\underline{\beta}-3(\rho\underline{\widehat{\chi}}-\,{}^*\rho\,{}^*\underline{\widehat{\chi}}).
\end{aligned}
$$

[11] Note that this equation follows from expanding $\mathbf{R}_{34ab}$.

Here,

$$
\begin{aligned}
div\, \varrho &= -(\nabla\rho + {}^*\nabla\ {}^*\rho),\\
div\, \check{\varrho} &= -(\nabla\rho - {}^*\nabla\ {}^*\rho).
\end{aligned}
$$

Proof. The proof follows line by line from the derivation in section 7.3 of [23] except, once more, for keeping track of the lack of symmetry for $\chi, \underline{\chi}$. Note also that $\check{\varrho}_{ab} = \varrho_{ba}$ and that $(div\, \varrho)_b = \nabla^a \varrho_{ab}$. □

2.2.7 Commutation formulas

Lemma 2.2.7. *Let $U_A = U_{a_1 \ldots a_k}$ be a general k-horizontal tensorfield.*

1. *We have*

$$
\begin{aligned}
[\nabla_3, \nabla_b] U_A &= -\underline{\chi}_{bc} \nabla_c U_A + (\eta_b - \zeta_b)\nabla_3 U_A + \underline{\xi}_b \nabla_4 U_A\\
&+ \sum_{i=1}^k \Big(- \in_{a_i c}\ {}^*\underline{\beta}_b + \frac{1}{2}\mathbf{B}_{a_i c 3 b}\Big) U_{a_1 \ldots}{}^c{}_{\ldots a_k}.
\end{aligned}
\tag{2.2.12}
$$

2. *We have*

$$
\begin{aligned}
[\nabla_4, \nabla_b] U_A &= -\chi_{bc} \nabla_c U_a + (\underline{\eta}_b + \zeta_b)\nabla_4 U_a + \xi_b \nabla_3 U_a\\
&+ \sum_{i=1}^k \Big(\in_{a_i c}\ {}^*\beta_b + \frac{1}{2}\mathbf{B}_{a_i c 4 b}\Big) U_{a_1 \ldots}{}^c{}_{\ldots a_k}.
\end{aligned}
\tag{2.2.13}
$$

3. *We have*

$$
\begin{aligned}
[\nabla_4, \nabla_3] U_A &= 2(\underline{\eta}_b - \eta_b)\nabla_b U_A + 2\omega \nabla_3 U_A - 2\underline{\omega}\nabla_4 U_A\\
&+ \sum_{i=1}^k \Big(- \in_{a_i b}\ {}^*\rho + \frac{1}{2}\mathbf{B}_{a_i b 4 3}\Big) U_{a_1 \ldots}{}^b{}_{\ldots a_k}.
\end{aligned}
\tag{2.2.14}
$$

Proof. It suffices to consider the case $k = 1$. Using Proposition 2.1.27, we have

$$
\begin{aligned}
\dot{\mathbf{D}}_3 \dot{\mathbf{D}}_b U_a &= \nabla_3 \nabla_b U_a - \eta_b \nabla_3 U_a - \underline{\xi}_b \nabla_4 U_a,\\
\dot{\mathbf{D}}_b \dot{\mathbf{D}}_3 U_a &= \nabla_b \nabla_3 U_a - \underline{\chi}_{bc} \nabla_c U_a - \zeta_b \nabla_3 U_a,\\
\dot{\mathbf{D}}_3 \dot{\mathbf{D}}_b U_a - \dot{\mathbf{D}}_b \dot{\mathbf{D}}_3 U_a &= \dot{\mathbf{R}}_{ac3b} U_c = \mathbf{R}_{ac3b} U_c + \frac{1}{2}\mathbf{B}_{ac3b} U_c = - \in_{ac}\ {}^*\underline{\beta}_b U_c + \frac{1}{2}\mathbf{B}_{ac3b} U_c.
\end{aligned}
$$

Hence,

$$\begin{aligned}
[\nabla_3, \nabla_b]U_a &= \nabla_3\nabla_b U_a - \nabla_b\nabla_3 U_a \\
&= -\underline{\chi}_{bc}\nabla_c U_a + (\eta_b - \zeta_b)\nabla_3 U_a + \underline{\xi}_b\nabla_4 U_a - \in_{ac} {}^*\underline{\beta}_b U_c + \frac{1}{2}\mathbf{B}_{ac3b}U_c,
\end{aligned}$$

as stated. The commutator formula for $[\nabla_4, \nabla_b]U_a$ is derived easily by symmetry. Also,

$$\begin{aligned}
\dot{\mathbf{D}}_4\dot{\mathbf{D}}_3 U_a &= \nabla_4\nabla_3 U_a - 2\omega\nabla_3 U_a - 2\underline{\eta}^b\nabla_b U_a, \\
\dot{\mathbf{D}}_3\dot{\mathbf{D}}_4 U_a &= \nabla_3\nabla_4 U_a - 2\underline{\omega}\nabla_4 U_a - 2\eta^b\nabla_b U_a, \\
\dot{\mathbf{D}}_4\dot{\mathbf{D}}_3 U_a - \dot{\mathbf{D}}_3\dot{\mathbf{D}}_4 U_a &= \dot{\mathbf{R}}_{ab43}U_b = \mathbf{R}_{ab43}U_b + \frac{1}{2}\mathbf{B}_{ab43}U_b = -2\ {}^*\rho \in_{ab} U^b + \frac{1}{2}\mathbf{B}_{ab43}U_b.
\end{aligned}$$

Hence

$$[\nabla_4, \nabla_3]U_a = 2(\underline{\eta}_b - \eta_b)\nabla_b U_a + 2\omega\nabla_3 U_a - 2\underline{\omega}\nabla_4 U - 2\ {}^*\rho \in_{ab} U^b + \frac{1}{2}\mathbf{B}_{ab43}U_b,$$

as stated. □

Using the values of $\mathbf{B}$ given by Proposition 2.2.4, we obtain Corollary A.1.1. In the following Lemma we specialize to the case of $\mathfrak{s}_0$, $\mathfrak{s}_1$ and $\mathfrak{s}_2$.

Lemma 2.2.8. *The following commutation formulas hold true:*

1. *Given $f \in \mathfrak{s}_0$, we have*

$$\begin{aligned}
[\nabla_3, \nabla_a]f &= -\frac{1}{2}\left(tr\underline{\chi}\nabla_a f + {}^{(a)}tr\underline{\chi}\ {}^*\nabla_a f\right) + (\eta_a - \zeta_a)\nabla_3 f - \widehat{\underline{\chi}}_{ab}\nabla_b f \\
&\quad + \underline{\xi}_a\nabla_4 f, \\
[\nabla_4, \nabla_a]f &= -\frac{1}{2}\left(tr\,\chi\nabla_a f + {}^{(a)}tr\chi\ {}^*\nabla_a f\right) + (\underline{\eta}_a + \zeta_a)\nabla_4 f - \widehat{\chi}_{ab}\nabla_b f \\
&\quad + \xi_a\nabla_3 f, \\
[\nabla_4, \nabla_3]f &= 2(\underline{\eta} - \eta)\cdot\nabla f + 2\omega\nabla_3 f - 2\underline{\omega}\nabla_4 f.
\end{aligned} \tag{2.2.15}$$

2. *Given $u \in \mathfrak{s}_1$, we have*

$$\begin{aligned}
[\nabla_3, \nabla_a]u_b &= -\frac{1}{2}tr\,\underline{\chi}\big(\nabla_a u_b + \eta_b u_a - \delta_{ab}\eta\cdot u\big) \\
&\quad - \frac{1}{2}{}^{(a)}tr\underline{\chi}\big({}^*\nabla_a u_b + \eta_b\ {}^*u_a - \in_{ab}\eta\cdot u\big) \\
&\quad + (\eta - \zeta)_a\nabla_3 u_b + Err_{3ab}[u], \\
Err_{3ab}[u] &= -\ {}^*\underline{\beta}_a\ {}^*u_b + \underline{\xi}_a\nabla_4 u_b - \underline{\xi}_b\chi_{ac}u_c + \chi_{ab}\,\underline{\xi}\cdot u - \widehat{\underline{\chi}}_{ac}\nabla_c u_b - \eta_b\widehat{\underline{\chi}}_{ac}u_c \\
&\quad + \widehat{\underline{\chi}}_{ab}\eta\cdot u,
\end{aligned} \tag{2.2.16}$$

$$\begin{aligned}
[\nabla_4, \nabla_a] u_b =& -\frac{1}{2} tr\,\chi \big(\nabla_a u_b + \underline{\eta}_b u_a - \delta_{ab} \underline{\eta} \cdot u\big) \\
& - \frac{1}{2} {}^{(a)}tr\chi \big({}^*\nabla_a u_b + \underline{\eta}_b {}^*u_a - \in_{ab} \underline{\eta} \cdot u\big) + (\underline{\eta} + \zeta)_a \nabla_4 u_b \\
& + Err_{4ab}[u], \\
Err_{4ab}[u] =& {}^*\beta_a {}^*u_b + \xi_a \nabla_3 u_b - \xi_b \underline{\chi}_{ac} u_c + \underline{\chi}_{ab} \xi \cdot u - \widehat{\chi}_{ac} \nabla_c u_b - \underline{\eta}_b \widehat{\chi}_{ac} u_c \\
& + \widehat{\chi}_{ab} \underline{\eta} \cdot u,
\end{aligned} \tag{2.2.17}$$

$$\begin{aligned}
[\nabla_4, \nabla_3] u_a =& 2\omega \nabla_3 u_a - 2\underline{\omega} \nabla_4 u_a + 2(\underline{\eta}_b - \eta_b) \nabla_b u_a + 2(\underline{\eta} \cdot u)\eta_a - 2(\eta \cdot u)\underline{\eta}_a \\
& - 2\, {}^*\rho\, {}^*u_a + Err_{43a}[u], \\
Err_{43a}[u] =& 2\big(\underline{\xi}_a \xi_b - \xi_a \underline{\xi}_b\big) u^b.
\end{aligned} \tag{2.2.18}$$

3. Given $u \in \mathfrak{s}_2$, we have

$$\begin{aligned}
[\nabla_3, \nabla_a] u_{bc} =& -\frac{1}{2} tr\underline{\chi} \left(\nabla_a u_{bc} + \eta_b u_{ac} + \eta_c u_{ab} - \delta_{ab}(\eta \cdot u)_c - \delta_{ac}(\eta \cdot u)_b\right) \\
& - \frac{1}{2} {}^{(a)}tr\underline{\chi} \left({}^*\nabla_a u_{bc} + \eta_b {}^*u_{ac} + \eta_c {}^*u_{ab} - \in_{ab} (\eta \cdot u)_c - \in_{ac} (\eta \cdot u)_b\right) \\
& + (\eta_a - \zeta_a) \nabla_3 u_{bc} + Err_{3abc}[u], \\
Err_{3abc}[u] =& -2\, {}^*\underline{\beta}_a {}^*u_{bc} + \underline{\xi}_a \nabla_4 u_{bc} - \underline{\xi}_b \chi_{ad} u_{dc} - \underline{\xi}_c \chi_{ad} u_{bd} + \chi_{ab} \underline{\xi}_d u_{dc} \\
& + \chi_{ac} \underline{\xi}_d u_{bd} - \widehat{\underline{\chi}}_{ad} \nabla_d u_{bc} - \eta_b \widehat{\underline{\chi}}_{ad} u_{dc} - \eta_c \widehat{\underline{\chi}}_{ad} u_{bd} + \widehat{\underline{\chi}}_{ab} \eta_d u_{dc} + \widehat{\underline{\chi}}_{ac} \eta_d u_{bd},
\end{aligned} \tag{2.2.19}$$

$$\begin{aligned}
[\nabla_4, \nabla_a] u_{bc} =& -\frac{1}{2} tr\,\chi \left(\nabla_a u_{bc} + \underline{\eta}_b u_{ac} + \underline{\eta}_c u_{ab} - \delta_{ab}(\underline{\eta} \cdot u)_c - \delta_{ac}(\underline{\eta} \cdot u)_b\right) \\
& - \frac{1}{2} {}^{(a)}tr\chi \left({}^*\nabla_a u_{bc} + \underline{\eta}_b {}^*u_{ac} + \underline{\eta}_c {}^*u_{ab} - \in_{ab} (\underline{\eta} \cdot u)_c - \in_{ac} (\underline{\eta} \cdot u)_b\right) \\
& + (\underline{\eta}_a + \zeta_a) \nabla_4 u_{bc} + Err_{4abc}[u], \\
Err_{4abc}[u] =& 2\, {}^*\beta_a {}^*u_{bc} + \xi_a \nabla_3 u_{bc} - \xi_b \underline{\chi}_{ad} u_{dc} - \xi_c \underline{\chi}_{ad} u_{bd} + \underline{\chi}_{ab} \xi_d u_{dc} + \underline{\chi}_{ac} \xi_d u_{bd} \\
& - \widehat{\chi}_{ad} \nabla_d u_{bc} - \underline{\eta}_b \widehat{\chi}_{ad} u_{dc} - \underline{\eta}_c \widehat{\chi}_{ad} u_{bd} + \widehat{\chi}_{ab} \underline{\eta}_d u_{dc} + \widehat{\chi}_{ac} \underline{\eta}_d u_{bd},
\end{aligned} \tag{2.2.20}$$

$$\begin{aligned}
[\nabla_4, \nabla_3] u_{ab} =& 2\omega \nabla_3 u_{ab} - 2\underline{\omega} \nabla_4 u_{ab} + 2(\underline{\eta}_c - \eta_c) \nabla_c u_{ab} \\
& - 2\underline{\eta}_a \eta_c u_{bc} - 2\underline{\eta}_b \eta_c u_{ac} + 2\eta_a \underline{\eta}_c u_{bc} + 2\eta_b \underline{\eta}_c u_{ac} - 4\, {}^*\rho\, {}^*u_{ab} + Err_{43ab}[u] \\
=& 2\omega \nabla_3 u_{ab} - 2\underline{\omega} \nabla_4 u_{ab} + 2(\underline{\eta}_c - \eta_c) \nabla_c u_{ab} + 4\eta \widehat{\otimes} (\underline{\eta} \cdot u) \\
& - 4\underline{\eta} \widehat{\otimes} (\eta \cdot u) - 4\, {}^*\rho\, {}^*u_{ab} + Err_{43ab}[u], \\
Err_{43ab}[u] =& 2\big(\underline{\xi}_a \xi_c - \xi_a \underline{\xi}_c\big) u^c{}_b + 2\big(\underline{\xi}_b \xi_c - \xi_b \underline{\xi}_c\big) u_a{}^c.
\end{aligned} \tag{2.2.21}$$

We deduce the following corollary.

Corollary 2.2.9. *The following commutation formulas hold true:*

1. *Given* $u \in \mathfrak{s}_1$, *we have*

$$\begin{aligned}
[\nabla_3, div\,]u &= -\frac{1}{2} tr\,\underline{\chi}\left(div\, u - \eta\cdot u\right) + \frac{1}{2}{}^{(a)}tr\underline{\chi}\left(div\ {}^*u - \eta\cdot\ {}^*u\right) + (\eta-\zeta)\cdot\nabla_3 u\\
&\quad + Err_{3div}[u],\\
Err_{3div}[u] &= -\ {}^*\underline{\beta}\cdot\ {}^*u + \underline{\xi}\cdot\nabla_4 u - \underline{\xi}\cdot\widehat{\chi}\cdot u - \widehat{\underline{\chi}}\cdot\nabla u - \eta\cdot\widehat{\underline{\chi}}\cdot u,\\
[\nabla_4, div\,]u &= -\frac{1}{2} tr\,\chi\left(div\, u - \underline{\eta}\cdot u\right) + \frac{1}{2}{}^{(a)}tr\chi\left(div\ {}^*u - \underline{\eta}\cdot\ {}^*u\right) + (\underline{\eta}+\zeta)\cdot\nabla_4 u\\
&\quad + Err_{4div}[u],\\
Err_{4div}[u] &= \ {}^*\beta\cdot\ {}^*u + \xi\cdot\nabla_3 u - \xi\cdot\widehat{\underline{\chi}}\cdot u - \widehat{\chi}\cdot\nabla u - \underline{\eta}\cdot\widehat{\chi}\cdot u.
\end{aligned} \tag{2.2.22}$$

Also,

$$\begin{aligned}
[\nabla_3, \nabla\widehat{\otimes}]u &= -\frac{1}{2} tr\,\underline{\chi}\left(\nabla\widehat{\otimes}u + \eta\widehat{\otimes}u\right) - \frac{1}{2}{}^{(a)}tr\underline{\chi}\ {}^*\left(\nabla\widehat{\otimes}u + \underline{\eta}\widehat{\otimes}u\right) + (\eta-\zeta)\widehat{\otimes}\nabla_3 u\\
&\quad + Err_{3\widehat{\otimes}}[u],\\
Err_{3\widehat{\otimes}}[u] &= -\ {}^*\underline{\beta}\widehat{\otimes}\ {}^*u + \underline{\xi}\widehat{\otimes}\nabla_4 u - \underline{\xi}\widehat{\otimes}(\chi\cdot u) + \widehat{\chi}\,(\underline{\xi}\cdot u) - \widehat{\underline{\chi}}\cdot\nabla u - \eta\widehat{\otimes}(\widehat{\underline{\chi}}\cdot u)\\
&\quad + \widehat{\underline{\chi}}(\eta\cdot u),\\
[\nabla_4, \nabla\widehat{\otimes}]u &= -\frac{1}{2} tr\,\chi\left(\nabla\widehat{\otimes}u + \underline{\eta}\widehat{\otimes}u\right) - \frac{1}{2}{}^{(a)}tr\chi\ {}^*\left(\nabla\widehat{\otimes}u + \underline{\eta}\widehat{\otimes}u\right) + (\underline{\eta}+\zeta)\widehat{\otimes}\nabla_4 u\\
&\quad + Err_{4\widehat{\otimes}}[u],\\
Err_{4\widehat{\otimes}}[u] &= \ {}^*\beta\widehat{\otimes}\ {}^*u + \xi\widehat{\otimes}\nabla_3 u - \xi\widehat{\otimes}(\underline{\chi}\cdot u) + \widehat{\underline{\chi}}\,(\xi\cdot u) - \widehat{\chi}\cdot\nabla u - \underline{\eta}\widehat{\otimes}(\widehat{\chi}\cdot u)\\
&\quad + \widehat{\chi}(\underline{\eta}\cdot u).
\end{aligned} \tag{2.2.23}$$

2. *Given* $u \in \mathfrak{s}_2$, *we have*

$$\begin{aligned}
[\nabla_3, div\,]u &= -\frac{1}{2} tr\underline{\chi}\big(div\, u - 2\eta\cdot u\big) + \frac{1}{2}{}^{(a)}tr\underline{\chi}\big(div\ {}^*u - 2\eta\cdot\ {}^*u\big)\\
&\quad + (\eta-\zeta)\cdot\nabla_3 u + Err_{3div}[u],\\
Err_{3div}[u] &= -2\ {}^*\underline{\beta}\cdot\ {}^*u + \underline{\xi}\cdot\nabla_4 u - \underline{\xi}\cdot\chi\cdot u - (\chi\cdot u)\underline{\xi} + \underline{\xi}\cdot u\cdot\chi - \widehat{\underline{\chi}}\cdot\nabla u\\
&\quad - \eta\cdot\widehat{\underline{\chi}}\cdot u - (\widehat{\underline{\chi}}\cdot u)\eta + \eta\cdot u\cdot\widehat{\underline{\chi}},\\
[\nabla_4, div\,]u &= -\frac{1}{2} tr\,\chi\big(div\, u - 2\underline{\eta}\cdot u\big) + \frac{1}{2}{}^{(a)}tr\chi\big(div\ {}^*u - 2\underline{\eta}\cdot\ {}^*u\big)\\
&\quad + (\underline{\eta}+\zeta)\cdot\nabla_4 u + Err_{4div}[u],\\
Err_{4div}[u] &= 2\ {}^*\beta\cdot\ {}^*u + \xi\cdot\nabla_3 u - \xi\cdot\underline{\chi}\cdot u - (\underline{\chi}\cdot u)\xi + \xi\cdot u\cdot\underline{\chi} - \widehat{\chi}\cdot\nabla u\\
&\quad - \underline{\eta}\cdot\widehat{\chi}\cdot u - (\widehat{\chi}\cdot u)\underline{\eta} + \underline{\eta}\cdot u\cdot\widehat{\chi}.
\end{aligned} \tag{2.2.24}$$

Proof. We check (2.2.23). From (2.2.17) we have

$$\begin{aligned}
2[\nabla_4, \nabla\widehat{\otimes}]u_{ab} &= [\nabla_4, \nabla_a]u_b + [\nabla_4, \nabla_b]u_a - \delta_{ab}[\nabla_4, \mathrm{d}iv\,]u \\
&= -\mathrm{tr}\,\chi\left(\nabla\widehat{\otimes}u + \underline{\eta}\widehat{\otimes}u\right) + (\underline{\eta}+\zeta)_a\nabla_4 u_b - \frac{1}{2}{}^{(a)}\mathrm{tr}\chi H_{ab} \\
&\quad + \mathrm{Err}_{4ab}[u] + \mathrm{Err}_{4ba}[u] - \delta_{ab}\mathrm{Err}_{4\mathrm{d}iv}[u]
\end{aligned}$$

where H_{ab} denotes

$$\begin{aligned}
H_{ab} : &= \left({}^*\nabla_a u_b + \eta_b\,{}^*u_a - \in_{ab}\eta\cdot u\right) + \left({}^*\nabla_b u_a + \eta_b\,{}^*u_a - \in_{ba}\eta\cdot u\right) \\
&\quad -\delta_{ab}\left({}^*\nabla\cdot u + \eta\cdot{}^*u\right) \\
&= 2({}^*\nabla\widehat{\otimes}u)_{ab} + 2(\eta\widehat{\otimes}\,{}^*u)_{ab}.
\end{aligned}$$

Recalling that ${}^*\xi\widehat{\otimes}\eta = \xi\widehat{\otimes}\,{}^*\eta = {}^*\left(\xi\widehat{\otimes}\eta\right)$ we infer that $H = 2\,{}^*\left(\nabla\widehat{\otimes}u + \eta\widehat{\otimes}u\right)$. This proves the desired result.

We check the last statement in item 2. From (2.2.21)

$$\begin{aligned}
[\nabla_4, \nabla_a]u_{bc} &= -\frac{1}{2}\mathrm{tr}\,\chi\left(\nabla_a u_{bc} + \underline{\eta}_b u_{ac} + \underline{\eta}_c u_{ab} - \delta_{ab}(\underline{\eta}\cdot u)_c - \delta_{ac}(\underline{\eta}\cdot u)_b\right) \\
&\quad -\frac{1}{2}{}^{(a)}\mathrm{tr}\chi\left({}^*\nabla_a u_{bc} + \underline{\eta}_b\,{}^*u_{ac} + \underline{\eta}_c\,{}^*u_{ab} - \in_{ab}(\underline{\eta}\cdot u)_c - \in_{ac}(\underline{\eta}\cdot u)_b\right) \\
&\quad +(\underline{\eta}_a + \zeta_a)\nabla_4 u_{bc}
\end{aligned}$$

we deduce, recalling that $\delta_{ab}u_{ab} = 0$,

$$\begin{aligned}
[\nabla_4, \mathrm{d}iv\,]u_c &= \delta_{ab}[\nabla_4, \nabla_a]u_{bc} \\
&= -\frac{1}{2}\mathrm{tr}\,\chi\left(\mathrm{d}iv\,u_c + (\underline{\eta}\cdot u)_c - 2(\underline{\eta}\cdot u)_c - (\underline{\eta}\cdot u)_c\right) \\
&\quad -\frac{1}{2}{}^{(a)}\mathrm{tr}\chi\left(-\mathrm{d}iv\;{}^*u_c + (\underline{\eta}\cdot{}^*u)_c + {}^*(\underline{\eta}\cdot u)_c\right) + ((\underline{\eta}+\zeta)\cdot\nabla_4 u)_c
\end{aligned}$$

which proves the desired result. □

2.2.8 Commutation formulas with horizontal Lie derivatives

Recall that the Lie derivative of a k-covariant tensor U relative to a vectorfield X is given by

$$\mathcal{L}_X(Y_1, \ldots, Y_k) = XU(Y_1, \ldots, Y_k) - U(\mathcal{L}_X Y_1, \ldots, Y_k) - U(Y_1, \ldots, \mathcal{L}_X Y_k),$$

where $\mathcal{L}_X Y = [X, Y]$. In components relative to an arbitrary frame

$$\mathcal{L}_X U_{\alpha_1\ldots\alpha_k} : = \mathbf{D}_X U_{\alpha_1\ldots\alpha_k} + \mathbf{D}_{\alpha_1}X^\beta U_{\beta\alpha_1\ldots\alpha_k} + \cdots \mathbf{D}_{\alpha_k}X^\beta U_{\alpha_1\ldots\beta}.$$

Recall also the general commutation Lemma, see chapter 7 in [23].

Lemma 2.2.10. *The following formula*[12] *for a vectorfield X and a k-covariant tensor-field U holds true:*

$$\mathbf{D}_\beta(\mathcal{L}_X U_{\alpha_1\ldots\alpha_k}) - \mathcal{L}_X(\mathbf{D}_\beta U_{\alpha_1\ldots\alpha_k}) = \sum_{j=1}^{k} {}^{(X)}\Gamma_{\alpha_j\beta\rho} U_{\alpha_1\ldots}{}^{\rho}{}_{\ldots\alpha_k}, \tag{2.2.25}$$

where

$$^{(X)}\Gamma_{\alpha\beta\mu} = \frac{1}{2}(\mathbf{D}_\alpha {}^{(X)}\pi_{\beta\mu} + \mathbf{D}_\beta {}^{(X)}\pi_{\alpha\mu} - \mathbf{D}_\mu {}^{(X)}\pi_{\alpha\beta}). \tag{2.2.26}$$

The proof of the Lemma was given in [23], see Lemma 7.1.3, based on the following

Lemma 2.2.11. *Given an arbitrary vectorfield X we have the identity*

$$\mathbf{D}_\mu\mathbf{D}_\nu X_\beta = \mathbf{R}_{\beta\mu\nu\gamma}X^\gamma + {}^{(X)}\Gamma_{\mu\nu\beta}.$$

Proof. Consider the tensor $A_{\mu\nu\beta} = \mathbf{D}_\mu\mathbf{D}_\nu X_\beta - \mathbf{R}_{\beta\mu\nu\gamma}X^\gamma - {}^{(X)}\Gamma_{\mu\nu\beta}$ and observe that it verifies the symmetries

$$A_{\mu\nu\beta} = A_{\nu\mu\beta} = -A_{\mu\beta\nu}.$$

The proof of Lemma 2.2.11 follows by observing that any such tensor must vanish identically[13]. □

We are now ready to define the horizontal Lie derivative operator $\not{\mathcal{L}}$ as follows.

Definition 2.2.12 (Horizontal Lie derivatives)**.** *Given vectorfields X, Y, the horizontal Lie derivative $\not{\mathcal{L}}_X Y$ is given by*

$$\not{\mathcal{L}}_X Y := \mathcal{L}_X Y + \frac{1}{2}\mathbf{g}(\mathcal{L}_X Y, e_3)e_4 + \frac{1}{2}\mathbf{g}(\mathcal{L}_X Y, e_4)e_3.$$

Given a horizontal covariant k-tensor U, the horizontal Lie derivative $\not{\mathcal{L}}_X U$ is defined to be the projection of $\mathcal{L}_X U$ to the horizontal space. Thus, for horizontal indices $A = a_1 \ldots a_k$,

$$(\not{\mathcal{L}}_X U)_A := \nabla_X U_A + \mathbf{D}_{a_1}X^b U_{b\ldots a_k} + \cdots + \mathbf{D}_{a_k}X^b U_{a_1\ldots b}. \tag{2.2.27}$$

[12] This holds true for an arbitrary pseudo-riemannian space $(\mathcal{M}, \mathbf{g})$.

[13] Indeed $A_{\mu\nu\beta} = -A_{\mu\beta\nu} = -A_{\beta\mu\nu} = A_{\beta\nu\mu} = A_{\nu\beta\mu} = -A_{\nu\mu\beta} = -A_{\mu\nu\beta}$.

Lemma 2.2.13. *The following commutation formulas hold true for a horizontal covariant k-tensor U and a vectorfield X*

$$
\begin{aligned}
\nabla_b(\not{\mathcal{L}}_X U_A) - \not{\mathcal{L}}_X(\nabla_b U_A) &= \sum_{j=1}^{k} {}^{(X)}\not{\Gamma}_{a_j bc} U_{a_1\ldots}{}^{c}{}_{\ldots a_k},\\
\nabla_4(\not{\mathcal{L}}_X U_A) - \not{\mathcal{L}}_X(\nabla_4 U_A) + \nabla_{\not{\mathcal{L}}_X e_4} U_A &= \sum_{j=1}^{k} {}^{(X)}\not{\Gamma}_{a_j 4c} U_{a_1\ldots}{}^{c}{}_{\ldots a_k},\\
\nabla_3(\not{\mathcal{L}}_X U_A) - \not{\mathcal{L}}_X(\nabla_3 U_A) + \nabla_{\not{\mathcal{L}}_X e_3} U_A &= \sum_{j=1}^{k} {}^{(X)}\not{\Gamma}_{a_j 3c} U_{a_1\ldots}{}^{c}{}_{\ldots a_k},
\end{aligned}
\tag{2.2.28}
$$

with[14]

$$
\begin{aligned}
{}^{(X)}\not{\Gamma}_{abc} &= \frac{1}{2}(\nabla_a {}^{(X)}\pi_{bc} + \nabla_b {}^{(X)}\pi_{ac} - \nabla_c {}^{(X)}\pi_{ab}),\\
{}^{(X)}\not{\Gamma}_{a4b} &= \frac{1}{2}(\nabla_a {}^{(X)}\pi_{4b} + \nabla_4 {}^{(X)}\pi_{ab} - \nabla_b {}^{(X)}\pi_{a4}),\\
{}^{(X)}\not{\Gamma}_{a3b} &= \frac{1}{2}(\nabla_a {}^{(X)}\pi_{3b} + \nabla_3 {}^{(X)}\pi_{ab} - \nabla_b {}^{(X)}\pi_{a3}).
\end{aligned}
\tag{2.2.29}
$$

Proof. Follows easily by projecting formula (2.2.25) in Lemma 2.2.10, see also Lemma 9.1 in [21]. □

We now extend the definition of horizontal Lie derivative to any $U \in \mathbf{T}_k(\mathcal{M}) \otimes \mathbf{O}_l(\mathcal{M})$.

Definition 2.2.14. *We define the general horizontal derivatives as follows.*

1. *Given $X \in \mathbf{T}(\mathcal{M})$ and a general, horizontal tensor-field $U \in \mathbf{O}_k(\mathcal{M})$, we define*

$$\dot{\mathcal{L}}_X U \;:=\; \not{\mathcal{L}}_X U.$$

2. *Given a tensor in $U \in \mathbf{T}_k(\mathcal{M}) \otimes \mathbf{O}_l(\mathcal{M})$ and $X \in T(\mathcal{M})$ we define, for $Z = Z_1, \ldots, Z_k \in O(\mathbf{M})$ and $Y = Y_1, \ldots Y_l \in \mathbf{O}_1(\mathcal{M})$*

$$
\begin{aligned}
\dot{\mathcal{L}}_X U\big(Z,Y\big) = XU\big(Z,Y\big) \quad & -U\big(\mathcal{L}_X Z_1, \cdots Z_k, Y\big) - \ldots - U\big(Z_1, \cdots \mathcal{L}_X Z_k, Y\big)\\
& -U\big(Z, \dot{\mathcal{L}}_X Y_1, \ldots, Y_l\big) - \ldots - U\big(Z, Y_1, \ldots, \dot{\mathcal{L}}_X Y_l\big).
\end{aligned}
$$

[14] Here, ${}^{(X)}\pi_{ab}$ is treated as a horizontal symmetric 2-tensor, and ${}^{(X)}\pi_{a4}$, ${}^{(X)}\pi_{a3}$, as horizontal 1-forms.

3. *We have*

$$\dot{\mathcal{L}}_X(U\otimes V)=\dot{\mathcal{L}}_X U\otimes V+U\otimes\dot{\mathcal{L}}_X V.$$

4. *The definition can be extended by duality to any mixed tensors tensors in* $\mathbf{T}^{k_1}_{k_2}(\mathcal{M})\otimes\mathbf{O}^{l_1}_{l_2}(\mathcal{M})$.

Lemma 2.2.15. *The following commutation formulas hold true*[15] *for* $U\in\mathbf{O}_k(\mathcal{M})$ *and* $X\in\mathbf{T}(\mathcal{M})$,

$$\dot{\mathbf{D}}_\mu(\dot{\mathcal{L}}_X U_{a_1\ldots a_k})-\dot{\mathcal{L}}_X(\dot{\mathbf{D}}_\mu U_{a_1\ldots a_k}) = \sum_{j=1}^k {}^{(X)}\not{\Gamma}_{a_j\mu c}U_{a_1\ldots}{}^{c}{}_{\ldots a_k}.$$

The following commutation formula holds true for $U\in\mathbf{T}(\mathcal{M})\otimes\mathbf{O}_k(\mathcal{M})$ *and* $X\in\mathbf{T}(\mathcal{M})$,

$$\dot{\mathbf{D}}_\mu(\dot{\mathcal{L}}_X U_{\gamma a_1\ldots a_k})-\dot{\mathcal{L}}_X(\dot{\mathbf{D}}_\mu U_{\gamma a_1\ldots a_k}) = {}^{(X)}\Gamma_{\gamma\mu\rho}U^{\rho}{}_{a_1\ldots\ \ldots a_k}+\sum_{j=1}^k {}^{(X)}\not{\Gamma}_{a_j\mu c}U_{\gamma a_1\ldots}{}^{c}{}_{\ldots a_k}.$$

Proof. Follows easily by projecting formula (2.2.25) in Lemma 2.2.10, see also Lemma 9.1 in [21]. □

2.2.9 Main equations using conformally invariant derivatives

Consider frame transformations of the form

$$e_3'=\lambda^{-1}e_3,\qquad e_4'=\lambda e_4,\qquad e_a'=e_a.$$

Note that under the above mentioned frame transformation we have

$$\begin{aligned}
\mathrm{tr}\,\underline{\chi}' &= \lambda^{-1}\mathrm{tr}\,\underline{\chi},\quad {}^{(a)}\mathrm{tr}\underline{\chi}'=\lambda^{-1}\,{}^{(a)}\mathrm{tr}\underline{\chi},\quad \mathrm{tr}\,\chi'=\lambda\mathrm{tr}\,\chi,\quad {}^{(a)}\mathrm{tr}\chi'=\lambda\,{}^{(a)}\mathrm{tr}\chi,\\
\xi' &= \lambda^2\xi,\quad \eta'=\eta,\quad \underline{\eta}'=\underline{\eta},\quad \underline{\xi}'=\lambda^{-2}\underline{\xi},\\
\alpha' &= \lambda^2\alpha,\quad \beta'=\lambda\beta,\quad \rho'=\rho,\quad {}^*\rho'={}^*\rho,\quad \underline{\beta}'=\lambda^{-1}\underline{\beta},\quad \underline{\alpha}'=\lambda^{-2}\underline{\alpha},
\end{aligned}$$

and

$$\underline{\omega}' = \lambda^{-1}\left(\underline{\omega}+\frac{1}{2}e_3(\log\lambda)\right),\quad \omega'=\lambda\left(\omega-\frac{1}{2}e_4(\log\lambda)\right),\quad \zeta'=\zeta-\nabla(\log\lambda).$$

[15] With $\not{\Gamma}_X$ defined in (2.2.29).

Definition 2.2.16 (s-conformally invariants)**.** *We say that a horizontal tensor f is s-conformally invariant if, under the conformal frame transformation above, it changes as $f' = \lambda^s f$.*

Remark 2.2.17. *If f s-conformal invariant, then $\nabla_3 f, \nabla_4 f, \nabla_a f$ are not conformal invariant.*

We correct the lacking of being conformal invariant by making the following definition.

Lemma 2.2.18. *If f is s-conformal invariant, then:*

1. ${}^{(c)}\nabla_3 f := \nabla_3 f - 2s\underline{\omega} f$ *is $(s-1)$-conformally invariant.*
2. ${}^{(c)}\nabla_4 f := \nabla_4 f + 2s\omega f$ *is $(s+1)$-conformally invariant.*
3. ${}^{(c)}\nabla_A f := \nabla_A f + s\zeta_A f$ *is s-conformally invariant.*

Proof. Immediate verification. □

Remark 2.2.19. *Note that s is precisely what in [23] is called the signature of the tensor. In GHP formalism [32], the signature is related to the boost weights of the complex scalars.*

Using these definitions we rewrite the main equations as follows.

Proposition 2.2.20. *We have*

$$
\begin{aligned}
{}^{(c)}\nabla_3 tr\underline{\chi} &= -|\widehat{\underline{\chi}}|^2 - \frac{1}{2}\big(tr\underline{\chi}^2 - {}^{(a)}tr\underline{\chi}^2\big) + 2\,{}^{(c)}div\,\underline{\xi} + 2\underline{\xi}\cdot(\eta+\underline{\eta}),\\
{}^{(c)}\nabla_3\,{}^{(a)}tr\underline{\chi} &= -tr\,\underline{\chi}\,{}^{(a)}tr\underline{\chi} + 2\,{}^{(c)}curl\,\underline{\xi} + 2\underline{\xi}\wedge(-\eta+\underline{\eta}),\\
{}^{(c)}\nabla_3\widehat{\underline{\chi}} &= -tr\,\underline{\chi}\,\widehat{\underline{\chi}} + {}^{(c)}\nabla\widehat{\otimes}\underline{\xi} + \underline{\xi}\widehat{\otimes}(\eta+\underline{\eta}) - \underline{\alpha},
\end{aligned}
$$

$$
\begin{aligned}
{}^{(c)}\nabla_3 tr\,\chi &= -\widehat{\underline{\chi}}\cdot\widehat{\chi} - \frac{1}{2}tr\,\underline{\chi}tr\,\chi + \frac{1}{2}\,{}^{(a)}tr\underline{\chi}\,{}^{(a)}tr\chi + 2\,{}^{(c)}div\,\eta + 2\big(\xi\cdot\underline{\xi} + |\eta|^2\big) + 2\rho,\\
{}^{(c)}\nabla_3\,{}^{(a)}tr\chi &= -\widehat{\underline{\chi}}\wedge\widehat{\chi} - \frac{1}{2}\big({}^{(a)}tr\underline{\chi}tr\,\chi + tr\,\underline{\chi}\,{}^{(a)}tr\chi\big) + 2\,{}^{(c)}curl\,\eta + 2\underline{\xi}\wedge\xi - 2\,{}^*\rho,\\
{}^{(c)}\nabla_3\widehat{\chi} &= -\frac{1}{2}\big(tr\,\chi\widehat{\underline{\chi}} + tr\,\underline{\chi}\widehat{\chi}\big) - \frac{1}{2}\big(-\,{}^*\widehat{\underline{\chi}}\,{}^{(a)}tr\chi + {}^*\widehat{\chi}\,{}^{(a)}tr\underline{\chi}\big) + {}^{(c)}\nabla\widehat{\otimes}\eta + \underline{\xi}\widehat{\otimes}\xi + \eta\widehat{\otimes}\eta,
\end{aligned}
$$

$$
\begin{aligned}
{}^{(c)}\nabla_4 tr\underline{\chi} &= -\widehat{\chi}\cdot\widehat{\underline{\chi}} - \frac{1}{2}tr\,\chi tr\underline{\chi} + \frac{1}{2}\,{}^{(a)}tr\chi\,{}^{(a)}tr\underline{\chi} + 2\,{}^{(c)}div\,\underline{\eta} + 2\big(\xi\cdot\underline{\xi} + |\underline{\eta}|^2\big) + 2\rho,\\
{}^{(c)}\nabla_4\,{}^{(a)}tr\underline{\chi} &= -\widehat{\chi}\wedge\widehat{\underline{\chi}} - \frac{1}{2}\big({}^{(a)}tr\chi tr\underline{\chi} + tr\,\chi\,{}^{(a)}tr\underline{\chi}\big) + 2\,{}^{(c)}curl\,\underline{\eta} + 2\xi\wedge\underline{\xi} + 2\,{}^*\rho,\\
{}^{(c)}\nabla_4\widehat{\underline{\chi}} &= -\frac{1}{2}\big(tr\underline{\chi}\widehat{\chi} + tr\,\chi\widehat{\underline{\chi}}\big) - \frac{1}{2}\big(-\,{}^*\widehat{\chi}\,{}^{(a)}tr\underline{\chi} + {}^*\widehat{\underline{\chi}}\,{}^{(a)}tr\chi\big) + {}^{(c)}\nabla\widehat{\otimes}\underline{\eta} + \xi\widehat{\otimes}\underline{\xi} + \underline{\eta}\widehat{\otimes}\underline{\eta},
\end{aligned}
$$

$$\begin{aligned}
{}^{(c)}\nabla_4 tr\,\chi &= -|\widehat{\chi}|^2 - \frac{1}{2}\big(tr\,\chi^2 - {}^{(a)}tr\chi^2\big) + 2\,{}^{(c)}div\,\xi + 2\xi\cdot(\underline{\eta}+\eta),\\
{}^{(c)}\nabla_4\,{}^{(a)}tr\chi &= -tr\,\chi\,{}^{(a)}tr\chi + 2\,{}^{(c)}curl\,\xi + 2\xi\wedge(-\underline{\eta}+\eta),\\
{}^{(c)}\nabla_4\widehat{\chi} &= -tr\,\chi\,\widehat{\chi} + {}^{(c)}\nabla\widehat{\otimes}\xi + \xi\widehat{\otimes}(\underline{\eta}+\eta) - \alpha,
\end{aligned}$$

$$\begin{aligned}
{}^{(c)}\nabla_3\underline{\eta} - {}^{(c)}\nabla_4\underline{\xi} &= -\widehat{\underline{\chi}}\cdot(\underline{\eta}-\eta) - \frac{1}{2}tr\,\underline{\chi}(\underline{\eta}-\eta) + \frac{1}{2}\,{}^{(a)}tr\underline{\chi}({}^*\underline{\eta} - {}^*\eta) + \underline{\beta},\\
{}^{(c)}\nabla_4\eta - {}^{(c)}\nabla_3\xi &= -\widehat{\chi}\cdot(\eta-\underline{\eta}) - \frac{1}{2}tr\,\chi(\eta-\underline{\eta}) + \frac{1}{2}\,{}^{(a)}tr\chi({}^*\eta - {}^*\underline{\eta}) - \beta.
\end{aligned}$$

Also,

$$\begin{aligned}
{}^{(c)}div\,\widehat{\chi} &= \frac{1}{2}\,{}^{(c)}\nabla(tr\,\chi) - \frac{1}{2}\,{}^*\,{}^{(c)}\nabla({}^{(a)}tr\chi) - {}^{(a)}tr\chi\,{}^*\eta - {}^{(a)}tr\underline{\chi}\,{}^*\xi - \beta,\\
{}^{(c)}div\,\widehat{\underline{\chi}} &= \frac{1}{2}\,{}^{(c)}\nabla(tr\underline{\chi}) - \frac{1}{2}\,{}^*\,{}^{(c)}\nabla({}^{(a)}tr\underline{\chi}) - {}^{(a)}tr\underline{\chi}\,{}^*\underline{\eta} - {}^{(a)}tr\chi\,{}^*\underline{\xi} + \underline{\beta}.
\end{aligned}$$

Proposition 2.2.21. *We have*

$$\begin{aligned}
{}^{(c)}\nabla_3\alpha - {}^{(c)}\nabla\widehat{\otimes}\beta &= -\frac{1}{2}\big(tr\underline{\chi}\alpha + {}^{(a)}tr\underline{\chi}\,{}^*\alpha\big) + 4\eta\widehat{\otimes}\beta - 3(\rho\widehat{\chi} + {}^*\rho\,{}^*\widehat{\chi}),\\
{}^{(c)}\nabla_4\beta - {}^{(c)}div\,\alpha &= -2(tr\,\chi\beta - {}^{(a)}tr\chi\,{}^*\beta) + \alpha\cdot\underline{\eta} + 3(\xi\rho + {}^*\xi\,{}^*\rho),\\
{}^{(c)}\nabla_3\beta + {}^{(c)}div\,\varrho &= -(tr\underline{\chi}\beta + {}^{(a)}tr\underline{\chi}\,{}^*\beta) + 2\underline{\beta}\cdot\widehat{\chi} + 3(\rho\eta + {}^*\rho\,{}^*\eta) + \alpha\cdot\underline{\xi},\\
{}^{(c)}\nabla_4\rho - {}^{(c)}div\,\beta &= -\frac{3}{2}(tr\,\chi\rho + {}^{(a)}tr\chi\,{}^*\rho) + 2\underline{\eta}\cdot\beta - 2\xi\cdot\underline{\beta} - \frac{1}{2}\widehat{\underline{\chi}}\cdot\alpha,\\
{}^{(c)}\nabla_4\,{}^*\rho + {}^{(c)}curl\,\beta &= -\frac{3}{2}(tr\,\chi\,{}^*\rho - {}^{(a)}tr\chi\rho) - 2\underline{\eta}\cdot{}^*\beta - 2\xi\cdot{}^*\underline{\beta} + \frac{1}{2}\widehat{\underline{\chi}}\cdot{}^*\alpha,\\
{}^{(c)}\nabla_3\rho + {}^{(c)}div\,\underline{\beta} &= -\frac{3}{2}(tr\underline{\chi}\rho - {}^{(a)}tr\underline{\chi}\,{}^*\rho) - 2\eta\cdot\underline{\beta} + 2\underline{\xi}\cdot\beta - \frac{1}{2}\widehat{\chi}\cdot\underline{\alpha},\\
{}^{(c)}\nabla_3\,{}^*\rho + {}^{(c)}curl\,\underline{\beta} &= -\frac{3}{2}(tr\underline{\chi}\,{}^*\rho + {}^{(a)}tr\underline{\chi}\rho) - 2\eta\cdot{}^*\underline{\beta} - 2\underline{\xi}\cdot{}^*\beta - \frac{1}{2}\widehat{\chi}\cdot{}^*\underline{\alpha},\\
{}^{(c)}\nabla_4\underline{\beta} - {}^{(c)}div\,\check{\varrho} &= -(tr\,\chi\underline{\beta} + {}^{(a)}tr\chi\,{}^*\underline{\beta}) + 2\beta\cdot\widehat{\underline{\chi}} - 3(\rho\underline{\eta} - {}^*\rho\,{}^*\underline{\eta}) - \underline{\alpha}\cdot\xi,\\
{}^{(c)}\nabla_3\underline{\beta} + {}^{(c)}div\,\underline{\alpha} &= -2(tr\underline{\chi}\,\underline{\beta} - {}^{(a)}tr\underline{\chi}\,{}^*\underline{\beta}) - \underline{\alpha}\cdot\eta - 3(\underline{\xi}\rho - {}^*\underline{\xi}\,{}^*\rho),\\
{}^{(c)}\nabla_4\underline{\alpha} + {}^{(c)}\nabla\widehat{\otimes}\underline{\beta} &= -\frac{1}{2}\big(tr\,\chi\underline{\alpha} - {}^{(a)}tr\chi\,{}^*\underline{\alpha}\big) - 4\underline{\eta}\widehat{\otimes}\underline{\beta} - 3(\rho\widehat{\underline{\chi}} - {}^*\rho\,{}^*\widehat{\underline{\chi}}).
\end{aligned}$$

2.3 Commutations with horizontal wave operators

Consider a spacetime $(\mathcal{M}, \mathbf{g})$ with a horizontal structure induced by a null pair (e_3, e_4).

Definition 2.3.1. *We define the wave operator for tensor-fields $\psi \in \mathbf{O}_k(\mathcal{M})$ to be*

$$\dot{\Box}_k \psi := \mathbf{g}^{\mu\nu}\dot{\mathbf{D}}_\mu \dot{\mathbf{D}}_\nu \psi. \tag{2.3.1}$$

2.3.1 Commutation with $\dot{\mathcal{L}}_X$

Proposition 2.3.2. *The following commutation formula*[16] *holds true for $\psi \in \mathfrak{s}_2$ and $X \in \mathbf{T}(\mathcal{M})$,*

$$\begin{aligned}
[\dot{\mathcal{L}}_X, \dot{\Box}_2]\psi_{ab} &= -{}^{(X)}\pi^{\mu\nu}\dot{\mathbf{D}}_\mu\dot{\mathbf{D}}_\nu\psi_{ab} \\
&\quad -{}^{(X)}\Gamma^\mu{}_{\mu\rho}\dot{\mathbf{D}}^\rho\psi_{ab} - 2{}^{(X)}\not{\Gamma}_{a\mu c}\dot{\mathbf{D}}^\mu\psi^c{}_b - 2{}^{(X)}\not{\Gamma}_{b\mu c}\dot{\mathbf{D}}^\mu\psi_a{}^c \\
&\quad -\dot{\mathbf{D}}^\nu({}^{(X)}\not{\Gamma}_{a\nu c})\psi^c{}_b - \dot{\mathbf{D}}^\nu({}^{(X)}\not{\Gamma}_{b\nu c})\psi_a{}^c.
\end{aligned}$$

Proof. We have

$$\begin{aligned}
[\dot{\mathcal{L}}_X, \dot{\Box}_2]\psi_{ab} &= [\dot{\mathcal{L}}_X, \mathbf{g}^{\mu\nu}\dot{\mathbf{D}}_\mu\dot{\mathbf{D}}_\nu]\psi_{ab} \\
&= (\dot{\mathcal{L}}_X\mathbf{g}^{\mu\nu})\dot{\mathbf{D}}_\mu\dot{\mathbf{D}}_\nu\psi_{ab} + \mathbf{g}^{\mu\nu}[\dot{\mathcal{L}}_X, \dot{\mathbf{D}}_\mu]\dot{\mathbf{D}}_\nu\psi_{ab} + \mathbf{g}^{\mu\nu}\dot{\mathbf{D}}_\mu[\dot{\mathcal{L}}_X, \dot{\mathbf{D}}_\nu]\psi_{ab}.
\end{aligned}$$

Using Lemma 2.2.15 where recall the definition of ${}^{(X)}\Gamma, {}^{(X)}\not{\Gamma}$ in section 2.2.8, we obtain

$$\begin{aligned}
[\dot{\mathcal{L}}_X, \dot{\Box}_2]\psi_{ab} &= -{}^{(X)}\pi^{\mu\nu}\dot{\mathbf{D}}_\mu\dot{\mathbf{D}}_\nu\psi_{ab} - \dot{\mathbf{D}}^\nu\big({}^{(X)}\not{\Gamma}_{a\nu c}\psi^c{}_b + {}^{(X)}\not{\Gamma}_{b\nu c}\psi_a{}^c\big) \\
&\quad -{}^{(X)}\Gamma^\mu{}_{\mu\rho}\dot{\mathbf{D}}^\rho\psi_{ab} - {}^{(X)}\not{\Gamma}_{a\mu c}\dot{\mathbf{D}}^\mu\psi^c{}_b - {}^{(X)}\not{\Gamma}_{b\mu c}\dot{\mathbf{D}}^\mu\psi_a{}^c \\
&= -{}^{(X)}\pi^{\mu\nu}\dot{\mathbf{D}}_\mu\dot{\mathbf{D}}_\nu\psi_{ab} - {}^{(X)}\Gamma^\mu{}_{\mu\rho}\dot{\mathbf{D}}^\rho\psi_{ab} - 2{}^{(X)}\not{\Gamma}_{a\mu c}\dot{\mathbf{D}}^\mu\psi^c{}_b - 2{}^{(X)}\not{\Gamma}_{b\mu c}\dot{\mathbf{D}}^\mu\psi_a{}^c \\
&\quad -\dot{\mathbf{D}}^\nu({}^{(X)}\not{\Gamma}_{a\nu c})\psi^c{}_b - \dot{\mathbf{D}}^\nu({}^{(X)}\not{\Gamma}_{b\nu c})\psi_a{}^c,
\end{aligned}$$

as stated. $\Box$

2.3.2 Commutation with $\dot{\mathbf{D}}_X$

Lemma 2.3.3. *We have in a vacuum spacetime*

$$\begin{aligned}
\dot{\Box}(X^\beta\dot{\mathbf{D}}_\beta U_a) - X^\beta\dot{\mathbf{D}}_\beta\dot{\Box}U_a &= \pi^{\mu\nu}\dot{\mathbf{D}}_\mu\dot{\mathbf{D}}_\nu U_a + \big(\mathbf{D}^\mu\pi_\mu{}^\beta - \frac{1}{2}\mathbf{D}^\beta tr\pi)\dot{\mathbf{D}}_\beta U_a \\
&\quad -2X^\beta\mathbf{R}_{ac\beta\mu}\dot{\mathbf{D}}^\mu U_c + \mathbf{D}^\beta X^\mu\mathbf{R}_{ac\beta\mu}U^c \\
&\quad -X^\beta\mathbf{B}_{ac\beta\mu}\dot{\mathbf{D}}^\mu U_c + \frac{1}{2}\mathbf{D}^\beta X^\mu\mathbf{B}_{ac\beta\mu}U^c + \frac{1}{2}X^\beta\mathbf{D}^\mu\mathbf{B}_{ac\mu\beta}U^c.
\end{aligned}$$

[16] Recall that $\dot{\mathcal{L}}$ has been introduced in Definition 2.2.14.

Proof. Straightforward computation using Lemma 2.2.11 and Proposition 2.1.27, see section A.2. □

2.3.3 Killing tensor and Carter operator

Recall that the deformation tensor of a vectorfield ${}^{(X)}\pi$ is defined as

$${}^{(X)}\pi_{\mu\nu} := \mathbf{D}_{(\mu}X_{\nu)} = \mathbf{D}_\mu X_\nu + \mathbf{D}_\nu X_\mu.$$

The vectorfield is said to be Killing if ${}^{(X)}\pi \equiv 0$. The Kerr spacetime has, in addition to the symmetries generated by its two linearly independent Killing vectorfields $\mathbf{T}$ and $\mathbf{Z}$, a higher order symmetry defined by a Killing tensor.

Definition 2.3.4. *A symmetric 2-tensor $K_{\mu\nu}$ is said to be a Killing tensor if its deformation 3-tensor Π, defined below, vanishes identically.*

$$\Pi_{\mu\nu\rho} := \mathbf{D}_{(\mu}K_{\nu\rho)} = \mathbf{D}_\mu K_{\nu\rho} + \mathbf{D}_\nu K_{\rho\mu} + \mathbf{D}_\rho K_{\mu\nu}. \tag{2.3.2}$$

Remark 2.3.5. *Observe that if X, Y are Killing vectorfields then the symmetric 2-tensor $K = \frac{1}{2}(X \otimes Y + Y \otimes X)$ is a Killing tensor.*

We define the second order differential operator associated to a tensor-field $\psi \in \mathfrak{s}_k$.

Definition 2.3.6. *Given a symmetric tensor K its associated second order differential operator $\mathcal{K}$ applied to a tensor $\psi \in \mathfrak{s}_k$ is defined as*

$$\mathcal{K}(\psi) = \dot{\mathbf{D}}_\mu(K^{\mu\nu}\dot{\mathbf{D}}_\nu(\psi)). \tag{2.3.3}$$

We now compute the commutators of $\mathcal{K}$ with $\square_\mathbf{g}$ in terms of the symmetric tensor Π.

Proposition 2.3.7. *In a vacuum spacetime, the commutator between the differential operator $\mathcal{K}$ and the $\square_\mathbf{g}$ operator applied to a scalar function ϕ is given by*

$$[\mathcal{K}, \square_\mathbf{g}]\phi = Err[\Pi](\phi)$$

where $Err[\Pi](\phi)$ denotes terms involving Π given by

$$\begin{aligned} Err[\Pi](\phi) := \; & \mathbf{D}^\mu \left(\big(\mathbf{D}^\alpha \Pi_{\alpha\nu\mu} - \frac{1}{2}\mathbf{D}_\mu \Pi^\alpha{}_{\alpha\nu} + \frac{1}{2}\mathbf{D}_\nu \Pi^\alpha{}_{\alpha\mu}\big)\dot{\mathbf{D}}^\nu\phi - 2\Pi_{\mu\alpha\nu}\dot{\mathbf{D}}^\alpha\dot{\mathbf{D}}^\nu\phi \right) \\ & -2\left(\mathbf{D}^\alpha\Pi_{\alpha\nu\mu}\right)\dot{\mathbf{D}}^\mu\dot{\mathbf{D}}^\nu\phi. \end{aligned}$$

Proof. See section A.3. □

2.4 Main equations in complex notations

In this section we introduce complex notations for the Ricci coefficients and the curvature components with the objective of simplifying the main equations. From the real scalars, 1-tensors and symmetric traceless 2-tensors already introduced, we define their complexified version which results in anti-self dual tensors.

2.4.1 Complex notations

Recall Definition 2.1.11 of the set of real horizontal k-tensors $\mathfrak{s}_k = \mathfrak{s}_k(\mathcal{M}, \mathbb{R})$ on $\mathcal{M}$. For instance,

- $(a, b) \in \mathfrak{s}_0$ is a pair of real scalar function on $\mathcal{M}$,
- $f \in \mathfrak{s}_1$ is a real horizontal 1-tensor on $\mathcal{M}$,
- $u \in \mathfrak{s}_2$ is a real horizontal symmetric traceless 2-tensor on $\mathcal{M}$.

By Definition 2.1.7, the duals of real horizontal tensors are real horizontal tensors of the same type, i.e. ${}^*f \in \mathfrak{s}_1$ and ${}^*u \in \mathfrak{s}_2$.

We define the complexified version of horizontal tensors on $\mathcal{M}$.

Definition 2.4.1. *We denote by $\mathfrak{s}_k(\mathbb{C}) = \mathfrak{s}_k(\mathcal{M}, \mathbb{C})$ the set of complex anti-self dual k-tensors on $\mathcal{M}$. More precisely,*

- *$a + ib \in \mathfrak{s}_0(\mathbb{C})$ is a complex scalar function on $\mathcal{M}$ if $(a, b) \in \mathfrak{s}_0$,*
- *$F = f + i\,{}^*f \in \mathfrak{s}_1(\mathbb{C})$ is a complex anti-self dual 1-tensor on $\mathcal{M}$ if $f \in \mathfrak{s}_1$,*
- *$U = u + i\,{}^*u \in \mathfrak{s}_2(\mathbb{C})$ is a complex anti-self dual symmetric traceless 2-tensor on $\mathcal{M}$ if $u \in \mathfrak{s}_2$.*

Observe that $F \in \mathfrak{s}_1(\mathbb{C})$ and $U \in \mathfrak{s}_2(\mathbb{C})$ are indeed anti-self dual tensors, i.e.

$$ {}^*F = -iF, \qquad {}^*U = -iU. $$

More precisely

$$ U_{12} = U_{21} = i\,{}^*U_{12} = i \in_{12} U_{22} = -iU_{11}, \qquad U_{11} = iU_{12}. $$

Recall that the derivatives ∇_3, ∇_4 and ∇_a are real derivatives. We can use the dual operators to define the complexified version of the ∇_a derivative, which allows to simplify the notations in the main equations.

Definition 2.4.2. *We define the complexified version of the horizontal derivative as*

$$\mathcal{D} = \nabla + i\,{}^*\nabla, \qquad \overline{\mathcal{D}} = \nabla - i\,{}^*\nabla.$$

More precisely, we have:

- *for $a + ib \in \mathfrak{s}_0(\mathbb{C})$,*
$$\mathcal{D}(a+ib) \;:=\; (\nabla + i\,{}^*\nabla)(a+ib), \qquad \overline{\mathcal{D}}(a+ib) := (\nabla - i\,{}^*\nabla)(a+ib).$$
- *For $f + i\,{}^*f \in \mathfrak{s}_1(\mathbb{C})$,*
$$\begin{aligned}\mathcal{D}\cdot(f+i\,{}^*f) &:= (\nabla + i\,{}^*\nabla)\cdot(f+i\,{}^*f) = 0,\\ \overline{\mathcal{D}}\cdot(f+i\,{}^*f) &:= (\nabla - i\,{}^*\nabla)\cdot(f+i\,{}^*f),\\ \mathcal{D}\widehat{\otimes}(f+i\,{}^*f) &:= (\nabla + i\,{}^*\nabla)\widehat{\otimes}(f+i\,{}^*f).\end{aligned}$$
- *For $u + i\,{}^*u \in \mathfrak{s}_2(\mathbb{C})$,*
$$\begin{aligned}\mathcal{D}\cdot(u+i\,{}^*u) &:= (\nabla + i\,{}^*\nabla)\cdot(u+i\,{}^*u) = 0,\\ \overline{\mathcal{D}}\cdot(u+i\,{}^*u) &:= (\nabla - i\,{}^*\nabla)\cdot(u+i\,{}^*u).\end{aligned}$$

Note that ${}^*\mathcal{D} = -i\mathcal{D}$.

For $F = f + i\,{}^*f \in \mathfrak{s}_1(\mathbb{C})$ the operator $-\frac{1}{2}\mathcal{D}\widehat{\otimes}$ is formally adjoint to the operator $\overline{\mathcal{D}}\cdot U$ applied to $U \in \mathfrak{s}_2(\mathbb{C})$. For $h = a + ib \in \mathfrak{s}_0(\mathbb{C})$ the operator $-\mathcal{D}h$ is formally adjoint to the operator $\overline{\mathcal{D}}\cdot F$ applied to $F \in \mathfrak{s}_1(\mathbb{C})$. These notions makes sense literally only if the horizontal structure is integrable.

Lemma 2.4.3. *For $F = f + i\,{}^*f \in \mathfrak{s}_1(\mathbb{C})$ and $U = u + i\,{}^*u \in \mathfrak{s}_2(\mathbb{C})$, we have*

$$(\mathcal{D}\widehat{\otimes}F)\cdot\overline{U} \;=\; -2F\cdot(\mathcal{D}\cdot\overline{U}) - ((H+\underline{H})\widehat{\otimes}F)\cdot\overline{U} + 2\mathcal{D}\cdot(F\cdot\overline{U}). \tag{2.4.1}$$

Proof. We look at the real parts. Then

$$(\nabla\widehat{\otimes}f)\cdot u \;=\; \big(\nabla_a f_b + \nabla_b f_a - \delta_{ab}\mathrm{d}iv\, f\big)u_{ab} = 2(\nabla_a f_b)u_{ab} = 2\nabla_a(u_{ab}f_b) - 2(\mathrm{d}iv\, u)\cdot f.$$

Using Lemma 2.1.40 applied to $\xi = u\cdot f$ we obtain

$$\begin{aligned}(\nabla\widehat{\otimes}f)\cdot u &= 2\nabla^a(u_{ab}f_b) - 2(\eta+\underline{\eta})\cdot(u\cdot f) - 2(\mathrm{d}iv\, u)\cdot f\\ &= -2(\mathrm{d}iv\, u)\cdot f - ((\eta+\underline{\eta})\widehat{\otimes}f)\cdot u + 2\mathrm{d}iv\,(u\cdot f).\end{aligned}$$

By complexifying, we obtain the stated identity. □

Lemma 2.4.4. *The following holds:*

- *If* $\xi, \eta \in \mathfrak{s}_1$

$$\begin{aligned}\xi\cdot\eta + i\,{}^*\xi\cdot\eta &= \frac{1}{2}\Big((\xi+i\,{}^*\xi)\cdot(\overline{\eta+i\,{}^*\eta})\Big),\\ \xi\widehat{\otimes}\eta + i\,{}^*(\xi\widehat{\otimes}\eta) &= \frac{1}{2}\Big((\xi+i\,{}^*\xi)\widehat{\otimes}(\eta+i\,{}^*\eta)\Big).\end{aligned}$$

- *If* $\eta \in \mathfrak{s}_1$, $u \in \mathfrak{s}_2$

$$u\cdot\eta + i\,{}^*u\cdot\eta = \frac{1}{2}(u+i\,{}^*u)\cdot(\overline{\eta+i\,{}^*\eta}).$$

- *If* $u, v \in \mathfrak{s}_2$

$$u\cdot v + i\,{}^*u\cdot v = \frac{1}{2}(u+i\,{}^*u)\cdot(\overline{v+i\,{}^*v}).$$

- *If* $(a, b) \in \mathfrak{s}_0$

$$\nabla a - {}^*\nabla b + i({}^*\nabla a + \nabla b) = \mathcal{D}(a+ib).$$

- *If* $\xi \in \mathfrak{s}_1$

$$\begin{aligned}div\,\xi + i curl\,\xi &= \frac{1}{2}\overline{\mathcal{D}}\cdot(\xi+i\,{}^*\xi),\\ \nabla\widehat{\otimes}\xi + i\,{}^*(\nabla\widehat{\otimes}\xi) &= \frac{1}{2}\mathcal{D}\widehat{\otimes}(\xi+i\,{}^*\xi).\end{aligned}$$

- *If* $u \in \mathfrak{s}_2$

$$div\,u + i\,{}^*(div\,u) = \frac{1}{2}\overline{\mathcal{D}}\cdot(u+i\,{}^*u).$$

Proof. The first identities rely on Lemma 2.1.19. The other rely on the following identities, for $\xi \in \mathfrak{s}_1$, $u \in \mathfrak{s}_2$,

$$\begin{aligned}&\nabla\cdot\,{}^*\xi = curl\,\xi,\quad {}^*\nabla\cdot\xi = -curl\,\xi,\quad {}^*\nabla\cdot\,{}^*\xi = \nabla\xi,\ \nabla\widehat{\otimes}\,{}^*\xi = {}^*\nabla\widehat{\otimes}\xi = {}^*(\nabla\widehat{\otimes}\xi),\\ &{}^*\nabla\widehat{\otimes}\,{}^*\xi = -\nabla\widehat{\otimes}\xi,\ {}^*(\mathrm{d}iv\,u) = \nabla\cdot\,{}^*u,\quad {}^*\nabla\cdot u = -\,{}^*(\mathrm{d}iv\,u),\quad {}^*\nabla\cdot\,{}^*u = \nabla\cdot u.\end{aligned}$$

□

Lemma 2.4.5. *Let $E, F \in \mathfrak{s}_1(\mathbb{C})$ and $U \in \mathfrak{s}_2(\mathbb{C})$. Then*

$$E\widehat{\otimes}(\overline{F}\cdot U)+F\widehat{\otimes}(\overline{E}\cdot U) \;=\; 2(E\cdot\overline{F}+\overline{E}\cdot F)\, U. \tag{2.4.2}$$

Also, for $E = e + i\,{}^*e$, $F = f + i\,{}^*f$

$$E\widehat{\otimes}(\overline{F}\cdot U) \;=\; 4\big(e\cdot f - ie\wedge f\big)U. \tag{2.4.3}$$

Proof. Recall, see Lemma 2.1.20,

$$\xi\widehat{\otimes}(\eta\cdot u)+\eta\widehat{\otimes}(\xi\cdot u) = 2(\xi\cdot\eta)u.$$

For $E = \xi + i\,{}^*\xi$, $F = \eta + i\,{}^*\eta$ and $U = u + i\,{}^*u$, we have

$$\begin{aligned}
E\widehat{\otimes}(\overline{F}\cdot U) &= (\xi+i\,{}^*\xi)\widehat{\otimes}\big(\overline{(\eta+i\,{}^*\eta)}\cdot(u+i\,{}^*u)\big)\\
&= 2(\xi+i\,{}^*\xi)\widehat{\otimes}\big((u\cdot\eta)+i\,{}^*(u\cdot\eta)\big)\\
&= 4\Big(\xi\widehat{\otimes}(u\cdot\eta)+i\,{}^*(\xi\widehat{\otimes}(u\cdot\eta))\Big),\\
F\widehat{\otimes}(\overline{E}\cdot U) &= 4\Big(\eta\widehat{\otimes}(u\cdot\xi)+i\,{}^*(\eta\widehat{\otimes}(u\cdot\xi))\Big).
\end{aligned}$$

Therefore

$$\begin{aligned}
E\widehat{\otimes}(\overline{F}\cdot U)+F\widehat{\otimes}(\overline{E}\cdot U) &= 4(\xi\widehat{\otimes}(u\cdot\eta)+i\,{}^*(\xi\widehat{\otimes}(u\cdot\eta)))+4(\eta\widehat{\otimes}(u\cdot\xi)+i\,{}^*(\eta\widehat{\otimes}(u\cdot\xi)))\\
&= 4(\xi\widehat{\otimes}(u\cdot\eta)+\eta\widehat{\otimes}(u\cdot\xi))+4i\,{}^*(\xi\widehat{\otimes}(u\cdot\eta)+\eta\widehat{\otimes}(u\cdot\xi))\\
&= 8((\xi\cdot\eta)\;u)+8i\,{}^*((\xi\cdot\eta)u) = 8(\xi\cdot\eta)U
\end{aligned}$$

while

$$E\cdot\overline{F}+\overline{E}\cdot F \;=\; 2(\xi\cdot\eta+i\,{}^*\xi\cdot\eta)+2(\eta\cdot\xi+i\,{}^*\eta\cdot\xi) = 4(\xi\cdot\eta).$$

Hence,

$$E\widehat{\otimes}(\overline{F}\cdot U)+F\widehat{\otimes}(\overline{E}\cdot U) \;=\; 2\big(E\cdot\overline{F}+\overline{E}\cdot F\big)U$$

as stated. The second identity can be derived in the same manner. □

Leibniz formulas

We collect here Leibniz formulas involving the derivative operators defined above.

Lemma 2.4.6. *Let h be a scalar function, $F \in \mathfrak{s}_1(\mathbb{C})$, $U \in \mathfrak{s}_2(\mathbb{C})$. Then*

$$\begin{aligned} \overline{\mathcal{D}} \cdot (hF) &= h\overline{\mathcal{D}} \cdot F + \overline{\mathcal{D}}(h) \cdot F, \\ \mathcal{D}\widehat{\otimes}(hF) &= h\mathcal{D}\widehat{\otimes}F + \mathcal{D}(h)\widehat{\otimes}F, \\ \overline{\mathcal{D}} \cdot (hU) &= \overline{\mathcal{D}}(h) \cdot U + h(\overline{\mathcal{D}} \cdot U), \\ \mathcal{D}\widehat{\otimes}(\overline{F} \cdot U) &= 2(\mathcal{D} \cdot \overline{F})U + 2(\overline{F} \cdot \mathcal{D})U, \\ U \cdot \overline{\mathcal{D}}F &= U(\overline{\mathcal{D}} \cdot F). \end{aligned} \tag{2.4.4}$$

Also,

$$\begin{aligned} F\widehat{\otimes}(\overline{\mathcal{D}} \cdot U) &= 2(F \cdot \overline{\mathcal{D}})U = 4F \cdot \nabla U, \\ (F \cdot \overline{\mathcal{D}})U + (\overline{F} \cdot \mathcal{D})U &= 4f \cdot \nabla U = 2(F + \overline{F}) \cdot \nabla U. \end{aligned} \tag{2.4.5}$$

Proof. Straightforward verifications, see section A.5. □

Lemma 2.4.7. *As a corollary of* (2.4.5) *we derive the following formula for $U \in \mathfrak{s}_2(\mathbb{C})$*

$$\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}} \cdot U) \;=\; 2\triangle_2 U - 4\,{}^{(h)}KU - i\big({}^{(a)}tr\chi\nabla_3 + {}^{(a)}tr\underline{\chi}\nabla_4\big)U \tag{2.4.6}$$

where

$${}^{(h)}K = -\frac{1}{4}tr\,\chi tr\underline{\chi} - \frac{1}{4}\,{}^{(a)}tr\chi\,{}^{(a)}tr\underline{\chi} + \frac{1}{2}\widehat{\chi} \cdot \widehat{\underline{\chi}} - \frac{1}{4}\rho.$$

Proof. According to (2.4.5) we have

$$\frac{1}{2}\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}} \cdot U) \;= (\mathcal{D} \cdot \overline{\mathcal{D}})U = (\nabla^a + i\,{}^*\nabla^a)((\nabla^a - i\,{}^*\nabla^a)U = 2\triangle_2 U - 2i \in^{ab} \nabla_a\nabla_b U.$$

On the other hand, appealing to Proposition 2.1.43 we have

$$\begin{aligned} 2 \in^{ab} \nabla_a\nabla_b U \;&=\; \in^{ab} [\nabla_a, \nabla_b]A = 2\left(\frac{1}{2}({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4)\psi + 2\,{}^{(h)}K\,{}^*A\right) \\ &=\; ({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4)\psi - 4i\,{}^{(h)}KA \end{aligned}$$

Hence $\frac{1}{2}\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}} \cdot U) = 2\triangle_2 U - 4\,{}^{(h)}KU - i({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4)U$ as stated. □

2.4.2 Main equations in complex form

We now extend the definitions for the Ricci coefficients and curvature components given in Sections 2.2.1 and 2.2.2, to the complex case by using the anti-self dual tensors defined above.

Definition 2.4.8. *We define the following complex anti-self dual tensors:*

$$A := \alpha + i\,{}^*\alpha, \quad B := \beta + i\,{}^*\beta, \quad P := \rho + i\,{}^*\rho, \quad \underline{B} := \underline{\beta} + i\,{}^*\underline{\beta}, \quad \underline{A} := \underline{\alpha} + i\,{}^*\underline{\alpha},$$

and

$$\begin{aligned}&X = \chi + i\,{}^*\chi, \quad \underline{X} = \underline{\chi} + i\,{}^*\underline{\chi}, \quad H = \eta + i\,{}^*\eta, \quad \underline{H} = \underline{\eta} + i\,{}^*\underline{\eta}, \quad Z = \zeta + i\,{}^*\zeta,\\ &\Xi = \xi + i\,{}^*\xi, \quad \underline{\Xi} = \underline{\xi} + i\,{}^*\underline{\xi}.\end{aligned}$$

In particular, note that

$$trX = tr\,\chi - i\,{}^{(a)}tr\chi, \quad \widehat{X} = \widehat{\chi} + i\,{}^*\widehat{\chi}, \quad tr\underline{X} = tr\,\underline{\chi} - i\,{}^{(a)}tr\underline{\chi}, \quad \underline{\widehat{X}} = \underline{\widehat{\chi}} + i\,{}^*\underline{\widehat{\chi}}.$$

The complex notations allow us to rewrite the Ricci equations in a more compact form.

Proposition 2.4.9.

$$\begin{aligned}\nabla_3 tr\underline{X} + \frac{1}{2}(tr\underline{X})^2 + 2\underline{\omega}\, tr\underline{X} &= \mathcal{D}\cdot\overline{\underline{\Xi}} + \underline{\Xi}\cdot\overline{\underline{H}} + \overline{\underline{\Xi}}\cdot(H - 2Z) - \frac{1}{2}\underline{\widehat{X}}\cdot\overline{\underline{\widehat{X}}},\\ \nabla_3\underline{\widehat{X}} + \Re(tr\underline{X})\underline{\widehat{X}} + 2\underline{\omega}\,\underline{\widehat{X}} &= \frac{1}{2}\mathcal{D}\widehat{\otimes}\underline{\Xi} + \frac{1}{2}\underline{\Xi}\widehat{\otimes}(H + \underline{H} - 2Z) - \underline{A},\end{aligned}$$

$$\begin{aligned}\nabla_3 trX + \frac{1}{2}tr\underline{X}trX - 2\underline{\omega}trX &= \mathcal{D}\cdot\overline{\underline{H}} + H\cdot\overline{\underline{H}} + 2P + \underline{\Xi}\cdot\overline{\Xi} - \frac{1}{2}\underline{\widehat{X}}\cdot\overline{\widehat{X}},\\ \nabla_3\widehat{X} + \frac{1}{2}tr\underline{X}\,\widehat{X} - 2\underline{\omega}\widehat{X} &= \frac{1}{2}\mathcal{D}\widehat{\otimes}H + \frac{1}{2}H\widehat{\otimes}H - \frac{1}{2}\overline{trX}\underline{\widehat{X}} + \frac{1}{4}\underline{\Xi}\widehat{\otimes}\Xi,\end{aligned}$$

$$\begin{aligned}\nabla_4 tr\underline{X} + \frac{1}{2}trX\,tr\underline{X} - 2\omega tr\underline{X} &= \mathcal{D}\cdot\overline{\underline{H}} + \underline{H}\cdot\overline{\underline{H}} + 2\overline{P} + \Xi\cdot\overline{\underline{\Xi}} - \frac{1}{2}\widehat{X}\cdot\overline{\underline{\widehat{X}}},\\ \nabla_4\underline{\widehat{X}} + \frac{1}{2}trX\,\underline{\widehat{X}} - 2\omega\underline{\widehat{X}} &= \frac{1}{2}\mathcal{D}\widehat{\otimes}\,\underline{H} + \frac{1}{2}\,\underline{H}\widehat{\otimes}\,\underline{H} - \frac{1}{2}\overline{tr\underline{X}}\widehat{X} + \frac{1}{4}\Xi\widehat{\otimes}\underline{\Xi},\end{aligned}$$

$$\begin{aligned}\nabla_4 trX + \frac{1}{2}(trX)^2 + 2\omega trX &= \mathcal{D}\cdot\overline{\Xi} + \Xi\cdot\overline{H} + \overline{\Xi}\cdot(H + 2Z) - \frac{1}{2}\widehat{X}\cdot\overline{\widehat{X}},\\ \nabla_4\widehat{X} + \Re(trX)\widehat{X} + 2\omega\widehat{X} &= \frac{1}{2}\mathcal{D}\widehat{\otimes}\Xi + \frac{1}{2}\Xi\widehat{\otimes}(\underline{H} + H + 2Z) - A.\end{aligned}$$

Also,

$$\begin{aligned}
\nabla_3 Z+\frac{1}{2}tr\underline{X}(Z+H)-2\underline{\omega}(Z-H) &= -2\mathcal{D}\underline{\omega}-\frac{1}{2}\widehat{\underline{X}}\cdot(\overline{Z}+\overline{H})\\
&\quad+\frac{1}{2}trX\underline{\Xi}+2\omega\underline{\Xi}-\underline{B}+\frac{1}{2}\overline{\underline{\Xi}}\cdot\widehat{X},\\
\nabla_4 Z+\frac{1}{2}trX(Z-\underline{H})-2\omega(Z+\underline{H}) &= 2\mathcal{D}\omega+\frac{1}{2}\widehat{X}\cdot(-\overline{Z}+\overline{\underline{H}})\\
&\quad-\frac{1}{2}tr\underline{X}\Xi-2\underline{\omega}\Xi-B-\frac{1}{2}\overline{\Xi}\cdot\widehat{\underline{X}},\\
\nabla_3\underline{H}-\nabla_4\underline{\Xi} &= -\frac{1}{2}\overline{tr\underline{X}}(\underline{H}-H)-\frac{1}{2}\widehat{\underline{X}}\cdot(\overline{\underline{H}}-\overline{H})-4\omega\underline{\Xi}+\underline{B},\\
\nabla_4 H-\nabla_3\Xi &= -\frac{1}{2}\overline{trX}(H-\underline{H})-\frac{1}{2}\widehat{X}\cdot(\overline{H}-\overline{\underline{H}})-4\underline{\omega}\Xi-B,
\end{aligned}$$

and

$$\nabla_3\omega+\nabla_4\underline{\omega}-4\omega\underline{\omega}-\xi\cdot\underline{\xi}-(\eta-\underline{\eta})\cdot\zeta+\eta\cdot\underline{\eta} = \rho.$$

Also,

$$\begin{aligned}
\frac{1}{2}\overline{\mathcal{D}}\cdot\widehat{X}+\frac{1}{2}\widehat{X}\cdot\overline{Z} &= \frac{1}{2}\mathcal{D}\overline{trX}+\frac{1}{2}\overline{trX}Z-i\Im(trX)H-i\Im(tr\underline{X})\Xi-B,\\
\frac{1}{2}\overline{\mathcal{D}}\cdot\widehat{\underline{X}}-\frac{1}{2}\widehat{\underline{X}}\cdot\overline{Z} &= \frac{1}{2}\mathcal{D}\overline{tr\underline{X}}-\frac{1}{2}\overline{tr\underline{X}}Z-i\Im(tr\underline{X})\underline{H}-i\Im(trX)\underline{\Xi}+\underline{B},
\end{aligned}$$

and,

$$curl\,\zeta = -\frac{1}{2}\widehat{\chi}\wedge\widehat{\underline{\chi}}+\frac{1}{4}\big(tr\,\chi\,^{(a)}tr\underline{\chi}-tr\,\underline{\chi}\,^{(a)}tr\chi\big)+\omega\,^{(a)}tr\underline{\chi}-\underline{\omega}\,^{(a)}tr\chi+\,{}^*\rho.$$

We rewrite the Gauss equation in Proposition 2.1.43 for complex tensors.

Proposition 2.4.10. *The following identity holds true for $\Psi\in\mathfrak{s}_k(\mathbb{C})$ for $k=1,2$:*

$$[\nabla_a,\nabla_b]\Psi = \left(\frac{1}{2}(\,^{(a)}tr\chi\nabla_3+\,^{(a)}tr\underline{\chi}\nabla_4)\Psi-ik\,^{(h)}K\Psi\right)\in_{ab} \tag{2.4.7}$$

where

$$^{(h)}K = -\frac{1}{8}trX\overline{tr\underline{X}}-\frac{1}{8}tr\underline{X}\overline{trX}+\frac{1}{4}\widehat{X}\cdot\overline{\widehat{\underline{X}}}+\frac{1}{4}\overline{\widehat{X}}\cdot\widehat{\underline{X}}-\frac{1}{2}P-\frac{1}{2}\overline{P}.$$

The complex notations allow us to rewrite the Bianchi identities as follows.

Proposition 2.4.11. *We have,*

$$\begin{aligned}
\nabla_3 A - \frac{1}{2}\mathcal{D}\widehat{\otimes} B &= -\frac{1}{2}tr\underline{X} A + 4\underline{\omega} A + \frac{1}{2}(Z+4H)\widehat{\otimes} B - 3\overline{P}\widehat{X},\\
\nabla_4 B - \frac{1}{2}\overline{\mathcal{D}}\cdot A &= -2\overline{trX} B - 2\omega B + \frac{1}{2}A\cdot(\overline{2Z+\underline{H}}) + 3\overline{P}\,\Xi,\\
\nabla_3 B - \mathcal{D}\overline{P} &= -tr\underline{X} B + 2\underline{\omega} B + \underline{B}\cdot\widehat{X} + 3\overline{P} H + \frac{1}{2}A\cdot\underline{\Xi},\\
\nabla_4 P - \frac{1}{2}\mathcal{D}\cdot\overline{B} &= -\frac{3}{2}trX P + \frac{1}{2}(2\,\underline{H} + Z)\cdot\overline{B} - \Xi\cdot\underline{B} - \frac{1}{4}\widehat{\underline{X}}\cdot\overline{A},\\
\nabla_3 P + \frac{1}{2}\overline{\mathcal{D}}\cdot\underline{B} &= -\frac{3}{2}\overline{tr\underline{X}} P - \frac{1}{2}(\overline{2H-Z})\cdot\underline{B} + \underline{\Xi}\cdot\overline{B} - \frac{1}{4}\overline{\widehat{X}}\cdot\underline{A},\\
\nabla_4\underline{B} + \mathcal{D}P &= -trX\underline{B} + 2\omega\underline{B} + \overline{B}\cdot\widehat{\underline{X}} - 3P\,\underline{H} - \frac{1}{2}\underline{A}\cdot\Xi,\\
\nabla_3\underline{B} + \frac{1}{2}\overline{\mathcal{D}}\cdot\underline{A} &= -2\overline{tr\underline{X}}\,\underline{B} - 2\underline{\omega}\,\underline{B} - \frac{1}{2}\underline{A}\cdot(\overline{-2Z+H}) - 3P\,\underline{\Xi},\\
\nabla_4\underline{A} + \frac{1}{2}\mathcal{D}\widehat{\otimes}\underline{B} &= -\frac{1}{2}trX\underline{A} + 4\omega\underline{A} + \frac{1}{2}(Z - 4\,\underline{H})\widehat{\otimes}\underline{B} - 3P\widehat{\underline{X}}.
\end{aligned}$$

Proof. We derive the equations for A and B. Observe that from the Bianchi identity in Proposition 2.2.6,

$$\nabla_3\alpha - \nabla\widehat{\otimes}\beta = -\frac{1}{2}\big(\mathrm{tr}\,\underline{\chi}\alpha + {}^{(a)}\mathrm{tr}\underline{\chi}\,{}^*\alpha\big) + 4\underline{\omega}\alpha + (\zeta+4\eta)\widehat{\otimes}\beta - 3(\rho\widehat{\underline{\chi}} + {}^*\rho\,{}^*\widehat{\underline{\chi}}),$$

we obtain

$${}^*\nabla_3\alpha = {}^*(\nabla\widehat{\otimes}\beta) - \frac{1}{2}\big(\mathrm{tr}\,\underline{\chi}\,{}^*\alpha - {}^{(a)}\mathrm{tr}\underline{\chi}\alpha\big) + 4\underline{\omega}\,{}^*\alpha + {}^*((\zeta+4\eta)\widehat{\otimes}\beta) - 3(\rho\,{}^*\widehat{\underline{\chi}} - {}^*\rho\widehat{\underline{\chi}}).$$

This implies

$$\begin{aligned}
\nabla_3 A &= \nabla_3(\alpha + i\,{}^*\alpha)\\
&= \nabla\widehat{\otimes}\beta + i\,{}^*(\nabla\widehat{\otimes}\beta) - \frac{1}{2}\big(\mathrm{tr}\,\underline{\chi}\alpha + {}^{(a)}\mathrm{tr}\underline{\chi}\,{}^*\alpha\big) - \frac{1}{2}i\big(\mathrm{tr}\,\underline{\chi}\,{}^*\alpha - {}^{(a)}\mathrm{tr}\underline{\chi}\alpha\big) + 4\underline{\omega}(\alpha + i\,{}^*a)\\
&\quad +(\zeta+4\eta)\widehat{\otimes}\beta + i\,{}^*((\zeta+4\eta)\widehat{\otimes}\beta) - 3(\rho\widehat{\underline{\chi}} + {}^*\rho\,{}^*\widehat{\underline{\chi}}) - 3i(\rho\,{}^*\widehat{\underline{\chi}} - {}^*\rho\widehat{\underline{\chi}})\\
&= \frac{1}{2}\mathcal{D}\widehat{\otimes}(\beta + i\,{}^*\beta) - \frac{1}{2}(\mathrm{tr}\,\underline{\chi} - i\,{}^{(a)}\mathrm{tr}\underline{\chi})\alpha - \frac{1}{2}(\mathrm{tr}\,\underline{\chi} - i\,{}^{(a)}\mathrm{tr}\underline{\chi})i\,{}^*\alpha + 4\underline{\omega}(\alpha + i\,{}^*a)\\
&\quad +\frac{1}{2}((\zeta+4\eta+i\,{}^*(\zeta+4\eta))\widehat{\otimes}(\beta+i\,{}^*\beta)) - 3(\rho - i\,{}^*\rho)\widehat{\underline{\chi}} - 3(\rho - i\,{}^*\rho)i\,{}^*\widehat{\underline{\chi}}
\end{aligned}$$

which finally gives

$$\nabla_3 A = \frac{1}{2}\mathcal{D}\widehat{\otimes} B - \frac{1}{2}\mathrm{tr}\underline{X} A + 4\underline{\omega} A + \frac{1}{2}(Z+4H)\widehat{\otimes} B - 3\overline{P}\hat{\underline{X}},$$

as stated. From the equation

$$\nabla_4\beta - \mathrm{d}iv\,\alpha = -2(\mathrm{tr}\,\chi\beta - {}^{(a)}\mathrm{tr}\chi\,{}^*\beta) - 2\omega\beta + \alpha\cdot(2\zeta+\underline{\eta}) + 3(\xi\rho + {}^*\xi\,{}^*\rho),$$

we obtain

$${}^*\nabla_4\beta = {}^*\mathrm{d}iv\,\alpha - 2(\mathrm{tr}\,\chi\,{}^*\beta + {}^{(a)}\mathrm{tr}\chi\beta) - 2\omega\,{}^*\beta + {}^*(\alpha\cdot(2\zeta+\underline{\eta})) + 3({}^*\xi\rho - \xi\,{}^*\rho).$$

This implies

$$\begin{aligned}
\nabla_4 B &= \nabla_4(\beta + i\,{}^*\beta)\\
&= \mathrm{d}iv\,\alpha + i\,{}^*\mathrm{d}iv\,\alpha - 2(\mathrm{tr}\,\chi\beta - {}^{(a)}\mathrm{tr}\chi\,{}^*\beta) - 2i(\mathrm{tr}\,\chi\,{}^*\beta + {}^{(a)}\mathrm{tr}\chi\beta) - 2\omega(\beta + i\,{}^*\beta)\\
&\quad +\alpha\cdot(2\zeta+\underline{\eta}) + i\,{}^*(\alpha\cdot(2\zeta+\underline{\eta})) + 3(\xi\rho + {}^*\xi\,{}^*\rho) + 3i({}^*\xi\rho - \xi\,{}^*\rho)\\
&= \frac{1}{2}\overline{\mathcal{D}}\cdot(\alpha + i\,{}^*\alpha) - 2(\mathrm{tr}\,\chi + i\,{}^{(a)}\mathrm{tr}\chi)\beta - 2(\mathrm{tr}\,\chi + i\,{}^{(a)}\mathrm{tr}\chi)i\,{}^*\beta - 2\omega(\beta + i\,{}^*\beta)\\
&\quad +\frac{1}{2}(\alpha + i\,{}^*\alpha)\cdot(2\zeta + \underline{\eta} - i\,{}^*(2\zeta+\underline{\eta})) + 3(\rho - i\,{}^*\rho)\xi + 3(\rho - i\,{}^*\rho)i\,{}^*\xi
\end{aligned}$$

which finally gives

$$\nabla_4 B = \frac{1}{2}\overline{\mathcal{D}}\cdot A - 2\overline{\mathrm{tr}X}B - 2\omega B + \frac{1}{2}A\cdot(2\overline{Z} + \overline{\underline{H}}) + 3\overline{P}\Xi$$

as stated. The remaining equations are checked in the same fashion. □

2.4.3 Main complex equations using conformal derivatives

Definition 2.4.12. *We define the following conformal angular derivatives in the complex notation:*

- *For $a + ib \in \mathfrak{s}_0(\mathbb{C})$ we define*

$${}^{(c)}\mathcal{D}(a+ib) := \left({}^{(c)}\nabla + i\,{}^*\,{}^{(c)}\nabla\right)(a+ib).$$

- *For $f + i\,{}^*f \in \mathfrak{s}_1(\mathbb{C})$ we define*

$$\begin{aligned}
{}^{(c)}\mathcal{D}(f + i\,{}^*f) &:= \left({}^{(c)}\nabla + i\,{}^*\,{}^{(c)}\nabla\right)\cdot(f + i\,{}^*f),\\
{}^{(c)}\mathcal{D}\widehat{\otimes}(f + i\,{}^*f) &:= ({}^{(c)}\nabla + i\,{}^*\,{}^{(c)}\nabla)\widehat{\otimes}(f + i\,{}^*f).
\end{aligned}$$

- *For $u + i\,{}^*u \in \mathfrak{s}_2(\mathbb{C})$ we define*

$${}^{(c)}\mathcal{D}\cdot(u + i\,{}^*u) := \left({}^{(c)}\nabla + i\,{}^*\,{}^{(c)}\nabla\right)\cdot(u + i\,{}^*u).$$

- *In all the above cases we set*

$$\overline{{}^{(c)}\mathcal{D}} \quad := \quad {}^{(c)}\nabla - i\,{}^{(c)}\nabla.$$

These complex notations allow us to rewrite the null structure equations as follows.

Proposition 2.4.13. *We have*

$$\begin{aligned}
{}^{(c)}\nabla_3 tr\underline{X} + \frac{1}{2}(tr\underline{X})^2 &= {}^{(c)}\mathcal{D}\cdot\overline{\underline{\Xi}} + \underline{\Xi}\cdot\overline{\underline{H}} + \overline{\underline{\Xi}}\cdot H - \frac{1}{2}\underline{\widehat{X}}\cdot\overline{\underline{\widehat{X}}},\\
{}^{(c)}\nabla_3\underline{\widehat{X}} + \Re(tr\underline{X})\underline{\widehat{X}} &= \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\underline{\Xi} + \frac{1}{2}\underline{\Xi}\widehat{\otimes}(H + \underline{H}) - \underline{A},
\end{aligned}$$

$$\begin{aligned}
{}^{(c)}\nabla_3 trX + \frac{1}{2}tr\underline{X}trX &= {}^{(c)}\mathcal{D}\cdot\overline{H} + H\cdot\overline{H} + 2P + \underline{\Xi}\cdot\overline{\Xi} - \frac{1}{2}\underline{\widehat{X}}\cdot\overline{\widehat{X}},\\
{}^{(c)}\nabla_3\widehat{X} + \frac{1}{2}tr\underline{X}\,\widehat{X} &= \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}H + \frac{1}{2}H\widehat{\otimes}H - \frac{1}{2}\overline{trX}\underline{\widehat{X}} + \frac{1}{4}\underline{\Xi}\widehat{\otimes}\Xi,
\end{aligned}$$

$$\begin{aligned}
{}^{(c)}\nabla_4 tr\underline{X} + \frac{1}{2}trX\,tr\underline{X} &= {}^{(c)}\mathcal{D}\cdot\overline{\underline{H}} + \underline{H}\cdot\overline{\underline{H}} + 2\overline{P} + \Xi\cdot\overline{\underline{\Xi}} - \frac{1}{2}\widehat{X}\cdot\overline{\underline{\widehat{X}}},\\
{}^{(c)}\nabla_4\underline{\widehat{X}} + \frac{1}{2}trX\,\underline{\widehat{X}} &= \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\underline{H} + \frac{1}{2}\underline{H}\widehat{\otimes}\underline{H} - \frac{1}{2}\overline{tr\underline{X}}\widehat{X} + \frac{1}{4}\Xi\widehat{\otimes}\underline{\Xi},
\end{aligned}$$

$$\begin{aligned}
{}^{(c)}\nabla_4 trX + \frac{1}{2}(trX)^2 &= {}^{(c)}\mathcal{D}\cdot\overline{\Xi} + \Xi\cdot\overline{H} + \overline{\Xi}\cdot H - \frac{1}{2}\widehat{X}\cdot\overline{\widehat{X}},\\
{}^{(c)}\nabla_4\widehat{X} + \Re(trX)\widehat{X} &= \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\Xi + \frac{1}{2}\Xi\widehat{\otimes}(\underline{H} + H) - A,
\end{aligned}$$

$$\begin{aligned}
{}^{(c)}\nabla_3\underline{H} - {}^{(c)}\nabla_4\underline{\Xi} &= -\frac{1}{2}\overline{tr\underline{X}}(\underline{H} - H) - \frac{1}{2}\underline{\widehat{X}}\cdot(\overline{\underline{H}} - \overline{H}) + \underline{B},\\
{}^{(c)}\nabla_4 H - {}^{(c)}\nabla_3\Xi &= -\frac{1}{2}\overline{trX}(H - \underline{H}) - \frac{1}{2}\widehat{X}\cdot(\overline{H} - \overline{\underline{H}}) - B.
\end{aligned}$$

Also,

$$\begin{aligned}
\frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot\widehat{X} &= \frac{1}{2}\,{}^{(c)}\mathcal{D}\overline{trX} - i\Im(trX)H - i\Im(tr\underline{X})\Xi - B,\\
\frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot\underline{\widehat{X}} &= \frac{1}{2}\,{}^{(c)}\mathcal{D}\overline{tr\underline{X}} - i\Im(tr\underline{X})\,\underline{H} - i\Im(trX)\underline{\Xi} + \underline{B}.
\end{aligned}$$

The complex notations allow us to rewrite the Bianchi identities as follows.

Proposition 2.4.14. *We have*

$$\begin{aligned}
{}^{(c)}\nabla_3 A - \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes} B &= -\frac{1}{2}tr\underline{X}A + 2H\widehat{\otimes} B - 3\overline{P}\widehat{X},\\
{}^{(c)}\nabla_4 B - \frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot A &= -2\overline{trX}B + \frac{1}{2}A\cdot\overline{\underline{H}} + 3\overline{P}\,\Xi,\\
{}^{(c)}\nabla_3 B - {}^{(c)}\mathcal{D}\overline{P} &= -tr\underline{X}B + \overline{\underline{B}}\cdot\widehat{X} + 3\overline{P}H + \frac{1}{2}A\cdot\overline{\underline{\Xi}},\\
{}^{(c)}\nabla_4 P - \frac{1}{2}\,{}^{(c)}\mathcal{D}\cdot\overline{B} &= -\frac{3}{2}trXP + \underline{H}\cdot\overline{B} - \overline{\Xi}\cdot\underline{B} - \frac{1}{4}\underline{\widehat{X}}\cdot\overline{A},\\
{}^{(c)}\nabla_3 P + \frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot\underline{B} &= -\frac{3}{2}\overline{tr\underline{X}}P - \overline{H}\cdot\underline{B} + \underline{\Xi}\cdot\overline{B} - \frac{1}{4}\overline{\widehat{X}}\cdot\underline{A},\\
{}^{(c)}\nabla_4\underline{B} + {}^{(c)}\mathcal{D}P &= -trX\underline{B} + \overline{B}\cdot\underline{\widehat{X}} - 3P\,\underline{H} - \frac{1}{2}\underline{A}\cdot\overline{\Xi},\\
{}^{(c)}\nabla_3\underline{B} + \frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A} &= -2\overline{tr\underline{X}}\,\underline{B} - \frac{1}{2}\underline{A}\cdot\overline{H} - 3P\,\underline{\Xi},\\
{}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\underline{B} &= -\frac{1}{2}trX\underline{A} - 2\,\underline{H}\widehat{\otimes}\underline{B} - 3P\underline{\widehat{X}}.
\end{aligned}$$

2.4.4 Renormalized Bianchi identities

We define renormalized derivatives for the curvature components $A, B, P, \underline{B}, \underline{A}$ in the spirit of Definition 7.3.2 of [23]. These renormalizations play an important role in the derivation of the generalized Regge Wheeler equation for the quantity $\underline{\mathfrak{q}}$ in section 5.3.

Definition 2.4.15. *Given a conformally invariant curvature component Ψ (i.e. either $A, \underline{A}, B, \underline{B}, P$) with signature s we define the operators*

$$\begin{aligned}
\Psi_3 &:= {}^{(c)}\nabla_3\Psi + \frac{1}{2}\big(3-s\big)\overline{tr\underline{X}}\Psi,\\
\Psi_4 &:= {}^{(c)}\nabla_4\Psi + \frac{1}{2}\big(3+s\big)trX\Psi.
\end{aligned}$$

Proposition 2.4.16. *Using this definition the Bianchi identities take the form*[17]

$$\begin{aligned}
\overline{A}_3 - \frac{1}{2}\overline{{}^{(c)}\mathcal{D}\widehat{\otimes}B} &= 2\overline{H}\widehat{\otimes}\overline{B} - 3P\overline{\widehat{X}},\\
\overline{B}_4 - \frac{1}{2}\,{}^{(c)}\mathcal{D}\cdot\overline{A} &= \frac{1}{2}\overline{A}\cdot\underline{H} + 3P\,\overline{\Xi},\\
\overline{B}_3 - \overline{{}^{(c)}\mathcal{D}}P &= \underline{B}\cdot\overline{\widehat{X}} + 3P\overline{H} + \frac{1}{2}\overline{A}\cdot\underline{\Xi},\\
P_4 - \frac{1}{2}\,{}^{(c)}\mathcal{D}\cdot\overline{B} &= \underline{H}\cdot\overline{B} - \overline{\Xi}\cdot\underline{B} - \frac{1}{4}\underline{\widehat{X}}\cdot\overline{A},
\end{aligned}$$

[17]Note that B was changed to $\overline{B}$ to maintain the correct definition.

$$
\begin{aligned}
P_3 + \frac{1}{2}\overline{{}^{(c)}\mathcal{D}} \cdot \underline{B} &= -\overline{H} \cdot \underline{B} + \underline{\Xi} \cdot \overline{B} - \frac{1}{4}\overline{\widehat{X}} \cdot \underline{A}, \\
\underline{B}_4 + {}^{(c)}\mathcal{D}P &= \overline{B} \cdot \underline{\widehat{X}} - 3P\,\underline{H} - \frac{1}{2}\underline{A} \cdot \overline{\Xi}, \\
\underline{B}_3 + \frac{1}{2}\overline{{}^{(c)}\mathcal{D}} \cdot \underline{A} &= -\frac{1}{2}\underline{A} \cdot \overline{H} - 3P\,\underline{\Xi}, \\
\underline{A}_4 + \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\underline{B} &= -2\,\underline{H}\widehat{\otimes}\underline{B} - 3P\underline{\widehat{X}}.
\end{aligned}
$$

Chapter 3

The Kerr spacetime

In this chapter, we provide basic facts concerning the Kerr spacetime.

3.1 Boyer-Lindquist coordinates

We consider the Kerr metric in standard Boyer-Lindquist coordinates (t, r, θ, ϕ),

$$\mathbf{g}_{a,m} = -\frac{|q|^2\Delta}{\Sigma^2}(dt)^2 + \frac{\Sigma^2(\sin\theta)^2}{|q|^2}\left(d\phi - \frac{2amr}{\Sigma^2}dt\right)^2 + \frac{|q|^2}{\Delta}(dr)^2 + |q|^2(d\theta)^2,$$

where

$$q = r + ia\cos\theta, \tag{3.1.1}$$

and

$$\begin{cases} \Delta & = & r^2 - 2mr + a^2, \\ |q|^2 & = & r^2 + a^2(\cos\theta)^2, \\ \Sigma^2 & = & (r^2+a^2)|q|^2 + 2mra^2(\sin\theta)^2 = (r^2+a^2)^2 - a^2(\sin\theta)^2\Delta. \end{cases}$$

Observe that

$$(2mr - |q|^2)\Sigma^2 = -|q|^4\Delta + 4a^2m^2r^2(\sin\theta)^2.$$

The metric $\mathbf{g} = \mathbf{g}_{a,m}$ can also be written in the form

$$\mathbf{g} \quad = \quad -\frac{\left(\Delta - a^2\sin^2\theta\right)}{|q|^2}dt^2 - \frac{4amr}{|q|^2}\sin^2\theta dt d\phi + \frac{|q|^2}{\Delta}dr^2 + |q|^2 d\theta^2 + \frac{\Sigma^2}{|q|^2}\sin^2\theta d\phi^2.$$

Note that $\mathbf{g}_{tt}\mathbf{g}_{\phi\phi} - \mathbf{g}_{t\phi}^2 = -\Delta \sin^2\theta$ and that the non-vanishing components of the inverse metric are given by

$$\begin{aligned} \mathbf{g}^{00} &= -\frac{\Sigma^2}{|q|^2\Delta}, \qquad \mathbf{g}^{0\phi} = -\frac{2amr}{|q|^2\Delta}, \qquad \mathbf{g}^{\phi\phi} = \frac{\Delta - a^2\sin^2\theta}{|q|^2\Delta\sin^2\theta}, \\ \mathbf{g}^{rr} &= \frac{\Delta}{|q|^2}, \qquad \mathbf{g}^{\theta\theta} = \frac{1}{|q|^2}. \end{aligned} \tag{3.1.2}$$

The volume element $d\mu$ of $\mathbf{g}$ is given by

$$d\mu = |q|^2 \sin\theta dt dr d\theta d\phi, \qquad \sqrt{|g|} = |q|^2 \sin\theta.$$

We also note that

$$\mathbf{T} = \partial_t, \qquad \mathbf{Z} = \partial_\phi, \tag{3.1.3}$$

are both Killing and $\mathbf{T}$ is only time-like in the complement of the ergoregion, i.e. $|q|^2 > 2Mr$. The domain of outer communication of the Kerr metric is given by,

$$\mathcal{R} = \{(\theta, r, t, \phi) \in (0, \pi) \times (r_+, \infty) \times \mathbb{R} \times \mathbb{S}^1\},$$

where $r_+ := m + \sqrt{m^2 - a^2}$, the larger root of Δ, corresponds to the event horizon.

3.2 Vectorfields $\widehat{T}, \widehat{R}$

Definition 3.2.1. *We introduce the vectorfields $\widehat{T}, \widehat{R}$ as follows:*

$$\widehat{T} := \partial_t + \frac{a}{r^2 + a^2}\partial_\phi, \tag{3.2.1}$$

$$\widehat{R} := \frac{\Delta}{r^2 + a^2}\partial_r. \tag{3.2.2}$$

Note that $\widehat{R}$ is regular at the horizon, as opposed to ∂_r. Unlike $\mathbf{T}$, which is spacelike in the ergoregion, the vectorfield $\widehat{T}$, to which we refer as the Hawking vectorfield, is time-like in the domain of outer communication. More precisely we have

Proposition 3.2.2. *The vectorfield $\widehat{T}$ is timelike for $r > r_+$ and null on the horizon $r = r_+$. More precisely*

$$\mathbf{g}(\widehat{T}, \widehat{T}) = -\Delta \frac{|q|^2}{(r^2 + a^2)^2}. \tag{3.2.3}$$

Proof. The proof follows from the following more general computation.

Lemma 3.2.3. *The vectorfield $T_\lambda = \mathbf{T} + \lambda \mathbf{Z}$, for a scalar function λ, verifies*

$$\begin{aligned} \mathbf{g}(T_\lambda, T_\lambda) &= -\frac{\Delta}{|q|^2}\Big(1 + a^2\lambda^2(\sin\theta)^4\Big) + \frac{\sin^2\theta}{|q|^2} E_\lambda(r), \\ E_\lambda(r) &= a^2 - 4amr\lambda + \lambda^2(r^2+a^2)^2. \end{aligned}$$

To check (3.2.3) we replace λ by $\frac{a}{r^2+a^2}$ to deduce

$$\begin{aligned} \mathbf{g}(\widehat{T}, \widehat{T}) &= -\frac{\Delta}{|q|^2}\Big(1 + \frac{a^4}{(a^2+r^2)^2}(\sin\theta)^4\Big) + \frac{\sin^2\theta}{|q|^2}\Big(a^2 - 4amr\frac{a}{a^2+r^2} + \frac{a^2}{(a^2+r^2)^2}(r^2+a^2)^2\Big) \\ &= -\frac{\Delta}{|q|^2}\Big(1 + \frac{a^4}{(a^2+r^2)^2}(\sin\theta)^4\Big) + \frac{2a^2\sin^2\theta}{|q|^2}\Big(1 - 2mr\frac{1}{a^2+r^2}\Big) \\ &= -\frac{\Delta}{|q|^2}\Big(1 + \frac{a^4}{(a^2+r^2)^2}(\sin\theta)^4\Big) + \frac{2a^2\sin^2\theta}{|q|^2}\frac{\Delta}{a^2+r^2} \\ &= -\frac{\Delta}{|q|^2}\Big(1 + \frac{a^4}{(a^2+r^2)^2}(\sin\theta)^4 - 2\frac{a^2}{r^2+a^2}\sin^2\theta\Big) \\ &= -\frac{\Delta}{|q|^2}\Big(1 - \frac{a^2}{r^2+a^2}\sin^2\theta\Big)^2 = -\frac{\Delta}{|q|^2}\frac{|q|^4}{(r^2+a^2)^2} \\ &= -\Delta\frac{|q|^2}{(r^2+a^2)^2} \end{aligned}$$

as stated. To check Lemma 3.2.3 we write in BL coordinates $\mathbf{T} + \lambda\mathbf{Z} = \partial_t + \lambda\partial_\phi$. Hence

$$\begin{aligned} \mathbf{g}(\mathbf{T}+\lambda\mathbf{Z}, \mathbf{T}+\lambda\mathbf{Z}) &= \mathbf{g}(\partial_t + \lambda\partial_\phi, \partial_t + \lambda\partial_\phi) = \mathbf{g}_{tt} + 2\lambda\mathbf{g}_{t\phi} + \lambda^2\mathbf{g}_{\phi\phi} \\ &= -\frac{\big(\Delta - a^2\sin^2\theta\big)}{|q|^2} + 2\lambda\big(\frac{-2amr}{|q|^2}\sin^2\theta\big) + \lambda^2\frac{\Sigma^2}{|q|^2}\sin^2\theta \\ &= \frac{1}{|q|^2}\Big(- \Delta + a^2\sin^2\theta - 4amr\lambda\sin^2\theta + \lambda^2\Sigma^2\sin^2\theta\Big) \\ &= -\frac{\Delta}{|q|^2} + \frac{\sin^2\theta}{|q|^2}\Big(a^2 - 4amr\lambda + \lambda^2(r^2+a^2)^2 - a^2\lambda^2(\sin\theta)^2\Delta\Big) \\ &= -\frac{\Delta}{|q|^2}\Big(1 + a^2\lambda^2(\sin\theta)^4\Big) + \frac{\sin^2\theta}{|q|^2}\Big(a^2 - 4amr\lambda + \lambda^2(r^2+a^2)^2\Big) \\ &= -\frac{\Delta}{|q|^2}\Big(1 + a^2\lambda^2(\sin\theta)^4\Big) + \frac{\sin^2\theta}{|q|^2}E_\lambda(r), \end{aligned}$$

as stated. ☐

Remark 3.2.4. *As a consequence of the lemma, we also deduce that the Killing vectorfield $T_{\mathcal{H}} = \mathbf{T} + \omega_{\mathcal{H}}\mathbf{Z}$, with $\omega_{\mathcal{H}} = \frac{a}{r_+^2+a^2}$ the angular velocity of the horizon, is null on the horizon and timelike in a small neighborhood of it in $r > r_+$.*

3.3 Principal null frames

The Kerr metric is a spacetime of Petrov Type D, i.e. its Weyl curvature can be diagonalized with two linearly independent eigenvectors, the so-called principal null (PN) directions. We now present the ingoing PN frame $(e_4^{(in)}, e_3^{(in)})$ and the outgoing PN frame $(e_4^{(out)}, e_3^{(out)})$.

3.3.1 Ingoing PN frame

The ingoing PN frame (with $\mathbf{D}_3 e_3 = 0$), regular towards the future for all $r > 0$, is given by

$$\begin{aligned} e_4^{(in)} &= \frac{r^2+a^2}{|q|^2}\partial_t + \frac{\Delta}{|q|^2}\partial_r + \frac{a}{|q|^2}\partial_\phi, \\ e_3^{(in)} &= \frac{r^2+a^2}{\Delta}\partial_t - \partial_r + \frac{a}{\Delta}\partial_\phi. \end{aligned} \tag{3.3.1}$$

Note that

$$e_4^{(in)}(r) = \frac{\Delta}{|q|^2}, \qquad e_3^{(in)}(r) = -1. \tag{3.3.2}$$

We complete the PN frame with the following specific choice of horizontal frames e_1, e_2,

$$e_1 = \frac{1}{|q|}\partial_\theta, \qquad e_2 = \frac{a\sin\theta}{|q|}\partial_t + \frac{1}{|q|\sin\theta}\partial_\phi. \tag{3.3.3}$$

We refer to (3.3.3) as the *canonical horizontal basis* of Kerr.

Using the Hawking vectorfield $\widehat{T} = \partial_t + \frac{a}{r^2+a^2}\partial_\phi$, we have

$$e_4^{(in)} = \frac{r^2+a^2}{|q|^2}\left(\widehat{T} + \frac{\Delta}{r^2+a^2}\partial_r\right), \qquad e_3^{(in)} = \frac{r^2+a^2}{\Delta}\left(\widehat{T} - \frac{\Delta}{r^2+a^2}\partial_r\right) \tag{3.3.4}$$

from which we deduce

$$e_4^{(in)} = \frac{r^2+a^2}{|q|^2}\Big(\widehat{T} + \widehat{R}\Big), \qquad e_3^{(in)} = \frac{r^2+a^2}{\Delta}\Big(\widehat{T} - \widehat{R}\Big). \tag{3.3.5}$$

Lemma 3.3.1. *The following identities hold true.*

$$\begin{aligned} \widehat{T} &= \frac{1}{2}\left(\frac{|q|^2}{r^2+a^2}e_4^{(in)} + \frac{\Delta}{r^2+a^2}e_3^{(in)}\right), \\ \widehat{R} &= \frac{1}{2}\left(\frac{|q|^2}{r^2+a^2}e_4^{(in)} - \frac{\Delta}{r^2+a^2}e_3^{(in)}\right), \end{aligned} \tag{3.3.6}$$

and

$$\begin{aligned}
\mathbf{T} &= \frac{r^2+a^2}{|q|^2}\widehat{T} - \frac{a\sin\theta}{|q|}e_2,\\
\mathbf{Z} &= -\frac{a(r^2+a^2)\sin^2\theta}{|q|^2}\widehat{T} + \frac{(r^2+a^2)\sin\theta}{|q|}e_2.
\end{aligned} \tag{3.3.7}$$

Combining we derive

$$\begin{aligned}
\mathbf{T} &= \frac{1}{2}\left(e_4^{(in)} + \frac{\Delta}{|q|^2}e_3^{(in)}\right) - \frac{a\sin\theta}{|q|}e_2,\\
\mathbf{Z} &= \frac{(r^2+a^2)\sin\theta}{|q|}e_2 - \frac{1}{2}a\sin^2\theta\left(e_4^{(in)} + \frac{\Delta}{|q|^2}e_3^{(in)}\right).
\end{aligned} \tag{3.3.8}$$

Note also that $\widehat{T}, \widehat{R}$ are perpendicular to the horizontal structure and $\mathbf{g}(\widehat{T}, \widehat{R}) = 0$.

Proof. Straightforward verification. □

Ingoing Ricci and curvature coefficients

The real Ricci coefficients in the ingoing PN frame are given by

$$\begin{aligned}
&\widehat{\chi} = \widehat{\underline{\chi}} = \xi = \underline{\xi} = \underline{\omega} = 0,\\
&\mathrm{tr}\,\chi = \frac{2\Delta r}{|q|^4}, \quad {}^{(a)}\mathrm{tr}\chi = \frac{2a\Delta\cos\theta}{|q|^4}, \quad \mathrm{tr}\,\underline{\chi} = -\frac{2r}{|q|^2}, \quad {}^{(a)}\mathrm{tr}\underline{\chi} = \frac{2a\cos\theta}{|q|^2},\\
&\eta = \zeta, \quad \omega = -\frac{a^2\cos^2\theta(r-m) + mr^2 - a^2 r}{|q|^4} = -\frac{1}{2}\partial_r\left(\frac{\Delta}{|q|^2}\right).
\end{aligned}$$

Also, we have

$$\begin{aligned}
\eta_1 &= -\frac{a^2\sin\theta\cos\theta}{|q|^3}, \quad \eta_2 = \frac{ar\sin\theta}{|q|^3}, \quad {}^*\eta_1 = \frac{ar\sin\theta}{|q|^3}, \quad \eta_2 = \frac{a^2\sin\theta\cos\theta}{|q|^3},\\
\underline{\eta}_1 &= -\frac{a^2\sin\theta\cos\theta}{|q|^3}, \quad \underline{\eta}_2 = -\frac{ar\sin\theta}{|q|^3}, \quad {}^*\underline{\eta}_1 = -\frac{ar\sin\theta}{|q|^3}, \quad {}^*\underline{\eta}_2 = \frac{a^2\sin\theta\cos\theta}{|q|^3}.
\end{aligned}$$

The complex Ricci coefficients in the ingoing PN frame are given by

$$\widehat{X} = \widehat{\underline{X}} = \Xi = \underline{\Xi} = \underline{\omega} = 0, \quad \mathrm{tr}X = \frac{2\Delta\overline{q}}{|q|^4}, \quad \mathrm{tr}\underline{X} = -\frac{2}{\overline{q}}, \quad H = Z.$$

The real curvature components in any PN frame[1] are given by

$$\alpha = \beta = \underline{\beta} = \underline{\alpha} = 0, \quad \rho = -\frac{2m}{|q|^6}(r^3 - 3ra^2\cos^2\theta), \quad {}^*\rho = \frac{2am\cos\theta}{|q|^6}(3r^2 - a^2\cos^2\theta).$$

The complex curvature components are given by

$$A = B = \underline{B} = \underline{A} = 0, \quad P = -\frac{2m}{q^3}.$$

In any PN frame the components of $\mathbf{B}$ are given by, see Proposition 2.2.4,

$$\begin{aligned}
\mathbf{B}_{abc3} &= -\mathbf{B}_{ab3c} = -\mathrm{tr}\,\underline{\chi}\big(\delta_{ca}\eta_b - \delta_{cb}\eta_a\big) - {}^{(a)}\mathrm{tr}\underline{\chi}\big(\in_{ca}\eta_b - \in_{cb}\eta_a\big),\\
\mathbf{B}_{abc4} &= -\mathbf{B}_{ab4c} = -\mathrm{tr}\,\chi\big(\delta_{ca}\underline{\eta}_b - \delta_{cb}\underline{\eta}_a\big) - {}^{(a)}\mathrm{tr}\chi\big(\in_{ca}\underline{\eta}_b - \in_{cb}\underline{\eta}_a\big),\\
\mathbf{B}_{ab34} &= -\mathbf{B}_{ab43} = -4\big(\eta_a\underline{\eta}_b - \underline{\eta}_a\eta_b\big),\\
\mathbf{B}_{1212} &= -\mathbf{B}_{1221} = \mathbf{B}_{2121} = \frac{1}{2}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} + \frac{1}{2}{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi}.
\end{aligned} \tag{3.3.9}$$

Ingoing Eddington-Finkelstein coordinates

Let r_0 be a constant $r_0 > r_+$. We introduce the adapted ingoing Eddington-Finkelstein function $\underline{u}$ defined by[2]

$$\underline{u} \;=\; t + \int_{r_0}^{r} \frac{r'^2 + a^2}{\Delta(r')} dr'.$$

Note that

$$e_4^{(in)}(\underline{u}) = \frac{2(r^2+a^2)}{|q|^2}, \qquad e_3^{(in)}(\underline{u}) = 0, \qquad e_1(\underline{u}) = 0, \qquad e_2(\underline{u}) = \frac{a\sin\theta}{|q|}.$$

Remark 3.3.2. *Note that the non-vanishing of $e_2(\underline{u})$ in Kerr is connected with the lack of integrability of the null pair $(e_3^{(in)}, e_4^{(in)})$.*

Definition 3.3.3. *The principal null pair $(e_3^{(in)}, e_4^{(in)})$ together with the BL function r, such that $e_3^{(in)}(r) = 1$, is called the canonical, ingoing, principal geodesic structure (PG) of Kerr. The associated, ingoing, Eddington-Finkelstein coordinates $(\underline{u}, r, \theta, \varphi_+)$ are given by*

$$\underline{u} := t + f(r), \quad f'(r) = \frac{r^2+a^2}{\Delta}, \qquad \varphi_+ := \phi + h(r), \quad h'(r) = \frac{a}{\Delta},$$

such that,

$$e_3^{(in)}(r) = 1, \qquad e_3^{(in)}(\underline{u}) = e_3^{(in)}(\theta) = e_3^{(in)}(\varphi_+) = 0.$$

[1] By definition of principal null frame, $\alpha = \beta = \underline{\beta} = \underline{\alpha} = 0$; on the other hand, ρ and ${}^*\rho$ are gauge invariant quantities.

[2] Note that the choice of u and $\underline{u}$ is such that we have $u = \underline{u} = t$ on the timelike hypersurface $r = r_0$.

Calculations in the canonical horizontal basis

Remind that we call (3.3.3) the canonical horizontal basis of Kerr. Note that

$$e_1(r) = 0, \qquad e_2(r) = 0.$$

Also

$$\mathbf{g}(\mathbf{D}_4 e_1, e_2) \;=\; \chi_{12} = \frac{1}{2}\,{}^{(a)}\mathrm{tr}\chi, \qquad \mathbf{g}(\mathbf{D}_3 e_1, e_2) = \underline{\chi}_{12} = \frac{1}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi},$$

which gives

$$\nabla_4 e_1 \;=\; \frac{1}{2}\,{}^{(a)}\mathrm{tr}\chi e_2, \qquad \nabla_3 e_1 = \frac{1}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi} e_2. \tag{3.3.10}$$

Also

$$\begin{aligned}
(\Lambda_1)_{21} := \mathbf{g}(\mathbf{D}_1 e_1, e_2) &= 0,\\
(\Lambda_2)_{21} := \mathbf{g}(\mathbf{D}_2 e_1, e_2) &= \frac{r^2+a^2}{|q|^3}\cot\theta,\\
(\Lambda_1)_{12} := \mathbf{g}(\mathbf{D}_1 e_2, e_1) &= 0,\\
(\Lambda_2)_{12} := \mathbf{g}(\mathbf{D}_2 e_2, e_1) &= -\frac{r^2+a^2}{|q|^3}\cot\theta,
\end{aligned}$$

or

$$\nabla_{e_1} e_1 = \nabla_{e_1} e_2 = 0, \quad \nabla_{e_2} e_1 = \Lambda e_2, \quad \nabla_{e_2} e_2 = -\Lambda e_1, \quad \Lambda := \frac{r^2+a^2}{|q|^3}\cot\theta. \tag{3.3.11}$$

3.3.2 Outgoing PN frame

The outgoing PN frame (with $\mathbf{D}_4 e_4 = 0$), regular towards the future for all $r > r_+$, is given by

$$\begin{aligned}
e_4^{(out)} &= \frac{|q|^2}{\Delta} e_4^{(in)} = \frac{r^2+a^2}{\Delta}\partial_t + \partial_r + \frac{a}{\Delta}\partial_\phi,\\
e_3^{(out)} &= \frac{\Delta}{|q|^2} e_3^{(in)} = \frac{r^2+a^2}{|q|^2}\partial_t - \frac{\Delta}{|q|^2}\partial_r + \frac{a}{|q|^2}\partial_\phi,
\end{aligned} \tag{3.3.12}$$

with e_1, e_2 the canonical horizontal basis (3.3.3). Note that we have

$$e_4^{(out)}(r) = 1, \qquad e_3^{(out)}(r) = -\frac{\Delta}{|q|^2}, \qquad e_1(r) = 0, \qquad e_2(r) = 0.$$

Using the Hawking vectorfield $\widehat{T} = \partial_t + \frac{a}{r^2+a^2}\partial_\phi$ we have,

$$e_4^{(out)} = \frac{r^2+a^2}{\Delta}\left(\widehat{T} + \frac{\Delta}{r^2+a^2}\partial_r\right), \qquad e_3^{(out)} = \frac{r^2+a^2}{|q|^2}\left(\widehat{T} - \frac{\Delta}{r^2+a^2}\partial_r\right),$$

and, using also the definition of the vectorfield $\widehat{R}$,

$$e_4^{(out)} = \frac{r^2+a^2}{\Delta}\big(\widehat{T}+\widehat{R}\big), \qquad e_3^{(out)} = \frac{r^2+a^2}{|q|^2}\big(\widehat{T}-\widehat{R}\big). \tag{3.3.13}$$

Lemma 3.3.4. *The following identities hold true.*

$$\begin{aligned}\widehat{T} &= \frac{1}{2}\left(\frac{\Delta}{r^2+a^2}e_4^{(out)} + \frac{|q|^2}{r^2+a^2}e_3^{(out)}\right),\\ \widehat{R} &= \frac{1}{2}\left(\frac{\Delta}{r^2+a^2}e_4^{(out)} - \frac{|q|^2}{r^2+a^2}e_3^{(out)}\right).\end{aligned} \tag{3.3.14}$$

Also,

$$\begin{aligned}\mathbf{T} &= \frac{1}{2}\left(\frac{\Delta}{|q|^2}e_4^{(out)} + e_3^{(out)}\right) - \frac{a\sin\theta}{|q|}e_2,\\ \mathbf{Z} &= \frac{(r^2+a^2)\sin\theta}{|q|}e_2 - \frac{1}{2}a\sin^2\theta\left(\frac{\Delta}{|q|^2}e_4^{(out)} + e_3^{(out)}\right).\end{aligned} \tag{3.3.15}$$

Proof. Straightforward verification. □

The real Ricci coefficients in the outgoing PN frame are given by

$$\begin{aligned}&\widehat{\chi} = \widehat{\underline{\chi}} = \xi = \underline{\xi} = \omega = 0,\\ &\mathrm{tr}\,\chi = \frac{2r}{|q|^2}, \quad {}^{(a)}\mathrm{tr}\chi = \frac{2a\cos\theta}{|q|^2}, \quad \mathrm{tr}\,\underline{\chi} = -\frac{2r\Delta}{|q|^4}, \quad {}^{(a)}\mathrm{tr}\underline{\chi} = \frac{2a\Delta\cos\theta}{|q|^4},\\ &\underline{\eta} = -\zeta, \quad \underline{\omega} = \frac{a^2\cos^2\theta(r-m)+mr^2-a^2r}{|q|^4} = \frac{1}{2}\partial_r\left(\frac{\Delta}{|q|^2}\right).\end{aligned}$$

Also, we have

$$\begin{aligned}\eta_1 &= -\frac{a^2\sin\theta\cos\theta}{|q|^3}, \quad \eta_2 = \frac{ar\sin\theta}{|q|^3}, \quad {}^*\eta_1 = \frac{ar\sin\theta}{|q|^3}, \quad \eta_2 = \frac{a^2\sin\theta\cos\theta}{|q|^3},\\ \underline{\eta}_1 &= -\frac{a^2\sin\theta\cos\theta}{|q|^3}, \quad \underline{\eta}_2 = -\frac{ar\sin\theta}{|q|^3}, \quad {}^*\underline{\eta}_1 = -\frac{ar\sin\theta}{|q|^3}, \quad {}^*\underline{\eta}_2 = \frac{a^2\sin\theta\cos\theta}{|q|^3}.\end{aligned}$$

The complex Ricci coefficients in the outgoing PN frame are given by

$$\widehat{X} = \underline{\widehat{X}} = \Xi = \underline{\Xi} = \omega = 0, \quad \mathrm{tr}X = \frac{2}{q}, \quad \mathrm{tr}\underline{X} = -\frac{2\Delta q}{|q|^4}, \quad \underline{H} = -Z,$$

and

$$H_1 = \frac{aqi\sin\theta}{|q|^3}, \quad H_2 = \frac{aq\sin\theta}{|q|^3}, \quad Z_1 = \frac{a\overline{q}i\sin\theta}{|q|^3}, \qquad Z_2 = \frac{a\overline{q}\sin\theta}{|q|^3}.$$

Remark 3.3.5. *Note the identities*

$$H_1 = -\overline{Z_1}, \quad H_2 = \overline{Z_2}, \qquad H_1 = \overline{\underline{H}_1}, \quad H_2 = -\overline{\underline{H}_2}.$$

Outgoing Eddington-Finkelstein coordinates

Let r_0 be a constant $r_0 > r_+$. We introduce the adapted outgoing Eddington-Finkelstein function u defined by

$$u \;:=\; t - \int_{r_0}^{r} \frac{r'^2 + a^2}{\Delta(r')} dr'.$$

Note that

$$e_4^{(out)}(u) = 0, \quad e_3^{(out)}(u) = \frac{2(r^2+a^2)}{|q|^2}, \quad e_1(u) = 0, \quad e_2(u) = \frac{a\sin\theta}{|q|},$$

and

$$\mathbf{g}(\mathbf{D}u, \mathbf{D}u) = \frac{a^2\sin^2\theta}{|q|^2}.$$

Definition 3.3.6. *The principal null pair* $(e_3^{(out)}, e_4^{(out)})$ *together with the BL function* r, *such that* $e_4^{(out)}(r) = 1$, *is called the canonical, outgoing, PG structure of Kerr. The associated, outgoing, Eddington-Finkelstein coordinates* $(u, r, \theta, \varphi_-)$ *are given by*

$$u := t - f(r), \quad f'(r) = \frac{r^2+a^2}{\Delta}, \qquad \varphi_- := \phi - h(r), \quad h'(r) = \frac{a}{\Delta},$$

such that,

$$e_4^{(out)}(r) = 1, \qquad e_4^{(out)}(u) = e_4^{(out)}(\theta) = e_4^{(out)}(\varphi_-) = 0.$$

Lemma 3.3.7. *Relative to the outgoing Eddington-Finkelstein coordinates* $(u, r, \theta, \varphi_-)$ *we have:*

1. *The action of the outgoing PG frame on the coordinates* $(u, r, \theta, \varphi_-)$ *is given by*

$$\begin{array}{llll} e_4(r) = 1, & e_4(u) = 0, & e_4(\theta) = 0, & e_4(\varphi_-) = 0, \\ e_3(r) = -\frac{\Delta}{|q|^2}, & e_3(u) = \frac{2(r^2+a^2)}{|q|^2}, & e_3(\theta) = 0, & e_3(\varphi_-) = \frac{2a}{|q|^2}, \\ e_1(r) = 0, & e_1(u) = 0, & e_1(\theta) = \frac{1}{|q|}, & e_1(\varphi_-) = 0, \\ e_2(r) = 0, & e_2(u) = \frac{a\sin\theta}{|q|}, & e_2(\theta) = 0, & e_2(\varphi_-) = \frac{1}{|q|\sin\theta}. \end{array} \tag{3.3.16}$$

2. *In particular*

$$\begin{pmatrix} e_4 \\ e_3 \\ e_2 \\ e_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{\Delta}{|q|^2} & \frac{2(r^2+a^2)}{|q|^2} & 0 & \frac{2a}{|q|^2} \\ 0 & \frac{a\sin\theta}{|q|} & 0 & \frac{1}{|q|\sin\theta} \\ 0 & 0 & \frac{1}{|q|} & 0 \end{pmatrix} \begin{pmatrix} \partial_r \\ \partial_u \\ \partial_\theta \\ \partial_{\varphi_-} \end{pmatrix}.$$

3. *In the outgoing EF coordinates, the metric takes the form*

$$\begin{aligned} \mathbf{g} \; = \; & -\left(1-\frac{2mr}{|q|^2}\right)(du)^2 - 2drdu + 2a(\sin\theta)^2 dr d\varphi_- \\ & -\frac{4mra(\sin\theta)^2}{|q|^2} du d\varphi_- + |q|^2(d\theta)^2 + \frac{\Sigma^2(\sin\theta)^2}{|q|^2}(d\varphi_-)^2. \end{aligned}$$

Proof. Straightforward verification. □

3.4 Additional relations

Observe that, as a consequence of the null structure equations and Bianchi identities, in Kerr we have

$$\begin{aligned} \mathcal{D}P + 3P\,\underline{H} = 0, \qquad \mathcal{D}\overline{P} + 3\overline{P}H = 0, \\ \mathcal{D}\widehat{\otimes}\,\underline{H} + \underline{H}\widehat{\otimes}\,\underline{H} = 0, \qquad \mathcal{D}\widehat{\otimes}H + H\widehat{\otimes}H = 0, \end{aligned} \tag{3.4.1}$$

which are valid in any frame.

We also have

$$\begin{aligned} \mathrm{tr}\,\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} + \mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\chi &= 0, \\ |\eta|^2 - |\underline{\eta}|^2 &= 0, \\ \mathrm{d}iv\,(\eta - \underline{\eta}) &= 0, \\ \mathrm{d}iv\,({}^*\eta + {}^*\underline{\eta}) &= 0, \end{aligned} \tag{3.4.2}$$

which is immediate from the values given in the previous section.

We also have the following relations, valid in any frame:

$$\begin{aligned}
{}^{(a)}\mathrm{tr}\chi e_3 + {}^{(a)}\mathrm{tr}\underline{\chi} e_4 &= \frac{4a\cos\theta(r^2+a^2)}{|q|^4}\widehat{T},\\
{}^{(a)}\mathrm{tr}\chi e_3 + {}^{(a)}\mathrm{tr}\underline{\chi} e_4 + 2(\eta+\underline{\eta})\cdot {}^*\nabla &= \frac{4a\cos\theta}{|q|^2}\mathbf{T},\\
{}^{(a)}\mathrm{tr}\chi e_3 + {}^{(a)}\mathrm{tr}\underline{\chi} e_4 - 4\Lambda e_2 &= -\frac{4\cos\theta}{|q|^2\sin^2\theta}\mathbf{Z}.
\end{aligned} \tag{3.4.3}$$

Indeed, for example in the outgoing frame we have

$$\begin{aligned}
{}^{(a)}\mathrm{tr}\chi e_3^{(out)} + {}^{(a)}\mathrm{tr}\underline{\chi} e_4^{(out)} &= \frac{2a\cos\theta}{|q|^2}e_3^{(out)} + \frac{2a\Delta\cos\theta}{|q|^4}e_4^{(out)}\\
&= \frac{4a\cos\theta(r^2+a^2)}{|q|^4}\frac{1}{2}\left(\frac{\Delta}{r^2+a^2}e_4^{(out)} + \frac{|q|^2}{r^2+a^2}e_3^{(out)}\right)\\
&= \frac{4a\cos\theta(r^2+a^2)}{|q|^4}\widehat{T}.
\end{aligned}$$

Also,

$$\begin{aligned}
2(\eta+\underline{\eta})\cdot {}^*\nabla &= 2(\eta+\underline{\eta})_1\, {}^*e_1 + 2(\eta+\underline{\eta})_2\, {}^*e_2 = 2(\eta+\underline{\eta})_1 e_2\\
&= -\frac{4a^3\sin^2\theta\cos\theta}{|q|^4}\mathbf{T} - \frac{4a^2\cos\theta}{|q|^4}\mathbf{Z},
\end{aligned}$$

which gives

$$\begin{aligned}
&{}^{(a)}\mathrm{tr}\chi e_3^{(out)} + {}^{(a)}\mathrm{tr}\underline{\chi} e_4^{(out)} + 2(\eta+\underline{\eta})\cdot {}^*\nabla\\
= \;& \frac{4a\cos\theta(r^2+a^2)}{|q|^4}\big(\mathbf{T} + \frac{a}{r^2+a^2}\mathbf{Z}\big) - \frac{4a^3\sin^2\theta\cos\theta}{|q|^4}\mathbf{T} - \frac{4a^2\cos\theta}{|q|^4}\mathbf{Z} = \frac{4a\cos\theta}{|q|^2}\mathbf{T}.
\end{aligned}$$

We also have, since $\Lambda = \frac{r^2+a^2}{|q|^3}\cot\theta$,

$$\begin{aligned}
{}^{(a)}\mathrm{tr}\chi e_3 + {}^{(a)}\mathrm{tr}\underline{\chi} e_4 - 4\Lambda e_2 &= \frac{4a\cos\theta(r^2+a^2)}{|q|^4}\big(\mathbf{T} + \frac{a}{r^2+a^2}\mathbf{Z}\big)\\
&\quad -4\frac{r^2+a^2}{|q|^3}\cot\theta\big(\frac{a\sin\theta}{|q|}\mathbf{T} + \frac{1}{|q|\sin\theta}\mathbf{Z}\big)\\
&= \frac{4a^2\cos\theta}{|q|^4}\mathbf{Z} - 4\frac{r^2+a^2}{|q|^4}\frac{\cos\theta}{\sin^2\theta}\mathbf{Z} = -\frac{4\cos\theta}{|q|^2\sin^2\theta}\mathbf{Z}.
\end{aligned}$$

3.4.1 The scalar quantity q

Recall the definition (3.1.1) of $q = r + ia\cos\theta$. We have the following equations for q.

Lemma 3.4.1. *The scalar q satisfies for both the outgoing and incoming PN frames,*

$$\begin{aligned}\nabla_4 q = \frac{1}{2} trX q, \quad \nabla_3 q = \frac{1}{2}\overline{tr\underline{X}} q, \quad \mathcal{D} q = q\,\underline{H}, \quad \overline{\mathcal{D}} q = q\overline{H},\\ \nabla q = \frac{1}{2}(\,\underline{H} + \overline{H})q, \quad \nabla(|q|^2) = (\eta + \underline{\eta})|q|^2, \quad q\,\underline{H} = -\overline{q}H.\end{aligned} \tag{3.4.4}$$

In particular $|H|^2 = |\,\underline{H}|^2$.

Proof. In the outgoing frame, from the value of $\mathrm{tr}X = \frac{2}{q}$, and the reduced equation $\nabla_4 \mathrm{tr}X + \frac{1}{2}(\mathrm{tr}X)^2 = 0$ we deduce the equation for $\nabla_4 q$. From the value of $P = -\frac{2m}{q^3}$ and the reduced Bianchi identity $\nabla_3 P = -\frac{3}{2}\overline{\mathrm{tr}\underline{X}}P$ we deduce the equation for $\nabla_3 q$. Similarly, equation (3.4.1) becomes $\mathcal{D}\overline{q} = H\overline{q}$ or $\overline{\mathcal{D}} q = q\overline{H}$. The last equation in (3.4.4) follows in the same manner from (3.4.1). Similarly in the incoming frame.

We write, recalling that $q = r + ia\cos\theta$ and using $e_a(r) = 0$, i.e. $\mathcal{D}r = 0$,

$$\begin{aligned} q\,\underline{H} &= \mathcal{D}q = \mathcal{D}(r + ia\cos\theta) = \nabla(r + ia\cos\theta) + i\,{}^*\nabla(r + ia\cos\theta)\\ &= ia\nabla\cos\theta - a\,{}^*\nabla\cos\theta,\\ q\overline{H} &= \overline{\mathcal{D}}q = \overline{\mathcal{D}}(r + ia\cos\theta) = \nabla(r + ia\cos\theta) - i\,{}^*\nabla(r + ia\cos\theta)\\ &= ia\nabla\cos\theta + a\,{}^*\nabla\cos\theta.\end{aligned}$$

We deduce $\overline{q\,\underline{H}} = -q\overline{H}$ i.e. $q\,\underline{H} = -\overline{q}H$ as stated. □

3.4.2 The canonical complex 1-form $\mathfrak{J}$

Definition 3.4.2. *We define the following complex horizontal* 1*-tensor in Kerr, given in components relative to e_1, e_2 by*

$$\mathfrak{J}_1 = \frac{i\sin\theta}{|q|}, \qquad \mathfrak{J}_2 = \frac{\sin\theta}{|q|}.$$

Note that $\mathfrak{J}$ is regular (even at the axis), as well as anti-selfadjoint, i.e. ${}^*\mathfrak{J} = -i\mathfrak{J}$.

Remark 3.4.3. *The complex 1-tensor $\mathfrak{J}$ can also be written as*

$$\mathfrak{J} \;=\; j + i\,{}^*j$$

with

$$j_1 = -\,{}^*j_2 = 0, \qquad j_2 = \,{}^*j_1 = \frac{\sin\theta}{|q|}.$$

Using the form $\mathfrak{J}$ we can rewrite the expressions for H, $\underline{H}$ in the form

$$\underline{H} \quad = \quad -\frac{a}{q}\mathfrak{J} = -\frac{a\overline{q}}{|q|^2}\mathfrak{J}, \qquad H = \frac{a}{\overline{q}}\mathfrak{J} = \frac{aq}{|q|^2}\mathfrak{J}. \tag{3.4.5}$$

In particular,

$$q\,\underline{H} + \overline{q}H = 0,$$

as obtained in (3.4.4).

Lemma 3.4.4. *The complex* 1*-form* $\mathfrak{J}$ *verifies*

$${}^*\mathfrak{J} = -i\mathfrak{J}, \qquad \mathfrak{J}\cdot\overline{\mathfrak{J}} = \frac{2(\sin\theta)^2}{|q|^2}, \quad |\Re(\mathfrak{J})|^2 = \frac{(\sin\theta)^2}{|q|^2}.$$

Also

$$\nabla_4\mathfrak{J} + \frac{1}{2}trX\mathfrak{J} = 0, \qquad \nabla_3\mathfrak{J} + \frac{1}{2}tr\underline{X}\mathfrak{J} = 0, \tag{3.4.6}$$

or

$$\nabla_4(q\mathfrak{J}) = \nabla_3(\overline{q}\mathfrak{J}) = 0.$$

Also

$$\overline{\mathcal{D}}\cdot\mathfrak{J} = \frac{4i(r^2+a^2)\cos\theta}{|q|^4}, \qquad \mathcal{D}\widehat{\otimes}\mathfrak{J} = 0,$$

and

$$\mathcal{D}(q) = -a\mathfrak{J}, \qquad \mathcal{D}(\overline{q}) = a\mathfrak{J}.$$

Proof. Straightforward verification. One can check (3.4.6) using the complex structure equations in Kerr

$$\nabla_3\,\underline{H} = -\frac{1}{2}\overline{\mathrm{tr}\underline{X}}(\,\underline{H} - H), \qquad \nabla_4 H = -\frac{1}{2}\overline{\mathrm{tr}X}(H - \,\underline{H}),$$

and (3.4.4) together with the expressions (3.4.5) of H, $\underline{H}$ in terms of $\mathfrak{J}$. Note that (3.4.6) holds true in both the incoming and outgoing frame. In the outgoing frame it becomes

$$\nabla_4\mathfrak{J} = -\frac{1}{q}\mathfrak{J}, \qquad \nabla_3\mathfrak{J} = \frac{\Delta q}{|q|^4}\mathfrak{J}.$$

From the complex structure equations in Kerr (3.4.1) and (3.4.4), we deduce that $\mathcal{D}\widehat{\otimes}\mathfrak{J} = 0$. We then compute

$$\begin{aligned}
\frac{1}{2}\overline{\mathcal{D}}\cdot\mathfrak{J} &= \mathrm{d}iv\, j + icurl\, j = \nabla_1 j_1 + \nabla_2 j_2 + i(\nabla_1 j_2 - \nabla_2 j_1)\\
&= -j_{\nabla_1 1} - j_{\nabla_2 2} + i(e_1(j_2) - j_{\nabla_1 2} + j_{\nabla_2 1}) = i(e_1(j_2) + \Lambda j_2)\\
&= i\left(\frac{1}{|q|}\partial_\theta\left(\frac{\sin\theta}{|q|}\right) + \frac{r^2+a^2}{|q|^3}\cot\theta\frac{\sin\theta}{|q|}\right)\\
&= i\left(\frac{\cos\theta}{|q|^4}(r^2+a^2) + \frac{r^2+a^2}{|q|^4}\cos\theta\right) = 2i\frac{r^2+a^2}{|q|^4}\cos\theta,
\end{aligned}$$

as stated. □

Using the canonical form $\mathfrak{J}$, we can also deduce the following identities, which are not immediately implied by the null structure equations.

Lemma 3.4.5. *In Kerr, relative to any frame, the following relations hold true:*

$$\begin{aligned}
\nabla_4\underline{H} + trX\,\underline{H} &= 0, & \nabla_3 H + tr\underline{X}H &= 0,\\
\overline{trX}\,\underline{H} + trX H &= 0, & \overline{tr\underline{X}}H + tr\underline{X}\,\underline{H} &= 0,\\
{}^{(c)}\mathcal{D}trX + 2trX\,\underline{H} &= 0, & {}^{(c)}\mathcal{D}tr\underline{X} + 2tr\underline{X}H &= 0.
\end{aligned} \tag{3.4.7}$$

The above can also be written as

$$\begin{aligned}
\nabla_4\underline{\eta} + tr\,\chi\underline{\eta} + {}^{(a)}tr\chi\,{}^*\underline{\eta} &= 0, & \nabla_3\eta + tr\,\underline{\chi}\eta + {}^{(a)}tr\underline{\chi}\,{}^*\eta &= 0,\\
tr\,\chi(\eta+\underline{\eta}) + {}^{(a)}tr\chi({}^*\eta - {}^*\underline{\eta}) &= 0, & tr\underline{\chi}(\eta+\underline{\eta}) + {}^{(a)}tr\underline{\chi}({}^*\underline{\eta} - {}^*\eta) &= 0,
\end{aligned} \tag{3.4.8}$$

and

$$\begin{aligned}
{}^{(c)}\nabla tr\,\chi + {}^*\,{}^{(c)}\nabla\,{}^{(a)}tr\chi + 2tr\,\chi\underline{\eta} + 2\,{}^{(a)}tr\chi\,{}^*\underline{\eta} &= 0,\\
{}^{(c)}\nabla tr\,\underline{\chi} + {}^*\,{}^{(c)}\nabla\,{}^{(a)}tr\underline{\chi} + 2tr\,\underline{\chi}\eta + 2\,{}^{(a)}tr\underline{\chi}\,{}^*\eta &= 0.
\end{aligned} \tag{3.4.9}$$

We also have

$$\nabla tr\underline{X} = -\left(\frac{1}{2}\underline{H} + \frac{1}{2}\overline{H} + \overline{\underline{H}} + H\right)tr\underline{X},$$

which can be written as

$$\begin{aligned}\nabla(tr\underline{\chi}) &= -\frac{3}{2}tr\underline{\chi}\left(\underline{\eta}+\eta\right)-\frac{1}{2}{}^{(a)}tr\underline{\chi}\left({}^*\eta-{}^*\underline{\eta}\right),\\ \nabla({}^{(a)}tr\underline{\chi}) &= -\frac{3}{2}{}^{(a)}tr\underline{\chi}(\underline{\eta}+\eta)+\frac{1}{2}tr\underline{\chi}({}^*\eta-{}^*\underline{\eta}).\end{aligned} \tag{3.4.10}$$

Proof. One can check, using (3.4.5) and (3.4.6), that

$$\nabla_4(q\,\underline{H}) \;=\; -a\nabla_4\mathfrak{J}=\frac{1}{2}a\mathrm{tr}X\mathfrak{J}=-\frac{1}{2}\mathrm{tr}Xq\,\underline{H}.$$

On the other hand, using (3.4.4) we have

$$\nabla_4(q\,\underline{H}) \;=\; \frac{1}{2}\mathrm{tr}Xq\,\underline{H}+q\nabla_4\,\underline{H},$$

which implies the first relation. Similarly for H. Also, we have

$$\begin{aligned}\overline{\mathrm{tr}X}\,\underline{H} &= \frac{2}{\overline{q}}\left(-\frac{a\overline{q}}{|q|^2}\mathfrak{J}\right)=-\frac{2a}{|q|^2}\mathfrak{J},\\ \mathrm{tr}XH &= \frac{2}{q}\left(\frac{aq}{|q|^2}\mathfrak{J}\right)=\frac{2a}{|q|^2}\mathfrak{J},\end{aligned}$$

which implies the second relation, and similarly for $\mathrm{tr}\underline{X}$. Finally, using (3.4.4) we compute

$${}^{(c)}\mathcal{D}(\mathrm{tr}X) \;=\; \mathcal{D}\left(\frac{2}{q}\right)+\mathrm{tr}XZ=-\frac{2}{q^2}\mathcal{D}(q)+\frac{2a}{q^2}\mathfrak{J}=\frac{4a}{q^2}\mathfrak{J}=-2\mathrm{tr}X\,\underline{H},$$

as desired. Similarly for $\mathrm{tr}\underline{X}$. Taking the real parts of the expressions we obtain (3.4.8).

To obtain the last relation, we use that $\nabla q=\frac{1}{2}q(\,\underline{H}+\overline{H})$, and since $\mathrm{tr}\underline{X}=-\frac{2\Delta}{q\overline{q}^2}$ we have

$$\begin{aligned}\nabla\mathrm{tr}\underline{X} &= \frac{2\Delta}{q^2\overline{q}^2}\nabla q+\frac{4\Delta}{q\overline{q}^3}\nabla\overline{q}=\frac{2\Delta}{q\overline{q}^2}\frac{1}{2}(\,\underline{H}+\overline{H})+\frac{2\Delta}{q\overline{q}^2}(\,\overline{\underline{H}}+H)\\ &= \frac{2\Delta}{q\overline{q}^2}(\frac{1}{2}\,\underline{H}+\frac{1}{2}\overline{H}+\,\overline{\underline{H}}+H)=-\mathrm{tr}\underline{X}\left(\frac{1}{2}\,\underline{H}+\frac{1}{2}\overline{H}+\,\overline{\underline{H}}+H\right).\end{aligned}$$

By taking the real and the imaginary part we obtain the stated identities. □

3.5 Inverse Kerr metric and Killing tensors

We summarize here a computation to write the inverse of the Kerr metric which will be crucial in Chapter 8.

Lemma 3.5.1. *The inverse Kerr metric can be written in the form*

$$|q|^2\mathbf{g}^{\alpha\beta} = \Delta\partial_r^\alpha\partial_r^\beta + \frac{1}{\Delta}\mathcal{R}^{\alpha\beta} \tag{3.5.1}$$

with

$$\begin{aligned}\mathcal{R}^{\alpha\beta} &= -(r^2+a^2)^2\partial_t^\alpha\partial_t^\beta - 2a(r^2+a^2)\partial_t^{(\alpha}\partial_\phi^{\beta)} - a^2\partial_\phi^\alpha\partial_\phi^\beta + \Delta O^{\alpha\beta},\\ O^{\alpha\beta} &= \partial_\theta^\alpha\partial_\theta^\beta + \frac{1}{\sin^2\theta}\partial_\phi^\alpha\partial_\phi^\beta + 2a\partial_t^{(\alpha}\partial_\phi^{\beta)} + a^2\sin^2\theta\partial_t^\alpha\partial_t^\beta.\end{aligned} \tag{3.5.2}$$

Note that

$$\mathcal{R}^{\alpha\beta} = -(r^2+a^2)^2\widehat{T}^\alpha\widehat{T}^\beta + \Delta O^{\alpha\beta}, \qquad O^{\alpha\beta} = |q|^2\big(e_1^\alpha e_1^\beta + e_2^\alpha e_2^\beta\big), \tag{3.5.3}$$

thus the inverse metric can also be written in the form

$$|q|^2\mathbf{g}^{\alpha\beta} \;=\; \frac{(r^2+a^2)^2}{\Delta}\big(-\widehat{T}^\alpha\widehat{T}^\beta + \widehat{R}^\alpha\widehat{R}^\beta\big) + O^{\alpha\beta}. \tag{3.5.4}$$

Proof. From the expression of the Kerr metric, the inverse metric can be written in the form

$$|q|^2\mathbf{g}^{\alpha\beta} = \Delta\partial_r^\alpha\partial_r^\beta + \frac{1}{\Delta}\mathcal{R}^{\alpha\beta}$$

with

$$\mathcal{R}^{\alpha\beta} \;=\; -\Sigma^2\partial_t^\alpha\partial_t^\beta - 2amr\partial_t^\alpha\partial_\phi^\beta - 2amr\partial_\phi^\alpha\partial_t^\beta + \Delta\partial_\theta^\alpha\partial_\theta^\beta + \frac{\Delta - a^2\sin^2\theta}{\sin^2\theta}\partial_\phi^\alpha\partial_\phi^\beta$$

which establishes (3.5.2). According to the definition (3.2.1) of $\widehat{T}$, we can write

$$\begin{aligned}\mathcal{R}^{\alpha\beta} &= -(r^2+a^2)^2\left(\partial_t^\alpha\partial_t^\beta + \frac{2a}{r^2+a^2}\partial_t^{(\alpha}\partial_\phi^{\beta)} + \frac{a^2}{(r^2+a^2)^2}\partial_\phi^\alpha\partial_\phi^\beta\right) + \Delta O^{\alpha\beta}\\ &= -(r^2+a^2)^2\widehat{T}^\alpha\widehat{T}^\beta + \Delta O^{\alpha\beta}\end{aligned}$$

which establishes the first expression in (3.5.3). Finally the second expression (3.5.3) can be easily checked from the expressions of e_1, e_2 in terms of the BL coordinates in (3.3.12). □

Remark 3.5.2. *Observe that in [4], Andersson-Blue use instead the following expression for $\mathcal{R}$:*

$$\mathcal{R}^{\alpha\beta} = -(r^2+a^2)^2\partial_t^\alpha\partial_t^\beta - 4amr\partial_t^{(\alpha}\partial_\phi^{\beta)} + (\Delta - a^2)\partial_\phi^\alpha\partial_\phi^\beta + \Delta Q^{\alpha\beta} \tag{3.5.5}$$

where

$$Q^{\alpha\beta} \;=\; O^{\alpha\beta} - 2a\partial_t^\alpha\partial_\phi^\beta - \partial_\phi^\alpha\partial_\phi^\beta = \partial_\theta^\alpha\partial_\theta^\beta + \frac{\cos^2\theta}{\sin^2\theta}\partial_\phi^\alpha\partial_\phi^\beta + a^2\sin^2\theta\partial_t^\alpha\partial_t^\beta. \tag{3.5.6}$$

The relevance of the decomposition of the metric in (3.5.1) is in the fact that the operator $\mathcal{R}^{\alpha\beta}$ can be written in terms of $\partial_t^\alpha \partial_t^\beta$, $a\partial_t^{(\alpha}\partial_\phi^{\beta)}$, $a^2\partial_\phi^\alpha\partial_\phi^\beta$ and $O^{\alpha\beta}$.

Definition 3.5.3. *We define the following symmetric spacetime 2-tensors*

$$\begin{aligned} S_1^{\alpha\beta} &:= \mathbf{T}^\alpha\mathbf{T}^\beta = \partial_t^\alpha\partial_t^\beta, \\ S_2^{\alpha\beta} &:= a\mathbf{T}^{(\alpha}\mathbf{Z}^{\beta)} = a\partial_t^{(\alpha}\partial_\phi^{\beta)}, \\ S_3^{\alpha\beta} &:= a^2\mathbf{Z}^\alpha\mathbf{Z}^\beta = a^2\partial_\phi^\alpha\partial_\phi^\beta, \\ S_4^{\alpha\beta} &:= O^{\alpha\beta} = |q|^2\big(e_1^\alpha e_1^\beta + e_2^\alpha e_2^\beta\big). \end{aligned}$$

We denote the set of the above tensors as $S_{\underline{a}}$, for $\underline{a} = 1, 2, 3, 4$.

Remark 3.5.4. *Observe that the tensor S_2 and S_3 are defined with a factor of a and a^2 respectively. The reason for this choice, which differs from the definition in [4], is our application in Chapter 8 of the method to 2-tensors as opposed to scalars. Note also that $S_1 - S_3$ are Killing tensor while S_4 is related to the Carter tensor K, see section 3.7.*

With the above definition, from (3.5.2) we write

$$\mathcal{R}^{\alpha\beta} = -(r^2+a^2)^2 S_1^{\alpha\beta} - 2(r^2+a^2)S_2^{\alpha\beta} - S_3^{\alpha\beta} + \Delta S_4^{\alpha\beta}.$$

More compactly, using the repetition in $\underline{a}$ to signify summation over $\underline{a} = 1, 2, 3, 4$, we denote

$$\mathcal{R}^{\alpha\beta} = \mathcal{R}^{\underline{a}} S_{\underline{a}}^{\alpha\beta}, \tag{3.5.7}$$

with $\mathcal{R}^{\underline{a}}$, $\underline{a} = 1, 2, 3, 4$, given by

$$\mathcal{R}^1 = -(r^2+a^2)^2, \quad \mathcal{R}^2 = -2(r^2+a^2), \quad \mathcal{R}^3 = -1, \quad \mathcal{R}^4 = \Delta. \tag{3.5.8}$$

3.6 Commutation properties for T and Z

We start by collecting the following commutation property between $\nabla_{\mathbf{T}}$ and $\nabla_{\mathbf{Z}}$.

Lemma 3.6.1. *In Kerr spacetime, for $\psi \in \mathfrak{s}_2$ we have*

$$[\nabla_{\mathbf{T}}, \nabla_{\mathbf{Z}}]\psi = 0.$$

Proof. Recall, see Corollary 2.1.29, that for X and Y vectorfield

$$\big(\nabla_X\nabla_Y-\nabla_Y\nabla_X\big)\psi_{ab}=\nabla_{[X,Y]}\psi_{ab}+X^\mu Y^\nu(\dot{\mathbf{R}}_{ac\mu\nu}\psi^c{}_b+\dot{\mathbf{R}}_{bc\mu\nu}\psi^c{}_a).$$

Since $[\mathbf{T},\mathbf{Z}]=0$, we obtain

$$\big(\nabla_{\mathbf{T}}\nabla_{\mathbf{Z}}-\nabla_{\mathbf{Z}}\nabla_{\mathbf{T}}\big)\psi_{ab}=\mathbf{T}^\mu\mathbf{Z}^\nu(\dot{\mathbf{R}}_{ac\mu\nu}\psi^c{}_b+\dot{\mathbf{R}}_{bc\mu\nu}\psi^c{}_a),$$

with $\dot{\mathbf{R}}_{ab\mu\nu}=\mathbf{R}_{ab\mu\nu}+\frac{1}{2}\mathbf{B}_{ab\mu\nu}$. Using that the only non-vanishing Riemann curvature terms are

$$\mathbf{R}_{a3b4}=-\rho\delta_{ab}+{}^*\rho\in_{ab},\qquad \mathbf{R}_{ab34}=2\in_{ab}{}^*\rho,\qquad \mathbf{R}_{abcd}=-\in_{ab}\in_{cd}\rho,$$

we obtain, using from (3.3.8) that $\mathbf{T}^3\mathbf{Z}^4=\mathbf{T}^4\mathbf{Z}^3$,

$$\begin{aligned}\mathbf{T}^\mu\mathbf{Z}^\nu\mathbf{R}_{ac\mu\nu}\psi^c{}_b &= \mathbf{T}^d\mathbf{Z}^e\mathbf{R}_{acde}\psi^c{}_b+\mathbf{T}^3\mathbf{Z}^4\mathbf{R}_{ac34}\psi^c{}_b+\mathbf{T}^4\mathbf{Z}^3\mathbf{R}_{ac43}\psi^c{}_b\\ &= -2\rho\in_{de}\mathbf{T}^d\mathbf{Z}^e\,{}^*\psi_{ab}+4\,{}^*\rho(\mathbf{T}^3\mathbf{Z}^4-\mathbf{T}^4\mathbf{Z}^3)\,{}^*\psi_{ab}=0,\end{aligned}$$

and similarly $\mathbf{T}^\mu\mathbf{Z}^\nu\mathbf{R}_{bc\mu\nu}\psi^c{}_a=0$. We also have

$$\begin{aligned}\mathbf{T}^\mu\mathbf{Z}^\nu\mathbf{B}_{ac\mu\nu}\psi^c{}_b &= \mathbf{T}^d\mathbf{Z}^e\mathbf{B}_{acde}\psi^c{}_b+\mathbf{T}^d\mathbf{Z}^3\mathbf{B}_{acd3}\psi^c{}_b+\mathbf{T}^d\mathbf{Z}^4\mathbf{B}_{acd4}\psi^c{}_b\\ &\quad+\mathbf{T}^3\mathbf{Z}^d\mathbf{B}_{ac3d}\psi^c{}_b+\mathbf{T}^3\mathbf{Z}^4\mathbf{B}_{ac34}\psi^c{}_b+\mathbf{T}^4\mathbf{Z}^d\mathbf{B}_{ac4d}\psi^c{}_b+\mathbf{T}^4\mathbf{Z}^3\mathbf{B}_{ac43}\psi^c{}_b\\ &= (\mathbf{T}^d\mathbf{Z}^3-\mathbf{T}^3\mathbf{Z}^d)\mathbf{B}_{acd3}\psi^c{}_b+(\mathbf{T}^d\mathbf{Z}^4-\mathbf{T}^4\mathbf{Z}^d)\mathbf{B}_{acd4}\psi^c{}_b\\ &= (\mathbf{T}^d\mathbf{Z}^3-\mathbf{T}^3\mathbf{Z}^d)\Big(-\mathrm{tr}\,\underline{\chi}\big(\delta_{da}\eta_c-\delta_{dc}\eta_a\big)-{}^{(a)}\mathrm{tr}\underline{\chi}\big(\in_{da}\eta_c-\in_{dc}\eta_a\big)\Big)\psi^c{}_b\\ &\quad+(\mathbf{T}^d\mathbf{Z}^4-\mathbf{T}^4\mathbf{Z}^d)\Big(-\mathrm{tr}\,\chi\big(\delta_{da}\underline{\eta}_c-\delta_{dc}\underline{\eta}_a\big)-{}^{(a)}\mathrm{tr}\chi\big(\in_{da}\underline{\eta}_c-\in_{dc}\underline{\eta}_a\big)\Big)\psi^c{}_b.\end{aligned}$$

Using that, in the ingoing frame,

$$\begin{aligned}&\mathbf{T}^2\mathbf{Z}^3-\mathbf{T}^3\mathbf{Z}^2=-\frac{\Delta\sin\theta}{2|q|},\qquad \mathbf{T}^2\mathbf{Z}^4-\mathbf{T}^4\mathbf{Z}^2=-\frac{\sin\theta}{2}|q|,\\ &\mathrm{tr}\,\chi=\frac{2\Delta r}{|q|^4},\quad {}^{(a)}\mathrm{tr}\chi=\frac{2a\Delta\cos\theta}{|q|^4},\quad \mathrm{tr}\,\underline{\chi}=-\frac{2r}{|q|^2},\quad {}^{(a)}\mathrm{tr}\underline{\chi}=\frac{2a\cos\theta}{|q|^2},\end{aligned}$$

we have

$$\begin{aligned}\mathbf{T}^\mu\mathbf{Z}^\nu\mathbf{B}_{ac\mu\nu}\psi^c{}_b &= -\frac{\Delta r\sin\theta}{|q|^3}\big(\delta_{2a}(\eta_c-\underline{\eta}_c)-\delta_{2c}(\eta_a-\underline{\eta}_a)\big)\psi^c{}_b\\ &\quad+\frac{\Delta a\cos\theta\sin\theta}{|q|^3}\big(\in_{2a}(\eta_c+\underline{\eta}_c)-\in_{2c}(\eta_a+\underline{\eta}_a)\big)\psi^c{}_b.\end{aligned}$$

In components, recalling that $\eta_1 - \underline{\eta}_1 = 0$, $\eta_2 + \underline{\eta}_2 = 0$, this gives

$$\begin{aligned}
\mathbf{T}^\mu \mathbf{Z}^\nu \mathbf{B}_{1c\mu\nu}\psi^c{}_b &= \frac{\Delta a\cos\theta\sin\theta}{|q|^3}\big(\in_{21} (\eta_c + \underline{\eta}_c)\psi^c{}_b - \in_{2c} (\eta_1 + \underline{\eta}_1)\psi^c{}_b\big)\\
&= \frac{\Delta a\cos\theta\sin\theta}{|q|^3}\big(\in_{21} (\eta_1 + \underline{\eta}_1)\psi^1{}_b - \in_{21} (\eta_1 + \underline{\eta}_1)\psi^1{}_b\big) = 0,\\
\mathbf{T}^\mu \mathbf{Z}^\nu \mathbf{B}_{2c\mu\nu}\psi^c{}_b &= -\frac{\Delta r\sin\theta}{|q|^3}\big((\eta_c - \underline{\eta}_c)\psi^c{}_b - \delta_{2c}(\eta_2 - \underline{\eta}_2)\psi^c{}_b\big)\\
&= -\frac{\Delta r\sin\theta}{|q|^3}\big((\eta_2 - \underline{\eta}_2)\psi^2{}_b - (\eta_2 - \underline{\eta}_2)\psi^2{}_b\big) = 0.
\end{aligned}$$

We therefore infer $\mathbf{T}^\mu\mathbf{Z}^\nu(\dot{\mathbf{R}}_{ac\mu\nu}\psi^c{}_b + \dot{\mathbf{R}}_{bc\mu\nu}\psi^c{}_a) = 0$, proving the lemma. For a different proof in perturbations of Kerr spacetime, see also Corollary 9.2.2. $\square$

As Killing vectorfields, **T** and **Z** also have favorable properties regarding commutation with the D'Alembertian operator $\square_{\mathbf{g}}$ for scalars. Nevertheless, the commutation with the D'Alembertian operator $\dot{\square}_2$ for tensors presents in addition terms involving the Riemann curvature. In the case of **Z**, those term do not vanish even in Schwarzschild.

Proposition 3.6.2. *The first order differential operators $\nabla_{\mathbf{T}}$ and $\nabla_{\mathbf{Z}}$ satisfy the following commutation relations with $\dot{\square}_2$ for $\psi \in \mathfrak{s}_2$:*

$$\begin{aligned}
[\nabla_{\mathbf{T}}, \dot{\square}_2]\psi &= O(ar^{-4})\mathfrak{d}^{\leq 1}\psi,\\
[\nabla_{\mathbf{Z}}, \dot{\square}_2]\psi &= O(mr^{-2})\mathfrak{d}^{\leq 1}\psi.
\end{aligned}$$

Proof. We first derive the following general computation, which specializes Lemma 2.3.3 to the case of Kerr spacetime.

Lemma 3.6.3. *In Kerr spacetime we have for $\psi \in \mathfrak{s}_2$:*

$$\begin{aligned}
[\dot{\square}_2, \nabla_X]\psi &= \pi^{\mu\nu}\dot{\mathbf{D}}_\mu\dot{\mathbf{D}}_\nu\psi + \left(\mathbf{D}^\mu\pi_\mu{}^\beta - \frac{1}{2}\mathbf{D}^\beta tr\pi\right)\dot{\mathbf{D}}_\beta\psi\\
&\quad + O(amr^{-4})\big(X^3\nabla_3 - X^4\nabla_4\big)\ {}^*\psi + O(mr^{-3})X^a\nabla\ {}^*\psi\\
&\quad + O(amr^{-4})\big(\mathbf{D}^3X^4 - \mathbf{D}^4X^3\big)\ {}^*\psi + O(mr^{-3}) \in_{ab} \mathbf{D}^aX^b\ {}^*\psi\\
&\quad + O(ar^{-3})(X^3 + X^4)\big(\nabla\ {}^*\psi + r^{-1}\ {}^*\psi\big) + O(r^{-2})X^a\big(\nabla\ {}^*\psi + r^{-1}\ {}^*\psi\big)\\
&\quad + O(ar^{-3})(\mathbf{D}^3X^d + \mathbf{D}^4X^d)\ {}^*\psi.
\end{aligned}$$

Proof. According to Lemma 2.3.3 we have for $\psi \in \mathfrak{s}_2$,

$$
\begin{aligned}
[\dot{\square}_2, \nabla_X]\psi_{ab} &= \pi^{\mu\nu}\dot{\mathbf{D}}_\mu\dot{\mathbf{D}}_\nu\psi_{ab} + \left(\mathbf{D}^\mu\pi_\mu{}^\beta - \frac{1}{2}\mathbf{D}^\beta\mathrm{tr}\pi\right)\dot{\mathbf{D}}_\beta\psi_{ab}\\
&-2X^\beta\mathbf{R}_{ac\beta\mu}\dot{\mathbf{D}}^\mu\psi_{cb} - 2X^\beta\mathbf{R}_{bc\beta\mu}\dot{\mathbf{D}}^\mu\psi_{ac} + \mathbf{D}^\beta X^\mu\mathbf{R}_{ac\beta\mu}\psi^{cb} + \mathbf{D}^\beta X^\mu\mathbf{R}_{bc\beta\mu}\psi^{ac}\\
&-X^\beta\mathbf{B}_{ac\beta\mu}\dot{\mathbf{D}}^\mu\psi_{cb} - X^\beta\mathbf{B}_{bc\beta\mu}\dot{\mathbf{D}}^\mu\psi_{ac} + \frac{1}{2}\mathbf{D}^\beta X^\mu\mathbf{B}_{ac\beta\mu}\psi^{cb} + \frac{1}{2}\mathbf{D}^\beta X^\mu\mathbf{B}_{bc\beta\mu}\psi^{ac}\\
&+\frac{1}{2}X^\beta\mathbf{D}^\mu\mathbf{B}_{ac\mu\beta}\psi^{cb} + \frac{1}{2}X^\beta\mathbf{D}^\mu\mathbf{B}_{bc\mu\beta}\psi^{ac}.
\end{aligned}
$$

Using that the only non-vanishing Riemann curvature terms are

$$
\begin{aligned}
\mathbf{R}_{a3b4} &= O(mr^{-3})\delta_{ab} + O(amr^{-4})\in_{ab}, \qquad \mathbf{R}_{ab34} = O(amr^{-4})\in_{ab},\\
\mathbf{R}_{abcd} &= O(mr^{-3})\in_{ab}\in_{cd},
\end{aligned}
$$

and the only non-vanishing $\mathbf{B}$ terms are

$$
\mathbf{B}_{abc3} = O(ar^{-3}), \qquad \mathbf{B}_{abc4} = O(ar^{-3}), \qquad \mathbf{B}_{ab34} = O(a^2r^{-4}), \qquad \mathbf{B}_{1212} = O(r^{-2}),
$$

we have

$$
\begin{aligned}
[\dot{\square}_2, \nabla_X]\psi &= \pi^{\mu\nu}\dot{\mathbf{D}}_\mu\dot{\mathbf{D}}_\nu\psi + \left(\mathbf{D}^\mu\pi_\mu{}^\beta - \frac{1}{2}\mathbf{D}^\beta\mathrm{tr}\pi\right)\dot{\mathbf{D}}_\beta\psi\\
&+O(amr^{-4})\big(X^3\nabla_3 - X^4\nabla_4\big)\,{}^*\psi + O(mr^{-3})X^a\nabla\,{}^*\psi\\
&+O(amr^{-4})\big(\mathbf{D}^3X^4 - \mathbf{D}^4X^3\big)\,{}^*\psi + O(mr^{-3})\in_{ab}\mathbf{D}^aX^b\,{}^*\psi\\
&+O(ar^{-3})(X^3+X^4)\big(\nabla\,{}^*\psi + r^{-1}\,{}^*\psi\big) + O(r^{-2})X^a\big(\nabla\,{}^*\psi + r^{-1}\,{}^*\psi\big)\\
&+O(ar^{-3})(\mathbf{D}^3X^d + \mathbf{D}^4X^d)\,{}^*\psi,
\end{aligned}
$$

as stated. This concludes the proof of Lemma 3.6.3. □

Using Lemma 3.6.3, with vanishing ${}^{(\mathbf{T})}\pi$ and ${}^{(\mathbf{Z})}\pi$, and writing that $\mathbf{T}^3\nabla_3 - \mathbf{T}^4\nabla_4 = \nabla_{\widehat{R}}$ and $\mathbf{Z}^3\nabla_3 - \mathbf{Z}^4\nabla_4 = a\nabla_{\widehat{R}}$, and $\mathbf{T}^a = O(ar^{-1})$, $\mathbf{Z}^a = O(r)$, we obtain

$$
\begin{aligned}
[\nabla_{\mathbf{T}}, \dot{\square}_2]\psi &= O(amr^{-4})\nabla_{\widehat{R}}\,{}^*\psi + O(amr^{-4})\nabla\,{}^*\psi + O(amr^{-5})\,{}^*\psi\\
&+O(ar^{-3})\big(\nabla\,{}^*\psi + r^{-1}\,{}^*\psi\big),
\end{aligned}
$$

and

$$
\begin{aligned}
[\nabla_{\mathbf{Z}}, \dot{\square}_2]\psi &= O(a^2mr^{-4})\nabla_{\widehat{R}}\,{}^*\psi + O(mr^{-2})\nabla\,{}^*\psi + O(amr^{-5})\,{}^*\psi\\
&+O(r^{-1})\big(\nabla\,{}^*\psi + r^{-1}\,{}^*\psi\big),
\end{aligned}
$$

which can be schematically written as stated. This concludes the proof of Proposition 3.6.2. □

3.7 Carter tensor and Carter operator

Recall, see Definition 2.3.4, that a Killing 2-tensor K is a symmetric 2-tensor satisfying $\mathbf{D}_{(\mu}K_{\alpha\beta)} = 0$. One of the fundamental properties of the Kerr metric is that it admits a Killing tensor K which cannot be written in terms of the Killing vectorfields $\mathbf{T}$ and $\mathbf{Z}$. This 2-tensor is called the Carter tensor [13].

Definition 3.7.1 (Carter tensor)**.** *In Kerr spacetime, the Carter tensor is defined as*

$$K^{\alpha\beta} = -(a^2\cos^2\theta)\mathbf{g}^{\alpha\beta} + O^{\alpha\beta} \tag{3.7.1}$$

where the tensor O is defined in (3.5.2)*.*

Note that the only non-vanishing components of K are:

$$K_{ab} = r^2\delta_{ab}, \qquad K_{34} = 2a^2\cos^2\theta.$$

Proposition 3.7.2. *The Carter tensor defined in* (3.7.1) *is a Killing tensor of the Kerr metric, i.e.* $\mathbf{D}_{(\mu}K_{\nu\rho)} = 0$*.*

Proof. See Section B.1. □

Associated to a Killing tensor $K^{\alpha\beta}$, one can construct a corresponding second order operator $\mathcal{K}$ for horizontal tensors $\psi \in \mathfrak{s}_k$ according to Definition 2.3.6, given by

$$\mathcal{K}(\psi) := \dot{\mathbf{D}}_\beta(K^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi),$$

which has the fundamental property of commuting with the D'Alembertian operator for scalars in a vacuum spacetime, see Proposition 2.3.7. Its explicit expression in Kerr for K as in (3.7.1) is given in the following.

Proposition 3.7.3. *In Kerr spacetime, the Carter operator $\mathcal{K}$ for $\psi \in \mathfrak{s}_k$ is given by*

$$\mathcal{K} = -(a^2\cos^2\theta)\dot{\square}_k + \mathcal{O} \tag{3.7.2}$$

where $\mathcal{O}$ is the following second order angular operator:

$$\mathcal{O}(\psi) := |q|^2\big(\triangle_k\psi + (\eta + \underline{\eta})\cdot\nabla\psi\big). \tag{3.7.3}$$

Proof. See Section B.2. □

Lemma 3.7.4. *In Kerr, the second order operator $\mathcal{O}$ defined in* (3.7.3) *is equivalent to*

$$\mathcal{O}(\psi) \;=\; |q|^2\left(\triangle_k\psi - \frac{2a^2\cos\theta}{|q|^2}\,{}^*\Re(\mathfrak{J})^b\nabla_b\psi\right). \tag{3.7.4}$$

Also, the operator $\mathcal{O}$ can be written in the following ways, all equivalent to (3.7.3)*:*

$$\begin{aligned}
\mathcal{O}(\psi) &= |q|^2\left(\triangle_k\psi + \frac{2\nabla(|q|)}{|q|}\cdot\nabla\psi\right), && (3.7.5)\\
\mathcal{O}(\psi) &= \nabla\cdot\big(|q|^2\nabla\psi\big), && (3.7.6)\\
\mathcal{O}(\psi) &= \dot{\mathbf{D}}_\beta(O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi) - \nabla(a^2\cos^2\theta)\cdot\nabla\psi, && (3.7.7)\\
\mathcal{O}(\psi) &= |q|^2\dot{\mathbf{D}}_\beta(|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi). && (3.7.8)
\end{aligned}$$

Proof. Using (3.4.5), we see that in Kerr

$$\eta + \underline{\eta} = \Re\big(H + \underline{H}\big) = \Re\left(\frac{aq}{|q|^2}\mathfrak{J} - \frac{a\overline{q}}{|q|^2}\mathfrak{J}\right) = \frac{a}{|q|^2}\Re\big(2ia\cos\theta\mathfrak{J}\big) = \frac{2a^2\cos\theta}{|q|^2}\Re\big(i\mathfrak{J}\big),$$

and since ${}^*\mathfrak{J} = -i\mathfrak{J}$, we obtain

$$\eta + \underline{\eta} = -\frac{2a^2\cos\theta}{|q|^2}\Re(\,{}^*\mathfrak{J}) = -\frac{2a^2\cos\theta}{|q|^2}\,{}^*\Re(\mathfrak{J}),$$

which gives (3.7.4). Using from (3.4.4) that $\nabla(|q|) = \frac{1}{2}(\eta+\underline{\eta})|q|$ and $\nabla(|q|^2) = (\eta+\underline{\eta})|q|^2$, we obtain (3.7.5) and (3.7.6).

Using the expression of the Carter tensor $K^{\alpha\beta}$ given by (3.7.1), we obtain

$$\begin{aligned}
\mathcal{K}(\psi) &= \dot{\mathbf{D}}_\beta(-(a^2\cos^2\theta)\mathbf{g}^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi + O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi)\\
&= -(a^2\cos^2\theta)\mathbf{g}^{\alpha\beta}\dot{\mathbf{D}}_\beta\dot{\mathbf{D}}_\alpha\psi - \dot{\mathbf{D}}_\beta(a^2\cos^2\theta)\mathbf{g}^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi + \dot{\mathbf{D}}_\beta(O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi)\\
&= -(a^2\cos^2\theta)\dot{\square}_k\psi + \dot{\mathbf{D}}_\beta(O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi) - \mathbf{g}^{\alpha\beta}\dot{\mathbf{D}}_\beta(a^2\cos^2\theta)\dot{\mathbf{D}}_\alpha\psi.
\end{aligned}$$

By comparing the above with (3.7.2) we deduce (3.7.7). Using that $\nabla(a^2\cos^2\theta) = \nabla(r^2 + a^2\cos^2\theta) = \nabla(|q|^2)$, we deduce from the above

$$\begin{aligned}
\mathcal{O}(\psi) &= |q|^2\dot{\mathbf{D}}_\beta(|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi) - |q|^2O^{\alpha\beta}\dot{\mathbf{D}}_\beta(|q|^{-2})\dot{\mathbf{D}}_\alpha\psi - \nabla(a^2\cos^2\theta)\cdot\nabla\psi\\
&= |q|^2\dot{\mathbf{D}}_\beta(|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi) + |q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\beta(|q|^2)\dot{\mathbf{D}}_\alpha\psi - \nabla(a^2\cos^2\theta)\cdot\nabla\psi\\
&= |q|^2\dot{\mathbf{D}}_\beta(|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi),
\end{aligned}$$

which proves (3.7.8). □

Remark 3.7.5. *Even though the above definitions are all equivalent in Kerr, in perturbations of Kerr it is important to define the operator $\mathcal{O}$ using relation* (3.7.4) *in order to obtain acceptable error terms, see Definition 4.5.2.*

From Proposition 2.3.7, we deduce that the operator $\mathcal{K}$ commutes with the D'Alembertian for scalars, as $\Pi = 0$. In the case of horizontal 2-tensors, the commutator between $\mathcal{K}$ and $\dot{\square}_2$ gives rise to lower order terms involving the Riemann curvature which vanish in Schwarzschild. Using the relation between $\mathcal{K}$ and $\mathcal{O}$ given by (3.7.2), one can prove that the modified Laplacian $\mathcal{O}$ inherits the commutation properties with the conformal D'Alembertian.

Proposition 3.7.6. *In Kerr, the following commutation formula holds true for $\psi \in \mathfrak{s}_2$,*

$$[\mathcal{O}, |q|^2\dot{\square}_2]\psi = |q|^2 \left[\nabla\left(\frac{8a(r^2+a^2)\cos\theta}{|q|^2}\right)\cdot\nabla\nabla_{\widehat{T}}\,{}^*\psi + O(ar^{-2})\nabla_{\widehat{R}}^{\leq 1}\mathfrak{d}^{\leq 1}\psi\right], \quad (3.7.9)$$

where $\mathfrak{d} = (\nabla_3, r\nabla_4, r\nabla)$ denotes weighted derivatives as in [50] and [53].

Proof. See Section B.3. □

Remark 3.7.7. *Observe that the terms on the right hand side of* (3.7.9) *come from Riemann curvature terms as commutators for a 2-tensor ψ. In the proof of* (3.7.9)*, we use the following relation, see Lemma 9.2.1,*

$$\nabla_{\mathbf{T}}\psi \quad = \quad \mathcal{L}\!\!\!/_{\mathbf{T}}\psi + \frac{4amr\cos\theta}{|q|^4}\,{}^*\psi,$$

where the second term is only present for tensors due to curvature.

3.7.1 The symmetry operators

We are now ready to define the set of second order symmetry operators which have remarkable commutation properties with the D'Alembertian operator $\dot{\square}_2$ for horizontal 2-tensors.

Definition 3.7.8. *We define the following second order differential operators, acting on $\mathfrak{s}_2$ tensors, as*

$$\mathcal{S}_1 = \nabla_{\mathbf{T}}\nabla_{\mathbf{T}}, \quad \mathcal{S}_2 = a\nabla_{\mathbf{T}}\nabla_{\mathbf{Z}}, \quad \mathcal{S}_3 = a^2\nabla_{\mathbf{Z}}\nabla_{\mathbf{Z}}, \quad \mathcal{S}_4 = \mathcal{O}. \quad (3.7.10)$$

Observe that since $\mathbf{T}$ and $\mathbf{Z}$ are Killing vectorfields which commute, as obtained in Lemma 3.6.1, and because of relation (3.7.8), we can also write[3] as a unique formula

$$\mathcal{S}_{\underline{a}}\psi = |q|^2\dot{\mathbf{D}}_\alpha(|q|^{-2}S_{\underline{a}}^{\alpha\beta}\dot{\mathbf{D}}_\beta\psi) \qquad \text{for } \underline{a}=1,2,3,4, \tag{3.7.11}$$

where $S_{\underline{a}}^{\alpha\beta}$ are given in Definition 3.5.3.

The commutation properties of the symmetry operators in Kerr can be deduced from the more general ones for approximate symmetry operators in perturbations, see Section 4.6.

3.8 Null geodesics in Kerr

In this section, we derive estimates for null geodesics in Kerr using the vectorfield ∂_r as multiplier. The computations appearing in these estimates present similar calculations to the Morawetz estimates for solutions to the wave equations, and for this reason we derive here their precise forms.

3.8.1 The constants of motion for geodesics

Let $\gamma(\lambda)$ be a null geodesic in Kerr. Using the expression for the inverse of the metric given by (3.5.1), along $\gamma(\lambda)$, since $\mathbf{g}(\dot{\gamma},\dot{\gamma})=0$ we have, with $\dot{\gamma}_r=\partial_r^\alpha\dot{\gamma}_\alpha$, $\dot{\gamma}_t=\partial_t^\alpha\dot{\gamma}_\alpha$, $\dot{\gamma}_\varphi=\partial_\varphi^\alpha\dot{\gamma}_\alpha$

$$0=|q|^2\mathbf{g}^{\alpha\beta}\dot{\gamma}_\alpha\dot{\gamma}_\beta=\left(\Delta\partial_r^\alpha\partial_r^\beta+\frac{1}{\Delta}\mathcal{R}^{\alpha\beta}\right)\dot{\gamma}_\alpha\dot{\gamma}_\beta=\Delta\dot{\gamma}_r\dot{\gamma}_r+\frac{1}{\Delta}\mathcal{R}^{\alpha\beta}\dot{\gamma}_\alpha\dot{\gamma}_\beta$$

with

$$\mathcal{R}^{\alpha\beta}\dot{\gamma}_\alpha\dot{\gamma}_\beta = -(r^2+a^2)^2\dot{\gamma}_t\dot{\gamma}_t-2a(r^2+a^2)\dot{\gamma}_t\dot{\gamma}_\varphi-a^2\dot{\gamma}_\varphi\dot{\gamma}_\varphi+\Delta O^{\alpha\beta}\dot{\gamma}_\alpha\dot{\gamma}_\beta.$$

Since $\partial_t=T$ and $\partial_\varphi=Z$ are Killing vectorfields, we deduce that $\dot{\gamma}_t=\mathbf{g}(\dot{\gamma},T)$ and $\dot{\gamma}_\varphi=\mathbf{g}(\dot{\gamma},Z)$ are constants of the motion, i.e. constants along γ, and respectively called the energy and the azimuthal angular momentum. We write,

$$\mathbf{e}:=-\mathbf{g}(\dot{\gamma},T), \qquad \ell_{\mathbf{z}}:=-\mathbf{g}(\dot{\gamma},Z).$$

[3]Note that we use the calligraphic $\mathcal{S}$ for the differential operators, and the normal $S^{\alpha\beta}$ for the symmetric tensors.

We also define[4]

$$\mathbf{k}^2 \;\; := \;\; K^{\alpha\beta}\dot{\gamma}_\alpha\dot{\gamma}_\beta$$

for the Carter tensor K in Kerr. Since K is Killing, $\mathbf{k}^2$ is also a constant of motion. Indeed, we have

$$\frac{d}{d\lambda}\mathbf{k}^2\big(\gamma(\lambda)\big) \;\; = \;\; \mathbf{D}_\lambda K_{\alpha\beta}\,\dot{\gamma}^\lambda\dot{\gamma}^\alpha\dot{\gamma}^\beta = 0.$$

Since from (3.7.1), $K = -(a^2\cos^2\theta)\mathbf{g} + O$ and γ is null we deduce, with $\dot{\gamma}_a = \mathbf{g}(\dot{\gamma}, e_a)$

$$\mathbf{k}^2 \;\; = \;\; O^{\alpha\beta}\dot{\gamma}_\alpha\dot{\gamma}_\beta = |q|^2\big(e_1^\alpha e_1^\beta + e_2^\alpha e_2^\beta\big)\dot{\gamma}_\alpha\dot{\gamma}_\beta = |q|^2\big(|\dot{\gamma}_1|^2 + |\dot{\gamma}_2|^2\big).$$

We summarize the result in the following.

Proposition 3.8.1. *The quantities*

$$\mathbf{e} = -\mathbf{g}(\dot{\gamma}, T), \qquad \ell_{\mathbf{z}} = -\mathbf{g}(\dot{\gamma}, Z), \qquad \mathbf{k}^2 = K^{\alpha\beta}\dot{\gamma}_\alpha\dot{\gamma}_\beta,$$

are constant along null geodesics. Moreover, relative to the null frame

$$\mathbf{k}^2 \;\; = |q|^2\big(|\dot{\gamma}_1|^2 + |\dot{\gamma}_2|^2\big).$$

With these constants we have

$$\begin{aligned}\mathcal{R}^{\alpha\beta}\dot{\gamma}_\alpha\dot{\gamma}_\beta \;\; &= \;\; -(r^2+a^2)^2\dot{\gamma}_t\dot{\gamma}_t - 2a(r^2+a^2)\dot{\gamma}_t\dot{\gamma}_\varphi - a^2\dot{\gamma}_\varphi\dot{\gamma}_\varphi + \Delta O^{\alpha\beta}\dot{\gamma}_\alpha\dot{\gamma}_\beta \\ &= \;\; -(r^2+a^2)^2\mathbf{e}^2 - 2a(r^2+a^2)\mathbf{e}\,\ell_{\mathbf{z}} - a^2{\ell_{\mathbf{z}}}^2 + \Delta\mathbf{k}^2\end{aligned}$$

which is only a function of r along any fixed null geodesic. We introduce the notation

$$\mathcal{R}(r; a, m, \mathbf{e}, \ell_{\mathbf{z}}, \mathbf{k}) := -(r^2+a^2)^2\mathbf{e}^2 - 2a(r^2+a^2)\mathbf{e}\,\ell_{\mathbf{z}} - a^2{\ell_{\mathbf{z}}}^2 + \Delta\mathbf{k}^2.$$

Note that we have the identity

$$-\mathcal{R}(r; a, m, \mathbf{e}, \ell_{\mathbf{z}}, \mathbf{k}) \;\; = \big((r^2+a^2)\mathbf{e} + a\ell_{\mathbf{z}}\big)^2 - \Delta\mathbf{k}^2. \tag{3.8.1}$$

In view of the above, we infer that

$$0 = \Delta\dot{\gamma}_r\dot{\gamma}_r + \frac{1}{\Delta}\mathcal{R}^{\alpha\beta}\dot{\gamma}_\alpha\dot{\gamma}_\beta = \Delta\dot{\gamma}_r\dot{\gamma}_r + \frac{1}{\Delta}\mathcal{R}(r; a, m, \mathbf{e}, \ell_{\mathbf{z}}, \mathbf{k}).$$

Since

$$\frac{dr}{d\lambda} \;\; = \;\; \dot{\gamma}^\alpha\frac{\partial r}{\partial\alpha} = \dot{\gamma}^r = \mathbf{g}^{rr}\dot{\gamma}_r = \frac{\Delta}{|q|^2}\dot{\gamma}_r,$$

we finally obtain

$$|q|^4\Big(\frac{dr}{d\lambda}\Big)^2 = -\mathcal{R}(r; a, m, \mathbf{e}, \ell_{\mathbf{z}}, \mathbf{k})$$

which is the equation for a null geodesic with constants of motion $\mathbf{e}$, $\ell_{\mathbf{z}}$, $\mathbf{k}$.

[4]Observe that $\mathbf{k}^2$ is a positive constant of motion.

3.8.2 Trapped null geodesics

There exist null geodesics along which $\mathcal{R}(r; a, m, \mathbf{e}, \ell_{\mathbf{z}}, \mathbf{q}) = 0$ i.e. r remains constant. These are called orbital null geodesics.

Remark 3.8.2. *Trapped null geodesics correspond to null geodesics that stay in a region $[r_1, r_2]$ of r with $r_+ < r_1 < r_2 < +\infty$ for all values of λ, and are thus a priori more general than orbital null geodesics. As it turns out, see for example Proposition 2 in [14], all trapped null geodesics in Kerr are in fact orbital null geodesics. Thus, from now on, we do not distinguish between trapped and orbital null geodesics.*

If r is constant we also have

$$-\partial_r \mathcal{R}(r; a, m, \mathbf{e}, \ell_{\mathbf{z}}, \mathbf{k}) = \partial_r(|q|^4)\left(\frac{dr}{d\lambda}\right)^2 + 2|q|^4 \frac{dr}{d\lambda}\partial_r\left(\frac{dr}{d\lambda}\right) = 0.$$

The r values for which such solutions are possible must then verify the equations

$$\mathcal{R}(r; a, m, \mathbf{e}, \ell_{\mathbf{z}}, \mathbf{k}) = \partial_r \mathcal{R}(r; a, m, \mathbf{e}, \ell_{\mathbf{z}}, \mathbf{k}) = 0.$$

Thus, introducing

$$\Pi := (r^2 + a^2)\mathbf{e} + a\ell_{\mathbf{z}}$$

we write from (3.8.1)

$$\begin{aligned} -\mathcal{R}(r; a, m, \mathbf{e}, \ell_{\mathbf{z}}, \mathbf{k}) &= \Pi^2 - \Delta \mathbf{k}^2 = 0, \\ -\partial_r \mathcal{R}(r; a, m, \mathbf{e}, \ell_{\mathbf{z}}, \mathbf{k}) &= 2\Pi(\partial_r \Pi) - (\partial_r \Delta)\mathbf{k}^2 = 0. \end{aligned}$$

From the second equation, we deduce

$$\mathbf{k}^2 = 2\Pi \frac{\partial_r \Pi}{\partial_r \Delta}. \tag{3.8.2}$$

Thus, substituting in the first equation,

$$\Pi^2 - 2\Pi \frac{\Delta \partial_r \Pi}{\partial_r \Delta} = 0$$

or, if $\Pi \neq 0$,

$$\Pi(\partial_r \Delta) - 2(\partial_r \Pi)\Delta = 0. \tag{3.8.3}$$

We make use of the following calculation.

Lemma 3.8.3. *We have the identity*

$$\Pi(\partial_r\Delta) - 2(\partial_r\Pi)\Delta = -2\mathcal{T}_{\mathbf{e},\ell_{\mathbf{z}}} \tag{3.8.4}$$

where

$$\mathcal{T}_{\mathbf{e},\ell_{\mathbf{z}}} := \big(r^3 - 3mr^2 + ra^2 + ma^2\big)\mathbf{e} - (r-m)a\ell_{\mathbf{z}}.$$

Proof. We have

$$\begin{aligned}
(\partial_r\Delta)\Pi - 2\Delta(\partial_r\Pi) &= 2(r-m)\big((r^2+a^2)\mathbf{e} + a\ell_{\mathbf{z}}\big) - 4r\big(r^2+a^2-2rm\big)\mathbf{e}\\
&= 2\Big((r-m)\big((r^2+a^2)\mathbf{e} + a\ell_{\mathbf{z}}\big) - 2r\big(r^2+a^2-2rm\big)\mathbf{e}\Big)\\
&= 2\Big(\big(-r^3 + 3mr^2 - ra^2 - ma^2\big)\mathbf{e} + (r-m)a\ell_{\mathbf{z}}\Big)\\
&= -2\mathcal{T}_{\mathbf{e},\ell_{\mathbf{z}}}
\end{aligned}$$

as stated. □

As a consequence of the Lemma we deduce that all orbital null geodesics are given by the equation

$$\mathcal{T}_{\mathbf{e},\ell_{\mathbf{z}}} = \big(r^3 - 3mr^2 + ra^2 + ma^2\big)\mathbf{e} - (r-m)a\ell_{\mathbf{z}} = 0.$$

Remark 3.8.4. *The following hold true.*

1. *There are no trapped null geodesics perpendicular to $T = \partial_t$ in the exterior of a non-extremal Kerr.*

2. *The values of r for which trapped null geodesics exist depends on the ratio $\ell_{\mathbf{z}}/\mathbf{e}$. More precisely, at trapped null geodesics, we have*
$$\frac{r^3 - 3mr^2 + ra^2 + ma^2}{r-m} = \frac{a\ell_{\mathbf{z}}}{\mathbf{e}}.$$
In particular, for $\ell_{\mathbf{z}} = 0$, the trapped null geodesics are given by the equation
$$\mathcal{T} := r^3 - 3mr^2 + a^2r + a^2m = 0. \tag{3.8.5}$$

Remark 3.8.5. *Note that one may specify possible values of r for which trapped null geodesics exists. Indeed, let*

$$\begin{aligned}
\hat{r}_1 &:= 2m\left(1 + \cos\left(\frac{2}{3}\arccos\left(-\frac{|a|}{m}\right)\right)\right),\\
\hat{r}_2 &:= 2m\left(1 + \cos\left(\frac{2}{3}\arccos\left(\frac{|a|}{m}\right)\right)\right).
\end{aligned} \tag{3.8.6}$$

Then, a trapped null geodesics satisfies $r \in [\hat{r}_1, \hat{r}_2]$, see for example [71].

We now show that the trapped null geodesics are unstable, i.e. that $\partial_r^2\mathcal{R}(r; a, m, \mathbf{e}, \ell_{\mathbf{z}}, \mathbf{k}) \leq 0$. We have

$$\begin{aligned} \Pi &= (r^2+a^2)\mathbf{e} + a\ell_{\mathbf{z}}, \\ \partial_r\Pi &= 2r\mathbf{e}, \\ \partial_r^2\Pi &= 2\mathbf{e}, \end{aligned}$$

and using (3.8.2) to write $\mathbf{k}^2 = 4r\frac{\Pi}{\partial_r\Delta}\mathbf{e}$, we have

$$\begin{aligned} -\partial_r^2\mathcal{R}(r; a, m, \mathbf{e}, \ell_{\mathbf{z}}, \mathbf{k}) &= 2(\partial_r\Pi)^2 + 2\Pi(\partial_r^2\Pi) - 2\mathbf{k}^2 \\ &= 8r^2\mathbf{e}^2 + 4\Pi\mathbf{e} - 8r\frac{\Pi}{\partial_r\Delta}\mathbf{e} \\ &= \frac{4}{\partial_r\Delta}\left(2r^2\partial_r\Delta\mathbf{e}^2 + (\partial_r\Delta)\Pi\mathbf{e} - 2r\Pi\mathbf{e}\right) \\ &= \frac{4\mathbf{e}}{\partial_r\Delta}\left(4r^2(r-m)\mathbf{e} - 2m\Pi\right). \end{aligned}$$

Using (3.8.3) to write $\Pi = \frac{4r\Delta\mathbf{e}}{\partial_r\Delta}$ we deduce

$$\begin{aligned} -\partial_r^2\mathcal{R}(r; a, m, \mathbf{e}, \ell_{\mathbf{z}}, \mathbf{k}) &= \frac{4\mathbf{e}}{\partial_r\Delta}\left(4r^2(r-m)\mathbf{e} - 2m\frac{4r\Delta\mathbf{e}}{\partial_r\Delta}\right) \\ &= \frac{32r\mathbf{e}^2}{(\partial_r\Delta)^2}\left(r(r-m)^2 - m\Delta\right) \\ &= \frac{8r}{(r-m)^2}\mathbf{e}^2\Big(r(r-m)^2 - m(r^2+a^2-2mr)\Big) \\ &= \frac{8r}{(r-m)^2}\mathbf{e}^2\Big((r-m)^3 + m(m^2-a^2)\Big), \end{aligned}$$

which is positive since $r \geq m$ and $|a| \leq m$.

3.8.3 Morawetz estimates for geodesics

We now derive estimates for null geodesics in Kerr using the vectorfield $X = \mathcal{F}(r)\partial_r$, for some function $\mathcal{F}(r)$ as multiplier. The energy[5] relative to X is given by

$$e_X[\gamma](\tau) := -\mathbf{g}^{\alpha\beta}X_\alpha\dot{\gamma}_\beta.$$

[5] Observe that this energy is not positive definite since X is not causal.

Our goal is to find a function $\mathcal{F}(r)$ such that the energy $e_X[\gamma]$ is non-increasing for all τ. We have, recalling that $\mathbf{D}_{\dot\gamma}\dot\gamma = 0$ and $\mathbf{g}^{\alpha\beta}\dot\gamma_\alpha\dot\gamma_\beta = 0$,

$$\frac{d}{d\tau}e_X[\gamma](\tau) = -\mathbf{D}_\alpha X_\beta \dot\gamma^\alpha\dot\gamma^\beta = -\frac{1}{2}(\mathcal{L}_X\mathbf{g}_{\alpha\beta})\dot\gamma^\alpha\dot\gamma^\beta = \frac{1}{2}\mathcal{L}_X(\mathbf{g}^{\alpha\beta})\dot\gamma_\alpha\dot\gamma_\beta = \frac{1}{2}|q|^{-2}\mathcal{L}_X(|q|^2\mathbf{g}^{\alpha\beta})\dot\gamma_\alpha\dot\gamma_\beta.$$

We perform the following computations as done in [4].

Lemma 3.8.6. *We have, for $X = \mathcal{F}(r)\partial_r$ and $\mathcal{R}^{\alpha\beta}$ defined in* (3.5.2),

$$\mathcal{L}_X(|q|^2\mathbf{g}^{\alpha\beta}) = \big(\mathcal{F}\partial_r\Delta - 2\Delta\partial_r\mathcal{F}\big)\partial_r^\alpha\partial_r^\beta + \mathcal{F}\partial_r\left(\frac{1}{\Delta}\mathcal{R}^{\alpha\beta}\right).$$

Proof. Recall from (3.5.1) that we have $|q|^2\mathbf{g}^{\alpha\beta} = \Delta\partial_r^\alpha\partial_r^\beta + \frac{1}{\Delta}\mathcal{R}^{\alpha\beta}$. Hence,

$$\begin{aligned}
\mathcal{L}_X(|q|^2\mathbf{g}^{\alpha\beta}) &= \mathcal{L}_X\big(\Delta\partial_r^\alpha\partial_r^\beta\big) + \mathcal{L}_X\left(\frac{1}{\Delta}\mathcal{R}^{\alpha\beta}\right)\\
&= X(\Delta)\partial_r^\alpha\partial_r^\beta + \Delta[X,\partial_r]^\alpha\partial_r^\beta + \Delta\partial_r^\alpha[X,\partial_r]^\beta + \mathcal{L}_X\left(\frac{1}{\Delta}\mathcal{R}^{\alpha\beta}\right)\\
&= \mathcal{F}\partial_r\Delta\partial_r^\alpha\partial_r^\beta - 2\Delta\partial_r\mathcal{F}\partial_r^\alpha\partial_r^\beta + \mathcal{L}_{\mathcal{F}\partial_r}\left(\frac{1}{\Delta}\mathcal{R}^{\alpha\beta}\right).
\end{aligned}$$

For a tensor $U^{\alpha\beta}$ we have,

$$\mathcal{L}_X U^{\alpha\beta} = X^\lambda\partial_\lambda U^{\alpha\beta} - U^{\lambda\beta}\partial_\lambda X^\alpha - U^{\alpha\lambda}\partial_\lambda X^\beta.$$

For $X = \mathcal{F}(r)\partial_r$ and $U^{r\beta} = 0 = U^{\alpha r}$, we deduce

$$\mathcal{L}_X U^{\alpha\beta} = \mathcal{F}\partial_r U^{\alpha\beta}.$$

Hence

$$\begin{aligned}
\mathcal{L}_X(|q|^2\mathbf{g}^{\alpha\beta}) &= \mathcal{F}\partial_r\Delta\partial_r^\alpha\partial_r^\beta - 2\Delta\partial_r\mathcal{F}\partial_r^\alpha\partial_r^\beta + \mathcal{F}\partial_r\left(\frac{1}{\Delta}\mathcal{R}^{\alpha\beta}\right)\\
&= \big(\mathcal{F}\partial_r\Delta - 2\Delta\partial_r\mathcal{F}\big)\partial_r^\alpha\partial_r^\beta + \mathcal{F}\partial_r\left(\frac{1}{\Delta}\mathcal{R}^{\alpha\beta}\right)
\end{aligned}$$

as stated. □

Using the lemma we deduce,

$$|q|^2\frac{d}{d\tau}e_{\mathcal{F}\partial_r}[\gamma](\tau) = \frac{1}{2}\Big(\mathcal{F}\partial_r\Delta - 2\Delta\partial_r\mathcal{F}\Big)\dot\gamma_r\dot\gamma_r + \frac{1}{2}\mathcal{F}\partial_r\left(\frac{1}{\Delta}\mathcal{R}^{\alpha\beta}\right)\dot\gamma_\alpha\dot\gamma_\beta.$$

We now recall that,

$$\mathcal{R} = \mathcal{R}^{\alpha\beta}\dot{\gamma}_\alpha\dot{\gamma}_\beta,$$
$$\mathcal{R}^{\alpha\beta} = -(r^2+a^2)^2\partial_t^\alpha\partial_t^\beta - 2a(r^2+a^2)\partial_t^{(\alpha}\partial_\varphi^{\beta)} - a^2\partial_\varphi^\alpha\partial_\varphi^\beta + \Delta O^{\alpha\beta}$$

Thus, we have

$$\partial_r\mathcal{R}^{\alpha\beta} = -4r(r^2+a^2)\partial_t^\alpha\partial_t^\beta - 4ar\partial_t^{(\alpha}\partial_\varphi^{\beta)} + 2(r-m)O^{\alpha\beta}.$$

This yields

$$\begin{aligned}\mathcal{F}\partial_r\left(\frac{1}{\Delta}\mathcal{R}^{\alpha\beta}\right)\dot{\gamma}_\alpha\dot{\gamma}_\beta &= -\mathcal{F}\frac{\partial_r\Delta}{\Delta^2}\mathcal{R} + \mathcal{F}\frac{1}{\Delta}(\partial_r\mathcal{R}^{\alpha\beta})\dot{\gamma}_\alpha\dot{\gamma}_\beta\\ &= -\mathcal{F}\frac{\partial_r\Delta}{\Delta^2}\mathcal{R} + \mathcal{F}\frac{1}{\Delta}\left(-4r(r^2+a^2)\mathbf{e}^2 - 4ar\mathbf{e}\,\ell_{\mathbf{z}} + 2(r-m)\mathbf{k}^2\right)\\ &= \mathcal{F}\partial_r\left(\frac{1}{\Delta}\mathcal{R}\right)\end{aligned}$$

and therefore

$$|q|^2\frac{d}{d\tau}e_{\mathcal{F}\partial_r}[\gamma](\tau) = \frac{1}{2}\Big(\mathcal{F}\partial_r\Delta - 2\Delta\partial_r\mathcal{F}\Big)\dot{\gamma}_r^2 + \frac{1}{2}\mathcal{F}\partial_r\left(\frac{1}{\Delta}\mathcal{R}\right). \tag{3.8.7}$$

Remark 3.8.7. *In the particular case of trapped null geodesics, for which $\mathcal{R} = \partial_r\mathcal{R} = 0$, the non-increasing of the above energy can be obtained by choosing $\mathcal{F} = -\partial_r\mathcal{R}$. Indeed, for that choice, evaluating on a trapped null geodesic γ:*

$$\begin{aligned}|q|^2\frac{d}{d\tau}e_{\mathcal{F}\partial_r}[\gamma](\tau) &= \frac{1}{2}\Big(-\partial_r\mathcal{R}\partial_r\Delta + 2\Delta\partial_r^2\mathcal{R}\Big)\dot{\gamma}_r^2 - \frac{1}{2}(\partial_r\mathcal{R})\partial_r\left(\frac{1}{\Delta}\mathcal{R}\right)\\ &= \Delta(\partial_r^2\mathcal{R})\dot{\gamma}_r^2 \le 0.\end{aligned}$$

In order to obtain a similar result for all null geodesics, we need to add more freedom in the choice of additional scalar functions.

We obtain the following proposition.

Proposition 3.8.8. *The following identity holds true for any scalar functions $\mathcal{F}$, $w = w_{red}$,*

$$|q|^2\frac{d}{d\tau}e_{\mathcal{F}\partial_r}[\gamma](\tau) = \frac{1}{2}\Big(\mathcal{F}\partial_r\Delta - 2\Delta\partial_r\mathcal{F} + \Delta w_{red}\Big)\dot{\gamma}_r^2 + \frac{1}{2}\mathcal{F}\partial_r\left(\frac{1}{\Delta}\mathcal{R}\right) + \frac{1}{2}w_{red}\frac{1}{\Delta}\mathcal{R}, \tag{3.8.8}$$

We write the above in the form

$$|q|^2\frac{d}{d\tau}e_{\mathcal{F}\partial_r}[\gamma](\tau) = -\mathcal{A}\dot{\gamma}_r^2 - \mathcal{U} \tag{3.8.9}$$

where

$$\mathcal{A} := -\frac{1}{2}\mathcal{F}\partial_r\Delta + \Delta\partial_r\mathcal{F} - \frac{1}{2}\Delta w_{red}, \tag{3.8.10}$$

$$\mathcal{U} := -\frac{1}{2}\mathcal{F}\partial_r\left(\frac{1}{\Delta}\mathcal{R}\right) - \frac{1}{2}w_{red}\frac{1}{\Delta}\mathcal{R}. \tag{3.8.11}$$

Proof. Using again (3.5.1), and the fact that γ is null we can rewrite (3.8.7), for some scalar w_{red}

$$\begin{aligned}
|q|^2\frac{d}{d\tau}e_{\mathcal{F}\partial_r}[\gamma](\tau) &= \frac{1}{2}\Big(\mathcal{F}\partial_r\Delta - 2\Delta\partial_r\mathcal{F}\Big)\dot{\gamma}_r^2 + \frac{1}{2}\mathcal{F}\partial_r\left(\frac{1}{\Delta}\mathcal{R}\right) + \frac{1}{2}|q|^2 w_{red}\mathbf{g}^{\alpha\beta}\,\dot{\gamma}_\alpha\dot{\gamma}_\beta\\
&= \frac{1}{2}\Big(\mathcal{F}\partial_r\Delta - 2\Delta\partial_r\mathcal{F}\Big)\dot{\gamma}_r^2 + \frac{1}{2}\mathcal{F}\partial_r\left(\frac{1}{\Delta}\mathcal{R}\right) + \frac{1}{2}w_{red}\left(\Delta\,\dot{\gamma}_r^2 + \frac{1}{\Delta}\mathcal{R}\right)\\
&= \frac{1}{2}\Big(\mathcal{F}\partial_r\Delta - 2\Delta\partial_r\mathcal{F} + \Delta w_{red}\Big)\dot{\gamma}_r^2 + \frac{1}{2}\mathcal{F}\partial_r\left(\frac{1}{\Delta}\mathcal{R}\right) + \frac{1}{2}w_{red}\frac{1}{\Delta}\mathcal{R}
\end{aligned}$$

as stated. □

In order to obtain that the energy $e_X[\gamma]$ is non-increasing for all τ, our goal is to choose $\mathcal{F}$ and w_{red} so that $\mathcal{A} \geq 0$ and $\mathcal{U} \geq 0$. We collect here the conditions we need.

Calculation of $\mathcal{U}$. Given a scalar function z we write

$$-\frac{1}{2}\mathcal{F}\partial_r\left(\frac{1}{\Delta}\mathcal{R}\right) = -\frac{1}{2}\mathcal{F}z^{-1}\partial_r\left(\frac{z}{\Delta}\mathcal{R}\right) + \frac{1}{2}\mathcal{F}z^{-1}\partial_r z\frac{\mathcal{R}}{\Delta}.$$

We then have from (3.8.11)

$$\mathcal{U} = -\frac{1}{2}\mathcal{F}\partial_r\left(\frac{1}{\Delta}\mathcal{R}\right) - \frac{1}{2}w_{red}\frac{1}{\Delta}\mathcal{R} = -\frac{1}{2}\mathcal{F}z^{-1}\partial_r\left(\frac{z}{\Delta}\mathcal{R}\right) + \frac{1}{2}\left(\mathcal{F}z^{-1}\partial_r z - w_{red}\right)\frac{\mathcal{R}}{\Delta}.$$

By choosing

$$w_{red} = \mathcal{F}z^{-1}\partial_r z, \tag{3.8.12}$$

the coefficient of $\frac{\mathcal{R}}{\Delta}$ cancels out. Thus

$$\mathcal{U} = -\frac{1}{2}\mathcal{F}z^{-1}\partial_r\left(\frac{z}{\Delta}\mathcal{R}\right) = -\frac{1}{2}\mathcal{F}z^{-1}\tilde{\mathcal{R}}', \qquad \tilde{\mathcal{R}}' := \partial_r\left(\frac{z}{\Delta}\mathcal{R}\right).$$

Finally, given a scalar function h, by choosing[6]

$$\mathcal{F} = -zh\tilde{\mathcal{R}}', \qquad w_{red} = -(\partial_r z)h\tilde{\mathcal{R}}', \tag{3.8.13}$$

[6]The relation for w_{red} is implied by the previous dependence on $\mathcal{F}$.

we deduce

$$\mathcal{U} = -\frac{1}{2}\mathcal{F}z^{-1}\tilde{\mathcal{R}}' = \frac{1}{2}h(\tilde{\mathcal{R}}')^2.$$

In particular, $\mathcal{U}$ is positive as long as h is positive.

Calculation of $\mathcal{A}$. With the choices of $\mathcal{F}$ and w_{red} given by (3.8.13), we compute from (3.8.10)

$$\begin{aligned}
\mathcal{A} &= -\frac{1}{2}\mathcal{F}\partial_r\Delta + \Delta\partial_r\mathcal{F} - \frac{1}{2}\Delta w_{red} = \partial_r\left(\frac{\mathcal{F}}{\Delta^{1/2}}\right)\Delta^{3/2} - \frac{1}{2}\Delta w_{red} \\
&= \partial_r\left(\frac{-zh\tilde{\mathcal{R}}'}{\Delta^{1/2}}\right)\Delta^{3/2} - \frac{1}{2}\Delta(-(\partial_r z)h\tilde{\mathcal{R}}') = -\partial_r\left(\frac{z^{1/2}z^{1/2}h\tilde{\mathcal{R}}'}{\Delta^{1/2}}\right)\Delta^{3/2} + \frac{1}{2}\Delta(\partial_r z)h\tilde{\mathcal{R}}' \\
&= -\frac{1}{2}\partial_r z\left(\frac{h\tilde{\mathcal{R}}'}{\Delta^{1/2}}\right)\Delta^{3/2} - z^{1/2}\partial_r\left(\frac{z^{1/2}h\tilde{\mathcal{R}}'}{\Delta^{1/2}}\right)\Delta^{3/2} + \frac{1}{2}\Delta(\partial_r z)h\tilde{\mathcal{R}}' \\
&= -z^{1/2}\Delta^{3/2}\partial_r\left(h\frac{z^{1/2}\tilde{\mathcal{R}}'}{\Delta^{1/2}}\right).
\end{aligned}$$

We obtain

$$\mathcal{A} = -z^{1/2}\Delta^{3/2}\tilde{\mathcal{R}}'', \qquad \tilde{\mathcal{R}}'' := \partial_r\left(h\frac{z^{1/2}}{\Delta^{1/2}}\tilde{\mathcal{R}}'\right).$$

We summarize the result in the following

Lemma 3.8.9. *With the choice of the functions $\mathcal{F} = -zh\tilde{\mathcal{R}}'$, $w_{red} = -(\partial_r z)h\tilde{\mathcal{R}}'$, for functions z and h, the identity (3.8.9) takes the form*

$$|q|^2\frac{d}{d\tau}e_{\mathcal{F}\partial_r}[\gamma](\tau) = \left(z^{1/2}\Delta^{3/2}\tilde{\mathcal{R}}''\right)\dot{\gamma}_r^2 - \frac{1}{2}h(\tilde{\mathcal{R}}')^2$$

where $\tilde{\mathcal{R}}' = \partial_r\left(\frac{z}{\Delta}\mathcal{R}\right)$ and $\tilde{\mathcal{R}}'' = \partial_r\left(h\frac{z^{1/2}}{\Delta^{1/2}}\tilde{\mathcal{R}}'\right)$.

To obtain a non-increasing energy $e_{\mathcal{F}\partial_r}[\gamma]$, we are therefore left to choose positive scalar functions z and h for which $\tilde{\mathcal{R}}'' \leq 0$.

Choice of z and h

From (3.8.1), we write

$$\mathcal{R} = -(r^2+a^2)^2\mathbf{e}^2 - 2a(r^2+a^2)\mathbf{e}\,\ell_{\mathbf{z}} - a^2\ell_{\mathbf{z}}{}^2 + \Delta\mathbf{k}^2,$$

and therefore we have

$$\begin{aligned}\tilde{\mathcal{R}}' &= \partial_r\left(\frac{z}{\Delta}\mathcal{R}\right)\\ &= -\partial_r\left(\frac{z}{\Delta}(r^2+a^2)^2\right)\mathbf{e}^2 - 2a\partial_r\left(\frac{z}{\Delta}(r^2+a^2)\right)\mathbf{e}\,\ell_{\mathbf{z}} - a^2\partial_r\left(\frac{z}{\Delta}\right){\ell_{\mathbf{z}}}^2 + (\partial_r z)\mathbf{k}^2.\end{aligned}$$

We choose z to cancel the coefficient of $\mathbf{e}^2$ in $\tilde{\mathcal{R}}'$, i.e.

$$z = \frac{\Delta}{(r^2+a^2)^2}. \tag{3.8.14}$$

This choice implies

$$\begin{aligned}\tilde{\mathcal{R}}' &= -2a\partial_r\left(\frac{1}{(r^2+a^2)}\right)\mathbf{e}\cdot\ell_{\mathbf{z}} - a^2\partial_r\left(\frac{1}{(r^2+a^2)^2}\right){\ell_{\mathbf{z}}}^2 + \partial_r\left(\frac{\Delta}{(r^2+a^2)^2}\right)\mathbf{k}^2\\ &= \frac{4ar}{(r^2+a^2)^2}\mathbf{e}\cdot\ell_{\mathbf{z}} + \frac{4a^2r}{(r^2+a^2)^3}{\ell_{\mathbf{z}}}^2 - 2\left(\frac{r^3-3mr^2+a^2r+ma^2}{(r^2+a^2)^3}\right)\mathbf{k}^2.\end{aligned}$$

Remark 3.8.10. *Observe that*

$$\partial_r z = \partial_r\left(\frac{\Delta}{(r^2+a^2)^2}\right) = -2\left(\frac{r^3-3mr^2+a^2r+ma^2}{(r^2+a^2)^3}\right) = -\frac{2\mathcal{T}}{(r^2+a^2)^3} \tag{3.8.15}$$

where $\mathcal{T}$ is defined as in (3.8.5) *to be the locus of trapped geodesics with $\ell_{\mathbf{z}} = 0$, i.e. the trapped set in the axially symmetric case.*

We now compute $\tilde{\mathcal{R}}''$:

$$\begin{aligned}\tilde{\mathcal{R}}'' &= \partial_r\left(h\frac{z^{1/2}}{\Delta^{1/2}}\tilde{\mathcal{R}}'\right) = \partial_r\left(\frac{h}{r^2+a^2}\tilde{\mathcal{R}}'\right)\\ &= \partial_r\left(h\frac{4ar}{(r^2+a^2)^3}\right)\mathbf{e}\cdot\ell_{\mathbf{z}} + \partial_r\left(h\frac{4a^2r}{(r^2+a^2)^4}\right){\ell_{\mathbf{z}}}^2 - 2\partial_r\left(h\frac{r^3-3mr^2+a^2r+ma^2}{(r^2+a^2)^4}\right)\mathbf{k}^2.\end{aligned}$$

We choose[7] h to cancel the coefficient of $\mathbf{e}\cdot\ell_{\mathbf{z}}$ in $\tilde{\mathcal{R}}''$, i.e.

$$h = \frac{(r^2+a^2)^3}{r}. \tag{3.8.16}$$

This choice implies

$$\begin{aligned}\tilde{\mathcal{R}}'' &= \partial_r\left(\frac{4a^2}{(r^2+a^2)}\right){\ell_{\mathbf{z}}}^2 - 2\partial_r\left(\frac{r^3-3mr^2+a^2r+ma^2}{r(r^2+a^2)}\right)\mathbf{k}^2\\ &= -\frac{8a^2r}{(r^2+a^2)^2}{\ell_{\mathbf{z}}}^2 - \frac{2m\big(3r^4-6a^2r^2-a^4\big)}{r^2(r^2+a^2)^2}\mathbf{k}^2.\end{aligned}$$

[7]In the Morawetz estimates for the wave equation we will choose a different h, i.e. $h = \frac{(r^2+a^2)^4}{r(r^2-a^2)}$. See Remark 3.8.11.

Observe that $\tilde{\mathcal{R}}'' \leq 0$ as long as $3r^4 - 6a^2r^2 - a^4 \geq 0$. Note that $3r^4 - 6a^2r^2 - a^4 \geq 0$ if

$$r^2 \geq \frac{a^2(3+2\sqrt{3})}{3}, \qquad \text{for} \quad r \geq r_+.$$

It thus suffices to check the sign on the horizon $r = r_+ = m + \sqrt{m^2 - a^2}$. We need therefore, $\big(m + \sqrt{m^2-a^2}\big)^2 \geq \frac{a^2(3+2\sqrt{3})}{3}$ or, taking $\lambda = \frac{a}{m}$,

$$\big(1+\sqrt{1-\lambda^2}\big)^2 \geq \lambda^2 \frac{(3+2\sqrt{3})}{3}$$

This implies $\sqrt{1-\lambda^2} \geq |\lambda|\sqrt{\frac{(3+2\sqrt{3})}{3}} - 1$, which is verified for

$$\frac{|a|}{m} = |\lambda| \leq \frac{3}{3+\sqrt{3}}\sqrt{\frac{(3+2\sqrt{3})}{3}} \simeq 0.9306.$$

In particular, for Kerr spacetime with $|\frac{a}{m}| \leq 0.93$, we have

$$-\tilde{\mathcal{R}}'' \geq 0 \qquad \text{for } r \geq r_+.$$

Remark 3.8.11. *Observe that if we restrict our analysis to axially symmetric geodesics, for which $\ell_{\mathbf{z}} = 0$, then the term $\tilde{\mathcal{R}}''$ reduces to*

$$\tilde{\mathcal{R}}'' = -2\partial_r\left(h\frac{r^3 - 3mr^2 + a^2r + ma^2}{(r^2+a^2)^4}\right)\mathbf{k}^2$$

for any positive function h. In particular, by choosing

$$h = \frac{(r^2+a^2)^4}{r(r^2-a^2)} \tag{3.8.17}$$

we obtain

$$\tilde{\mathcal{R}}'' = -2\partial_r\left(\frac{r^3 - 3mr^2 + a^2r + ma^2}{r(r^2-a^2)}\right)\mathbf{k}^2 = -2\frac{3mr^4 - 4a^2r^3 + ma^4}{r^2(r^2-a^2)^2}\mathbf{k}^2$$

which is negative in the exterior region in the full sub-extremal range $|a| < m$. This can be seen by setting $a^2 = \gamma m^2$ and $r = (m + \sqrt{m^2-a^2})x = (1+\sqrt{1-\gamma})mx$, where the exterior region is given by $x > 1$. We then obtain

$$3mr^4 - 4a^2r^3 + ma^4 = m^5\big(3(1+\sqrt{1-\gamma})^4x^4 - 4\gamma(1+\sqrt{1-\gamma})^3x^3 + \gamma^2\big)$$

which is positive for $x \geq 1$ and $0 \leq \gamma \leq 1$.

The above choices allow to prove the Morawetz estimates for geodesics in Kerr, as summarized in the following proposition.

Proposition 3.8.12. *Let γ be a null geodesics in a Kerr spacetime with $|\frac{a}{m}| \leq 0.93$. Then, with the choices*

$$\mathcal{F} = -zh\tilde{\mathcal{R}}', \qquad z = \frac{\Delta}{(r^2+a^2)^2}, \qquad h = \frac{(r^2+a^2)^3}{r},$$

the energy $e_{\mathcal{F}\partial_r}[\gamma]$ is non-increasing for all τ, i.e.

$$\begin{aligned}
|q|^2 \frac{d}{d\tau} e_{\mathcal{F}\partial_r}[\gamma](\tau) &= \left(z^{1/2}\Delta^{3/2}\tilde{\mathcal{R}}''\right)\dot{\gamma}_r^2 - \frac{1}{2}h(\tilde{\mathcal{R}}')^2 \\
&= -\frac{\Delta^2}{(r^2+a^2)}(-\tilde{\mathcal{R}}'')\dot{\gamma}_r^2 - \frac{(r^2+a^2)^3}{2r}(\tilde{\mathcal{R}}')^2 \leq 0.
\end{aligned}$$

Chapter 4

Perturbations of Kerr

4.1 Set-up and linearized quantities

We consider a given vacuum spacetime $(\mathcal{M}, \mathbf{g})$ together with a null pair (e_3, e_4) and its corresponding horizontal structure as in section 2.1.1. We will use the complexified Ricci and curvature coefficients of Definition 2.4.8. To be able to talk about small perturbations of Kerr we also need:

- $\mathcal{M}$ is endowed with a pair of constants (a, m).
- $\mathcal{M}$ is endowed with a pair of scalar functions (r, θ).
- The complex valued scalar function q is defined as $q := r + ia\cos\theta$.
- $\mathcal{M}$ is endowed with a complex horizontal 1-form $\mathfrak{J}$.
- Define linearized quantities, such as $\check{\Gamma} = \Gamma - \Gamma_{Kerr}$, $\check{R} = R - R_{Kerr}$, i.e. subtract from the Ricci and curvature coefficients the corresponding values in Kerr, expressed relative to $(a, m, r, \cos\theta, \mathfrak{J})$.

4.1.1 Definition of linearized quantities

Recall from section 3.3 that the following Ricci and curvature coefficients vanish in Kerr

$$\widehat{X}, \quad \underline{\widehat{X}}, \quad \Xi, \quad \underline{\Xi}, \quad A, \quad B, \quad \underline{B}, \quad \underline{A},$$

as well as the following additional quantities

$$\nabla(r), \qquad e_4(\theta), \qquad e_3(\theta), \qquad \mathcal{D}\widehat{\otimes}\mathfrak{J}.$$

We renormalize below all other quantities, not vanishing in Kerr. We start with quantities which are 0-conformally invariant in the sense of Definition 2.2.16.

Definition 4.1.1 (Renormalization for 0-conformally invariant quantities)**.** *We define the following renormalizations.*

1. *Linearization of 0-conformally invariant Ricci and curvature coefficients:*

$$\check{H} := H - \frac{aq}{|q|^2}\mathfrak{J}, \qquad \check{\underline{H}} := \underline{H} + \frac{a\overline{q}}{|q|^2}\mathfrak{J}, \qquad \check{P} := P + \frac{2m}{q^3}. \tag{4.1.1}$$

2. *Linearization of 0-conformally invariant derivatives of r, $\cos\theta$ and q:*

$$\widecheck{\mathcal{D}q} := \mathcal{D}q + a\mathfrak{J}, \qquad \widecheck{\mathcal{D}\overline{q}} := \mathcal{D}\overline{q} - a\mathfrak{J}, \qquad \widecheck{\mathcal{D}(\cos\theta)} := \mathcal{D}(\cos\theta) - i\mathfrak{J}. \tag{4.1.2}$$

3. *Linearization of 0-conformally invariant derivatives of $\mathfrak{J}$:*

$$\widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}} := \overline{\mathcal{D}}\cdot\mathfrak{J} - \frac{4i(r^2+a^2)\cos\theta}{|q|^4}. \tag{4.1.3}$$

Definition 4.1.2 (Outgoing renormalization for the remaining quantities)**.** *We define the following renormalizations.*[1]

1. *Linearization of the remaining Ricci and curvature coefficients:*

$$\begin{aligned} \widecheck{trX} &:= trX - \frac{2}{q}, & \widecheck{tr\underline{X}} &:= tr\underline{X} + \frac{2q\Delta}{|q|^4}, \\ \check{Z} &:= Z - \frac{a\overline{q}}{|q|^2}\mathfrak{J}, & \check{\underline{\omega}} &:= \underline{\omega} - \frac{1}{2}\partial_r\left(\frac{\Delta}{|q|^2}\right). \end{aligned} \tag{4.1.4}$$

2. *Linearization of the remaining derivatives of r:*

$$\widecheck{e_3(r)} := e_3(r) + \frac{\Delta}{|q|^2}, \qquad \widecheck{e_4(r)} := e_4(r) - 1. \tag{4.1.5}$$

[1]Note that in Kerr, we have $\omega = 0$ in the outgoing principal frame so that this quantity does not need to be renormalized. By convention, we thus define $\check{\omega} = \omega$ if the normalization of the null pair (e_3, e_4) is outgoing. In addition, note that in Kerr we have $e_3(q) = e_3(r)$ and $e_4(q) = e_4(r)$ so that it suffices to linearize $e_3(r)$ and $e_4(r)$.

3. Linearization of the remaining derivatives of $\mathfrak{J}$*:*

$$\widetilde{\nabla_3\mathfrak{J}} := \nabla_3\mathfrak{J} - \frac{\Delta q}{|q|^4}\mathfrak{J}, \qquad \widetilde{\nabla_4\mathfrak{J}} := \nabla_4\mathfrak{J} + \frac{1}{q}\mathfrak{J}. \tag{4.1.6}$$

Definition 4.1.3 (Ingoing renormalization for the remaining quantities)**.** *We define the following renormalizations.*[2]

1. Linearization of the remaining Ricci and curvature coefficients.

$$\begin{aligned} \widetilde{trX} &:= trX - \frac{2\overline{q}\Delta}{|q|^4}, & \widetilde{tr\underline{X}} &:= tr\underline{X} + \frac{2}{\overline{q}}, \\ \check{Z} &:= Z - \frac{aq}{|q|^2}\mathfrak{J}, & \check{\underline{\omega}} &:= \underline{\omega} + \frac{1}{2}\partial_r\left(\frac{\Delta}{|q|^2}\right). \end{aligned} \tag{4.1.7}$$

2. Linearization of remaining derivatives of r*.*

$$\widetilde{e_3(r)} := e_3(r) + 1, \qquad \widetilde{e_4(r)} := e_4(r) - \frac{\Delta}{|q|^2}. \tag{4.1.8}$$

3. Linearization of remaining derivatives of $\mathfrak{J}$*.*

$$\widetilde{\nabla_3\mathfrak{J}} := \nabla_3\mathfrak{J} - \frac{1}{\overline{q}}\mathfrak{J}, \qquad \widetilde{\nabla_4\mathfrak{J}} := \nabla_4\mathfrak{J} + \frac{\Delta\overline{q}}{|q|^4}\mathfrak{J}. \tag{4.1.9}$$

Remark 4.1.4. *In Part II and Part III, we will always consider normalizations which are either outgoing, i.e. for which* $\underline{\omega}$ *is small in a suitable sense, or ingoing, i.e. for which* ω *is small in a suitable sense. Also, in the region* $r \leq r_+ + \delta_{\mathcal{H}}$*, we will only consider the ingoing renormalization.*

4.1.2 Definition of the notations Γ_b and Γ_g for error terms

Definition 4.1.5. *The set of all linearized quantities is of the form* $\Gamma_g \cup \Gamma_b$ *with* Γ_g, Γ_b *defined as follows.*

1. The set $\Gamma_g = \Gamma_{g,1} \cup \Gamma_{g,2}$ *with*

$$\begin{aligned} \Gamma_{g,1} &= \Big\{\Xi, \quad \check{\omega}, \quad \widetilde{trX}, \quad \widehat{X}, \quad \check{Z}, \quad \check{\underline{H}}, \quad \widetilde{tr\underline{X}}, \quad r\check{P}, \quad rB, \quad rA\Big\}, \\ \Gamma_{g,2} &= \Big\{\widetilde{e_4(r)}, \quad r^{-1}\nabla(r), \quad e_4(\cos\theta), \quad r\widetilde{\nabla_4\mathfrak{J}}\Big\}. \end{aligned} \tag{4.1.10}$$

[2]Note that in Kerr, we have $\underline{\omega} = 0$ in the ingoing principal frame so that this quantity does not need to be renormalized. By convention, we thus define $\check{\underline{\omega}} = \underline{\omega}$ if the normalization of the null pair (e_3, e_4) is ingoing. In addition, note that in Kerr we have $e_3(q) = e_3(r)$ and $e_4(q) = e_4(r)$ so that it suffices to linearize $e_3(r)$ and $e_4(r)$.

2. The set $\Gamma_b = \Gamma_{b,1} \cup \Gamma_{b,2} \cup \Gamma_{b,3}$ *with*

$$\begin{aligned}
\Gamma_{b,1} &= \left\{ \check{H}, \quad \widehat{\underline{X}}, \quad \check{\underline{\omega}}, \quad \underline{\Xi}, \quad r\underline{B}, \quad \underline{A} \right\}, \\
\Gamma_{b,2} &= \left\{ r^{-1}\widetilde{e_3(r)}, \quad \widetilde{\mathcal{D}(\cos\theta)}, \quad e_3(\cos\theta) \right\}, \\
\Gamma_{b,3} &= \left\{ r\,\widetilde{\overline{\mathcal{D}}\cdot\mathfrak{J}}, \quad r\,\mathcal{D}\widehat{\otimes}\mathfrak{J}, \quad r\,\widetilde{\nabla_3\mathfrak{J}} \right\}.
\end{aligned} \tag{4.1.11}$$

Remark 4.1.6. *The justification for the above decompositions has to do with the expected decay properties of the linearized components in perturbations of Kerr. More precisely, we will consider perturbations of Kerr for which* Γ_g *and* Γ_b *satisfy the following estimates, see section 6.1.2 for details,*

$$\begin{aligned}
\left|\mathfrak{d}^{\leq s}\Gamma_g\right| &\lesssim \epsilon \min\left\{ r^{-2}\tau^{-1/2-\delta_{dec}},\, r^{-1}\tau^{-1-\delta_{dec}} \right\}, \\
\left|\nabla_3\mathfrak{d}^{\leq s-1}\Gamma_g\right| &\lesssim \epsilon r^{-2}\tau^{-1-\delta_{dec}}, \\
\left|\mathfrak{d}^{\leq s}\Gamma_b\right| &\lesssim \epsilon r^{-1}\tau^{-1-\delta_{dec}},
\end{aligned} \tag{4.1.12}$$

for a small constant $\delta_{dec} > 0$*, where* $\mathfrak{d} = \{\nabla_3, r\nabla_4, \not{\mathfrak{d}} = r\nabla\}$ *denotes weighted derivatives, and* τ *is a scalar function on* $\mathcal{M}$ *whose properties are given in section 6.1.3.*

In addition to the above, as a consequence of the definition of the linearized quantities and of (Γ_b, Γ_g), as well as of the relations (3.4.1) and (3.4.7), we also have the following:

$$\begin{aligned}
\mathcal{D}P + 3P\,\underline{H} &\in r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g, \\
\mathcal{D}\overline{P} + 3\overline{P}H &\in r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g, \\
\mathcal{D}\widehat{\otimes}\,\underline{H} + \underline{H}\widehat{\otimes}\,\underline{H} &\in r^{-1}\mathfrak{d}^{\leq 1}\Gamma_g, \\
\nabla_4\,\underline{H} + \mathrm{tr}X\,\underline{H} &\in r^{-1}\mathfrak{d}^{\leq 1}\Gamma_g, \\
{}^{(c)}\mathcal{D}\mathrm{tr}X + 2\mathrm{tr}X\,\underline{H} &\in r^{-1}\mathfrak{d}^{\leq 1}\Gamma_g, \\
\mathcal{D}\widehat{\otimes}H + H\widehat{\otimes}H &\in r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b, \\
\overline{\mathrm{tr}X}\,\underline{H} + \mathrm{tr}XH &\in r^{-1}\Gamma_b.
\end{aligned} \tag{4.1.13}$$

In view of (3.4.3) we also have

$$\begin{aligned}
{}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4 &= \frac{4a\cos\theta(r^2+a^2)}{|q|^4}\widehat{T} + \Gamma_g\cdot\mathfrak{d}, \\
{}^{(a)}\mathrm{tr}\chi e_3 + {}^{(a)}\mathrm{tr}\underline{\chi}e_4 + 2(\eta+\underline{\eta})\cdot\ {}^*\nabla &= \frac{4a\cos\theta}{|q|^2}\mathbf{T} + \Gamma_g\cdot\mathfrak{d}.
\end{aligned} \tag{4.1.14}$$

4.2 Commutation formulas revisited

We revisit the commutation formulas of section 2.2.7. In some cases, we write the schematic structure of the error terms by making use of the definition of Γ_b and Γ_g introduced in section 4.1.2, by keeping track of different level of precision for the error terms, as they will be useful in different contexts.

Lemma 4.2.1. *The following commutation formulas hold true.*

1. *Let $h \in \mathfrak{s}_0(\mathbb{C})$ s-conformally invariant. Then*

$$\begin{aligned}
[\nabla_4, \mathcal{D}]h &= -\frac{1}{2}trX\mathcal{D}h + (\,\underline{H} + Z)\nabla_4 h - \frac{1}{2}\widehat{X}\cdot\overline{\mathcal{D}}h + \Xi\nabla_3 h,\\
[\nabla_3, \mathcal{D}]h &= -\frac{1}{2}tr\underline{X}\mathcal{D}h + (H - Z)\nabla_3 h - \frac{1}{2}\underline{\widehat{X}}\cdot\overline{\mathcal{D}}h + \underline{\Xi}\nabla_4 h.
\end{aligned} \tag{4.2.1}$$

2. *Let $F \in \mathfrak{s}_1(\mathbb{C})$. Then*

$$\begin{aligned}
[\nabla_4, \mathcal{D}\widehat{\otimes}]F &= -\frac{1}{2}trX\left(\mathcal{D}\widehat{\otimes}F + \,\underline{H}\widehat{\otimes}F\right) + (\,\underline{H} + Z)\widehat{\otimes}\nabla_4 F + \Xi\widehat{\otimes}\nabla_3 F\\
&\quad - B\widehat{\otimes}F - \frac{1}{2}tr\underline{X}\Xi\widehat{\otimes}F - \frac{1}{2}\widehat{X}\cdot\overline{\mathcal{D}}F + \frac{1}{2}\widehat{X}(\,\overline{\underline{H}}\cdot F) + (\Gamma_b\cdot\Gamma_g)F,\\
[\nabla_3, \mathcal{D}\widehat{\otimes}]F &= -\frac{1}{2}tr\underline{X}\left(\mathcal{D}\widehat{\otimes}F + H\widehat{\otimes}F\right) + (H - Z)\widehat{\otimes}\nabla_3 F + \underline{\Xi}\widehat{\otimes}\nabla_4 F\\
&\quad + \underline{B}\widehat{\otimes}F - \frac{1}{2}trX\underline{\Xi}\widehat{\otimes}F - \frac{1}{2}\underline{\widehat{X}}\cdot\overline{\mathcal{D}}F + \frac{1}{2}\underline{\widehat{X}}(\overline{H}\cdot F) + (\Gamma_b\cdot\Gamma_g)F.
\end{aligned} \tag{4.2.2}$$

Using the schematic structure of the error terms, the above can be written as

$$\begin{aligned}
[\nabla_4, \mathcal{D}\widehat{\otimes}]F &= -\frac{1}{2}trX(\mathcal{D}\widehat{\otimes}F + \underline{H}\widehat{\otimes}F) + (\underline{H} + Z)\widehat{\otimes}\nabla_4 F\\
&\quad + \Xi\cdot{}^{(c)}\nabla_3 F + r^{-1}\Gamma_g\cdot\mathfrak{d}^{\le 1}F,\\
[\nabla_3, \mathcal{D}\widehat{\otimes}]F &= -\frac{1}{2}tr\underline{X}(\mathcal{D}\widehat{\otimes}F + H\widehat{\otimes}F) + (H - Z)\widehat{\otimes}\nabla_3 F + r^{-1}\Gamma_b\cdot\mathfrak{d}^{\le 1}F.
\end{aligned} \tag{4.2.3}$$

3. *Let $U \in \mathfrak{s}_2(\mathbb{C})$. Then*

$$\begin{aligned}
[\nabla_4, \overline{\mathcal{D}}\cdot]U &= -\frac{1}{2}\overline{trX}\big(\overline{\mathcal{D}}\cdot U - 2\,\overline{\underline{H}}\cdot U\big) + \overline{(\,\underline{H} + Z)}\cdot\nabla_4 U + \overline{\Xi}\cdot\nabla_3 U\\
&\quad + 2\overline{B}\cdot U - \frac{1}{2}\overline{tr\underline{X}\Xi}\cdot U - \frac{1}{2}\widehat{X}\cdot\overline{\mathcal{D}}U - \frac{1}{2}(\overline{\widehat{X}}\cdot U)\,\overline{\underline{H}} + (\Gamma_b\cdot\Gamma_g)U,\\
[\nabla_3, \overline{\mathcal{D}}\cdot]U &= -\frac{1}{2}\overline{tr\underline{X}}\big(\overline{\mathcal{D}}\cdot U - 2\overline{H}\cdot U\big) + \overline{(H - Z)}\cdot\nabla_3 U + \overline{\underline{\Xi}}\cdot\nabla_4 U\\
&\quad - 2\overline{\underline{B}}\cdot U - \frac{1}{2}\overline{trX\underline{\Xi}}\cdot U - \frac{1}{2}\underline{\widehat{X}}\cdot\overline{\mathcal{D}}U - \frac{1}{2}(\overline{\underline{\widehat{X}}}\cdot U)\overline{H} + (\Gamma_b\cdot\Gamma_g)U.
\end{aligned} \tag{4.2.4}$$

Using the schematic structure of the error terms, the above can be written as

$$\begin{aligned}
[\nabla_4, \overline{\mathcal{D}}\cdot]U &= -\frac{1}{2}\overline{trX}\,(\overline{\mathcal{D}}\cdot U - 2\,\underline{\overline{H}}\cdot U) + \overline{(\,\underline{H} + Z)}\cdot\nabla_4 U \\
&\quad + \Xi\cdot {}^{(c)}\nabla_3 U + r^{-1}\Gamma_g\cdot\mathfrak{d}^{\leq 1}U, \\
[\nabla_3, \overline{\mathcal{D}}\cdot]U &= -\frac{1}{2}\overline{tr\underline{X}}\,(\overline{\mathcal{D}}\cdot U - 2\overline{H}\cdot U) + \overline{(H - Z)}\cdot\nabla_3 U + r^{-1}\Gamma_b\cdot\mathfrak{d}^{\leq 1}U.
\end{aligned} \tag{4.2.5}$$

Similarly, for $F \in \mathfrak{s}_1(\mathbb{C})$ we have

$$\begin{aligned}
[\nabla_4, \overline{\mathcal{D}}\cdot]F &= -\frac{1}{2}\overline{trX}\,(\overline{\mathcal{D}}\cdot F - \,\underline{\overline{H}}\cdot F) + \overline{(\,\underline{H} + Z)}\cdot\nabla_4 F \\
&\quad + \overline{\Xi}\cdot {}^{(c)}\nabla_3 F + r^{-1}\Gamma_g\cdot\mathfrak{d}^{\leq 1}F, \\
[\nabla_3, \overline{\mathcal{D}}\cdot]F &= -\frac{1}{2}\overline{tr\underline{X}}\,(\overline{\mathcal{D}}\cdot F - \overline{H}\cdot F) + \overline{(H - Z)}\cdot\nabla_3 F + r^{-1}\Gamma_b\cdot\mathfrak{d}^{\leq 1}F.
\end{aligned} \tag{4.2.6}$$

4. *Let $U \in \mathfrak{s}_2(\mathbb{C})$. Then*

$$\begin{aligned}
[\nabla_3, \nabla_4]U &= -2\omega\nabla_3 U + 2\underline{\omega}\nabla_4 U + 2(\eta_c - \underline{\eta}_c)\nabla_c U + 4i\left(-\,{}^*\rho + \eta\wedge\underline{\eta}\right)U \\
&\quad + (\Gamma_b\cdot\Gamma_g)\,U.
\end{aligned} \tag{4.2.7}$$

Also,

$$\begin{aligned}
[\nabla_3, \nabla_a]U_{bc} &= -\frac{1}{2}tr\underline{\chi}\left(\nabla_a U_{bc} + \eta_b U_{ac} + \eta_c U_{ab} - \delta_{ab}(\eta\cdot U)_c - \delta_{ac}(\eta\cdot U)_b\right) \\
&\quad - \frac{1}{2}{}^{(a)}tr\underline{\chi}\left({}^*\nabla_a U_{bc} + \eta_b\,{}^*U_{ac} + \eta_c\,{}^*U_{ab} - \in_{ab}(\eta\cdot U)_c - \in_{ac}(\eta\cdot U)_b\right) \\
&\quad + (\eta_a - \zeta_a)\nabla_3 U_{bc} + r^{-1}\Gamma_b\cdot\mathfrak{d}^{\leq 1}U,
\end{aligned} \tag{4.2.8}$$

where the above error terms may also contain terms which are quadratic in the perturbation and enjoy better decay properties, or are higher order and decay at least as good.

Proof. See section C.1. □

We collect here the commutation formulas for the conformal derivatives introduced in Lemma 2.2.18.

Lemma 4.2.2. *The following commutation formulas hold true.*

1. *Let $h \in \mathfrak{s}_0(\mathbb{C})$ s-conformally invariant. Then*

$$
\begin{aligned}
[\,^{(c)}\nabla_4, \,^{(c)}\mathcal{D}]h &= -\frac{1}{2}trX\,^{(c)}\mathcal{D}h + \underline{H}\,^{(c)}\nabla_4 h - \frac{1}{2}\widehat{X}\cdot\overline{\,^{(c)}\mathcal{D}}h + \Xi\,^{(c)}\nabla_3 h\\
&+ s\left(\frac{1}{2}trX\,\underline{H} + \frac{1}{2}\widehat{X}\cdot\overline{\underline{H}} - \frac{1}{2}tr\underline{X}\Xi - B\right)h + (\Gamma_b\cdot\Gamma_g)h,\\
[\,^{(c)}\nabla_3, \,^{(c)}\mathcal{D}]h &= -\frac{1}{2}tr\underline{X}\,^{(c)}\mathcal{D}h + H\,^{(c)}\nabla_3 h - \frac{1}{2}\widehat{\underline{X}}\cdot\overline{\,^{(c)}\mathcal{D}}h + \underline{\Xi}\,^{(c)}\nabla_4 h\\
&- s\left(\frac{1}{2}tr\underline{X}H + \frac{1}{2}\widehat{\underline{X}}\cdot\overline{H} - \frac{1}{2}trX\underline{\Xi} + \underline{B}\right)h + (\Gamma_b\cdot\Gamma_g)h.
\end{aligned}
\tag{4.2.9}
$$

Using the schematic structure of the error terms, the above can be written as

$$
\begin{aligned}
[\,^{(c)}\nabla_4, \,^{(c)}\mathcal{D}]h &= -\frac{1}{2}trX\,^{(c)}\mathcal{D}h + s\frac{1}{2}trX\,\underline{H}h + \underline{H}\,^{(c)}\nabla_4 h\\
&+ \Xi\,^{(c)}\nabla_3 h + r^{-1}\Gamma_g\cdot\mathfrak{d}^{\leq 1}h,\\
[\,^{(c)}\nabla_3, \,^{(c)}\mathcal{D}]h &= -\frac{1}{2}tr\underline{X}\,^{(c)}\mathcal{D}h - s\frac{1}{2}tr\underline{X}Hh + H\,^{(c)}\nabla_3 h + r^{-1}\Gamma_b\cdot\mathfrak{d}^{\leq 1}h.
\end{aligned}
\tag{4.2.10}
$$

We also have

$$
[\,^{(c)}\nabla_3, \,^{(c)}\nabla_4]h \;=\; 2(\eta-\underline{\eta})\cdot\,^{(c)}\nabla h + 2s(\rho - \eta\cdot\underline{\eta})h + (\Gamma_b\cdot\Gamma_g)h. \tag{4.2.11}
$$

2. *Let $F \in \mathfrak{s}_1(\mathbb{C})$ s-conformally invariant. Then*

$$
\begin{aligned}
[\,^{(c)}\nabla_4, \,^{(c)}\mathcal{D}\widehat{\otimes}]F &= -\frac{1}{2}trX\left(\,^{(c)}\mathcal{D}\widehat{\otimes}F + (1-s)\,\underline{H}\widehat{\otimes}F\right) + \underline{H}\widehat{\otimes}\,^{(c)}\nabla_4 F + \Xi\widehat{\otimes}\,^{(c)}\nabla_3 F\\
&- (s+1)B\widehat{\otimes}F - (s+1)\frac{1}{2}tr\underline{X}\Xi\widehat{\otimes}F - \frac{1}{2}\widehat{X}\cdot\overline{\,^{(c)}\mathcal{D}}F\\
&+ \frac{1}{2}\widehat{X}(\overline{\underline{H}}\cdot F) + s\frac{1}{2}(\widehat{X}\cdot\overline{\underline{H}})\widehat{\otimes}F + (\Gamma_b\cdot\Gamma_g)F,\\
[\,^{(c)}\nabla_3, \,^{(c)}\mathcal{D}\widehat{\otimes}]F &= -\frac{1}{2}tr\underline{X}\left(\,^{(c)}\mathcal{D}\widehat{\otimes}F + (s+1)H\widehat{\otimes}F\right) + H\widehat{\otimes}\,^{(c)}\nabla_3 F + \underline{\Xi}\widehat{\otimes}\,^{(c)}\nabla_4 F\\
&- (s-1)\underline{B}\widehat{\otimes}F + (s-1)\frac{1}{2}trX\underline{\Xi}\widehat{\otimes}F - \frac{1}{2}\widehat{\underline{X}}\cdot\overline{\,^{(c)}\mathcal{D}}F\\
&+ \frac{1}{2}\widehat{\underline{X}}(\overline{H}\cdot F) - s\frac{1}{2}(\widehat{\underline{X}}\cdot\overline{H})\widehat{\otimes}F + (\Gamma_b\cdot\Gamma_g)F.
\end{aligned}
\tag{4.2.12}
$$

Using the schematic structure of the error terms, the above can be written as

$$
\begin{aligned}
[\,^{(c)}\nabla_4, \,^{(c)}\mathcal{D}\widehat{\otimes}]F &= -\frac{1}{2}trX\left(\,^{(c)}\mathcal{D}\widehat{\otimes}F + (1-s)\,\underline{H}\widehat{\otimes}F\right) + \underline{H}\widehat{\otimes}\,^{(c)}\nabla_4 F\\
&+ \Xi\widehat{\otimes}\,^{(c)}\nabla_3 F + r^{-1}\Gamma_g\cdot\mathfrak{d}^{\leq 1}F,\\
[\,^{(c)}\nabla_3, \,^{(c)}\mathcal{D}\widehat{\otimes}]F &= -\frac{1}{2}tr\underline{X}\left(\,^{(c)}\mathcal{D}\widehat{\otimes}F + (1+s)H\widehat{\otimes}F\right) + H\widehat{\otimes}\,^{(c)}\nabla_3 F + r^{-1}\Gamma_b\cdot\mathfrak{d}^{\leq 1}F.
\end{aligned}
\tag{4.2.13}
$$

3. *Let $U \in \mathfrak{s}_2(\mathbb{C})$ s-conformally invariant. Then*

$$
\begin{aligned}
[\,^{(c)}\nabla_4, \overline{^{(c)}\mathcal{D}}\cdot]U &= -\frac{1}{2}\overline{trX}\big(\overline{^{(c)}\mathcal{D}}\cdot U-(s+2)\overline{\underline{H}}\cdot U\big)+\overline{\underline{H}}\cdot\,^{(c)}\nabla_4U\\
&+\overline{\Xi}\cdot\,^{(c)}\nabla_3U-(s-2)\overline{B}\cdot U-(s+1)\frac{1}{2}\overline{tr\underline{X}\Xi}\cdot U-\frac{1}{2}\widehat{X}\cdot\overline{^{(c)}\mathcal{D}}U\\
&-\frac{1}{2}(\overline{\widehat{X}}\cdot U)\overline{\underline{H}}+s\frac{1}{2}(\overline{\widehat{X}}\cdot\,\underline{H})\cdot U+(\Gamma_b\cdot\Gamma_g)U,\\
[\,^{(c)}\nabla_3, \overline{^{(c)}\mathcal{D}}\cdot]U &= -\frac{1}{2}\overline{tr\underline{X}}\big(\overline{^{(c)}\mathcal{D}}\cdot U+(s-2)\overline{H}\cdot U\big)+\overline{H}\cdot\,^{(c)}\nabla_3U\\
&+\overline{\underline{\Xi}}\cdot\,^{(c)}\nabla_4U+(s+2)\overline{\underline{B}}\cdot U+(s-1)\frac{1}{2}\overline{trX\underline{\Xi}}\cdot U-\frac{1}{2}\underline{\widehat{X}}\cdot\overline{^{(c)}\mathcal{D}}U\\
&-\frac{1}{2}(\overline{\underline{\widehat{X}}}\cdot U)\overline{H}-s\frac{1}{2}(\overline{\underline{\widehat{X}}}\cdot H)\cdot U+(\Gamma_b\cdot\Gamma_g)U.
\end{aligned}
\tag{4.2.14}
$$

Using the schematic structure of the error terms, the above can be written as

$$
\begin{aligned}
[\,^{(c)}\nabla_4, \overline{^{(c)}\mathcal{D}}\cdot]U &= -\frac{1}{2}\overline{trX}\,(\overline{^{(c)}\mathcal{D}}\cdot U-(s+2)\overline{\underline{H}}\cdot U)+\overline{\underline{H}}\cdot\,^{(c)}\nabla_4U\\
&+\overline{\Xi}\cdot\,^{(c)}\nabla_3U+r^{-1}\Gamma_g\cdot\mathfrak{d}^{\leq 1}U,\\
[\,^{(c)}\nabla_3, \overline{^{(c)}\mathcal{D}}\cdot]U &= -\frac{1}{2}\overline{tr\underline{X}}\,(\overline{^{(c)}\mathcal{D}}\cdot U+(s-2)\overline{H}\cdot U)+\overline{H}\cdot\,^{(c)}\nabla_3U+r^{-1}\Gamma_b\cdot\mathfrak{d}^{\leq 1}U.
\end{aligned}
\tag{4.2.15}
$$

Similarly, for $F \in \mathfrak{s}_1(\mathbb{C})$,

$$
\begin{aligned}
[\,^{(c)}\nabla_4, \overline{^{(c)}\mathcal{D}}\cdot]F &= -\frac{1}{2}\overline{trX}\,\Big(\overline{^{(c)}\mathcal{D}}\cdot F-(s+1)\overline{\underline{H}}\cdot F\Big)+\overline{\underline{H}}\cdot\,^{(c)}\nabla_4F\\
&+\overline{\Xi}\cdot\,^{(c)}\nabla_3F+r^{-1}\Gamma_g\cdot\mathfrak{d}^{\leq 1}F,\\
[\,^{(c)}\nabla_3, \overline{^{(c)}\mathcal{D}}\cdot]F &= -\frac{1}{2}\overline{tr\underline{X}}\,(\overline{^{(c)}\mathcal{D}}\cdot F+(s-1)\overline{H}\cdot F)+\overline{H}\cdot\,^{(c)}\nabla_3F+r^{-1}\Gamma_b\cdot\mathfrak{d}^{\leq 1}F.
\end{aligned}
\tag{4.2.16}
$$

4. *Let $U \in \mathfrak{s}_2(\mathbb{C})$ s-conformally invariant. Then*

$$
\begin{aligned}
[\,^{(c)}\nabla_3, \,^{(c)}\nabla_4]U &= 2(\eta-\underline{\eta})\cdot\,^{(c)}\nabla U+\Big(2s\left(\rho-\eta\cdot\underline{\eta}\right)+4i\left(-\,^*\rho+\eta\wedge\underline{\eta}\right)\Big)U\\
&\quad+(\Gamma_b\cdot\Gamma_g)U,\\
&= 2(\eta-\underline{\eta})\cdot\,^{(c)}\nabla U+\Big((s-2)P+(s+2)\overline{P}-2s\eta\cdot\underline{\eta}+4i\eta\wedge\underline{\eta}\Big)U\\
&\quad+(\Gamma_b\cdot\Gamma_g)U.
\end{aligned}
$$

Also,

$$
\begin{aligned}
[\,{}^{(c)}\nabla_3,\, {}^{(c)}\nabla_a]U_{bc} &= \eta_a\, {}^{(c)}\nabla_3 U_{bc} - \frac{1}{2}tr\underline{\chi}\Big[\,{}^{(c)}\nabla_a U_{bc} + s(\eta_a)U_{bc} + \eta_b U_{ac} + \eta_c U_{ab} \\
&\quad -\delta_{ab}(\eta\cdot U)_c - \delta_{ac}(\eta\cdot U)_b\Big] \\
&\quad -\frac{1}{2}\,{}^{(a)}tr\underline{\chi}\Big[\,{}^{*\,(c)}\nabla_a U_{bc} + s({}^*\eta_a)U_{bc} + \eta_b\,{}^*U_{ac} + \eta_c\,{}^*U_{ab} \\
&\quad - \in_{ab}(\eta\cdot U)_c - \in_{ac}(\eta\cdot U)_b\Big] + r^{-1}\Gamma_b\cdot\mathfrak{d}^{\leq 1}U.
\end{aligned}
$$

Proof. See section C.2. □

4.3 Approximate Killing vectorfields T and Z

Definition 4.3.1. *In $\mathcal{M}$, we define* **T** *and* **Z** *as follows:*

- *If the normalization of (e_3, e_4) is ingoing, we have*

$$
\begin{aligned}
\mathbf{T} &:= \frac{1}{2}\left(e_4 + \frac{\Delta}{|q|^2}e_3 - 2a\Re(\mathfrak{J})^b e_b\right), \\
\mathbf{Z} &:= \frac{1}{2}\left(2(r^2+a^2)\Re(\mathfrak{J})^b e_b - a(\sin\theta)^2 e_4 - \frac{a(\sin\theta)^2\Delta}{|q|^2}e_3\right).
\end{aligned}
$$

- *If the normalization of (e_3, e_4) is outgoing, we have*

$$
\begin{aligned}
\mathbf{T} &:= \frac{1}{2}\left(e_3 + \frac{\Delta}{|q|^2}e_4 - 2a\Re(\mathfrak{J})^b e_b\right), \\
\mathbf{Z} &:= \frac{1}{2}\left(2(r^2+a^2)\Re(\mathfrak{J})^b e_b - a(\sin\theta)^2 e_3 - \frac{a(\sin\theta)^2\Delta}{|q|^2}e_4\right).
\end{aligned}
$$

From relations (3.3.8) and (3.3.15) and the values of $\mathfrak{J}$ as in Definition 3.4.2, one can see that the above vectorfields reduce to the Killing vectorfields **T** and **Z** in Kerr.

The following lemma shows that **T** and **Z** are almost Killing vectorfields.

Lemma 4.3.2. *Let ${}^{(\mathbf{T})}\pi$ and ${}^{(\mathbf{Z})}\pi$ be the deformation tensors of* **T** *and* **Z** *as defined above. We have*

$$
{}^{(\mathbf{T})}\pi_{44},\, {}^{(\mathbf{T})}\pi_{4a} \in \Gamma_g, \qquad {}^{(\mathbf{T})}\pi_{33},\, {}^{(\mathbf{T})}\pi_{34},\, {}^{(\mathbf{T})}\pi_{3a},\, {}^{(\mathbf{T})}\pi_{ab} \in \Gamma_b,
$$

and

$$^{(\mathbf{Z})}\pi_{44},\ r^{-1}\,{}^{(\mathbf{Z})}\pi_{4a}\in\Gamma_g,\qquad {}^{(\mathbf{Z})}\pi_{33},\ {}^{(\mathbf{Z})}\pi_{34},\ {}^{(\mathbf{Z})}\pi_{3a},\ {}^{(\mathbf{Z})}\pi_{ab}\in r\Gamma_b.$$

Moreover,

$$tr^{(\mathbf{T})}\pi\in\Gamma_g,\qquad tr^{(\mathbf{Z})}\pi\in r\Gamma_b,$$

and

$$\begin{array}{ll}(Div^{(\mathbf{T})}\pi)_3\in r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b, & (Div^{(\mathbf{T})}\pi)_4, (Div^{(\mathbf{T})}\pi)_a\in\mathfrak{d}^{\leq 1}\Gamma_g,\\ (Div^{(\mathbf{Z})}\pi)_4\in\mathfrak{d}^{\leq 1}\Gamma_g, & (Div^{(\mathbf{Z})}\pi)_3\in\mathfrak{d}^{\leq 1}\Gamma_b,\\ (Div^{(\mathbf{Z})}\pi)_a\in r\mathfrak{d}^{\leq 1}\Gamma_g. & \end{array}$$

Proof. See section C.3. □

We collect here the following commutator identities for **T** and **Z** and the D'Alembertian, for scalar functions and for horizontal symmetric 2-tensors. For horizontal symmetric 2-tensors additional terms coming from curvature appear.

Proposition 4.3.3. *The following commutation formulas hold true for a scalar ψ:*

$$\begin{aligned}[][\mathbf{T},\square_{\mathbf{g}}]\psi &= \mathfrak{d}\big(\Gamma_g\cdot\mathfrak{d}\psi\big)+\Gamma_b\cdot\square_{\mathbf{g}}\psi,\\ [\mathbf{Z},\square_{\mathbf{g}}]\psi &= \mathfrak{d}\big(\Gamma_g\cdot\mathfrak{d}\psi\big)+r\Gamma_b\cdot\square_{\mathbf{g}}\psi.\end{aligned}\tag{4.3.1}$$

The following commutation formulas hold true for $\psi\in\mathfrak{s}_2$:

$$\begin{aligned}[][\nabla_{\mathbf{T}},\dot{\square}_2]\psi &= O(ar^{-4})\mathfrak{d}^{\leq 1}\psi+\mathfrak{d}\big(\Gamma_g\cdot\mathfrak{d}\psi\big)+\Gamma_b\cdot\dot{\square}_2\psi,\\ [\nabla_{\mathbf{Z}},\dot{\square}_2]\psi &= O(r^{-2})\mathfrak{d}^{\leq 1}\psi+\mathfrak{d}\big(\Gamma_g\cdot\mathfrak{d}\psi\big)+r\Gamma_b\cdot\dot{\square}_2\psi.\end{aligned}\tag{4.3.2}$$

Proof. See section C.4 for the proof of (4.3.1). By putting together Proposition 3.6.2 and (4.3.1) we deduce (4.3.2). □

Recall the definition of horizontal Lie derivative $\dot{\mathcal{L}}_X$ for $X\in\mathbf{T}(\mathcal{M})$ as given in Definition 2.2.14. We collect here the commutator identities for $\dot{\mathcal{L}}_{\mathbf{T}}$ and $\dot{\mathcal{L}}_{\mathbf{Z}}$ and the D'Alembertian operator.

Corollary 4.3.4. *The following commutation formulas holds true for $\psi\in\mathfrak{s}_2$:*

$$\begin{aligned}[][\dot{\mathcal{L}}_{\mathbf{T}},\dot{\square}_2]\psi_{ab} &= \mathfrak{d}\big(\Gamma_g\cdot\mathfrak{d}\psi\big)+\Gamma_b\cdot\dot{\square}_2\psi,\\ [\dot{\mathcal{L}}_{\mathbf{Z}},\dot{\square}_2]\psi_{ab} &= \mathfrak{d}\big(\Gamma_g\cdot\mathfrak{d}\psi\big)+r\Gamma_b\cdot\dot{\square}_2\psi.\end{aligned}$$

Proof. This is a corollary of Proposition 2.3.2, according to which

$$
\begin{aligned}
[\dot{\mathcal{L}}_X, \dot{\square}_2]\psi_{ab} &= -{}^{(X)}\pi^{\mu\nu}\dot{\mathbf{D}}_\mu\dot{\mathbf{D}}_\nu\psi_{ab} - {}^{(X)}\Gamma^\mu{}_{\mu\rho}\dot{\mathbf{D}}^\rho\psi_{ab} \\
& \quad -2{}^{(X)}\not{\Gamma}_{a\mu c}\dot{\mathbf{D}}^\mu\psi^c{}_b - 2{}^{(X)}\not{\Gamma}_{b\mu c}\dot{\mathbf{D}}^\mu\psi_a{}^c - \dot{\mathbf{D}}^\nu({}^{(X)}\not{\Gamma}_{a\nu c})\psi^c{}_b - \dot{\mathbf{D}}^\nu({}^{(X)}\not{\Gamma}_{b\nu c})\psi_a{}^c
\end{aligned}
$$

where

$$
{}^{(X)}\Gamma_{\alpha\beta\mu} = \frac{1}{2}(\mathbf{D}_\alpha\, {}^{(X)}\pi_{\beta\mu} + \mathbf{D}_\beta\, {}^{(X)}\pi_{\alpha\mu} - \mathbf{D}_\mu\, {}^{(X)}\pi_{\alpha\beta}),
$$

and similarly for ${}^{(X)}\not{\Gamma}$. In particular, we can write

$$
\begin{aligned}
[\dot{\mathcal{L}}_X, \dot{\square}_2]\psi_{ab} &= -{}^{(X)}\pi^{\mu\nu}\dot{\mathbf{D}}_\mu\dot{\mathbf{D}}_\nu\psi_{ab} - \Big(\mathrm{Div}\, {}^{(X)}\pi_\rho - \frac{1}{2}\mathbf{D}_\rho\, {}^{(X)}\pi\Big)\dot{\mathbf{D}}^\rho\psi_{ab} \\
& \quad -2{}^{(X)}\not{\Gamma}_{a\mu c}\dot{\mathbf{D}}^\mu\psi^c{}_b - 2{}^{(X)}\not{\Gamma}_{b\mu c}\dot{\mathbf{D}}^\mu\psi_a{}^c - \dot{\mathbf{D}}^\nu({}^{(X)}\not{\Gamma}_{a\nu c})\psi^c{}_b - \dot{\mathbf{D}}^\nu({}^{(X)}\not{\Gamma}_{b\nu c})\psi_a{}^c.
\end{aligned}
$$

Observe that the first line gives the same expression as for $[X, \square_{\mathbf{g}}]\psi$, i.e. (C.4.1), and can then be computed as in Proposition 4.3.3. For the second line we compute

$$
\begin{aligned}
\dot{\mathbf{D}}^\nu({}^{(X)}\not{\Gamma}_{a\nu c}) &= \frac{1}{2}(\mathbf{D}^\mu\nabla_a\, {}^{(X)}\pi_{\mu c} + \mathbf{D}^\mu\nabla_\mu\, {}^{(X)}\pi_{ac} - \mathbf{D}^\mu\nabla_c\, {}^{(X)}\pi_{a\mu}) \\
&= \frac{1}{2}(\nabla_a\mathrm{Div}\, {}^{(X)}\pi_c - \nabla_c\mathrm{Div}\, {}^{(X)}\pi_a + \square_{\mathbf{g}}\, {}^{(X)}\pi_{ac}) + O(r^{-2})\, {}^{(X)}\pi,
\end{aligned}
$$

and

$$
{}^{(X)}\not{\Gamma}_{a\mu c}\dot{\mathbf{D}}^\mu\psi^c{}_b = \Big(r^{-1}\big({}^{(X)}\not{\Gamma}_{adc}, {}^{(X)}\not{\Gamma}_{a3c}\big), {}^{(X)}\not{\Gamma}_{a4c}\Big)\mathfrak{d}\psi_{cb}.
$$

Using Lemma 4.3.2 and the improved decay for ${}^{(\mathbf{Z})}\pi$ in (C.4.3), we obtain

$$
\begin{aligned}
-\dot{\mathbf{D}}^\nu({}^{(\mathbf{T})}\not{\Gamma}_{a\nu c})\psi^c{}_b - \dot{\mathbf{D}}^\nu({}^{(\mathbf{T})}\not{\Gamma}_{b\nu c})\psi_a{}^c &= r^{-1}\mathfrak{d}^{\leq 2}\Gamma_b\cdot\psi, \\
-\dot{\mathbf{D}}^\nu({}^{(\mathbf{Z})}\not{\Gamma}_{a\nu c})\psi^c{}_b - \dot{\mathbf{D}}^\nu({}^{(\mathbf{Z})}\not{\Gamma}_{b\nu c})\psi_a{}^c &= \mathfrak{d}^{\leq 2}\Gamma_g\cdot\psi,
\end{aligned}
$$

and

$$
\begin{aligned}
{}^{(\mathbf{T})}\not{\Gamma}_{adc} &= \frac{1}{2}(\nabla_a{}^{(\mathbf{T})}\pi_{dc} + \nabla_d{}^{(\mathbf{T})}\pi_{ac} - \nabla_c{}^{(\mathbf{T})}\pi_{ad}) = r^{-1}\mathfrak{d}\Gamma_b, \\
{}^{(\mathbf{T})}\not{\Gamma}_{a3c} &= \frac{1}{2}(\nabla_a{}^{(\mathbf{T})}\pi_{3c} + \nabla_3{}^{(\mathbf{T})}\pi_{ac} - \nabla_c{}^{(\mathbf{T})}\pi_{a3}) = \mathfrak{d}\Gamma_b, \\
{}^{(\mathbf{T})}\not{\Gamma}_{a4c} &= \frac{1}{2}(\nabla_a{}^{(\mathbf{T})}\pi_{4c} + \nabla_4{}^{(\mathbf{T})}\pi_{ac} - \nabla_c{}^{(\mathbf{T})}\pi_{a4}) = r^{-1}\mathfrak{d}\Gamma_b, \\
{}^{(\mathbf{Z})}\not{\Gamma}_{adc} &= \frac{1}{2}(\nabla_a\, {}^{(\mathbf{Z})}\pi_{dc} + \nabla_d\, {}^{(\mathbf{Z})}\pi_{ac} - \nabla_c\, {}^{(\mathbf{Z})}\pi_{ad}) = \mathfrak{d}\Gamma_b, \\
{}^{(\mathbf{Z})}\not{\Gamma}_{a3c} &= \frac{1}{2}(\nabla_a\, {}^{(\mathbf{Z})}\pi_{3c} + \nabla_3\, {}^{(\mathbf{Z})}\pi_{ac} - \nabla_c\, {}^{(\mathbf{Z})}\pi_{a3}) = \Gamma_b, \\
{}^{(\mathbf{Z})}\not{\Gamma}_{a4c} &= \frac{1}{2}(\nabla_a\, {}^{(\mathbf{Z})}\pi_{4c} + \nabla_4\, {}^{(\mathbf{Z})}\pi_{ac} - \nabla_c\, {}^{(\mathbf{Z})}\pi_{a4}) = \Gamma_g.
\end{aligned}
$$

Putting the above together, this concludes the proof. □

4.4 Inverse metric and $\mathcal{R}$, O tensors

In $\mathcal{M}$, we define the vectorfields $\widehat{T}$ and $\widehat{R}$ as follows:

- If the normalization of (e_3, e_4) is ingoing, we have

$$\begin{aligned}\widehat{T} &:= \frac{1}{2}\left(\frac{|q|^2}{r^2+a^2}e_4+\frac{\Delta}{r^2+a^2}e_3\right),\\ \widehat{R} &:= \frac{1}{2}\left(\frac{|q|^2}{r^2+a^2}e_4-\frac{\Delta}{r^2+a^2}e_3\right).\end{aligned}$$

- If the normalization of (e_3, e_4) is outgoing, we have

$$\begin{aligned}\widehat{T} &:= \frac{1}{2}\left(\frac{\Delta}{r^2+a^2}e_4+\frac{|q|^2}{r^2+a^2}e_3\right),\\ \widehat{R} &:= \frac{1}{2}\left(\frac{\Delta}{r^2+a^2}e_4-\frac{|q|^2}{r^2+a^2}e_3\right).\end{aligned}$$

From relations (3.3.14) and (3.3.6), one can see that the above vectorfields reduce to $\widehat{T}$ and $\widehat{R}$ in Kerr. Note also that $\widehat{T}, \widehat{R}$ are perpendicular to the horizontal structure and $\mathbf{g}(\widehat{T}, \widehat{R}) = 0$.

Lemma 4.4.1. *The vectorfields $\widehat{R}$ and $\widehat{T}$ satisfy*

$$[\widehat{R}, \widehat{T}] = -\frac{ar\Delta}{(r^2+a^2)^3}\mathbf{Z} + r^{-1}\Gamma_b \cdot \mathfrak{d}.$$

Proof. Using the definition in the outgoing normalization, we have

$$\begin{aligned}[\widehat{R}, \widehat{T}] &= \frac{1}{4}\left[\frac{\Delta}{r^2+a^2}e_4-\frac{|q|^2}{r^2+a^2}e_3, \frac{\Delta}{r^2+a^2}e_4+\frac{|q|^2}{r^2+a^2}e_3\right]\\ &= \frac{1}{2}\left[\frac{\Delta}{r^2+a^2}e_4, \frac{|q|^2}{r^2+a^2}e_3\right]\\ &= \frac{1}{2}\frac{|q|^2\Delta}{(r^2+a^2)^2}[e_4, e_3] + \frac{1}{2}\frac{\Delta}{r^2+a^2}e_4\left(\frac{|q|^2}{r^2+a^2}\right)e_3 - \frac{1}{2}\frac{|q|^2}{r^2+a^2}e_3\left(\frac{\Delta}{r^2+a^2}\right)e_4.\end{aligned}$$

Using Lemma 2.2.8 to write in the outgoing normalization

$$\begin{aligned}[e_4, e_3] &= 2(\underline{\eta}-\eta)\cdot\nabla + 2\omega e_3 - 2\underline{\omega}e_4\\ &= -\frac{4ar}{|q|^2}\Re(\mathfrak{J}^b)e_b - \partial_r\left(\frac{\Delta}{|q|^2}\right)e_4 + r^{-1}\Gamma_b \cdot \mathfrak{d},\end{aligned}$$

and using (C.7.1) we obtain

$$\begin{aligned}
[\widehat{R}, \widehat{T}] &= -\frac{ar\Delta}{(r^2+a^2)^3}\Big(2(r^2+a^2)\Re(\mathfrak{J}^b)e_b - a\sin^2\theta e_3 - \frac{\Delta a\sin^2\theta}{|q|^2}e_4\Big) + r^{-1}\Gamma_b\cdot\mathfrak{d}\\
&= -\frac{ar\Delta}{(r^2+a^2)^3}\mathbf{Z} + r^{-1}\Gamma_b\cdot\mathfrak{d},
\end{aligned}$$

as stated. □

Observe that with the above definitions we have

$$\widehat{T} = \mathbf{T} + \frac{a}{r^2+a^2}\mathbf{Z}. \tag{4.4.1}$$

Indeed, for example in the outgoing normalization,

$$\begin{aligned}
\mathbf{T} + \frac{a}{r^2+a^2}\mathbf{Z} &= \frac{1}{2}\left(e_3 + \frac{\Delta}{|q|^2}e_4 - 2a\Re(\mathfrak{J})^b e_b\right)\\
&\quad + \frac{a}{r^2+a^2}\frac{1}{2}\left(2(r^2+a^2)\Re(\mathfrak{J})^b e_b - a(\sin\theta)^2 e_3 - \frac{a(\sin\theta)^2\Delta}{|q|^2}e_4\right)\\
&= \frac{1}{2}\left(\frac{\Delta}{r^2+a^2}e_4 + \frac{|q|^2}{r^2+a^2}e_3\right) = \widehat{T}.
\end{aligned}$$

We also define, in terms of the horizontal structure,

$$O^{\alpha\beta} := |q|^2\gamma^{ab}e_a^\alpha e_b^\beta, \tag{4.4.2}$$

where we recall that $\gamma_{ab} = \mathbf{g}(e_a, e_b)$, i.e. γ is the metric induced by $\mathbf{g}$ on the horizontal structure. From (3.5.3), one can see that the above tensor reduce to $O^{\alpha\beta}$ in Kerr.

The definition $O^{\alpha\beta}$ allows us to express the inverse metric in perturbations of Kerr as follows, see Lemma 3.5.1 for the case of Kerr.

Lemma 4.4.2. *Let $(\mathcal{M}, \mathbf{g})$ be a spacetime, with $\widehat{R}$, $\widehat{T}$ and O defined as above. Then*

$$|q|^2\mathbf{g}^{\alpha\beta} = \frac{(r^2+a^2)^2}{\Delta}\widehat{R}^\alpha\widehat{R}^\beta + \frac{1}{\Delta}\mathcal{R}^{\alpha\beta} \tag{4.4.3}$$

where

$$\mathcal{R}^{\alpha\beta} := -(r^2+a^2)^2\widehat{T}^\alpha\widehat{T}^\beta + \Delta O^{\alpha\beta}. \tag{4.4.4}$$

Thus the inverse metric can also be written in the form[3]

$$|q|^2\mathbf{g}^{\alpha\beta} = \frac{(r^2+a^2)^2}{\Delta}\big(-\widehat{T}^\alpha\widehat{T}^\beta + \widehat{R}^\alpha\widehat{R}^\beta\big) + O^{\alpha\beta}. \tag{4.4.5}$$

[3]Observe that this expression of the metric is not regular at the horizon.

Proof. Using the definition of frames in $(\mathcal{M}, \mathbf{g})$, we can write

$$\mathbf{g}^{\alpha\beta} = -\frac{1}{2}\big(e_3^\alpha e_4^\beta + e_3^\beta e_4^\alpha\big) + \gamma^{ab} e_a^\alpha e_b^\beta.$$

Using (4.4.2), we obtain

$$|q|^2\mathbf{g}^{\alpha\beta} = -\frac{1}{2}|q|^2\big(e_3^\alpha e_4^\beta + e_3^\beta e_4^\alpha\big) + O^{\alpha\beta}.$$

We then compute, for example in the ingoing normalization,

$$\begin{aligned}
\widehat{R}^\alpha\widehat{R}^\beta - \widehat{T}^\alpha\widehat{T}^\beta &= \frac{1}{4}\left(\frac{|q|^2}{r^2+a^2}e_4^\alpha - \frac{\Delta}{r^2+a^2}e_3^\alpha\right)\left(\frac{|q|^2}{r^2+a^2}e_4^\beta - \frac{\Delta}{r^2+a^2}e_3^\beta\right)\\
&\quad -\frac{1}{4}\left(\frac{|q|^2}{r^2+a^2}e_4^\alpha + \frac{\Delta}{r^2+a^2}e_3^\alpha\right)\left(\frac{|q|^2}{r^2+a^2}e_4^\beta + \frac{\Delta}{r^2+a^2}e_3^\beta\right)\\
&= \frac{\Delta}{(r^2+a^2)^2}\left(-\frac{1}{2}|q|^2 e_3^\alpha e_4^\beta - \frac{1}{2}|q|^2 e_3^\beta e_4^\alpha\right),
\end{aligned}$$

which proves the lemma. □

As in Definition 3.5.3, we define the following approximate symmetric tensors.

Definition 4.4.3. *We define the following symmetric spacetime 2-tensors:*

$$\begin{aligned}
S_1^{\alpha\beta} &:= \mathbf{T}^\alpha\mathbf{T}^\beta,\\
S_2^{\alpha\beta} &:= a\mathbf{T}^{(\alpha}\mathbf{Z}^{\beta)},\\
S_3^{\alpha\beta} &:= a^2\mathbf{Z}^\alpha\mathbf{Z}^\beta,\\
S_4^{\alpha\beta} &:= O^{\alpha\beta} = |q|^2\gamma^{ab}e_a^\alpha e_b^\beta.
\end{aligned}$$

We denote the set of the above tensors as $S_{\underline{a}}$*, for* $\underline{a} = 1, 2, 3, 4$.

Using (4.4.1) to write $\widehat{T} = \mathbf{T} + \frac{a}{r^2+a^2}\mathbf{Z}$ in (4.4.4), we can write

$$\mathcal{R}^{\alpha\beta} = \mathcal{R}^{\underline{a}} S_{\underline{a}}^{\alpha\beta}, \tag{4.4.6}$$

with $\mathcal{R}^{\underline{a}}$, $\underline{a} = 1, 2, 3, 4$ given by

$$\mathcal{R}^1 = -(r^2+a^2)^2, \quad \mathcal{R}^2 = -2(r^2+a^2), \quad \mathcal{R}^3 = -1, \quad \mathcal{R}^4 = \Delta. \tag{4.4.7}$$

4.5 Approximate Carter tensor and Carter operator

We now extend the definition of Carter tensor as given in Definition 3.7.1 to the case of perturbations of Kerr.

Definition 4.5.1. *In* $(\mathcal{M}, \mathbf{g})$ *the Carter tensor is defined as the following symmetric 2-tensor* K*:*

$$K^{\alpha\beta} \;=\; -(a^2\cos^2\theta)\mathbf{g}^{\alpha\beta} + O^{\alpha\beta}, \tag{4.5.1}$$

where the tensor O *is defined in* (4.4.2).

Recall the definition of Carter operator $\mathcal{K}$ for horizontal tensors $\psi \in \mathfrak{s}_k$ according to Definition 2.3.6, given by

$$\mathcal{K}(\psi) := \dot{\mathbf{D}}_\beta(K^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi).$$

Associated to the Carter operator $\mathcal{K}$ we define the following.

Definition 4.5.2. *In* $(\mathcal{M}, \mathbf{g})$ *we define the following second order angular operator for* $\psi \in \mathfrak{s}_k$*:*

$$\mathcal{O}(\psi) \;:=\; |q|^2\left(\triangle_k\psi - \frac{2a^2\cos\theta}{|q|^2}\;{}^*\Re(\mathfrak{J})^b\nabla_b\psi\right). \tag{4.5.2}$$

Note that the above definition reduces to the operator $\mathcal{O}$ in Kerr, see (3.7.4).

We now show that $\mathcal{O}$ is an approximate symmetry operator for scalars and, up to Riemann curvature terms, for tensors in perturbations of Kerr.

Proposition 4.5.3. *The operator* $\mathcal{O}$ *defined in* (4.5.2) *for a scalar function* ψ *satisfies the following commutation formula:*

$$[\mathcal{O}, |q|^2\square_{\mathbf{g}}]\psi \;=\; |q|^2\Big[\mathfrak{d}^2\big(\Gamma_g\cdot\mathfrak{d}\psi\big) + \dot{\mathbf{D}}_3\mathfrak{d}\big(|q|^2\xi\cdot\dot{\mathbf{D}}_a\psi\big)\Big]. \tag{4.5.3}$$

The operator $\mathcal{O}$ *for* $\psi \in \mathfrak{s}_2$ *satisfies the following commutation formula:*

$$\begin{aligned}
[\mathcal{O}, |q|^2\dot{\square}_2]\psi = |q|^2\Bigg[&\nabla\left(\frac{8a(r^2+a^2)\cos\theta}{|q|^2}\right)\cdot\nabla\nabla_{\widehat{T}}\;{}^*\psi + O(ar^{-2})\nabla_{\widehat{R}}^{\leq 1}\mathfrak{d}^{\leq 1}\psi\\
&+\mathfrak{d}^2\big(\Gamma_g\cdot\mathfrak{d}\psi\big) + \dot{\mathbf{D}}_3\mathfrak{d}\big(|q|^2\xi\cdot\dot{\mathbf{D}}_a\psi\big)\Bigg].
\end{aligned} \tag{4.5.4}$$

Proof. See Section C.5. Observe that in order to obtain acceptable error terms like the ones on the right hand side of (4.5.3) we need to derive the commutator using the decomposition in null frames of $\square_{\mathbf{g}}$ as in Lemma 4.7.4. Similarly for $\dot{\square}_2$. The curvature terms on the first line of (4.5.4) can be obtained as in Proposition 3.7.6. $\square$

Corollary 4.5.4. *The following commutation formula holds true for a scalar ψ:*

$$[|q|^2\triangle, |q|^2\square_{\mathbf{g}}]\psi = |q|^2\Big[O(a^2r^{-3})\mathfrak{d}^{\leq 2}\psi + \mathfrak{d}^2\big(\Gamma_g\cdot\mathfrak{d}\psi\big) + \dot{\mathbf{D}}_3\mathfrak{d}\big(|q|^2\xi\cdot\dot{\mathbf{D}}_a\psi\big)\Big].$$

Proof. Using the definition (4.5.2) to write that $|q|^2\triangle = \mathcal{O} + O(a^2r^{-2})\not{\mathfrak{d}} + r^{-1}\Gamma_b\cdot\mathfrak{d}$, we deduce from Proposition 4.5.3,

$$\begin{aligned}[|q|^2\triangle, |q|^2\square_{\mathbf{g}}]\psi &= [\mathcal{O}, |q|^2\square_{\mathbf{g}}] + O(a^2r^{-2})[\not{\mathfrak{d}}, |q|^2\square_{\mathbf{g}}]\psi + [r^{-1}\Gamma_b\cdot\mathfrak{d}, |q|^2\square_{\mathbf{g}}]\psi\\ &= |q|^2\Big[O(a^2r^{-3})\mathfrak{d}^{\leq 2}\psi + \mathfrak{d}^2\big(\Gamma_g\cdot\mathfrak{d}\psi\big) + \dot{\mathbf{D}}_3\mathfrak{d}\big(|q|^2\xi\cdot\dot{\mathbf{D}}_a\psi\big)\Big],\end{aligned}$$

as stated. $\square$

4.6 Approximate symmetry operators

We can finally define in $\mathcal{M}$ the following approximate symmetry operators as in Definition 3.7.8.

Definition 4.6.1. *We define the following second order differential operators, acting on $\mathfrak{s}_2$ tensors,*

$$\mathcal{S}_1 = \nabla_{\mathbf{T}}\nabla_{\mathbf{T}}, \quad \mathcal{S}_2 = a\nabla_{\mathbf{T}}\nabla_{\mathbf{Z}}, \quad \mathcal{S}_3 = a^2\nabla_{\mathbf{Z}}\nabla_{\mathbf{Z}}, \quad \mathcal{S}_4 = \mathcal{O}.$$

Recall that in Kerr we had also the alternative definition of $\mathcal{S}_{\underline{a}}$ give by (3.7.11). In perturbations, they differ by error terms in the following way.

Lemma 4.6.2. *Let*

$$\widetilde{\mathcal{S}}_{\underline{a}}\psi = |q|^2\dot{\mathbf{D}}_\alpha(|q|^{-2}S_{\underline{a}}^{\alpha\beta}\dot{\mathbf{D}}_\beta\psi), \qquad \text{for } \underline{a} = 1,2,3,4,$$

where $S_{\underline{a}}^{\alpha\beta}$ are given in Definition 4.4.3. Then, we have the following comparison with the approximate symmetry operators of Definition 4.6.1:

$$\begin{aligned}\widetilde{\mathcal{S}}_1 &= \mathcal{S}_1 + \Gamma_b\cdot\mathfrak{d},\\ \widetilde{\mathcal{S}}_2 &= \mathcal{S}_2 + r\Gamma_b\cdot\mathfrak{d},\\ \widetilde{\mathcal{S}}_3 &= \mathcal{S}_3 + r\Gamma_b\cdot\mathfrak{d},\\ \widetilde{\mathcal{S}}_4 &= \mathcal{O} + r\Gamma_b\cdot\mathfrak{d}.\end{aligned}$$

Proof. See Section C.6. □

We now collect the formula for the commutators of the approximate symmetry operators $\mathcal{S}_{\underline{a}}$ for $\underline{a} = 1, 2, 3, 4$ with $\dot{\square}_2$.

Proposition 4.6.3. *The following commutation formulas hold true for $\psi \in \mathfrak{s}_2$:*

$$\begin{aligned}
[\mathcal{S}_1, \dot{\square}_2]\psi &= O(ar^{-2})\mathfrak{d}^{\leq 2}\psi + \mathfrak{d}^2\big(\Gamma_g \cdot \mathfrak{d}\psi\big) + \mathfrak{d}\big(\Gamma_b \cdot \dot{\square}_2\psi\big),\\
[\mathcal{S}_2, \dot{\square}_2]\psi &= O(ar^{-2})\mathfrak{d}^{\leq 2}\psi + \mathfrak{d}^2\big(\Gamma_g \cdot \mathfrak{d}\psi\big) + r\mathfrak{d}\big(\Gamma_b \cdot \dot{\square}_2\psi\big),\\
[\mathcal{S}_3, \dot{\square}_2]\psi &= O(ar^{-2})\mathfrak{d}^{\leq 2}\psi + \mathfrak{d}^2\big(\Gamma_g \cdot \mathfrak{d}\psi\big) + r\mathfrak{d}\big(\Gamma_b \cdot \dot{\square}_2\psi\big),
\end{aligned}$$

and

$$[\mathcal{S}_4, |q|^2\dot{\square}_2]\psi = |q|^2\Big[O(ar^{-2})\ \mathfrak{d}^{\leq 2}\psi + \mathfrak{d}^2\big(\Gamma_g \cdot \mathfrak{d}\psi\big) + \dot{\mathbf{D}}_3\mathfrak{d}\big(|q|^2\xi \cdot \dot{\mathbf{D}}_a\psi\big)\Big].$$

Proof. The first three relations are straightforward from (4.3.2) in Proposition 4.3.3. The commutator for $\mathcal{S}_4 = \mathcal{O}$ is obtained from Proposition 4.5.3. □

4.7 Wave operator and Energy Momentum tensor

Consider variational wave equations for real-valued tensors $\psi \in \mathfrak{s}_k$, of the form

$$\dot{\square}_k\psi - V\psi = N, \tag{4.7.1}$$

where V is a real potential. The variational wave equation (4.7.1) has Lagrangian

$$\mathcal{L}[\psi] = \mathbf{g}^{\mu\nu}\dot{\mathbf{D}}_\mu\psi \cdot \dot{\mathbf{D}}_\nu\psi + V\psi \cdot \psi,$$

where the dot product here denotes full contraction with respect to the horizontal indices.

The corresponding energy-momentum tensor associated to (4.7.1) is given by

$$\mathcal{Q}_{\mu\nu} := \dot{\mathbf{D}}_\mu\psi \cdot \dot{\mathbf{D}}_\nu\psi - \frac{1}{2}\mathbf{g}_{\mu\nu}\Big(\dot{\mathbf{D}}_\lambda\psi \cdot \dot{\mathbf{D}}^\lambda\psi + V\psi \cdot \psi\Big) = \dot{\mathbf{D}}_\mu\psi \cdot \dot{\mathbf{D}}_\nu\psi - \frac{1}{2}\mathbf{g}_{\mu\nu}\mathcal{L}[\psi]. \tag{4.7.2}$$

Lemma 4.7.1. *Given a solution $\psi \in \mathfrak{s}_k$ of equation* (4.7.1) *we have*

$$\mathbf{D}^\nu\mathcal{Q}_{\mu\nu} = \dot{\mathbf{D}}_\mu\psi \cdot \big(\dot{\square}_k\psi - V\psi\big) + \dot{\mathbf{D}}^\nu\psi^A\dot{\mathbf{R}}_{AB\nu\mu}\psi^B - \frac{1}{2}\mathbf{D}_\mu V|\psi|^2.$$

Proof. We have, making us of Proposition 2.1.27

$$\begin{aligned}
\mathbf{D}^\nu \mathcal{Q}_{\mu\nu} &= \dot{\mathbf{D}}^\nu \dot{\mathbf{D}}_\nu \psi \cdot \dot{\mathbf{D}}_\mu \psi + \dot{\mathbf{D}}^\nu \psi \cdot \Big(\dot{\mathbf{D}}_\nu \dot{\mathbf{D}}_\mu - \dot{\mathbf{D}}_\mu \dot{\mathbf{D}}_\nu \Big) \psi - V \mathbf{D}_\mu \psi \cdot \psi - \frac{1}{2} \mathbf{D}_\mu V \psi \cdot \psi \\
&= \dot{\mathbf{D}}_\mu \psi \cdot \dot{\mathbf{D}}^\nu \dot{\mathbf{D}}_\nu \psi + \dot{\mathbf{D}}^\nu \psi^a \dot{\mathbf{R}}_{ab\nu\mu} \psi^b - V \mathbf{D}_\mu \psi \cdot \psi - \frac{1}{2} \mathbf{D}_\mu V \psi \cdot \psi \\
&= \dot{\mathbf{D}}_\mu \psi \cdot \big(\dot{\square}_k \psi - V\psi \big) + \dot{\mathbf{D}}^\nu \psi^a \dot{\mathbf{R}}_{ab\nu\mu} \psi^b - \frac{1}{2} \mathbf{D}_\mu V |\psi|^2
\end{aligned}$$

with $|\psi|^2 := \psi \cdot \psi$. □

Standard calculation for generalized currents

We collect here some general calculations for generalized currents associated to equation (4.7.1).

Proposition 4.7.2. *Let $\psi \in \mathfrak{s}_k$ be a solution of (4.7.1) and X be a vectorfield. Then,*

1. *The 1-form $\mathcal{P}_\mu = \mathcal{Q}_{\mu\nu} X^\nu$ verifies*

$$\mathbf{D}^\mu \mathcal{P}_\mu = \frac{1}{2} \mathcal{Q} \cdot {}^{(X)}\pi + X(\psi) \cdot \big(\dot{\square}_k \psi - V\psi \big) - \frac{1}{2} X(V) |\psi|^2 + X^\mu \dot{\mathbf{D}}^\nu \psi^a \dot{\mathbf{R}}_{ab\nu\mu} \psi^b.$$

2. *Let X as above, w a scalar and M a one form. Define*

$$\mathcal{P}_\mu[X, w, M] := \mathcal{Q}_{\mu\nu} X^\nu + \frac{1}{2} w \psi \cdot \dot{\mathbf{D}}_\mu \psi - \frac{1}{4} |\psi|^2 \partial_\mu w + \frac{1}{4} |\psi|^2 M_\mu.$$

Then,

$$\begin{aligned}
\mathbf{D}^\mu \mathcal{P}_\mu[X, w, M] &= \frac{1}{2} \mathcal{Q} \cdot {}^{(X)}\pi - \frac{1}{2} X(V) |\psi|^2 + \frac{1}{2} w \mathcal{L}[\psi] - \frac{1}{4} |\psi|^2 \square_{\mathbf{g}} w \\
&+ X^\mu \dot{\mathbf{D}}^\nu \psi^a \dot{\mathbf{R}}_{ab\nu\mu} \psi^b + \frac{1}{4} Div(|\psi|^2 M) \\
&+ \left(X(\psi) + \frac{1}{2} w \psi \right) \cdot \big(\dot{\square}_k \psi - V\psi \big).
\end{aligned}$$

Proof. Let $\mathcal{P}_\mu[X, 0, 0] = \mathcal{Q}_{\mu\nu} X^\nu$. Then,

$$\begin{aligned}
\mathbf{D}^\mu \mathcal{P}_\mu[X, 0, 0] &= \mathcal{Q}_{\mu\nu} \mathbf{D}^\mu X^\nu + X^\nu \mathbf{D}^\mu \mathcal{Q}_{\mu\nu} \\
&= \frac{1}{2} \mathcal{Q} \cdot {}^{(X)}\pi + X^\mu \dot{\mathbf{D}}_\mu \psi \cdot \big(\dot{\square}_k \psi - V\psi \big) + X^\mu \dot{\mathbf{D}}^\nu \psi^a \dot{\mathbf{R}}_{ab\nu\mu} \psi^b - \frac{1}{2} X^\mu \mathbf{D}_\mu V |\psi|^2
\end{aligned}$$

where we used Lemma 4.7.1. This proves the first part of the proposition. To prove the second part we write

$$\begin{aligned}
\mathbf{D}^\mu\mathcal{P}_\mu[X,w,M] &= \frac{1}{2}\mathcal{Q}\cdot{}^{(X)}\pi + X(\psi)\cdot\big(\dot\square\psi - V\psi\big) - \frac{1}{2}X(V)|\psi|^2 + X^\mu\dot{\mathbf{D}}^\nu\psi^a\dot{\mathbf{R}}_{ab\nu\mu}\psi^b\\
&+ \frac{1}{2}\mathbf{D}^\mu w\,\psi\cdot\dot{\mathbf{D}}_\mu\psi + \frac{1}{2}w\,\dot{\mathbf{D}}^\mu\psi\cdot\dot{\mathbf{D}}_\mu\psi\\
&+ \frac{1}{2}w\psi\dot\square_k\psi - \frac{1}{2}\psi\cdot\dot{\mathbf{D}}^\mu\psi\partial_\mu w - \frac{1}{4}|\psi|^2\square_{\mathbf{g}}w + \frac{1}{4}\mathrm{Div}(|\psi|^2M\big)\\
&= \frac{1}{2}\mathcal{Q}\cdot{}^{(X)}\pi - \frac{1}{2}X(V)|\psi|^2 + \frac{1}{2}w\,\dot{\mathbf{D}}^\mu\psi\cdot\dot{\mathbf{D}}_\mu\psi + \frac{1}{2}w\psi\,\big(\dot\square\psi\big)\\
&+ X^\mu\dot{\mathbf{D}}^\nu\psi^a\mathbf{R}_{ab\nu\mu}\psi^b - \frac{1}{4}|\psi|^2\square_{\mathbf{g}}w + \frac{1}{4}\mathrm{Div}(|\psi|^2M\big) + X(\psi)\cdot\big(\dot\square_k\psi - V\psi\big)
\end{aligned}$$

which gives the desired result. □

We now specialize Proposition 4.7.2 to the case of equation (4.7.1) in perturbations of Kerr.

Proposition 4.7.3. *Let $\psi\in\mathfrak{s}_k(\mathcal{M})$ be a solution of (4.7.1) and X be a vectorfield of the form*

$$X = X^3e_3 + X^4e_4.$$

Then,

1. *The 1-form $\mathcal{P}_\mu = \mathcal{Q}_{\mu\nu}X^\nu$ verifies*

$$\begin{aligned}
\mathbf{D}^\mu\mathcal{P}_\mu &= \frac{1}{2}\mathcal{Q}\cdot{}^{(X)}\pi + X(\psi)\cdot N - \frac{1}{2}X(V)|\psi|^2 - \big({}^*\rho + \underline{\eta}\wedge\eta\big)\nabla_{X^4e_4 - X^3e_3}\psi\cdot{}^*\psi\\
&- \frac{1}{2}\Im\Big(tr\underline{X}HX^3 + trX\,\underline{H}X^4\Big)\cdot\nabla\psi\cdot{}^*\psi + r^{-2}\big(X^3\Gamma_b + X^4\Gamma_g\big)\mathfrak{d}\psi\cdot\psi.
\end{aligned}$$

2. *Let X as above, w a scalar and M a one form. Define*

$$\mathcal{P}_\mu[X,w,M] := \mathcal{Q}_{\mu\nu}X^\nu + \frac{1}{2}w\psi\cdot\dot{\mathbf{D}}_\mu\psi - \frac{1}{4}|\psi|^2\partial_\mu w + \frac{1}{4}|\psi|^2M_\mu.$$

Then,

$$\begin{aligned}
\mathbf{D}^\mu\mathcal{P}_\mu[X,w,M] &= \frac{1}{2}\mathcal{Q}\cdot{}^{(X)}\pi - \frac{1}{2}X(V)|\psi|^2 + \frac{1}{2}w\mathcal{L}[\psi] - \frac{1}{4}|\psi|^2\square_{\mathbf{g}}w\\
&- \big({}^*\rho + \underline{\eta}\wedge\eta\big)\nabla_{X^4e_4 - X^3e_3}\psi\cdot{}^*\psi\\
&- \frac{1}{2}\Im\Big(tr\underline{X}HX^3 + trX\,\underline{H}X^4\Big)\cdot\nabla\psi\cdot{}^*\psi\\
&+ \frac{1}{4}Div(|\psi|^2M\big) + \left(X(\psi) + \frac{1}{2}w\psi\right)\cdot N\\
&+ r^{-2}\big(X^3\Gamma_b + X^4\Gamma_g\big)\mathfrak{d}\psi\cdot\psi.
\end{aligned}$$

Proof. By Proposition 4.7.2, we only need to specialize the computation of the term $X^\mu \dot{\mathbf{D}}^\nu \psi^A \dot{\mathbf{R}}_{AB\nu\mu}\psi^B$ to the case of $X = X^3 e_3 + X^4 e_4$ and perturbations of Kerr. Since $\dot{\mathbf{R}}_{ab\nu\mu}$ is antisymmetric with respect to (a,b), we have

$$X^\mu \dot{\mathbf{D}}^\nu \psi^a \dot{\mathbf{R}}_{ab\nu\mu}\psi^b = \frac{1}{2} \in^{bc} X^\mu \dot{\mathbf{R}}_{bc\nu\mu}\dot{\mathbf{D}}^\nu \psi \cdot {}^*\psi.$$

Introducing the spacetime 1-form

$$F_\nu := \in^{bc} \dot{\mathbf{R}}_{bc\nu\mu} X^\mu, \tag{4.7.3}$$

we infer

$$X^\mu \dot{\mathbf{D}}^\nu \psi^a \dot{\mathbf{R}}_{ab\nu\mu}\psi^b = \frac{1}{2} F_\mu \mathbf{D}^\mu \psi \cdot {}^*\psi.$$

Next, we rewrite F_μ as

$$F_\mu = \in^{bc} \dot{\mathbf{R}}_{bc\mu 4} X^4 + \in^{bc} \dot{\mathbf{R}}_{bc\mu 3} X^3$$

and we compute the various components of F_μ. To this end, recall that we have the following decomposition of the curvature:

$$\begin{aligned}
\mathbf{R}_{ab34} &= 2 \in_{ab} {}^*\rho,\\
\mathbf{R}_{abc3} &= \in_{ab} {}^*\underline{\beta}_c = r^{-1}\Gamma_b,\\
\mathbf{R}_{abc4} &= - \in_{ab} {}^*\beta_c = r^{-1}\Gamma_g.
\end{aligned}$$

Also, recalling Proposition 2.2.4, the components of $\mathbf{B}$ are given by the formula

$$\begin{aligned}
\mathbf{B}_{abc3} &= -\mathrm{tr}\,\underline{\chi}\big(\delta_{ca}\eta_b - \delta_{cb}\eta_a\big) - {}^{(a)}\mathrm{tr}\underline{\chi}\big(\in_{ca} \eta_b - \in_{cb} \eta_a\big)\\
&\quad + 2\big(- \underline{\widehat{\chi}}_{ca}\eta_b + \underline{\widehat{\chi}}_{cb}\eta_a - \chi_{ca}\underline{\xi}_b + \chi_{cb}\underline{\xi}_a\big)\\
&= -\mathrm{tr}\,\underline{\chi}\big(\delta_{ca}\eta_b - \delta_{cb}\eta_a\big) - {}^{(a)}\mathrm{tr}\underline{\chi}\big(\in_{ca} \eta_b - \in_{cb} \eta_a\big) + r^{-1}\Gamma_b,\\
\mathbf{B}_{abc4} &= -\mathrm{tr}\,\chi\big(\delta_{ca}\underline{\eta}_b - \delta_{cb}\underline{\eta}_a\big) - {}^{(a)}\mathrm{tr}\chi\big(\in_{ca} \underline{\eta}_b - \in_{cb} \underline{\eta}_a\big)\\
&\quad + 2\big(- \widehat{\chi}_{ca}\underline{\eta}_b + \widehat{\chi}_{cb}\underline{\eta}_a - \underline{\chi}_{ca}\xi_b + \underline{\chi}_{cb}\xi_a\big)\\
&= -\mathrm{tr}\,\chi\big(\delta_{ca}\underline{\eta}_b - \delta_{cb}\underline{\eta}_a\big) - {}^{(a)}\mathrm{tr}\chi\big(\in_{ca} \underline{\eta}_b - \in_{cb} \underline{\eta}_a\big) + r^{-1}\Gamma_g,\\
\mathbf{B}_{ab34} &= 4\big(- \underline{\xi}_a\xi_b + \xi_a\underline{\xi}_b - \eta_a\underline{\eta}_b + \underline{\eta}_a\eta_b\big)\\
&= 4\big(- \eta_a\underline{\eta}_b + \underline{\eta}_a\eta_b\big) + \Gamma_b \cdot \Gamma_g.
\end{aligned}$$

Using the definition (2.1.13) of $\dot{\mathbf{R}}$, we infer

$$\begin{aligned}
\dot{\mathbf{R}}_{ab34} &= 2\Big(\in_{ab} {}^*\rho + \big(- \eta_a\underline{\eta}_b + \underline{\eta}_a\eta_b\big)\Big),\\
\dot{\mathbf{R}}_{abc3} &= -\frac{1}{2}\mathrm{tr}\,\underline{\chi}\big(\delta_{ca}\eta_b - \delta_{cb}\eta_a\big) - \frac{1}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi}\big(\in_{ca} \eta_b - \in_{cb} \eta_a\big) + r^{-1}\Gamma_b,\\
\dot{\mathbf{R}}_{abc4} &= -\frac{1}{2}\mathrm{tr}\,\chi\big(\delta_{ca}\underline{\eta}_b - \delta_{cb}\underline{\eta}_a\big) - \frac{1}{2}\,{}^{(a)}\mathrm{tr}\chi\big(\in_{ca} \underline{\eta}_b - \in_{cb} \underline{\eta}_a\big) + r^{-1}\Gamma_g.
\end{aligned}$$

We deduce from the definition for F_μ and the above identities for the components of $B_{ab\mu\nu}$

$$\begin{aligned} F_4 &= \in^{bc} \dot{\mathbf{R}}_{bc43} X^3 \\ &= \in^{bc} \Big(-2 \in_{bc} {}^*\rho - 2(\underline{\eta}_b \eta_c - \eta_b \underline{\eta}_c) \Big) X^3 \\ &= -4 \ {}^*\rho X^3 - 4(\underline{\eta} \wedge \eta) X^3, \end{aligned}$$

$$F_3 = \in^{bc} \dot{\mathbf{R}}_{bc34} X^4 = 4 \ {}^*\rho X^4 + 4(\underline{\eta} \wedge \eta) X^4,$$

and

$$\begin{aligned} F_e &= \in^{bc} \dot{\mathbf{R}}_{bce3} X^3 + \in^{bc} \dot{\mathbf{R}}_{bce4} X^4 \\ &= \frac{1}{2} \in^{bc} \Big(-\mathrm{tr}\, \underline{\chi} \big(\delta_{eb} \eta_c - \delta_{ec} \eta_b \big) - {}^{(a)}\mathrm{tr} \underline{\chi} \big(\in_{eb} \eta_c - \in_{ec} \eta_b \big) + r^{-1} \Gamma_b \Big) X^3 \\ &\quad + \frac{1}{2} \in^{bc} \Big(-\mathrm{tr}\, \chi \big(\delta_{eb} \underline{\eta}_c - \delta_{ec} \underline{\eta}_b \big) - {}^{(a)}\mathrm{tr} \chi \big(\in_{eb} \underline{\eta}_c - \in_{ec} \underline{\eta}_b \big) + r^{-1} \Gamma_g \Big) X^4 \\ &= \Big(-\mathrm{tr}\, \underline{\chi} \ {}^*\eta_e + {}^{(a)}\mathrm{tr} \underline{\chi} \eta_e \Big) X^3 + \Big(-\mathrm{tr}\, \chi \ {}^*\underline{\eta}_e + {}^{(a)}\mathrm{tr} \chi \underline{\eta}_e \Big) X^4 + r^{-1} (X^3 \Gamma_b + X^4 \Gamma_g), \end{aligned}$$

i.e.

$$\begin{aligned} F_4 &= -4 \ {}^*\rho X^3 - 4(\underline{\eta} \wedge \eta) X^3, \\ F_3 &= 4 \ {}^*\rho X^4 + 4(\underline{\eta} \wedge \eta) X^4, \\ F_e &= \Big(-\mathrm{tr}\, \underline{\chi} \ {}^*\eta_e + {}^{(a)}\mathrm{tr} \underline{\chi} \eta_e \Big) X^3 + \Big(-\mathrm{tr}\, \chi \ {}^*\underline{\eta}_e + {}^{(a)}\mathrm{tr} \chi \underline{\eta}_e \Big) X^4 + r^{-1} (X^3 \Gamma_b + X^4 \Gamma_g). \end{aligned}$$

Since we have

$$\begin{aligned} X^\mu \dot{\mathbf{D}}^\nu \psi^a \dot{\mathbf{R}}_{ab\nu\mu} \psi^b &= \frac{1}{2} F_\mu \dot{\mathbf{D}}^\mu \psi \cdot {}^*\psi \\ &= \frac{1}{2} F_4 \dot{\mathbf{D}}^4 \psi \cdot {}^*\psi + \frac{1}{2} F_3 \dot{\mathbf{D}}^3 \psi \cdot {}^*\psi + \frac{1}{2} F_b \dot{\mathbf{D}}^b \psi \cdot {}^*\psi \\ &= -\frac{1}{4} F_4 \nabla_3 \psi \cdot {}^*\psi - \frac{1}{4} F_3 \nabla_4 \psi \cdot {}^*\psi + \frac{1}{2} F_b \nabla^b \psi \cdot {}^*\psi, \end{aligned}$$

we infer

$$\begin{aligned} & X^\mu \dot{\mathbf{D}}^\nu \psi^a \dot{\mathbf{R}}_{ab\nu\mu} \psi^b \\ = \ & -\big({}^*\rho + \underline{\eta} \wedge \eta \big) \nabla_{X^4 e_4 - X^3 e_3} \psi \cdot {}^*\psi \\ & + \frac{1}{2} \Big(\big(-\mathrm{tr}\, \underline{\chi} \ {}^*\eta + {}^{(a)}\mathrm{tr} \underline{\chi} \eta \big) X^3 + \big(-\mathrm{tr}\, \chi \ {}^*\underline{\eta} + {}^{(a)}\mathrm{tr} \chi \underline{\eta} \big) X^4 \Big) \cdot \nabla \psi \cdot {}^*\psi \\ & + r^{-2} \big(X^3 \Gamma_b + X^4 \Gamma_g \big) \mathfrak{d} \psi \cdot \psi \end{aligned}$$

and hence

$$\begin{aligned} X^\mu \dot{\mathbf{D}}^\nu \psi^a \dot{\mathbf{R}}_{ab\nu\mu} \psi^b &= -\big({}^*\rho + \underline{\eta} \wedge \eta \big) \nabla_{X^4 e_4 - X^3 e_3} \psi \cdot {}^*\psi \\ & \quad - \frac{1}{2} \Im \Big(\mathrm{tr} \underline{X} H X^3 + \mathrm{tr} X \, \underline{H} X^4 \Big) \cdot \nabla \psi \cdot {}^*\psi + r^{-2} \big(X^3 \Gamma_b + X^4 \Gamma_g \big) \mathfrak{d} \psi \cdot \psi \end{aligned}$$

which concludes the proof of Proposition 4.7.2. □

4.7.1 Decomposition of the wave operator in null frames

Lemma 4.7.4. *The wave operator for* $\psi \in \mathfrak{s}_k$ *is given by*

$$\dot{\square}_k \psi = -\frac{1}{2}\big(\nabla_3\nabla_4\psi + \nabla_4\nabla_3\psi\big) + \left(\underline{\omega} - \frac{1}{2}tr\,\underline{\chi}\right)\nabla_4\psi + \left(\omega - \frac{1}{2}tr\,\chi\right)\nabla_3\psi \\ + \triangle_k\psi + (\eta + \underline{\eta})\cdot\nabla\psi, \tag{4.7.4}$$

where $\triangle = \nabla^a\nabla_a$ *denotes the horizontal Laplacian for* k*-tensors.*

Proof. By definition

$$\dot{\square}_k\psi = \mathbf{g}^{34}\dot{\mathbf{D}}_3\dot{\mathbf{D}}_4\psi + \mathbf{g}^{43}\dot{\mathbf{D}}_4\dot{\mathbf{D}}_3\psi + \mathbf{g}^{cd}\dot{\mathbf{D}}_c\dot{\mathbf{D}}_d\psi.$$

We write, using (2.2.3),

$$\begin{aligned}
\dot{\mathbf{D}}_4\psi &= \nabla_4\psi,\\
\dot{\mathbf{D}}_3\dot{\mathbf{D}}_4\psi &= \nabla_3\nabla_4\psi - 2\underline{\omega}\nabla_4\psi - 2\eta\cdot\nabla\psi,\\
\dot{\mathbf{D}}_4\dot{\mathbf{D}}_3\psi &= \nabla_4\nabla_3\psi - 2\omega\nabla_3\psi - 2\underline{\eta}\cdot\nabla\psi,\\
\dot{\mathbf{D}}_d\psi &= \nabla_d\psi,\\
\dot{\mathbf{D}}_c\dot{\mathbf{D}}_d\psi &= \nabla_c\nabla_d\psi - \frac{1}{2}\chi_{cd}\nabla_3\psi - \frac{1}{2}\underline{\chi}_{cd}\nabla_4\psi.
\end{aligned}$$

Hence

$$\begin{aligned}
\dot{\square}_k\psi &= -\frac{1}{2}\dot{\mathbf{D}}_3\dot{\mathbf{D}}_4\psi - \frac{1}{2}\dot{\mathbf{D}}_4\dot{\mathbf{D}}_3\psi + \mathbf{g}^{cd}\dot{\mathbf{D}}_c\dot{\mathbf{D}}_d\psi\\
&= -\frac{1}{2}\big(\nabla_3\nabla_4\psi + \nabla_4\nabla_3\psi\big) + \mathbf{g}^{cd}\left(\nabla_c\nabla_d\psi - \frac{1}{2}\chi_{cd}\nabla_3\psi - \frac{1}{2}\underline{\chi}_{cd}\nabla_4\psi\right)\\
&+ \underline{\omega}\nabla_4\psi + \eta\cdot\nabla\psi + \underline{\omega}\nabla_3\psi + \underline{\eta}\cdot\nabla\psi\\
&= -\frac{1}{2}\big(\nabla_3\nabla_4\psi + \nabla_4\nabla_3\psi\big) + \triangle_2\psi - \frac{1}{2}\mathrm{tr}\,\chi\nabla_3\psi - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\nabla_4\psi\\
&+ \underline{\omega}\nabla_4\psi + \eta\cdot\nabla\psi + \underline{\omega}\nabla_3\psi + \underline{\eta}\cdot\nabla\psi.
\end{aligned}$$

Hence

$$\begin{aligned}
\dot{\square}_k\psi &= -\frac{1}{2}\big(\nabla_3\nabla_4\psi + \nabla_4\nabla_3\psi\big) + \triangle_k\psi + \left(\underline{\omega} - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\right)\nabla_4\psi + \left(\omega - \frac{1}{2}\mathrm{tr}\,\chi\right)\nabla_3\psi\\
&+(\eta + \underline{\eta})\cdot\nabla\psi,
\end{aligned}$$

as stated. ∎

Lemma 4.7.5. *The wave operator for $\psi \in \mathfrak{s}_2(\mathbb{C})$ is given by*

$$\begin{aligned}\dot{\square}_2\psi &= -\nabla_4\nabla_3\psi - \frac{1}{2}tr\underline{\chi}\nabla_4\psi + \left(2\omega - \frac{1}{2}tr\,\chi\right)\nabla_3\psi + \triangle_2\psi + 2\underline{\eta}\cdot\nabla\psi \\ &+ 2i\left({}^*\rho - \eta\wedge\underline{\eta}\right)\psi + (\Gamma_b\cdot\Gamma_g)\cdot\psi.\end{aligned} \tag{4.7.5}$$

Moreover, if $\psi \in \mathfrak{s}_2(\mathbb{C})$ is 0-conformally invariant, the above can be written as

$$\begin{aligned}\dot{\square}_2\psi &= -\,{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3\psi - \frac{1}{2}tr\underline{\chi}\,{}^{(c)}\nabla_4\psi - \frac{1}{2}tr\,\chi\,{}^{(c)}\nabla_3\psi + {}^{(c)}\triangle_2\psi + 2\underline{\eta}\cdot{}^{(c)}\nabla\psi \\ &+ 2i\left({}^*\rho - \eta\wedge\underline{\eta}\right)\psi + (\Gamma_b\cdot\Gamma_g)\cdot\psi.\end{aligned} \tag{4.7.6}$$

Proof. Using Lemma 4.7.4 for $\psi \in \mathfrak{s}_2(\mathbb{C})$ and using (4.2.7) , we obtain

$$\begin{aligned}\dot{\square}_2\psi &= -\nabla_4\nabla_3\psi - \frac{1}{2}[\nabla_3,\nabla_4]\psi + \triangle_2\psi + \left(\underline{\omega} - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\right)\nabla_4\psi + \left(\omega - \frac{1}{2}\mathrm{tr}\,\chi\right)\nabla_3\psi \\ &\quad +(\eta+\underline{\eta})\cdot\nabla\psi \\ &= -\nabla_4\nabla_3\psi + \triangle_2\psi - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\nabla_4\psi + \left(2\omega - \frac{1}{2}\mathrm{tr}\,\chi\right)\nabla_3\psi + 2\underline{\eta}\cdot\nabla\psi \\ &\quad +2i\left({}^*\rho - \eta\wedge\underline{\eta}\right)\psi + (\Gamma_b\cdot\Gamma_g)\cdot\psi,\end{aligned}$$

as stated. The second relation for ψ of signature 0 is straightforward. □

4.7.2 Representation of the wave operator using $\widehat{T}$, $\widehat{R}$

Lemma 4.7.6. *We have[4] for $\psi \in \mathfrak{s}_k$,*

$$\begin{aligned}|q|^2\dot{\square}_k\psi &= \frac{(r^2+a^2)^2}{\Delta}\left(-\nabla_{\widehat{T}}\nabla_{\widehat{T}}\psi + \nabla_{\widehat{R}}\nabla_{\widehat{R}}\psi\right) + 2r\nabla_{\widehat{R}}\psi \\ &+ |q|^2\triangle_k\psi + |q|^2(\eta+\underline{\eta})\cdot\nabla\psi + r^2\Gamma_g\cdot\mathfrak{d}\psi,\end{aligned} \tag{4.7.7}$$

where $\triangle_k$ denotes the horizontal Laplacian for k-tensors.

Proof. See Section C.7. □

[4] Observe that the expression in (4.7.7) is not regular at the horizon, i.e. for $\Delta = 0$.

4.7.3 The wave operator using complex derivatives

We now express the laplacian in terms of complex derivatives. We summarize the result in the following.

Lemma 4.7.7. *We have for $\psi \in \mathfrak{s}_2(\mathbb{C})$,*

$$\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot\psi) = 4\triangle_2\psi - 2i\left({}^{(a)}tr\chi\nabla_3 + {}^{(a)}tr\underline{\chi}\nabla_4\right)\psi - 8\,{}^{(h)}K\psi \tag{4.7.8}$$

where ${}^{(h)}K$ is defined in (2.2.9). *In particular, in perturbations of Kerr we have*

$$\begin{aligned}\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot\psi) &= 4\triangle_2\psi - 2i\left({}^{(a)}tr\chi\nabla_3 + {}^{(a)}tr\underline{\chi}\nabla_4\right)\psi\\ &\quad +2\left(tr\,\chi tr\underline{\chi} + {}^{(a)}tr\chi\,{}^{(a)}tr\underline{\chi} + 4\rho\right)\psi + (\Gamma_g\cdot\Gamma_b)\cdot\psi, \qquad (4.7.9)\\ \mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot\psi) &= 4\triangle_2\psi - 2i\left({}^{(a)}tr\chi\nabla_3 + {}^{(a)}tr\underline{\chi}\nabla_4\right)\psi\\ &\quad +2\left(\frac{1}{2}trX\overline{tr\underline{X}} + \frac{1}{2}tr\underline{X}\overline{trX} + 2P + 2\overline{P}\right)\psi + (\Gamma_g\cdot\Gamma_b)\cdot\psi. \quad (4.7.10)\end{aligned}$$

Proof. See section C.8. □

We rewrite the above using the conformal derivatives introduced in Lemma 2.2.18.

Lemma 4.7.8. *We have for $\psi \in \mathfrak{s}_2(\mathbb{C})$ s-conformally invariant,*

$$\begin{aligned}{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot\psi) &= 4\,{}^{(c)}\triangle_2\psi - 2i\left({}^{(a)}tr\chi\,{}^{(c)}\nabla_3 + {}^{(a)}tr\underline{\chi}\,{}^{(c)}\nabla_4\right)\psi\\ &\quad + 2\Big[\left(tr\,\chi tr\underline{\chi} + {}^{(a)}tr\chi\,{}^{(a)}tr\underline{\chi} + 4\rho\right)\\ &\quad - is\left(\frac{1}{2}\left(tr\,\chi\,{}^{(a)}tr\underline{\chi} - tr\,\underline{\chi}\,{}^{(a)}tr\chi\right) + 2\,{}^*\rho\right)\Big]\psi\\ &\quad + (\Gamma_g\cdot\Gamma_b)\cdot\psi\end{aligned} \tag{4.7.11}$$

where ${}^{(c)}\triangle_2 := \gamma^{ab}\,{}^{(c)}\nabla_a\,{}^{(c)}\nabla_b$ is the conformal Laplacian operator for horizontal 2-tensors.

Proof. See section C.9. □

By putting together the canonical expression for the wave operator given in Lemma 4.7.4 and the expression for the Laplacian given in Lemma 4.7.7, we obtain the following.

Corollary 4.7.9. *We have, for $\psi \in \mathfrak{s}_2(\mathbb{C})$,*

$$
\begin{aligned}
\dot{\square}_2\psi = & -\nabla_4\nabla_3\psi + \frac{1}{4}\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot\psi) + \left(2\omega - \frac{1}{2}trX\right)\nabla_3\psi - \frac{1}{2}tr\underline{X}\nabla_4\psi + 2\underline{\eta}\cdot\nabla\psi \\
& + \left(-\frac{1}{2}tr\,\chi tr\underline{\chi} - \frac{1}{2}{}^{(a)}tr\chi\,{}^{(a)}tr\underline{\chi} - 2\rho\right)\psi + 2i\left({}^*\rho - \eta\wedge\underline{\eta}\right)\psi + (\Gamma_b\cdot\Gamma_g)\cdot\psi,
\end{aligned}
\tag{4.7.12}
$$

which can be rewritten as

$$
\begin{aligned}
\dot{\square}_2\psi = & -\nabla_4\nabla_3\psi + \frac{1}{4}\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot\psi) + \left(2\omega - \frac{1}{2}trX\right)\nabla_3\psi - \frac{1}{2}tr\underline{X}\nabla_4\psi + 2\underline{\eta}\cdot\nabla\psi \\
& + \left(-\frac{1}{4}trX\overline{tr\underline{X}} - \frac{1}{4}tr\underline{X}\overline{trX} - 2\overline{P}\right)\psi - 2i\left(\eta\wedge\underline{\eta}\right)\psi + (\Gamma_b\cdot\Gamma_g)\cdot\psi.
\end{aligned}
\tag{4.7.13}
$$

4.7.4 Commutators with the D'Alembertian

We collect here some additional commutators with the horizontal laplacian, the operator $\mathcal{O}$ and the D'Alembertian.

Lemma 4.7.10. *The following commutation formulas hold true for a 2-tensor $\psi \in \mathfrak{s}_2$:*

$$
\begin{aligned}
[\nabla_3, |q|^2\triangle]\psi &= (\eta-\zeta)\cdot|q|^2\nabla_3\nabla\psi + (\eta-\zeta)\cdot|q|^2\nabla\nabla_3\psi + div\,(\eta-\zeta)|q|^2\nabla_3\psi \\
& \quad -\frac{1}{2}|q|^2\left(\nabla tr\,\underline{\chi}\cdot\nabla\psi + \nabla\,{}^{(a)}tr\underline{\chi}\cdot\,{}^*\nabla\psi\right) + O(ar^{-4})\mathfrak{d}^{\le 1}\psi + \mathfrak{d}\big(\Gamma_b\cdot\mathfrak{d}\psi\big), \\
[\nabla_4, |q|^2\triangle]\psi &= (\underline{\eta}+\zeta)\cdot|q|^2\nabla_4\nabla\psi + (\underline{\eta}+\zeta)\cdot|q|^2\nabla\nabla_4\psi + div\,(\underline{\eta}+\zeta)|q|^2\nabla_4\psi \\
& \quad -\frac{1}{2}|q|^2\left(\nabla tr\,\chi\cdot\nabla\psi + \nabla\,{}^{(a)}tr\chi\cdot\,{}^*\nabla\psi\right) + O(ar^{-4})\mathfrak{d}^{\le 1}\psi \\
& \quad + r^2\dot{\mathbf{D}}_3\big(\xi\cdot\dot{\mathbf{D}}_a\psi\big) + \mathfrak{d}\big(\Gamma_g\cdot\mathfrak{d}\psi\big).
\end{aligned}
$$

Similarly we have for a 2-tensor $\psi \in \mathfrak{s}_2$:

$$
\begin{aligned}
[\nabla_3, \mathcal{O}]\psi &= (\eta-\zeta)\cdot|q|^2\nabla_3\nabla\psi + (\eta-\zeta)\cdot|q|^2\nabla\nabla_3\psi \\
& \quad + \big(div\,(\eta-\zeta) + (\eta+\underline{\eta})\cdot(\eta-\zeta)\big)|q|^2\nabla_3\psi + O(ar^{-2})\mathfrak{d}^{\le 1}\psi + \mathfrak{d}\big(\Gamma_b\cdot\mathfrak{d}\psi\big), \\
[\nabla_4, \mathcal{O}]\psi &= (\underline{\eta}+\zeta)\cdot|q|^2\nabla_4\nabla\psi + (\underline{\eta}+\zeta)\cdot|q|^2\nabla\nabla_4\psi \\
& \quad + \big(div\,(\underline{\eta}+\zeta) + (\eta+\underline{\eta})\cdot(\underline{\eta}+\zeta)\big)|q|^2\nabla_4\psi + O(ar^{-3})\mathfrak{d}^{\le 1}\psi \\
& \quad + r^2\dot{\mathbf{D}}_3\big(\xi\cdot\dot{\mathbf{D}}_a\psi\big) + \mathfrak{d}\big(\Gamma_g\cdot\mathfrak{d}\psi\big).
\end{aligned}
$$

Proof. See section C.10. □

Lemma 4.7.11. *The following commutation formulas hold true for a scalar ψ:*

$$\begin{aligned}
[\nabla_3, \square_{\mathbf{g}}]\psi &= 2\omega\nabla_3\nabla_3\psi - (tr\underline{\chi} + 2\underline{\omega})\nabla_3\nabla_4\psi - tr\underline{\chi}\square_{\mathbf{g}}\psi \\
&\quad + r^{-2}\mathfrak{d}^{\le 1}\psi + O(ar^{-2})\nabla_3\nabla\psi + r^{-2}\mathfrak{d}\big(\Gamma_b \cdot \mathfrak{d}\psi\big), \\
[\nabla_4, \square_{\mathbf{g}}]\psi &= 2\underline{\omega}\nabla_4\nabla_4\psi - (tr\,\chi + 2\omega)\nabla_4\nabla_3\psi - tr\,\chi\square_{\mathbf{g}}\psi \\
&\quad + r^{-2}\mathfrak{d}^{\le 1}\psi + O(ar^{-2})\nabla_4\nabla\psi + r^{-1}\mathfrak{d}\big(\Gamma_g \cdot \mathfrak{d}\psi\big).
\end{aligned}$$

The following commutation formulas hold true for $\psi \in \mathfrak{s}_2$:

$$\begin{aligned}
[\nabla_4, \dot{\square}_2]\psi &= 2\underline{\omega}\nabla_4\nabla_4\psi - (tr\,\chi + 2\omega)\nabla_4\nabla_3\psi + 2(\eta + \zeta)\cdot\nabla_4\nabla\psi - tr\,\chi\dot{\square}_2\psi \\
&\quad - \frac{1}{4}(tr\,\chi)^2\nabla_3\psi + O(r^{-3})\mathfrak{d}^{\le 1}\psi + \dot{\mathbf{D}}_3\big(\xi\cdot\dot{\mathbf{D}}_a\psi\big) + r^{-2}\mathfrak{d}\big(\Gamma_g \cdot \mathfrak{d}\psi\big).
\end{aligned}$$

In particular, the above can be written as

$$\begin{aligned}
[\nabla_4, \dot{\square}_2]\psi &= -(tr\,\chi + 2\omega)\nabla_4\nabla_3\psi - tr\,\chi\dot{\square}_2\psi - \frac{1}{4}(tr\,\chi)^2\nabla_3\psi \\
&\quad + O(r^{-3})\mathfrak{d}^{\le 1}\psi + O(r^{-4})\mathfrak{d}^{\le 2}\psi + \dot{\mathbf{D}}_3\big(\xi\cdot\dot{\mathbf{D}}_a\psi\big) + r^{-2}\mathfrak{d}\big(\Gamma_g \cdot \mathfrak{d}\psi\big).
\end{aligned}$$

We can also deduce for $\psi \in \mathfrak{s}_2$:

$$\begin{aligned}
[r\nabla_4, \dot{\square}_2]\psi = {}& -\nabla_4\nabla_4\psi - r\Big(\frac{1}{2}tr\,\chi - 2\omega\Big)\dot{\square}_2\psi - r\Big(\frac{1}{2}tr\,\chi + 2\omega\Big)\triangle_2\psi \\
& + O(r^{-2})\mathfrak{d}^{\le 1}\psi + O(r^{-3})\mathfrak{d}^{\le 2}\psi + r\dot{\mathbf{D}}_3\big(\xi\cdot\dot{\mathbf{D}}_a\psi\big) + r^{-1}\mathfrak{d}\big(\Gamma_g \cdot \mathfrak{d}\psi\big).
\end{aligned} \tag{4.7.14}$$

Proof. See section C.11. □

Lemma 4.7.12. *The following commutation formula holds true for a scalar ψ:*

$$\begin{aligned}
[\nabla_{\widehat{R}}, |q|^2\square_{\mathbf{g}}]\psi &= O(r)\square_{\mathbf{g}}\psi + O(r)\triangle\psi + O(ar^{-1})\mathfrak{d}^{\le 2}\psi + O(1)\nabla_{\widehat{R}}\psi + O(ar^{-1})\nabla\psi \\
&\quad + r\mathfrak{d}\big(\Gamma_b \cdot \mathfrak{d}\psi\big).
\end{aligned}$$

We also have for a scalar ψ:

$$[\nabla_{\widehat{R}}, |q|^2\triangle]\psi = O(ar^{-3})\,\not{\mathfrak{d}}\psi + r\mathfrak{d}\big(\Gamma_g \cdot \mathfrak{d}\psi\big). \tag{4.7.15}$$

In particular,

$$[\nabla_{\widehat{R}}, \triangle]\psi = O(r^{-3}\Delta)\triangle\psi + O(ar^{-5})\,\not{\mathfrak{d}}\psi + r^{-1}\mathfrak{d}\big(\Gamma_g \cdot \mathfrak{d}\psi\big).$$

Proof. See Section C.12. □

Lemma 4.7.13. *The following commutation formulas hold true for $\psi \in \mathfrak{s}_2$:*

$$\begin{aligned}|q|\,{\not{\mathcal{D}}}_2\dot{\square}_2 - \dot{\square}_1|q|\,{\not{\mathcal{D}}}_2\psi &= 3\,{}^{(h)}K|q|\,{\not{\mathcal{D}}}_2\psi - \frac{2a\cos\theta}{|q|}\,{}^*{\not{\mathcal{D}}}_2\not{\mathcal{L}}_{\mathbf{T}}\psi - |q|(\eta+\underline{\eta})\cdot\dot{\square}_2\psi\\ &+O(ar^{-2})\mathfrak{d}^{\leq 1}\psi + O(ar^{-3})\mathfrak{d}^{\leq 2}\psi\\ &+\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\psi) + r\check{H}\cdot\dot{\square}_2\psi + \dot{\mathbf{D}}_3(r\xi\cdot\nabla_3\psi),\end{aligned}$$

and similarly for $|q|\,{\not{\mathcal{D}}}_1^*$. In particular,

$$\begin{aligned}|q|\,{\not{\mathcal{D}}}_2\dot{\square}_2\psi - \dot{\square}_1|q|\,{\not{\mathcal{D}}}_2\psi &= \frac{3}{r^2}|q|\,{\not{\mathcal{D}}}_2\psi + O(ar^{-2})\mathfrak{d}^{\leq 2}\psi + \mathfrak{d}^{\leq 2}(\Gamma_g\cdot\psi)\\ &+r\check{H}\cdot\dot{\square}_2\psi + \dot{\mathbf{D}}_3(r\xi\cdot\nabla_3\psi),\\ |q|\,{\not{\mathcal{D}}}_1^*\dot{\square}_0\psi - \dot{\square}_1|q|\,{\not{\mathcal{D}}}_1^*\psi &= -\frac{1}{r^2}|q|\,{\not{\mathcal{D}}}_1^*\psi + O(ar^{-2})\mathfrak{d}^{\leq 2}\psi + \mathfrak{d}^{\leq 2}(\Gamma_g\cdot\psi)\\ &+r\check{H}\cdot\dot{\square}_2\psi + \dot{\mathbf{D}}_3(r\xi\cdot\nabla_3\psi).\end{aligned}$$

The following commutation formulas hold true for $\psi \in \mathfrak{s}_1$:

$$\begin{aligned}|q|\,{\not{\mathcal{D}}}_2^*\dot{\square}_1\psi - \dot{\square}_2|q|\,{\not{\mathcal{D}}}_2^*\psi &= -\frac{3}{r^2}|q|\,{\not{\mathcal{D}}}_2^*\psi + O(ar^{-2})\mathfrak{d}^{\leq 2}\psi + \mathfrak{d}^{\leq 2}(\Gamma_g\cdot\psi)\\ &+r\check{H}\cdot\dot{\square}_1\psi + \dot{\mathbf{D}}_3(r\xi\cdot\nabla_3\psi),\\ |q|\mathcal{D}_1\dot{\square}_1\psi - \dot{\square}_0|q|\mathcal{D}_1\psi &= \frac{1}{r^2}|q|\mathcal{D}_1\psi + O(ar^{-2})\mathfrak{d}^{\leq 2}\psi + \mathfrak{d}^{\leq 2}(\Gamma_g\cdot\psi)\\ &+r\check{H}\cdot\dot{\square}_1\psi + \dot{\mathbf{D}}_3(r\xi\cdot\nabla_3\psi).\end{aligned}$$

Proof. See section C.13. □

4.8 Spacetime elliptic identities

We adapt the spacetime elliptic identities in Proposition 2.1.47 to the case of perturbations of Kerr.

Lemma 4.8.1. *Given a not necessarily integrable horizontal structure, the following pointwise relations hold:*

i.) *The following identity holds for $f \in \mathfrak{s}_1$:*

$$\begin{aligned}|\nabla f|^2 + {}^{(h)}K|f|^2 = |\,{\not{\mathcal{D}}}_1 f|^2 + \frac{2a\cos\theta}{|q|^2}\,{}^*\nabla_{\mathbf{T}}f\cdot f\\ + \mathbf{D}_\alpha\big(\nabla^\alpha f\cdot f - (div\, f)f^\alpha - (curl\, f)({}^*f)^\alpha\big) + \Gamma_g\cdot\mathfrak{d}f\cdot f,\end{aligned} \tag{4.8.1}$$

and

$$|\nabla f|^2 + {}^{(h)}K|f|^2 = |\not{\mathcal{D}}_1 f|^2 + \frac{2a\cos\theta(r^2+a^2)}{|q|^4}\, {}^*\nabla_{\widehat{T}} f\cdot f$$
$$+\nabla_a\Big(\nabla^a f\cdot f - (div\, f) f^a - (curl\, f)({}^*f)^a\Big) + \Gamma_g\cdot\mathfrak{d}f\cdot f.$$

ii.) *The following identity holds for $f\in\mathfrak{s}_2$:*

$$\begin{aligned}|\nabla f|^2 + 2\,{}^{(h)}K|f|^2 &= 2|\not{\mathcal{D}}_2 f|^2 + \frac{2a\cos\theta}{|q|^2}\, {}^*\nabla_{\mathbf{T}} f\cdot f\\ &+\mathbf{D}_\alpha\big(\nabla^\alpha f\cdot f - 2(div\, f)_\beta f^{\alpha\beta}\big) + \Gamma_g\cdot\mathfrak{d}f\cdot f,\end{aligned}\tag{4.8.2}$$

and

$$|\nabla f|^2 + 2\,{}^{(h)}K|f|^2 = 2|\not{\mathcal{D}}_2 f|^2 + \frac{2a\cos\theta(r^2+a^2)}{|q|^4}\, {}^*\nabla_{\widehat{T}} f\cdot f$$
$$+\nabla_a\big(\nabla^a f\cdot f - 2(div\, f)_b f^{ab}\big) + \Gamma_g\cdot\mathfrak{d}f\cdot f.$$

iii.) *The following identity holds for $f\in\mathfrak{s}_1$:*

$$\begin{aligned}|\nabla f|^2 - {}^{(h)}K|f|^2 &= 2|\not{\mathcal{D}}_2^* f|^2 - \frac{2a\cos\theta}{|q|^2}\, {}^*\nabla_{\mathbf{T}} f\cdot f\\ &+\mathbf{D}_\alpha\big(\nabla^\alpha f\cdot f + 2(\not{\mathcal{D}}_2^* f)^{\alpha\beta} f_\beta\big) + \Gamma_g\cdot\mathfrak{d}f\cdot f,\end{aligned}\tag{4.8.3}$$

and

$$|\nabla f|^2 - {}^{(h)}K|f|^2 = 2|\not{\mathcal{D}}_2^* f|^2 - \frac{2a\cos\theta(r^2+a^2)}{|q|^4}\, {}^*\nabla_{\widehat{T}} f\cdot f$$
$$+\nabla_a\Big(\nabla^a f\cdot f + 2(\not{\mathcal{D}}_2^* f)^{ab} f_b\Big) + \Gamma_g\cdot\mathfrak{d}f\cdot f.$$

Proof. Straightforward application of Proposition 2.1.47 and (4.1.14). □

We collect here some elliptic identities involving the operator $\mathcal{O}$ and $\triangle_2$.

Lemma 4.8.2. *We have for $\psi\in\mathfrak{s}_2$,*

$$\begin{aligned}\triangle_2\psi\cdot\psi &= -|\nabla\psi|^2 - ((\eta+\underline{\eta})\cdot\nabla\psi)\cdot\psi + \mathbf{D}^\alpha(\nabla_\alpha\psi\cdot\psi),\\ \mathcal{O}(\psi)\cdot\psi &= -|q|^2|\nabla\psi|^2 - |q|^2((\eta+\underline{\eta})\cdot\nabla\psi)\cdot\psi + \mathbf{D}^\alpha(|q|^2\nabla_\alpha\psi\cdot\psi) + \Gamma_b\cdot\mathfrak{d}\psi\cdot\psi.\end{aligned}\tag{4.8.4}$$

We have for $\psi\in\mathfrak{s}_2(\mathbb{C})$,

$$\begin{aligned}\triangle_2\psi\cdot\overline{\psi} &= -|\nabla\psi|^2 - ((H+\underline{H})\cdot\nabla\psi)\cdot\overline{\psi} + \mathbf{D}^\alpha(\nabla_\alpha\psi\cdot\overline{\psi}),\\ \mathcal{O}(\psi)\cdot\overline{\psi} &= -|q|^2|\nabla\psi|^2 - |q|^2((H+\underline{H})\cdot\nabla\psi)\cdot\overline{\psi} + \mathbf{D}^\alpha(|q|^2\nabla_\alpha\psi\cdot\overline{\psi}) + \Gamma_b\cdot\mathfrak{d}\psi\cdot\psi.\end{aligned}\tag{4.8.5}$$

Proof. Using Lemma 2.1.40, we obtain

$$\begin{aligned}\triangle_2\psi\cdot\psi &= \nabla^a\nabla_a\psi\cdot\psi=\nabla^a(\nabla_a\psi\cdot\psi)-|\nabla\psi|^2\\ &= \mathbf{D}^\alpha(\nabla_\alpha\psi\cdot\psi)-((\eta+\underline{\eta})\cdot\nabla\psi)\cdot\psi-|\nabla\psi|^2\end{aligned}$$

which gives the first identity. We then deduce, using $\nabla(|q|^2)=(\eta+\underline{\eta})|q|^2+r\Gamma_b$,

$$\begin{aligned}\mathcal{O}(\psi)\cdot\psi &= |q|^2\left(\triangle_2\psi+(\eta+\underline{\eta})\cdot\nabla\psi\right)\cdot\psi\\ &= -|q|^2|\nabla\psi|^2+|q|^2\mathbf{D}^\alpha(\nabla_\alpha\psi\cdot\psi)\\ &= -|q|^2|\nabla\psi|^2-|q|^2((\eta+\underline{\eta})\cdot\nabla\psi)\cdot\psi+\mathbf{D}^\alpha(|q|^2\nabla_\alpha\psi\cdot\psi)+\Gamma_b\cdot\mathfrak{d}\psi\cdot\psi\end{aligned}$$

as stated. ∎

Chapter 5

Derivation of the main equations

5.1 Teukolsky equation for A

It is known that the curvature components A and $\underline{A}$ satisfy wave equations which decouple from all other components at the linear level, the celebrated Teukolsky equations. In this section we derive, using our formalism, the corresponding Teukolsky equation for A while keeping track of the error terms generated by the perturbation from Kerr expressed in terms of (Γ_b, Γ_g).

5.1.1 The Teukolsky equation for A

Proposition 5.1.1. *The complex tensor $A \in \mathfrak{s}_2(\mathbb{C})$ satisfies the following equation:*

$$\mathcal{L}(A) \quad = \quad Err[\mathcal{L}(A)] \tag{5.1.1}$$

where

$$\begin{aligned}\mathcal{L}(A) = &-{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3 A + \frac{1}{4}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A) + \left(-\frac{1}{2}trX - 2\overline{trX}\right){}^{(c)}\nabla_3 A \\ &-\frac{1}{2}tr\underline{X}\,{}^{(c)}\nabla_4 A + \big(4H + \underline{H} + \overline{\underline{H}}\big)\cdot{}^{(c)}\nabla A + \big(-\overline{trX}\,tr\underline{X} + 2\overline{P}\big)A + H\widehat{\otimes}(\overline{\underline{H}}\cdot A),\end{aligned} \tag{5.1.2}$$

with error term expressed schematically

$$Err[\mathcal{L}(A)] \quad = \quad r^{-1}\mathfrak{d}^{\leq 1}\big(\Gamma_g\cdot B\big) + {}^{(c)}\nabla_3\Xi\cdot B + \Gamma_b\cdot\Gamma_g\cdot A. \tag{5.1.3}$$

Proof. See section D.1. □

For completeness, we collect here the real and imaginary part of the Teukolsky operator $\mathcal{L}(A)$ defined in (5.1.2).

Corollary 5.1.2. *The Teukolsky operator in* (5.1.2) *can also be written as*

$$\mathcal{L}(A) = \Re(\mathcal{L})(A) + i\Im(\mathcal{L})(A)$$

where

$$\begin{aligned}
\Re(\mathcal{L})(A) &= -{}^{(c)}\nabla_4{}^{(c)}\nabla_3 A + {}^{(c)}\triangle_2 A + (4\eta + 2\underline{\eta})\cdot{}^{(c)}\nabla A - \frac{1}{2}tr\underline{\chi}\,{}^{(c)}\nabla_4 A - \frac{5}{2}tr\,\chi\,{}^{(c)}\nabla_3 A\\
&+ \left(-\frac{1}{2}tr\,\chi tr\underline{\chi} - \frac{1}{2}{}^{(a)}tr\chi\,{}^{(a)}tr\underline{\chi} + 4\rho + 4\eta\cdot\underline{\eta}\right)A,\\
\Im(\mathcal{L})(A) &= -2\,{}^{(a)}tr\chi\,{}^{(c)}\nabla_3 A + 4\,{}^*\eta\cdot{}^{(c)}\nabla A\\
&+ \left(\frac{1}{2}tr\,\chi\,{}^{(a)}tr\underline{\chi} - \frac{1}{2}tr\underline{\chi}\,{}^{(a)}tr\chi - 4\,{}^*\rho - 4\eta\wedge\underline{\eta}\right)A.
\end{aligned}$$

Proof. See section D.2. □

5.1.2 Connection to the classical Teukolsky equation

The Teukolsky equation (5.1.1) is a tensorial equation for $A \in \mathfrak{s}_2(\mathbb{C})$, as defined in our formalism. The standard derivation of the equation, in linear theory, is done instead with respect to the Newman-Penrose formalism, see section 2.2.4. To relate the Teukolsky equation in our formalism to the classical one in NP formalism we have to project it with respect to the standard horizontal frame e_1, e_2 of Kerr, see (3.3.3),

$$e_1 = \frac{1}{|q|}\partial_\theta, \qquad e_2 = \frac{a\sin\theta}{|q|}\partial_t + \frac{1}{|q|\sin\theta}\partial_\phi,$$

for which the relations (3.3.10) are verified.

One can check in fact that the standard Teukolsky variable, which we denote by $\alpha^{[+2]}$, is related to our A via the formula

$$\alpha^{[+2]} := -\frac{\overline{q}}{q}A_{11}, \qquad A_{11} := A(e_1, e_1). \tag{5.1.4}$$

Indeed, in the physics literature, see [16], the curvature component $\alpha^{[+2]}$ is a complex scalar defined in Newman-Penrose formalism as

$$\alpha^{[+2]} = -W(l,m,l,m)$$

where l and m are related to our (ingoing) frame e_4, e_3, e_1, e_3 by

$$l = e_4, \qquad m = \frac{|q|}{\sqrt{2}q}\left(e_1 + ie_2\right),$$

and W coincides with the Riemann curvature tensor $\mathbf{R}$ for vacuum spacetimes. We therefore deduce

$$\begin{aligned}\alpha^{[+2]} &= -\frac{|q|^2}{2q^2}W(e_4, (e_1+ie_2), e_4, (e_1+ie_2)) = -\frac{\overline{q}}{2q}\left(W_{4141} + iW_{4142} + iW_{4241} - W_{4242}\right)\\ &= -\frac{\overline{q}}{2q}\left(W_{4141} - W_{4242} + 2iW_{4142}\right) = -\frac{\overline{q}}{q}\left(W_{4141} + iW_{4142}\right).\end{aligned}$$

On the other hand

$$A_{11} = W_{4141} + iW_{4142}$$

and therefore $\alpha^{[+2]} = -\frac{\overline{q}}{q}A_{11}$.

One can check that in the particular case of the Kerr metric we have

$$\begin{aligned}|q|^2\square_{m,a}\alpha^{[+2]} &= -4(r-m)\partial_r\alpha^{[+2]} - 4\left(\frac{m(r^2-a^2)}{\Delta} - r - ia\cos\theta\right)\partial_t\alpha^{[+2]}\\ &\quad - 4\left(\frac{a(r-m)}{\Delta} + i\frac{\cos\theta}{\sin^2\theta}\right)\partial_\varphi\alpha^{[+2]} + (4\cot^2\theta - 2)\alpha^{[+2]},\end{aligned} \tag{5.1.5}$$

where $\square_{m,a}$ is the D'Alembertian relative to the Kerr metric. This is the standard form of the Teukolsky equation in Boyer-Lindquist coordinates, see [72].

5.2 Generalized Regge-Wheeler equation for $\mathfrak{q}$

In this section we derive the generalized Regge-Wheeler-type equation.

5.2.1 The invariant quantities Q and $\mathfrak{q}$

We start with the following lemma.

Lemma 5.2.1. *Let C_1 and C_2 be scalar functions. The expression*

$$Q(A) \;=\; {}^{(c)}\nabla_3\,{}^{(c)}\nabla_3 A + C_1\,{}^{(c)}\nabla_3 A + C_2 A \in \mathfrak{s}_2(\mathbb{C}) \tag{5.2.1}$$

is 0-conformally invariant provided C_1 is -1-conformally invariant and C_2 is -2-conformally invariant.

Proof. Direct verification in view of the definition of the conformal derivative ${}^{(c)}\nabla_3$. □

Definition 5.2.2. *Given a fixed null pair (e_3, e_4) and scalar functions r and θ as in Section 4.1, we define our main quantity $\mathfrak{q} \in \mathfrak{s}_2(\mathbb{C})$ as*

$$\mathfrak{q} \;=\; q\overline{q}^3 Q(A) = q\overline{q}^3\left({}^{(c)}\nabla_3\,{}^{(c)}\nabla_3 A + C_1\,{}^{(c)}\nabla_3 A + C_2 A\right) \tag{5.2.2}$$

where $q = r + ia\cos\theta$, and the scalar function C_1, C_2 are given by

$$\begin{aligned} C_1 &= 2tr\underline{\chi} - 2\frac{{}^{(a)}tr\underline{\chi}^2}{tr\underline{\chi}} - 4i\,{}^{(a)}tr\underline{\chi},\\ C_2 &= \frac{1}{2}tr\underline{\chi}^2 - 4\,{}^{(a)}tr\underline{\chi}^2 + \frac{3}{2}\frac{{}^{(a)}tr\underline{\chi}^4}{tr\underline{\chi}^2} + i\left(-2tr\underline{\chi}\,{}^{(a)}tr\underline{\chi} + 4\frac{{}^{(a)}tr\underline{\chi}^3}{tr\underline{\chi}}\right). \end{aligned} \tag{5.2.3}$$

Remark 5.2.3. *Note that $\mathfrak{q}$ is independent of the particular normalization. More precisely if $e_3' = \lambda^{-1}e_3, e_4' = \lambda e_4$ and $A' = \lambda^2 A$ then $\mathfrak{q}' = \mathfrak{q}$.*

In perturbations of Kerr, the quantity $\mathfrak{q}$ defined above can be factorized as follows[1].

Proposition 5.2.4. *In perturbations of Kerr, the quantity $\mathfrak{q}$ defined in (5.2.2) with C_1, C_2 given by (5.2.3) can be factorized as*

$$r\,{}^{(c)}\nabla_3\left(r^2\left({}^{(c)}\nabla_3\left(r\frac{\overline{q}^4}{r^4}A\right)\right)\right) \;=\; \frac{\overline{q}}{q}\mathfrak{q} + r^4\Gamma_b\cdot\nabla_3^{\leq 1}A, \tag{5.2.4}$$

or also as

$$r^2\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_3\left(\frac{\overline{q}^4}{r^2}A\right) \;=\; \frac{\overline{q}}{q}\mathfrak{q} + r^4\Gamma_b\cdot\nabla_3^{\leq 1}A. \tag{5.2.5}$$

In particular, in Kerr we have

$$r\,{}^{(c)}\nabla_3\left(r^2\left({}^{(c)}\nabla_3\left(r\frac{\overline{q}^4}{r^4}A\right)\right)\right) \;=\; \frac{\overline{q}}{q}\mathfrak{q}, \tag{5.2.6}$$

$$r^2\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_3\left(\frac{\overline{q}^4}{r^2}A\right) \;=\; \frac{\overline{q}}{q}\mathfrak{q}. \tag{5.2.7}$$

Proof. See section D.3. □

[1] In practice, we will rely on a more precise factorization, see Lemma 11.1.3.

5.2.2 Comparison of $\mathfrak{q}$ with Ma's quantity

The quantity in Ma, [57], which corresponds to our $\mathfrak{q}$, is chosen to be a complex scalar $\alpha^{[+2]}$, obtained by a Chadrasekhar type transformation from what Ma denotes as $\alpha^{[+2]}$, i.e. the complex scalar verifying the standard Teukolsky equation (5.1.5) as derived in [72] by using the NP formalism. Thus $\alpha^{[+2]}$ relates to our A according to (5.1.4), i.e.

$$\alpha^{[+2]} = -\frac{\overline{q}}{q} A_{11}, \qquad A_{11} = A(e_1, e_2).$$

Proposition 5.2.5. *We have*

$$\mathfrak{q}_{11} \quad = \quad -r\nabla_3 \left(r^2 \nabla_3 \left(r \frac{|q|^4}{r^4} \alpha^{[+2]} \right) \right). \tag{5.2.8}$$

Proof. The proof is based on the following lemma.

Lemma 5.2.6. *Let U be an anti-selfadjoint complex 2 tensor $U \in \mathfrak{s}_2(\mathbb{C})$ in Kerr. Then, setting $p = q\overline{q}^{-1}$ we have*

$$(\nabla_3 U)_{11} = p^{-1} e_3(pU_{11}).$$

Proof. We calculate, using the relations (3.3.10) and $U_{12} = U_{21} = ({}^*U)_{11} = -iU_{11}$,

$$\nabla_3 U_{11} \quad = \quad e_3(U_{11}) - 2U_{\nabla_3 11} = e_3 U_{11} - {}^{(a)}\mathrm{tr}\underline{\chi} U_{12} = e_3(U_{11}) + i\,{}^{(a)}\mathrm{tr}\underline{\chi} U_{11}.$$

Note that[2] $e_3 p = i\,{}^{(a)}\mathrm{tr}\underline{\chi} p$. Hence

$$\nabla_3 U_{11} \quad = \quad e_3(U_{11}) + e_3(p) U_{11} = p^{-1} e_3(pU_{11})$$

as stated. □

According to Proposition 5.2.4 we have, in the ingoing normalization,

$$\frac{\overline{q}}{q}\mathfrak{q} = r\nabla_3 \left(r^2 \left(\nabla_3 \left(r \frac{\overline{q}^4}{r^4} A \right) \right) \right).$$

Therefore, using the above lemma,

$$\frac{\overline{q}}{q}\mathfrak{q}_{11} \quad = \quad r(p^{-1}\nabla_3 p) \left(r^2 \left((p^{-1}\,{}^{(c)}\nabla_3 p) \left(r \frac{\overline{q}^4}{r^4} A_{11} \right) \right) \right) = p^{-1} r\nabla_3 \left(r^2 \nabla_3 \left(rp \frac{\overline{q}^4}{r^4} A_{11} \right) \right).$$

[2] We have $e_3(p) = e_3(q\overline{q}^{-1}) = -\overline{q}^{-1} + q\overline{q}^{-2} = \overline{q}^{-2}\big(-\overline{q} + q \big) = 2ai\cos\theta \overline{q}^{-2} = pi\,{}^{(a)}\mathrm{tr}\underline{\chi}$.

We deduce, using $A_{11} = -p\alpha^{[+2]}$,

$$\begin{aligned}\mathfrak{q}_{11} &= r\nabla_3\left(r^2\nabla_3\left(r\frac{q}{\overline{q}}\frac{\overline{q}^4}{r^4}A_{11}\right)\right) = -r\nabla_3\left(r^2\nabla_3\left(r\frac{q}{\overline{q}}\frac{\overline{q}^4}{r^4}\frac{q}{\overline{q}}\alpha^{[+2]}\right)\right)\\ &= -r\nabla_3\left(r^2\nabla_3\left(r\frac{|q|^4}{r^4}\alpha^{[+2]}\right)\right)\end{aligned}$$

as stated. □

Remark 5.2.7. *Note that all quantities in Proposition 5.2.5 are defined with respect to the ingoing normalization. If we replace $\alpha^{[+2]}$ with $\alpha^{[+2]}_{(out)} = \frac{\Delta^2}{|q|^4}\alpha^{[+2]}$, corresponding to the outgoing normalization $e_4^{(out)} = \frac{|q|^2}{\Delta}e_4^{(in)}$, then* (5.2.8) *becomes*

$$\mathfrak{q}_{11} = -r\nabla_3\left(r^2\nabla_3\left(r\frac{\Delta^2}{r^4}\alpha^{[+2]}_{(out)}\right)\right).$$

Ma denotes $\phi^0_{+2} = \frac{\Delta^2}{r^4}\alpha^{[+2]}_{(out)}$ and defines

$$\phi^2_{+2} = (rYr)(rYr)(\phi^0_{+2}) \tag{5.2.9}$$

where Y is precisely $e_3 = \frac{r^2+a^2}{\Delta}\partial_t + \frac{a}{\Delta}\partial_\phi - \partial_r$ in BL coordinates. Thus $\phi^2_{+2} = -\mathfrak{q}_{11}$.

In [57], Ma states the following proposition.

Proposition 5.2.8 (Equation (24.c) in [57])**.** *The quantity*

$$\phi^2_{+2} := r\nabla_3\Big(r^2\nabla_3\big(r\phi^0_{+2}\big)\Big)$$

verifies the following equation in Kerr

$$\begin{aligned}&|q|^2\square_{a,m}\phi^2_{+2} + 4i\left(\frac{\cos\theta}{\sin^2\theta}\partial_\phi - a\cos\theta\partial_t\right)\phi^2_{+2} - 4\left(\cot^2\theta + \frac{r^2-2mr+2a^2}{r^2}\right)\phi^2_{+2}\\ =\;& -8(a^2\partial_t + a\partial_\phi)\phi^1_{+2} - 12a^2\phi^0_{+2},\end{aligned} \tag{5.2.10}$$

where $\square_{a,m}$ is the standard D'Alembertian relative to the Kerr metric and

$$\phi^0_{+2} = \frac{\Delta^2\alpha^{[+2]}_{(out)}}{r^4}, \qquad \phi^1_{+2} = re_3r(\phi^0_{+2}).$$

5.2.3 The derivation of the gRW equation for $\mathfrak{q}$

We now state the first main result of Part I concerning the wave equation satisfied by $\mathfrak{q}$. To start with we note that that we have decided to use as definition of $\mathfrak{q}$ the more complicated expression in (5.2.2) rather than the more direct formula appearing on the left hand side of (5.2.6), which is directly comparable (by projection) with the quantity ϕ^2_{+2} of Ma, due to the fact that the second formula generates un-acceptable error terms.

Theorem 5.2.9. *The invariant symmetric traceless 2-tensor $\mathfrak{q}\in\mathfrak{s}_2(\mathbb{C})$ in Definition 5.2.2 satisfies the equation*

$$\dot{\square}_2\mathfrak{q} - i\frac{4a\cos\theta}{|q|^2}\nabla_{\mathbf{T}}\mathfrak{q} - V\mathfrak{q} \quad = \quad L_{\mathfrak{q}}[A] + Err[\dot{\square}_2\mathfrak{q}], \tag{5.2.11}$$

where:

- $\mathbf{T}$ *is the vectorfield given by Definition 4.3.1, see also Remark 5.2.10 below.*
- *The potential V is the* ***real*** *scalar function given by*

$$V \quad = \quad \frac{4}{|q|^2}\frac{r^2-2mr+2a^2}{r^2} - \frac{4a^2\cos^2\theta}{|q|^6}(r^2+6mr+a^2\cos^2\theta), \tag{5.2.12}$$

  which for $a=0$ coincides with the potential of the Regge-Wheeler equation in Schwarzschild, i.e. $V=-\mathrm{tr}\,\chi\mathrm{tr}\underline{\chi}+O(\frac{|a|}{r^4})$, see also Remark 5.2.11 below.
- *$L_{\mathfrak{q}}[A]$ is a linear second order operator in A, given in the outgoing frame by*

$$\begin{aligned} L_{\mathfrak{q}}[A] \quad = \quad q\overline{q}^3\Bigg(& -\frac{8a^2\Delta}{r^2|q|^4}\nabla_{\mathbf{T}}\nabla_3 A - \frac{8a\Delta}{r^2|q|^4}\nabla_{\mathbf{Z}}\nabla_3 A \\ & + W_4\ \nabla_4 A + W_3\nabla_3 A + W\cdot\nabla A + W_0 A\Bigg), \end{aligned}$$

  where W_4, W_3, W_0 are complex functions of (r,θ) and W is the product of a complex function of (r,θ) with ${}^*\mathfrak{R}(\mathfrak{J})$, with the following fall-off in r

$$q\overline{q}^3W_4 = q\overline{q}^3W_3 = q\overline{q}^3W = O(a), \qquad q\overline{q}^3W_0 = O\left(\frac{a}{r}\right).$$

- *$Err[\dot{\square}_2\mathfrak{q}]$ is the nonlinear correction term, which under the additional condition[3]*

$$\check{H}\in\Gamma_g$$

[3]This additional condition makes the structure of Err$[\dot{\square}_2\mathfrak{q}]$ in (5.2.13) possible. This structure is essential in the control of the nonlinear term in Chapter 11, see also Chapter 5 in [50] in the particular case of perturbations of Schwarzschild.

is given schematically by the expression

$$\begin{aligned} Err[\dot{\square}_2\mathfrak{q}] = & \, r^2\mathfrak{d}^{\leq 3}(\Gamma_g\cdot(A,B)) + \nabla_3(r^2\mathfrak{d}^{\leq 2}(\Gamma_b\cdot(A,B))) \\ & + \mathfrak{d}^{\leq 1}(\Gamma_g\cdot\mathfrak{q}) + r^3\mathfrak{d}^{\leq 2}\big(\Gamma_b\cdot\Gamma_g\cdot\Gamma_g\big). \end{aligned} \tag{5.2.13}$$

We now collect some remarks on Theorem 5.2.9.

Remark 5.2.10. *The first order term in $\nabla_{\mathbf{T}}$ on the LHS of* (5.2.11) *presents good divergence properties because of its structure. More precisely, since it is given by an imaginary function multiplied by $\nabla_{\mathbf{T}}\mathfrak{q}$, this term cancels out in the derivation of the energy estimates in the trapping region.*

Notice that there is a conformally invariant definition of $\mathbf{T}$ (where in Definition 4.3.1 the vectors are replaced by their conformally invariant counterparts), but since $\mathfrak{q}$ is 0-conformally invariant the two definitions coincide in this case, and equation (5.2.11) *is fully conformally invariant.*

Remark 5.2.11. *The potential term V is a real function which coincides, for zero angular momentum, with the potential in Schwarzschild in [50], given by*

$$V_0 = -tr\,\chi tr\underline{\chi} = \frac{4\Delta}{(r^2+a^2)|q|^2}.$$

*Observe that the fact that the potential is **real** is crucial in the derivation of the estimates for the Regge-Wheeler equation. This is obtained through the choice of the imaginary part of the scalar function C_1 which gives $\Im(V)=0$. This is obtained in Proposition D.4.5.*

Remark 5.2.12. *The scalar functions C_1 and C_2 are chosen in order to obtain cancellation of terms in the derivation of the gRW equation. The real parts of C_1 and C_2 are chosen to obtain the cancellation of the highest order terms in the commutator $[Q,\mathcal{L}]$, see Proposition D.4.1, and they coincide with the values in Schwarzschild in [50]. As mentioned above, the imaginary part of C_1 is chosen so to cancel the imaginary part of the potential $\Im(V)$ and the imaginary part of the functions appearing as coefficients of the highest order terms in $L_{\mathfrak{q}}[A]$. Finally, the imaginary part of C_2 is chosen in order to have a good transport relation between $\mathfrak{q}$ and A, see Lemma 11.1.3, a fact used in the derivation of the estimates.*

Remark 5.2.13. *In the derivation of the estimates for the Regge-Wheeler equation, the linear second order operator $L_{\mathfrak{q}}[A]$, which contains at most two derivatives of A, will be treated as a lower order term through transport estimates. The form of the highest order terms in $L_{\mathfrak{q}}[A]$, i.e. $\nabla_{\mathbf{T}}(\nabla_3 A)$ and $\nabla_{\mathbf{Z}}(\nabla_3 A)$, is crucial for the estimates in the trapping and relies on an important cancellation obtained in Proposition D.4.5.*

We now describe the steps of the proof of Theorem 5.2.9, relying on the computations collected in Appendix D.4.

1. In Step 1, obtained in section D.4.1, we apply to the Teukolsky equation $\mathcal{L}(A) = \text{Err}[\mathcal{L}(A)]$ the operator $Q = {}^{(c)}\nabla_3 {}^{(c)}\nabla_3 + C_1 {}^{(c)}\nabla_3 + C_2$. We then compute the commutator $[Q, \mathcal{L}]$, and find conditions on the real part of C_1 and C_2 in order to have lower order terms in the commutator, denoted $L_Q(A)$, which are $O(|a|)$, see Proposition D.4.1.

2. In Step 2, obtained in section D.4.2, we derive the wave equation for $Q(A)$ and $\mathfrak{q}$. We first obtain, see Proposition D.4.3, the wave equation for $Q(A)$, and we then rescale $Q(A)$ through the function $f = q\overline{q}^3$ by defining $\mathfrak{q} = fQ(A)$, which satisfies an equation of the form, see Proposition D.4.4,
$$\dot{\square}_2\mathfrak{q} - i\frac{4a\cos\theta}{|q|^2}\nabla_{\mathbf{T}}\mathfrak{q} - V_1\mathfrak{q} \;\; = \;\; \widetilde{L_{\mathfrak{q}}[A]} + f\Big(\text{Err}[\dot{\square}_2 Q] + \Gamma_g \cdot \mathfrak{d}^{\leq 1} Q\Big).$$

3. In Step 3, obtained in section D.4.3, we derive additional conditions on the scalar functions C_1 and C_2 by imposing the reality of the potential of the equation for $\mathfrak{q}$ as well as a specific structure for the lower order terms. More precisely, this is obtained by first imposing that the potential and the terms in $L_{\mathfrak{q}}[A]$ involving two derivatives of A should be real, see Proposition D.4.5, and then by combining the potential term with the lower order term to have only $\mathbf{T}$ and $\mathbf{Z}$ derivatives of $\nabla_3 A$.

5.2.4 The real part of the gRW equation

Since $\mathfrak{q} \in \mathfrak{s}_2(\mathbb{C})$ is a complex anti-self dual tensor, we can decompose it as
$$\mathfrak{q} = \psi + i\,{}^*\psi \tag{5.2.14}$$
for some $\psi = \Re(\mathfrak{q}) \in \mathfrak{s}_2(\mathbb{R})$. Taking the real part of (5.2.11), since V is real, we then obtain an equation for ψ, which is given by
$$\dot{\square}_2\psi + \frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_{\mathbf{T}}\psi - V\psi \;\; = \;\; \Re(L_{\mathfrak{q}}[A]) + \Re(\text{Err}[\dot{\square}_2\mathfrak{q}]).$$
We summarize in the following.

Proposition 5.2.14. *The tensor $\psi \in \mathfrak{s}_2(\mathbb{R})$ satisfies*
$$\dot{\square}_2\psi - V_0\psi = -\frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_T\psi + N, \qquad V_0 = \frac{4\Delta}{(r^2+a^2)|q|^2}, \tag{5.2.15}$$

with the right hand side N being given by

$$\begin{aligned} N &:= \big(V - V_0\big)\psi + \Re(L_{\mathfrak{q}}[A]) + \Re(Err[\dot{\square}_2\mathfrak{q}]) \\ &= N_0 + N_L + N_{Err} \end{aligned} \tag{5.2.16}$$

where:

- N_0 *denotes the zero-th order term in* ψ*, i.e.*

$$N_0 := \left(V - \frac{4\Delta}{(r^2+a^2)|q|^2}\right)\psi = O\left(\frac{a}{r^4}\right)\psi. \tag{5.2.17}$$

- N_L *denotes the lower order terms in* ψ*, i.e.*

$$\begin{aligned} N_L := \Re\Bigg(q\overline{q}^3\Bigg[&- \frac{8a^2\Delta}{r^2|q|^4}\nabla_{\mathbf{T}}\nabla_3 A - \frac{8a\Delta}{r^2|q|^4}\nabla_{\mathbf{Z}}\nabla_3 A \\ &+ W_4\nabla_4 A + W_3\nabla_3 A + W\cdot\nabla A + W_0 A\Bigg]\Bigg) \end{aligned} \tag{5.2.18}$$

where W_4*,* W_3*,* W_0 *are complex functions of* (r,θ)*, and* W *is the product of a complex function of* (r,θ) *with* $^*\Re(\mathfrak{J})$*, having the following fall-off in* r

$$q\overline{q}^3 W_4 = q\overline{q}^3 W_3 = q\overline{q}^3 W = O(a), \qquad q\overline{q}^3 W_0 = O\left(\frac{a}{r}\right).$$

- $N_{Err}[\psi]$ *denotes the error terms, i.e.*

$$N_{Err}[\psi] := \Re(Err[\dot{\square}_2\mathfrak{q}]) \tag{5.2.19}$$

which are schematically given by

$$\begin{aligned} N_{Err}[\psi] &= r^2\mathfrak{d}^{\leq 2}(\Gamma_g\cdot(\alpha,\beta)) + \nabla_3(r^2\mathfrak{d}^{\leq 2}(\Gamma_b\cdot(\alpha,\beta))) \\ &\quad + \mathfrak{d}^{\leq 1}(\Gamma_g\psi) + r^3\mathfrak{d}^{\leq 2}\big(\Gamma_b\cdot\Gamma_g\cdot\Gamma_g\big). \end{aligned}$$

Also, recall that ψ *and* A *are related by the differential relation:*

$$\psi = \Re\Big(q\overline{q}^3\big({}^{(c)}\nabla_3\,{}^{(c)}\nabla_3 A + C_1\,{}^{(c)}\nabla_3 A + C_2 A\big)\Big),$$

with

$$\begin{aligned} C_1 &= 2tr\underline{\chi} - 2\frac{{}^{(a)}tr\underline{\chi}^2}{tr\underline{\chi}} - 4i\,{}^{(a)}tr\underline{\chi}, \\ C_2 &= \frac{1}{2}tr\underline{\chi}^2 - 4\,{}^{(a)}tr\underline{\chi}^2 + \frac{3}{2}\frac{{}^{(a)}tr\underline{\chi}^4}{tr\underline{\chi}^2} + i\left(-2tr\underline{\chi}\,{}^{(a)}tr\underline{\chi} + 4\frac{{}^{(a)}tr\underline{\chi}^3}{tr\underline{\chi}}\right). \end{aligned}$$

5.3 Generalized Regge-Wheeler equation for $\underline{\mathfrak{q}}$

In this section, we derive the generalized Regge-Wheeler equation for $\underline{\mathfrak{q}}$.

5.3.1 The Teukolsky equation for $\underline{A}$

Here we derive the Teukolsky equation for $\underline{A}$. In order to capture correctly the non linear terms in the equation, we express the Bianchi identity for $\underline{A}$ in terms of

$$\underline{A}_4 = {}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A},$$

see Definition 2.4.15, which has an improved decay rate as compared to ${}^{(c)}\nabla_4\underline{A}$. In the derivation of the Teukolsky equation below, we express explicitly the error terms which decay less than $r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b)$.

Proposition 5.3.1. *We have*

$$\begin{aligned}\left({}^{(c)}\nabla_3 + 2\overline{tr\underline{X}} + \frac{1}{2}tr\underline{X}\right)\underline{A}_4 &= \frac{1}{4}\big({}^{(c)}\mathcal{D} + H + 4\,\underline{H}\big)\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A} + \overline{H}\cdot\underline{A}\big)\\ &\quad + 3P\underline{A} + Err_{TE}\end{aligned} \tag{5.3.1}$$

where Err_{TE} is given schematically by

$$Err_{TE} \;=\; trX\underline{\Xi}\widehat{\otimes}\underline{B} + \big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{B}\big)\widehat{\underline{X}} + (\widehat{\underline{X}}\cdot\overline{\check{H}})\underline{B} + (\Gamma_b\cdot\Gamma_b)\cdot(A,B) + r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b).$$

Proof. See section D.5. □

Remark 5.3.2. *Observe that, if we were to obtain the Teukolsky equation for $\underline{A}$ in the symmetric way as the Teukolsky equation for A in Proposition 5.1.1, we would obtain*

$$\underline{\mathcal{L}}(\underline{A}) \;=\; Err[\underline{\mathcal{L}}(\underline{A})]$$

where

$$\begin{aligned}\underline{\mathcal{L}}(\underline{A}) &= -\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_4\underline{A} + \frac{1}{4}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}\big) + \left(-\frac{1}{2}tr\underline{X} - 2\overline{tr\underline{X}}\right){}^{(c)}\nabla_4\underline{A}\\ &\quad - \frac{1}{2}trX\,{}^{(c)}\nabla_3\underline{A} + \big(4\,\underline{H} + H + \overline{H}\big)\cdot{}^{(c)}\nabla\underline{A} + \big(-\overline{tr\underline{X}}trX + 2P\big)\underline{A} + \underline{H}\widehat{\otimes}(\overline{H}\cdot\underline{A})\end{aligned}$$

with error term expressed schematically as

$$Err[\underline{\mathcal{L}}(\underline{A})] \;=\; r^{-1}\mathfrak{d}^{\leq 1}\big(\Gamma_b\cdot\underline{B}\big) + \Gamma_b\cdot\Gamma_b\cdot\Gamma_g.$$

The above error terms are not acceptable in the forthcoming derivation of the generalized Regge-Wheeler equation for $\underline{\mathfrak{q}}$, and we therefore rely instead on Proposition 5.3.1, i.e. we express the Teukolsky equation in terms of $\underline{A}_4 = {}^{(c)}\nabla_4\underline{A} + \frac{1}{2}trX\underline{A}$, which has an improved decay rate as compared to ${}^{(c)}\nabla_4\underline{A}$.

5.3.2 The invariant quantities $\underline{Q}$ and $\underline{\mathfrak{q}}$

In this section we consider the analog $\underline{\mathfrak{q}}$ of $\mathfrak{q}$ and derive its corresponding gRW equation.

Definition 5.3.3. *Given a fixed null pair (e_3, e_4) and scalar functions r and θ as in Section 4.1, we define our second main quantity $\underline{\mathfrak{q}} \in \mathfrak{s}_2(\mathbb{C})$ as*

$$\underline{\mathfrak{q}} \quad = \quad \overline{q}q^3\underline{Q}(\underline{A}) = \overline{q}q^3\left({}^{(c)}\nabla_4\,{}^{(c)}\nabla_4\underline{A} + \underline{C}_1\,{}^{(c)}\nabla_4\underline{A} + \underline{C}_2\underline{A}\right), \tag{5.3.2}$$

with complex scalars

$$\begin{aligned}\underline{C}_1 &= 2tr\,\chi - 2\frac{{}^{(a)}tr\chi^2}{tr\,\chi} - 4i\,{}^{(a)}tr\chi,\\ \underline{C}_2 &= \frac{1}{2}tr\,\chi^2 - 4\,{}^{(a)}tr\chi^2 + \frac{3}{2}\frac{{}^{(a)}tr\chi^4}{tr\,\chi^2} + i\left(-2tr\,\chi\,{}^{(a)}tr\chi + 4\frac{{}^{(a)}tr\chi^3}{tr\,\chi}\right).\end{aligned} \tag{5.3.3}$$

In the particular case of Kerr, the quantity $\underline{\mathfrak{q}}$ defined above can be factorized as follows.

Proposition 5.3.4. *In Kerr, the quantity $\underline{\mathfrak{q}}$ defined in* (5.3.2) *with $\underline{C}_1, \underline{C}_2$ given by* (5.3.3) *can be factorized as*

$$r\,{}^{(c)}\nabla_4\left(r^2\left({}^{(c)}\nabla_4\left(r\frac{q^4}{r^4}\underline{A}\right)\right)\right) = \frac{q}{\overline{q}}\underline{\mathfrak{q}}. \tag{5.3.4}$$

Alternatively,

$${}^{(c)}\nabla_4\,{}^{(c)}\nabla_4\left(\frac{q^4}{r^2}\underline{A}\right) \quad = \quad \frac{q}{\overline{q}}r^{-2}\underline{\mathfrak{q}}. \tag{5.3.5}$$

Proof. We proceed as in the proof of Proposition 5.2.4. Since the formulas are manifestly scale invariant we chose the outgoing normalization of e_4 such that $\omega = 0$ and $e_4(r) = 1$ and $\mathrm{tr}\,\chi = \frac{2r}{|q|^2}$, ${}^{(a)}\mathrm{tr}\chi = \frac{2a\cos\theta}{|q|^2}$. Thus to check (5.3.4) we have to show that

$$I := r\nabla_4\Big(r^2\big(\nabla_4(rf\underline{A})\big)\Big) = r^4 f\Big(\nabla_4\nabla_4\underline{A} + I_1\nabla_4\underline{A} + I_2\underline{A}\Big)$$

with $f = \frac{q^4}{r^4} = f_1 f_2^2$, $f_1 = \frac{|q|^4}{r^4}$, $f_2 = \frac{q}{\overline{q}}$ and

$$\begin{aligned} I_1 &= 2f^{-1}(e_4 f + 2r^{-1} f), \\ I_2 &= f^{-1}\Big(\nabla_4 e_4(f) + 4r^{-1}e_4 f + 2r^{-2} f\Big). \end{aligned}$$

Note that $e_4 f_1 = -\frac{4a^2\cos^2\theta}{r|q|^2} f_1$, $e_4 f_2 = -i\,{}^{(a)}\mathrm{tr}\chi f_2$ and therefore

$$e_4 f = \left(-\frac{4a^2\cos^2\theta}{r|q|^2} - 2i\,{}^{(a)}\mathrm{tr}\chi\right) f.$$

Thus,

$$\begin{aligned} I_1 &= 2\big(f^{-1}e_4 f + 2r^{-1}\big) = 2\left(-\frac{4a^2\cos^2\theta}{r|q|^2} + \frac{2r}{|q|^2} - 2i\,{}^{(a)}\mathrm{tr}\chi\right) \\ &= 2\left(-\frac{2a^2\cos^2\theta}{r|q|^2} + \frac{2r}{|q|^2} - 2i\,{}^{(a)}\mathrm{tr}\chi\right) \\ &= 2\mathrm{tr}\,\chi - 2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi} - 4i\,{}^{(a)}\mathrm{tr}\chi = \underline{C_1}. \end{aligned}$$

Similarly, see the proof of Proposition 5.2.4, $I_2 = \underline{C}_2$ which establishes (5.3.4). □

In the derivation of the gRW equation for $\underline{\mathfrak{q}}$, we will rely on the following more involved factorization of $\underline{\mathfrak{q}}$.

Lemma 5.3.5. *Let*

$$\widetilde{\underline{Q}(\underline{A})} := \left({}^{(c)}\nabla_4 + \frac{5}{2}trX - \overline{trX} - 2\frac{{}^{(a)}tr\chi^2}{tr\,\chi}\right)\underline{A}_4. \tag{5.3.6}$$

Then, we the following holds

$$\underline{\mathfrak{q}} = \overline{q}q^3\widetilde{\underline{Q}(\underline{A})} + O(a^2)\underline{A} + r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b). \tag{5.3.7}$$

Proof. We have, see Lemma 12.3.7,

$$\begin{aligned} &\overline{q}q^3\left({}^{(c)}\nabla_4 + \frac{3}{2}\mathrm{tr}X - 2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi} - 2i\,{}^{(a)}\mathrm{tr}\chi\right)\left({}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A}\right) \\ &= \underline{\mathfrak{q}} + O(a^2)\underline{A} + r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b), \end{aligned}$$

which together with the fact that $\overline{\mathrm{tr}X} = \mathrm{tr}\,\chi + i\,{}^{(a)}\mathrm{tr}\chi$ and $\underline{A}_4 = {}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A}$ proves the lemma. □

5.3.3 The derivation of the gRW equation for $\underline{\mathfrak{q}}$

We state below the gRW equation satisfied by $\underline{\mathfrak{q}}$.

Theorem 5.3.6. *The invariant symmetric traceless 2-tensor $\underline{\mathfrak{q}} \in \mathfrak{s}_2(\mathbb{C})$ in Definition 5.3.3 satisfies the equation*

$$\dot{\square}_2\underline{\mathfrak{q}} + i\frac{4a\cos\theta}{|q|^2}\nabla_{\mathbf{T}}\underline{\mathfrak{q}} - V\underline{\mathfrak{q}} \quad = \quad L_{\underline{\mathfrak{q}}}[\underline{A}] + Err[\dot{\square}_2\underline{\mathfrak{q}}] \tag{5.3.8}$$

where:

- *The potential V is the **real** scalar function given by*

$$V \quad = \quad \frac{4}{|q|^2}\frac{r^2-2mr+2a^2}{r^2} - \frac{4a^2\cos^2\theta}{|q|^6}(r^2+6mr+a^2\cos^2\theta), \tag{5.3.9}$$

  which for $a=0$ coincides with the potential of the Regge-Wheeler equation in Schwarzschild, i.e. $V = -tr\,\chi tr\underline{\chi} + O(\frac{|a|}{r^4})$.

- *$L_{\underline{\mathfrak{q}}}[\underline{A}]$ is a linear second order operator in $\underline{A}$ given in the ingoing frame by*

$$L_{\underline{\mathfrak{q}}}[\underline{A}] = q\overline{q}^3\left(\frac{8a^2\Delta}{r^2|q|^4}\nabla_{\mathbf{T}}\underline{A}_4 + \frac{8a\Delta}{r^2|q|^4}\nabla_{\mathbf{Z}}\underline{A}_4 + \underline{W}_4\underline{A}_4 + \underline{W}_3\nabla_3\underline{A} + \underline{W}\cdot\nabla\underline{A} + \underline{W}_0\underline{A}\right),$$

  where $\underline{W}_4$, $\underline{W}_3$, $\underline{W}_0$ are complex functions of (r,θ) and $\underline{W}$ is the product of a complex function of (r,θ) with ${}^*\mathfrak{R}(\mathfrak{J})$, with the following fall-off[4] in r

$$q\overline{q}^3\underline{W}_4, q\overline{q}^3\underline{W} = O\left(\frac{a^2}{r}\right), \qquad q\overline{q}^3\underline{W}_3, q\overline{q}^3\underline{W}_0 = O\left(\frac{a^2}{r^2}\right).$$

- *$Err[\dot{\square}_2\underline{\mathfrak{q}}]$ is the nonlinear correction term, which under the additional conditions[5]*

$$\Xi = 0, \qquad \underline{\widecheck{H}} = 0, \quad for \quad r \geq r_0,$$

  is given schematically by the expression

$$Err[\square_2\underline{\mathfrak{q}}] \quad = \quad r^2\mathfrak{d}^{\leq 2}(\Gamma_b\cdot(A,B)) + \mathfrak{d}^{\leq 3}(\Gamma_g\cdot\Gamma_b).$$

[4]Note that the fall-off provided here for $\underline{W}_4$, $\underline{W}_3$, $\underline{W}_0$ and $\underline{W}$ is stronger than the one for W_4, W_3, W_0 and W in $L_{\mathfrak{q}}[A]$, see section 5.2.3. In fact, provided we replace $\nabla_3 A$ with A_3 in the definition of $L_{\mathfrak{q}}[A]$, W_4, W_3, W_0 and W satisfy the same fall-off is r, but this stronger fall-off is unnecessary for the weighted estimates derived for $\mathfrak{q}$ in Chapter 11.

[5]In fact, it suffices to assume that $\Xi \in r^{-2}\Gamma_g$ and $\underline{\widecheck{H}} \in r^{-1}\Gamma_g$. These additional conditions make the structure of Err$[\dot{\square}_2\underline{\mathfrak{q}}]$ in (5.3.10) possible. This structure is essential in the control of the nonlinear term in Chapter 12.

We now describe the steps of the proof of Theorem 5.3.6, relying on the computations collected in Appendix D.6.

1. In Step 1, we take the first derivative in the ${}^{(c)}\nabla_4$ direction of the Teukolsky equation for $\underline{A}$. We express explicitly the error terms which decay less than $r^{-3}\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\Gamma_b)$. We obtain, see Proposition D.6.1,

$$\begin{aligned}&\left({}^{(c)}\nabla_4+\mathrm{tr}X+\frac{1}{2}\overline{\mathrm{tr}X}\right)\left({}^{(c)}\nabla_3+2\overline{\mathrm{tr}\underline{X}}+\frac{1}{2}\mathrm{tr}\underline{X}\right)\underline{A}_4\\ =\;&\frac{1}{4}\big({}^{(c)}\mathcal{D}+H+5\,\underline{H}\big)\widehat{\otimes}\Big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\Big)+3P\underline{A}_4\\ &+3\left(\frac{1}{2}\overline{\mathrm{tr}X}-\mathrm{tr}X\right)P\underline{A}+\mathscr{J}_{434}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)+\mathrm{Err}_{434},\end{aligned}$$

for a one-form $\mathscr{J}_{434}$ and error terms given by

$$\begin{aligned}\mathrm{Err}_{434}\;=\;&{}^{(c)}\nabla_4\mathrm{Err}_{TE}+\left(\frac{1}{2}\mathrm{tr}X+\overline{\mathrm{tr}X}\right)\mathrm{Err}_{TE}\\ &+{}^{(c)}\mathcal{D}\widehat{\otimes}(\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}\underline{A})+{}^{(c)}\mathcal{D}\widehat{\otimes}(({}^{(c)}\mathcal{D}\cdot\overline{\widehat{X}})\cdot\underline{A})\\ &+r^{-1}\mathfrak{d}^{\leq 1}((A,B)\cdot\Gamma_b)+r^{-3}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b).\end{aligned}$$

2. In Step 2, we take the second derivative in the ${}^{(c)}\nabla_4$ direction of the Teukolsky equation for $\underline{A}$. Again, we express explicitly the error terms which decay less than $r^{-4}\mathfrak{d}^{\leq 3}(\Gamma_g\cdot\Gamma_b)$. We obtain, see Proposition D.6.2,

$$\begin{aligned}&\left({}^{(c)}\nabla_4+3\mathrm{tr}X-\frac{1}{2}\overline{\mathrm{tr}X}-2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}\right)\left({}^{(c)}\nabla_4+\mathrm{tr}X+\frac{1}{2}\overline{\mathrm{tr}X}\right)\left({}^{(c)}\nabla_3+2\overline{\mathrm{tr}\underline{X}}+\frac{1}{2}\mathrm{tr}\underline{X}\right)\underline{A}_4\\ &+\big(2\mathrm{tr}X-\overline{\mathrm{tr}X}\big)3P\underline{A}_4\\ =\;&\frac{1}{4}\big({}^{(c)}\mathcal{D}+H+6\,\underline{H}\big)\widehat{\otimes}\Big(\overline{{}^{(c)}\mathcal{D}}\cdot\widetilde{\underline{Q}(\underline{A})}+(\overline{H}+2\,\overline{\underline{H}})\cdot\widetilde{\underline{Q}(\underline{A})}\Big)+3P\,\widetilde{\underline{Q}(\underline{A})}\\ &+\mathscr{L}[\underline{A},\underline{A}_4,\mathcal{D}\underline{A}_4,\mathcal{D}\underline{A}]+\mathrm{Err}_{4434}\end{aligned}$$

where $\widetilde{\underline{Q}(\underline{A})}$ is defined in (5.3.6), and is such that

$$\underline{\mathfrak{q}}=\overline{q}q^3\widetilde{\underline{Q}(\underline{A})}+O(a^2)\underline{A}+r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b).$$

We denote by $\mathscr{L}[\underline{A},\underline{A}_4,\mathcal{D}\underline{A}_4,\mathcal{D}\underline{A}]$ linear order terms in $\underline{A},\underline{A}_4,\mathcal{D}\underline{A},\mathcal{D}\underline{A}_4$ schematically given by (D.6.11), i.e.

$$\begin{aligned}\mathscr{L}[\underline{A},\underline{A}_4,\mathcal{D}\underline{A},\mathcal{D}\underline{A}_4]\;=\;&O(ar^{-3})\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\big)\\ &+O(a^2r^{-4})\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)+O(a^2r^{-7})\underline{A}.\end{aligned}$$

The error terms are given by

$$\begin{aligned}
\mathrm{Err}_{4434} &= {}^{(c)}\nabla_4 \mathrm{Err}_{434} + \left(\mathrm{tr}X + \frac{3}{2}\overline{\mathrm{tr}X} + 2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}\right)\mathrm{Err}_{434} \\
&\quad + r^{-2}\mathfrak{d}^{\leq 1}((A, B)\cdot \Gamma_b) + r^{-4}\mathfrak{d}^{\leq 2}(\Gamma_g \cdot \Gamma_b) \\
&\quad + r^{-1}\left({}^{(c)}\nabla_4(\widehat{X}\cdot \check{\underline{H}}) + \frac{3}{2}\mathrm{tr}X(\widehat{X}\cdot\check{\underline{H}})\right)\cdot \mathfrak{d}^{\leq 1}\underline{A}.
\end{aligned}$$

By commuting the operators on the left hand side of (D.6.6) and using the expression of $\underline{\mathfrak{q}}$ in terms of $\widetilde{\underline{Q}(\underline{A})}$ given in (5.3.7), we can deduce the form of the equation for $\underline{\mathfrak{q}}$.

3. In Step 4, we show that the error terms Err_{4434} can be simplified to obtain, see Proposition D.6.3,

$$\mathrm{Err}_{4434} = r^{-2}\mathfrak{d}^{\leq 2}((A, B)\cdot\Gamma_b) + r^{-4}\mathfrak{d}^{\leq 3}(\Gamma_g\cdot\Gamma_b).$$

We show that the terms which behave worse that $r^{-4}\mathfrak{d}^{\leq 3}(\Gamma_g\cdot\Gamma_b)$ get improved once applied the differential operators $({}^{(c)}\nabla_4 + \mathrm{tr}X + \frac{1}{2}\overline{\mathrm{tr}X})$ and ${}^{(c)}\nabla_4 + 2\mathrm{tr}X + \frac{1}{2}\overline{\mathrm{tr}X}$ by making use of the renormalized Bianchi identities and improved decay in the null structure equations. This ends the proof of Theorem 5.3.6.

As in section 5.2.4 in the case of $\mathfrak{q}$, we infer from Theorem 5.3.6, for the real part of $\underline{\mathfrak{q}}$ denoted $\underline{\psi} = \Re(\underline{\mathfrak{q}})$, the following real equation:

$$\dot{\square}_2\underline{\psi} - V_0\underline{\psi} = \frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_T\underline{\psi} + N, \qquad V_0 = \frac{4\Delta}{(r^2+a^2)|q|^2}, \tag{5.3.10}$$

where the right hand side N is given by $N = N_0 + N_L + N_{\mathrm{Err}}$, with

$$N_0 = O\left(\frac{a}{r^4}\right)\underline{\psi}, \qquad N_L = \Re(L_{\underline{\mathfrak{q}}}[\underline{A}]), \qquad N_{\mathrm{Err}} = \Re(\mathrm{Err}[\square_2\underline{\mathfrak{q}}]).$$

5.4 Teukolsky-Starobinski identity

We state here, in the context of perturbations of Kerr, one of the Teukolsky-Starobinski identities, which relate the complex curvature components A and $\underline{A}$ through fourth-order differential operators.

Proposition 5.4.1. *Assume that $\Xi = 0$ in $r \leq r_0$. The complex tensors A, $\underline{A} \in \mathfrak{s}_2(\mathbb{C})$ satisfy the following relation in the region $r \leq r_0$*

$$\left({}^{(c)}\nabla_4 + 2trX\right)^4\underline{A} = r^{-4}\mathfrak{d}^{\leq 4}A + \mathfrak{d}^{\leq 3}\big(\Gamma_b\cdot\Gamma_g\big). \tag{5.4.1}$$

Proof. See Appendix D.7. □

Remark 5.4.2. *Proposition 5.4.1 is stated without proof in Chapter 7 of [53]. Both the assumption $\Xi = 0$ and the restriction to $r \leq r_0$ are unnecessary and assumed only for convenience, as they hold when applying Proposition 5.4.1 in Chapter 7 of [53]. In particular, the restriction to $r \leq r_0$ allows us to avoid having to track the precise powers of r in the nonlinear terms.*

Remark 5.4.3. *Choosing a normalization such that $\omega \in \Gamma_g$, and hence $trX = \frac{2}{q} + \Gamma_g$, we infer from Proposition 5.4.1, for $\Xi = 0$ in the region $r \leq r_0$,*

$$\frac{1}{q^7}\nabla_4\left(q^2\nabla_4\left(q^2\nabla_4\left(q^2\nabla_4(q\underline{A})\right)\right)\right) \quad = \quad r^{-4}\mathfrak{d}^{\leq 4}A + \mathfrak{d}^{\leq 3}\big(\Gamma_b \cdot \Gamma_g\big).$$

5.5 The wave equation for P

Here we derive the wave equation satisfied by the curvature component P.

Lemma 5.5.1. *The curvature component P satisfies the following scalar wave equation:*

$$\begin{aligned} \square_{\mathbf{g}} P &= trX\nabla_3 P + \overline{tr\underline{X}}\,\nabla_4 P - \overline{H}\cdot\mathcal{D}P - \underline{H}\cdot\overline{\mathcal{D}}P \\ &\quad + \frac{3}{2}\Big[\overline{tr\underline{X}}trX + 2P - 2\,\underline{H}\cdot\overline{H}\Big]P + Err[\square_{\mathbf{g}}P], \end{aligned} \tag{5.5.1}$$

with error terms given by

$$\begin{aligned} Err[\square_{\mathbf{g}}P] &= -\,{}^{(c)}\nabla_3(\overline{\Xi}\cdot\underline{B}) - \frac{1}{4}\,{}^{(c)}\nabla_3(\widehat{\underline{X}}\cdot\overline{A}) + \frac{1}{2}\,{}^{(c)}\mathcal{D}\cdot\left(\underline{B}\cdot\overline{\widehat{X}} + \frac{1}{2}\overline{A}\cdot\underline{\Xi}\right) \\ &\quad + \underline{H}\cdot\left(\underline{B}\cdot\overline{\widehat{X}} + \frac{1}{2}\overline{A}\cdot\underline{\Xi}\right) + \frac{1}{2}\Big(-\overline{\underline{B}}\cdot B + \underline{\Xi}\cdot\nabla_4\overline{B} - \overline{\widehat{\underline{X}}}\cdot{}^{(c)}\mathcal{D}\overline{B} - \overline{H}\cdot\overline{\widehat{\underline{X}}}\cdot B\Big) \\ &\quad - \frac{1}{2}\big(2\overline{tr\underline{X}} + tr\underline{X}\big)\left(\overline{\Xi}\cdot\underline{B} + \frac{1}{4}\widehat{\underline{X}}\cdot\overline{A}\right) + \frac{1}{2}H\cdot\left(\underline{B}\cdot\overline{\widehat{X}} + \frac{1}{2}\overline{A}\cdot\underline{\Xi}\right) \\ &\quad + \left(-\frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot\widehat{\underline{X}} - i\Im(trX)\underline{\Xi} + \underline{B} + {}^{(c)}\nabla_4\underline{\Xi} - \frac{1}{2}\widehat{\underline{X}}\cdot(\overline{\underline{H}} - \overline{H}) + \underline{B}\right)\cdot\overline{B} \\ &\quad - \frac{3}{2}\big(\underline{\Xi}\cdot\overline{\Xi} - \frac{1}{2}\widehat{\underline{X}}\cdot\overline{\widehat{X}}\big)P. \end{aligned} \tag{5.5.2}$$

Proof. See Appendix D.8. □

Remark 5.5.2. *From the expression in* (5.5.2) *and using the null structure equation for ${}^{(c)}\nabla_3\widehat{\underline{X}}$, observe that the error terms $Err[\square_{\mathbf{g}}P]$ can be schematically written as*

$$Err[\square_{\mathbf{g}}P] \quad = \quad r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\underline{B}) + \nabla_3(\Xi\cdot\underline{B}) + r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b\cdot(A,B)) + r^{-3}(\Gamma_g\cdot\Gamma_b) - \underline{A}\cdot\overline{A}.$$

The above structure will be used in Chapter 14, see (14.2.1).

5.5.1 A renormalized wave equation

Lemma 5.5.3. *Let Ψ a scalar function solution to the following wave equation*

$$\square_{\mathbf{g}}\Psi = trX\nabla_3\Psi + \overline{tr\underline{X}}\nabla_4\Psi - \overline{H}\cdot\mathcal{D}\Psi - \underline{H}\cdot\overline{\mathcal{D}}\Psi + V\Psi + F, \tag{5.5.3}$$

where V is a potential and F a scalar function. Then, we have

$$\square_{\mathbf{g}}(q^2\Psi) = \Big[V + q^{-2}\square_{\mathbf{g}}(q^2)\Big]q^2\Psi + r\Gamma_b\cdot\mathfrak{d}\Psi + q^2F.$$

Proof. We have

$$\begin{aligned}\square_{\mathbf{g}}(q^2\Psi) &= q^2\square_{\mathbf{g}}(\Psi) + \square_{\mathbf{g}}(q^2)\Psi + 2\mathbf{g}^{\alpha\beta}\partial_\alpha(q^2)\partial_\beta(\Psi)\\ &= q^2\square_{\mathbf{g}}(\Psi) + \square_{\mathbf{g}}(q^2)\Psi - e_3(q^2)e_4\Psi - e_4(q^2)e_3\Psi + 2\nabla(q^2)\nabla\Psi\\ &= q^2\left[\square_{\mathbf{g}}(\Psi) + q^{-2}\square_{\mathbf{g}}(q^2)\Psi - \frac{2e_3(q)}{q}e_4\Psi - \frac{2e_4(q)}{q}e_3\Psi + 4\frac{\nabla(q)}{q}\cdot\nabla\Psi\right].\end{aligned}$$

Since

$$\begin{aligned}\frac{\overline{\mathcal{D}}(q)}{q}\cdot\mathcal{D}\Psi + \frac{\mathcal{D}(q)}{q}\cdot\overline{\mathcal{D}}\Psi &= \frac{(\nabla - i\,{}^*\nabla)(q)}{q}\cdot(\nabla + i\,{}^*\nabla)\Psi + \frac{(\nabla + i\,{}^*\nabla)(q)}{q}\cdot(\nabla - i\,{}^*\nabla)\Psi\\ &= 4\frac{\nabla q}{q}\cdot\nabla\Psi,\end{aligned}$$

we infer

$$\square_{\mathbf{g}}(q^2\Psi) = q^2\left[\square_{\mathbf{g}}(\Psi) + q^{-2}\square_{\mathbf{g}}(q^2)\Psi - \frac{2e_3(q)}{q}e_4\Psi - \frac{2e_4(q)}{q}e_3\Psi + \frac{\overline{\mathcal{D}}(q)}{q}\cdot\mathcal{D}\Psi + \frac{\mathcal{D}(q)}{q}\cdot\overline{\mathcal{D}}\Psi\right].$$

Using

$$\frac{e_4(q)}{q} = \frac{1}{2}\mathrm{tr}X + r^{-1}\Gamma_b, \quad \frac{e_3(q)}{q} = \frac{1}{2}\overline{\mathrm{tr}\underline{X}} + \Gamma_b, \quad H = \frac{\mathcal{D}(\overline{q})}{\overline{q}} + r^{-1}\Gamma_b, \quad \underline{H} = \frac{\mathcal{D}(q)}{q} + \Gamma_b,$$

we deduce

$$\begin{aligned}\square_{\mathbf{g}}(q^2\Psi) &= q^2\left[\square_{\mathbf{g}}(\Psi) + q^{-2}\square_{\mathbf{g}}(q^2)\Psi - \overline{\mathrm{tr}\underline{X}}e_4\Psi - \mathrm{tr}X e_3\Psi + \overline{H}\cdot\mathcal{D}\Psi + \underline{H}\cdot\overline{\mathcal{D}}\Psi\right]\\ &\quad + r\Gamma_b\cdot\mathfrak{d}\Psi + q^2F.\end{aligned}$$

Since Ψ satisfies (5.5.3) we infer

$$\square_{\mathbf{g}}(q^2\Psi) = \Big[V + q^{-2}\square_{\mathbf{g}}(q^2)\Big]q^2\Psi + r\Gamma_b\cdot\mathfrak{d}\Psi + q^2F$$

as stated. $\square$

5.5.2 Wave equations for $q^2(\mathbf{T}, \mathbf{Z})P$

Lemma 5.5.4. *The linearized quantities $q^2\mathbf{T}P$, $q^2\mathbf{Z}P$ verify the following wave equations*

$$\begin{aligned}\square_{\mathbf{g}}(q^2\mathbf{T}P) &= Wq^2\mathbf{T}P + r\Gamma_b\cdot\mathfrak{d}^2P + r^2\mathfrak{d}Err[\square_{\mathbf{g}}P] + r^2\mathfrak{d}^{\le 1}\big(\Gamma_g\cdot\mathfrak{d}^{\le 1}P\big) + r^2\Gamma_b\cdot\square_{\mathbf{g}}P,\\ \square_{\mathbf{g}}(q^2\mathbf{Z}P) &= Wq^2\mathbf{Z}P + r\Gamma_b\cdot\mathfrak{d}^2P + r^2\mathfrak{d}Err[\square_{\mathbf{g}}P] + r^2\mathfrak{d}^{\le 1}\big(\Gamma_g\cdot\mathfrak{d}^{\le 1}P\big) + r^3\Gamma_b\cdot\square_{\mathbf{g}}P,\end{aligned} \tag{5.5.4}$$

where the potential W, given by

$$W = \frac{3}{2}\Big[\overline{tr\underline{X}}trX + 2P - 2\,\underline{H}\cdot\overline{H}\Big] + q^{-2}\square_{\mathbf{g}}(q^2),$$

is complex, and satisfies

$$\Re(W) = W_{Sch} + O(ar^{-4}), \qquad \Im(W) = O(ar^{-3}).$$

where $W_{Sch} = O(mr^{-3})$ is the respective real potential in Schwarzschild.

Proof. We commute the wave equation for P (5.5.1)

$$\square_{\mathbf{g}}P \;= \mathrm{tr}X\nabla_3P + \overline{\mathrm{tr}\underline{X}}\nabla_4P - \overline{H}\cdot\mathcal{D}P - \underline{H}\cdot\overline{\mathcal{D}}P + VP + F,$$

with $\mathbf{T}$. Using (4.3.1) and, as a consequence of Lemma 4.3.2,

$$[\mathbf{T}, e_3] = [\mathbf{T}, e_4] = [\mathbf{T}, e_a] = \Gamma_b\cdot\mathfrak{d}$$

we deduce

$$\begin{aligned}\square_{\mathbf{g}}(\mathbf{T}P) &= \mathbf{T}\square_{\mathbf{g}}P + \mathfrak{d}\big(\Gamma_g\cdot\mathfrak{d}P\big) + \Gamma_b\cdot\square_{\mathbf{g}}P\\ &= \mathrm{tr}X\nabla_3(\mathbf{T}P) + \overline{\mathrm{tr}\underline{X}}\nabla_4(\mathbf{T}P) - \overline{H}\cdot\mathcal{D}(\mathbf{T}P) - \underline{H}\cdot\overline{\mathcal{D}}(\mathbf{T}P) + V\mathbf{T}P + \mathbf{T}F\\ &+ \mathbf{T}(\mathrm{tr}X)\nabla_3P + \mathbf{T}(\overline{\mathrm{tr}\underline{X}})\nabla_4P - \mathbf{T}(\overline{H})\cdot\mathcal{D}P - \mathbf{T}(\underline{H})\cdot\overline{\mathcal{D}}P + \mathbf{T}(V)P\\ &+ \mathrm{tr}X[\mathbf{T},\nabla_3]P + \overline{\mathrm{tr}\underline{X}}[\mathbf{T},\nabla_4]P + \mathfrak{d}\big(\Gamma_g\cdot\mathfrak{d}P\big) + \Gamma_b\cdot\square_{\mathbf{g}}P\\ &= \mathrm{tr}X\nabla_3(\mathbf{T}P) + \overline{\mathrm{tr}\underline{X}}\nabla_4(\mathbf{T}P) - \overline{H}\cdot\mathcal{D}(\mathbf{T}P) - \underline{H}\cdot\overline{\mathcal{D}}(\mathbf{T}P) + V\mathbf{T}P\\ &\quad+\mathfrak{d}\mathrm{Err}[\square_{\mathbf{g}}P] + \mathfrak{d}^{\le 1}\big(\Gamma_g\cdot\mathfrak{d}^{\le 1}P\big) + \Gamma_b\cdot\square_{\mathbf{g}}P.\end{aligned}$$

To the above we can apply Lemma 5.5.3 and deduce

$$\square_{\mathbf{g}}(q^2\mathbf{T}P) \;=\; Wq^2\mathbf{T}P + r\Gamma_b\cdot\mathfrak{d}^2P + r^2\mathfrak{d}\mathrm{Err}[\square_{\mathbf{g}}P] + r^2\mathfrak{d}^{\le 1}\big(\Gamma_g\cdot\mathfrak{d}^{\le 1}P\big) + r^2\Gamma_b\cdot\square_{\mathbf{g}}P$$

where

$$W = V + q^{-2}\square_{\mathbf{g}}(q^2).$$

The second equation in (14.3.1) can be derived in the same manner. □

5.6 An identity for ${}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}P$

The following identity will be used in Chapter 14.

Proposition 5.6.1. *The following relation holds true:*

$$\begin{aligned}
&{}^{(c)}\nabla_4\,{}^{(c)}\nabla_4\underline{A} + 2trX\,{}^{(c)}\nabla_4\underline{A} + \frac{1}{2}(trX)^2\underline{A}\\
&= \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}P + \big(4\,{}^{(c)}\mathcal{D}P + 6P\,\underline{H} + trX\underline{B}\big)\widehat{\otimes}\,\underline{H} + \frac{3}{2}P\big(trX\,\widehat{\underline{X}} + \overline{tr\underline{X}}\widehat{X}\big)\\
&+ \Xi\widehat{\otimes}\,{}^{(c)}\nabla_3\underline{B} + r^{-1}\mathfrak{d}^{\leq 1}(\Xi\cdot\underline{A}) + r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\underline{B}) + r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b\cdot B).
\end{aligned} \tag{5.6.1}$$

From the above, we deduce the following relation between $\underline{\mathfrak{q}}$ and $\check{P}$

$$\begin{aligned}
\underline{\mathfrak{q}} &= \frac{1}{2}\overline{q}q^3\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\check{P} + \not{\mathfrak{d}}^{\leq 1}(\Gamma_b, r\Gamma_g) + O(ar)\mathfrak{d}^{\leq 1}\underline{B} + O(a^2)\underline{A} + O(ar)\mathfrak{d}^{\leq 1}\check{P}\\
&+ r^3\mathfrak{d}^{\leq 1}(\Gamma_b\cdot(\check{P}, B)) + r^2\mathfrak{d}^{\leq 1}(\Gamma_g\cdot(r\underline{B}, \underline{A})) + r^4\Xi\widehat{\otimes}\,{}^{(c)}\nabla_3\underline{B} + r^3\mathfrak{d}^{\leq 1}(\Xi\cdot\underline{A})
\end{aligned} \tag{5.6.2}$$

where the linear term $\not{\mathfrak{d}}^{\leq 1}(\Gamma_b, r\Gamma_g)$ does not contain $\underline{\Xi}$ or $\widecheck{tr\underline{X}}$.

Proof. See Appendix D.9. □

Part II

Analysis of the wave equations

Chapter 6

Estimates for the model gRW equation in perturbations of Kerr

6.1 Preliminaries

In this chapter we introduce the model problem for the full generalized Regge-Wheeler equation in perturbations of Kerr obtained in Proposition 5.2.14. The model problem consists in the following gRW equation for a real tensor $\psi \in \mathfrak{s}_2$ in the spacetime $\mathcal{M}$ of section 4.1:

$$\dot{\square}_2\psi - V\psi = -\frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_T\psi + N, \qquad V = \frac{4\Delta}{(r^2+a^2)|q|^2}, \tag{6.1.1}$$

for some right-hand side N. Here $\dot{\square}_2$ denotes the D'Alembertian operator for horizontal 2-tensors in $\mathcal{M}$.

Remark 6.1.1. *We note that in applications to the gRW equation for $\mathfrak{q}$, N contains linear terms, in particular $\Re(L_{\mathfrak{q}}[A])$, as well as quadratic $\Re(Err[\square_2\mathfrak{q}])$, see Proposition 5.2.14.*

6.1.1 The spacetime $\mathcal{M}$

We consider a given vacuum spacetime $\mathcal{M}$ satisfying the properties in section 4.1:

- $\mathcal{M}$ comes together with a null pair (e_4, e_3) and its corresponding horizontal structure as in section 2.1.1.

- $\mathcal{M}$ is endowed with a pair of constants (a, m).
- $\mathcal{M}$ is endowed with a pair of scalar functions (r, θ).
- The complex valued scalar function q is defined as

$$q := r + i\cos\theta.$$

- $\mathcal{M}$ is endowed with a complex horizontal 1-form $\mathfrak{J}$.

In addition, we assume:

1. $\mathcal{M}$ is also endowed with a scalar function τ whose level sets $\Sigma(\tau)$ are spacelike. $\tau \in [1, \tau_*]$ on $\mathcal{M}$ for some arbitrary large constant τ_*.
2. The boundary of $\mathcal{M}$ is given by

$$\partial\mathcal{M} = \mathcal{A} \cup \Sigma_* \cup \Sigma(1) \cup \Sigma(\tau_*) \tag{6.1.2}$$

where

$$\mathcal{A} := \Big\{ r = r_+ - \delta_{\mathcal{H}},\ 1 \le \tau \le \tau_* \Big\}, \tag{6.1.3}$$

and Σ_* is a spacelike hypersurface on which τ takes the values $[1, \tau_*]$ and $r \ge r_*$ with $r_* \gg \tau_*$.
3. Let r_0 a large enough fixed constant. We decompose $\mathcal{M}$ as follows

$$^{(int)}\mathcal{M} := \mathcal{M} \cap \{r \le r_0\}, \qquad ^{(ext)}\mathcal{M} := \mathcal{M} \cap \{r \ge r_0\}. \tag{6.1.4}$$

6.1.2 Admissible perturbations of Kerr

Recall that $\mathcal{M}$ comes together three scalar functions (r, θ, τ), and with a null pair (e_4, e_3) and its corresponding horizontal structure as in section 2.1.1. Then:

- We use the complexified Ricci and curvature coefficients of Definition 2.4.8.
- We define the linearized quantities corresponding to these complexified coefficients as in Definition 4.1.1 and 4.1.3, i.e. we consider that the normalization of (e_3, e_4) is ingoing.

- With respect to these linearized quantities, we the notations Γ_g and Γ_b for error terms are given by Definition 4.1.5.

Remark 6.1.2. *We define the following linearized quantity*

$$\widetilde{|\Re(\mathfrak{J})|^2} := |\Re(\mathfrak{J})|^2 - \frac{(\sin\theta)^2}{|q|^2}.$$

Note that $\widetilde{|\Re(\mathfrak{J})|^2} = 0$ *in Kerr. We assume in addition that* $\mathfrak{d}^{\leq 1}(\widetilde{|\Re(\mathfrak{J})|^2}) \in r^{-2}\Gamma_b$.

We this definition of Γ_g and Γ_b, we can now state our main assumptions on $\mathcal{M}$. Let k_L a large enough integer. We make assumptions on decay and on boundedness on (Γ_b, Γ_g).

Assumptions on $\mathcal{M}$

On $\mathcal{M}$, we will prove energy Morawetz estimates. To this end, we introduce the scalar function τ_{trap} defined by

$$\tau_{trap} := \begin{cases} 1+\tau & \text{on} \quad \mathcal{M}_{trap}, \\ 1 & \text{on} \quad \mathcal{M}_{\not{trap}}. \end{cases} \tag{6.1.5}$$

Then, we assume that the linearized quantities satisfy the following estimates on $\mathcal{M}$

$$\begin{aligned} r^3|\mathfrak{d}^{\leq k}\xi| + r^2|\mathfrak{d}^{\leq k}\Gamma_g| + r|\mathfrak{d}^{\leq k}\Gamma_b| &\leq \epsilon, & k &\leq k_L, \\ r^3|\mathfrak{d}^{\leq k}\xi| + r^2|\mathfrak{d}^{\leq k}\Gamma_g| + r|\mathfrak{d}^{\leq k}\Gamma_b| &\leq \frac{\epsilon}{\tau_{trap}^{1+\delta_{dec}}}, & k &\leq \frac{k_L}{2}. \end{aligned} \tag{6.1.6}$$

Remark 6.1.3. *In this section k_L is an unspecified large positive integer. The bounds* (6.1.6) *will be assumed in all results and proofs of Chapters*[1] *6, 9 and 10. In applications to chapters 11 and 12 we will specify k_L and make additional assumptions.*

Remark 6.1.4. *Note that the assumptions for ξ in* (6.1.6) *are consistent with $\xi \in r^{-1}\Gamma_g$, while ξ is a priori only in Γ_g according to Definition 4.1.5. These stronger assumptions* (6.1.6) *for ξ will always hold whenever we apply the results of Chapter 6:*

- *In Chapter 11, this follows from the assumptions* (11.1.4) *and the fact that $\xi \in \Gamma_g$.*
- *In Chapter 12, this follows from the assumptions* (12.1.3).
- *In Part III, this follows from the assumptions* (13.6.5).

[1]Chapters 7 and 8 concern proofs in Kerr and do thus not require assumptions on (Γ_g, Γ_b).

6.1.3 Basic properties of the τ function

Choice of τ

Recall that $\mathcal{M}$ is also endowed with a scalar function τ whose level sets $\Sigma(\tau)$ are spacelike. We provide in this section the basic properties of the τ function that will be used later. Recall that, given a time function τ, the vectorfield $= -\mathbf{g}^{\alpha\beta}\partial_\beta\tau\partial_\alpha$ is timelike future oriented. Given a level hypersurface $\Sigma = \Sigma(\tau)$, we denote

$$N_\Sigma := -\mathbf{g}^{\alpha\beta}\partial_\beta\tau\partial_\alpha.$$

Definition 6.1.5 (Choice of τ)**.** *Let $\delta_{\mathcal{H}} > 0$ small enough. We choose the smooth scalar function τ on $r \geq r_+(1-\delta_{\mathcal{H}})$ such that we have on $r \geq r_+(1-\delta_{\mathcal{H}})$*

$$\mathbf{g}(N_\Sigma, N_\Sigma) \leq -\frac{m^2}{8r^2}, \qquad e_4(\tau) > 0, \qquad e_3(\tau) > 0, \qquad |\nabla\tau|^2 \leq \frac{8}{9}e_4(\tau)e_3(\tau).$$

In addition, we have the following asymptotic behavior for r large

$$\frac{m^2}{r^2} \lesssim e_4(\tau) \lesssim \frac{m^2}{r^2}, \qquad 1 \lesssim e_3(\tau) \lesssim 1.$$

Finally, we assume on $\mathcal{M}$

$$\mathbf{T}(\tau) = 1 + r\Gamma_b, \qquad \nabla(\tau) = a\Re(\mathfrak{J}) + \Gamma_b,$$

where $\mathbf{T}$ is the vectorfield introduced below in Definition 6.1.10.

Remark 6.1.6. *We refer to the statement and proof of Proposition 9.3.5 in [53] for an explicit example of function τ verifying the above properties.*

Coordinates systems on $S(\tau, r)$

We assume that the spheres $S(\tau, r)$ are covered by three coordinates systems.

Definition 6.1.7 (Coordinates systems on $S(\tau, r)$)**.** *On each $S(\tau, r)$, let*

- (x^1_S, x^2_S) *a coordinates system defined on $\frac{2\pi}{3} < \theta \leq \pi$,*
- (x^1_E, x^2_E) *a coordinates system defined on $\frac{\pi}{4} < \theta \leq \frac{3\pi}{4}$,*
- (x^1_N, x^2_N) *a coordinates system defined on $0 \leq \theta < \pi$,*

so that we have the following control on each corresponding coordinate chart

$$\max_{b,c=1,2}|\partial^{\leq 2}(g_{bc}-(g_{a,m})_{bc})| \lesssim r^2\epsilon,$$

where g_{bc} denotes the induced metric coefficients in these coordinates systems, $(g_{a,m})_{bc}$ the corresponding expression in Kerr, and $\partial^{\leq 2}$ at most two coordinates derivatives.

Remark 6.1.8. *In [53], these coordinates systems are constructed using the scalar function θ and an auxiliary scalar function φ as follows*

$$(x^1_S, x^2_S) = (x^1_N, x^2_N) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi), \qquad (x^1_E, x^2_E) = (\theta, \varphi).$$

We refer to Propositions 4.1 and 4.2 of [53] for the control of such coordinates systems. Also, see Lemma 2.4.10 in [53] for the form of $(g_{a,m})_{bc}$ in the (x^1_E, x^2_E) coordinates system, and see Lemma 2.4.24 in [53] for the form of $(g_{a,m})_{bc}$ in the (x^1_S, x^2_S) and (x^1_N, x^2_N) coordinates systems.

6.1.4 Regions of integration and basic vectorfields

Regions of integration

Recall the time function τ introduced in Definition 6.1.5. We denote by Σ_τ the level sets of the function τ.

Definition 6.1.9. *We define the following regions of $\mathcal{M}$.*

1. *We define the trapping region of $\mathcal{M}$ to be the set*
$$\mathcal{M}_{trap}(\delta_{trap}) = \mathcal{M}\cap\left\{\frac{|\mathcal{T}|}{r^3}\leq\delta_{trap}\right\}, \qquad \delta_{trap}=\frac{1}{10}, \tag{6.1.7}$$
where $\mathcal{T}$ is the polynomial in r defined in (3.8.5), i.e.
$$\mathcal{T} = r^3-3mr^2+a^2r+ma^2.$$

2. *We denote $\mathcal{M}_{\not{trap}}$ the complement to the trapping region $\mathcal{M}_{trap}$.*

3. *We denote $\mathcal{M}_{red}$ the be the region*
$$\mathcal{M}_{red} := \mathcal{M}\cap\big\{r\leq r_+(1+2\delta_{red})\big\}. \tag{6.1.8}$$
where the small enough constant $\delta_{red}>0$ depends only on $m-|a|$ and is such that $\delta_{\mathcal{H}}\ll\delta_{red}^{20}$.

4. *We define the domain $\mathcal{M}(\tau_1,\tau_2)$ to be the region of $\mathcal{M}$ where $\tau_1\leq\tau\leq\tau_2$, where τ is the time function defined in Definition 6.1.5.*

Basic vectorfields

We start with the definition of $\mathbf{T}$ and $\mathbf{Z}$ as in section 4.6, taking into account that the normalization of (e_3, e_4) is ingoing.

Definition 6.1.10. *In $\mathcal{M}$, we define $\mathbf{T}$ and $\mathbf{Z}$ as follows*

$$\begin{aligned}
\mathbf{T} &:= \frac{1}{2}\left(e_4 + \frac{\Delta}{|q|^2}e_3 - 2a\Re(\mathfrak{J})^b e_b\right), \\
\mathbf{Z} &:= \frac{1}{2}\left(2(r^2+a^2)\Re(\mathfrak{J})^b e_b - a(\sin\theta)^2 e_4 - \frac{a(\sin\theta)^2\Delta}{|q|^2}e_3\right).
\end{aligned}$$

Remark 6.1.11. *Note that we have*

$$\begin{aligned}
\mathbf{g}(\mathbf{T},\mathbf{T}) &= -\frac{\Delta}{|q|^2} + a^2|\Re(\mathfrak{J})|^2 = -\frac{r^2+a^2(\cos\theta)^2-2mr}{|q|^2} + a^2\left(|\Re(\mathfrak{J})|^2 - \frac{(\sin\theta)^2}{|q|^2}\right) \\
&= -\frac{r^2+a^2(\cos\theta)^2-2mr}{|q|^2} + r^{-1}\Gamma_b = -\frac{r^2+a^2(\cos\theta)^2-2mr}{|q|^2} + O(r^{-2}\epsilon)
\end{aligned}$$

where we have used the definition of Γ_b and its control by (6.1.6). *In particular, we have the following non sharp estimate for any $|a| \leq m$*

$$\mathbf{g}(\mathbf{T},\mathbf{T}) < 0 \quad on \quad r \geq \frac{5m}{2}$$

so that $\mathbf{T}$ is timelike on the region $r \geq \frac{5m}{2}$ of $\mathcal{M}$.

Lemma 6.1.12. *For $|a|/m$ sufficiently small and $\delta_{trap} = \frac{1}{10}$, the vectorfield $\mathbf{T}$ is strictly timelike in $\mathcal{M}_{trap}$.*

Proof. Observe that $\mathcal{M}_{trap} = [\tilde{r}_-, \tilde{r}_+]$ where $\tilde{r}_\pm$ is the unique root to $\frac{\mathcal{T}}{r^3} = \pm\delta_{trap}$. By writing $a^2 = \gamma m^2$, for $0 \leq \gamma \ll 1$, and defining $\tilde{x}_\pm = \frac{\tilde{r}_\pm}{m}$, $\tilde{x}_\pm$ is the unique root of the following equation

$$(1 \mp \delta_{trap})(\tilde{x}_\pm)^3 - 3(\tilde{x}_\pm)^2 + \gamma\tilde{x}_\pm + \gamma = 0.$$

Since $0 \leq \gamma \ll 1$, we easily infer

$$\tilde{x}_\pm = \frac{3}{1 \mp \delta_{trap}} + O(\gamma), \qquad \tilde{r}_\pm = \left(\frac{3}{1 \mp \delta_{trap}} + O(\gamma)\right)m.$$

In particular, fixing $\delta_{trap} = \frac{1}{10}$, we have clearly $\tilde{r}_- > \frac{5m}{2}$ for γ small enough and hence $\mathbf{T}$ is strictly timelike in $\mathcal{M}_{trap}(\delta_{trap})$ in view of Remark 6.1.11. $\square$

Also, we define the following vectorfields

$$\widehat{T} \ = \ \frac{1}{2}\left(\frac{|q|^2}{r^2+a^2}e_4+\frac{\Delta}{r^2+a^2}e_3\right), \qquad \widehat{R}=\frac{1}{2}\left(\frac{|q|^2}{r^2+a^2}e_4-\frac{\Delta}{r^2+a^2}e_3\right). \quad (6.1.9)$$

Finally, we introduce the vectorfield $\widehat{T}_\delta$ that will be used for energy estimates.

Definition 6.1.13. *We define the vectorfield*

$$\widehat{T}_\delta := \mathbf{T} + \frac{a}{r^2+a^2}\chi_0\left(\delta^{-1}\frac{\mathcal{T}}{r^3}\right)\mathbf{Z} \quad (6.1.10)$$

with $\delta=\delta_{trap}$ *and with* χ_0 *the smooth bump function*

$$\chi_0(x) = \begin{cases} 0 & \textit{if} \quad |x|\leq 1, \\ 1 & \textit{if} \quad |x|\geq 2. \end{cases} \quad (6.1.11)$$

We also write

$$\widehat{T}_\delta := \mathbf{T}+\chi_\delta\mathbf{Z}, \qquad \chi_\delta := \frac{a}{r^2+a^2}\chi_0\left(\delta^{-1}\frac{\mathcal{T}}{r^3}\right).$$

6.1.5 Main norms

We introduce in this section the main norms needed to state the main results of this chapter concerning combined Energy-Morawetz and r^p-weighted estimates for solutions to (6.1.1):

1. Reduced basic Morawetz norms.

$$\begin{aligned}\text{Mor}[\psi](\tau_1,\tau_2) &:= \int_{\mathcal{M}(\tau_1,\tau_2)} r^{-2}|\nabla_{\widehat{R}}\psi|^2+r^{-3}|\psi|^2 \\ &\quad + \int_{\mathcal{M}_{\not{trap}}(\tau_1,\tau_2)}\left(r^{-2}|\nabla_3\psi|^2+r^{-1}|\nabla\psi|^2\right), \\ \text{Morr}[\psi](\tau_1,\tau_2) &:= \text{Mor}[\psi](\tau_1,\tau_2)+\int_{\mathcal{M}_{r\geq 4m}(\tau_1,\tau_2)} r^{-1-\delta}|\nabla_3\psi|^2.\end{aligned} \quad (6.1.12)$$

2. Basic Energy norm.

$$E[\psi](\tau) := \int_{\Sigma(\tau)}\Big(|\nabla_4\psi|^2+r^{-2}|\nabla_3\psi|^2+|\nabla\psi|^2+r^{-2}|\psi|^2\Big). \quad (6.1.13)$$

3. Basic Flux norm.

$$\begin{aligned}F[\psi](\tau_1,\tau_2) :=& F_{\mathcal{A}}[\psi](\tau_1,\tau_2) + F_{\Sigma_*}[\psi](\tau_1,\tau_2),\\ F_{\mathcal{A}}[\psi](\tau_1,\tau_2) :=& \int_{\mathcal{A}(\tau_1,\tau_2)} \Big(|\nabla_4\psi|^2 + |\nabla_3\psi|^2 + |\nabla\psi|^2 + r^{-2}|\psi|^2\Big),\\ F_{\Sigma_*}[\psi](\tau_1,\tau_2) :=& \int_{\Sigma_*(\tau_1,\tau_2)} \Big(|\nabla_4\psi|^2 + |\nabla_3\psi|^2 + |\nabla\psi|^2 + r^{-2}|\psi|^2\Big).\end{aligned} \tag{6.1.14}$$

4. Basic N- norm.

$$\begin{aligned}\mathcal{N}[\psi,N](\tau_1,\tau_2) :=& \int_{\mathcal{M}(\tau_1,\tau_2)} \big(|\nabla_{\widehat{R}}\psi| + r^{-1}|\psi|\big)|N| + \left|\int_{\mathcal{M}_{trap}} \nabla_{\widehat{T}_\delta}\psi\cdot N\right| + \int_{\mathcal{M}_{\not{trap}}} |D\psi||N|\\ &+ \int_{\mathcal{M}(\tau_1,\tau_2)} |N|^2 + \sup_{\tau\in[\tau_1,\tau_2]}\int_{\Sigma(\tau)} |N|^2 + \int_{\Sigma_*(\tau_1,\tau_2)} |N|^2.\end{aligned} \tag{6.1.15}$$

5. Weighted bulk norm. For $0 < p < 2$, we define

$$B_p[\psi](\tau_1,\tau_2) \;:=\; \text{Morr}[\psi](\tau_1,\tau_2) + \int_{\mathcal{M}_{r\geq 4m}(\tau_1,\tau_2)} r^{p-3}\Big(|\mathfrak{d}\psi|^2 + |\psi|^2\Big). \tag{6.1.16}$$

6. Weighted energy norm. For $0 < p < 2$, we define

$$E_p[\psi](\tau) := \begin{cases} E[\psi] + \displaystyle\int_{\Sigma_{r\geq 4m}(\tau)} r^p\Big(|\nabla_4\psi|^2 + r^{-2}|\psi|^2\Big) & \text{for } p\leq 1-\delta,\\ E[\psi] + \displaystyle\int_{\Sigma_{r\geq 4m}(\tau)} r^p\Big(|r^{-1}\nabla_4(r\psi)|^2 + r^{-p-1-\delta}|\psi|^2\Big) & \text{for } p > 1-\delta.\end{cases} \tag{6.1.17}$$

Remark 6.1.14. *By a slight abuse of notation, we will often identify $E_p[\psi](\tau_2)$ with $\sup_{\tau\in[\tau_1,\tau_2]} E_p[\psi](\tau)$.*

7. Weighted flux norm. For $0 < p < 2$, we define

$$\begin{aligned}F_p[\psi](\tau_1,\tau_2) :\;=&\; F[\psi](\tau_1,\tau_2)\\ &+ \int_{\Sigma_*(\tau_1,\tau_2)} r^p\Big(|\nabla_4\psi|^2 + |\nabla\psi|^2 + r^{-2}|\psi|^2\Big).\end{aligned} \tag{6.1.18}$$

8. Combined norms. We denote the combined norm

$$BEF_p[\psi](\tau_1,\tau_2) := \sup_{\tau\in[\tau_1,\tau_2]} E_p[\psi](\tau) + B_p[\psi](\tau_1,\tau_2) + F_p[\psi](\tau_1,\tau_2). \tag{6.1.19}$$

and

$$EF_p[\psi](\tau_1,\tau_2) := \sup_{\tau\in[\tau_1,\tau_2]} E_p[\psi](\tau) + F_p[\psi](\tau_1,\tau_2). \tag{6.1.20}$$

9. Weighted N- norm. For $0 < p < 2$, we define

$$\mathcal{N}_p[\psi, N](\tau_1,\tau_2) \quad = \quad \mathcal{N}[\psi, N](\tau_1,\tau_2) + \left|\int_{\mathcal{M}_{r\geq 4m}} r^{p-1}\,\nabla_4(r\psi)\cdot N\right|. \tag{6.1.21}$$

10. $^{(ext)}\mathcal{M}$ norms. We denote by $^{(ext)}B_p$, $^{(ext)}E_p$, $^{(ext)}\mathcal{N}_p$ the restrictions of the norms $B_p, E_p, \mathcal{N}_p$ to $^{(ext)}\mathcal{M}$, i.e. the region in $\mathcal{M}$ where $r \geq r_0$.

11. Higher order norms. We define the higher derivative norms $\text{Mor}^s[\psi]$, $E^s[\psi]$, $F^s[\psi]$, $\mathcal{N}^s[\psi, N]$, $B^s_p[\psi]$, $E^s_p[\psi]$, $F^s_p[\psi]$, $\mathcal{N}^s_p[\psi]$, by the general procedure for a norm $Q[\psi]$, i.e.

$$Q^s[\psi] = \sum_{k\leq s} Q[\mathfrak{d}^k\psi].$$

Remark 6.1.15. *The $\int_{\mathcal{M}}\big(|\nabla_{\widehat{R}}\psi| + r^{-1}|\psi|\big)|N|$ part in the definition of of $\mathcal{N}[\psi, N]$ is obtained in the proof of the Morawetz estimate while the $\left|\int_{\mathcal{M}}\nabla_{\widehat{T}_\delta}\psi\cdot N\right|$ part is needed for the energy estimate. We stress here the presence of the term involving $\nabla_{\widehat{T}_\delta}$, as its specific form will be needed in Chapter 11 to treat the full Regge-Wheeler equation.*

6.2 Main theorems for the model gRW

We start by stating the main results of this chapter concerning combined Energy-Morawetz and r^p-weighted estimates for solutions to (6.1.1) on a spacetime $\mathcal{M}$ which is an admissible perturbation of Kerr in the sense that (6.1.6) holds.

Theorem 6.2.1 (Basic r^p-weighted estimates)**.** *The following estimates hold true for solutions $\psi\in\mathfrak{s}_2$ of (6.1.1) on $\mathcal{M}$, for all $\delta\leq p\leq 2-\delta$ and $2\leq s\leq k_L$,*

$$\sup_{\tau\in[\tau_1,\tau_2]} E^s_p[\psi](\tau) + B^s_p[\psi](\tau_1,\tau_2) + F^s_p[\psi](\tau_1,\tau_2) \lesssim E^s_p[\psi](\tau_1) + \mathcal{N}^s_p[\psi, N](\tau_1,\tau_2). \tag{6.2.1}$$

To state the second theorem we need to introduce the quantity

$$\check{\psi} := f_2\left(e_4\psi + \frac{r}{|q|^2}\psi\right). \tag{6.2.2}$$

with $f_2 = r^2$ for $r \geq R$ and $f_2 = 0$ for $r \leq R/2$.

Theorem 6.2.2 (Improved r^p-weighted estimates). *The following estimates hold true for the quantity $\check{\psi}$ for solutions $\psi \in \mathfrak{s}_2$ of* (6.1.1) *on $\mathcal{M}$, for all $-1+\delta < q \leq 1-\delta$, $s \leq k_L - 1$,*

$$BEF^s_q[\check{\psi}](\tau_1,\tau_2) \lesssim \widetilde{E}^s_q[\check{\psi}](\tau_1) + \widetilde{\mathcal{N}}^s_q[\check{\psi}, N](\tau_1,\tau_2) + \mathcal{N}^{s+1}_{\max\{q,\delta\}}[\psi, N](\tau_1,\tau_2). \tag{6.2.3}$$

where[2] *the norms on the right are given by*

$$\widetilde{E}^s_q[\check{\psi}](\tau) = E^s_q[\check{\psi}](\tau) + E^{s+1}_{\max\{q,\delta\}}[\psi](\tau) \tag{6.2.4}$$

and

$$\widetilde{\mathcal{N}}^s_q[\check{\psi}, N](\tau_1,\tau_2) \;\; = \;\; \left|\int_{\mathcal{M}_{\geq R}(\tau_1,\tau_2)} r^{q+2}\mathfrak{d}^{\leq s}\check{\psi}\cdot\left(\nabla_4\mathfrak{d}^{\leq s}N + \frac{3}{r}\mathfrak{d}^{\leq s}N\right)\right|. \tag{6.2.5}$$

Remark 6.2.3. *These results are the analog in perturbations of Kerr to Theorem 5.17 and Theorem 5.18. in [50] for perturbations of Schwarzschild.*

Theorems 6.2.1 and 6.2.2 will be proved in Chapter 10 by relying on r^p-weighted estimates, and Morawetz-Energy estimates derived in Chapter 9. In the next section, we discuss these Morawetz-Energy estimates.

6.3 Main Morawetz-Energy results

The following theorem is our main Morawetz-Energy result for solutions to (6.1.1) on a spacetime $\mathcal{M}$ which is an admissible perturbation of Kerr in the sense that (6.1.6) holds.

Theorem 6.3.1 (Morawetz-Energy). *Let ψ an $\mathfrak{s}_2$ solution of* (6.1.1) *in $\mathcal{M}$. For $|a|/m \ll 1$ sufficiently small, we have, for all $2 \leq s \leq k_L$, and for any $\delta > 0$,*

$$\begin{aligned}
& Mor[(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{\leq s}\psi](\tau_1,\tau_2) + E[(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{\leq s}\psi](\tau_2) + F[(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{\leq s}\psi](\tau_1,\tau_2) \\
\lesssim\;& E^s[\psi](\tau_1) + \mathcal{N}^s[\psi, N](\tau_1,\tau_2) \\
& + \left(\frac{|a|}{m} + \epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^s[\psi] + B^s_\delta[\psi](\tau_1,\tau_2) + F^s[\psi](\tau_1,\tau_2)\right).
\end{aligned} \tag{6.3.1}$$

[2]Recall that $BEF^s_q[\check{\psi}](\tau_1,\tau_2) = \sup_{\tau\in[\tau_1,\tau_2]} E^s_q[\check{\psi}](\tau) + B^s_q[\check{\psi}](\tau_1,\tau_2) + F^s_q[\check{\psi}](\tau_1,\tau_2)$.

Remark 6.3.2. *This result is the analog in perturbations of Kerr to Theorem 10.1 of [50] for perturbations of Schwarzschild.*

Theorem 6.3.1 will be proved in section 9.5. In the rest of this section, we introduce norms needed to state intermediary Morawetz-Energy estimates. Then we state these results while providing the outline of the proof of Theorem 6.3.1.

6.3.1 Additional energy flux and bulk quantities

In this section, we introduce additional energy flux and bulk quantities that are needed for the proof of the Energy-Morawetz estimates.

Together with the above gRW equation (6.1.1), we consider the commuted gRW equations, given by

$$\square_2\psi_{\underline{a}} - V\psi_{\underline{a}} = -\frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_T\psi_{\underline{a}} + N_{\underline{a}}, \qquad V = \frac{4\Delta}{(r^2+a^2)|q|^2}, \tag{6.3.2}$$

where $\psi_{\underline{a}}$ is defined by

$$\psi_{\underline{a}} := \mathcal{S}_{\underline{a}}\psi \quad \text{for} \quad \underline{a} = 1, 2, 3, 4, \tag{6.3.3}$$

with $\mathcal{S}_{\underline{a}}$ denoting the set of second order differential operators, see Definition 4.6.1,

$$\begin{aligned} \mathcal{S}_1\psi &= \nabla_T\nabla_T\psi, \\ \mathcal{S}_2\psi &= a\nabla_T\nabla_Z\psi, \\ \mathcal{S}_3\psi &= a^2\nabla_Z\nabla_Z\psi, \\ \mathcal{S}_4\psi &= \mathcal{O}(\psi), \end{aligned}$$

and where the right-hand sides $N_{\underline{a}}$ can be explicitly computed from N and ψ, see Lemma 9.2.4.

Pointwise notation

Definition 6.3.3. *We introduce the following pointwise notation for $\psi \in \mathfrak{s}_2$.*

1. *We denote*

$$|\psi|^2_{\mathcal{S}} := \sum_{\underline{a}=1}^{4}|\psi_{\underline{a}}|^2.$$

2. *Given a vectorfield Y we denote*

$$|\nabla_Y\psi|^2_{\mathcal{S}} \;:=\; \sum_{\underline{a}=1}^{4}|\nabla_Y\psi_{\underline{a}}|^2.$$

Degenerate energy norm

Definition 6.3.4 (Degenerate energy norm)**.** *We define the following degenerate energy for $\psi \in \mathfrak{s}_2$ along $\Sigma(\tau)$:*

$$E_{deg}[\psi](\tau) := \int_{\Sigma(\tau)} \left(|\nabla_4\psi|^2 + \frac{|\Delta|}{r^4}|\nabla_3\psi|^2 + |\nabla\psi|^2 + r^{-2}|\psi|^2\right).$$

Refined Morawetz norms

Definition 6.3.5 (Refined Morawetz norms)**.** *We define the following Morawetz norms for $\psi \in \mathfrak{s}_2$.*

1. *The degenerate axially symmetric Morawetz norms in $\mathcal{M} = \mathcal{M}(\tau_1, \tau_2)$:*

$$\dot{Mor}^{ax}_{deg}[\psi](\tau_1,\tau_2) \;:=\; \int_{\mathcal{M}(\tau_1,\tau_2)} \frac{m}{r^2}|\nabla_{\widehat{R}}\psi|^2 + \frac{\mathcal{T}^2}{r^6}\left(\frac{m}{r^2}|\nabla_{\widehat{T}}\psi|^2 + r^{-1}|\nabla\psi|^2\right),$$
$$Mor^{ax}_{deg}[\psi](\tau_1,\tau_2) \;:=\; \int_{\mathcal{M}(\tau_1,\tau_2)} \frac{m}{r^2}|\nabla_{\widehat{R}}\psi|^2 + r^{-3}|\psi|^2 + \frac{\mathcal{T}^2}{r^6}\left(\frac{m}{r^2}|\nabla_{\widehat{T}}\psi|^2 + r^{-1}|\nabla\psi|^2\right).$$

2. *We also define the higher degenerate and non-degenerate Morawetz norms in $\mathcal{M} = \mathcal{M}(\tau_1, \tau_2)$, for a scalar function z:*

$$\dot{Mor}_{\mathcal{S},z,deg}[\psi](\tau_1,\tau_2) \;:=\; \int_{\mathcal{M}} \frac{m}{r^2}|\nabla_{\widehat{R}}\psi|^2_{\mathcal{S}} + r^3\Big(\big|\nabla_{\widehat{T}}\Psi_z\big|^2 + r^2\big|\nabla\Psi_z\big|^2\Big),$$
$$Mor_{\mathcal{S},z,deg}[\psi](\tau_1,\tau_2) \;:=\; \int_{\mathcal{M}} \frac{m}{r^2}|\nabla_{\widehat{R}}\psi|^2_{\mathcal{S}} + r^{-3}|\psi|^2_{\mathcal{S}} + r^3\Big(\big|\nabla_{\widehat{T}}\Psi_z\big|^2 + r^2\big|\nabla\Psi_z\big|^2\Big),$$

where[3]

$$\Psi_z = \Psi_z[\psi] \;:=\; \tilde{\mathcal{R}}'^{\underline{a}}[z]\psi_{\underline{a}}, \qquad \tilde{\mathcal{R}}'^{\underline{a}}[z] := \partial_r\left(\frac{z}{\Delta}\mathcal{R}^{\underline{a}}\right),$$

[3]Note that z, $\mathcal{R}^{\underline{a}}$ and Δ are functions depending only on r so that ∂_r in the formula for $\tilde{\mathcal{R}}'^{\underline{a}}[z]$ simply denotes differentiation w.r.t. r.

with z a suitable function of r to be chosen later, and with the scalar functions $\mathcal{R}^{\underline{a}}$ given by (3.5.8)*, i.e.*

$$\mathcal{R}^1 \;=\; -(r^2+a^2)^2, \qquad \mathcal{R}^2 = -2(r^2+a^2), \qquad \mathcal{R}^3 = -1, \qquad \mathcal{R}^4 = \Delta.$$

Remark 6.3.6. *Observe the following:*

1. *The axially symmetric norms $\dot{Mor}^{ax}$ and Mor^{ax} have trapped $\nabla_{\widehat{T}}$ and ∇ derivatives at $\mathcal{T}=0$, which describes the trapping region for axially symmetric solutions. For general solutions, those norms cannot be bounded by the initial energy. Nevertheless those Morawetz energies will be used in Part 1 of our proof.*

2. *The higher norms $\dot{Mor}_{\mathcal{S},z}$ and $Mor_{\mathcal{S},z}$ are positive definite norms where the trapping properties are encoded in the term Ψ_z defined above. For our choice $z=z_0-\delta_0 z_0^2$, for $z_0=\frac{\Delta}{(r^2+a^2)^2}$, the term Ψ_z is given by, see* (8.2.11)*,*

$$\Psi_z = -\frac{2\mathcal{T}}{(r^2+a^2)^3}\big(\delta_0\mathcal{S}_1\psi + (1+O(r^{-2}\delta_0))\mathcal{O}\psi\big) + \frac{4ar}{(r^2+a^2)^2}\nabla_{\widehat{T}}\nabla_Z\psi\big(1+O(r^{-2}\delta_0)\big).$$

   The overall expression of Ψ_z describes the trapping structure of general solutions.

3. *The norms $\dot{Mor}_{\mathcal{S},z}$ and $Mor_{\mathcal{S},z}$ involve a sum of positive terms, each of which is given in terms of a linear combination of derivatives of ψ. In order to bound them by a sum of positive terms involving ψ directly, one needs to have a full degeneracy in the trapping region $\mathcal{M}_{trap}$, i.e. for small $|a|/m$,*

$$\begin{aligned}
&\int_{\mathcal{M}(\tau_1,\tau_2)}\frac{m}{r^2}\big(|\nabla_{\widehat{R}}\psi|^2_{\mathcal{S}}+r^{-2}|\psi|^2_{\mathcal{S}}\big) + \int_{\mathcal{M}_{\not{trap}}(\tau_1,\tau_2)} r^{-1}\Big(\frac{m}{r}|\nabla_T\psi|^2_{\mathcal{S}}+|\nabla\psi|^2_{\mathcal{S}}\Big)\\
\lesssim\;& \sup_{\tau\in[\tau_1,\tau_2]} E_{deg}[(\nabla_T,\not{\mathfrak{d}})^{\leq 2}\psi](\tau) + Mor_{\mathcal{S},z,deg}[\psi](\tau_1,\tau_2) + \delta_{\mathcal{H}}F^2_{\mathcal{A}}[\psi](\tau_1,\tau_2)\\
&+F_{\Sigma_*}[(\nabla_T,\not{\mathfrak{d}})^{\leq 2}\psi](\tau_1,\tau_2) + \frac{|a|}{m}B^2_{\delta}[\psi](\tau_1,\tau_2).
\end{aligned}$$

   This will be achieved thanks to Lemma 6.3.11.

6.3.2 Outline of the proof of Theorem 6.3.1

In what follows, we give a description of the main steps in the proof of Theorem 6.3.1.

Part 1: conditional Morawetz and Energy estimates in Kerr

In the first part of the proof of the Energy-Morawetz estimates, we prove bounds for the first derivatives of the solution in what we call *conditional* Morawetz and energy estimates. Such estimates are conditional as they depend on the control of a derivative of the solution with respect to Z on the right hand side of the estimate. In the case of axially symmetric solutions, those would become unconditional.

To ease the exposition, the proof of the conditional Morawetz and energy estimates are first derived in the case of Kerr in Chapter 7, and will then be extended to perturbations of Kerr in section 9.2.10.

First, we derive the following basic degenerate, conditional, Morawetz estimate, see Proposition 9.2.12 for the extension to perturbations of Kerr.

Proposition 6.3.7. *The following estimates hold true in Kerr:*

1. *For all $|a|/m < 1$, we have*

$$\dot{Mor}^{ax}_{deg}[\psi](\tau_1,\tau_2) \lesssim \int_{\partial\mathcal{M}(\tau_1,\tau_2)} |M(\psi)| + \int_{\mathcal{M}(\tau_1,\tau_2)} \left(a^2 r^{-4}|\nabla_Z\psi|^2 + r^{-3}|\psi|^2\right) + \int_{\mathcal{M}(\tau_1,\tau_2)} \left(|\nabla_{\widehat{R}}\psi| + r^{-1}|\psi|\right)|N|. \tag{6.3.4}$$

2. *For $|a|/m \ll 1$ sufficiently small, we have*

$$Mor^{ax}_{deg}[\psi](\tau_1,\tau_2) \lesssim \int_{\partial\mathcal{M}(\tau_1,\tau_2)} |M(\psi)| + \int_{\mathcal{M}(\tau_1,\tau_2)} ar^{-1}\left(|\nabla\psi|^2 + r^{-2}|\nabla_t\psi|^2\right) + \int_{\mathcal{M}(\tau_1,\tau_2)} \left(|\nabla_{\widehat{R}}\psi| + r^{-1}|\psi|\right)|N|. \tag{6.3.5}$$

In both cases $M(\psi)$ denotes a quadratic expression in ψ and its first derivatives for which we have

$$\int_{\partial\mathcal{M}(\tau_1,\tau_2)} |M(\psi)| \ \ \lesssim \ \ \sup_{[\tau_1,\tau_2]} E_{deg}[\psi](\tau) + \delta_{\mathcal{H}} F_{\mathcal{A}}[\psi](\tau_1,\tau_2) + F_{\Sigma_*}[\psi](\tau_1,\tau_2).$$

Remark 6.3.8. *Observe that the above conditional estimates contain respectively the integrals*

$$\int_{\mathcal{M}(\tau_1,\tau_2)} a^2 r^{-4}|\nabla_Z\psi|^2, \qquad \int_{\mathcal{M}(\tau_1,\tau_2)} ar^{-1}\left(|\nabla\psi|^2 + r^{-2}|\nabla_t\psi|^2\right),$$

on the right hand side. These term can be absorbed only by considering higher derivative estimates, as described in Part 2. Also, the difference in the interval for the parameter $|a|/m$ in the two estimates of Proposition 6.3.7 is related to the control of the zero-th order term. To be able to show that this term comes with the right sign, and thus is only on the left hand side, we make use of a Poincaré and a Hardy type inequality, whose validity is restricted to small angular momentum.

We summarize here the main ideas of the proof of Proposition 6.3.7, which is obtained in Section 7.2.

1. The Morawetz estimate is obtained by applying the vector field method with the vectorfield $X = \mathcal{F}(r)\partial_r$ as multiplier. The choice of the function $\mathcal{F}$ is inspired by [4], and the current relative to the vectorfield X has the following form (see (7.2.1)):

$$|q|^2 \mathbf{D}^\mu \mathcal{P}_\mu[X, w] \;\;=\;\; \mathcal{A}|\nabla_r \psi|^2 + \mathcal{U}^{\alpha\beta}(\dot{\mathbf{D}}_\alpha \psi)(\dot{\mathbf{D}}_\beta \psi) + \mathcal{V}|\psi|^2.$$

   The function $\mathcal{F}$ is chosen so that the term $\mathcal{U}^{\alpha\beta}(\dot{\mathbf{D}}_\alpha \psi)(\dot{\mathbf{D}}_\beta \psi)$ vanishes at $\mathcal{T} = 0$ and the coefficient $\mathcal{A}$ of $\nabla_r \psi$ is positive for all $|a|/m < 1$. This is done in Section 7.2.1.

2. By making use of the Lagrangian of the wave equation, the trapped term $\mathcal{U}^{\alpha\beta}(\dot{\mathbf{D}}_\alpha \psi)(\dot{\mathbf{D}}_\beta \psi)$ can be upgraded to contain also the time derivative of ψ. This requires to absorb a term by $\mathcal{A}$, which is still positive in the full sub-extremal range $|a|/m < 1$. This is done in Section 7.2.2. These steps imply estimate (6.3.4), as shown in Section 7.2.3.

3. To prove (6.3.5), we need to obtain positivity of the coefficient $\mathcal{V}$ of $|\psi|^2$. This is obtained for sufficiently small $|a|/m$ through a combined Poincaré inequality (which extracts extra positivity from the trapped term) and a Hardy inequality (which extract extra positivity from the $\mathcal{A}$ term). This is done in Section 7.2.4, and completes the proof of Proposition 6.3.7.

Next, we derive a conditional degenerate energy estimate.

We define the vectorfield $\widehat{T}_\delta$ such that $\widehat{T}_\delta = T$ at the trapped set and $\widehat{T}_\delta = \widehat{T}$ away from it, with $\delta_{trap} = \frac{1}{10}$ so that, in view of Lemma 6.1.12, T is strictly timelike in $\mathcal{M}_{trap}$. We then prove the following, see Proposition 9.2.13 for the extension to perturbations of Kerr.

Proposition 6.3.9. *The following estimate holds true in Kerr for solutions of* (6.1.1) *in*

$\mathcal{M}$*, for* $|a|/m \ll 1$ *sufficiently small,*

$$
\begin{aligned}
E_{deg}[\psi](\tau_2) + F_{\Sigma_*}[\psi](\tau_1,\tau_2) \;\lesssim\;& E_{deg}[\psi](\tau_1) + \delta_{\mathcal{H}}\big(E_{r\leq r_+(1+\delta_{\mathcal{H}})}(\tau_2)[\psi] + F_{\mathcal{A}}[\psi](\tau_1,\tau_2)\big)\\
&+\frac{|a|}{m} Mor^{ax}_{deg}[\psi](\tau_1,\tau_2) + \left|\int_{\mathcal{M}(\tau_1,\tau_2)} \nabla_{\widehat{T}_\delta}\psi\cdot N\right|\\
&+\int_{\mathcal{M}(\tau_1,\tau_2)} |N|^2. \qquad (6.3.6)
\end{aligned}
$$

The proof of this proposition is obtained in Section 7.3, and is an application of the vectorfield method applied with the multiplier $\widehat{T}_\delta$.

Part 2: $\mathcal{S}$-derivatives Morawetz estimates in Kerr

The main limitation of the results of Proposition 6.3.7 is the presence of $\nabla_Z\psi$ on the right hand side of the estimates. To correct for this, we follow the approach of Andersson and Blue in [4] based on a remarkable extension of the classical vectorfield method to include commutation with the second order Carter operator. To ease the exposition, the proof of the $\mathcal{S}$-derivatives Morawetz estimates are first derived in the case of Kerr in Chapter 8, and will then be extended to perturbations of Kerr in section 9.2.11.

We have the following, see Proposition 9.2.15 for the extension to perturbations of Kerr.

Proposition 6.3.10 ($\mathcal{S}$-derivatives Morawetz estimates)**.** *Let the scalar function* z *given by*

$$
z = z_0 - \delta_0 z_0^2, \qquad z_0 = \frac{\Delta}{(r^2+a^2)^2}.
$$

Then, for $|a|/m \ll 1$ *sufficiently small, the following estimate holds true for solutions of* (6.1.1) *in Kerr:*

$$
\begin{aligned}
Mor_{\mathcal{S},z,deg}[\psi](\tau_1,\tau_2) \lesssim& \int_{\partial\mathcal{M}(\tau_1,\tau_2)} |M_{\mathcal{S}}(\psi)| + \frac{|a|}{m} B^2_\delta[\psi](\tau_1,\tau_2)\\
&+\sum_{\underline{a}=1}^4 \int_{\mathcal{M}(\tau_1,\tau_2)} \big(|\nabla_{\widehat{R}}\psi_{\underline{a}}| + r^{-1}|\psi_{\underline{a}}|\big)|N_{\underline{a}}|
\end{aligned}
\qquad (6.3.7)
$$

where $M_{\mathcal{S}}(\psi)$ *denotes an expression in* ψ *for which we have a bound of the form*

$$\begin{aligned}
&\int_{\partial\mathcal{M}(\tau_1,\tau_2)} |M_{\mathcal{S}}(\psi)| \\
\lesssim\ & \sum_{\underline{a}=1}^{4}\left(\sup_{[\tau_1,\tau_2]} E_{deg}[\psi_{\underline{a}}](\tau)+\delta_{\mathcal{H}}F_{\mathcal{A}}[\psi_{\underline{a}}](\tau_1,\tau_2)+F_{\Sigma_*}[\psi_{\underline{a}}](\tau_1,\tau_2)\right) \\
+\ & \left(\sup_{[\tau_1,\tau_2]} E_{deg}[(\nabla_T,\, \not{\mathfrak{d}})^{\leq 1}\psi](\tau)+\delta_{\mathcal{H}}F_{\mathcal{A}}[(\nabla_T,\, \not{\mathfrak{d}})^{\leq 1}\psi](\tau_1,\tau_2)+F_{\Sigma_*}[(\nabla_T,\, \not{\mathfrak{d}})^{\leq 1}\psi](\tau_1,\tau_2)\right)^{\frac{1}{2}} \\
\times\ & \left(\sup_{[\tau_1,\tau_2]} E_{deg}[(\nabla_T,\, \not{\mathfrak{d}})^{\leq 2}\psi](\tau)+\delta_{\mathcal{H}}F_{\mathcal{A}}[(\nabla_T,\, \not{\mathfrak{d}})^{\leq 2}\psi](\tau_1,\tau_2)+F_{\Sigma_*}[(\nabla_T,\, \not{\mathfrak{d}})^{\leq 2}\psi](\tau_1,\tau_2)\right)^{\frac{1}{2}}.
\end{aligned}$$

We summarize here the main ideas in the proof of Proposition 6.3.10, which is obtained in Section 8.2 and Section 8.3.

1. The Morawetz estimate is obtained by applying the generalized vector field method with the double-indexed vectorfield $X^{\underline{ab}} = \mathcal{F}^{\underline{ab}}(r)\partial_r$ as multiplier, to obtain the current, see (8.1.4),
$$|q|^2\mathbf{D}^\mu\mathcal{P}_\mu[X,w] \ = \ \mathcal{A}^{\underline{ab}}\nabla_r\psi_{\underline{a}}\nabla_r\psi_{\underline{b}}+\mathcal{U}^{\alpha\beta\underline{ab}}\,\mathbf{D}_\alpha\psi_{\underline{a}}\,\mathbf{D}_\beta\psi_{\underline{b}}+\mathcal{V}^{\underline{ab}}\psi_{\underline{a}}\psi_{\underline{b}}.$$
The double-indexed function $\mathcal{F}^{\underline{ab}}$ is chosen so that the term $\mathcal{U}^{\alpha\beta\underline{ab}}\,\mathbf{D}_\alpha\psi_{\underline{a}}\,\mathbf{D}_\beta\psi_{\underline{b}}$ presents a quadratic expressions in higher derivatives and the coefficients $\mathcal{A}^{\underline{ab}}$ are positive for all $|a|/m<1$. More precisely, by performing a crucial integration by parts (see Lemma 8.1.4), we can write
$$\mathcal{U}^{\alpha\beta\underline{ab}}\,\mathbf{D}_\alpha\psi_{\underline{a}}\,\mathbf{D}_\beta\psi_{\underline{b}} \ = \ \frac{1}{2}hL^{\alpha\beta}\mathbf{D}_\alpha\Psi\ \mathbf{D}_\beta\Psi+\text{boundary terms}, \qquad \Psi:=\tilde{\mathcal{R}}'^{\underline{a}}\psi_{\underline{a}},$$
for some constant coefficients $L^{\alpha\beta}$. This is done in Section 8.1.
2. By using another integration by parts lemma, see Lemma 8.2.3, we prove that the term $\mathcal{A}^{\underline{ab}}\nabla_r\psi_{\underline{a}}\nabla_r\psi_{\underline{b}}$ is positive. This is done in Section 8.2.
3. Finally, to prove (6.3.7), we apply a combined Poincaré inequality and Hardy inequality to obtain positivity for the term $\mathcal{V}^{\underline{ab}}\psi_{\underline{a}}\psi_{\underline{b}}$, for sufficiently small $|a|/m$. This is done in Section 8.3, and completes the proof of Proposition 6.3.10.

Finally, to show that $\text{Mor}_{\mathcal{S},z}$ controls ψ in $\mathcal{M}_{tr\not{p}}$, we will rely on the following lemma proved in section 8.4, see Lemma 9.2.16 for the extension to perturbations of Kerr.

Lemma 6.3.11. *For* $\delta_0>0$ *small enough*[4] *and* $|a|/m\ll\delta_0$*, there exists a universal*

[4]Recall that the constant $\delta_0>0$ is involved in the definition of $z=z_0-\delta_0 z_0^2$.

constant $c_0 > 0$ such that the following holds on $\mathcal{M}_{tr\not{a}p}$ in Kerr:

$$
\begin{aligned}
r^3\Big(|\nabla_T\Psi_z|^2 + r^2|\nabla\Psi_z|^2\Big) + r^{-3}|\psi|^2_{\mathcal{S}} \;\; \geq \;\; & c_0 r^{-3}\Big(|\nabla_T\psi|^2_{\mathcal{S}} + |\nabla_Z\psi|^2_{\mathcal{S}} + r^2|\nabla\psi|^2_{\mathcal{S}}\Big)\\
& -O(ar^{-3})\big|(\nabla_T, \not{\mathfrak{d}})^{\leq 1}\mathfrak{d}^{\leq 2}\psi\big|^2 + \dot{\mathbf{D}}_\alpha F^\alpha
\end{aligned}
$$

where the 1-form F denotes an expression in ψ for which we have a bound of the form

$$
\begin{aligned}
& \left|\int_{\partial\mathcal{M}(\tau_1,\tau_2)} F^\mu N_\mu\right|\\
\lesssim\;\; & \left(\sup_{[\tau_1,\tau_2]} E_{deg}[(\nabla_T, \not{\mathfrak{d}})^{\leq 1}\psi](\tau) + \delta_{\mathcal{H}} F_{\mathcal{A}}[(\nabla_T, \not{\mathfrak{d}})^{\leq 1}\psi](\tau_1,\tau_2) + F_{\Sigma_*}[(\nabla_T, \not{\mathfrak{d}})^{\leq 1}\psi](\tau_1,\tau_2)\right)^{\frac{1}{2}}\\
\times\;\; & \left(\sup_{[\tau_1,\tau_2]} E_{deg}[(\nabla_T, \not{\mathfrak{d}})^{\leq 2}\psi](\tau) + \delta_{\mathcal{H}} F_{\mathcal{A}}[(\nabla_T, \not{\mathfrak{d}})^{\leq 2}\psi](\tau_1,\tau_2) + F_{\Sigma_*}[(\nabla_T, \not{\mathfrak{d}})^{\leq 2}\psi](\tau_1,\tau_2)\right)^{\frac{1}{2}}.
\end{aligned}
$$

Part 3: Outline of the proof of Theorem 6.3.1

Theorem 6.3.1 is proved in Chapter 9 according to the following steps:

1. First, we revisit the proof of Propositions 6.3.7, 6.3.9 and 6.3.10, and of Lemma 6.3.11, by exhibiting the extra terms in perturbations of Kerr, and prove that the conclusions of Propositions 6.3.7, 6.3.9 and 6.3.10, and of Lemma 6.3.11, also hold in perturbations of Kerr up to the addition of suitable error terms see sections 9.2.10 and 9.2.11.

2. Next, we prove redshift estimates to remove the degeneracy on the horizon, see section 9.4.

3. Then, we derive the conclusions of Theorem 6.3.1 in the particular case $s = 2$, see section 9.5.1.

4. Finally, we argue by iteration from $s = 2$ to recover higher order derivatives which concludes the proof of Theorem 6.3.1, see section 9.5.2.

Chapter 7

Proof of conditional Morawetz and Energy estimates in Kerr

In this chapter we prove the conditional Morawetz and energy estimates of Propositions 6.3.7 and 6.3.9. Recall that we are in Kerr throughout this chapter and that the results of this chapter will be extended to perturbations of Kerr in section 9.2.10.

7.1 Preliminaries

In this section we collect preliminary results to apply the vector field method to obtain the desired energy-Morawetz estimates.

7.1.1 Deformation tensors of basic vectorfields

Recall the Hawking timelike vectorfield $\widehat{T} = \partial_t + \frac{a}{r^2+a^2}\partial_\phi$ defined in (3.2.1).

Definition 7.1.1. *We define the vectorfield*

$$\widehat{T}_\delta := \partial_t + \frac{a}{r^2+a^2}\chi_0\left(\delta^{-1}\frac{\mathcal{T}}{r^3}\right)\partial_\phi$$

with $\delta = \delta_{trap}$ and with χ_0 the smooth bump function

$$\chi_0(x) = \begin{cases} 0 & \text{if} \quad |x| \le 1, \\ 1 & \text{if} \quad |x| \ge 2. \end{cases} \tag{7.1.1}$$

We also write

$$\widehat{T}_\delta := \partial_t + \chi_\delta \partial_\phi, \qquad \chi_\delta := \frac{a}{r^2+a^2}\chi_0\left(\delta^{-1}\frac{\mathcal{T}}{r^3}\right).$$

According to the definition (9.1.2) of the trapped set, we have that $\widehat{T}_\delta = T$ at the trapped set and $\widehat{T}_\delta = \widehat{T}$ away from it. In view of Lemma 6.1.12 we can fix $\delta = \frac{1}{10}$ such that $T = \partial_t$ is strictly timelike in $\mathcal{M}_{trap}$.

In the derivation of the Energy-Morawetz inequalities we make use of the following vectorfields:

1. The radial vectorfield $X = \mathcal{F}(r)\partial_r$, for a well chosen function $\mathcal{F}$.
2. The modified timelike vectorfield $\widehat{T}_\delta = \partial_t + \chi_\delta(r)\partial_\phi$ as defined in Definition 7.1.1 with $\delta = \frac{1}{10}$.

Lemma 7.1.2. *The vectorfields $\widehat{T}$ and $\widehat{R}$ are dual to each other in the following sense.*

1. *If $X = \widehat{R}$ then*
$$X^4 e_4 - X^3 e_3 = \widehat{T}.$$
2. *If $X = \widehat{T}$ then*
$$X^4 e_4 - X^3 e_3 = \widehat{R}.$$
3. *If $X = \partial_\phi$ then*
$$X^4 e_4 - X^3 e_3 = -\frac{a\sin^2\theta(r^2+a^2)}{|q|^2}\widehat{R}.$$
4. *If $X = \widehat{T}_\delta$ then, with $\widetilde{\chi}_\delta = (\chi_\delta - \frac{a}{r^2+a^2}) = \frac{a}{r^2+a^2}\Big(\chi_0\big(\frac{\mathcal{T}}{r^3}\delta^{-1}\big) - 1\Big)$,*
$$\widehat{T}_\delta^4 e_4 - \widehat{T}_\delta^3 e_3 \;=\; \left(1 - \widetilde{\chi}_\delta\frac{a\sin^2\theta(r^2+a^2)}{|q|^2}\right)\widehat{R}.$$

Proof. The first two identities follow from, see (3.3.6),

$$\widehat{T} \;=\; \frac{1}{2}\left(\frac{|q|^2}{r^2+a^2}e_4 + \frac{\Delta}{r^2+a^2}e_3\right), \qquad \widehat{R} = \frac{1}{2}\left(\frac{|q|^2}{r^2+a^2}e_4 - \frac{\Delta}{r^2+a^2}e_3\right).$$

To check the third identity we note in view of the formula for $Z = \partial_\phi$ in (3.3.7) that

$$\partial_\phi^4 = -\frac{a(\sin\theta)^2}{2}, \qquad \partial_\phi^3 = -\frac{\Delta}{|q|^2}\frac{a(\sin\theta)^2}{2}.$$

Thus,

$$\partial_\phi^4 e_4 - \partial_\phi^3 e_3 = -\frac{a\sin^2\theta(r^2+a^2)}{2|q|^2}\left(\frac{|q|^2}{r^2+a^2}e_4 - \frac{\Delta}{r^2+a^2}e_3\right) = -\frac{a\sin^2\theta(r^2+a^2)}{|q|^2}\widehat{R}$$

as stated. Finally, with $\widetilde{\chi}_\delta = (\chi_\delta - \frac{a}{r^2+a^2})$,

$$\begin{aligned}\widehat{T}_\delta^4 e_4 - \widehat{T}_\delta^3 e_3 &= \widehat{T}^4 e_4 - \widehat{T}^3 e_3 + \widetilde{\chi}_\delta\big(\partial_\phi^4 e_4 - \partial_\phi^3 e_3\big) = \widehat{R} - \widetilde{\chi}_\delta\frac{a\sin^2\theta(r^2+a^2)}{|q|^2}\widehat{R}\\ &= \left(1 - \widetilde{\chi}_\delta\frac{a\sin^2\theta(r^2+a^2)}{|q|^2}\right)\widehat{R}\end{aligned}$$

as stated. □

In the lemma below we calculate the deformation tensors of these vectorfields.

Lemma 7.1.3. *The following identities hold true.*

1. *For $X = \mathcal{F}\partial_r$ we have*

$$\mathcal{L}_X(|q|^2\mathbf{g}^{\alpha\beta}) = \big(\mathcal{F}\partial_r\Delta - 2\Delta\partial_r\mathcal{F}\big)\partial_r^\alpha\partial_r^\beta + \mathcal{F}\partial_r\left(\frac{1}{\Delta}\mathcal{R}^{\alpha\beta}\right),$$

and

$${}^{(X)}\pi^{\alpha\beta} = -|q|^{-2}\left(\big(\mathcal{F}\partial_r\Delta - 2\Delta\partial_r\mathcal{F}\big)\partial_r^\alpha\partial_r^\beta + \mathcal{F}\partial_r\left(\frac{1}{\Delta}\mathcal{R}^{\alpha\beta}\right)\right) + |q|^{-2}X\big(|q|^2\big)\mathbf{g}^{\alpha\beta}.$$

2. *For $\widehat{T}_\delta = \partial_t + \chi_\delta\partial_\phi$ we have*

$$\mathcal{L}_{\widehat{T}_\delta}(|q|^2\mathbf{g}^{\alpha\beta}) = -2\Delta(\partial_r\chi_\delta)\,\partial_\phi^\alpha\partial_r^\beta$$

and

$${}^{(\widehat{T}_\delta)}\pi^{\alpha\beta} = \frac{2\Delta(\partial_r\chi_\delta)}{|q|^2}\partial_\phi^\alpha\partial_r^\beta, \tag{7.1.2}$$

with

$$\partial_r\chi_\delta = -\frac{2ar}{(r^2+a^2)^2}\chi_0 + O(a\delta^{-1})\chi_0'. \tag{7.1.3}$$

3. In particular, for $\widehat{T} = \partial_t + \frac{a}{r^2+a^2}\partial_\phi$, *we have*

$$\mathcal{L}_{\widehat{T}}(|q|^2\mathbf{g}^{\alpha\beta}) = \frac{4ar\Delta}{(r^2+a^2)^2}\partial_\phi^{(\alpha}\partial_r^{\beta)}$$

and

$$^{(\widehat{T})}\pi^{\alpha\beta} = -\frac{4ar\Delta}{(r^2+a^2)^2|q|^2}\partial_\phi^{(\alpha}\partial_r^{\beta)}. \tag{7.1.4}$$

Proof. The first part of the lemma has already been established in Lemma 3.8.6. Since $\partial_t, \partial_\phi$ are Killing, since $[\widehat{T}_\delta, \partial_\theta] = 0$, and since $[\widehat{T}_\delta, \partial_r] = -\partial_r\chi_\delta\partial_\phi$, we have, using the formula (3.5.1) for $|q|^2\mathbf{g}^{\alpha\beta}$,

$$\begin{aligned}
\mathcal{L}_{\widehat{T}_\delta}(|q|^2\mathbf{g}^{\alpha\beta}) &= \mathcal{L}_{\widehat{T}_\delta}\left(\Delta\partial_r^\alpha\partial_r^\beta + \frac{1}{\Delta}\mathcal{R}^{\alpha\beta}\right)\\
&= \mathcal{L}_{\widehat{T}_\delta}\Bigg(\Delta\partial_r^\alpha\partial_r^\beta + \partial_\theta^\alpha\partial_\theta^\beta\\
&\quad + \frac{1}{\Delta}\left(-\Sigma^2\partial_t^\alpha\partial_t^\beta - 2amr\partial_t^\alpha\partial_\phi^\beta - 2amr\partial_\phi^\alpha\partial_t^\beta + \frac{\Delta - a^2\sin^2\theta}{\sin^2\theta}\partial_\phi^\alpha\partial_\phi^\beta\right)\Bigg)\\
&= \Delta[\widehat{T}_\delta, \partial_r]^\alpha\partial_r^\beta + \Delta\partial_r^\alpha[\widehat{T}_\delta, \partial_r^\beta] = -2\Delta(\partial_r\chi_\delta)\partial_\phi^\alpha\partial_r^\beta
\end{aligned}$$

as stated.

To calculate the deformation tensors we simply remark that, for any vectorfield X,

$$\begin{aligned}
^{(X)}\pi^{\alpha\beta} &= -\mathcal{L}_X\big(|q|^{-2}|q|^2\mathbf{g}^{\alpha\beta}\big) = -|q|^{-2}\mathcal{L}_X\big(|q|^2\mathbf{g}^{\alpha\beta}\big) - |q|^2\mathcal{L}_X\big(|q|^{-2}\big)\mathbf{g}^{\alpha\beta}\\
&= -|q|^{-2}\mathcal{L}_X\big(|q|^2\mathbf{g}^{\alpha\beta}\big) + |q|^{-2}X\big(|q|^2\big)\mathbf{g}^{\alpha\beta}.
\end{aligned}$$

Note that the second term vanishes for $X = \widehat{T}_\delta$. The identities for $\widehat{T}$ correspond to the ones for $\widehat{T}_\delta$ in the particular case $\chi_0 = 1$. □

Recall the energy-momentum tensor as defined in Section 4.7, i.e.

$$\mathcal{Q}_{\mu\nu} := \dot{\mathbf{D}}_\mu\psi\cdot\dot{\mathbf{D}}_\nu\psi - \frac{1}{2}\mathbf{g}_{\mu\nu}\left(\dot{\mathbf{D}}_\lambda\psi\cdot\dot{\mathbf{D}}^\lambda\psi + V\psi\cdot\psi\right) = \dot{\mathbf{D}}_\mu\psi\cdot\dot{\mathbf{D}}_\nu\psi - \frac{1}{2}\mathbf{g}_{\mu\nu}\mathcal{L}[\psi].$$

As a corollary to the above lemma we derive the following.

Lemma 7.1.4. *The following identities hold true.*

1. *For $X = \mathcal{F}\partial_r$ we have the identity*

$$|q|^2 \mathcal{Q} \cdot {}^{(X)}\pi = \big(2\Delta\partial_r\mathcal{F} - \mathcal{F}\partial_r\Delta\big)|\nabla_r\psi|^2 - \mathcal{F}\partial_r\left(\frac{1}{\Delta}\mathcal{R}^{\alpha\beta}\right)\dot{\mathbf{D}}_\alpha\psi\cdot\dot{\mathbf{D}}_\beta\psi + X\big(|q|^2\big)\big(\mathcal{L}[\psi] - V|\psi|^2\big) - \mathcal{L}[\psi]|q|^2 Div_{\mathbf{g}}X. \tag{7.1.5}$$

2. *For $\widehat{T}_\delta$ we have the identity,*

$$\begin{aligned} |q|^2\mathcal{Q}\cdot {}^{(\widehat{T}_\delta)}\pi &= 2(r^2+a^2)(\partial_r\chi_\delta)\nabla_\phi\psi\cdot\nabla_{\widehat{R}}\psi \\ &= \left(-\frac{4ar}{r^2+a^2}\chi_0 + O(a\delta^{-1})\chi_0'\right)\nabla_\phi\psi\cdot\nabla_{\widehat{R}}\psi. \end{aligned} \tag{7.1.6}$$

3. *In particular*

$$|q|^2\mathcal{Q}\cdot {}^{(\widehat{T})}\pi = -\frac{4ar}{r^2+a^2}\nabla_\phi\psi\cdot\nabla_{\widehat{R}}\psi. \tag{7.1.7}$$

Proof. We compute

$$\begin{aligned} \mathcal{Q}\cdot {}^{(X)}\pi &= {}^{(X)}\pi^{\alpha\beta}\left(\dot{\mathbf{D}}_\alpha\psi\dot{\mathbf{D}}_\beta\psi - \frac{1}{2}\mathbf{g}_{\alpha\beta}\mathcal{L}[\psi]\right) = {}^{(X)}\pi^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot\dot{\mathbf{D}}_\beta\psi - \frac{1}{2}\mathcal{L}[\psi]\mathbf{g}^{\alpha\beta}\,{}^{(X)}\pi_{\alpha\beta} \\ &= {}^{(X)}\pi^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot\dot{\mathbf{D}}_\beta\psi - \mathcal{L}[\psi]\mathrm{Div}_{\mathbf{g}}X. \end{aligned}$$

According to Lemma 7.1.3, we have[1]

$$\begin{aligned} |q|^2\,{}^{(X)}\pi^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\dot{\mathbf{D}}_\beta\psi &= \big(2\Delta\partial_r\mathcal{F} - \mathcal{F}\partial_r\Delta\big)|\nabla_r\psi|^2 - \mathcal{F}\partial_r\left(\frac{1}{\Delta}\mathcal{R}^{\alpha\beta}\right)\dot{\mathbf{D}}_\alpha\psi\cdot\dot{\mathbf{D}}_\beta\psi \\ &\quad + X\big(|q|^2\big)\mathbf{g}^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot\dot{\mathbf{D}}_\beta\psi \\ &= \big(2\Delta\partial_r\mathcal{F} - \mathcal{F}\partial_r\Delta\big)|\nabla_r\psi|^2 - \mathcal{F}\partial_r\left(\frac{1}{\Delta}\mathcal{R}^{\alpha\beta}\right)\dot{\mathbf{D}}_\alpha\psi\cdot\dot{\mathbf{D}}_\beta\psi \\ &\quad + X\big(|q|^2\big)\big(\mathcal{L}[\psi] - V|\psi|^2\big) \end{aligned}$$

and hence

$$\begin{aligned} |q|^2\mathcal{Q}\cdot {}^{(X)}\pi &= \big(2\Delta\partial_r\mathcal{F} - \mathcal{F}\partial_r\Delta\big)|\nabla_r\psi|^2 - \mathcal{F}\partial_r\left(\frac{1}{\Delta}\mathcal{R}^{\alpha\beta}\right)\dot{\mathbf{D}}_\alpha\psi\cdot\dot{\mathbf{D}}_\beta\psi \\ &\quad + X\big(|q|^2\big)\big(\mathcal{L}[\psi] - V\psi^2\big) - \mathcal{L}[\psi]|q|^2\mathrm{Div}_{\mathbf{g}}X \end{aligned}$$

as stated.

[1] Here $\nabla_r = \nabla_{\partial_r}$

Similarly, we have

$$
\begin{aligned}
|q|^2\mathcal{Q}\cdot {}^{(\widehat{T}_\delta)}\pi &= |q|^2\, {}^{(\widehat{T}_\delta)}\pi^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot\dot{\mathbf{D}}_\beta\psi - \frac{1}{2}|q|^2\mathbf{g}_{\alpha\beta}\, {}^{(\widehat{T}_\delta)}\pi^{\alpha\beta}\mathcal{L}[\psi]\\
&= \Big(2\Delta(\partial_r\chi_\delta)\partial_\phi^\alpha\partial_r^\beta\Big)\dot{\mathbf{D}}_\alpha\psi\cdot\dot{\mathbf{D}}_\beta\psi - \frac{1}{2}\Big(2\Delta(\partial_r\chi_\delta)\partial_\phi^\alpha\partial_r^\beta\Big)\mathbf{g}_{\alpha\beta}\mathcal{L}[\psi]\\
&= 2\Delta(\partial_r\chi_\delta)\nabla_\phi\psi\cdot\nabla_r\psi = 2(r^2+a^2)(\partial_r\chi_\delta)\nabla_\phi\psi\cdot\nabla_{\widehat{R}}\psi
\end{aligned}
$$

since $\nabla_{\widehat{R}} = \frac{\Delta}{r^2+a^2}\nabla_r$. Finally, the expression for $\widehat{T}$ corresponds to the ones for $\widehat{T}_\delta$ in the particular case $\chi_0 = 1$. $\square$

7.1.2 Basic spacetime identity for $X = \mathcal{F}\partial_r$

In this section we prove a fundamental spacetime identity for $X = \mathcal{F}(r)\partial_r$, as summarized in Proposition 7.1.5. The computations in this section follow closely the corresponding ones in [4].

Recall, see Proposition 4.7.3, that for X a linear combination of e_3, e_4 we can write in Kerr

$$
\begin{aligned}
\mathbf{D}^\mu\mathcal{P}_\mu[X,w,M] &= \frac{1}{2}\mathcal{Q}\cdot {}^{(X)}\pi - \frac{1}{2}X(V)|\psi|^2 + \frac{1}{2}w\mathcal{L}[\psi] - \frac{1}{4}|\psi|^2\square_{\mathbf{g}}w + \frac{1}{4}\mathrm{Div}(|\psi|^2M)\\
&- \big({}^*\rho + \underline{\eta}\wedge\eta\big)\nabla_{X^4e_4-X^3e_3}\psi\cdot{}^*\psi\\
&- \frac{1}{2}\Im\Big(\mathrm{tr}\underline{X}HX^3 + \mathrm{tr}X\,\underline{H}X^4\Big)\cdot\nabla\psi\cdot{}^*\psi\\
&+ \left(\nabla_X\psi + \frac{1}{2}w\psi\right)\cdot\big(\dot{\square}_k\psi - V\psi\big).
\end{aligned}
$$

We introduce the expression

$$
\begin{aligned}
\mathcal{E}[X,w,M] := {}& \mathbf{D}^\mu\mathcal{P}_\mu[X,w,M] - \left(\nabla_X\psi + \frac{1}{2}w\psi\right)\cdot\big(\dot{\square}_k\psi - V\psi\big)\\
&+ \big({}^*\rho + \underline{\eta}\wedge\eta\big)\nabla_{X^4e_4-X^3e_3}\psi\cdot{}^*\psi + \frac{1}{2}\Im\Big(\mathrm{tr}\underline{X}HX^3 + \mathrm{tr}X\,\underline{H}X^4\Big)\cdot\nabla\psi\cdot{}^*\psi
\end{aligned}
\tag{7.1.8}
$$

which represents the current for the Morawetz bulk.

In what follows, we assume, in BL coordinates[2], $X = \mathcal{F}\partial_r = \mathcal{F}\frac{r^2+a^2}{\Delta}\widehat{R}$. In that case, in view of Lemma 7.1.2,

$$
\nabla_{X^4e_4-X^3e_3} = \mathcal{F}\frac{r^2+a^2}{\Delta}\nabla_{\widehat{T}}\psi.
$$

[2]Or, relative to the ingoing null frame, $X = \frac{1}{2}\mathcal{F}\Big(\frac{|q|^2}{\Delta}e_4 - e_3\Big)$.

Also, since $X^4=\mathcal{F}\frac{|q|^2}{2\Delta}$ and $X^3=-\frac{1}{2}\mathcal{F}$, we have

$$\begin{aligned}\Im\Big(\mathrm{tr}\underline{X}HX^3+\mathrm{tr}X\,\underline{H}X^4\Big) &= \Im\left(\frac{1}{\overline{q}}\frac{aq}{|q|^2}\mathcal{F}\mathfrak{J}-\frac{1}{q}\frac{a\overline{q}}{|q|^2}\mathcal{F}\mathfrak{J}\right)=\frac{a\mathcal{F}(r)}{|q|^4}\Im((q^2-\overline{q}^2)\mathfrak{J})\\ &= \frac{4a^2r\cos\theta\mathcal{F}(r)}{|q|^4}\Re(\mathfrak{J})\\ &= \frac{4a^2r\cos\theta\mathcal{F}(r)}{(r^2+a^2)|q|^4}\left(\partial_\phi+a(\sin\theta)^2\frac{r^2+a^2}{|q|^2}\widehat{T}\right)\\ &= \frac{4a^2r\cos\theta\mathcal{F}(r)}{(r^2+a^2)|q|^4}\partial_\phi+\frac{4a^3r\cos\theta(\sin\theta)^2\mathcal{F}(r)}{|q|^6}\widehat{T}.\end{aligned}$$

Thus (7.1.8) takes the form

$$\begin{aligned}\mathcal{E}[X,w,M] &= \mathbf{D}^\mu\mathcal{P}_\mu[X,w,M]\\ &-\left(\nabla_X\psi+\frac{1}{2}w\psi\right)\cdot\left(\dot{\square}_k\psi-V\psi\right)+\frac{2a^2r\cos\theta\mathcal{F}(r)}{(r^2+a^2)|q|^4}\nabla_\phi\psi\cdot{}^*\psi\\ &+\left(\left({}^*\rho+\underline{\eta}\wedge\eta\right)\frac{r^2+a^2}{\Delta}+\frac{2a^3r\cos\theta(\sin\theta)^2}{|q|^6}\right)\mathcal{F}\nabla_{\widehat{T}}\psi\cdot{}^*\psi.\end{aligned}\tag{7.1.9}$$

With this definition of $\mathcal{E}$ we write, using (7.1.5),

$$\begin{aligned}|q|^2\mathcal{E}[X,w,M] &= \frac{1}{2}|q|^2\mathcal{Q}\cdot{}^{(X)}\pi+|q|^2\left(-\frac{1}{2}X(V)|\psi|^2+\frac{1}{2}w\mathcal{L}[\psi]-\frac{1}{4}|\psi|^2\square_{\mathbf{g}}w\right)\\ &\quad+\frac{1}{4}|q|^2\mathrm{Div}(|\psi|^2M)\\ &= \left(\Delta\partial_r\mathcal{F}-\frac{1}{2}\mathcal{F}\partial_r\Delta\right)|\nabla_r\psi|^2-\frac{1}{2}\mathcal{F}\partial_r\left(\frac{1}{\Delta}\mathcal{R}^{\alpha\beta}\right)\dot{\mathbf{D}}_\alpha\psi\cdot\dot{\mathbf{D}}_\beta\psi\\ &\quad+\frac{1}{2}\Big(X\big(|q|^2\big)-|q|^2\mathrm{Div}_{\mathbf{g}}X+|q|^2w\Big)\mathcal{L}[\psi]\\ &\quad-\frac{1}{2}\left(X\big(|q|^2\big)V+|q|^2X(V)+\frac{1}{2}|q|^2\square_{\mathbf{g}}w\right)|\psi|^2+\frac{1}{4}|q|^2\mathrm{Div}\big(|\psi|^2M\big).\end{aligned}$$

To simplify the coefficient of $\mathcal{L}[\psi]$ we introduce a reduced function w_{red}, given by

$$w_{red} := |q|^2\mathbf{D}_\alpha\big(|q|^{-2}X^\alpha\big)-w=\mathrm{Div}_{\mathbf{g}}X-|q|^{-2}X(|q|^2)-w$$

and therefore we write the coefficient of $\mathcal{L}[\psi]$ as

$$\frac{1}{2}\Big(X\big(|q|^2\big)-|q|^2\mathrm{Div}_{\mathbf{g}}X+|q|^2w\Big)=\frac{1}{2}|q|^2\Big(|q|^{-2}X\big(|q|^2\big)-\mathrm{Div}_{\mathbf{g}}X+w\Big)=-\frac{1}{2}|q|^2w_{red}.$$

Hence

$$\begin{aligned}
|q|^2\mathcal{E}[X,w,M] &= \left(\Delta\partial_r\mathcal{F} - \frac{1}{2}\mathcal{F}\partial_r\Delta\right)|\nabla_r\psi|^2 - \frac{1}{2}\mathcal{F}\partial_r\left(\frac{1}{\Delta}\mathcal{R}^{\alpha\beta}\right)\dot{\mathbf{D}}_\alpha\psi\cdot\dot{\mathbf{D}}_\beta\psi \\
&\quad -\frac{1}{2}|q|^2 w_{red}\mathcal{L}[\psi] - \frac{1}{2}\Big(X\big(|q|^2\big)V + |q|^2X(V) + \frac{1}{2}|q|^2\square_{\mathbf{g}}w\Big)|\psi|^2 \\
&\quad +\frac{1}{4}|q|^2\text{Div}(|\psi|^2M\big).
\end{aligned}$$

Finally writing

$$\begin{aligned}
|q|^2\mathcal{L}[\psi] &= |q|^2\mathbf{g}^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot\dot{\mathbf{D}}_\beta\psi + |q|^2V|\psi|^2 = \left(\Delta\partial_r^\alpha\partial_r^\beta + \frac{1}{\Delta}\mathcal{R}^{\alpha\beta}\right)\dot{\mathbf{D}}_\alpha\psi\cdot\dot{\mathbf{D}}_\beta\psi + |q|^2V|\psi|^2 \\
&= \Delta|\nabla_r\psi|^2 + \frac{1}{\Delta}\mathcal{R}^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot\dot{\mathbf{D}}_\beta\psi + |q|^2V|\psi|^2
\end{aligned}$$

we obtain[3]

$$\begin{aligned}
|q|^2\mathcal{E}[X,w,M] &= \left(\Delta\partial_r\mathcal{F} - \frac{1}{2}\mathcal{F}\partial_r\Delta\right)|\nabla_r\psi|^2 - \frac{1}{2}\mathcal{F}\partial_r\left(\frac{1}{\Delta}\mathcal{R}^{\alpha\beta}\right)\dot{\mathbf{D}}_\alpha\psi\cdot\dot{\mathbf{D}}_\beta\psi \\
&\quad -\frac{1}{2}w_{red}\left(\Delta|\nabla_r\psi|^2 + \frac{1}{\Delta}\mathcal{R}^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot\dot{\mathbf{D}}_\beta\psi + |q|^2V|\psi|^2\right) \\
&\quad -\frac{1}{2}\Big(X\big(|q|^2\big)V + |q|^2X(V) + \frac{1}{2}|q|^2\square_{\mathbf{g}}w\Big)|\psi|^2 + \frac{1}{4}|q|^2\text{Div}(|\psi|^2M\big) \\
&= \left(\Delta\partial_r\mathcal{F} - \frac{1}{2}\mathcal{F}\partial_r\Delta - \frac{1}{2}\Delta w_{red}\right)|\nabla_r\psi|^2 \\
&\quad -\frac{1}{2}\left(\mathcal{F}\partial_r\left(\frac{1}{\Delta}\mathcal{R}^{\alpha\beta}\right) + w_{red}\frac{1}{\Delta}\mathcal{R}^{\alpha\beta}\right)\dot{\mathbf{D}}_\alpha\psi\cdot\dot{\mathbf{D}}_\beta\psi \\
&\quad -\frac{1}{2}\Big(X\big(|q|^2\big)V + |q|^2X(V) + \frac{1}{2}|q|^2\square_{\mathbf{g}}w + |q|^2w_{red}V\Big)|\psi|^2 \\
&\quad +\frac{1}{4}|q|^2\text{Div}(|\psi|^2M\big).
\end{aligned}$$

We summarize the result in the first part of the following.

Proposition 7.1.5. *The following statements hold true.*

1. *Let $\mathcal{F}, w_{red}$ given functions of r. With the choice of vectorfield $X = \mathcal{F}\partial_r$ and scalar function $w = |q|^2\mathbf{D}_\alpha\big(|q|^{-2}X^\alpha\big) - w_{red}$, the generalized current $\mathcal{E}[X,w,M]$ defined in*

[3] Observe that the first line is identical (with opposite sign) to the computations obtained in (3.8.8) in Proposition 3.8.8 in the case of geodesics.

(7.1.8) *verifies*

$$|q|^2\mathcal{E}[X,w,M] = \mathcal{A}|\nabla_r\psi|^2 + \mathcal{U}^{\alpha\beta}(\dot{\mathbf{D}}_\alpha\psi)\cdot(\dot{\mathbf{D}}_\beta\psi) + \mathcal{V}|\psi|^2 + \frac{1}{4}|q|^2\mathbf{D}^\mu(|\psi|^2 M_\mu) \tag{7.1.10}$$

where[4]

$$\begin{aligned}
\mathcal{A} &= \Delta\partial_r\mathcal{F} - \frac{1}{2}\mathcal{F}\partial_r\Delta - \frac{1}{2}\Delta w_{red},\\
\mathcal{U}^{\alpha\beta} &= -\frac{1}{2}\mathcal{F}\partial_r\left(\frac{1}{\Delta}\mathcal{R}^{\alpha\beta}\right) - \frac{1}{2}w_{red}\frac{1}{\Delta}\mathcal{R}^{\alpha\beta},\\
\mathcal{V} &= -\frac{1}{2}\Big(X\big(|q|^2\big)V + |q|^2X(V) + \frac{1}{2}|q|^2\square_{\mathbf{g}}w + |q|^2 w_{red}V\Big).
\end{aligned}$$

2. If in addition we choose, for fixed functions z, f, h depending on r,

$$\mathcal{F} = -zhf, \qquad w_{red} = \mathcal{F}z^{-1}\partial_r z = -(\partial_r z)hf, \qquad w = -z\partial_r\big(hf\big), \tag{7.1.11}$$

then

$$\begin{aligned}
\mathcal{A} &= -z^{1/2}\Delta^{3/2}\partial_r\left(h\frac{z^{1/2}f}{\Delta^{1/2}}\right),\\
\mathcal{U}^{\alpha\beta} &= \frac{1}{2}hf\partial_r\left(\frac{z}{\Delta}\mathcal{R}^{\alpha\beta}\right),\\
\mathcal{V} &= \frac{1}{4}\left(\partial_r\left(\Delta\partial_r\Big(z\partial_r\big(hf\big)\Big)\right) + 8h\partial_r\left(\frac{z\Delta}{r^2+a^2}\right)f\right).
\end{aligned} \tag{7.1.12}$$

3. If $M = v(r)\partial_r$, for some function $v = v(r)$, we have

$$\frac{1}{4}|q|^2 Div(|\psi|^2 M) = \frac{1}{4}|q|^2\left(2v(r)\psi\cdot\nabla_r\psi + \left(\partial_r v + \frac{2r}{|q|^2}v\right)|\psi|^2\right). \tag{7.1.13}$$

Proof. It remains to check the last two parts of the proposition.

Calculation of $\mathcal{U}^{\alpha\beta}$. As in the computations after Proposition 3.8.8, we have

$$\mathcal{U}^{\alpha\beta} = -\frac{1}{2}\mathcal{F}\partial_r\left(\frac{1}{\Delta}\mathcal{R}^{\alpha\beta}\right) - \frac{1}{2}w_{red}\frac{1}{\Delta}\mathcal{R}^{\alpha\beta} = -\frac{1}{2}\mathcal{F}z^{-1}\partial_r\left(\frac{z}{\Delta}\mathcal{R}^{\alpha\beta}\right) + \frac{1}{2}\left(\mathcal{F}z^{-1}\partial_r z - w_{red}\right)\frac{\mathcal{R}^{\alpha\beta}}{\Delta}.$$

[4]Observe that the expressions for $\mathcal{U}^{\alpha\beta}$ and for $\mathcal{A}$ are the same as the ones for geodesics, see Proposition 3.8.8.

Choosing $w_{red} = \mathcal{F} z^{-1}\partial_r z$, the coefficient of $\frac{\mathcal{R}^{\alpha\beta}}{\Delta}$ cancels out, and setting $\mathcal{F} = -zhf$, we deduce the stated expression for $\mathcal{U}^{\alpha\beta}$.

Calculation of $\mathcal{A}$. As in the computations after Proposition 3.8.8, with the choices of $\mathcal{F}$ and w_{red} given by (7.1.11), we compute

$$\begin{aligned}
\mathcal{A} &= -\frac{1}{2}\mathcal{F}\partial_r\Delta + \Delta\partial_r\mathcal{F} - \frac{1}{2}\Delta w_{red} = \partial_r\left(\frac{\mathcal{F}}{\Delta^{1/2}}\right)\Delta^{3/2} - \frac{1}{2}\Delta w_{red}\\
&= \partial_r\left(\frac{-zhf}{\Delta^{1/2}}\right)\Delta^{3/2} - \frac{1}{2}\Delta(-(\partial_r z)hf) = -\frac{1}{2}\partial_r z\left(\frac{hf}{\Delta^{1/2}}\right)\Delta^{3/2}\\
&\quad -z^{1/2}\partial_r\left(\frac{z^{1/2}hf}{\Delta^{1/2}}\right)\Delta^{3/2} + \frac{1}{2}\Delta(\partial_r z)hf\\
&= -z^{1/2}\Delta^{3/2}\partial_r\left(h\frac{z^{1/2}f}{\Delta^{1/2}}\right)
\end{aligned}$$

as stated.

Calculation of $\mathcal{V}$. We calculate $\mathcal{V}$ with the choices we have made so far. We have

$$\mathcal{V} = -\frac{1}{2}\Big(X\big(|q|^2\big)V + |q|^2X(V) + \frac{1}{2}|q|^2\square_{\mathbf{g}}w + |q|^2 w_{red}V\Big) = \mathcal{V}_0 + \mathcal{V}_1$$

with

$$\mathcal{V}_0 := -\frac{1}{4}|q|^2\square_{\mathbf{g}}w, \qquad \mathcal{V}_1 := -\frac{1}{2}\Big(X\big(|q|^2\big)V + |q|^2X(V) + |q|^2 w_{red}V\Big).$$

We first calculate $\mathcal{V}_0$. Recalling the definition of w and $w_{red} = \mathcal{F} z^{-1}\partial_r z$,

$$\begin{aligned}
w &= |q|^2\mathbf{D}_\alpha\big(|q|^{-2}(X)^\alpha\big) - w_{red} = |q|^2\mathbf{D}_\alpha\big(|q|^{-2}\mathcal{F}\partial_r^\alpha\big) - \mathcal{F} z^{-1}\partial_r z\\
&= |q|^2\partial_r\big(|q|^{-2}\mathcal{F}\big) + \mathcal{F}(\mathbf{D}_\alpha\partial_r^\alpha) - \mathcal{F} z^{-1}\partial_r z.
\end{aligned}$$

We write, for $Y = \partial_r$,

$$\mathbf{D}_\alpha Y^\alpha = \frac{1}{\sqrt{|g|}}\partial_\alpha\big(\sqrt{|\mathbf{g}|}Y^\alpha\big) = \frac{1}{\sqrt{|g|}}\partial_r\big(\sqrt{|\mathbf{g}|}\big) = \frac{1}{|q|^2}\partial_r\big(|q|^2\big).$$

Hence

$$w = |q|^2\partial_r\big(|q|^{-2}\mathcal{F}\big) + \frac{1}{|q|^2}\partial_r\big(|q|^2\big)\mathcal{F} - \mathcal{F} z^{-1}\partial_r z = \partial_r\mathcal{F} - \mathcal{F} z^{-1}\partial_r z = z\partial_r\left(\frac{\mathcal{F}}{z}\right).$$

Thus, in view of our choice for $\mathcal{F} = -zhf$ in (7.1.11)

$$w = z\partial_r\left(\frac{\mathcal{F}}{z}\right) = -z\partial_r\big(hf\big). \tag{7.1.14}$$

Now, for a function $H = H(r)$,

$$\square_{\mathbf{g}} H = \frac{1}{\sqrt{|\mathbf{g}|}}\partial_\alpha\big(\sqrt{|\mathbf{g}|}\mathbf{g}^{\alpha\beta}\partial_\beta\big)H = \frac{1}{\sqrt{|\mathbf{g}|}}\partial_r\big(\sqrt{|\mathbf{g}|}\mathbf{g}^{rr}\partial_r\big)H = \frac{1}{|q|^2}\partial_r\big(\Delta\partial_r H\big). \quad (7.1.15)$$

Thus

$$|q|^2\square_{\mathbf{g}} w = \partial_r\Big(\Delta\partial_r\big(w\big)\Big) = \partial_r\left(\Delta\partial_r\Big(-z\partial_r\big(hf\big)\Big)\right).$$

We deduce,

$$\mathcal{V}_0 := -\frac{1}{4}|q|^2\square_{\mathbf{g}} w = \frac{1}{4}\partial_r\left(\Delta\partial_r\Big(z\partial_r\big(hf\big)\Big)\right).$$

It remains to calculate

$$\mathcal{V}_1 \;=\; -\frac{1}{2}\Big(X\big(|q|^2\big)V + |q|^2X(V) + |q|^2 w_{red}V\Big) = -\frac{1}{2}\Big(X\big(|q|^2V\big) + |q|^2 w_{red}V\Big).$$

Recalling that $|q|^2V = \frac{4\Delta}{r^2+a^2}$, $w_{red} = \mathcal{F}z^{-1}\partial_r z$ and $\mathcal{F} = -zhf$ we deduce

$$\begin{aligned}\mathcal{V}_1 \;&=\; -2\left(X\left(\frac{\Delta}{r^2+a^2}\right) + w_{red}\frac{\Delta}{r^2+a^2}\right) = -2\left(\mathcal{F}\partial_r\left(\frac{\Delta}{r^2+a^2}\right) + \mathcal{F}z^{-1}\partial_r z\frac{\Delta}{r^2+a^2}\right)\\ &=\; -2z^{-1}\partial_r\left(\frac{z\Delta}{r^2+a^2}\right)\mathcal{F} = -2z^{-1}\partial_r\left(\frac{z\Delta}{r^2+a^2}\right)\big(-zhf\big) = 2hf\partial_r\left(\frac{z\Delta}{r^2+a^2}\right).\end{aligned}$$

Thus, as stated,

$$\mathcal{V} \;=\; \mathcal{V}_0 + \mathcal{V}_1 = \frac{1}{4}\partial_r\left(\Delta\partial_r\Big(z\partial_r\big(hf\big)\Big)\right) + 2h\partial_r\left(\frac{z\Delta}{r^2+a^2}\right)f.$$

Calculation of $\frac{1}{4}|q|^2\mathbf{Div}(|\psi|^2M\big)$. We choose $M = v(r)\partial_r$ for some function $v = v(r)$ and write

$$\mathbf{D}^\mu(|\psi|^2M_\mu) \;=\; 2v(r)\psi\cdot\nabla_r\psi + |\psi|^2\mathrm{Div}M.$$

On the other hand

$$\mathrm{Div}M \;=\; \frac{1}{\sqrt{|\mathbf{g}|}}\partial_\alpha\big(\mathbf{g}^{\alpha\beta}\sqrt{|\mathbf{g}|}M_\beta\big) = \frac{1}{\sqrt{|\mathbf{g}|}}\partial_r\Big(\sqrt{|\mathbf{g}|}v\Big) = \frac{1}{|q|^2}\partial_r\big(|q|^2v\big) = \partial_r v + \frac{2r}{|q|^2}v.$$

Hence

$$\frac{1}{4}|q|^2\mathrm{Div}(|\psi|^2M\big) \;=\; \frac{1}{4}|q|^2\left(2v(r)\psi\cdot\nabla_r\psi + \left(\partial_r v + \frac{2r}{|q|^2}v\right)|\psi|^2\right) \quad (7.1.16)$$

as stated. $\square$

We collect here a lemma which will be used in the derivation of the estimates to analyze the right hand side of the main equation (6.1.1).

Lemma 7.1.6. *The following identity holds true with X and w as in Proposition 7.1.5.*

1. *We have, with $T = \partial_t$ in BL coordinates*

$$\begin{aligned}\left(\nabla_X\psi + \frac{1}{2}w\psi\right)\cdot\nabla_T({}^*\psi) &= \frac{1}{2}(\partial_r z)hf\psi\cdot\nabla_T{}^*\psi + zhf\,{}^*\rho\frac{|q|^2}{\Delta}|\psi|^2\\ &-\frac{1}{2}\nabla_r\Big(zhf\psi\cdot\nabla_T{}^*\psi\Big) + \frac{1}{2}\nabla_T\Big(zhf\psi\cdot\nabla_r{}^*\psi\Big).\end{aligned} \tag{7.1.17}$$

2. *We have with $Z = \partial_\phi$ in BL coordinates*

$$\begin{aligned}\left(\nabla_X\psi + \frac{1}{2}w\psi\right)\cdot\nabla_Z({}^*\psi) &= \frac{1}{2}(\partial_r z)hf\psi\cdot\nabla_Z{}^*\psi - zhf\,{}^*\rho\frac{a(\sin\theta)^2|q|^2}{\Delta}|\psi|^2\\ &-\frac{1}{2}\nabla_r\Big(zhf\psi\cdot\nabla_Z{}^*\psi\Big) + \frac{1}{2}\nabla_Z\Big(zhf\psi\cdot\nabla_r{}^*\psi\Big).\end{aligned} \tag{7.1.18}$$

Proof. We set recalling (7.1.11)

$$J := -\left(\nabla_X\psi + \frac{1}{2}w\psi\right)\cdot\nabla_T({}^*\psi) = \left(zhf\nabla_r\psi + \frac{1}{2}z\partial_r\big(hf\big)\psi\right)\cdot\nabla_T{}^*\psi$$

and proceed as follows

$$\begin{aligned}J &= zhf\nabla_r\psi\cdot\nabla_T{}^*\psi + \frac{1}{2}z\partial_r\big(hf\big)\psi\cdot\nabla_T{}^*\psi\\ &= zhf\nabla_r\psi\cdot\nabla_T{}^*\psi + \frac{1}{2}\nabla_r\Big(zhf\psi\cdot\nabla_T{}^*\psi\Big) - \frac{1}{2}zhf\nabla_r\psi\cdot\nabla_T{}^*\psi\\ &\quad -\frac{1}{2}(\partial_r z)hf\psi\cdot\nabla_T{}^*\psi - \frac{1}{2}zhf\psi\cdot\nabla_r\nabla_t{}^*\psi\\ &= \frac{1}{2}zhf\nabla_r\psi\cdot\nabla_T{}^*\psi - \frac{1}{2}(\partial_r z)hf\psi\cdot\nabla_T{}^*\psi + \frac{1}{2}\partial_r\Big(zhf\psi\cdot\nabla_T{}^*\psi\Big)\\ &\quad -\frac{1}{2}zhf\psi\cdot\nabla_r\nabla_t{}^*\psi.\end{aligned}$$

For the last term we write

$$\begin{aligned}-\frac{1}{2}zhf\psi\cdot\nabla_r\nabla_t{}^*\psi &= -\frac{1}{2}zhf\psi\cdot\nabla_t\nabla_r{}^*\psi - \frac{1}{2}zhf\psi\cdot[\nabla_r,\nabla_t]{}^*\psi\\ &= -\frac{1}{2}\partial_t\Big(zhf\psi\cdot\nabla_r{}^*\psi\Big) + \frac{1}{2}zhf\nabla_T\psi\cdot\nabla_r{}^*\psi\\ &\quad -\frac{1}{2}zhf\psi\cdot\left(-2\,{}^*\rho\frac{|q|^2}{\Delta}{}^*({}^*\psi)\right)\\ &= -\frac{1}{2}\partial_t\Big(zhf\psi\cdot\nabla_r{}^*\psi\Big) + \frac{1}{2}zhf\nabla_T\psi\cdot\nabla_r{}^*\psi - zhf\,{}^*\rho\frac{|q|^2}{\Delta}|\psi|^2.\end{aligned}$$

Hence

$$\begin{aligned} J &= \frac{1}{2}zhf\nabla_r\psi\cdot\nabla_T\,{}^*\psi+\frac{1}{2}zhf\nabla_T\psi\cdot\nabla_r\,{}^*\psi-\frac{1}{2}(\partial_r z)hf\psi\cdot\nabla_T\,{}^*\psi-zhf\,\,{}^*\rho\frac{|q|^2}{\Delta}|\psi|^2\\ &+\frac{1}{2}\partial_r\Big(zhf\psi\cdot\nabla_T\,{}^*\psi\Big)-\frac{1}{2}\partial_t\Big(zhf\psi\cdot\nabla_r\,{}^*\psi\Big). \end{aligned}$$

Note that $\nabla_r\psi\cdot\nabla_T\,{}^*\psi+\nabla_T\psi\cdot\nabla_r\,{}^*\psi=0$. Therefore,

$$\begin{aligned} J &= -\frac{1}{2}(\partial_r z)hf\psi\cdot\nabla_T\,{}^*\psi-zhf\,\,{}^*\rho\frac{|q|^2}{\Delta}|\psi|^2+\frac{1}{2}\nabla_r\Big(zhf\psi\cdot\nabla_T\,{}^*\psi\Big)\\ &-\frac{1}{2}\partial_t\Big(zhf\psi\cdot\nabla_r\,{}^*\psi\Big). \end{aligned}$$

The second statement is proved in the same way. □

7.1.3 Choice of z, f and h

In this section, we present a choice for the functions z, h, f. Our goal is to choose such functions so that the generalized current $\mathcal{E}[X,w,M]$ as given in Proposition 7.1.5 is positive definite. According to (7.1.10), we have

$$|q|^2\mathcal{E}[X,w,M]=\mathcal{A}|\nabla_r\psi|^2+\mathcal{U}^{\alpha\beta}(\dot{\mathbf{D}}_\alpha\psi)\cdot(\dot{\mathbf{D}}_\beta\psi)+\mathcal{V}|\psi|^2+\frac{1}{4}|q|^2\mathbf{D}^\mu(|\psi|^2M_\mu).$$

We start by looking at what we call *principal term* in derivative of ψ, i.e.

$$P:=\mathcal{U}^{\alpha\beta}(\dot{\mathbf{D}}_\alpha\psi)\cdot(\dot{\mathbf{D}}_\beta\psi)=\frac{1}{2}hf\tilde{\mathcal{R}}'^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot\dot{\mathbf{D}}_\beta\psi,\quad \tilde{\mathcal{R}}'^{\alpha\beta}:=\partial_r\left(\frac{z}{\Delta}\mathcal{R}^{\alpha\beta}\right). \tag{7.1.19}$$

From (3.5.2), we write

$$\mathcal{R}^{\alpha\beta}=-(r^2+a^2)^2\partial_t^\alpha\partial_t^\beta-2a(r^2+a^2)\partial_t^{(\alpha}\partial_\phi^{\beta)}-a^2\partial_\phi^\alpha\partial_\phi^\beta+\Delta O^{\alpha\beta}.$$

Hence

$$\begin{aligned} \tilde{\mathcal{R}}'^{\alpha\beta} &= -\partial_r\left(\frac{z}{\Delta}(r^2+a^2)^2\right)\partial_t^\alpha\partial_t^\beta-2a\partial_r\left(\frac{z}{\Delta}(r^2+a^2)\right)\partial_t^{(\alpha}\partial_\phi^{\beta)}-a^2\partial_r\left(\frac{z}{\Delta}\right)\partial_\phi^\alpha\partial_\phi^\beta\\ &+(\partial_r z)O^{\alpha\beta}. \end{aligned}$$

Just as in (3.8.14) in the case of geodesics, we similarly choose z to cancel the coefficient of $\partial_t^\alpha\partial_t^\beta$ in $\tilde{\mathcal{R}}'^{\alpha\beta}$, i.e.

$$z=\frac{\Delta}{(r^2+a^2)^2}. \tag{7.1.20}$$

Recall (3.8.15), i.e.

$$\partial_r z = -\frac{2\mathcal{T}}{(r^2+a^2)^3},$$

and thus

$$\tilde{\mathcal{R}}'^{\alpha\beta} = \frac{4ar}{(r^2+a^2)^2}\partial_t^{(\alpha}\partial_\phi^{\beta)} + \frac{4a^2r}{(r^2+a^2)^3}\partial_\phi^\alpha\partial_\phi^\beta - \frac{2\mathcal{T}}{(r^2+a^2)^3}O^{\alpha\beta}. \tag{7.1.21}$$

The principal term P then becomes

$$\begin{aligned}
P &= \frac{1}{2}hf\left(-\frac{2\mathcal{T}}{(r^2+a^2)^3}O^{\alpha\beta} + \frac{4ar}{(r^2+a^2)^2}\partial_t^{(\alpha}\partial_\phi^{\beta)} + \frac{4a^2r}{(r^2+a^2)^3}\partial_\phi^\alpha\partial_\phi^\beta\right)\dot{\mathbf{D}}_\alpha\psi\cdot\dot{\mathbf{D}}_\beta\psi \\
&= \frac{1}{2}hf\left(-\frac{2\mathcal{T}}{(r^2+a^2)^3}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot\dot{\mathbf{D}}_\beta\psi\right) \\
&\quad +\frac{1}{2}hf\frac{4ar}{(r^2+a^2)^2}\left(\nabla_t\psi\cdot\nabla_\phi\psi + \frac{a}{(r^2+a^2)}\nabla_\phi\psi\cdot\nabla_\phi\psi\right) \\
&= \frac{1}{2}hf\left(-\frac{2\mathcal{T}}{(r^2+a^2)^3}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot\dot{\mathbf{D}}_\beta\psi + \frac{4ar}{(r^2+a^2)^2}\left(\nabla_t\psi + \frac{a}{r^2+a^2}\nabla_\phi\psi\right)\cdot\nabla_\phi\psi\right),
\end{aligned}$$

i.e. using the Hawking vector field $\widehat{T} = \partial_t + \frac{a}{r^2+a^2}\partial_\phi$,

$$P = \frac{1}{2}hf\left(-\frac{2\mathcal{T}}{(r^2+a^2)^3}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot\dot{\mathbf{D}}_\beta\psi + \frac{4ar}{(r^2+a^2)^2}\nabla_{\widehat{T}}\psi\cdot\nabla_\phi\psi\right). \tag{7.1.22}$$

Remark 7.1.7. *In the particular case of axial symmetry, i.e.* $\nabla_\phi\psi = 0$, (7.1.22) *becomes*

$$P = -\frac{1}{2}hf\frac{2\mathcal{T}}{(r^2+a^2)^3}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot\dot{\mathbf{D}}_\beta\psi$$

and thus, to have P non-negative, it makes sense to choose, for a non-negative h,

$$f = \partial_r z = -\frac{2\mathcal{T}}{(r^2+a^2)^3}. \tag{7.1.23}$$

In the general case, we keep the choice of f given by (7.1.23). With these choices, we compute according to formula (7.1.12) for $\mathcal{A}$

$$\mathcal{A} = -z^{1/2}\Delta^{3/2}\partial_r\left(\frac{hf}{r^2+a^2}\right) = \frac{2\Delta^2}{r^2+a^2}\partial_r\left(h\frac{\mathcal{T}}{(r^2+a^2)^4}\right).$$

For the choice of h, in order to obtain a bound for $\mathcal{A}$ which is valid in the full sub-extremal range $|a| < m$, we choose h to be (as in (3.8.17) in the case of axially symmetric geodesics)

$$h = \frac{(r^2+a^2)^4}{r(r^2-a^2)}. \tag{7.1.24}$$

Thus, with this choice,

$$\mathcal{A} \;=\; \frac{2\Delta^2}{r^2+a^2}\partial_r\left(\frac{\mathcal{T}}{r(r^2-a^2)}\right) = \frac{2\Delta^2}{r^2(r^2-a^2)^2(r^2+a^2)}\big(3mr^4-4a^2r^3+ma^4\big),$$

which is positive in the exterior region in the sub-extremal range.

It remains to calculate the coefficient $\mathcal{V}$ of the lower order term. According to formula (7.1.12) for $\mathcal{V}$, in view of our choices of z, f, h we derive

$$\begin{aligned}
\mathcal{V}_1 &= 2hf\partial_r\left(z\frac{\Delta}{r^2+a^2}\right) = -4\frac{(r^2+a^2)\mathcal{T}}{r(r^2-a^2)}\partial_r\left(\frac{\Delta^2}{(r^2+a^2)^3}\right)\\
&= -4\frac{(r^2+a^2)\mathcal{T}}{r(r^2-a^2)}\Delta\frac{-2r^3+8mr^2-2a^2r-4ma^2}{(r^2+a^2)^4}\\
&= 8\Delta\frac{r^3-4mr^2+a^2r+2ma^2}{r(r^2+a^2)^4}\frac{r^2+a^2}{r^2-a^2}\mathcal{T},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{V}_0 &= \frac{1}{4}\partial_r\left(\Delta\partial_r\left(z\partial_r\left(h\frac{-2\mathcal{T}}{(r^2+a^2)^3}\right)\right)\right) = \frac{1}{4}\partial_r\left(\Delta\partial_r\left(z\partial_r\left(\frac{-2\mathcal{T}}{r}\frac{r^2+a^2}{r^2-a^2}\right)\right)\right)\\
&= \frac{1}{2}\partial_r\left(\Delta\partial_r\left(z\left(-2r+3m-\frac{5ma^2}{r^2}+\frac{8a^4}{r(r^2-a^2)}-\frac{12ma^4}{r^2(r^2-a^2)}+8a^6\frac{r-m}{r^2(r^2-a^2)^2}\right)\right)\right).
\end{aligned}$$

Continuing to differentiate we find

$$\mathcal{V}_0 = \frac{9mr^6-46m^2r^5+54m^3r^4}{(r^2+a^2)^4} + O(a^2r^{-3}).$$

We summarize the results above in the following.

Proposition 7.1.8. *The generalized current induced by*

$$X = -zhf\partial_r, \qquad w = -z\partial_r\big(hf\big),$$

with the choices

$$z = \frac{\Delta}{(r^2+a^2)^2}, \qquad f = -\frac{2\mathcal{T}}{(r^2+a^2)^3}, \qquad h = \frac{(r^2+a^2)^4}{r(r^2-a^2)},$$

satisfies the following:

1. *We have*

$$\mathcal{F} = \frac{2\Delta\mathcal{T}}{r(r^2-a^2)(r^2+a^2)}, \quad X = \frac{2\Delta\mathcal{T}}{r(r^2-a^2)(r^2+a^2)}\partial_r = \frac{2\mathcal{T}}{r(r^2-a^2)}\widehat{R}, \tag{7.1.25}$$

and

$$w_{red} = -hf\partial_r z = -\frac{4\mathcal{T}^2}{r(r^2-a^2)(r^2+a^2)^2}.$$

2. *The principal term* $P = \mathcal{U}^{\alpha\beta}(\dot{\mathbf{D}}_\alpha\psi)(\dot{\mathbf{D}}_\beta\psi)$ *is given by*

$$P = \frac{\mathcal{T}}{r}\frac{r^2+a^2}{r^2-a^2}\left(\frac{2\mathcal{T}}{(r^2+a^2)^3}O^{\alpha\beta}\nabla_\alpha\psi\cdot\nabla_\beta\psi - \frac{4ar}{(r^2+a^2)^2}\nabla_{\widehat{T}}\psi\cdot\nabla_\phi\psi\right). \tag{7.1.26}$$

3. *The coefficients* $\mathcal{A}, \mathcal{V}$ *in equation* (7.1.12) *of Proposition 7.1.5 take the form*

$$\mathcal{A} = \frac{2\Delta^2}{r^2(r^2-a^2)^2(r^2+a^2)}\big(3mr^4 - 4a^2r^3 + ma^4\big), \tag{7.1.27}$$

and

$$\begin{aligned} \mathcal{V} &= \mathcal{V}_0 + \mathcal{V}_1, \\ \mathcal{V}_1 &= 8\Delta\frac{r^3-4mr^2+a^2r+2ma^2}{r(r^2+a^2)^4}\frac{r^2+a^2}{r^2-a^2}\mathcal{T}, \\ \mathcal{V}_0 &= \frac{9mr^6-46m^2r^5+54m^3r^4}{(r^2+a^2)^4} + O(a^2r^{-3}). \end{aligned} \tag{7.1.28}$$

4. *For small a, we have*

$$\mathcal{V} \;=\; \frac{8r^3-63mr^2+162m^2r-138m^3}{r^4} + O(a^2r^{-3}).$$

7.2 Conditional Morawetz estimates

In this section we derive the first conditional Morawetz estimates, proving Proposition 6.3.7.

7.2.1 A first lower bound

According to Proposition 7.1.8, for $M = 0$ (i.e. $v = 0$), the generalized current is given by

$$
\begin{aligned}
|q|^2\mathcal{E}[X, w, M = 0] &= \mathcal{A}|\nabla_r\psi|^2 + P + \mathcal{V}|\psi|^2,\\
\mathcal{A} &= \frac{2\Delta^2}{r^2(r^2-a^2)^2(r^2+a^2)}\big(3mr^4 - 4a^2r^3 + ma^4\big),\\
P &= \frac{\mathcal{T}}{r}\frac{r^2+a^2}{r^2-a^2}\left(\frac{2\mathcal{T}}{(r^2+a^2)^3}O^{\alpha\beta}\nabla_\alpha\psi\cdot\nabla_\beta\psi - \frac{4ar}{(r^2+a^2)^2}\nabla_{\widehat{T}}\psi\cdot\nabla_\phi\psi\right),
\end{aligned}
\tag{7.2.1}
$$

with $\mathcal{V}$ given by (7.1.28), for which one easily checks that $\mathcal{V} = O(r^{-1})$.

In Remark 3.8.11 we have shown that the polynomial $3mr^4 - 4a^2r^3 + ma^4 \geq 0$ for all values of $r \geq r_+$ in the range $|a|/m \leq 1$. More precisely, there exists a small constant $\delta_0 > 0$ such that[5] for $|a|/m \leq 1$

$$
\mathcal{A} \geq \delta_0\frac{m\Delta^2}{r^4}.
$$

We therefore obtain for $|a|/m < 1$

$$
|q|^2\mathcal{E}[X, w, M = 0] \;\; \geq \;\; \delta_0\frac{m\Delta^2}{r^4}|\nabla_r\psi|^2 + P - O(r^{-1})|\psi|^2
$$

which implies

$$
\mathcal{E}[X, w, M = 0] \;\; \geq \;\; \delta_0\frac{m\Delta^2}{r^6}|\nabla_r\psi|^2 + r^{-2}P - O(r^{-3})|\psi|^2. \tag{7.2.2}
$$

7.2.2 A second lower bound containing $\nabla_{\widehat{T}}$

We now want to incorporate the identity (7.2.2) with a (trapped) control for the $\nabla_{\widehat{T}}$ derivative as well. We achieve this with the help of a new current of the form

$$
\mathcal{P}'_\mu \;\; = \;\; \frac{1}{2}w'\psi\cdot\dot{\mathbf{D}}_\mu\psi - \frac{1}{4}\psi^2\partial_\mu w', \qquad w' = -w'_{red},
$$

[5] In particular, note the following computation at $r = r_+$ with $\gamma = |a|/m$,

$$
\frac{3mr^4 - 4a^2r^3 + ma^4}{(r^2-a^2)^2} \;\; = \;\; \frac{m(2(3-2\gamma) + (6-\gamma)\sqrt{1-\gamma})}{(1+\sqrt{1-\gamma})^2} > 0, \qquad 0 \leq \gamma \leq 1.
$$

which corresponds to a current associated to a zero vector field $X = 0$ and a scalar function w' to be chosen. In view of Proposition 7.1.5 we derive

$$|q|^2\mathcal{E}'[0,w',0] = \frac{1}{2}\Delta\, w'|\nabla_r\psi|^2 + \frac{1}{2}w'\frac{1}{\Delta}\mathcal{R}^{\alpha\beta}\,\dot{\mathbf{D}}_\alpha\psi\cdot\dot{\mathbf{D}}_\beta\psi - \frac{1}{2}\left(\frac{1}{2}|q|^2\square_{\mathbf{g}}w' - |q|^2w'V\right)|\psi|^2.$$

Recall that, in view of (3.5.3)

$$\mathcal{R}^{\alpha\beta}\,\dot{\mathbf{D}}_\alpha\psi\cdot\dot{\mathbf{D}}_\beta\psi = -(r^2+a^2)^2|\nabla_{\widehat{T}}\psi|^2 + \Delta O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot\dot{\mathbf{D}}_\beta\psi.$$

Thus, we have

$$\begin{aligned}|q|^2\mathcal{E}'[0,w',0] &= \frac{1}{2}\Delta\, w'|\nabla_r\psi|^2 - \frac{w'(r^2+a^2)^2}{2\Delta}|\nabla_{\widehat{T}}\psi|^2 + \frac{1}{2}w'O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot\dot{\mathbf{D}}_\beta\psi\\ &\quad -\frac{1}{2}\Big(\frac{1}{2}|q|^2\square_{\mathbf{g}}w' - |q|^2w'V\Big)|\psi|^2.\end{aligned}$$

By summing the above to (7.2.1) we obtain

$$\begin{aligned}|q|^2\Big(\mathcal{E}+\mathcal{E}'\Big) &= -\frac{w'(r^2+a^2)^2}{2\Delta}|\nabla_{\widehat{T}}\psi|^2 + \left(\mathcal{A}+\frac{1}{2}\Delta w'\right)|\nabla_r\psi|^2\\ &\quad+\left(\frac{2\mathcal{T}^2}{r(r^2+a^2)^2(r^2-a^2)}+\frac{1}{2}w'\right)O^{\alpha\beta}\nabla_\alpha\psi\cdot\nabla_\beta\psi\\ &\quad-\frac{\mathcal{T}}{r}\frac{4ar}{(r^2+a^2)(r^2-a^2)}\nabla_{\widehat{T}}\psi\cdot\nabla_\phi\psi\\ &\quad+\left(\mathcal{V}-\frac{1}{2}\left(\frac{1}{2}|q|^2\square_{\mathbf{g}}w' - |q|^2w'V\right)\right)|\psi|^2.\end{aligned}$$

We choose for some $\delta_1 > 0$

$$w' = -\delta_1\frac{4m\Delta\mathcal{T}^2}{r^2(r^2+a^2)^4}$$

so that

$$\begin{aligned}|q|^2\Big(\mathcal{E}+\mathcal{E}'\Big) &= \delta_1\frac{2m\mathcal{T}^2}{r^2(r^2+a^2)^2}|\nabla_{\widehat{T}}\psi|^2\\ &\quad+\left(1-\delta_1\frac{m\Delta(r^2-a^2)}{r(r^2+a^2)^2}\right)\frac{2\mathcal{T}^2}{r(r^2+a^2)^2(r^2-a^2)}O^{\alpha\beta}\nabla_\alpha\psi\cdot\nabla_\beta\psi\\ &\quad+\left(\mathcal{A}+\frac{1}{2}\Delta w'\right)|\nabla_r\psi|^2 - \frac{\mathcal{T}}{r}\frac{4ar}{(r^2+a^2)(r^2-a^2)}\nabla_{\widehat{T}}\psi\cdot\nabla_\phi\psi\\ &\quad+\left(\mathcal{V}-\frac{1}{2}\left(\frac{1}{2}|q|^2\square_{\mathbf{g}}w' - |q|^2w'V\right)\right)|\psi|^2.\end{aligned}$$

Observe that for $\delta_1 < 1$, the coefficient $1 - \delta_1 \frac{m\Delta(r^2-a^2)}{r(r^2+a^2)^2}$ is positive in the exterior in the full subextremal range. Therefore, we can write

$$
\begin{aligned}
& |q|^2\Big(\mathcal{E}+\mathcal{E}'\Big)\\
= & \left(\mathcal{A} - \delta_1 \frac{2m\Delta^2\mathcal{T}^2}{r^2(r^2+a^2)^4}\right)|\nabla_r\psi|^2\\
& + \frac{2\mathcal{T}^2}{r(r^2+a^2)^2}\left(\delta_1\frac{m}{r}|\nabla_{\widehat{T}}\psi|^2 + \left(1-\delta_1\frac{m\Delta(r^2-a^2)}{r(r^2+a^2)^2}\right)\frac{1}{(r^2-a^2)}O^{\alpha\beta}\nabla_\alpha\psi\cdot\nabla_\beta\psi\right)\\
& - \frac{\mathcal{T}}{r}\frac{4ar}{(r^2+a^2)(r^2-a^2)}\nabla_{\widehat{T}}\psi\cdot\nabla_\phi\psi + \left(\mathcal{V} - \frac{1}{2}\left(\frac{1}{2}|q|^2\square_{\mathbf{g}}w' - |q|^2w'V\right)\right)|\psi|^2.
\end{aligned}
$$

Note also that $\mathcal{E}+\mathcal{E}'$ is in fact the generalized current associated to $X = \mathcal{F}\partial_r$ and w the sum between the old $w = -z\partial_r\big(hf\big)$ (see Proposition 7.1.8) and $w' = -\delta_1\frac{4m\Delta\mathcal{T}^2}{r^2(r^2+a^2)^4}$, i.e.

$$
w = w_X - \delta_1\frac{4m\Delta\mathcal{T}^2}{r^2(r^2+a^2)^4}, \qquad w_X = -z\partial_r\big(hf\big). \tag{7.2.3}
$$

We replace the vectorfield ∂_r with $\widehat{R} = \frac{\Delta}{r^2+a^2}\partial_r$, see (3.2.2), to deduce

$$
\begin{aligned}
|q|^2(\mathcal{E}+\mathcal{E}') = |q|^2\mathcal{E}[X,w] \ \geq\ & \mathcal{A}_{\delta_1}\frac{(r^2+a^2)^2}{\Delta^2}|\nabla_{\widehat{R}}\psi|^2 + P'_{\delta_1} - \frac{\mathcal{T}}{r}\frac{4ar}{(r^2+a^2)^2}\nabla_{\widehat{T}}\psi\cdot\nabla_\phi\psi\\
& + \mathcal{V}_{\delta_1}|\psi|^2.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathcal{E}[X,w] \ \geq\ & \mathcal{A}_{\delta_1}\frac{(r^2+a^2)^2}{|q|^2\Delta^2}|\nabla_{\widehat{R}}\psi|^2 + \frac{1}{|q|^2}P'_{\delta_1} - \frac{\mathcal{T}}{r|q|^2}\frac{4ar}{(r^2+a^2)(r^2-a^2)}\nabla_{\widehat{T}}\psi\cdot\nabla_\phi\psi\\
& + \frac{1}{|q|^2}\mathcal{V}_{\delta_1}|\psi|^2,
\end{aligned}
$$

with

$$
\begin{aligned}
\mathcal{A}_{\delta_1} &= \mathcal{A} - \delta_1\frac{2m\Delta^2\mathcal{T}^2}{r^2(r^2+a^2)^4}, \qquad \mathcal{V}_{\delta_1} = \mathcal{V} - \frac{1}{2}\left(\frac{1}{2}|q|^2\square_{\mathbf{g}}w' - |q|^2w'V\right),\\
P'_{\delta_1} &= \frac{2\mathcal{T}^2}{r(r^2+a^2)^2}\left(\delta_1\frac{m}{r}|\nabla_{\widehat{T}}\psi|^2 + \left(1-\delta_1\frac{m\Delta}{r(r^2+a^2)}\right)\frac{1}{(r^2-a^2)}O^{\alpha\beta}\nabla_\alpha\psi\cdot\nabla_\beta\psi\right).
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\mathcal{A}_{\delta_1} &= \frac{2\Delta^2}{r^2(r^2-a^2)^2(r^2+a^2)}\big(3mr^4 - 4a^2r^3 + ma^4\big) - \delta_1\frac{2m\Delta^2\mathcal{T}^2}{r^2(r^2+a^2)^4}\\
&= \frac{2\Delta^2}{r^2(r^2+a^2)}\left(\frac{\big(3mr^4 - 4a^2r^3 + ma^4\big)}{(r^2-a^2)^2} - \delta_1\frac{m\mathcal{T}^2}{(r^2+a^2)^3}\right).
\end{aligned}
$$

Observe that

$$r \to \frac{\left(3mr^4 - 4a^2r^3 + ma^4\right)}{(r^2-a^2)^2} - \delta_1 \frac{m\mathcal{T}^2}{(r^2+a^2)^3}$$

is increasing on $r \geq r+$ for $\delta_1 > 0$ small enough. In particular, we have

$$\mathcal{A}_{\delta_1} \geq \frac{2\Delta^2}{r^2(r^2+a^2)} \left[\left(\frac{\left(3mr^4 - 4a^2r^3 + ma^4\right)}{(r^2-a^2)^2} - \delta_1 \frac{m\mathcal{T}^2}{(r^2+a^2)^3}\right)\right]\Big|_{r=r_+}.$$

To compute the expression in bracket on $r = r_+$, we used $r_+^2 = 2mr_+ - a^2$, and therefore, for $r = r_+$, we have

$$\begin{aligned}
\frac{\left(3mr^4 - 4a^2r^3 + ma^4\right)}{(r^2-a^2)^2} &= \frac{\left(3m(2mr-a^2)^2 - 4a^2r(2mr-a^2) + ma^4\right)}{(2mr-2a^2)^2} \\
&= \frac{\left(3m(4m^2r^2 - 4ma^2r + a^4) - 4a^2r(2mr-a^2) + ma^4\right)}{4(mr-a^2)^2} \\
&= \frac{(3m^3 - 2ma^2)r^2 - (3m^2a^2 - a^4)r + ma^4}{(mr-a^2)^2}.
\end{aligned}$$

Using that at the horizon $(mr-a^2)^2 = (2mr-a^2)(m^2-a^2)$, we have for $r = r_+$

$$\begin{aligned}
&= \frac{(6m^4 - 7m^2a^2 + a^4)r - 3a^2m^3 + 3ma^4}{(mr-a^2)^2} = \frac{(6m^2-a^2)(m^2-a^2)r - 3a^2m(m^2-a^2)}{(2mr-a^2)(m^2-a^2)} \\
&= \frac{(6m^2-a^2)r - 3a^2m}{(2mr-a^2)}.
\end{aligned}$$

On the other hand, we have on $r = r_+$

$$\frac{m\mathcal{T}^2}{(r^2+a^2)^3} = \frac{(mr-a^2)^2}{2r^3} = \frac{r^2(m^2-a^2)}{2r^3} = \frac{m^2-a^2}{2r}$$

and hence, with the notation $\gamma = a^2/m^2$, we infer

$$\begin{aligned}
&\left[\left(\frac{\left(3mr^4 - 4a^2r^3 + ma^4\right)}{(r^2-a^2)^2} - \delta_1 \frac{m\mathcal{T}^2}{(r^2+a^2)^3}\right)\right]\Big|_{r=r_+} \\
&= \frac{(6m^2-a^2)r_+ - 3a^2m}{(2mr_+ - a^2)} - \delta_1 \frac{m^2-a^2}{2r_+} \\
&= m\left(\frac{(6-\gamma)(1+\sqrt{1-\gamma}) - 3\gamma}{2(1+\sqrt{1-\gamma}) - \gamma} - \delta_1 \frac{1-\gamma}{2(1+\sqrt{1-\gamma})}\right) \\
&= m\left(\frac{(6-4\gamma) + (6-\gamma)\sqrt{1-\gamma}}{(2-\gamma) + 2\sqrt{1-\gamma}} - \delta_1 \frac{1-\gamma}{2(1+\sqrt{1-\gamma})}\right).
\end{aligned}$$

Since $0 \leq \gamma \leq 1$, we infer the existence of a constant $\delta_* > 0$ such that for $|a| < m$

$$\left[\left(\frac{\left(3mr^4 - 4a^2r^3 + ma^4\right)}{(r^2-a^2)^2} - \delta_1 \frac{m\mathcal{T}^2}{(r^2+a^2)^3}\right)\right]\Big|_{r=r_+} \geq \delta_*$$

and hence, for $|a| < m$, we obtain

$$\mathcal{A}_{\delta_1} \geq \delta_* \frac{m\Delta^2}{(r^2+a^2)^4} r^4.$$

Also, as before we have $\frac{1}{|q|^2}\mathcal{V}_{\delta_1} = O(r^{-3})$. Finally, the term $-\frac{4a\mathcal{T}}{|q|^2(r^2+a^2)^2}\nabla_{\widehat{T}}\psi\nabla_\phi\psi$ can be bounded by Cauchy-Schwarz by a term containing $\nabla_{\widehat{T}}\psi$ which can be absorbed by P'_{δ_1} and a term of the form $O(a^2r^{-4})|\nabla_\phi\psi|^2$.

Using (3.5.3) to write $O^{\alpha\beta}\nabla_\alpha\psi \cdot \nabla_\beta\psi = |q|^2|\nabla\psi|^2$, we summarize the final estimate in the following.

Proposition 7.2.1. *The generalized current induced by the vectorfield $X = \mathcal{F}\partial_r$ and the scalar function $w = w_X - \delta_1 \frac{4m\Delta\mathcal{T}^2}{r^2(r^2+a^2)^4}$ with $\mathcal{F}$ and w_X defined as in Proposition 7.1.8 satisfies the following estimate, for all $|a|/m < 1$ and for a constant $\delta_* > 0$ depending*[6] *on $1 - \frac{|a|}{m}$,*

$$\begin{aligned}\mathcal{E}[X,w] \geq{}& \delta_* \frac{m}{r^2}|\nabla_{\widehat{R}}\psi|^2 + \delta_* \frac{\mathcal{T}^2}{r^6}\left(\frac{m}{r^2}|\nabla_{\widehat{T}}\psi|^2 + r^{-1}|\nabla\psi|^2\right) \\ &- O(a^2r^{-4})|\nabla_\phi\psi|^2 - O(r^{-3})|\psi|^2.\end{aligned} \tag{7.2.4}$$

Observe that the first line on the right hand side of the above is precisely the density in $\dot{Mor}^{ax}_{deg}[\psi](\tau_1,\tau_2)$ as defined in Definition 6.3.5.

7.2.3 Proof of Proposition 6.3.7, estimate (6.3.4)

We are now ready to prove the first part of Proposition 6.3.7, i.e. estimate (6.3.4).

For $X = \mathcal{F}\partial_r$ and $w = w_X - \delta_1 \frac{4m\Delta\mathcal{T}^2}{r^2(r^2+a^2)^4}$ with $\mathcal{F}$ and w_X defined as in Proposition 7.1.8,

[6]The term $|\nabla_\phi\psi|^2$ is multiplied in particular by $r^4(r^2-a^2)^{-2} \lesssim \left(1 - \frac{|a|}{m}\right)^{-1}$ so that we need $\delta_* \ll 1 - \frac{|a|}{m}$ for $|a|$ close to m.

using (7.1.9), we have

$$\begin{aligned}\mathcal{E}[X,w] &= \mathbf{D}^\mu \mathcal{P}_\mu[X,w] - \left(\nabla_X \psi + \frac{1}{2} w\psi\right) \cdot (\dot{\square}_2 \psi - V\psi) + \frac{2a^2 r \cos\theta \mathcal{F}(r)}{(r^2+a^2)|q|^4} \nabla_\phi \psi \cdot {}^*\psi \\ &+ \left(\left({}^*\rho + \underline{\eta} \wedge \eta\right) \frac{r^2+a^2}{\Delta} + \frac{2a^3 r \cos\theta (\sin\theta)^2}{|q|^6}\right) \mathcal{F} \nabla_{\widehat{T}} \psi \cdot {}^*\psi.\end{aligned}$$

We now show how to absorb the last three terms on the right hand side in the following.

Lemma 7.2.2. *We have, for arbitrarily small positive constants δ_2, δ_3, to be fixed later:*

$$\begin{aligned}\left(\nabla_X \psi + \frac{1}{2} w\psi\right) \cdot (\dot{\square}_2 \psi - V\psi) \geq & -\delta_2 \frac{m}{r^2} |\nabla_{\widehat{R}} \psi|^2 - \delta_2 \frac{\mathcal{T}^2}{r^6} \frac{m}{r^2} |\nabla_{\widehat{T}} \psi|^2 \\ & + O(1)(|\nabla_{\widehat{R}} \psi| + r^{-1}|\psi|)|N| \\ & + O(a^3 r^{-6}) |\nabla_\phi \psi|^2 + O(a r^{-4}) |\psi|^2 \\ & + \mathbf{D}_\mu \left(\frac{2a\cos\theta}{|q|^2} (\partial_r)^\mu z h f \psi \cdot \nabla_T {}^*\psi\right) \\ & - \partial_t \left(\frac{2a\cos\theta}{|q|^2} z h f \psi \cdot \nabla_r {}^*\psi\right)\end{aligned} \tag{7.2.5}$$

and

$$\begin{aligned}& \left|\left(\left({}^*\rho + \underline{\eta} \wedge \eta\right) \frac{r^2+a^2}{\Delta} + \frac{2a^3 r\cos\theta(\sin\theta)^2}{|q|^6}\right) \mathcal{F} \nabla_{\widehat{T}} \psi \cdot {}^*\psi\right| \\ & + \left|\frac{2a^2 r\cos\theta \mathcal{F}(r)}{(r^2+a^2)|q|^4} \nabla_\phi \psi \cdot {}^*\psi\right| \\ \leq\ & \delta_3 \frac{\mathcal{T}^2}{r^6} \left(\frac{m}{r^2} |\nabla_{\widehat{T}} \psi|^2 + \frac{1}{r^4} |\nabla_\phi \psi|^2\right) + O(a^2 r^{-6}) |\psi|^2.\end{aligned} \tag{7.2.6}$$

Proof. According to equation (6.1.1), we have

$$\left(\nabla_X \psi + \frac{1}{2} w\psi\right) \cdot (\dot{\square}_2 \psi - V\psi) = \left(\nabla_X \psi + \frac{1}{2} w\psi\right) \cdot \left(-\frac{4a\cos\theta}{|q|^2} {}^*\nabla_T \psi + N\right).$$

We consider the first order term. Using (7.2.3), we write

$$\begin{aligned}\left(\nabla_X \psi + \frac{1}{2} w\psi\right) \cdot \left(-\frac{4a\cos\theta}{|q|^2} {}^*\nabla_T \psi\right) &= -\frac{4a\cos\theta}{|q|^2} \left(\nabla_X \psi + \frac{1}{2} w_X \psi\right) \cdot \left({}^*\nabla_T \psi\right) \\ & - \frac{4a\cos\theta}{|q|^2} \frac{1}{2} w' \psi \cdot \left({}^*\nabla_T \psi\right).\end{aligned}$$

According to Lemma 7.1.6 and recalling that $\partial_r z = f = -\frac{2\mathcal{T}}{(r^2+a^2)^3}$, we deduce

$$\begin{aligned}
& -\frac{4a\cos\theta}{|q|^2}\left(\nabla_X\psi+\frac{1}{2}w_X\psi\right)\cdot\left({}^*\nabla_T\psi\right)\\
= & -\frac{2a\cos\theta}{|q|^2}\Bigg(hf^2\psi\cdot\nabla_T{}^*\psi+2zhf\,{}^*\rho\frac{|q|^2}{\Delta}|\psi|^2-\partial_r\Big(zhf\psi\cdot\nabla_T{}^*\psi\Big)\\
& +\partial_t\Big(zhf\psi\cdot\nabla_r{}^*\psi\Big)\Bigg).
\end{aligned}$$

Hence,

$$\begin{aligned}
& -\frac{4a\cos\theta}{|q|^2}\left(\nabla_X\psi+\frac{1}{2}w\psi\right)\cdot\left({}^*\nabla_T\psi\right)\\
= & -\frac{2a\cos\theta}{|q|^2}hf^2\psi\cdot\nabla_T{}^*\psi+\delta_1\frac{4a\cos\theta}{|q|^2}\frac{2m\Delta\mathcal{T}^2}{r^2(r^2+a^2)^4}\psi\cdot\nabla_T{}^*\psi-\frac{4a\cos\theta}{|q|^2}zhf\,{}^*\rho\frac{|q|^2}{\Delta}|\psi|^2\\
& +\frac{2a\cos\theta}{|q|^2}\Bigg(\partial_r\Big(zhf\psi\cdot\nabla_T{}^*\psi\Big)-\partial_t\Big(zhf\psi\cdot\nabla_r{}^*\psi\Big)\Bigg),
\end{aligned}$$

i.e.

$$\begin{aligned}
& -\frac{4a\cos\theta}{|q|^2}\left(\nabla_X\psi+\frac{1}{2}w\psi\right)\cdot\left({}^*\nabla_T\psi\right)\\
= & -\frac{8a\cos\theta}{|q|^2}\frac{\mathcal{T}^2}{r(r^2+a^2)^2(r^2-a^2)}\left(1-\delta_1\frac{m\Delta}{r(r^2+a^2)}\right)\psi\cdot\nabla_T{}^*\psi\\
& -\frac{4a\cos\theta}{|q|^2}zhf\,{}^*\rho\frac{|q|^2}{\Delta}|\psi|^2\\
& +\frac{2a\cos\theta}{|q|^2}\Bigg(\partial_r\Big(zhf\psi\cdot\nabla_T{}^*\psi\Big)-\partial_t\Big(zhf\psi\cdot\nabla_r{}^*\psi\Big)\Bigg).
\end{aligned}$$

Also, notice that[7] $\mathbf{D}_\mu(\cos\theta|q|^{-2}(\partial_r)^\mu)=0$ which implies

$$\frac{2a\cos\theta}{|q|^2}\partial_r\Big(zhf\psi\cdot\nabla_T{}^*\psi\Big) = \mathbf{D}_\mu\left(\frac{2a\cos\theta}{|q|^2}(\partial_r)^\mu zhf\psi\cdot\nabla_T{}^*\psi\right)$$

[7]Indeed, we have $\sqrt{|\mathbf{g}|}=\sin\theta|q|^2$ and hence

$$\mathbf{D}_\mu\Big(\cos\theta|q|^{-2}(\partial_r)^\mu\Big)=\frac{1}{\sqrt{|\mathbf{g}|}}\partial_\mu(\sqrt{|\mathbf{g}|}\cos\theta|q|^{-2}(\partial_r)^\mu)=\frac{1}{\sin\theta|q|^2}\partial_r(\sin\theta|q|^2\cos\theta|q|^{-2})=0.$$

and hence

$$
\begin{aligned}
&-\frac{4a\cos\theta}{|q|^2}\left(\nabla_X\psi+\frac{1}{2}w\psi\right)\cdot\left({}^*\nabla_T\psi\right)\\
=\;&-\frac{8a\cos\theta}{|q|^2}\frac{\mathcal{T}^2}{r(r^2+a^2)^2(r^2-a^2)}\left(1-\delta_1\frac{m\Delta}{r(r^2+a^2)}\right)\psi\cdot\nabla_T{}^*\psi\\
&-\frac{4a\cos\theta}{|q|^2}zhf\,{}^*\rho\frac{|q|^2}{\Delta}|\psi|^2\\
&+\mathbf{D}_\mu\left(\frac{2a\cos\theta}{|q|^2}(\partial_r)^\mu zhf\psi\cdot\nabla_T{}^*\psi\right)-\partial_t\left(\frac{2a\cos\theta}{|q|^2}zhf\psi\cdot\nabla_r{}^*\psi\right).
\end{aligned}
$$

The first two lines on the right hand side can be bounded as follows:

$$
\begin{aligned}
&\left|\frac{8a\cos\theta}{|q|^2}\frac{\mathcal{T}^2}{r(r^2+a^2)^3}\left(1+\delta_1\frac{m\Delta}{r(r^2+a^2)}\right)\psi\cdot\nabla_T{}^*\psi-\frac{4a\cos\theta}{|q|^2}zhf\,{}^*\rho\frac{|q|^2}{\Delta}|\psi|^2\right|\\
\leq\;&\frac{a}{|q|^2}\frac{\mathcal{T}^2}{r(r^2+a^2)^3}\left(\delta_2 r|\nabla_T\psi|^2+\delta_2^{-1}r^{-1}|\psi|^2\right)+O(ar^{-6})|\psi|^2\\
\leq\;&\frac{a}{|q|^2}\frac{\mathcal{T}^2}{r(r^2+a^2)^3}\left(\delta_2 r|\nabla_{\widehat{T}}\psi|^2-\delta_2\frac{2ar}{r^2+a^2}\nabla_{\widehat{T}}\psi\cdot\nabla_\phi\psi+\delta_2\frac{a^2r}{(r^2+a^2)^2}|\nabla_\phi\psi|^2\right)\\
&+O(ar^{-4})|\psi|^2
\end{aligned}
$$

which finally gives

$$
\begin{aligned}
\left(\nabla_X\psi+\frac{1}{2}w\psi\right)\cdot\left(-\frac{4a\cos\theta}{|q|^2}{}^*\nabla_T\psi\right)\;\geq\;&-\delta_2\frac{\mathcal{T}^2}{r^6}\frac{m}{r^2}|\nabla_{\widehat{T}}\psi|^2+O(a^3r^{-6})|\nabla_\phi\psi|^2\\
&+O(ar^{-4})|\psi|^2+\mathbf{D}_\mu\left(\frac{2a\cos\theta}{|q|^2}(\partial_r)^\mu zhf\psi\cdot\nabla_T{}^*\psi\right)\\
&-\partial_t\left(\frac{2a\cos\theta}{|q|^2}zhf\psi\cdot\nabla_r{}^*\psi\right).
\end{aligned}
$$

Since $X=\mathcal{F}\partial_r=\frac{2\mathcal{T}}{r(r^2-a^2)}\widehat{R}=O(1)\widehat{R}$ and $w=O(r^{-1})$, we can bound the second product by

$$
\left|\left(\nabla_X\psi+\frac{1}{2}w\psi\right)\cdot N\right|\;\lesssim\;\Big(|\nabla_{\widehat{R}}\psi|+r^{-1}|\psi|\Big)|N|.
$$

By putting together with the previous bound we obtain the first desired estimate.

Next, notice that

$$
|\,{}^*\rho\,|\lesssim\frac{am}{r^4},\qquad |\eta|+|\underline{\eta}|\lesssim\frac{a}{r^2},\qquad |\mathcal{F}|\lesssim\frac{|\Delta|}{r^2}\frac{|\mathcal{T}|}{r^3}.
$$

We therefore deduce

$$\begin{aligned}
&\left|\left(\left({}^*\rho+\underline{\eta}\wedge\eta\right)\frac{r^2+a^2}{\Delta}+\frac{2a^3r\cos\theta(\sin\theta)^2}{|q|^6}\right)\mathcal{F}\nabla_{\widehat{T}}\psi\cdot{}^*\psi\right|\\
&+\left|\frac{2a^2r\cos\theta\mathcal{F}(r)}{(r^2+a^2)|q|^4}\nabla_\phi\psi\cdot{}^*\psi\right|\\
\leq\ &\frac{am}{r^7}|\mathcal{T}||\nabla_{\widehat{T}}\psi|\,|\psi|+\frac{am}{r^8}|\mathcal{T}||\nabla_\phi\psi|\,|\psi|
\end{aligned}$$

and hence

$$\begin{aligned}
&\left|\left(\left({}^*\rho+\underline{\eta}\wedge\eta\right)\frac{r^2+a^2}{\Delta}+\frac{2a^3r\cos\theta(\sin\theta)^2}{|q|^6}\right)\mathcal{F}\nabla_{\widehat{T}}\psi\cdot{}^*\psi\right|\\
&+\left|\frac{2a^2r\cos\theta\mathcal{F}(r)}{(r^2+a^2)|q|^4}\nabla_\phi\psi\cdot{}^*\psi\right|\\
\leq\ &\delta_3\frac{\mathcal{T}^2}{r^6}\left(\frac{m}{r^2}|\nabla_{\widehat{T}}\psi|^2+\frac{1}{r^4}|\nabla_\phi\psi|^2\right)+O(a^2r^{-6})|\psi|^2
\end{aligned}$$

as stated. □

By putting together Proposition 7.2.1 and Lemma 7.2.2, we obtain from (7.1.9)

$$\begin{aligned}
\mathbf{D}^\mu\mathcal{P}_\mu[X,w] &= \mathcal{E}[X,w]+\left(\nabla_X\psi+\frac{1}{2}w\psi\right)\cdot\left(\dot{\square}_2\psi-V\psi\right)-\frac{2a^2r\cos\theta\mathcal{F}(r)}{(r^2+a^2)|q|^4}\nabla_\phi\psi\cdot{}^*\psi\\
&-\left(\left({}^*\rho+\underline{\eta}\wedge\eta\right)\frac{r^2+a^2}{\Delta}+\frac{2a^3r\cos\theta(\sin\theta)^2}{|q|^6}\right)\mathcal{F}\nabla_{\widehat{T}}\psi\cdot{}^*\psi\\
&\geq \delta_*\frac{m}{r^2}|\nabla_{\widehat{R}}\psi|^2+(\delta_*-\delta_2-\delta_3)\frac{\mathcal{T}^2}{r^6}\left(\frac{m}{r^2}|\nabla_{\widehat{T}}\psi|^2+r^{-1}|\nabla\psi|^2\right)\\
&-O(a^2r^{-4})|\nabla_\phi\psi|^2-O(r^{-3})|\psi|^2-O(1)\Big(|\nabla_{\widehat{R}}\psi|+r^{-1}|\psi|\Big)|N|\\
&+\mathbf{D}_\mu\left(\frac{2a\cos\theta}{|q|^2}(\partial_r)^\mu zhf\psi\cdot\nabla_T{}^*\psi\right)-\partial_t\left(\frac{2a\cos\theta}{|q|^2}zhf\psi\cdot\nabla_r{}^*\psi\right).
\end{aligned}$$

By choosing δ_2 and δ_3 sufficiently small, integrating the above inequality on the region $\mathcal{M}(\tau_1,\tau_2)$ and applying the divergence theorem we deduce

$$\begin{aligned}
&\int_{\mathcal{M}(\tau_1,\tau_2)}\frac{m}{r^2}|\nabla_{\widehat{R}}\psi|^2+\frac{\mathcal{T}^2}{r^6}\left(\frac{m}{r^2}|\nabla_{\widehat{T}}\psi|^2+r^{-1}|\nabla\psi|^2\right)\\
\lesssim\ &\int_{\partial\mathcal{M}(\tau_1,\tau_2)}|M(\psi)|+\int_{\mathcal{M}(\tau_1,\tau_2)}\left(r^{-4}a^2|\nabla_Z\psi|^2+r^{-3}|\psi|^2\right)+\int_{\mathcal{M}(\tau_1,\tau_2)}\Big(|\nabla_{\widehat{R}}\psi|+r^{-1}|\psi|\Big)|N|
\end{aligned}$$

where

$$M(\psi):=\mathcal{P}\cdot N+O(ar^{-2})zhf\psi\cdot\nabla_T{}^*\psi+O(ar^{-2})zhf\psi\cdot\nabla_r{}^*\psi.$$

Recalling that $X = \frac{2\mathcal{T}}{r(r^2+a^2)}\widehat{R} = O(1)\widehat{R}$, $w = O(r^{-1})$, the properties of N_Σ in Definition 6.1.5 and

$$\mathcal{P}_\mu = \mathcal{P}_\mu[X, w, M] = \mathcal{Q}_{\mu\nu}X^\nu + \frac{1}{2}w\psi \cdot \dot{\mathbf{D}}_\mu\psi - \frac{1}{4}|\psi|^2\partial_\mu w + \frac{1}{4}|\psi|^2 M_\mu,$$

we easily deduce

$$\int_{\partial\mathcal{M}(\tau_1,\tau_2)} |M(\psi)| \lesssim \sup_{[\tau_1,\tau_2]} E_{deg}[\psi](\tau) + \delta_{\mathcal{H}} F_{\mathcal{A}}[\psi](\tau_1, \tau_2) + F_{\Sigma_*}[\psi](\tau_1, \tau_2).$$

This ends the proof of estimate (6.3.4) of Proposition 6.3.7.

7.2.4 Proof of Proposition 6.3.7, estimate (6.3.5)

In this section we provide the proof for the estimate in the second part of Proposition 6.3.7.

The main difference between estimate (6.3.4) and estimate (6.3.5) in Proposition 6.3.7 is in the control on the left hand side of the zero-th order term $|\psi|^2$. To extend the estimate to control such lower order term we will make use of a Poincaré inequality and the one-form M in the definition of the current through a Hardy estimate.

Poincaré inequality

Recall that according to Proposition 7.1.5, for choices of functions z, f, h and v in the definition of $M = v\partial_r$, the generalized current associated to the Morawetz vectorfield in (7.1.10) is given by

$$|q|^2\mathcal{E}[X, w, M] = \mathcal{A}|\nabla_r\psi|^2 + P + \mathcal{V}|\psi|^2 + \frac{1}{4}|q|^2\mathbf{D}^\mu(|\psi|^2 M_\mu).$$

Here we consider the choices for z, f, h made in Proposition 7.1.8 and show that there exists a choice of v such that the above expression is positive definite.

With the choices of z, f, h mentioned above, the coefficients $\mathcal{A}$ and $\mathcal{V}$ are given by (7.1.27) and (7.1.28) respectively, and the principal term P given by (7.1.26), i.e.

$$P = \mathcal{U}^{\alpha\beta}(\dot{\mathbf{D}}_\alpha\psi) \cdot (\dot{\mathbf{D}}_\beta\psi) = \frac{\mathcal{T}}{r}\frac{r^2+a^2}{r^2-a^2}\left(\frac{2\mathcal{T}}{(r^2+a^2)^3}O^{\alpha\beta}\nabla_\alpha\psi \cdot \nabla_\beta\psi - \frac{4ar}{(r^2+a^2)^2}\nabla_{\widehat{T}}\psi \cdot \nabla_\phi\psi\right).$$

We rewrite P as

$$P = r^2 H|\nabla\psi|^2 - O(a)\big(r|\nabla\psi|^2 + r^{-1}|\nabla_t\psi|^2\big), \qquad H := \frac{2\mathcal{T}^2}{r(r^2+a^2)^2(r^2-a^2)}.$$

To obtain a lower bound for $\int_S P$, we will rely on the following Poincaré inequality.

Lemma 7.2.3. *For $\psi \in \mathfrak{s}_2$, we have*

$$\int_S |\nabla\psi|^2 \;\geq\; \frac{2}{r^2}\int_S |\psi|^2 - O(a)\int_S \big(|\nabla\psi|^2 + r^{-2}|\nabla_t\psi|^2 + r^{-4}|\psi|^2\big).$$

Proof. Denoting by ∇^S the covariant derivative for the induced metric on S, one easily checks

$$\nabla^S \;=\; \big(1+O(a^2r^{-2})\big)\nabla - \frac{a\sin\theta}{r}\big(1+O(a^2r^{-2})\big)\nabla_t$$

and hence

$$|\nabla^S\psi|^2 = |\nabla\psi|^2 - O(a)\big(|\nabla\psi|^2 + r^{-2}|\nabla_t\psi|^2\big)$$

so that

$$\int_S |\nabla^S\psi|^2 \;=\; \int_S |\nabla\psi|^2 - O(a)\int_S \big(|\nabla\psi|^2 + r^{-2}|\nabla_t\psi|^2\big).$$

Also, note that in the three coordinates systems of Remark 6.1.8, we have, in view of Lemma 2.4.10 and Lemma 2.4.24 in [53]

$$\max_{b,c=1,2}\big|\partial^{\leq 2}((g_S)_{x^a x^b} - r^2(\gamma_{\mathbb{S}^2})_{x^a x^b})\big| \;\lesssim\; a^2$$

where $\partial^{\leq 2}$ denotes at most 2 coordinates derivatives, and $(\gamma_{\mathbb{S}^2})_{x^a x^b}$ the metric coefficients on $\mathbb{S}^2$ in the corresponding coordinates system. We deduce in particular

$$K_S \;=\; \frac{1}{r^2}\big(1+O(a^2r^{-2})\big), \qquad r_S = r\big(1+O(a^2r^{-2})\big).$$

Applying the effective uniformization result of Corollary 3.8 in [52], we obtain a map $\Phi : \mathbb{S}^2 \to S$ and a scalar function u on $\mathbb{S}^2$ such that

$$\Phi^\#(g_S) \;=\; (r_S)^2 e^{2u}\gamma_{\mathbb{S}^2}, \qquad \|\not{\mathfrak{d}}^{\leq 2}(u\circ\Phi^{-1})\|_{L^2(S)} \lesssim (a^2r^{-2})r_S.$$

We infer, relying on the well known Poincaré inequality for $\psi \in \mathfrak{s}_2(\mathbb{S}^2)$, see for example [36],

$$\begin{aligned}\int_S |\nabla^S\psi|^2 &= \frac{1}{r^2}\big(1+O(a^2r^{-2})\big)\int_{\mathbb{S}^2}|\nabla_{\mathbb{S}^2}\Phi^\#\psi|^2+\frac{1}{r^2}O(a^2r^{-2})\int_{\mathbb{S}^2}|\Phi^\#\psi|^2\\ &\geq \frac{2}{r^2}\big(1+O(a^2r^{-2})\big)\int_{\mathbb{S}^2}|\Phi^\#\psi|^2\\ &\geq \frac{2}{r^2}\int_S\big(1+O(a^2r^{-2})\big)|\psi|^2\end{aligned}$$

and hence

$$\int_S|\nabla\psi|^2 \;\geq\; \frac{2}{r^2}\int_S|\psi|^2-O(a)\int_S\big(|\nabla\psi|^2+r^{-2}|\nabla_t\psi|^2+r^{-4}|\psi|^2\big)$$

as stated. □

Next, we define, for $\delta > 0$ sufficiently small chosen later, the following quadratic form

$$\begin{aligned}\mathrm{Qr}_\delta[\psi] =&(1-\delta)\mathcal{A}|\nabla_r\psi|^2+\left(\mathcal{V}+(1-\delta)\frac{2\mathcal{T}^2}{(r^2+a^2)^2(r^2-a^2)}\right)|\psi|^2\\ &+\frac{1}{4}|q|^2\mathbf{D}^\mu(|\psi|^2M_\mu)\end{aligned} \tag{7.2.7}$$

so that, in view of the above, we have

$$\begin{aligned}|q|^2\mathcal{E}[X,w,M] &= \mathcal{A}|\nabla_r\psi|^2+P+\mathcal{V}|\psi|^2+\frac{1}{4}|q|^2\mathbf{D}^\mu(|\psi|^2M_\mu)\\ &= \delta\mathcal{A}|\nabla_r\psi|^2+\delta P+(1-\delta)\left(P-\frac{2\mathcal{T}^2}{r(r^2+a^2)^2(r^2-a^2)}|\psi|^2\right)+\mathrm{Qr}_\delta[\psi].\end{aligned}$$

Now, since

$$P=\frac{2r\mathcal{T}^2}{(r^2+a^2)^2(r^2-a^2)}|\nabla\psi|^2-O(a)\big(r|\nabla\psi|^2+r^{-1}|\nabla_t\psi|^2\big),$$

we infer, in view of Lemma 7.2.3,

$$\begin{aligned}&\int_S\frac{1}{|q|^2}\left(\delta P+(1-\delta)\left(P-\frac{2\mathcal{T}^2}{r(r^2+a^2)^3}|\psi|^2\right)\right)\\ \gtrsim\;&\delta\frac{\mathcal{T}^2}{r^7}\int_S|\nabla\psi|^2-O(ar^{-1})\int_S\big(|\nabla\psi|^2+r^{-2}|\nabla_t\psi|^2+r^{-2}|\psi|^2\big)\end{aligned}$$

and hence

$$\begin{aligned}\int_S \mathcal{E}[X,w,M] &\gtrsim \delta\int_S\left(\frac{\mathcal{A}}{|q|^2}|\nabla_r\psi|^2+\frac{\mathcal{T}^2}{r^7}|\nabla\psi|^2\right)+\int_S\frac{1}{|q|^2}\mathrm{Qr}_\delta[\psi]\\ &\quad -O(ar^{-1})\int_S\left(|\nabla\psi|^2+r^{-2}|\nabla_t\psi|^2+r^{-2}|\psi|^2\right),\end{aligned}$$

or

$$\begin{aligned}\int_S \mathcal{E}[X,w,M] &\geq \delta\int_S\left(\frac{m}{r^2}|\nabla_{\widehat{R}}\psi|^2+\frac{\mathcal{T}^2}{r^7}|\nabla\psi|^2\right)+\int_S\frac{1}{|q|^2}\mathrm{Qr}_\delta[\psi]\\ &\quad -O(ar^{-1})\int_S\left(|\nabla\psi|^2+r^{-2}|\nabla_t\psi|^2+r^{-2}|\psi|^2\right).\end{aligned}\tag{7.2.8}$$

The Hardy inequality

Note that, using (7.1.13) and (7.2.7), we have

$$\begin{aligned}\mathrm{Qr}_\delta[\psi] =&(1-\delta)\mathcal{A}|\nabla_r\psi|^2+\left(\mathcal{V}+(1-\delta)\frac{2\mathcal{T}^2}{(r^2+a^2)^2(r^2-a^2)}\right)|\psi|^2\\ &+\frac{1}{4}|q|^2\left(2v(r)\psi\cdot\nabla_r\psi+\left(\partial_r v+\frac{2r}{|q|^2}v\right)|\psi|^2\right).\end{aligned}$$

In view of (7.2.8), it remains to derive a lower bound for the term $\mathrm{Qr}_\delta[\psi]$ which is done in the following lemma.

Lemma 7.2.4 (Half Poincaré + Hardy)**.** *For $|a|/m \ll 1$, there exists a function $v(r)$ with $v(r) = O(m^{1/2}\Delta r^{-9/2})$, and $\delta > 0$ sufficiently small, such that for all $r \geq r_+(1-\delta_{\mathcal{H}})$,*

$$Qr_\delta[\Phi] \;\geq\; O(\delta)\left(\frac{m\Delta^2}{r^4}|\nabla_r\Phi|^2+r^{-1}|\Phi|^2\right).$$

Proof. We have

$$\begin{aligned}\mathrm{Qr}_\delta[\Phi] =&(1-\delta)\mathcal{A}\left|\nabla_r\Phi+\frac{|q|^2}{4(1-\delta)\mathcal{A}}v(r)\Phi\right|^2-\frac{|q|^4}{16(1-\delta)\mathcal{A}}v^2|\Phi|^2\\ &+\left(\mathcal{V}+(1-\delta)\frac{2\mathcal{T}^2}{(r^2+a^2)^2(r^2-a^2)}\right)|\Phi|^2+\frac{1}{4}|q|^2\left(\partial_r v+\frac{2r}{|q|^2}v\right)|\Phi|^2\end{aligned}$$

and hence

$$\mathrm{Qr}_\delta[\Phi] \geq \left(\mathcal{V}+(1-\delta)\frac{2\mathcal{T}^2}{(r^2+a^2)^2(r^2-a^2)}+\frac{1}{4}|q|^2\left(\partial_r v+\frac{2r}{|q|^2}v\right)-\frac{|q|^4}{16(1-\delta)\mathcal{A}}v^2\right)|\Phi|^2$$

Thus, it suffices to prove that for $|a|/m \ll 1$ and $\delta > 0$ small enough, there exists a function v such that

$$E := \mathcal{V} + \frac{2(1-\delta)\mathcal{T}^2}{r(r^2+a^2)^2(r^2-a^2)} + \frac{1}{4}|q|^2\left(\partial_r v + \frac{2r}{|q|^2}v\right) - \frac{1}{16(1-\delta)\mathcal{A}}|q|^4 v^2 > 0. \quad (7.2.9)$$

By continuity it remains to prove that there exists a function $v(r)$ for which the condition (7.2.9) is valid for $a = 0$ and $\delta = 0$. For $a = 0$, i.e. for Schwarzschild spacetime, we have, with $\Upsilon := 1 - \frac{2m}{r}$,

$$\begin{aligned}
\mathcal{V} &= \mathcal{V}_0 + \mathcal{V}_1,\\
\mathcal{V}_1 &= 8\Delta\frac{r^3-4mr^2}{r^9}(r^3-3mr^2) = 8r^{-1}\Upsilon\left(1-\frac{4m}{r}\right)\left(1-\frac{3m}{r}\right),\\
\mathcal{V}_0 &= \frac{9mr^6-46m^2r^5+54m^3r^4}{r^8} = \frac{9mr^2-46m^2r+54m^3}{r^4},\\
\mathcal{V} &= \frac{8r^3-63mr^2+162m^2r-138m^3}{r^4},\\
\mathcal{A} &= \frac{2m\Delta^2}{r^8}3r^4 = 6m\Upsilon^2.
\end{aligned}$$

The expression E in (7.2.9) becomes, setting $\widetilde{v} = r^2 v$,

$$\begin{aligned}
E &= \mathcal{V} + \frac{2}{r}\left(1-\frac{3m}{r}\right)^2 + \frac{1}{4}r^2\left(\partial_r v + \frac{2r}{r^2}v\right) - \frac{1}{16\cdot 6m}\Upsilon^{-2}r^4v^2\\
&= \mathcal{V} + \frac{2}{r}\left(1-\frac{3m}{r}\right)^2 + \frac{1}{4}\partial_r(\widetilde{v}) - \frac{1}{96m}\Upsilon^{-2}\widetilde{v}^2\\
&= \frac{8r^3-63mr^2+162m^2r-138m^3}{r^4} + \frac{2}{r}\left(1-\frac{3m}{r}\right)^2 + \frac{1}{4}\partial_r\widetilde{v} - \frac{1}{96m}\Upsilon^{-2}\widetilde{v}^2\\
&= \frac{10r^3-75mr^2+180m^2r-138m^3}{r^4} + \frac{1}{4}\partial_r\widetilde{v} - \frac{1}{96m}\Upsilon^{-2}\widetilde{v}^2.
\end{aligned}$$

Introducing $x = \frac{r}{2m}$ and assuming that $\widetilde{v} = \widetilde{v}\big(\frac{r}{2m}\big) = \widetilde{v}_0(x)$, we derive

$$\begin{aligned}
E &= \frac{10(2mx)^3-75m(2mx)^2+180m^2(2mx)-138m^3}{(2mx)^4} + \frac{1}{8m}\widetilde{v}_0' - \frac{1}{96m}\frac{x^2}{(x-1)^2}\widetilde{v}_0^2\\
&= \frac{40x^3-150x^2+180x-69}{8mx^4} + \frac{1}{8m}\widetilde{v}_0' - \frac{1}{96m}\frac{x^2}{(x-1)^2}\widetilde{v}_0^2.
\end{aligned}$$

By setting $\widetilde{v}_0(x) = (x-1)k_0(x)$, and $\widetilde{v}_0' = k_0 + (x-1)k_0'$, then

$$8mE = 40x^{-1} - 150x^{-2} + 180x^{-3} - 69x^{-4} + k_0 + (x-1)k_0' - \frac{1}{12}x^2k_0^2.$$

In order to have that $x^4 k_0^2$ with same degree in x than x^{-1} we assume

$$k_0(x) = Ax^{-3/2}, \qquad k_0'(x) = -\frac{3}{2}Ax^{-5/2}.$$

This yields

$$\begin{aligned}8mE &= 40x^{-1} - 150x^{-2} + 180x^{-3} - 69x^{-4} + Ax^{-3/2} - \frac{3}{2}A(x-1)x^{-5/2} - \frac{1}{12}x^2A^2x^{-3}\\ &= \left(40 - \frac{1}{12}A^2\right)x^{-1} - \frac{1}{2}Ax^{-3/2} - 150x^{-2} + \frac{3}{2}Ax^{-5/2} + 180x^{-3} - 69x^{-4}.\end{aligned}$$

For example for $A = 2$, the above is positive for $x \geq 1$. This results in

$$\begin{aligned}E &= \frac{119}{12}\frac{1}{r} - \frac{1}{8m}\left(\frac{2m}{r}\right)^{3/2} - 150\frac{1}{8m}\left(\frac{2m}{r}\right)^2 + 3\frac{1}{8m}\left(\frac{2m}{r}\right)^{5/2} + 180\frac{1}{8m}\left(\frac{2m}{r}\right)^3\\ &\quad -69\frac{1}{8m}\left(\frac{2m}{r}\right)^4.\end{aligned}$$

Note also that in this case

$$v = r^{-2}\widetilde{v} = 2\frac{(2m)^{3/2}}{r^{7/2}}\left(\frac{r}{2m} - 1\right),$$

where v satisfies $v(r) = O(m^{1/2}\Delta r^{-9/2})$. This ends the proof of Lemma 7.2.4. □

Proposition 7.2.5. *There exists a choice of $v(r) = O\big(m^{1/2}\Delta r^{-9/2}\big)$ and a small universal constant $c_0 > 0$ such that, for $|a|/m \ll 1$ and $r \geq r_+(1-\delta_{\mathcal{H}})$,*

$$\begin{aligned}\int_S \mathcal{E}[X, w, M] &\geq c_0 \int_S \left(\frac{m}{r^2}|\nabla_{\widehat{R}}\psi|^2 + \frac{\mathcal{T}^2}{r^7}|\nabla\psi|^2 + r^{-3}|\psi|^2\right)\\ &\quad -O(ar^{-1})\int_S \big(|\nabla\psi|^2 + r^{-2}|\nabla_t\psi|^2\big).\end{aligned}$$

Proof. Immediate consequence of (7.2.8) and Lemma 7.2.4. □

We can proceed as in Section 7.2.2 to derive a version of Proposition 7.2.5 which also contains $\nabla_{\widehat{T}}\psi$. We obtain the following proposition.

Proposition 7.2.6. *There exists a choice of $v(r) = O\big(m^{1/2}\Delta r^{-9/2}\big)$, a redefinition of w and a small universal constant $c_0 > 0$ such that, for all a verifying $|a|/m \ll 1$ and all $r \geq r_+(1-\delta_{\mathcal{H}})$,*

$$\begin{aligned}\int_S \mathcal{E}[X, w, M] &\geq c_0 \int_S \left(\frac{m}{r^2}|\nabla_{\widehat{R}}\psi|^2 + \frac{\mathcal{T}^2}{r^6}\left(\frac{m}{r^2}|\nabla_{\widehat{T}}\psi|^2 + r^{-1}|\nabla\psi|^2\right) + r^{-3}|\psi|^2\right)\\ &\quad -O(ar^{-1})\int_S \big(|\nabla\psi|^2 + r^{-2}|\nabla_t\psi|^2\big).\end{aligned}$$

End of the proof

We now proceed as in Section 7.2.3 and we apply Lemma 7.2.2 to control the right hand side in the divergence of $\mathcal{P}[X, w, M]$. Since the zero-th order terms on the right hand side appear with higher decay in r and are multiplied by the angular momentum a they can be absorbed by the positive Morawetz bulk.

By applying the divergence theorem we then obtain the following estimate

$$\begin{aligned}
&\int_{\mathcal{M}(\tau_1,\tau_2)} \frac{m}{r^2}|\nabla_{\widehat{R}}\psi|^2 + r^{-3}|\psi|^2 + \frac{\mathcal{T}^2}{r^6}\left(\frac{m}{r^2}|\nabla_{\widehat{T}}\psi|^2 + r^{-1}|\nabla\psi|^2\right)\\
\lesssim\ &\int_{\partial\mathcal{M}(\tau_1,\tau_2)} |\mathcal{P}\cdot N_\Sigma| + \int_{\mathcal{M}(\tau_1,\tau_2)} ar^{-1}\big(|\nabla\psi|^2 + r^{-2}|\nabla_t\psi|^2\big)\\
&+\int_{\mathcal{M}(\tau_1,\tau_2)} \Big(|\nabla_{\widehat{R}}\psi| + r^{-1}|\psi|\Big)|N|
\end{aligned}$$

with the same bounds as before for $M(\psi) := \mathcal{P}\cdot N_\Sigma$, which is precisely (6.3.5), and hence concludes the proof of Proposition 6.3.7.

7.3 Energy estimates

We start with the following lemma.

Lemma 7.3.1. *The following hold true with a sufficiently small $c_0 > 0$, for any $|a| \ll m$,*

$$\begin{aligned}
\int_{\Sigma(\tau)} \mathcal{Q}(\widehat{T}, N_\Sigma) &\geq c_0 E_{deg}[\psi](\tau) - O(\delta_{\mathcal{H}})E_{r\leq r_+}[\psi](\tau),\\
\int_{\Sigma_*(\tau_1,\tau_2)} \mathcal{Q}(\widehat{T}, N_{\Sigma_*}) &\geq c_0 F_{\Sigma_*}[\psi](\tau_1,\tau_2),\\
\int_{\mathcal{A}(\tau_1,\tau_2)} \mathcal{Q}(\widehat{T}, N_{\mathcal{A}}) &\gtrsim -\delta_{\mathcal{H}} F_{\mathcal{A}}[\psi](\tau_1,\tau_2).
\end{aligned} \tag{7.3.1}$$

Proof. We have

$$\begin{aligned}
\mathcal{Q}(e_4, N_\Sigma) &= \mathcal{Q}\left(e_4, \frac{1}{2}e_3(\tau)e_4 + \frac{1}{2}e_4(\tau)e_3 - \nabla(\tau)\right)\\
&= \frac{1}{2}e_3(\tau)\mathcal{Q}_{44} + \frac{1}{2}e_4(\tau)\mathcal{Q}_{34} - \nabla^a(\tau)\mathcal{Q}_{4a},\\
\mathcal{Q}(e_3, N_\Sigma) &= \frac{1}{2}e_3(\tau)\mathcal{Q}_{34} + \frac{1}{2}e_4(\tau)\mathcal{Q}_{33} - \nabla^a(\tau)\mathcal{Q}_{3a}.
\end{aligned}$$

In view of (4.7.2), we have

$$\mathcal{Q}_{33} = |\nabla_3\psi|^2, \qquad \mathcal{Q}_{44} = |\nabla_4\psi|^2, \qquad \mathcal{Q}_{34} = |\nabla\psi|^2 + V|\psi|^2,$$
$$\mathcal{Q}_{4a} = \nabla_4\Psi\cdot\nabla_a\Psi, \qquad \mathcal{Q}_{3a} = \nabla_3\Psi\cdot\nabla_a\Psi.$$

We infer

$$\begin{aligned}
\mathcal{Q}(e_4, N_\Sigma) &\geq \frac{1}{2}e_3(\tau)|\nabla_4\psi|^2 + \frac{1}{2}e_4(\tau)\Big(|\nabla\psi|^2 + V|\psi|^2\Big) - |\nabla\tau||\nabla_4\psi||\nabla\psi|,\\
\mathcal{Q}(e_3, N_\Sigma) &\geq \frac{1}{2}e_3(\tau)\Big(|\nabla\psi|^2 + V|\psi|^2\Big) + \frac{1}{2}e_4(\tau)|\nabla_3\psi|^2 - |\nabla\tau||\nabla_3\psi||\nabla\psi|.
\end{aligned}$$

Next, since we have in view of the choice of τ

$$e_4(\tau) > 0, \qquad e_3(\tau) > 0, \qquad |\nabla\tau|^2 \leq \frac{8}{9}e_4(\tau)e_3(\tau),$$

we infer

$$\begin{aligned}
|\nabla\tau||\nabla_4\psi||\nabla\psi| &\leq \sqrt{\frac{8}{9}e_4(\tau)e_3(\tau)}|\nabla_4\psi||\nabla\psi| \leq \sqrt{\frac{8}{9}}\left(\frac{1}{2}e_3(\tau)|\nabla_4\psi|^2 + \frac{1}{2}e_4(\tau)|\nabla\psi|^2\right),\\
|\nabla\tau||\nabla_3\psi||\nabla\psi| &\leq \sqrt{\frac{8}{9}e_4(\tau)e_3(\tau)}|\nabla_3\psi||\nabla\psi| \leq \sqrt{\frac{8}{9}}\left(\frac{1}{2}e_3(\tau)|\nabla\psi|^2 + \frac{1}{2}e_4(\tau)|\nabla_3\psi|^2\right),
\end{aligned}$$

and thus

$$\begin{aligned}
\mathcal{Q}(e_4, N_\Sigma) &\geq \left(1 - \sqrt{\frac{8}{9}}\right)\left(\frac{1}{2}e_3(\tau)|\nabla_4\psi|^2 + \frac{1}{2}e_4(\tau)|\nabla\psi|^2\right) + \frac{1}{2}e_4(\tau)V|\psi|^2,\\
\mathcal{Q}(e_3, N_\Sigma) &\geq \left(1 - \sqrt{\frac{8}{9}}\right)\left(\frac{1}{2}e_3(\tau)|\nabla\psi|^2 + \frac{1}{2}e_4(\tau)|\nabla_3\psi|^2\right) + \frac{1}{2}e_3(\tau)V|\psi|^2.
\end{aligned}$$

Using the fact that we have[8] on $\mathcal{M}$

$$e_4(\tau) \gtrsim \frac{m^2}{r^2}, \qquad e_3(\tau) \gtrsim 1, \qquad V \geq -O(\delta_{\mathcal{H}})\mathbb{1}_{r\leq r_+},$$

we infer the existence of a constant $c_0 > 0$ such that, on $\mathcal{M}$,

$$\begin{aligned}
\mathcal{Q}(e_4, N_\Sigma) &\geq c_0\Big(|\nabla_4\psi|^2 + r^{-2}|\nabla\psi|^2\Big) - O(\delta_{\mathcal{H}})|\psi|^2\mathbb{1}_{r\leq r_+},\\
\mathcal{Q}(e_3, N_\Sigma) &\geq c_0\Big(r^{-2}|\nabla_3\psi|^2 + |\nabla\psi|^2\Big) - O(\delta_{\mathcal{H}})|\psi|^2\mathbb{1}_{r\leq r_+}.
\end{aligned}$$

[8]Recall that we consider the explicit potential $V = \frac{4\Delta}{(r^2+a^2)|q|^2}$, see (6.1.1), which indeed satisfies $V \geq -O(\delta_{\mathcal{H}})\mathbb{1}_{r\leq r_+}$.

We infer on $\mathcal{M}$

$$\begin{aligned}\mathcal{Q}(\widehat{T}, N_\Sigma) &= \frac{1}{2}\frac{|q|^2}{r^2+a^2}\mathcal{Q}(e_4, N_\Sigma) + \frac{1}{2}\frac{\Delta}{r^2+a^2}\mathcal{Q}(e_3, N_\Sigma)\\ &\geq c_0\left(|\nabla_4\psi|^2 + \frac{|\Delta|}{r^4}|\nabla_3\psi|^2 + |\nabla\psi|^2\right) - O(\delta_{\mathcal{H}})\Big(|\nabla_3\psi|^2 + |\psi|^2\Big)\mathbb{1}_{r\leq r_+}.\end{aligned}$$

Similarly, using $\Delta \gtrsim r^{-2}$ and $V \gtrsim r^{-2}$ on Σ_*,

$$\mathcal{Q}(\widehat{T}, N_{\Sigma_*}) \geq c_0\Big(|\nabla_4\psi|^2 + |\nabla_3\psi|^2 + |\nabla\psi|^2 + r^{-2}|\psi|^2\Big).$$

Also, we have on $\mathcal{A}$

$$\mathcal{Q}(\widehat{T}, N_{\mathcal{A}}) \geq -O(\delta_{\mathcal{H}})\mathcal{Q}_{34} - O(\delta_{\mathcal{H}}^2)\mathcal{Q}_{33} \geq -O(\delta_{\mathcal{H}})|\nabla\psi|^2 - O(\delta_{\mathcal{H}}^2)(|\nabla_3\psi|^2 + |\psi|^2).$$

This yields

$$\begin{aligned}\int_{\Sigma(\tau)}\mathcal{Q}(\widehat{T}, N_\Sigma) &\geq c_0\int_{\Sigma(\tau)}\left(|\nabla_4\psi|^2 + \frac{|\Delta|}{r^4}|\nabla_3\psi|^2 + |\nabla\psi|^2\right) - O(\delta_{\mathcal{H}})E_{r\leq r_+}[\psi](\tau),\\ \int_{\Sigma_*(\tau_1,\tau_2)}\mathcal{Q}(\widehat{T}, N_{\Sigma_*}) &\geq c_0 F_{\Sigma_*}[\psi](\tau_1,\tau_2),\\ \int_{\mathcal{A}(\tau_1,\tau_2)}\mathcal{Q}(\widehat{T}, N_{\mathcal{A}}) &\gtrsim -\delta_{\mathcal{H}}F_{\mathcal{A}}[\psi](\tau_1,\tau_2).\end{aligned}$$

In particular, we have obtained the desired estimates on Σ_* and $\mathcal{A}$.

Also, we have in view of Lemma 7.2.3, for $\psi\in\mathfrak{s}_2$ and $|a|\ll m$,

$$\begin{aligned}\frac{1}{r^2}\int_S|\psi|^2 &\lesssim \int_S\left(|\nabla\psi|^2 + r^{-2}|\nabla_t\psi|^2\right)\\ &\lesssim \int_S\left(|\nabla_4\psi|^2 + \frac{|\Delta|}{r^4}|\nabla_3\psi|^2 + |\nabla\psi|^2\right).\end{aligned}$$

Together with the above, we infer

$$\begin{aligned}\int_{\Sigma(\tau)}\mathcal{Q}(\widehat{T}, N_\Sigma) \geq\ & c_0\int_{\Sigma(\tau)}\left(|\nabla_4\psi|^2 + \frac{|\Delta|}{r^4}|\nabla_3\psi|^2 + |\nabla\psi|^2 + r^{-2}|\psi|^2\right)\\ & - O(\delta_{\mathcal{H}})E_{r\leq r_+}[\psi](\tau),\end{aligned}$$

and hence

$$\int_{\Sigma(\tau)}\mathcal{Q}(\widehat{T}, N_\Sigma) \geq c_0 E_{deg}[\psi](\tau) - O(\delta_{\mathcal{H}})E_{r\leq r_+}[\psi](\tau)$$

as stated. This concludes the proof of Lemma 7.3.1. □

We consider the energy current associated to the modified timelike vectorfield $\widehat{T}_\delta = \partial_t + \chi_\delta(r)\partial_\phi$, as defined in Definition 7.1.1, with $\delta = \frac{1}{10}$ and $|a|/m \ll 1$ small enough. Recall that $\chi_\delta = \frac{a}{r^2+a^2}\chi_0\Big(\delta^{-1}\frac{\mathcal{T}}{r^3}\Big)$, with $\chi_0 = 0$ in $\mathcal{M}_{trap}$.

From Proposition 4.7.2, we have for the current associated to $\widehat{T}_\delta$:

$$\mathbf{D}^\mu\mathcal{P}_\mu[\widehat{T}_\delta, 0, 0] = \frac{1}{2}\mathcal{Q}\cdot {}^{(\widehat{T}_\delta)}\pi + \widehat{T}_\delta^\mu\dot{\mathbf{D}}^\nu\psi^a\dot{\mathbf{R}}_{ab\nu\mu}\psi^b + \nabla_{\widehat{T}_\delta}\psi\cdot\big(\dot{\square}_2\psi - V\psi\big)$$

and hence

$$\begin{aligned}\mathbf{D}^\mu\mathcal{P}_\mu[\widehat{T}_\delta, 0, 0] &= \frac{1}{2}\mathcal{Q}\cdot {}^{(\widehat{T}_\delta)}\pi + \partial_t^\mu\dot{\mathbf{D}}^\nu\psi^a\dot{\mathbf{R}}_{ab\nu\mu}\psi^b\\ &\quad +(\widehat{T}_\delta - \partial_t)^\mu\dot{\mathbf{D}}^\nu\psi^a\dot{\mathbf{R}}_{ab\nu\mu}\psi^b + \nabla_{\widehat{T}_\delta}\psi\cdot\big(\dot{\square}_2\psi - V\psi\big).\end{aligned}$$

Since $\dot{\mathbf{R}}_{ab\nu\mu}$ is antisymmetric with respect to (a,b), we rewrite

$$\begin{aligned}\mathbf{D}^\mu\mathcal{P}_\mu[\widehat{T}_\delta, 0, 0] &= \frac{1}{2}\mathcal{Q}\cdot {}^{(\widehat{T}_\delta)}\pi + \frac{1}{2}\partial_t^\mu \in^{ab} \dot{\mathbf{R}}_{ab\nu\mu}\ {}^*\psi\cdot\dot{\mathbf{D}}^\nu\psi\\ &\quad +(\widehat{T}_\delta - \partial_t)^\mu\dot{\mathbf{D}}^\nu\psi^a\dot{\mathbf{R}}_{ab\nu\mu}\psi^b + \nabla_{\widehat{T}_\delta}\psi\cdot\big(\dot{\square}_2\psi - V\psi\big).\end{aligned}$$

Introducing the following spacetime 1-form

$$A_\mu := \in^{bc} \dot{\mathbf{R}}_{bc\mu\nu}\partial_t^\nu, \tag{7.3.2}$$

we infer

$$\begin{aligned}\mathbf{D}^\mu\mathcal{P}_\mu[\widehat{T}_\delta, 0, 0] &= \frac{1}{2}\mathcal{Q}\cdot {}^{(\widehat{T}_\delta)}\pi + \frac{1}{2}A_\nu\ {}^*\psi\cdot\dot{\mathbf{D}}^\nu\psi + (\widehat{T}_\delta - \partial_t)^\mu\dot{\mathbf{D}}^\nu\psi^a\dot{\mathbf{R}}_{ab\nu\mu}\psi^b\\ &\quad +\nabla_{\widehat{T}_\delta}\psi\cdot\big(\dot{\square}_2\psi - V\psi\big).\end{aligned} \tag{7.3.3}$$

Next, we compute the components of A.

Lemma 7.3.2. *Let A the spacetime 1-form given by* (7.3.2)*. Then, we have*

$$\begin{aligned}A_4 &= -4\ {}^*\rho\partial_t^3 - 4(\underline{\eta}\wedge\eta)\partial_t^3 + tr\,\chi\big({}^{(h)}\partial_t\wedge\underline{\eta}\big) - {}^{(a)}tr\chi\big(\underline{\eta}\cdot{}^{(h)}\partial_t\big),\\ A_3 &= 4\ {}^*\rho\partial_t^4 + 4(\underline{\eta}\wedge\eta)\partial_t^4 + tr\underline{\chi}\big({}^{(h)}\partial_t\wedge\eta\big) - {}^{(a)}tr\underline{\chi}\big(\eta\cdot{}^{(h)}\partial_t\big),\\ A_e &= \Big(-tr\underline{\chi}\ {}^*\eta_e + {}^{(a)}tr\underline{\chi}\eta_e\Big)\partial_t^3 + \Big(-tr\,\chi\ {}^*\underline{\eta}_e + {}^{(a)}tr\chi\underline{\eta}_e\Big)\partial_t^4\\ &\quad -\frac{1}{2}\Big(4\rho + tr\,\chi tr\underline{\chi} + {}^{(a)}tr\chi\,{}^{(a)}tr\underline{\chi}\Big)\ {}^*({}^{(h)}\partial_t)_e.\end{aligned}$$

Proof. We rewrite A_μ as

$$A_\mu \;=\; \in^{bc} \dot{\mathbf{R}}_{bc\mu 3}\partial_t^3 + \in^{bc} \dot{\mathbf{R}}_{bc\mu 4}\partial_t^4 + \in^{bc} \dot{\mathbf{R}}_{bc\mu d}\partial_t^d.$$

Next, we compute the various components of A_μ. We have in Kerr, using the horizontal tensor ${}^{(h)}\partial_t$ defined by $({}^{(h)}\partial_t)_b = (\partial_t)_b$, the definition (2.1.13) of $\dot{\mathbf{R}}$, and Proposition 2.2.4,

$$\begin{aligned}
A_4 &= \in^{bc} \dot{\mathbf{R}}_{bc43}\partial_t^3 + \in^{bc} \dot{\mathbf{R}}_{bc4d}\partial_t^d,\\
&= \in^{bc}\Big(-2 \in_{bc} {}^*\rho - 2(\underline{\eta}_b\eta_c - \eta_b\underline{\eta}_c)\Big)\partial_t^3\\
&\quad + \frac{1}{2}\in^{bc}\Big(\mathrm{tr}\,\chi(\delta_{db}\underline{\eta}_c - \delta_{dc}\underline{\eta}_b) + {}^{(a)}\mathrm{tr}\chi(\in_{db}\underline{\eta}_c - \in_{dc}\underline{\eta}_b)\Big)\partial_t^d\\
&= -4\, {}^*\rho\partial_t^3 - 4(\underline{\eta}\wedge\eta)\partial_t^3 + \frac{1}{2}\in^{bc}\Big(\mathrm{tr}\,\chi(\underline{\eta}_c\mathbf{T}_b - \underline{\eta}_b\mathbf{T}_c) + {}^{(a)}\mathrm{tr}\chi(-\underline{\eta}_c\, {}^*({}^{(h)}\partial_t)_b + {}^*({}^{(h)}\partial_t)_c\underline{\eta}_b)\Big)\\
&= -4\, {}^*\rho\partial_t^3 - 4(\underline{\eta}\wedge\eta)\partial_t^3 + \mathrm{tr}\,\chi\big({}^{(h)}\partial_t\wedge\underline{\eta}\big) - {}^{(a)}\mathrm{tr}\chi\big(\underline{\eta}\cdot{}^{(h)}\partial_t\big),
\end{aligned}$$

$$\begin{aligned}
A_3 &= \in^{bc} \dot{\mathbf{R}}_{bc34}\partial_t^4 + \in^{bc} \dot{\mathbf{R}}_{bc3d}\partial_t^d\\
&= 4\, {}^*\rho\partial_t^4 + 4(\underline{\eta}\wedge\eta)\partial_t^4 + \mathrm{tr}\,\underline{\chi}\big({}^{(h)}\partial_t\wedge\eta\big) - {}^{(a)}\mathrm{tr}\underline{\chi}\big(\eta\cdot{}^{(h)}\partial_t\big),
\end{aligned}$$

and

$$\begin{aligned}
A_e &= \in^{bc} \dot{\mathbf{R}}_{bce3}\partial_t^3 + \in^{bc} \dot{\mathbf{R}}_{bce4}\partial_t^4 + \in^{bc} \dot{\mathbf{R}}_{bced}\partial_t^d\\
&= \frac{1}{2}\in^{bc}\Big(-\mathrm{tr}\,\underline{\chi}\big(\delta_{eb}\eta_c - \delta_{ec}\eta_b\big) - {}^{(a)}\mathrm{tr}\underline{\chi}\big(\in_{eb}\eta_c - \in_{ec}\eta_b\big)\Big)\partial_t^3\\
&\quad + \frac{1}{2}\in^{bc}\Big(-\mathrm{tr}\,\chi\big(\delta_{eb}\underline{\eta}_c - \delta_{ec}\underline{\eta}_b\big) - {}^{(a)}\mathrm{tr}\chi\big(\in_{eb}\underline{\eta}_c - \in_{ec}\underline{\eta}_b\big)\Big)\partial_t^4\\
&\quad + \frac{1}{2}\in^{bc}\left(-2\in_{bc}\in_{ed}\rho - \frac{1}{2}\left(\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} + {}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi}\right)\in_{bc}\in_{ed}\right)\partial_t^d\\
&= \Big(-\mathrm{tr}\,\underline{\chi}\,{}^*\eta_e + {}^{(a)}\mathrm{tr}\underline{\chi}\eta_e\Big)\partial_t^3 + \Big(-\mathrm{tr}\,\chi\,{}^*\underline{\eta}_e + {}^{(a)}\mathrm{tr}\chi\underline{\eta}_e\Big)\partial_t^4\\
&\quad - \frac{1}{2}\Big(4\rho + \mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} + {}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi}\Big)\,{}^*({}^{(h)}\partial_t)_e
\end{aligned}$$

as stated. This concludes the proof of Lemma 7.3.2. □

We infer the following corollary.

Corollary 7.3.3. *We have*

$$A_\mu = -\mathbf{D}_\mu\left(\Im\left(\frac{2m}{q^2}\right)\right). \tag{7.3.4}$$

Proof. We have in Kerr

$$\partial_t^4 = \frac{1}{2}, \qquad \partial_t^3 = \frac{1}{2}\frac{\Delta}{|q|^2}, \qquad (\partial_t)_b = ({}^{(h)}\partial_t)_b = -a\Re(\mathfrak{J})_b.$$

Plugging in the identities of Lemma 7.3.2, we infer

$$\begin{aligned} A_4 &= -2\,{}^*\rho\frac{\Delta}{|q|^2} - 2(\underline{\eta}\wedge\eta)\frac{\Delta}{|q|^2} - a tr\,\chi\big(\Re(\mathfrak{J})\wedge\underline{\eta}\big) + a\,{}^{(a)}\mathrm{tr}\chi\big(\underline{\eta}\cdot\Re(\mathfrak{J})\big),\\ A_3 &= 2\,{}^*\rho + 2(\underline{\eta}\wedge\eta) - a\mathrm{tr}\,\underline{\chi}\big(\Re(\mathfrak{J})\wedge\eta\big) + a\,{}^{(a)}\mathrm{tr}\underline{\chi}\big(\eta\cdot\Re(\mathfrak{J})\big),\\ A_e &= \frac{1}{2}\Big(-\mathrm{tr}\,\underline{\chi}\,{}^*\eta_e + {}^{(a)}\mathrm{tr}\underline{\chi}\eta_e\Big)\frac{\Delta}{|q|^2} + \frac{1}{2}\Big(-\mathrm{tr}\,\chi\,{}^*\underline{\eta}_e + {}^{(a)}\mathrm{tr}\chi\underline{\eta}_e\Big)\\ &\quad + \frac{a}{2}\Big(4\rho + \mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} + {}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi}\Big)\,{}^*\Re(\mathfrak{J})_e. \end{aligned}$$

Next, we rewrite $2A_e$ as

$$\begin{aligned} 2A_e &= \Big(-\Re(\mathrm{tr}\underline{X})\Im(H_e) - \Im(\mathrm{tr}\underline{X})\Re(H_e)\Big)\frac{\Delta}{|q|^2} + \Big(-\Re(\mathrm{tr}X)\Im(\underline{H}_e) - \Im(\mathrm{tr}X)\Re(\underline{H}_e)\Big)\\ &\quad + a\Big(4\Re(P) + \Re(\mathrm{tr}X)\Re(\mathrm{tr}\underline{X}) + \Im(\mathrm{tr}X)\Im(\mathrm{tr}\underline{X})\Big)\,{}^*\Re(\mathfrak{J})_e\\ &= -\Im(\mathrm{tr}\underline{X}H_e)\frac{\Delta}{|q|^2} - \Im(\mathrm{tr}X\,\underline{H}_e) + a\Re\big(4P + \mathrm{tr}X\overline{\mathrm{tr}\underline{X}}\big)\,{}^*\Re(\mathfrak{J})_e\\ &= -\Im\left(-\frac{2}{\overline{q}}\frac{aq}{|q|^2}\mathfrak{J}_e\right)\frac{\Delta}{|q|^2} - \Im\left(\frac{2}{q}\left(-\frac{\Delta}{|q|^2}\frac{a\overline{q}}{|q|^2}\mathfrak{J}_e\right)\right) + a\Re\left(-\frac{8m}{q^3} - \frac{\Delta}{|q|^2}\frac{4}{q^2}\right)\,{}^*\Re(\mathfrak{J})_e\\ &= \frac{2a\Delta}{|q|^2}\left(\Im\left(\left(\frac{1}{\overline{q}^2} + \frac{1}{q^2}\right)\mathfrak{J}_e\right) - \Re\left(\frac{2}{q^2}\right)\,{}^*\Re(\mathfrak{J})_e\right) - a\Re\left(\frac{8m}{q^3}\right)\,{}^*\Re(\mathfrak{J})_e \end{aligned}$$

and hence

$$\begin{aligned} 2A_e &= \frac{2a\Delta}{|q|^2}\left(\Re\left(\frac{1}{\overline{q}^2} + \frac{1}{q^2}\right)\Im(\mathfrak{J}_e) - \Re\left(\frac{2}{q^2}\right)\,{}^*\Re(\mathfrak{J})_e\right) - a\Re\left(\frac{8m}{q^3}\right)\,{}^*\Re(\mathfrak{J}_e)\\ &= \frac{4a\Delta}{|q|^2}\left(\Re\left(\frac{1}{q^2}\right)\Im(\mathfrak{J}_e) - \Re\left(\frac{1}{q^2}\right)\,{}^*\Re(\mathfrak{J}_e)\right) - a\Re\left(\frac{8m}{q^3}\right)\,{}^*\Re(\mathfrak{J})_e\\ &= -a\Re\left(\frac{8m}{q^3}\right)\,{}^*\Re(\mathfrak{J})_e. \end{aligned}$$

Since

$$\nabla(\cos\theta) = \Re(i\mathfrak{J}) = -\,{}^*\Re(\mathfrak{J}),$$

we infer

$$\begin{aligned} A_e &= a\Re\left(\frac{4m}{q^3}\right)\nabla_e(\cos\theta) = \Re\left(\frac{4m}{q^3}\nabla_e(a\cos\theta)\right) \\ &= \Im\left(\frac{4m}{q^3}\nabla_e(ia\cos\theta)\right) = \Im\left(\frac{4m}{q^3}\nabla_e(q)\right) \end{aligned}$$

and thus

$$A_e = -\nabla_e\left(\Im\left(\frac{2m}{q^2}\right)\right).$$

Next, we rewrite A_4 as

$$\begin{aligned} A_4 &= -2\ {}^*\rho\frac{\Delta}{|q|^2} - 2(\underline{\eta}\cdot\ {}^*\eta)\frac{\Delta}{|q|^2} - a tr\,\chi\big(\Re(\mathfrak{J})\cdot\ {}^*\underline{\eta}\big) + a\,{}^{(a)}\mathrm{tr}\chi\big(\underline{\eta}\cdot\Re(\mathfrak{J})\big) \\ &= -2\ {}^*\rho\frac{\Delta}{|q|^2} - 2(\Re(\underline{H})\cdot\Im(H))\frac{\Delta}{|q|^2} - a\Re(\mathrm{tr}X)\big(\Re(\mathfrak{J})\cdot\Im(\underline{H})\big) - a\Im(\mathrm{tr}X)\big(\Re(\underline{H})\cdot\Re(\mathfrak{J})\big) \\ &= -2\ {}^*\rho\frac{\Delta}{|q|^2} - 2(\Re(\underline{H})\cdot\Im(H))\frac{\Delta}{|q|^2} - a\Big(\Re(\mathrm{tr}X)\Im(\underline{H}) + \Im(\mathrm{tr}X)\Re(\underline{H})\Big)\cdot\Re(\mathfrak{J}) \\ &= -2\ {}^*\rho\frac{\Delta}{|q|^2} - 2(\Re(\underline{H})\cdot\Im(H))\frac{\Delta}{|q|^2} - a\Im(\mathrm{tr}X\,\underline{H})\cdot\Re(\mathfrak{J}) \end{aligned}$$

and hence, since $\Im(\mathfrak{J}) = {}^*\Re(\mathfrak{J})$,

$$\begin{aligned} A_4 &= -2\ {}^*\rho\frac{\Delta}{|q|^2} + 2a^2\frac{\Delta}{|q|^6}\Re(\overline{q}\mathfrak{J})\cdot\Im(q\mathfrak{J}) - a\Im\left(\frac{2}{q}\frac{\Delta}{|q|^2}\left(-\frac{a\overline{q}}{|q|^2}\mathfrak{J}\right)\right)\cdot\Re(\mathfrak{J}) \\ &= -2\ {}^*\rho\frac{\Delta}{|q|^2} + 2a^2\frac{\Delta}{|q|^6}\Big(\Re(q)\Re(\mathfrak{J}) + \Im(q)\Im(\mathfrak{J})\Big)\cdot\Big(\Im(q)\Re(\mathfrak{J}) + \Re(q)\Im(\mathfrak{J})\Big) \\ &\quad +2a^2\frac{\Delta}{|q|^6}\Im(\overline{q}^2\mathfrak{J})\cdot\Re(\mathfrak{J}) \\ &= -2\ {}^*\rho\frac{\Delta}{|q|^2} + 4a^2\frac{\Delta}{|q|^6}\Re(q)\Im(q)|\Re(\mathfrak{J})|^2 + 2a^2\frac{\Delta}{|q|^6}\Im(\overline{q}^2)|\Re(\mathfrak{J})|^2 \\ &= -2\ {}^*\rho\frac{\Delta}{|q|^2} + 4a^2\frac{\Delta}{|q|^6}\Re(q)\Im(q)|\Re(\mathfrak{J})|^2 + 4a^2\frac{\Delta}{|q|^6}\Re(\overline{q})\Im(\overline{q})|\Re(\mathfrak{J})|^2 \\ &= -2\ {}^*\rho\frac{\Delta}{|q|^2}. \end{aligned}$$

Since $e_4(q) = \frac{\Delta}{|q|^2}$, this yields

$$\begin{aligned} A_4 &= 4\Im\left(\frac{m}{q^3}\right)\frac{\Delta}{|q|^2} = 4\Im\left(\frac{m}{q^3}\right)e_4(q) \\ &= -e_4\left(\Im\left(\frac{2m}{q^2}\right)\right) \end{aligned}$$

and similarly

$$A_3 \quad = \quad -e_3\left(\Im\left(\frac{2m}{q^2}\right)\right).$$

Thus, we have obtained

$$A_e = \nabla_e\left(\Im\left(\frac{2m}{q^2}\right)\right), \qquad A_4 = e_4\left(\Im\left(\frac{2m}{q^2}\right)\right), \qquad A_3 = e_3\left(\Im\left(\frac{2m}{q^2}\right)\right),$$

and hence

$$A_\mu = -\mathbf{D}_\mu\left(\Im\left(\frac{2m}{q^2}\right)\right)$$

as stated. This concludes the proof of Corollary 7.3.3. □

(7.3.3) and Corollary 7.3.3 imply

$$\begin{aligned}
\mathbf{D}^\mu \mathcal{P}_\mu[\widehat{T}_\delta, 0, 0] \quad = \quad & \frac{1}{2}\mathcal{Q}\cdot {}^{(\widehat{T}_\delta)}\pi - \mathbf{D}_\nu\left(\Im\left(\frac{m}{q^2}\right)\right)\ {}^*\psi\cdot\dot{\mathbf{D}}^\nu\psi + (\widehat{T}_\delta - \partial_t)^\mu \dot{\mathbf{D}}^\nu\psi^a \dot{\mathbf{R}}_{ab\nu\mu}\psi^b \\
& + \nabla_{\widehat{T}_\delta}\psi\cdot\big(\dot{\square}_2\psi - V\psi\big). \qquad (7.3.5)
\end{aligned}$$

Next, we modify the identity (7.3.5) to cancel the second term on the RHS. To this end, we consider the following modified current

$$\widetilde{\mathcal{P}}_\mu \quad := \quad \mathcal{P}_\mu[\widehat{T}_\delta, 0, 0] + \tilde{w}\ {}^*\psi\cdot\dot{\mathbf{D}}_\mu\psi, \qquad (7.3.6)$$

for a scalar function $\tilde{w} = \tilde{w}(r, \cos\theta)$ to be chosen below. Since

$$\begin{aligned}
\mathbf{D}^\mu\Big[\tilde{w}\ {}^*\psi\cdot\dot{\mathbf{D}}_\mu\psi\Big] &= \tilde{w}\ {}^*\psi\cdot\dot{\mathbf{D}}^\mu\dot{\mathbf{D}}_\mu\psi + \tilde{w}\ {}^*\dot{\mathbf{D}}^\mu\psi\cdot\dot{\mathbf{D}}_\mu\psi + \mathbf{D}^\mu(\tilde{w})\ {}^*\psi\cdot\dot{\mathbf{D}}_\mu\psi \\
&= \tilde{w}\ {}^*\psi\cdot\dot{\square}_2\psi + \mathbf{D}^\mu(\tilde{w})\ {}^*\psi\cdot\dot{\mathbf{D}}_\mu\psi \\
&= \tilde{w}\ {}^*\psi\cdot\left(V\psi - \frac{4a\cos\theta}{|q|^2}\ {}^*\nabla_t\psi + N\right) + \mathbf{D}^\mu(\tilde{w})\ {}^*\psi\cdot\dot{\mathbf{D}}_\mu\psi \\
&= -\tilde{w}\frac{4a\cos\theta}{|q|^2}\nabla_t(|\ {}^*\psi|^2) + \tilde{w}\ {}^*\psi\cdot N + \mathbf{D}^\mu(\tilde{w})\ {}^*\psi\cdot\dot{\mathbf{D}}_\mu\psi \\
&= \mathbf{D}^\mu(\tilde{w})\ {}^*\psi\cdot\dot{\mathbf{D}}_\mu\psi - \mathbf{D}_\mu\left(\partial_t^\mu\tilde{w}\frac{4a\cos\theta}{|q|^2}|\psi|^2\right) + \tilde{w}\ {}^*\psi\cdot N
\end{aligned}$$

we infer

$$\begin{aligned}
\mathcal{D}^\mu\left(\widetilde{\mathcal{P}}_\mu + \partial_t^\mu\tilde{w}\frac{4a\cos\theta}{|q|^2}|\psi|^2\right) =& \frac{1}{2}\mathcal{Q}\cdot {}^{(\widehat{T}_\delta)}\pi + \mathbf{D}_\nu\left(-\Im\left(\frac{m}{q^2}\right) + \tilde{w}\right)\ {}^*\psi\cdot\dot{\mathbf{D}}^\nu\psi \\
& + (\widehat{T}_\delta - \partial_t)^\mu\dot{\mathbf{D}}^\nu\psi^a\dot{\mathbf{R}}_{ab\nu\mu}\psi^b \\
& + \nabla_{\widehat{T}_\delta}\psi\cdot\big(\dot{\square}_2\psi - V\psi\big) + \tilde{w}\ {}^*\psi\cdot N.
\end{aligned}$$

Next, we make the following choice for $\tilde{w}$

$$\tilde{w} := \Im\left(\frac{m}{q^2}\right) = -\frac{2amr\cos\theta}{|q|^4} \tag{7.3.7}$$

which yields

$$\begin{aligned}\mathcal{D}^\mu\left(\widetilde{\mathcal{P}}_\mu + \partial_t^\mu \tilde{w}\frac{4a\cos\theta}{|q|^2}|\psi|^2\right) =& \frac{1}{2}\mathcal{Q}\cdot {}^{(\widehat{T}_\delta)}\pi + (\widehat{T}_\delta - \partial_t)^\mu \dot{\mathbf{D}}^\nu\psi^a\dot{\mathbf{R}}_{ab\nu\mu}\psi^b \\ & + \nabla_{\widehat{T}_\delta}\psi\cdot\left(\dot{\square}_2\psi - V\psi\right) + \tilde{w}\ {}^*\psi\cdot N.\end{aligned} \tag{7.3.8}$$

Using (7.1.6), we have

$$\begin{aligned}\left|\mathcal{Q}\cdot {}^{(\widehat{T}_\delta)}\pi\right| &\lesssim \left(\frac{4|a|r}{|q|^2(r^2+a^2)}|\chi_0| + O(|a|\delta^{-1})|\chi_0'|\right)|\nabla_\phi\psi||\nabla_{\widehat{R}}\psi| \\ &\lesssim \mathbb{1}_{\mathcal{M}_{trap\!\!\!/}}\delta^{-1}\frac{|a|}{r^3}|\nabla_\phi\psi||\nabla_{\widehat{R}}\psi|.\end{aligned}$$

Also, we have $\widehat{T}_\delta - \partial_t = \chi_\delta(r)\partial_\phi$ and hence

$$\begin{aligned}& (\widehat{T}_\delta - \partial_t)^\mu \dot{\mathbf{D}}^\nu\psi^a\dot{\mathbf{R}}_{ab\nu\mu}\psi^b \\ =\ & \frac{1}{2}\chi_\delta(r)\partial_\phi^\mu \in^{ab}\dot{\mathbf{R}}_{ab\nu\mu}\ {}^*\psi\cdot\dot{\mathbf{D}}^\nu\psi \\ =\ & \frac{1}{2}\chi_\delta(r)\left(-\frac{1}{2}\partial_\phi^\mu \in^{ab}\dot{\mathbf{R}}_{ab3\mu}\ {}^*\psi\cdot\nabla_4\psi - \frac{1}{2}\partial_\phi^\mu \in^{ab}\dot{\mathbf{R}}_{ab4\mu}\ {}^*\psi\cdot\nabla_3\psi + \partial_\phi^\mu \in^{ab}\dot{\mathbf{R}}_{abc\mu}\ {}^*\psi\cdot\nabla^c\psi\right).\end{aligned}$$

Since we have

$$\partial_\phi^3 = O(ar^{-2}\Delta), \qquad \partial_\phi^4 = O(a), \qquad \partial_\phi^c = O(r^2)\Re(\mathfrak{J})^c,$$

we infer, together with the definition (2.1.13) of $\dot{\mathbf{R}}$, and Proposition 2.2.4,

$$\begin{aligned}\partial_\phi^\mu \in^{ab}\dot{\mathbf{R}}_{ab3\mu} &= \partial_\phi^4 \in^{ab}\dot{\mathbf{R}}_{ab34} + \partial_\phi^c \in^{ab}\dot{\mathbf{R}}_{ab3c} = O(a)\left({}^*\rho, \eta\underline{\eta}\right) + O(r)\underline{\chi}\eta \\ &= O(ar^{-2}), \\ \partial_\phi^\mu \in^{ab}\dot{\mathbf{R}}_{ab4\mu} &= \partial_\phi^3 \in^{ab}\dot{\mathbf{R}}_{ab43} + \partial_\phi^c \in^{ab}\dot{\mathbf{R}}_{ab4c} = O(ar^{-2}\Delta)\left({}^*\rho, \eta\underline{\eta}\right) + O(r)\chi\underline{\eta} \\ &= O(ar^{-4}\Delta), \\ \partial_\phi^\mu \in^{ab}\dot{\mathbf{R}}_{abc\mu} &= \partial_\phi^4 \in^{ab}\dot{\mathbf{R}}_{abc4} + \partial_\phi^3 \in^{ab}\dot{\mathbf{R}}_{abc3} + \partial_\phi^d \in^{ab}\dot{\mathbf{R}}_{abcd} \\ &= O(a)\chi\underline{\eta} + O(ar^{-2}\Delta)\underline{\chi}\eta + O(r)(\rho, \chi\underline{\chi}) \\ &= O(r^{-1}),\end{aligned}$$

which implies, since $\chi_\delta = \frac{a}{r^2+a^2}\chi_0\Big(\delta^{-1}\frac{\mathcal{T}}{r^3}\Big)$,

$$\begin{aligned}
&\Big|(\widehat{T}_\delta - \partial_t)^\mu \dot{\mathbf{D}}^\nu \psi^a \dot{\mathbf{R}}_{ab\nu\mu}\psi^b\Big| \\
\lesssim\ & \frac{|a|}{r^2}\mathbb{1}_{\mathcal{M}_{trap\!\!\!/}}\Big(|\partial_\phi^\mu \in^{ab} \dot{\mathbf{R}}_{ab3\mu}||\ {}^*\psi\cdot\nabla_4\psi| + |\partial_\phi^\mu \in^{ab} \dot{\mathbf{R}}_{ab4\mu}||\ {}^*\psi\cdot\nabla_3\psi| \\
&+|\partial_\phi^\mu \in^{ab} \dot{\mathbf{R}}_{abc\mu}||\ {}^*\psi\cdot\nabla^c\psi|\Big) \\
\lesssim\ & \frac{|a|}{r^3}\mathbb{1}_{\mathcal{M}_{trap\!\!\!/}}\Big(|\nabla_4\psi| + r^{-2}|\Delta||\nabla_3\psi| + |\nabla\psi|\Big)|\psi| \\
\lesssim\ & \frac{|a|}{r^3}\mathbb{1}_{\mathcal{M}_{trap\!\!\!/}}\Big(|\nabla_{\widehat{R}}\psi| + |\nabla_T\psi| + |\nabla\psi|\Big)|\psi|.
\end{aligned}$$

Finally, using equation (6.1.1), we have

$$\begin{aligned}
\nabla_{\widehat{T}_\delta}\psi\cdot\big(\dot{\square}_2\psi - V\psi\big) &= -\frac{4a\cos\theta}{|q|^2}\nabla_{\widehat{T}_\delta}\psi\cdot\ {}^*\nabla_T\psi + \nabla_{\widehat{T}_\delta}\psi\cdot N \\
&= -\frac{4a\cos\theta}{|q|^2}(\nabla_T\psi + \chi_\delta\nabla_\phi\psi)\cdot\ {}^*\nabla_T\psi + \nabla_{\widehat{T}_\delta}\psi\cdot N \\
&= -\frac{4a\cos\theta}{|q|^2}\chi_\delta\nabla_\phi\psi\cdot\ {}^*\nabla_T\psi + \nabla_{\widehat{T}_\delta}\psi\cdot N
\end{aligned}$$

where we have the crucial cancellation $\nabla_T\psi\cdot\ {}^*\nabla_T\psi = 0$.

We summarize the result in the following.

Lemma 7.3.4. *Consider the modified current*

$$\widetilde{\mathcal{P}}_\mu = \mathcal{P}_\mu[\widehat{T}_\delta, 0, 0] + \tilde{w}\ {}^*\psi\cdot\dot{\mathbf{D}}_\mu\psi, \qquad \tilde{w} = \Im\left(\frac{m}{q^2}\right),$$

where the vectorfield $\widehat{T}_\delta$ is given by $\widehat{T}_\delta = \partial_t + \chi_\delta(r)\partial_\phi$ for $\delta = \frac{1}{10}$ and $|a|/m \ll 1$ small enough. Then, for all $r \geq r_+$,

$$\begin{aligned}
&\left|\mathbf{D}^\mu\left(\widetilde{\mathcal{P}}_\mu + \partial_t^\mu\tilde{w}\frac{4a\cos\theta}{|q|^2}|\psi|^2\right) - \Big(\nabla_{\widehat{T}_\delta}\psi + \tilde{w}\ {}^*\psi\Big)\cdot N\right| \\
\lesssim\ & \mathbb{1}_{\mathcal{M}_{trap\!\!\!/}}\left(\delta^{-1}\frac{|a|}{r^3}|\nabla_{\widehat{R}}\psi||\nabla_\phi\psi| + \frac{|a|}{r^4}|\nabla_T\psi||\nabla_\phi\psi| + \frac{|a|}{r^3}\Big[|\nabla_{\widehat{R}}\psi| + |\nabla_T\psi| + |\nabla\psi|\Big]|\psi|\right).
\end{aligned}$$

Integrating the above inequality on $\mathcal{M} = \mathcal{M}(\tau_1, \tau_2)$ and applying the divergence theorem we deduce, in view of the definition of $\text{Mor}^{ax}_{deg}[\psi](\tau_1,\tau_2)$,

$$\begin{aligned}
&\int_{\Sigma(\tau_2)}\left(\widetilde{\mathcal{P}}_\mu+\partial_t^\mu\tilde{w}\frac{4a\cos\theta}{|q|^2}|\psi|^2\right)\cdot N_\Sigma+\int_{\Sigma_*(\tau_1,\tau_2)}\left(\widetilde{\mathcal{P}}_\mu+\partial_t^\mu\tilde{w}\frac{4a\cos\theta}{|q|^2}|\psi|^2\right)\cdot N_{\Sigma_*}\\
&+\int_{\mathcal{A}(\tau_1,\tau_2)}\left(\widetilde{\mathcal{P}}_\mu+\partial_t^\mu\tilde{w}\frac{4a\cos\theta}{|q|^2}|\psi|^2\right)\cdot N_{\mathcal{A}}\\
\lesssim\ &\int_{\Sigma(\tau_1)}\left(\widetilde{\mathcal{P}}_\mu+\partial_t^\mu\tilde{w}\frac{4a\cos\theta}{|q|^2}|\psi|^2\right)\cdot N_\Sigma+\frac{|a|}{m}\text{Mor}^{ax}_{deg}[\psi](\tau_1,\tau_2)+\left|\int_{\mathcal{M}(\tau_1,\tau_2)}\nabla_{\widehat{T}_\delta}\psi\cdot N\right|\\
&+\int_{\mathcal{M}(\tau_1,\tau_2)}|N|^2
\end{aligned}$$

where we used the fact that $|\tilde{w}|\lesssim amr^{-3}$ to control the term $\tilde{w}\ {}^*\psi\cdot N$.

Finally, since $|\tilde{w}|\lesssim amr^{-3}$, we have for a vectorfield N

$$\begin{aligned}
\left(\widetilde{\mathcal{P}}_\mu+\partial_t^\mu\tilde{w}\frac{4a\cos\theta}{|q|^2}|\psi|^2\right)\cdot N &= \mathcal{Q}(\widehat{T}_\delta,N)+\tilde{w}\ {}^*\psi\cdot\nabla_N\psi+\mathbf{g}(\partial_t,N)\tilde{w}\frac{4a\cos\theta}{|q|^2}|\psi|^2\\
&= \mathcal{Q}(\widehat{T}_\delta,N)-O(amr^{-3})|\psi||\nabla_N\psi|\\
&\quad -O(a^2mr^{-5})|\mathbf{g}(\partial_t,N)||\psi|^2.
\end{aligned}\tag{7.3.9}$$

Also, we have, in view of Lemma 7.3.1 and the definition of $E_{deg}[\psi]$, for some constant $c_0>0$

$$\begin{aligned}
\int_{\Sigma(\tau)}\mathcal{Q}(\widehat{T},N_\Sigma) &\geq c_0E_{deg}[\psi](\tau)-O(\delta_{\mathcal{H}})E_{r\leq r_+}[\psi](\tau),\\
\int_{\Sigma_*(\tau_1,\tau_2)}\mathcal{Q}(\widehat{T},N_{\Sigma_*}) &\geq c_0F_{\Sigma_*}[\psi](\tau_1,\tau_2),\\
\int_{\mathcal{A}(\tau_1,\tau_2)}\mathcal{Q}(\widehat{T},N_{\mathcal{A}}) &\gtrsim -\delta_{\mathcal{H}}F_{\mathcal{A}}[\psi](\tau_1,\tau_2).
\end{aligned}$$

Since we have, in view of the definition of $\widehat{T}_\delta$,

$$\widehat{T}_\delta \ = \ \widehat{T}+\frac{a}{r^2+a^2}(\chi_0-1)\left(\delta^{-1}\frac{\mathcal{T}}{r^3}\right)\partial_\phi,$$

we have $\widehat{T}_\delta=\widehat{T}$ on Σ_* and $\mathcal{A}$, and hence

$$\begin{aligned}
\int_{\Sigma_*(\tau_1,\tau_2)}\mathcal{Q}(\widehat{T}_\delta,N_{\Sigma_*}) &\geq c_0F_{\Sigma_*}[\psi](\tau_1,\tau_2),\\
\int_{\mathcal{A}(\tau_1,\tau_2)}\mathcal{Q}(\widehat{T}_\delta,N_{\mathcal{A}}) &\gtrsim -\delta_{\mathcal{H}}F_{\mathcal{A}}[\psi](\tau_1,\tau_2).
\end{aligned}$$

In view of (7.3.9), we deduce for $|a| \ll m, m\delta_{\mathcal{H}}$,

$$\int_{\Sigma_*(\tau_1,\tau_2)} \left(\widetilde{\mathcal{P}}_\mu + \partial_t^\mu \tilde{w}\frac{4a\cos\theta}{|q|^2}|\psi|^2\right)\cdot N_{\Sigma_*} \geq c_0 F_{\Sigma_*}[\psi](\tau_1,\tau_2),$$
$$\int_{\mathcal{A}(\tau_1,\tau_2)} \left(\widetilde{\mathcal{P}}_\mu + \partial_t^\mu \tilde{w}\frac{4a\cos\theta}{|q|^2}|\psi|^2\right)\cdot N_{\mathcal{A}} \gtrsim -\delta_{\mathcal{H}} F_{\mathcal{A}}[\psi](\tau_1,\tau_2).$$

Also, we have

$$\mathcal{P}[\widehat{T}_\delta]\cdot N_\Sigma = \mathcal{Q}(\widehat{T}, N_\Sigma) + \frac{a}{r^2+a^2}(\chi_0 - 1)\left(\delta^{-1}\frac{\mathcal{T}}{r^3}\right)\mathcal{Q}(\partial_\phi, N_\Sigma).$$

In view of the above and the properties of χ_0, this yields

$$\int_{\Sigma(\tau)} \mathcal{P}[\widehat{T}_\delta]\cdot N_\Sigma \geq c_0 E_{deg}[\psi](\tau) - O(a)\int_{\Sigma_+(\tau)} |\mathcal{Q}(\partial_\phi, N_\Sigma)| \mathbb{1}_{\frac{|\mathcal{T}|}{r^3}\leq 2\delta} - O(\delta_{\mathcal{H}}) E_{r\leq r_+}[\psi](\tau).$$

Since the set $\frac{|\mathcal{T}|}{r^3} \leq 2\delta$ is localized near $r = 3m$ for $|a| \ll m$ and $\delta > 0$ small enough, we infer, using also (7.3.9),

$$\int_{\Sigma(\tau)} \left(\widetilde{\mathcal{P}}_\mu + \partial_t^\mu \tilde{w}\frac{4a\cos\theta}{|q|^2}|\psi|^2\right)\cdot N_\Sigma \geq \frac{c_0}{2} E_{deg}[\psi](\tau) - O(\delta_{\mathcal{H}}) E_{r\leq r_+}[\psi](\tau),$$

and hence

$$\begin{aligned} E_{deg}[\psi](\tau_2) + F_{\Sigma_*}[\psi](\tau_1,\tau_2) \lesssim\; & E_{deg}[\psi](\tau_1) + \delta_{\mathcal{H}}\big(E_{r\leq r_+(1+\delta_{\mathcal{H}})}(\tau_2)[\psi] + F_{\mathcal{A}}[\psi](\tau_1,\tau_2)\big) \\ & + \frac{|a|}{m}\mathrm{Mor}^{ax}_{deg}[\psi](\tau_1,\tau_2) + \left|\int_{\mathcal{M}(\tau_1,\tau_2)} \nabla_{\widehat{T}_\delta}\psi\cdot N\right| \\ & + \int_{\mathcal{M}(\tau_1,\tau_2)} |N|^2. \end{aligned}$$

This concludes the proof of Proposition 6.3.9.

7.4 Conditional control of Energy–Morawetz estimate for r large enough

In this section, we prove the following conditional control of Energy–Morawetz estimate in $\mathcal{M}(r \geq r_1)$ for r_1 large enough.

Proposition 7.4.1. *Let ψ a solution to the gRW equation* (6.1.1)*. Also, recall the norms $E[\psi]$ and $Mor[\psi]$ introduced in section 6.1.5. Then, for $r_1 = r_1(m)$ large enough, we have*

$$\begin{aligned} Mor_{r\geq 2r_1}[\psi](\tau_1,\tau_2) \lesssim & \sup_{\tau\in[\tau_1,\tau_2]} E_{r\geq r_1}[\psi](\tau) + r_1 Mor_{r_1\leq r\leq 2r_1}[\psi](\tau_1,\tau_2) \\ & + \mathcal{N}_{r\geq r_1}[\psi, N](\tau_1,\tau_2), \end{aligned} \tag{7.4.1}$$

and

$$\begin{aligned} \sup_{\tau\in[\tau_1,\tau_2]} E_{r\geq 2r_1}[\psi](\tau) \lesssim & E_{r\geq r_1}[\psi](\tau_1) + \mathcal{N}_{r\geq r_1}[\psi, N](\tau_1,\tau_2) + r_1 Mor_{r_1\leq r\leq 2r_1}[\psi](\tau_1,\tau_2) \\ & + \frac{|a|}{r_1} Mor_{r\geq 2r_1}[\psi](\tau_1,\tau_2). \end{aligned} \tag{7.4.2}$$

Proof. Let a smooth cut-off function χ such that $\chi = 0$ for $r \leq 1$ and $\chi = 1$ for $r \geq 2$, and let χ_{r_1} and ψ_{r_1} defined by

$$\chi_{r_1}(r) := \left(\frac{r}{r_1}\right), \qquad \psi_{r_1} := \chi_{r_1}\psi,$$

so that the support of ψ_{r_1} is included in $r \geq r_1$ and $\psi_{r_1} = \psi$ for $r \geq 2r_1$. Then, since ψ satisfies (6.1.1), ψ_{r_1} satisfies the following gRW equation

$$\begin{aligned} \dot{\square}_2\psi_{r_1} - V\psi_{r_1} &= -\frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_T\psi_{r_1} + N_{r_1}, \\ N_{r_1} &:= \mathbf{g}^{rr}\chi'_{r_1}(r)\nabla_r\psi + \square(\chi_{r_1})\psi + \chi\left(\frac{r}{r_1}\right)N. \end{aligned} \tag{7.4.3}$$

We start by deriving a Morawetz estimate for ψ_{r_1}. According to Proposition 7.1.5, with the choice $M = 0$, the generalized current associated to the Morawetz vectorfield in (7.1.10) is given by

$$|q|^2\mathcal{E}_{r_1}[X, w, M = 0] \quad = \quad \mathcal{A}|\nabla_r\psi_{r_1}|^2 + P[\psi_{r_1}] + \mathcal{V}|\psi_{r_1}|^2.$$

Here we consider the choices for z, f, h made in Proposition 7.1.8 so that the coefficients $\mathcal{A}$ and $\mathcal{V}$ are given by (7.1.27) and (7.1.28) respectively, and the principal term $P[\psi_{r_1}]$ is given by

$$P \quad = \quad \frac{\mathcal{T}}{r}\frac{r^2+a^2}{r^2-a^2}\left(\frac{2\mathcal{T}}{(r^2+a^2)^3}|q|^2|\nabla\psi|^2 - \frac{4ar}{(r^2+a^2)^2}\nabla_{\widehat{T}}\psi\cdot\nabla_\phi\psi\right).$$

Note that we have for $r \geq r_1$

$$\begin{aligned} &\mathcal{A} = 6m(1+O(mr^{-1})), \qquad \mathcal{V} = \frac{8}{r}(1+O(mr^{-1})), \\ &P = 2r(1+O(mr^{-1}))|\nabla\psi_{r_1}|^2 + O(m)|\nabla_{\widehat{T}}\psi_{r_1}||\nabla\psi_{r_1}| + O(m^2r^{-1})|\nabla_{\widehat{T}}\psi_{r_1}|^2, \end{aligned}$$

where we used in particular (7.1.27) and (7.1.28). This implies, for $r \geq r_1$ with $r_1 = r_1(m)$ large enough,

$$\mathcal{E}_{r_1}[X, w, M=0] \;\; \geq \;\; \frac{m}{r^2}|\nabla_r\psi_{r_1}|^2 + \frac{1}{r}|\nabla\psi_{r_1}|^2 + \frac{1}{r^3}|\psi_{r_1}|^2 - O(m^2r^{-3})|\nabla_{\widehat{T}}\psi_{r_1}|^2.$$

Also, since we have for $r \geq r_1$

$$z = \frac{1}{r^2}(1+O(mr^{-1})), \qquad f = -\frac{2}{r^3}(1+O(mr^{-1})), \qquad h = r^5(1+O(a^2r^{-2})),$$

we easily obtain the following simpler analogs of the estimates of Lemma 7.2.2

$$\begin{aligned}
&\left(\nabla_X\psi_{r_1} + \frac{1}{2}w\psi_{r_1}\right)\cdot(\dot{\square}_2\psi_{r_1} - V\psi_{r_1})\\
=\;& O(ar^{-3})|\psi_{r_1}||\nabla_T\psi_{r_1}| + O(a^2mr^{-6})|\psi_{r_1}|^2 + O(1)(|\nabla_{\widehat{R}}\psi_{r_1}| + r^{-1}|\psi_{r_1}|)|N_{r_1}|\\
&+\mathbf{D}_\mu\left(\frac{2a\cos\theta}{|q|^2}(\partial_r)^\mu zhf\psi_{r_1}\cdot\nabla_T\,{}^*\psi_{r_1}\right) - \partial_t\left(\frac{2a\cos\theta}{|q|^2}zhf\psi_{r_1}\cdot\nabla_r\,{}^*\psi_{r_1}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\left|\left((\,{}^*\rho + \underline{\eta}\wedge\eta)\frac{r^2+a^2}{\Delta} + \frac{2a^3r\cos\theta(\sin\theta)^2}{|q|^6}\right)\mathcal{F}\nabla_{\widehat{T}}\psi_{r_1}\cdot\,{}^*\psi_{r_1}\right|\\
&+\left|\frac{2a^2r\cos\theta\mathcal{F}(r)}{(r^2+a^2)|q|^4}\nabla_\phi\psi_{r_1}\cdot\,{}^*\psi_{r_1}\right|\\
\leq\;& \frac{|a|m}{r^4}|\nabla_{\widehat{T}}\psi_{r_1}||\psi_{r_1}| + \frac{a^2}{r^5}|\nabla_\phi\psi_{r_1}||\psi_{r_1}|.
\end{aligned}$$

In view of the above, and since ψ_{r_1} is supported in $r \geq r_1$, we obtain from (7.1.9), for $r_1 = r_1(m)$ large enough,

$$\begin{aligned}
&\mathbf{D}^\mu(\mathcal{P}_{r_1})_\mu[X, w]\\
=\;& \mathcal{E}_{r_1}[X, w] + \left(\nabla_X\psi_{r_1} + \frac{1}{2}w\psi_{r_1}\right)\cdot(\dot{\square}_2\psi_{r_1} - V\psi_{r_1}) - \frac{2a^2r\cos\theta\mathcal{F}(r)}{(r^2+a^2)|q|^4}\nabla_\phi\psi_{r_1}\cdot\,{}^*\psi_{r_1}\\
&-\left((\,{}^*\rho + \underline{\eta}\wedge\eta)\frac{r^2+a^2}{\Delta} + \frac{2a^3r\cos\theta(\sin\theta)^2}{|q|^6}\right)\mathcal{F}\nabla_{\widehat{T}}\psi_{r_1}\cdot\,{}^*\psi_{r_1}\\
\geq\;& \frac{m}{r^2}|\nabla_r\psi_{r_1}|^2 + \frac{1}{r}|\nabla\psi_{r_1}|^2 + \frac{1}{2r^3}|\psi_{r_1}|^2 - O(m^2r^{-3})|\nabla_{\widehat{T}}\psi_{r_1}|^2 - O(1)\Big(|\nabla_{\widehat{R}}\psi_{r_1}| + r^{-1}|\psi_{r_1}|\Big)|N_{r_1}|\\
&+\mathbf{D}_\mu\left(\frac{2a\cos\theta}{|q|^2}(\partial_r)^\mu zhf\psi_{r_1}\cdot\nabla_T\,{}^*\psi_{r_1}\right) - \partial_t\left(\frac{2a\cos\theta}{|q|^2}zhf\psi_{r_1}\cdot\nabla_r\,{}^*\psi_{r_1}\right).
\end{aligned}$$

By applying the divergence theorem, and using that ψ_{r_1} is supported in $r \geq r_1$, we then easily infer

$$\int_{\mathcal{M}_+} \frac{m}{r^2}|\nabla_{\widehat{R}}\psi_{r_1}|^2 + r^{-3}|\psi_{r_1}|^2 + r^{-1}|\nabla\psi_{r_1}|^2$$
$$\lesssim \sup_{\tau\in[\tau_1,\tau_2]} E_{r\geq r_1}[\psi](\tau) + \int_{\mathcal{M}_+} \frac{m^2}{r^3}|\nabla_{\widehat{T}}\psi_{r_1}|^2 + \int_{\mathcal{M}_+}\Big(|\nabla_{\widehat{R}}\psi_{r_1}| + r^{-1}|\psi_{r_1}|\Big)|N_{r_1}|.$$

Also, given the form of N_{r_1} and ψ_{r_1}, we deduce

$$\int_{\mathcal{M}_+} \frac{m}{r^2}|\nabla_{\widehat{R}}\psi_{r_1}|^2 + r^{-3}|\psi_{r_1}|^2 + r^{-1}|\nabla\psi_{r_1}|^2$$
$$\lesssim \sup_{\tau\in[\tau_1,\tau_2]} E_{r\geq r_1}[\psi](\tau) + \int_{\mathcal{M}_+} \frac{m^2}{r^3}|\nabla_{\widehat{T}}\psi_{r_1}|^2 + \frac{r_1}{m}\mathrm{Mor}_{r_1\leq r\leq 2r_1}[\psi](\tau_1,\tau_2) + \mathcal{N}_{r\geq r_1}[\psi,N](\tau_1,\tau_2).$$

To complete the desired Morawetz estimate, we still need to recover the control of $\nabla_{\widehat{T}}\psi_{r_1}$. We compute

$$\begin{aligned}
\dot{\mathbf{D}}^\alpha\left(\frac{m}{r^2}\psi_{r_1}\cdot\dot{\mathbf{D}}_\alpha\psi_{r_1}\right) &= \frac{m}{r^2}\psi_{r_1}\cdot\dot{\square}_2\psi_{r_1} + \frac{m}{r^2}\dot{\mathbf{D}}^\alpha\psi_{r_1}\cdot\dot{\mathbf{D}}_\alpha\psi_{r_1} - \frac{2m}{r^3}\mathbf{g}^{rr}\psi_{r_1}\cdot\dot{\mathbf{D}}_r\psi_{r_1}\\
&= \frac{m}{r^2}\psi_{r_1}\cdot\left(V\psi_{r_1} - \frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_T\psi_{r_1} + N_{r_1}\right) + \frac{m}{r^2}\dot{\mathbf{D}}^\alpha\psi_{r_1}\cdot\dot{\mathbf{D}}_\alpha\psi_{r_1}\\
&\quad -\frac{2m}{r^3}\mathbf{g}^{rr}\psi_{r_1}\cdot\dot{\mathbf{D}}_r\psi_{r_1}.
\end{aligned}$$

Since ψ_{r_1} is supported in $r\geq r_1$, we deduce, for $r_1 = r_1(m)$ large enough,

$$\begin{aligned}
\dot{\mathbf{D}}^\alpha\left(\frac{m}{r^2}\psi_{r_1}\cdot\dot{\mathbf{D}}_\alpha\psi_{r_1}\right) &= -\frac{m}{r^2}(1+O(mr^{-1}))|\nabla_{\widehat{T}}\psi|^2\\
&\quad +O(1)\left(\frac{m}{r^2}|\nabla_{\widehat{R}}\psi_{r_1}|^2 + r^{-3}|\psi_{r_1}|^2 + r^{-1}|\nabla\psi_{r_1}|^2\right)\\
&\quad +O(mr^{-2})|\psi_{r_1}||N_{r_1}|.
\end{aligned}$$

Together with the above, we infer

$$\int_{\mathcal{M}_+} \frac{m}{r^2}|\nabla_{\widehat{R}}\psi_{r_1}|^2 + \frac{m}{r^2}|\nabla_{\widehat{T}}\psi_{r_1}|^2 + r^{-3}|\psi_{r_1}|^2 + r^{-1}|\nabla\psi_{r_1}|^2$$
$$\lesssim \sup_{\tau\in[\tau_1,\tau_2]} E_{r\geq r_1}[\psi](\tau) + \int_{\mathcal{M}_+} \frac{m^2}{r^3}|\nabla_{\widehat{T}}\psi_{r_1}|^2 + \frac{r_1}{m}\mathrm{Mor}_{r_1\leq r\leq 2r_1}[\psi](\tau_1,\tau_2) + \mathcal{N}_{r\geq r_1}[\psi,N](\tau_1,\tau_2).$$

Since ψ_{r_1} is supported in $r\geq r_1$, and since $\psi_{r_1}=\psi$ for $r\geq 2r_1$, we deduce, for $r_1=r_1(m)$ large enough,

$$\mathrm{Mor}_{r\geq 2r_1}[\psi](\tau_1,\tau_2) \lesssim \sup_{\tau\in[\tau_1,\tau_2]} E_{r\geq r_1}[\psi](\tau) + \frac{r_1}{m}\mathrm{Mor}_{r_1\leq r\leq 2r_1}[\psi](\tau_1,\tau_2) + \mathcal{N}_{r\geq r_1}[\psi,N](\tau_1,\tau_2)$$

as stated.

Finally, it remains to control the energy. In view of Lemma 7.3.4 applied to ψ_{r_1}, we have

$$\begin{aligned}
&\left|\mathbf{D}^\mu\left((\widetilde{\mathcal{P}_{r_1}})_\mu+\partial_t^\mu\tilde{w}\frac{4a\cos\theta}{|q|^2}|\psi_{r_1}|^2\right)-\left(\nabla_{\widehat{T}_\delta}\psi_{r_1}+\tilde{w}\ {}^*\psi_{r_1}\right)\cdot N_{r_1}\right|\\
\lesssim\ & \frac{|a|}{r^3}|\nabla_{\widehat{R}}\psi_{r_1}||\nabla_\phi\psi_{r_1}|+\frac{|a|}{r^4}|\nabla_T\psi_{r_1}||\nabla_\phi\psi_{r_1}|+\frac{|a|m}{r^4}\Big[|\nabla_{\widehat{R}}\psi_{r_1}|+|\nabla_T\psi_{r_1}|+|\nabla\psi_{r_1}|\Big]|\psi_{r_1}|
\end{aligned}$$

where

$$(\widetilde{\mathcal{P}_{r_1}})_\mu = (\mathcal{P}_\mu)_{r_1}[\widehat{T}_\delta,0,0]+\tilde{w}\ {}^*\psi_{r_1}\cdot\dot{\mathbf{D}}_\mu\psi_{r_1}, \qquad \tilde{w}=-\Im\left(\frac{m}{q^2}\right).$$

By applying the divergence theorem, since ψ_{r_1} is supported in $r\geq r_1$, and since $\psi_{r_1}=\psi$ for $r\geq 2r_1$, we easily infer

$$\begin{aligned}
\sup_{\tau\in[\tau_1,\tau_2]}E_{r\geq 2r_1}[\psi](\tau) \lesssim\ & E_{r\geq r_1}[\psi](\tau_1)+\mathcal{N}_{r\geq r_1}[\psi,N](\tau_1,\tau_2)\\
&+\frac{r_1}{m}\mathrm{Mor}_{r_1\leq r\leq 2r_1}[\psi](\tau_1,\tau_2)+\frac{|a|}{r_1}\mathrm{Mor}_{r\geq 2r_1}[\psi](\tau_1,\tau_2)
\end{aligned}$$

as stated. This concludes the proof of Proposition 7.4.1. □

Chapter 8

Proof of $\mathcal{S}$-derivative Morawetz estimates in Kerr

In this chapter we provide a complete proof of the $\mathcal{S}$-derivative Morawetz estimates as stated in Section 6.3.2. Recall that we are in Kerr throughout this chapter and that the results of this chapter will be extended to perturbations of Kerr in section 9.2.11.

8.1 Preliminaries

In this section we collect preliminary results to extend the vector field method to include commutation with the following second order differential operators, see Definition 3.7.8,

$$\mathcal{S}_1(\psi) = \nabla_T\nabla_T\psi, \qquad \mathcal{S}_2(\psi) = a\nabla_T\nabla_Z\psi, \qquad \mathcal{S}_3(\psi) = a^2\nabla_Z\nabla_Z\psi, \qquad \mathcal{S}_4\psi = \mathcal{O}(\psi),$$

which we denote $\mathcal{S}_{\underline{a}}$, for $\underline{a} = 1, 2, 3, 4$.

Lemma 8.1.1. *Given a $\mathfrak{s}_2$ tensor ψ solution of the equation* (6.1.1). *Then the commuted $\mathfrak{s}_2$ tensor $\psi_{\underline{a}} := \mathcal{S}_{\underline{a}}\psi$ satisfies*

$$\dot{\square}_2\psi_{\underline{a}} - V\psi_{\underline{a}} = -\frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_T\psi_{\underline{a}} + N_{\underline{a}}, \qquad \underline{a} = 1, 2, 3, 4, \tag{8.1.1}$$

where

$$|N_{\underline{a}}| \lesssim |\mathfrak{d}^{\leq 2}N| + ar^{-2}|\mathfrak{d}^{\leq 2}\psi|, \qquad \underline{a} = 1, 2, 3, 4.$$

Proof. Since ψ satisfies (6.1.1), i.e.

$$\dot{\Box}_2\psi - V\psi = -\frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_T\psi + N,$$

we infer

$$\begin{aligned}
\dot{\Box}_2\psi_{\underline{a}} - V\psi_{\underline{a}} &= -\frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_T\psi_{\underline{a}} + N_{\underline{a}},\\
N_{\underline{a}} &:= -[\mathcal{S}_{\underline{a}}, \dot{\Box}_2]\psi + [\mathcal{S}_{\underline{a}}, V]\psi - {}^*\left[\mathcal{S}_{\underline{a}}, \frac{4a\cos\theta}{|q|^2}\nabla_T\right]\psi + \mathfrak{d}^{\leq 2}N, \qquad \underline{a} = 1, 2, 3,\\
N_4 &:= -\frac{1}{|q|^2}[\mathcal{S}_4, |q|^2\dot{\Box}_2]\psi + \frac{1}{|q|^2}[\mathcal{S}_4, |q|^2 V]\psi - \frac{1}{|q|^2}\,{}^*[\mathcal{S}_4, 4a\cos\theta\nabla_T]\psi + \mathfrak{d}^{\leq 2}N,
\end{aligned}$$

and hence

$$\begin{aligned}
|N_{\underline{a}}| &\lesssim |\mathfrak{d}^{\leq 2}N| + |[\mathcal{S}_{\underline{a}}, \dot{\Box}_2]\psi| + |[\mathcal{S}_{\underline{a}}, V]\psi| + a\left|\left[\mathcal{S}_{\underline{a}}, \frac{\cos\theta}{|q|^2}\nabla_T\right]\psi\right|, \qquad \underline{a} = 1, 2, 3,\\
|N_4| &\lesssim |\mathfrak{d}^{\leq 2}N| + r^{-2}|[\mathcal{S}_4, |q|^2\dot{\Box}_2]\psi| + r^{-2}|[\mathcal{S}_4, |q|^2 V]\psi| + ar^{-2}|[\mathcal{S}_4, \cos\theta\nabla_T]\psi|.
\end{aligned}$$

Next, since

$$V = \frac{4\Delta}{(r^2+a^2)|q|^2}, \qquad |q|^2 V = \frac{4\Delta}{(r^2+a^2)},$$

see (6.1.1), we infer $[\mathcal{S}_{\underline{a}}, V]\psi = 0$, $\underline{a} = 1, 2, 3$, and $[\mathcal{S}_4, |q|^2 V]\psi = 0$, and hence

$$\begin{aligned}
|N_{\underline{a}}| &\lesssim |\mathfrak{d}^{\leq 2}N| + |[\mathcal{S}_{\underline{a}}, \dot{\Box}_2]\psi| + a\left|\left[\mathcal{S}_{\underline{a}}, \frac{\cos\theta}{|q|^2}\nabla_T\right]\psi\right|, \qquad \underline{a} = 1, 2, 3,\\
|N_4| &\lesssim |\mathfrak{d}^{\leq 2}N| + r^{-2}|[\mathcal{S}_4, |q|^2\dot{\Box}_2]\psi| + ar^{-2}|[\mathcal{S}_4, \cos\theta\nabla_T]\psi|.
\end{aligned}$$

Also, using Lemma 3.6.1, we have

$$\left[\mathcal{S}_{\underline{a}}, \frac{\cos\theta}{|q|^2}\nabla_T\right]\psi = 0, \quad \underline{a} = 1, 2, 3.$$

We also have

$$\begin{aligned}
r^{-2}|[\mathcal{S}_4, \cos\theta\nabla_T]\psi| &\lesssim r^{-1}|\mathfrak{d}^{\leq 1}\nabla(\cos\theta)||\mathfrak{d}^{\leq 1}\nabla_T\psi| + r^{-2}|[\mathcal{S}_4, \nabla_T]\psi|\\
&\lesssim r^{-2}|\mathfrak{d}^{\leq 2}\psi|.
\end{aligned}$$

We infer

$$\begin{aligned}
|N_{\underline{a}}| &\lesssim |\mathfrak{d}^{\leq 2}N| + |[\mathcal{S}_{\underline{a}}, \dot{\Box}_2]\psi|, \qquad \underline{a} = 1, 2, 3,\\
|N_4| &\lesssim |\mathfrak{d}^{\leq 2}N| + r^{-2}|[\mathcal{S}_4, |q|^2\dot{\Box}_2]\psi| + ar^{-2}|\mathfrak{d}^{\leq 2}\psi|.
\end{aligned}$$

Finally, we have

$$|[\mathcal{S}_{\underline{a}}, \dot{\square}_2]\psi| \lesssim ar^{-2}|\mathfrak{d}^{\leq 2}\psi|, \qquad \underline{a} = 1, 2, 3.$$

Also, in view of Proposition 3.7.6, we have

$$r^{-2}|[\mathcal{S}_4, |q|^2\dot{\square}_2]\psi| \lesssim ar^{-2}|\mathfrak{d}^{\leq 2}\psi|.$$

We infer

$$|N_{\underline{a}}| \lesssim |\mathfrak{d}^{\leq 2}N| + ar^{-2}|\mathfrak{d}^{\leq 2}\psi| \qquad \underline{a} = 1, 2, 3, 4,$$

as stated. This concludes the proof of Lemma 8.1.1. □

8.1.1 Basic spacetime $\mathcal{S}$-valued identity

We extend the definition of generalized current as given in Proposition 4.7.2 to the case of double-indexed vector fields, functions and 1-forms.

Definition 8.1.2 (Generalized Current). *Let* $\mathbf{X}$ *be a double-indexed collection of vector fields* $\mathbf{X} = \{\mathbf{X}^{\underline{\mathbf{ab}}}\}$*,* $\mathbf{w}$ *be a double-indexed collection of functions* $\mathbf{w} = \{w^{\underline{ab}}\}$*, and* $\mathbf{M} = \{M^{\underline{ab}}\}$ *a double-indexed collection of* 1*-forms, all symmetric in the indices* $\underline{a}, \underline{b}$*.*

Consider a solution $\psi \in \mathfrak{s}_2$ *of equation* (6.1.1) *and let* $\psi_{\underline{a}} = \mathcal{S}_{\underline{a}}\psi$ *verifying* (8.1.1)*, i.e.*

$$\dot{\square}_2\psi_{\underline{a}} - V\psi_{\underline{a}} = -\frac{4a\cos\theta}{|q|^2}\, {}^*\nabla_T\psi_{\underline{a}} + N_{\underline{a}}.$$

The generalized current $\mathcal{P}_\mu = \mathcal{P}_\mu^{(\mathbf{X},\mathbf{w},\mathbf{M})}[\psi]$ *associated to* $\psi_{\underline{a}}, \psi_{\underline{b}}$ *is given by*

$$\mathcal{P}_\mu = \mathcal{Q}[\psi]_{\underline{ab}\mu\nu}X^{\underline{ab}\nu} + \frac{1}{2}w^{\underline{ab}}\,\dot{\mathbf{D}}_\mu\psi_{\underline{a}} \cdot \psi_{\underline{b}} - \frac{1}{4}(\partial_\mu w^{\underline{ab}})\psi_{\underline{a}} \cdot \psi_{\underline{b}} + \frac{1}{4}M_\mu^{\underline{ab}}\psi_{\underline{a}} \cdot \psi_{\underline{b}}. \quad (8.1.2)$$

We also define

$$\begin{aligned}
\mathcal{Q}_{\mu\nu}(\psi_{\underline{a}}, \psi_{\underline{b}}) &= \dot{\mathbf{D}}_\mu\psi_{\underline{a}} \cdot \dot{\mathbf{D}}_\nu\psi_{\underline{b}} - \frac{1}{2}\mathbf{g}_{\mu\nu}\left(\mathbf{g}^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi_{\underline{a}} \cdot \dot{\mathbf{D}}_\beta\psi_{\underline{b}} + V\psi_{\underline{a}} \cdot \psi_{\underline{b}}\right) \\
&= \dot{\mathbf{D}}_\mu\psi_{\underline{a}} \cdot \dot{\mathbf{D}}_\nu\psi_{\underline{b}} - \frac{1}{2}\mathbf{g}_{\mu\nu}\mathcal{L}[\psi_{\underline{a}}, \psi_{\underline{b}}], \\
\mathcal{L}[\psi_{\underline{a}}, \psi_{\underline{b}}] &= \mathbf{g}^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi_{\underline{a}} \cdot \dot{\mathbf{D}}_\beta\psi_{\underline{b}} + V\psi_{\underline{a}} \cdot \psi_{\underline{b}}.
\end{aligned}$$

We also define, in analogy with (7.1.8), the modified divergence

$$\begin{aligned}\mathcal{E}[\mathbf{X},\mathbf{w},\mathbf{M}] &:= \mathbf{D}^\mu\mathcal{P}_\mu[\mathbf{X},\mathbf{w},\mathbf{M}] - \mathcal{N}[\mathbf{X},\mathbf{w}],\\ \mathcal{N}[\mathbf{X},\mathbf{w}] &:= \left(\nabla_{X^{\underline{ab}}}\psi_{\underline{a}} + \frac{1}{2}w^{\underline{ab}}\psi_{\underline{a}}\right)\cdot\left(\dot{\square}_2\psi_{\underline{b}} - V\psi_{\underline{b}}\right)\\ &\quad - \left({}^*\rho + \underline{\eta}\wedge\eta\right)\nabla_{(X^{\underline{ab}})^4 e_4 - (X^{\underline{ab}})^3 e_3}\psi_{\underline{a}}\cdot{}^*\psi_{\underline{b}}\\ &\quad - \frac{1}{2}\Im\Big(\mathrm{tr}\underline{X}H(X^{\underline{ab}})^3 + \mathrm{tr}X\,\underline{H}(X^{\underline{ab}})^4\Big)\cdot\nabla\psi_{\underline{a}}\cdot{}^*\psi_{\underline{b}}.\end{aligned} \tag{8.1.3}$$

Using the above notation we derive the following analogue of Proposition 7.1.5.

Proposition 8.1.3. *The following hold true*

1. *If we choose*

$$X^{\underline{ab}} = \mathcal{F}^{\underline{ab}}\partial_r, \qquad w^{\underline{ab}} = |q|^2 Div\big(|q|^{-2}X^{\underline{ab}}\big) - w^{\underline{ab}}_{red},$$

then the generalized current defined in (8.1.3) *verifies the identity*

$$\begin{aligned}|q|^2\mathcal{E}[\mathbf{X},\mathbf{w},\mathbf{M}] &= \mathcal{A}^{\underline{ab}}\nabla_r\psi_{\underline{a}}\cdot\nabla_r\psi_{\underline{b}} + \mathcal{U}^{\alpha\beta\underline{ab}}\,\dot{\mathbf{D}}_\alpha\psi_{\underline{a}}\cdot\dot{\mathbf{D}}_\beta\psi_{\underline{b}} + \mathcal{V}^{\underline{ab}}\psi_{\underline{a}}\cdot\psi_{\underline{b}}\\ &\quad + \frac{1}{4}|q|^2\mathbf{D}^\mu\Big(M^{\underline{ab}}_\mu\psi_{\underline{a}}\cdot\psi_{\underline{b}}\Big)\end{aligned} \tag{8.1.4}$$

where

$$\begin{aligned}\mathcal{A}^{\underline{ab}} &= \Delta\partial_r\mathcal{F}^{\underline{ab}} - \frac{1}{2}\mathcal{F}^{\underline{ab}}\partial_r\Delta - \frac{1}{2}\Delta w^{\underline{ab}}_{red},\\ \mathcal{U}^{\alpha\beta\underline{ab}} &= -\frac{1}{2}\mathcal{F}^{\underline{ab}}\partial_r\left(\frac{1}{\Delta}\mathcal{R}^{\alpha\beta}\right) - \frac{1}{2}w^{\underline{ab}}_{red}\frac{1}{\Delta}\mathcal{R}^{\alpha\beta},\\ \mathcal{V}^{\underline{ab}} &= -\frac{1}{2}\left(X^{\underline{ab}}(|q|^2)V + |q|^2X^{\underline{ab}}(V) + \frac{1}{2}|q|^2\square_{\mathbf{g}}w^{\underline{ab}} + |q|^2w^{\underline{ab}}_{red}V\right).\end{aligned}$$

2. *If in addition we choose, for functions z and h, and a double-indexed function $f^{\underline{ab}}$*

$$\mathcal{F}^{\underline{ab}} = -zhf^{\underline{ab}}, \qquad w^{\underline{ab}} = -z\partial_r\big(hf^{\underline{ab}}\big), \qquad w^{\underline{ab}}_{red} = \mathcal{F}^{\underline{ab}}z^{-1}\partial_r z, \tag{8.1.5}$$

then

$$\begin{aligned}\mathcal{U}^{\alpha\beta\underline{ab}} &= \frac{1}{2}hf^{\underline{ab}}\partial_r\left(\frac{z}{\Delta}\mathcal{R}^{\alpha\beta}\right),\\ \mathcal{A}^{\underline{ab}} &= -z^{1/2}\Delta^{3/2}\partial_r\left(h\frac{z^{1/2}f^{\underline{ab}}}{\Delta^{1/2}}\right),\\ \mathcal{V}^{\underline{ab}} &= \frac{1}{4}\partial_r\left(\Delta\partial_r\Big(z\partial_r\big(hf^{\underline{ab}}\big)\Big)\right) + 2hf^{\underline{ab}}\partial_r\left(z\frac{\Delta}{r^2+a^2}\right).\end{aligned}$$

3. If $M^{\underline{ab}} = v^{\underline{ab}}(r)\partial_r$, for a double-indexed function $v = v^{\underline{ab}}(r)$, we have

$$\begin{aligned}
&\frac{1}{4}|q|^2 Div\big((\psi_{\underline{a}}\cdot\psi_{\underline{b}})M^{\underline{ab}}\big)\\
= &\frac{1}{4}|q|^2\left(2v^{\underline{ab}}(r)\psi_{\underline{a}}\cdot\nabla_r\psi_{\underline{b}} + \left(\partial_r v^{\underline{ab}} + \frac{2r}{|q|^2}v^{\underline{ab}}\right)\psi_{\underline{a}}\cdot\psi_{\underline{b}}\right). \qquad (8.1.6)
\end{aligned}$$

Proof. The proof follows exactly the same steps as in the proof of Proposition 7.1.5. For the convenience of the reader we derive below the formula for $\mathcal{V}$. We write

$$\begin{aligned}
\mathcal{V}^{\underline{ab}} &= \mathcal{V}_0^{\underline{ab}} + \mathcal{V}_1^{\underline{ab}},\\
\mathcal{V}_0^{\underline{ab}} &= -\frac{1}{4}|q|^2\square_{\mathbf{g}} w^{\underline{ab}},\\
\mathcal{V}_1^{\underline{ab}} &= -\frac{1}{2}\Big(X^{\underline{ab}}\big(|q|^2\big)V + |q|^2 X^{\underline{ab}}(V) + |q|^2 w_{red}^{\underline{ab}} V\Big).
\end{aligned}$$

We first calculate $\mathcal{V}_0$. Recall that

$$\begin{aligned}
w^{\underline{ab}} &= |q|^2\mathbf{D}_\alpha\big(|q|^{-2}\big(X^{\underline{ab}}\big)^\alpha\big) - w_{red}^{\underline{ab}} = |q|^2\mathbf{D}_\alpha\big(|q|^{-2}\mathcal{F}^{\underline{ab}}\partial_r^\alpha\big) - \mathcal{F}^{\underline{ab}}z^{-1}\partial_r z\\
&= |q|^2\partial_r\big(|q|^{-2}\mathcal{F}^{\underline{ab}}\big) + \mathcal{F}^{\underline{ab}}(\mathbf{D}_\alpha\partial_r^\alpha) - \mathcal{F}^{\underline{ab}}z^{-1}\partial_r z.
\end{aligned}$$

Using that $\mathbf{D}_\alpha\partial_r^\alpha = \frac{1}{|q|^2}\partial_r\big(|q|^2\big)$, we have

$$w^{\underline{ab}} = |q|^2\partial_r\big(|q|^{-2}\mathcal{F}^{\underline{ab}}\big) + \frac{1}{|q|^2}\partial_r\big(|q|^2\big)\mathcal{F}^{\underline{ab}} - \mathcal{F}^{\underline{ab}}z^{-1}\partial_r z = \partial_r\mathcal{F}^{\underline{ab}} - \mathcal{F}^{\underline{ab}}z^{-1}\partial_r z = z\partial_r\left(\frac{\mathcal{F}^{\underline{ab}}}{z}\right).$$

Thus, in view of our choice for $\mathcal{F}^{\underline{ab}} = -zhf^{\underline{ab}}$ in (8.1.5),

$$w^{\underline{ab}} = z\partial_r\left(\frac{\mathcal{F}^{\underline{ab}}}{z}\right) = -z\partial_r\big(hf^{\underline{ab}}\big).$$

Recalling formula (7.1.15) for $\square H(r)$, we deduce

$$\mathcal{V}_0^{\underline{ab}} = -\frac{1}{4}|q|^2\square_{\mathbf{g}} w^{\underline{ab}} = -\frac{1}{4}\partial_r\Big(\Delta\partial_r\big(w^{\underline{ab}}\big)\Big) = \frac{1}{4}\partial_r\left(\Delta\partial_r\Big(z\partial_r\big(hf^{\underline{ab}}\big)\Big)\right).$$

Remains to calculate

$$\mathcal{V}_1^{\underline{ab}} = -\frac{1}{2}\Big(X^{\underline{ab}}\big(|q|^2\big)V + |q|^2 X^{\underline{ab}}(V) + |q|^2 w_{red}^{\underline{ab}} V\Big) = -\frac{1}{2}\Big(X^{\underline{ab}}\big(|q|^2 V\big) + |q|^2 w_{red}^{\underline{ab}} V\Big).$$

Recalling that $|q|^2 V = 4\frac{\Delta}{r^2+a^2}$, $w_{red}^{\underline{ab}} = \mathcal{F}^{\underline{ab}} z^{-1}\partial_r z$ and $\mathcal{F}^{\underline{ab}} = -zhf^{\underline{ab}}$ we deduce

$$\begin{aligned}
\mathcal{V}_1^{\underline{ab}} &= -2\left(X^{\underline{ab}}\left(\frac{\Delta}{r^2+a^2}\right) + w_{red}^{\underline{ab}}\frac{\Delta}{r^2+a^2}\right)\\
&= -2\left(\mathcal{F}^{\underline{ab}}\partial_r\left(\frac{\Delta}{r^2+a^2}\right) + \mathcal{F}^{\underline{ab}} z^{-1}\partial_r z\frac{\Delta}{r^2+a^2}\right)\\
&= -2z^{-1}\partial_r\left(z\frac{\Delta}{r^2+a^2}\right)\mathcal{F}^{\underline{ab}} = 2hf^{\underline{ab}}\partial_r\left(z\frac{\Delta}{r^2+a^2}\right).
\end{aligned}$$

This proves the proposition. □

We simplify the notations by writing (8.1.4) as

$$|q|^2\mathcal{E}[\mathbf{X},\mathbf{w},\mathbf{M}] = P + I + J + K \tag{8.1.7}$$

where

$$\begin{aligned}
P &:= \mathcal{U}^{\alpha\beta\underline{ab}}\,\dot{\mathbf{D}}_\alpha\psi_{\underline{a}}\cdot\dot{\mathbf{D}}_\beta\psi_{\underline{b}},\\
I &:= \mathcal{A}^{\underline{ab}}\nabla_r\psi_{\underline{a}}\cdot\nabla_r\psi_{\underline{b}},\\
J &:= \mathcal{V}^{\underline{ab}}\psi_{\underline{a}}\cdot\psi_{\underline{b}},\\
K &:= \frac{1}{4}|q|^2\mathbf{D}^\mu\Big(M_\mu^{\underline{ab}}\psi_{\underline{a}}\cdot\psi_{\underline{b}}\Big).
\end{aligned}$$

In what follows, we will choose functions z, h and the double-indexed function $f^{\underline{ab}}$ to obtain positivity of the generalized current. Each of the above term will produce lower order terms in derivatives and in angular momentum a.

8.1.2 Principal term

We consider now the principal term P, i.e. from Proposition 8.1.3

$$P = \mathcal{U}^{\alpha\beta\underline{ab}}\,\dot{\mathbf{D}}_\alpha\psi_{\underline{a}}\cdot\dot{\mathbf{D}}_\beta\psi_{\underline{b}} = \frac{1}{2}hf^{\underline{ab}}\partial_r\left(\frac{z}{\Delta}\mathcal{R}^{\alpha\beta}\right)\dot{\mathbf{D}}_\alpha\psi_{\underline{a}}\cdot\dot{\mathbf{D}}_\beta\psi_{\underline{b}}.$$

Recall that, see (3.5.7),

$$\mathcal{R}^{\alpha\beta} = \mathcal{R}^{\underline{a}} S_{\underline{a}}^{\alpha\beta}.$$

We similarly set

$$\tilde{\mathcal{R}}'^{\underline{a}} := \partial_r\left(\frac{z}{\Delta}\mathcal{R}^{\underline{a}}\right),$$

and thus write

$$\partial_r\left(\frac{z}{\Delta}\mathcal{R}^{\alpha\beta}\right) = \partial_r\left(\frac{z}{\Delta}\mathcal{R}^{\underline{a}}S_{\underline{a}}^{\alpha\beta}\right) = \partial_r\left(\frac{z}{\Delta}\mathcal{R}^{\underline{a}}\right)S_{\underline{a}}^{\alpha\beta} = \tilde{\mathcal{R}}'^{\underline{a}}S_{\underline{a}}^{\alpha\beta},$$

which gives

$$P = \frac{1}{2}hf^{\underline{ab}}\tilde{\mathcal{R}}'^{\underline{c}}S_{\underline{c}}^{\alpha\beta}\,\dot{\mathbf{D}}_\alpha\psi_{\underline{a}}\cdot\dot{\mathbf{D}}_\beta\psi_{\underline{b}}.$$

To create a quadratic expression in $\tilde{\mathcal{R}}'^{\underline{a}}$ in P, we choose

$$f^{\underline{ab}} = \tilde{\mathcal{R}}'^{(\underline{a}}\mathcal{L}^{\underline{b})} = \frac{1}{2}\big(\tilde{\mathcal{R}}'^{\underline{a}}\mathcal{L}^{\underline{b}} + \tilde{\mathcal{R}}'^{\underline{b}}\mathcal{L}^{\underline{a}}\big), \tag{8.1.8}$$

where $\mathcal{L}^{\underline{a}}$ are constant coefficients to be chosen later of a given 2-tensor of the form

$$L^{\alpha\beta} = \mathcal{L}^{\underline{a}}S_{\underline{a}}^{\alpha\beta} = \mathcal{L}^1T^\alpha T^\beta + \mathcal{L}^2aT^{(\alpha}Z^{\beta)} + \mathcal{L}^3a^2Z^\alpha Z^\beta + \mathcal{L}^4O^{\alpha\beta}. \tag{8.1.9}$$

With the choice (8.1.8) for $f^{\underline{ab}}$ we deduce

$$\begin{aligned} P &= \frac{1}{4}h\big(\tilde{\mathcal{R}}'^{\underline{a}}\mathcal{L}^{\underline{b}} + \tilde{\mathcal{R}}'^{\underline{b}}\mathcal{L}^{\underline{a}}\big)\tilde{\mathcal{R}}'^{\underline{c}}S_{\underline{c}}^{\alpha\beta}\,\dot{\mathbf{D}}_\alpha\psi_{\underline{a}}\cdot\dot{\mathbf{D}}_\beta\psi_{\underline{b}}\\ &= \frac{1}{4}h\big(\tilde{\mathcal{R}}'^{\underline{a}}\tilde{\mathcal{R}}'^{\underline{c}}\mathcal{L}^{\underline{b}} + \tilde{\mathcal{R}}'^{\underline{b}}\tilde{\mathcal{R}}'^{\underline{c}}\mathcal{L}^{\underline{a}}\big)S_{\underline{c}}^{\alpha\beta}\,\dot{\mathbf{D}}_\alpha\psi_{\underline{a}}\cdot\dot{\mathbf{D}}_\beta\psi_{\underline{b}}\\ &= \mathcal{U}^{\underline{ab}\,\alpha\beta}\,\dot{\mathbf{D}}_\alpha\psi_{\underline{a}}\cdot\dot{\mathbf{D}}_\beta\psi_{\underline{b}}, \end{aligned}$$

where

$$\mathcal{U}^{\underline{ab}} := \frac{1}{2}h\tilde{\mathcal{R}}'^{\underline{a}}\tilde{\mathcal{R}}'^{\underline{b}}, \qquad \mathcal{U}^{\underline{ab}\,\alpha\beta} := \frac{1}{2}\big(\mathcal{U}^{\underline{ac}}\mathcal{L}^{\underline{b}} + \mathcal{U}^{\underline{bc}}\mathcal{L}^{\underline{a}}\big)S_{\underline{c}}^{\alpha\beta} = \mathcal{U}^{\underline{c}(\underline{a}}\mathcal{L}^{\underline{b})}S_{\underline{c}}^{\alpha\beta}. \tag{8.1.10}$$

We now isolate a positive part in the principal term P from a divergence component. This is the crucial step which allows to obtain a positive trapped term in the case of higher derivative Morawetz estimates.

Lemma 8.1.4. *Let P the principal term defined as above. We then have the identity*

$$P = \frac{1}{2}hL^{\alpha\beta}\dot{\mathbf{D}}_\alpha\Psi\cdot\dot{\mathbf{D}}_\beta\Psi - \frac{1}{2}h\Psi\cdot(\tilde{\mathcal{R}}'^{\underline{c}}\mathcal{L}^{\underline{b}}[\mathcal{S}_{\underline{c}},\mathcal{S}_{\underline{b}}]\psi) + |q|^2\dot{\mathbf{D}}_\alpha\mathcal{B}^\alpha,$$

where Ψ is defined as

$$\Psi := \tilde{\mathcal{R}}'^{\underline{a}}\psi_{\underline{a}}, \tag{8.1.11}$$

and the boundary term $\mathcal{B}$ is given by

$$\mathcal{B}^\alpha := |q|^{-2}\frac{1}{2}h\Psi\tilde{\mathcal{R}}'^{\underline{c}}\mathcal{L}^{\underline{b}}\cdot\left(S_{\underline{c}}^{\alpha\beta}\,\dot{\mathbf{D}}_\beta\psi_{\underline{b}} - S_{\underline{b}}^{\alpha\beta}\dot{\mathbf{D}}_\beta\psi_{\underline{c}}\right). \tag{8.1.12}$$

Proof. We consider

$$
\begin{aligned}
|q|^{-2}P &= \frac{1}{2}|q|^{-2}\big(\mathcal{U}^{\underline{ac}}\mathcal{L}^{\underline{b}}+\mathcal{U}^{\underline{bc}}\mathcal{L}^{\underline{a}}\big)S_{\underline{c}}^{\alpha\beta}\,\dot{\mathbf{D}}_\alpha\psi_{\underline{a}}\cdot\dot{\mathbf{D}}_\beta\psi_{\underline{b}} = |q|^{-2}\mathcal{U}^{\underline{ac}}\mathcal{L}^{\underline{b}}S_{\underline{c}}^{\alpha\beta}\,\dot{\mathbf{D}}_\alpha\psi_{\underline{a}}\cdot\dot{\mathbf{D}}_\beta\psi_{\underline{b}}\\
&= -\mathcal{U}^{\underline{ac}}\mathcal{L}^{\underline{b}}\psi_{\underline{a}}\cdot\dot{\mathbf{D}}_\alpha(|q|^{-2}S_{\underline{c}}^{\alpha\beta}\,\dot{\mathbf{D}}_\beta\psi_{\underline{b}})+\dot{\mathbf{D}}_\alpha(|q|^{-2}\mathcal{U}^{\underline{ac}}\mathcal{L}^{\underline{b}}S_{\underline{c}}^{\alpha\beta}\psi_{\underline{a}}\cdot\dot{\mathbf{D}}_\beta\psi_{\underline{b}})\\
&\quad -\dot{\mathbf{D}}_\alpha(\mathcal{U}^{\underline{ac}}\mathcal{L}^{\underline{b}})|q|^{-2}S_{\underline{c}}^{\alpha\beta}\psi_{\underline{a}}\cdot\dot{\mathbf{D}}_\beta\psi_{\underline{b}}.
\end{aligned}
$$

Note that $\mathcal{L}^{\underline{a}}$ and $\mathcal{U}^{\underline{ab}}$ only depend on r. Therefore, since for all $\underline{c}$ the 2-tensors $S_{\underline{c}}^{\alpha\beta}$ do not contain r derivatives we deduce

$$
\dot{\mathbf{D}}_\alpha\big(\mathcal{U}^{\underline{ac}}\mathcal{L}^{\underline{b}}\big)|q|^{-2}S_{\underline{c}}^{\alpha\beta}\psi_{\underline{a}}\cdot\dot{\mathbf{D}}_\beta\psi_{\underline{b}} = \partial_r\big(\mathcal{U}^{\underline{ac}}\mathcal{L}^{\underline{b}}\big)|q|^{-2}S_{\underline{c}}^{r\beta}\psi_{\underline{a}}\cdot\dot{\mathbf{D}}_\beta\psi_{\underline{b}}=0.
$$

Now consider $\dot{\mathbf{D}}_\alpha(|q|^{-2}S_{\underline{c}}^{\alpha\beta}\,\dot{\mathbf{D}}_\beta\psi_{\underline{b}})$. Recall Definition 3.7.8, i.e. for $\psi\in\mathfrak{s}_2$,

$$
\mathcal{S}_{\underline{a}}\psi = |q|^2\dot{\mathbf{D}}_\alpha(|q|^{-2}S_{\underline{a}}^{\alpha\beta}\dot{\mathbf{D}}_\beta\psi) \qquad \text{for } a=1,2,3,4.
$$

Therefore we write

$$
\begin{aligned}
\dot{\mathbf{D}}_\alpha(|q|^{-2}S_{\underline{c}}^{\alpha\beta}\,\dot{\mathbf{D}}_\beta\psi_{\underline{b}}) &= \dot{\mathbf{D}}_\alpha(|q|^{-2}S_{\underline{c}}^{\alpha\beta}\,\dot{\mathbf{D}}_\beta\mathcal{S}_{\underline{b}}\psi) = |q|^{-2}\mathcal{S}_{\underline{c}}\mathcal{S}_{\underline{b}}\psi = |q|^{-2}\mathcal{S}_{\underline{b}}\mathcal{S}_{\underline{c}}\psi+|q|^{-2}[\mathcal{S}_{\underline{c}},\mathcal{S}_{\underline{b}}]\psi\\
&= \dot{\mathbf{D}}_\alpha(|q|^{-2}S_{\underline{b}}^{\alpha\beta}\,\dot{\mathbf{D}}_\beta\psi_{\underline{c}})+|q|^{-2}[\mathcal{S}_{\underline{c}},\mathcal{S}_{\underline{b}}]\psi.
\end{aligned}
$$

Thus, repeating the integration by parts procedure and noting, as above, that the last term vanishes, we obtain

$$
\begin{aligned}
&\mathcal{U}^{\underline{ac}}\mathcal{L}^{\underline{b}}\psi_{\underline{a}}\cdot\dot{\mathbf{D}}_\alpha\big(|q|^{-2}S_{\underline{c}}^{\alpha\beta}\dot{\mathbf{D}}_\beta\psi_{\underline{b}}\big)\\
&= \mathcal{U}^{\underline{ac}}\mathcal{L}^{\underline{b}}\psi_{\underline{a}}\cdot\dot{\mathbf{D}}_\alpha(|q|^{-2}S_{\underline{b}}^{\alpha\beta}\,\dot{\mathbf{D}}_\beta\psi_{\underline{c}})+|q|^{-2}\mathcal{U}^{\underline{ac}}\mathcal{L}^{\underline{b}}\psi_{\underline{a}}\cdot[\mathcal{S}_{\underline{c}},\mathcal{S}_{\underline{b}}]\psi\\
&= -|q|^{-2}\mathcal{U}^{\underline{ac}}\mathcal{L}^{\underline{b}}S_{\underline{b}}^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi_{\underline{a}}\cdot\dot{\mathbf{D}}_\beta\psi_{\underline{c}}+\dot{\mathbf{D}}_\alpha\Big(|q|^{-2}\mathcal{U}^{\underline{ac}}\mathcal{L}^{\underline{b}}S_{\underline{b}}^{\alpha\beta}\psi_{\underline{a}}\cdot\dot{\mathbf{D}}_\beta\psi_{\underline{c}}\Big)\\
&\quad +|q|^{-2}\mathcal{U}^{\underline{ac}}\mathcal{L}^{\underline{b}}\psi_{\underline{a}}\cdot[\mathcal{S}_{\underline{c}},\mathcal{S}_{\underline{b}}]\psi.
\end{aligned}
$$

Therefore, recalling that $\mathcal{L}^{\underline{b}}S_{\underline{b}}^{\alpha\beta}=L^{\alpha\beta}$,

$$
\begin{aligned}
P &= \mathcal{U}^{\underline{ac}}L^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi_{\underline{a}}\cdot\dot{\mathbf{D}}_\beta\psi_{\underline{c}}+|q|^2\dot{\mathbf{D}}_\alpha\left(|q|^{-2}\mathcal{U}^{\underline{ac}}\mathcal{L}^{\underline{b}}\psi_{\underline{a}}\cdot\Big(S_{\underline{c}}^{\alpha\beta}\,\dot{\mathbf{D}}_\beta\psi_{\underline{b}}-S_{\underline{b}}^{\alpha\beta}\dot{\mathbf{D}}_\beta\psi_{\underline{c}}\Big)\right)\\
&\quad -\mathcal{U}^{\underline{ac}}\mathcal{L}^{\underline{b}}\psi_{\underline{a}}\cdot[\mathcal{S}_{\underline{c}},\mathcal{S}_{\underline{b}}]\psi.
\end{aligned}
$$

Finally, using (8.1.10), and using that $\tilde{\mathcal{R}}'^{\underline{a}}$ only depend on r, we obtain

$$
\begin{aligned}
P &= \frac{1}{2}h\tilde{\mathcal{R}}'^{\underline{a}}\tilde{\mathcal{R}}'^{\underline{c}}L^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi_{\underline{a}}\cdot\dot{\mathbf{D}}_\beta\psi_{\underline{c}}-\frac{1}{2}h\tilde{\mathcal{R}}'^{\underline{a}}\tilde{\mathcal{R}}'^{\underline{c}}\mathcal{L}^{\underline{b}}\psi_{\underline{a}}\cdot[\mathcal{S}_{\underline{c}},\mathcal{S}_{\underline{b}}]\psi\\
&\quad +|q|^2\dot{\mathbf{D}}_\alpha\left(|q|^{-2}\frac{1}{2}h\tilde{\mathcal{R}}'^{\underline{a}}\tilde{\mathcal{R}}'^{\underline{c}}\mathcal{L}^{\underline{b}}\psi_{\underline{a}}\cdot\Big(S_{\underline{c}}^{\alpha\beta}\,\dot{\mathbf{D}}_\beta\psi_{\underline{b}}-S_{\underline{b}}^{\alpha\beta}\dot{\mathbf{D}}_\beta\psi_{\underline{c}}\Big)\right)\\
&= \frac{1}{2}hL^{\alpha\beta}\dot{\mathbf{D}}_\alpha(\tilde{\mathcal{R}}'^{\underline{a}}\psi_{\underline{a}})\cdot\dot{\mathbf{D}}_\beta(\tilde{\mathcal{R}}'^{\underline{c}}\psi_{\underline{c}})-\frac{1}{2}h\tilde{\mathcal{R}}'^{\underline{c}}\mathcal{L}^{\underline{b}}(\tilde{\mathcal{R}}'^{\underline{a}}\psi_{\underline{a}})\cdot[\mathcal{S}_{\underline{c}},\mathcal{S}_{\underline{b}}]\psi\\
&\quad +|q|^2\dot{\mathbf{D}}_\alpha\left(|q|^{-2}\frac{1}{2}h\tilde{\mathcal{R}}'^{\underline{c}}\mathcal{L}^{\underline{b}}(\tilde{\mathcal{R}}'^{\underline{a}}\psi_{\underline{a}})\cdot\Big(S_{\underline{c}}^{\alpha\beta}\,\dot{\mathbf{D}}_\beta\psi_{\underline{b}}-S_{\underline{b}}^{\alpha\beta}\dot{\mathbf{D}}_\beta\psi_{\underline{c}}\Big)\right).
\end{aligned}
$$

By denoting $\Psi = \tilde{\mathcal{R}}'^{\underline{a}}\psi_{\underline{a}}$ we obtain the stated expression. □

We summarize the result in the following proposition.

Proposition 8.1.5. *Consider the generalized bilinear current* (8.1.2) *with*

$$X^{\underline{ab}} = \mathcal{F}^{\underline{ab}}\partial_r, \qquad w^{\underline{ab}} = |q|^2\mathbf{D}_\alpha\big(|q|^{-2}\big(X^{\underline{ab}}\big)^\alpha\big) - w^{\underline{ab}}_{red}, \qquad w^{\underline{ab}}_{red} = \mathcal{F}^{\underline{ab}}z^{-1}\partial_r z,$$

and

$$\mathcal{F}^{\underline{ab}} = -zh\tilde{\mathcal{R}}'^{(\underline{a}}\mathcal{L}^{\underline{b})} = -\frac{1}{2}zh\big(\tilde{\mathcal{R}}'^{\underline{a}}\mathcal{L}^{\underline{b}} + \tilde{\mathcal{R}}'^{\underline{b}}\mathcal{L}^{\underline{a}}\big), \qquad \tilde{\mathcal{R}}'^{\underline{a}} = \partial_r\Big(\frac{z}{\Delta}\mathcal{R}^{\underline{a}}\Big),$$

for constant $\mathcal{L}^{\underline{a}}$. Then,

$$|q|^2\mathcal{E}[\mathbf{X}, \mathbf{w}, \mathbf{M}] - |q|^2\mathbf{D}^\mu\mathcal{B}_\mu = \widetilde{P} + P_{lot} + I + J + K \tag{8.1.13}$$

where

- *$\mathcal{B}$ is the boundary term defined in* (8.1.12)*,*
- *The principal term $\widetilde{P}$ is positive definite and given by*

$$\widetilde{P} := \frac{1}{2}hL^{\alpha\beta}\dot{\mathbf{D}}_\alpha\Psi\cdot\dot{\mathbf{D}}_\beta\Psi, \qquad \Psi = \tilde{\mathcal{R}}'^{\underline{a}}\psi_{\underline{a}}, \tag{8.1.14}$$

  with $L^{\alpha\beta}$ as in (8.1.9)*,*
- *The lower order term P_{lot} is given by*

$$P_{lot} = -\frac{1}{2}h\Psi\cdot(\tilde{\mathcal{R}}'^{\underline{c}}\mathcal{L}^{\underline{b}}[\mathcal{S}_{\underline{c}}, \mathcal{S}_{\underline{b}}]\psi). \tag{8.1.15}$$

- *The quadratic form I is given by*

$$I = \big(\mathcal{A}^{\underline{a}}[z]\nabla_r\psi_{\underline{a}}\big)\cdot\big(\mathcal{L}^{\underline{a}}\nabla_r\psi_{\underline{a}}\big), \tag{8.1.16}$$

  where

$$\mathcal{A}^{\underline{a}}[z] = -z^{1/2}\Delta^{3/2}\tilde{\mathcal{R}}''^{\underline{a}}, \qquad \tilde{\mathcal{R}}''^{\underline{a}} := \partial_r\Big(\frac{hz^{1/2}\tilde{\mathcal{R}}'^{\underline{a}}}{\Delta^{1/2}}\Big).$$

- *The quadratic form J is given by*

$$J = \big(\mathcal{V}^{\underline{a}}[z]\psi_{\underline{a}}\big)\cdot\big(\mathcal{L}^{\underline{a}}\psi_{\underline{a}}\big), \tag{8.1.17}$$

  where

$$\mathcal{V}^{\underline{a}} = \frac{1}{4}\partial_r\left(\Delta\partial_r\Big(z\partial_r\big(h\tilde{\mathcal{R}}'^{\underline{a}}\big)\Big)\right) + 2h\tilde{\mathcal{R}}'^{\underline{a}}\partial_r\left(\frac{z\Delta}{r^2+a^2}\right).$$

- *The quadratic form K is given by*

$$K = \frac{1}{4}|q|^2\mathbf{D}^\mu\Big((M_\mu^{\underline{a}}\psi_{\underline{a}})\cdot(\mathcal{L}^{\underline{b}}\psi_{\underline{b}})\Big). \tag{8.1.18}$$

Proof. The above follows from (8.1.7) and Lemma 8.1.4. Also from Proposition 8.1.3 and the choice (8.1.8) $f^{\underline{ab}} = \tilde{\mathcal{R}}'^{(\underline{a}}\mathcal{L}^{\underline{b})}$, we deduce

$$\mathcal{A}^{\underline{ab}} = -z^{1/2}\Delta^{3/2}\tilde{\mathcal{R}}''^{(\underline{a}}\mathcal{L}^{\underline{b})}, \qquad \tilde{\mathcal{R}}''^{\underline{a}} := \partial_r\left(\frac{hz^{1/2}\tilde{\mathcal{R}}'^{\underline{a}}}{\Delta^{1/2}}\right),$$

and

$$\mathcal{V}^{\underline{ab}} = \frac{1}{4}\partial_r\left(\Delta\partial_r\Big(z\partial_r\big(h\tilde{\mathcal{R}}'^{(\underline{a}}\mathcal{L}^{\underline{b})}\big)\Big)\right) + 2h\tilde{\mathcal{R}}'^{(\underline{a}}\mathcal{L}^{\underline{b})}\partial_r\left(\frac{z\Delta}{r^2+a^2}\right).$$

The expressions for I, J and K are implied by writing $\mathcal{A}^{\underline{ab}} = \mathcal{A}^{(\underline{a}}\mathcal{L}^{\underline{b})}$, $\mathcal{V}^{\underline{ab}} = \mathcal{V}^{(\underline{a}}\mathcal{L}^{\underline{b})}$ and $M^{\underline{ab}} = M^{(\underline{a}}\mathcal{L}^{\underline{b})}$. □

We denote the effective generalized current

$$|q|^2\widetilde{\mathcal{E}}[\mathbf{X},\mathbf{w},\mathbf{M}] := |q|^2\mathcal{E}[\mathbf{X},\mathbf{w},\mathbf{M}] - |q|^2\mathbf{D}^\mu\mathcal{B}_\mu \tag{8.1.19}$$

which differs from the generalized bilinear current by the boundary terms given by $\mathcal{B}$. Our goal in deriving the higher order Morawetz estimates is to show positivity of the effective quadratic form $|q|^2\widetilde{\mathcal{E}}[\mathbf{X},\mathbf{w},\mathbf{M}]$.

8.1.3 Choice of z

In this section, we present two choices for the function z.

1. The first is the same choice as in Section 7.1.3, i.e.

$$z = z_0 = \frac{\Delta}{(r^2+a^2)^2}.$$

   This choice, which was good enough to derive conditional Morawetz estimates for first derivatives, has its limitations in the case of higher derivatives estimates. In particular, the contribution in $\mathcal{S}_1 = \nabla_T\nabla_T$ vanishes with this choice. We therefore have to modify it, but we use it as reference for the next choice.

2. The crucial choice in the next section is the following (see also [4]):

$$z = z_0 - \delta_0 z_0^2, \qquad \delta_0 > 0. \tag{8.1.20}$$

The parameter δ_0 will be chosen later to be a small positive constant. This allows to obtain a contribution in $\mathcal{S}_1$ with this choice, and for δ_0 small enough we can transpose most information from the first choice of z to the second one.

We now collect the main coefficients for the two choices.

Main coefficients for the choice $z = z_0$

Lemma 8.1.6. *With the choice* $z = \frac{\Delta}{(r^2+a^2)^2}$ *and* $h = \frac{(r^2+a^2)^4}{r(r^2-a^2)}$ *(as in Section 7.1.3) we have the following coefficients.*

1. *The coefficients* $\tilde{\mathcal{R}}'^{\underline{a}} = \partial_r\left(\frac{z}{\Delta}\mathcal{R}^{\underline{a}}\right)$ *are given by*

$$\begin{aligned} &\tilde{\mathcal{R}}'^1 = 0, \qquad \tilde{\mathcal{R}}'^2 = \frac{4r}{(r^2+a^2)^2}, \\ &\tilde{\mathcal{R}}'^3 = \frac{4r}{(r^2+a^2)^3}, \qquad \tilde{\mathcal{R}}'^4 = f = -\frac{2\mathcal{T}}{(r^2+a^2)^3}. \end{aligned} \tag{8.1.21}$$

2. *The coefficients* $\tilde{\mathcal{R}}''^{\underline{a}} = \partial_r\left(\frac{hz^{1/2}\tilde{\mathcal{R}}'^{\underline{a}}}{\Delta^{1/2}}\right)$ *are given by*

$$\begin{aligned} &\tilde{\mathcal{R}}''^1 = 0, \qquad \tilde{\mathcal{R}}''^2 = -\frac{16a^2r}{(r^2-a^2)^2}, \qquad \tilde{\mathcal{R}}''^3 = -\frac{8r}{(r^2-a^2)^2}, \\ &\tilde{\mathcal{R}}''^4 = -2\frac{3mr^4 - 4a^2r^3 + ma^4}{r^2(r^2-a^2)^2}. \end{aligned}$$

3. *The coefficients* $\mathcal{A}^{\underline{a}} = -z^{1/2}\Delta^{3/2}\tilde{\mathcal{R}}''^{\underline{a}}$ *are given by*

$$\begin{aligned} &\mathcal{A}^1 = 0, \quad \mathcal{A}^2 = \Delta^2\frac{16a^2r}{(r^2+a^2)(r^2-a^2)^2}, \quad \mathcal{A}^3 = \Delta^2\frac{8r}{(r^2+a^2)(r^2-a^2)^2}, \\ &\mathcal{A}^4 = 2\Delta^2\frac{3mr^4 - 4a^2r^3 + ma^4}{r^2(r^2+a^2)(r^2-a^2)^2}. \end{aligned} \tag{8.1.22}$$

4. *The coefficients* $\mathcal{V}^{\underline{a}}$ *verify*

$$\mathcal{V}^1 = 0, \qquad \mathcal{V}^2 = O(r^{-1}), \qquad \mathcal{V}^3 = O(r^{-3}), \qquad \mathcal{V}^4 = \mathcal{V},$$

where $\mathcal{V}$ *is the scalar function given by* (7.1.28).

Proof. The values of $\tilde{\mathcal{R}}'$ are the same as the one computed in Section 3.8.3 in the case of geodesics. For $\tilde{\mathcal{R}}''$ we compute

$$\begin{aligned}
\tilde{\mathcal{R}}''^2 &= \partial_r\left(\frac{(r^2+a^2)^3}{r(r^2-a^2)}\tilde{\mathcal{R}}'^2\right) = 4\partial_r\left(\frac{(r^2+a^2)}{(r^2-a^2)}\right) = -\frac{16a^2r}{(r^2-a^2)^2},\\
\tilde{\mathcal{R}}''^3 &= \partial_r\left(\frac{(r^2+a^2)^3}{r(r^2-a^2)}\tilde{\mathcal{R}}'^3\right) = 4\partial_r\left(\frac{1}{(r^2-a^2)}\right) = -\frac{8r}{(r^2-a^2)^2},\\
\tilde{\mathcal{R}}''^4 &= \partial_r\left(\frac{(r^2+a^2)^3}{r(r^2-a^2)}\tilde{\mathcal{R}}'^4\right) = -2\partial_r\left(\frac{\mathcal{T}}{r(r^2-a^2)}\right) = -2\frac{3mr^4-4a^2r^3+ma^4}{r^2(r^2-a^2)^2},
\end{aligned}$$

as desired. The computations of $\mathcal{A}^{\underline{a}}$ are immediate. Finally, $\mathcal{V}^1 = 0$ follows from $\tilde{\mathcal{R}}'^1 = 0$, $\mathcal{V}^2 = O(r^{-1})$ from $\tilde{\mathcal{R}}'^2 = O(r^{-3})$, $\mathcal{V}^3 = O(r^{-3})$ from $\tilde{\mathcal{R}}'^3 = O(r^{-5})$, and $\mathcal{V}^4 = \mathcal{V}$ from $\tilde{\mathcal{R}}'^4 = f$ and Proposition 7.1.8. □

Main coefficients for the choice $z = z_0 - \delta_0 z_0^2$

Lemma 8.1.7. *With the choice $z = z_0 - \delta_0 z_0^2$ as in (8.1.20) with $z_0 = \frac{\Delta}{(r^2+a^2)^2}$ and $h = \frac{(r^2+a^2)^4}{r(r^2-a^2)}$ we have the following coefficients.*

1. *The coefficients $\tilde{\mathcal{R}}'^a[z] = \partial_r\left(\frac{z}{\Delta}\mathcal{R}^{\underline{a}}\right)$ are given by*

$$\begin{aligned}
\tilde{\mathcal{R}}'^1[z] &= \delta_0 f,\\
\tilde{\mathcal{R}}'^2[z] &= \frac{4r}{(r^2+a^2)^2}\big(1+O(r^{-2}\delta_0)\big),\\
\tilde{\mathcal{R}}'^3[z] &= \frac{4r}{(r^2+a^2)^3}\big(1+O(r^{-2}\delta_0)\big),\\
\tilde{\mathcal{R}}'^4[z] &= f(1-2\delta_0 z_0),
\end{aligned} \tag{8.1.23}$$

where, as in (7.1.23), $f = \partial_r z_0 = -\frac{2\mathcal{T}}{(r^2+a^2)^3}$.

2. *The coefficients* $\tilde{\mathcal{R}}''^{\underline{a}}[z] = \partial_r\Big(\frac{hz^{1/2}\tilde{\mathcal{R}}'^{\underline{a}}}{\Delta^{1/2}}\Big)$ *are given by*

$$\begin{aligned}\tilde{\mathcal{R}}''^1[z] &= -\frac{2(3mr^4-4a^2r^3+ma^4)}{r^2(r^2-a^2)^2}\delta_0 + O(\delta_0^2 r^{-3}),\\ \tilde{\mathcal{R}}''^2[z] &= -\frac{16a^2r}{(r^2-a^2)^2} + O(\delta_0 r^{-3}),\\ \tilde{\mathcal{R}}''^3[z] &= -\frac{8r}{(r^2-a^2)^2}\Big(1+O(\delta_0 r^{-2})\Big),\\ \tilde{\mathcal{R}}''^4[z] &= \frac{-2(3mr^4-4a^2r^3+ma^4)}{r^2(r^2-a^2)^2} + O(\delta_0 r^{-3}).\end{aligned} \tag{8.1.24}$$

3. *The coefficients* $\mathcal{A}^{\underline{a}}[z] = -z^{1/2}\Delta^{3/2}\tilde{\mathcal{R}}''^{\underline{a}}$ *are given by*

$$\begin{aligned}\mathcal{A}^1[z] &= \delta_0\Delta^2\left(\frac{2(3mr^4-4a^2r^3+ma^4)}{r^2(r^2+a^2)(r^2-a^2)^2} + O(\delta_0 r^{-5})\right),\\ \mathcal{A}^2[z] &= \frac{2\Delta^2}{r^2+a^2}\left(\frac{8a^2r}{(r^2-a^2)^2} + O(\delta_0 r^{-3})\right),\\ \mathcal{A}^3[z] &= \Delta^2\left(\frac{8r}{(r^2+a^2)(r^2-a^2)^2} + O(\delta_0 r^{-7})\right),\\ \mathcal{A}^4[z] &= \Delta^2\left(\frac{2(3mr^4-4a^2r^3+ma^4)}{r^2(r^2+a^2)(r^2-a^2)^2} + O(\delta_0 r^{-5})\right).\end{aligned} \tag{8.1.25}$$

4. *The coefficients* $\mathcal{V}^{\underline{a}}$ *verify*

$$\mathcal{V}^1 = \delta_0\mathcal{V}, \qquad \mathcal{V}^2 = O(r^{-1}), \qquad \mathcal{V}^3 = O(r^{-3}), \qquad \mathcal{V}^4 = \mathcal{V} + O(\delta_0 r^{-3}), \tag{8.1.26}$$

where $\mathcal{V}$ *is the scalar function given by* (7.1.28).

Proof. With this choice for z we deduce

$$\tilde{\mathcal{R}}'^a[z] \;=\; \partial_r\Big(\frac{z}{\Delta}\mathcal{R}^{\underline{a}}\Big) = \partial_r\Big(\frac{z_0(1-\delta z_0)}{\Delta}\mathcal{R}^{\underline{a}}\Big) = (1-\delta z_0)\tilde{\mathcal{R}}'^a[z_0] - \delta(\partial_r z_0)\frac{z_0}{\Delta}\mathcal{R}^{\underline{a}}.$$

Hence,

$$\tilde{\mathcal{R}}'^a[z] = (1-\delta_0 z_0)\tilde{\mathcal{R}}'^a[z_0] - \delta_0 f\frac{1}{(r^2+a^2)^2}\mathcal{R}^{\underline{a}}. \tag{8.1.27}$$

In particular, using Lemma 8.1.6 for the coefficients $\tilde{\mathcal{R}}'^a[z_0]$, and (3.5.8) for $\mathcal{R}^{\underline{a}}$, we obtain for $\delta_0 > 0$ sufficiently small and all $r \geq r_+$,

$$\begin{aligned}
\tilde{\mathcal{R}}'^1[z] &= \delta_0 f,\\
\tilde{\mathcal{R}}'^2[z] &= \frac{4r}{(r^2+a^2)^2} - \delta_0\frac{4r\Delta}{(r^2+a^2)^4} + \delta_0 f\frac{2}{(r^2+a^2)} = \frac{4r}{(r^2+a^2)^2}\big(1+O(r^{-2}\delta_0)\big),\\
\tilde{\mathcal{R}}'^3[z] &= \frac{4r}{(r^2+a^2)^3} - \delta_0\frac{4r\Delta}{(r^2+a^2)^5} + \delta_0 f\frac{1}{(r^2+a^2)^2} = \frac{4r}{(r^2+a^2)^3}\big(1+O(r^{-2}\delta_0)\big),\\
\tilde{\mathcal{R}}'^4[z] &= f - \delta_0\frac{2\Delta}{(r^2+a^2)^2}f = f(1-2\delta_0 z_0),
\end{aligned}$$

which ends the proof of part 1. To check part 2, we write

$$z^{1/2} = z_0^{1/2}(1-\delta_0 z_0)^{1/2} = \frac{\Delta^{1/2}}{r^2+a^2}(1-\delta_0 z_0)^{1/2}.$$

Relying on the same function h as before i.e. $h = \frac{(r^2+a^2)^4}{r(r^2+a^2)}$, we deduce

$$\begin{aligned}
\tilde{\mathcal{R}}''^{\underline{a}} &= \partial_r\left(\frac{hz^{1/2}\tilde{\mathcal{R}}'^{\underline{a}}[z]}{\Delta^{1/2}}\right) = \partial_r\left(\frac{(r^2+a^2)^3}{r(r^2-a^2)}(1-\delta_0 z_0)^{1/2}\tilde{\mathcal{R}}'^{\underline{a}}[z]\right)\\
&= \partial_r\left(\frac{(r^2+a^2)^3}{r(r^2-a^2)}\tilde{\mathcal{R}}'^{\underline{a}}[z]\right)(1-\delta_0 z_0)^{1/2} + \frac{(r^2+a^2)^3}{r(r^2-a^2)}\tilde{\mathcal{R}}'^{\underline{a}}[z]\left(-\frac{\delta_0}{2}\right)\frac{\partial_r z_0}{(1-\delta_0 z_0)^{1/2}}\\
&= \partial_r\left(\frac{(r^2+a^2)^3}{r(r^2-a^2)}\tilde{\mathcal{R}}'^{\underline{a}}[z]\right)(1-\delta_0 z_0)^{1/2} + \frac{\delta_0\mathcal{T}}{r(r^2-a^2)(1-\delta_0 z_0)^{1/2}}\tilde{\mathcal{R}}'^{\underline{a}}[z].
\end{aligned}$$

We therefore compute, relying on (8.1.23),

$$\begin{aligned}
\tilde{\mathcal{R}}''^1 &= \partial_r\left(\frac{(r^2+a^2)^3}{r(r^2-a^2)}\tilde{\mathcal{R}}'^1[z]\right)(1-\delta_0 z_0)^{1/2} + \frac{\delta_0\mathcal{T}}{r(r^2-a^2)(1-\delta_0 z_0)^{1/2}}\tilde{\mathcal{R}}'^1[z]\\
&= \delta_0\partial_r\left(\frac{(r^2+a^2)^3 f}{r(r^2-a^2)}\right)(1-\delta_0 z_0)^{1/2} + \frac{\delta_0^2 f\mathcal{T}}{r(r^2-a^2)(1-\delta_0 z_0)^{1/2}}\\
&= -\delta_0\partial_r\left(\frac{2\mathcal{T}}{r(r^2-a^2)}\right) + O(\delta_0^2 r^{-3}) = -2\frac{3mr^4-4a^2r^3+ma^4}{r^2(r^2-a^2)^2}\delta_0 + O(\delta_0^2 r^{-3}),\\
\tilde{\mathcal{R}}''^3 &= \partial_r\left(\frac{(r^2+a^2)^3}{r(r^2-a^2)}\tilde{\mathcal{R}}'^3[z]\right)(1-\delta_0 z_0)^{1/2} + \frac{\delta_0\mathcal{T}}{r(r^2-a^2)(1-\delta_0 z_0)^{1/2}}\tilde{\mathcal{R}}'^3[z]\\
&= \partial_r\left(\frac{4}{r^2-a^2}\right) + O(\delta_0 r^{-5}) = -\frac{8r}{(r^2-a^2)^2} + O(\delta_0 r^{-5}),\\
\tilde{\mathcal{R}}''^4 &= \partial_r\left(\frac{(r^2+a^2)^3}{r(r^2-a^2)}\tilde{\mathcal{R}}'^4[z]\right)(1-\delta_0 z_0)^{1/2} + \frac{\delta_0\mathcal{T}}{r(r^2-a^2)(1-\delta_0 z_0)^{1/2}}\tilde{\mathcal{R}}'^4[z]\\
&= -\partial_r\Big(\frac{2\mathcal{T}}{r(r^2-a^2)}\Big) + O(\delta_0 r^{-3}) = \frac{-2(3mr^4-4a^2r^3+ma^4)}{r^2(r^2-a^2)^2} + O(\delta_0 r^{-3}).
\end{aligned}$$

We also have

$$\begin{aligned}
\tilde{\mathcal{R}}''^2 &= \partial_r\left(\frac{(r^2+a^2)^3}{r(r^2-a^2)}\tilde{\mathcal{R}}'^2[z]\right)(1-\delta_0 z_0)^{1/2} + \frac{\delta_0\mathcal{T}}{r(r^2-a^2)(1-\delta_0 z_0)^{1/2}}\tilde{\mathcal{R}}'^2[z]\\
&= \partial_r\left(\frac{(r^2+a^2)^3}{r(r^2-a^2)}\left(\frac{4r}{(r^2+a^2)^2} - \delta_0\frac{4r\Delta}{(r^2+a^2)^4} + \delta_0 f\frac{2}{(r^2+a^2)}\right)\right)(1-\delta_0 z_0)^{1/2}\\
&\quad + \frac{4\delta_0 r\mathcal{T}}{r(r^2-a^2)(r^2+a^2)^2(1-\delta_0 z_0)^{1/2}} = -\frac{16a^2 r}{(r^2-a^2)^2} + O(\delta_0 r^{-3}).
\end{aligned}$$

The second part of the lemma is immediate from the definition

$$\mathcal{A}^{\underline{a}} = -z^{1/2}\Delta^{3/2}\tilde{\mathcal{R}}''^{\underline{a}} = -\frac{\Delta^2}{r^2+a^2}\Big(1+O(\delta_0 r^{-2})\Big)\tilde{\mathcal{R}}''^{\underline{a}}.$$

The last part follows easily from the definition of $\mathcal{V}^{\underline{a}}$ and the case $\delta_0 = 0$ in Lemma 8.1.6. □

Remark 8.1.8. *Note that we can write*

$$\begin{aligned}
\mathcal{A}^1 &= \delta_0\mathcal{A}\big(1+O(r^{-1}\delta_0)\big), \quad & \mathcal{A}^4 &= \mathcal{A}\big(1+O(r^{-1}\delta_0)\big),\\
\mathcal{A}^2 &= \widetilde{\mathcal{A}}\big(2a^2+O(\delta_0)\big), & \mathcal{A}^3 &= \tilde{\mathcal{A}}\big(1+O(r^{-2}\delta_0)\big),
\end{aligned} \tag{8.1.28}$$

where

$$\mathcal{A} = \frac{2\Delta^2}{r^2(r^2+a^2)(r^2-a^2)^2}(3mr^4 - 4a^2r^3 + ma^4), \qquad \widetilde{\mathcal{A}} = \frac{8\Delta^2 r}{(r^2+a^2)(r^2-a^2)^2},$$

are positive coefficients for $|a|/m < 1$. We also note the term $\mathcal{A}$ is precisely the one appearing in Proposition 7.1.8.

8.2 Computation of the effective generalized current

From now on, we consider the choices for z and h made in Lemma 8.1.7, i.e.

$$z = z_0 - \delta_0 z_0^2, \qquad z_0 = \frac{\Delta}{(r^2+a^2)^2}, \qquad \delta_0 > 0, \qquad h = \frac{(r^2+a^2)^4}{r(r^2-a^2)}.$$

Recall that according to Proposition 8.1.5, the effective generalized current $\widetilde{\mathcal{E}}[\mathbf{X},\mathbf{w},\mathbf{M}]$ introduced in (8.1.19) is given by, see (8.1.13),

$$|q|^2\widetilde{\mathcal{E}}[\mathbf{X},\mathbf{w},\mathbf{M}] = \widetilde{P} + P_{lot} + I + J + K.$$

In this section, we compute the terms $\widetilde{P}$, P_{lot}, I, J and K.

8.2.1 Principal trapping term

In this section, we analyze the principal trapping term $\widetilde{P}$ in (8.1.14), i.e.

$$\widetilde{P} = \frac{1}{2} h L^{\alpha\beta} \dot{\mathbf{D}}_\alpha \Psi_z \dot{\mathbf{D}}_\beta \Psi_z, \qquad \Psi_z = \tilde{\mathcal{R}}'^{\underline{a}}[z]\psi_{\underline{a}}.$$

Observe that with the choice $z = z_0 - \delta_0 z_0^2$, using (8.1.23), we have

$$\Psi_z = f\big(\delta_0 \mathcal{S}_1 \psi + (1 + O(r^{-2}\delta_0))\mathcal{O}\psi\big) + \frac{4r}{(r^2+a^2)^2}\Big(\mathcal{S}_2\psi + \frac{1}{r^2+a^2}\mathcal{S}_3\psi\Big)\big(1 + O(r^{-2}\delta_0)\big).$$

Observe that

$$\mathcal{S}_2\psi + \frac{1}{r^2+a^2}\mathcal{S}_3\psi = a\nabla_T\nabla_Z\psi + \frac{a^2}{r^2+a^2}\nabla_Z\nabla_Z\psi = a\nabla_{\widehat{T}}\nabla_Z\psi$$

so we can write

$$\Psi_z = f\big(\delta_0 \mathcal{S}_1 \psi + (1 + O(r^{-2}\delta_0))\mathcal{O}\psi\big) + \frac{4ar}{(r^2+a^2)^2}\nabla_{\widehat{T}}\nabla_Z\psi\big(1 + O(r^{-2}\delta_0)\big). \tag{8.2.1}$$

By defining[1]

$$\mathcal{L}^1 = \delta_0, \qquad \mathcal{L}^2 = 0, \qquad \mathcal{L}^3 = \mathcal{L}^4 = 1, \tag{8.2.2}$$

from (8.1.9) and (8.1.14), we obtain

$$\widetilde{P} = \frac{1}{2} h L^{\alpha\beta} \dot{\mathbf{D}}_\alpha \Psi_z \cdot \dot{\mathbf{D}}_\beta \Psi_z = \frac{1}{2} h \Big(\delta_0 \big|\nabla_T \Psi_z\big|^2 + a^2 \big|\nabla_Z \Psi_z\big|^2 + O^{\alpha\beta} \dot{\mathbf{D}}_\alpha \Psi_z \dot{\mathbf{D}}_\beta \Psi_z\Big). \tag{8.2.3}$$

We can also simplify the term P_{lot}. From (8.1.15) and (8.1.23), we can write

$$\begin{aligned}
P_{lot} &= -\frac{1}{2} h\Psi \cdot \big(\tilde{\mathcal{R}}'^{\underline{c}} \mathcal{L}^{\underline{b}} [\mathcal{S}_{\underline{c}}, \mathcal{S}_{\underline{b}}]\psi\big)\\
&= -\frac{1}{2} h\Psi \cdot \Big(\big(\tilde{\mathcal{R}}'^1\mathcal{L}^4 - \tilde{\mathcal{R}}'^4\mathcal{L}^1\big)[\mathcal{S}_1, \mathcal{O}]\psi + \tilde{\mathcal{R}}'^2\mathcal{L}^4[\mathcal{S}_2, \mathcal{O}]\psi + \big(\tilde{\mathcal{R}}'^3\mathcal{L}^4 - \tilde{\mathcal{R}}'^4\mathcal{L}^3\big)[\mathcal{S}_3, \mathcal{O}]\psi\Big)\\
&= -\frac{1}{2} h\Psi \cdot \Big(O(\delta_0^2 r^{-5})[\mathcal{S}_1, \mathcal{O}]\psi + O(r^{-3})[\mathcal{S}_2, \mathcal{O}]\psi + O(r^{-3})[\mathcal{S}_3, \mathcal{O}]\psi\Big)\\
&= -\frac{1}{2} h\Psi \cdot \Big(O(ar^{-2})\nabla\nabla_T\psi + O(a^2 r^{-2})\nabla\nabla_Z\psi + O(ar^{-3})\mathfrak{d}^{\leq 1}\psi\Big),
\end{aligned}$$

which, since $h = O(r^5)$ and $\Psi = O(r^{-3})\mathfrak{d}^{\leq 2}\psi$, can be summarized as

$$P_{lot} = O(ar^{-1})(\mathfrak{d}^{\leq 2}\psi)^2. \tag{8.2.4}$$

[1] The relevance of this particular choice will become evident in Section 8.3.1, where it will be used to derive a Poincaré inequality.

8.2.2 Integration by parts identities

In the next section, we compute the quadratic forms I, J, K as given in (8.1.16), (8.1.17), (8.1.18). In order to do that, we will make use of some integration by parts identities, which we collect in Lemma 8.2.3 below. Such computations are important to observe that the mixed products between the symmetry operators $\mathcal{S}_1$ and $\mathcal{O}$ contain positive definite norms, modulo lower order or boundary terms.

We start with the following definition that will be useful to take care of boundary terms in the integrations by parts.

Definition 8.2.1. *We denote by $M(\psi)$ quadratic terms of the following type*

$$\nabla_T\psi\cdot\mathcal{S}_{\underline{a}}\psi, \qquad |q|^2\nabla\psi\cdot\nabla\nabla_T\psi, \qquad |q|^2\nabla\psi\cdot\nabla\nabla_Z\psi,$$

Also, we denote by $M(\nabla_r\psi)$ denote quadratic terms of the following type

$$\nabla_T\nabla_r\psi\cdot\mathcal{S}_{\underline{a}}\nabla_r\psi, \qquad |q|^2\nabla\nabla_r\psi\cdot\nabla\nabla_T\nabla_r\psi, \qquad |q|^2\nabla\nabla_r\psi\cdot\nabla\nabla_Z\nabla_r\psi.$$

Remark 8.2.2. *Notice in view of Definition 8.2.1 the following pointwise estimates for $M(\psi)$ and $M(\nabla_r\psi)$:*

$$\begin{aligned}|M(\psi)| &\lesssim |(\nabla_T, \not{\mathfrak{d}})^{\leq 1}\psi||(\nabla_T, \not{\mathfrak{d}})^{\leq 2}\psi|,\\ |M(\nabla_r\psi)| &\lesssim |\nabla_r^{\leq 1}(\nabla_T, \not{\mathfrak{d}})^{\leq 1}\psi||\nabla_r^{\leq 1}(\nabla_T, \not{\mathfrak{d}})^{\leq 2}\psi|.\end{aligned} \tag{8.2.5}$$

Lemma 8.2.3. *For any function $H = H(r)$, the following identities hold true:*

$$\begin{aligned}
H\mathcal{O}(\psi)\cdot\mathcal{S}_1\psi &= H|q|^2|\nabla\nabla_T\psi|^2 - O(ar^{-2})H(\mathfrak{d}^{\leq 2}\psi)^2 + |q|^2\dot{\mathbf{D}}_\beta(H|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot\mathcal{S}_1\psi)\\
&\quad +\partial_t(HM(\psi)),\\
H\nabla_r\mathcal{O}\psi\cdot\nabla_r\mathcal{S}_1\psi &= H|q|^2|\nabla\nabla_T\nabla_r\psi|^2 - O(ar^{-2})H(\nabla_r\mathfrak{d}^{\leq 2}\psi)^2 - O(ar^{-2})H(\mathfrak{d}^{\leq 2}\psi)^2\\
&\quad +|q|^2\dot{\mathbf{D}}_\beta(H|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\nabla_r\psi\cdot\mathcal{S}_1\nabla_r\psi) + \partial_t(HM(\nabla_r\psi)),\\
H\nabla_r\mathcal{O}(\psi)\cdot\mathcal{S}_1\psi &= H|q|^2\nabla\nabla_T\nabla_r\psi\cdot\nabla\nabla_T\psi - O(ar^{-2})H(\nabla_r\mathfrak{d}^{\leq 2}\psi)^2 - O(ar^{-2})H(\mathfrak{d}^{\leq 2}\psi)^2\\
&\quad +|q|^2\dot{\mathbf{D}}_\beta(H|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\nabla_r\psi\cdot\mathcal{S}_1\psi) + \partial_t(HM(\nabla_r\psi)),\\
H\mathcal{O}(\psi)\cdot\nabla_r\mathcal{S}_1\psi &= H|q|^2\nabla\nabla_T\nabla_r\psi\cdot\nabla\nabla_T\psi - O(ar^{-2})H(\nabla_r\mathfrak{d}^{\leq 2}\psi)^2 - O(ar^{-2})H(\mathfrak{d}^{\leq 2}\psi)^2\\
&\quad +|q|^2\dot{\mathbf{D}}_\beta(H|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot\mathcal{S}_1\nabla_r\psi) + \partial_t(HM(\nabla_r\psi)).
\end{aligned}$$

In all the above, $M(\psi)$ and $M(\nabla_r\psi)$ denote the quadratic expressions in ψ and its derivatives of Definition 8.2.1.

Proof. Using (3.7.8), i.e. $\mathcal{O}(\psi) = |q|^2\dot{\mathbf{D}}_\beta(|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi)$, we obtain for any Φ,

$$\begin{aligned}|q|^{-2}\mathcal{O}(\psi)\cdot\Phi &= \dot{\mathbf{D}}_\beta(|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi)\cdot\Phi\\ &= \dot{\mathbf{D}}_\beta(|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot\Phi) - |q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot\dot{\mathbf{D}}_\beta\Phi.\end{aligned} \tag{8.2.6}$$

Applying the above to $\Phi = \mathcal{S}_1\psi = \nabla_T\nabla_T\psi$, and using that $[\nabla, \nabla_T]\psi = O(ar^{-4})\,\mathfrak{\not{\partial}}^{\leq 1}\psi$ we obtain

$$\begin{aligned} H\mathcal{O}(\psi)\cdot\mathcal{S}_1\psi &= -H|q|^2\nabla\psi\cdot\nabla\nabla_T\nabla_T\psi + |q|^2\dot{\mathbf{D}}_\beta(H|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot\mathcal{S}_1\psi)\\ &= -H|q|^2\nabla\psi\cdot\nabla_T\nabla\nabla_T\psi - O(ar^{-2})H(\mathfrak{d}^{\leq 2}\psi)^2\\ &\quad +|q|^2\dot{\mathbf{D}}_\beta(H|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot\mathcal{S}_1\psi)\\ &= |q|^2H|\nabla\nabla_T\psi|^2 - O(ar^{-2})H(\mathfrak{d}^{\leq 2}\psi)^2 + |q|^2\dot{\mathbf{D}}_\beta(H|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot\mathcal{S}_1\psi)\\ &\quad -\partial_t\big(H|q|^2\nabla\psi\cdot\nabla\nabla_T\psi\big)\\ &= H|q|^2|\nabla\nabla_T\psi|^2 - O(ar^{-2})H(\mathfrak{d}^{\leq 2}\psi)^2 + |q|^2\dot{\mathbf{D}}_\beta(H|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot\mathcal{S}_1\psi)\\ &\quad +\partial_t(HM(\psi)) \end{aligned}$$

as stated.

Using $[\nabla_r, \mathcal{O}]\psi = O(ar^{-2})\,\mathfrak{\not{\partial}}^{\leq 1}\psi$, we obtain

$$\begin{aligned} &H\nabla_r\mathcal{O}\psi\cdot\nabla_r\mathcal{S}_1\psi\\ =\;& H\mathcal{O}\nabla_r\psi\cdot\mathcal{S}_1\nabla_r\psi + HO(ar^{-2})\,\mathfrak{\not{\partial}}^{\leq 1}\psi\cdot\nabla_r\mathcal{S}_1\psi + H\mathcal{O}\nabla_r\psi\cdot O(ar^{-4})\mathfrak{d}^{\leq 1}\psi\\ =\;& H|q|^2|\nabla\nabla_T\nabla_r\psi|^2 - O(ar^{-2})H\nabla\nabla_r\psi\cdot\,\mathfrak{\not{\partial}}^{\leq 1}\nabla_T\nabla_r\psi + |q|^2\dot{\mathbf{D}}_\beta(H|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\nabla_r\psi\cdot\mathcal{S}_1\nabla_r\psi)\\ &-\partial_t(|q|^2H\nabla\nabla_r\psi\cdot\nabla\nabla_T\nabla_r\psi) + HO(ar^{-2})\,\mathfrak{\not{\partial}}^{\leq 1}\psi\cdot\nabla_r\mathcal{S}_1\psi + H\mathcal{O}\nabla_r\psi\cdot O(ar^{-4})\mathfrak{d}^{\leq 1}\psi \end{aligned}$$

which can be written as

$$\begin{aligned} H\nabla_r\mathcal{O}\psi\cdot\nabla_r\mathcal{S}_1\psi &= H|q|^2|\nabla\nabla_T\nabla_r\psi|^2 + |q|^2\dot{\mathbf{D}}_\beta(H|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\nabla_r\psi\cdot\mathcal{S}_1\nabla_r\psi)\\ &\quad +\partial_t(HM(\nabla_r\psi)) - O(ar^{-2})H(\nabla_r\mathfrak{d}^{\leq 2}\psi)^2 - O(ar^{-2})H(\mathfrak{d}^{\leq 2}\psi)^2, \end{aligned}$$

as stated.

The mixed products $H\nabla_r\mathcal{O}\psi\cdot\mathcal{S}_1\psi$ and $H\mathcal{O}\psi\cdot\nabla_r\mathcal{S}_1\psi$ are treated similarly. This concludes the proof of Lemma 8.2.3. □

We use the above lemma to derive the following general computation.

Lemma 8.2.4. *Let $\Phi_{\underline{a}} = \mathcal{S}_{\underline{a}}\Phi$ for some*[2] *Φ, and let $\mathcal{Y}^{\underline{a}}$ be some coefficients only depending on r, such that*

$$\mathcal{Y}^1 = \delta_0\mathcal{Y}, \qquad \mathcal{Y}^4 = \mathcal{Y}.$$

Then for $\mathcal{L}^{\underline{a}}$ given by (8.2.2)*, i.e.*

$$\mathcal{L}^1 = \delta_0, \qquad \mathcal{L}^2 = 0, \qquad \mathcal{L}^3 = \mathcal{L}^4 = 1,$$

[2]We will apply it to $\Phi = \psi$ and $\Phi = \nabla_r\psi$.

we have

$$\begin{aligned}\big(\mathcal{Y}^{\underline{a}}\Phi_{\underline{a}}\big)\cdot\big(\mathcal{L}^{\underline{a}}\Phi_{\underline{a}}\big) &= \mathcal{Y}\Big(\delta_0^2|\mathcal{S}_1\Phi|^2+|\mathcal{O}\Phi|^2+2\delta_0|q|^2|\nabla\nabla_T\Phi|^2\Big)\\ &\quad -O(a)(|\mathcal{Y}|+|\mathcal{Y}^2|+|\mathcal{Y}^3|)(\mathfrak{d}^{\leq 2}\psi)^2+Bdr[\Phi],\end{aligned}$$

where the boundary term is given by

$$Bdr[\Phi] = \partial_t\Big(\delta_0\mathcal{Y}r^2M(\Phi)\Big)+|q|^2\dot{\mathbf{D}}_\beta\Big(2\delta_0|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\Phi\cdot\mathcal{Y}\mathcal{S}_1\Phi\Big).$$

Proof. By writing

$$\begin{aligned}\big(\mathcal{Y}^{\underline{a}}\Phi_{\underline{a}}\big) &= \delta_0\mathcal{Y}(\mathcal{S}_1\Phi)+\mathcal{Y}(\mathcal{O}\Phi)+\mathcal{Y}^2(\mathcal{S}_2\Phi)+\mathcal{Y}^3(\mathcal{S}_3\Phi)\\ \big(\mathcal{L}^{\underline{a}}\Phi_{\underline{a}}\big) &= \delta_0\mathcal{S}_1\Phi+\mathcal{O}\Phi+\mathcal{S}_3\Phi\end{aligned}$$

we obtain

$$\begin{aligned}\big(\mathcal{Y}^{\underline{a}}\Phi_{\underline{a}}\big)\cdot\big(\mathcal{L}^{\underline{a}}\Phi_{\underline{a}}\big) &= \delta_0^2\mathcal{Y}|\mathcal{S}_1\Phi|^2+\mathcal{Y}|\mathcal{O}\Phi|^2+\mathcal{Y}^3|\mathcal{S}_3\Phi|^2+(\mathcal{Y}+\mathcal{Y}^3)\delta_0\mathcal{S}_1\Phi\cdot\mathcal{S}_3\Phi\\ &\quad +2\delta_0\mathcal{Y}(\mathcal{S}_1\Phi\cdot\mathcal{O}\Phi)+(\mathcal{Y}+\mathcal{Y}^3)\mathcal{S}_3\Phi\cdot\mathcal{O}\Phi\\ &\quad +(\mathcal{Y}^2\mathcal{S}_2\Phi)\cdot(\delta_0\mathcal{S}_1\Phi+\mathcal{O}\Phi+\mathcal{S}_3\Phi).\end{aligned}$$

Since $\mathcal{S}_1\psi, \mathcal{O}\psi = O(1)\mathfrak{d}^{\leq 2}\psi$, $\mathcal{S}_2\psi = O(a)\mathfrak{d}^{\leq 2}\psi$ and $\mathcal{S}_3\psi = O(a^2)\mathfrak{d}^{\leq 2}\psi$, we infer

$$\begin{aligned}\big(\mathcal{Y}^{\underline{a}}\Phi_{\underline{a}}\big)\cdot\big(\mathcal{L}^{\underline{a}}\Phi_{\underline{a}}\big) &= \delta_0^2\mathcal{Y}|\mathcal{S}_1\Phi|^2+\mathcal{Y}|\mathcal{O}\Phi|^2+2\delta_0\mathcal{Y}(\mathcal{S}_1\Phi\cdot\mathcal{O}\Phi)\\ &\quad +O(a)(|\mathcal{Y}|+|\mathcal{Y}^2|+|\mathcal{Y}^3|)(\mathfrak{d}^{\leq 2}\psi)^2.\end{aligned}$$

Using Lemma 8.2.3, we obtain the desired identity. □

8.2.3 The quadratic forms I, J and K

The goal of this section is to compute the quadratic forms I, J and K appearing in the computation (8.1.13) of the effective generalized current $\widetilde{\mathcal{E}}[\mathbf{X},\mathbf{w},\mathbf{M}]$ by making use of the integration by parts identities of section 8.2.2. Recall that I, J and K are given by (8.1.16), (8.1.17), (8.1.18) and that we make use of the choice of z and h in Lemma 8.1.7, i.e.

$$z = z_0-\delta_0 z_0^2, \qquad z_0 = \frac{\Delta}{(r^2+a^2)^2}, \qquad \delta_0>0, \qquad h = \frac{(r^2+a^2)^4}{r(r^2-a^2)}.$$

The quadratic form I

Recall that the quadratic form I is given by (8.1.16), i.e.

$$I = \big(\mathcal{A}^{\underline{a}}[z]\nabla_r\psi_{\underline{a}}\big)\cdot\big(\mathcal{L}^{\underline{a}}\nabla_r\psi_{\underline{a}}\big),$$

where $\mathcal{L}^{\underline{a}}$ is given by (8.2.2), and where $\mathcal{A}^{\underline{a}}[z]$ is given by (8.1.25), which may be rewritten under the form (8.1.28),i.e.

$$\mathcal{A}^1 = \delta_0\mathcal{A}\big(1+O(r^{-1}\delta_0)\big), \quad \mathcal{A}^4 = \mathcal{A}\big(1+O(r^{-1}\delta_0)\big),$$
$$\mathcal{A}^2 = \widetilde{\mathcal{A}}\big(2a^2+O(\delta_0)\big), \quad \mathcal{A}^3 = \tilde{\mathcal{A}}\big(1+O(r^{-2}\delta_0)\big),$$

where

$$\mathcal{A} = \frac{2\Delta^2}{r^2(r^2+a^2)(r^2-a^2)^2}(3mr^4-4a^2r^3+ma^4), \qquad \widetilde{\mathcal{A}} = \frac{8\Delta^2 r}{(r^2+a^2)(r^2-a^2)^2},$$

are positive coefficients for $|a|/m<1$. We may thus apply Lemma 8.2.4 to I with the choice $\mathcal{Y}^{\underline{a}} = \mathcal{A}^{\underline{a}}$. Using the structure or the commutators $[\nabla_r, \mathcal{S}_{\underline{a}}]$, we obtain

$$\begin{aligned} I =& \mathcal{A}(1+O(r^{-1}\delta_0))\Big(\delta_0^2|\nabla_r\mathcal{S}_1\psi|^2+|\nabla_r\mathcal{O}\psi|^2+2\delta_0|q|^2|\nabla\nabla_T\nabla_r\psi|^2\Big)\\ &-O(a)(|\mathcal{A}|+|\widetilde{\mathcal{A}}|)(1+O(r^{-1}\delta_0))\Big((\nabla_r\mathfrak{d}^{\leq 2}\psi)^2+r^{-2}(\mathfrak{d}^{\leq 2}\psi)^2\Big)+\mathrm{Bdr}[\psi]_I, \end{aligned} \tag{8.2.7}$$

where the boundary term is given by

$$\mathrm{Bdr}[\psi]_I = \partial_t\Big(\delta_0\mathcal{A}M(\nabla_r\psi)\Big)+|q|^2\dot{\mathbf{D}}_\beta\Big(2\delta_0|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\nabla_r\psi\cdot\mathcal{A}\mathcal{S}_1\nabla_r\psi\Big).$$

Remark 8.2.5. *Observe that the estimate for the quadratic form I for the choice $z=z_0$ can be deduced by plugging $\delta_0=0$ in (8.2.7). In that case one can see that one does not control the term $|\nabla_r\mathcal{S}_1\psi|^2$. This is the main reason one is led to use the choice $z=z_0-\delta_0 z_0^2$.*

The quadratic forms J and K

As a corollary of Lemma 8.2.4, we obtain the expressions for J and K in Proposition 8.1.5, i.e.

$$J = \big(\mathcal{V}^{\underline{a}}[z]\psi_{\underline{a}}\big)\cdot\big(\mathcal{L}^{\underline{a}}\psi_{\underline{a}}\big), \qquad K = \frac{1}{4}|q|^2\mathbf{D}^\mu\Big((M^{\underline{a}}_\mu\psi_{\underline{a}})\cdot(\mathcal{L}^{\underline{b}}\psi_{\underline{b}})\Big).$$

Lemma 8.2.6. *The term J is given by*

$$J = \big(\mathcal{V} + O(\delta_0 r^{-3})\big)\Big(\delta_0^2|\mathcal{S}_1\psi|^2 + |\mathcal{O}\psi|^2 + 2\delta_0|q|^2|\nabla\nabla_T\psi|^2\Big) - O(ar^{-1})(\mathfrak{d}^{\leq 2}\psi)^2 + Bdr[\psi]_J, \tag{8.2.8}$$

with boundary term

$$Bdr_J[\psi] = \partial_t\Big(\delta_0(\mathcal{V} + O(r^{-3}))r^2 M(\psi)\Big) + |q|^2\dot{\mathbf{D}}_\beta\Big(2\delta_0|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot(\mathcal{V} + O(a^2r^{-3}))\mathcal{S}_1\psi\Big).$$

For a one-form of the type $M^{\underline{a}} := v^{\underline{a}}\partial_r$, with $v^2 = v^3 = 0$, $v^1 = \delta_0 v$ and $v^4 = v$ for some given function $v = v(r)$, the term K is given by

$$\begin{aligned}
K &= \frac{|q|^2}{2}v\Big(\delta_0^2\nabla_r\mathcal{S}_1\psi\cdot\mathcal{S}_1\psi + \nabla_r\mathcal{O}\psi\cdot\mathcal{O}\psi + 2\delta_0|q|^2\nabla\nabla_T\nabla_r\psi\cdot\nabla\nabla_T\psi\Big)\\
&\quad + \frac{|q|^2}{4}v'\Big(\delta_0^2|\mathcal{S}_1\psi|^2 + |\mathcal{O}\psi|^2 + 2\delta_0|q|^2|\nabla\nabla_T\psi|^2\Big)\\
&\quad - vO(ar^{\frac{5}{2}})(\nabla_r\mathfrak{d}^{\leq 2}\psi)^2 - O(ar^{\frac{3}{2}})v(\mathfrak{d}^{\leq 2}\psi)^2 - O(ar^2)v'(\mathfrak{d}^{\leq 2}\psi)^2 + Bdr[\psi]_K,
\end{aligned}$$

where we denoted

$$v'^{\underline{a}} := \partial_r v^{\underline{a}} + \frac{2r}{|q|^2}v^{\underline{a}},$$

and with boundary term

$$\begin{aligned}
Bdr[\psi]_K &= \partial_t\big(vr^4 M(\nabla_r\psi)\big) + \frac{|q|^4}{4}\dot{\mathbf{D}}_\beta(2\delta_0 v|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot\mathcal{S}_1\nabla_r\psi)\\
&\quad + \frac{|q|^4}{4}\dot{\mathbf{D}}_\beta(2\delta_0 v|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\nabla_r\psi\cdot\mathcal{S}_1\psi) + \partial_t\Big(\delta_0 v' r^4 M(\psi)\Big)\\
&\quad + \frac{|q|^4}{4}\dot{\mathbf{D}}_\beta\Big(2\delta_0 v'|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot\mathcal{S}_1\psi\Big).
\end{aligned}$$

Proof. The expressions for J is a straightforward application of Lemma 8.2.4, using that, see (8.1.26),

$$\mathcal{V}^1 = \delta_0\mathcal{V}, \qquad \mathcal{V}^4 = \mathcal{V} + O(\delta_0 r^{-3}), \qquad \mathcal{V}^2 = O(r^{-1}), \qquad \mathcal{V}^3 = O(r^{-3}).$$

To write the term K, we first compute

$$\begin{aligned}
\frac{4}{|q|^2}K &= \mathbf{D}^\mu\Big((M_\mu^{\underline{a}}\psi_{\underline{a}})\cdot(\mathcal{L}^{\underline{b}}\psi_{\underline{b}})\Big)\\
&= (\mathbf{D}^\mu(M_\mu^{\underline{a}}\psi_{\underline{a}}))\cdot(\mathcal{L}^{\underline{b}}\psi_{\underline{b}}) + (M_\mu^{\underline{a}}\psi_{\underline{a}})\cdot\mathbf{D}^\mu(\mathcal{L}^{\underline{b}}\psi_{\underline{b}})\\
&= (\mathbf{D}^\mu M_\mu^{\underline{a}})\psi_{\underline{a}}\cdot(\mathcal{L}^{\underline{b}}\psi_{\underline{b}}) + (M_\mu^{\underline{a}}\mathbf{D}^\mu\psi_{\underline{a}})\cdot(\mathcal{L}^{\underline{b}}\psi_{\underline{b}}) + (M_\mu^{\underline{a}}\psi_{\underline{a}})\cdot\mathbf{D}^\mu(\mathcal{L}^{\underline{b}}\psi_{\underline{b}}).
\end{aligned}$$

For $M^{\underline{a}} = v^{\underline{a}}\partial_r$, we obtain

$$\begin{aligned}\frac{4}{|q|^2}K &= \left(\partial_r v^{\underline{a}} + \frac{2r}{|q|^2}v^{\underline{a}}\right)\psi_{\underline{a}}\cdot(\mathcal{L}^{\underline{b}}\psi_{\underline{b}}) + v^{\underline{a}}\nabla_r\psi_{\underline{a}}\cdot(\mathcal{L}^{\underline{b}}\psi_{\underline{b}}) + (v^{\underline{a}}\psi_{\underline{a}})\cdot\nabla_r(\mathcal{L}^{\underline{b}}\psi_{\underline{b}})\\ &= \left(\partial_r v^{\underline{a}} + \frac{2r}{|q|^2}v^{\underline{a}}\right)\psi_{\underline{a}}\cdot(\mathcal{L}^{\underline{b}}\psi_{\underline{b}}) + v^{\underline{a}}\nabla_r\psi_{\underline{a}}\cdot(\mathcal{L}^{\underline{b}}\psi_{\underline{b}}) + (v^{\underline{a}}\psi_{\underline{a}})\cdot\mathcal{L}^{\underline{b}}\nabla_r\psi_{\underline{b}}\\ &= (v'^{\underline{a}}\psi_{\underline{a}})\cdot(\mathcal{L}^{\underline{b}}\psi_{\underline{b}}) + v^{\underline{a}}\nabla_r\psi_{\underline{a}}\cdot(\mathcal{L}^{\underline{b}}\psi_{\underline{b}}) + (v^{\underline{a}}\psi_{\underline{a}})\cdot\mathcal{L}^{\underline{b}}\nabla_r\psi_{\underline{b}}\end{aligned}$$

where we wrote $v'^{\underline{a}} := \partial_r v^{\underline{a}} + \frac{2r}{|q|^2}v^{\underline{a}}$. By defining for some v

$$v^1 = \delta_0 v, \qquad v'^1 = \delta_0 v', \qquad v^4 = v, \qquad v'^4 = v', \qquad v^2 = v^3 = 0,$$

we can apply Lemma 8.2.4 to the first term, and obtain

$$\left(v'^{\underline{a}}\psi_{\underline{a}}\right)\cdot\left(\mathcal{L}^{\underline{a}}\psi_{\underline{a}}\right) = v'\left(\delta_0^2|\mathcal{S}_1\psi|^2 + |\mathcal{O}\psi|^2 + 2\delta_0|q|^2|\nabla\nabla_T\psi|^2\right) - O(a)v'(\mathfrak{d}^{\leq 2}\psi)^2 + \mathrm{Bdr}[\psi],$$

where the boundary term is given by

$$\mathrm{Bdr}[\psi] = \partial_t\Big(\delta_0 v' M(\psi)\Big) + |q|^2\dot{\mathbf{D}}_\beta\Big(2\delta_0|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot v'\mathcal{S}_1\psi\Big).$$

For the other two terms we write, using Lemma 8.2.3,

$$\begin{aligned}& v^{\underline{a}}\nabla_r\psi_{\underline{a}}\cdot(\mathcal{L}^{\underline{b}}\psi_{\underline{b}}) + (v^{\underline{a}}\psi_{\underline{a}})\cdot\mathcal{L}^{\underline{b}}\nabla_r\psi_{\underline{b}}\\ =\ & \big(\delta_0 v\nabla_r\mathcal{S}_1\psi + v\nabla_r\mathcal{O}\psi\big)\cdot(\delta_0\mathcal{S}_1\psi + \mathcal{S}_3\psi + \mathcal{O}\psi)\\ & +\big(\delta_0 v\mathcal{S}_1\psi + v\mathcal{O}\psi\big)\cdot(\delta_0\nabla_r\mathcal{S}_1\psi + \nabla_r\mathcal{S}_3\psi + \nabla_r\mathcal{O}\psi)\\ =\ & 2\delta_0^2 v\nabla_r\mathcal{S}_1\psi\cdot\mathcal{S}_1\psi + 2v\nabla_r\mathcal{O}\psi\cdot\mathcal{O}\psi + 2\delta_0 v\big(\nabla_r\mathcal{S}_1\psi\cdot\mathcal{O}\psi + \mathcal{S}_1\psi\cdot\nabla_r\mathcal{O}\psi\big)\\ & +O(a)v|\nabla_r\mathfrak{d}^{\leq 2}\psi||\mathfrak{d}^{\leq 2}\psi|\\ =\ & 2\delta_0^2 v\nabla_r\mathcal{S}_1\psi\cdot\mathcal{S}_1\psi + 2v\nabla_r\mathcal{O}\psi\cdot\mathcal{O}\psi + 4\delta_0 v|q|^2\nabla\nabla_T\nabla_r\psi\cdot\nabla\nabla_T\psi\\ & +2\delta_0 v\big(\nabla_r\mathcal{S}_2\psi\cdot\mathcal{S}_2\psi\big) + 2v|q|^2a^2\nabla\nabla_Z\nabla_r\psi\cdot\nabla\nabla_Z\psi\\ & -vO(ar^{\frac{1}{2}})(\nabla_r\mathfrak{d}^{\leq 2}\psi)^2 - vO(ar^{-\frac{1}{2}})(\mathfrak{d}^{\leq 2}\psi)^2 + \partial_t\big(vM(\nabla_r\psi)\big)\\ & +|q|^2\dot{\mathbf{D}}_\beta(2\delta_0 v|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot\mathcal{S}_1\nabla_r\psi) + |q|^2\dot{\mathbf{D}}_\beta(2\delta_0 v|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\nabla_r\psi\cdot\mathcal{S}_1\psi).\end{aligned}$$

Summing the two above, we obtain the lemma. □

8.2.4 Conclusion

We summarize the results obtained so far in the following proposition.

Proposition 8.2.7. *The effective generalized current is given by*

$$|q|^2\widetilde{\mathcal{E}}[\mathbf{X},\mathbf{w},\mathbf{M}] \;=\; \widetilde{P} + Qr_{\mathcal{S}_1,\mathcal{O},\nabla\nabla_T} + \mathcal{E}_{lot} + Bdr \tag{8.2.9}$$

with the following terms.

1. *The principal trapping term* $\widetilde{P}$ *is given by, see* (8.2.3),

$$\widetilde{P} \;=\; \frac{1}{2}h\Big(\delta_0\big|\nabla_T\Psi_z\big|^2 + a^2|\nabla_Z\Psi_z|^2 + O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\Psi_z\dot{\mathbf{D}}_\beta\Psi_z\Big), \tag{8.2.10}$$

where

$$\begin{aligned}\Psi_z = &-\frac{2\mathcal{T}}{(r^2+a^2)^3}\big(\delta_0\mathcal{S}_1\psi + (1+O(r^{-2}\delta_0))\mathcal{O}\psi\big)\\ &+\frac{4ar}{(r^2+a^2)^2}\nabla_{\widehat{T}}\nabla_Z\psi\big(1+O(r^{-2}\delta_0)\big).\end{aligned} \tag{8.2.11}$$

2. *The quadratic form* $Qr_{\mathcal{S}_1,\mathcal{O},\nabla\nabla_T}$ *is given by*

$$\begin{aligned}Qr_{\mathcal{S}_1,\mathcal{O},\nabla\nabla_T} \;:=\;& \mathcal{A}(1+O(r^{-1}\delta_0))\Big(\delta_0^2|\nabla_r\mathcal{S}_1\psi|^2 + |\nabla_r\mathcal{O}\psi|^2 + 2\delta_0|q|^2|\nabla\nabla_T\nabla_r\psi|^2\Big)\\ &+\frac{|q|^2}{2}v\Big(\delta_0^2\nabla_r\mathcal{S}_1\psi\cdot\mathcal{S}_1\psi + \nabla_r\mathcal{O}\psi\cdot\mathcal{O}\psi + 2\delta_0|q|^2\nabla\nabla_T\nabla_r\psi\cdot\nabla\nabla_T\psi\Big)\\ &+\left(\mathcal{V} + \frac{|q|^2}{4}v' + O(\delta_0 r^{-3})\right)\Big(\delta_0^2|\mathcal{S}_1\psi|^2 + |\mathcal{O}\psi|^2 + 2\delta_0|q|^2|\nabla\nabla_T\psi|^2\Big).\end{aligned}$$

3. *The terms* $\mathcal{E}_{lot}$ *are lower order terms in* a*, given by*

$$\begin{aligned}\mathcal{E}_{lot} = &-O(a)(\mathcal{A}+\widetilde{\mathcal{A}})(1+O(r^{-1}\delta_0))((\nabla_r\mathfrak{d}^{\leq 2}\psi)^2 + r^{-2}(\mathfrak{d}^{\leq 2}\psi)^2)\\ &-O(ar^{-1})(\mathfrak{d}^{\leq 2}\psi)^2 - vO(ar^{\frac{5}{2}})(\nabla_r\mathfrak{d}^{\leq 2}\psi)^2 - O(ar^{\frac{3}{2}})v(\mathfrak{d}^{\leq 2}\psi)^2\\ &-O(ar^2)v'(\mathfrak{d}^{\leq 2}\psi)^2.\end{aligned} \tag{8.2.12}$$

In particular, since $\mathcal{A},\widetilde{\mathcal{A}} = O(\Delta^2 r^{-4})$*, and for* $v = O(m^{1/2}\Delta r^{-9/2})$*, we can bound the above as*

$$\mathcal{E}_{lot} \geq -O(a)\big(|\nabla_{\widehat{R}}\mathfrak{d}^{\leq 2}\psi|^2 + r^{-1}|\mathfrak{d}^{\leq 2}\psi|^2\big). \tag{8.2.13}$$

4. *The boundary terms are given by*

$$\begin{aligned}Bdr \;=\;& Bdr[\psi]_I + Bdr[\psi]_J + Bdr[\psi]_K\\ =\;& \partial_t\Big(M(\nabla_{\widehat{R}}\psi) + M(\psi)\Big) + |q|^2\mathbf{D}_\beta\widehat{\mathcal{B}}^\beta\end{aligned}$$

with

$$\begin{aligned}\widehat{\mathcal{B}}^\beta &:= 2\delta_0|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\nabla_r\psi\cdot\mathcal{A}\mathcal{S}_1\nabla_r\psi+2\delta_0|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot(\mathcal{V}+O(a^2r^{-3}))\mathcal{S}_1\psi\\ &+\frac{1}{4}\delta_0 vO^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot\mathcal{S}_1\nabla_r\psi+\frac{1}{4}\delta_0 vO^{\alpha\beta}\dot{\mathbf{D}}_\alpha\nabla_r\psi\cdot\mathcal{S}_1\psi+\frac{1}{2}\delta_0 O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot v'\mathcal{S}_1\psi,\end{aligned}$$

where $M(\psi)$ denotes the quadratic expressions in ψ and $M(\nabla_{\widehat{R}}\psi)$ denotes the quadratic expressions in ψ and its derivatives of Definition 8.2.1.

Proof. Recall that according to Proposition 8.1.5, the effective generalized current $\widetilde{\mathcal{E}}[\mathbf{X},\mathbf{w},\mathbf{M}]$ introduced in (8.1.19) is given by, see (8.1.13),

$$|q|^2\widetilde{\mathcal{E}}[\mathbf{X},\mathbf{w},\mathbf{M}] = \widetilde{P}+P_{lot}+I+J+K.$$

The proof then follows immediately from the control for $\widetilde{P}$ in (8.2.3), for P_{lot} in (8.2.4), for I in (8.2.7), and for J and K in Lemma 8.2.6. □

8.3 Proof of $\mathcal{S}$-derivative Morawetz estimate

In this section we prove the $\mathcal{S}$-valued Morawetz estimates, i.e. we prove Proposition 6.3.10. Recall that we make use of the choices for z and h made in Lemma 8.1.7, i.e.

$$z=z_0-\delta_0 z_0^2,\qquad z_0=\frac{\Delta}{(r^2+a^2)^2},\qquad \delta_0>0,\qquad h=\frac{(r^2+a^2)^4}{r(r^2-a^2)}.$$

8.3.1 Control of the quadratic form $\mathrm{Qr}_{\mathcal{S}_1,\mathcal{O},\nabla\nabla_T}$

In this section, we prove that the quadratic form $\mathrm{Qr}_{\mathcal{S}_1,\mathcal{O},\nabla\nabla_T}$ appearing in Proposition 8.2.7 is positive definite. In order to do so, as in Section 7.2.4, we will apply Poincaré and Hardy inequalities.

Recalling that

$$\begin{aligned}\frac{2}{h}\widetilde{P} &= \delta_0\big|\nabla_T\Psi_z\big|^2+a^2|\nabla_Z\Psi_z|^2+O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\Psi_z\dot{\mathbf{D}}_\beta\Psi_z\\ &= \delta_0\big|\nabla_T\Psi_z\big|^2+a^2|\nabla_Z\Psi_z|^2+|q|^2|\nabla\Psi_z|^2\end{aligned}$$

we obtain, using the Poincaré inequality of Lemma 7.2.3, for $|a|\ll m$ and $|a|\ll\delta_0 m$,

$$\int_S\widetilde{P} \geq \int_S h|\Psi_z|^2-O(ar^7)\int_S|\nabla\Psi_z|^2. \tag{8.3.1}$$

Using the integration by parts identities as in Lemma 8.2.3, we can write

$$\begin{aligned}
|\Psi_z|^2 &= \big(\tilde{\mathcal{R}}'^{\underline{a}}[z]\psi_{\underline{a}}\big)\cdot\big(\tilde{\mathcal{R}}'^{\underline{b}}[z]\psi_{\underline{b}}\big)\\
&= (\tilde{\mathcal{R}}'^1)^2|\mathcal{S}_1\psi|^2+(\tilde{\mathcal{R}}'^2)^2|\mathcal{S}_2\psi|^2+(\tilde{\mathcal{R}}'^3)^2|\mathcal{S}_3\psi|^2+(\tilde{\mathcal{R}}'^4)^2|\mathcal{O}\psi|^2\\
&\quad+2\tilde{\mathcal{R}}'^1\tilde{\mathcal{R}}'^3\mathcal{S}_1\psi\cdot\mathcal{S}_3\psi+2\tilde{\mathcal{R}}'^1\tilde{\mathcal{R}}'^4\mathcal{S}_1\psi\cdot\mathcal{O}\psi+2\tilde{\mathcal{R}}'^3\tilde{\mathcal{R}}'^4\mathcal{S}_3\psi\cdot\mathcal{O}\psi\\
&\quad+2\tilde{\mathcal{R}}'^2\mathcal{S}_2\psi\cdot(\tilde{\mathcal{R}}'^1\mathcal{S}_1\psi+\tilde{\mathcal{R}}'^3\mathcal{S}_3\psi+\tilde{\mathcal{R}}'^4\mathcal{O}\psi)\\
&= (\tilde{\mathcal{R}}'^1)^2|\mathcal{S}_1\psi|^2+(\tilde{\mathcal{R}}'^4)^2|\mathcal{O}\psi|^2+2\tilde{\mathcal{R}}'^1\tilde{\mathcal{R}}'^4\mathcal{S}_1\psi\cdot\mathcal{O}\psi\\
&\quad+O(a)\Big((\tilde{\mathcal{R}}'^1)^2+(\tilde{\mathcal{R}}'^2)^2+(\tilde{\mathcal{R}}'^3)^2+(\tilde{\mathcal{R}}'^4)^2\Big)(\mathfrak{d}^{\leq 2}\psi)^2\\
&= (\tilde{\mathcal{R}}'^1)^2|\mathcal{S}_1\psi|^2+(\tilde{\mathcal{R}}'^4)^2|\mathcal{O}\psi|^2+2\tilde{\mathcal{R}}'^1\tilde{\mathcal{R}}'^4|q|^2|\nabla\nabla_T\psi|^2\\
&\quad+O(a)\Big((\tilde{\mathcal{R}}'^1)^2+(\tilde{\mathcal{R}}'^2)^2+(\tilde{\mathcal{R}}'^3)^2+(\tilde{\mathcal{R}}'^4)^2\Big)(\mathfrak{d}^{\leq 2}\psi)^2\\
&\quad+\partial_t\big(\tilde{\mathcal{R}}'^1\tilde{\mathcal{R}}'^4M(\psi)\big)+|q|^2\dot{\mathbf{D}}_\beta(|q|^{-2}\tilde{\mathcal{R}}'^1\tilde{\mathcal{R}}'^4O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot\mathcal{S}_1\psi).
\end{aligned}$$

Writing from (8.1.23),

$$\tilde{\mathcal{R}}'^1=\delta_0 f,\qquad \tilde{\mathcal{R}}'^4=f(1+O(\delta_0 r^{-2})),\qquad \tilde{\mathcal{R}}'^2=O(r^{-3}),\qquad \tilde{\mathcal{R}}'^3=O(r^{-5}),$$

we obtain

$$\begin{aligned}
|\Psi_z|^2 &= f^2\big(\delta_0^2|\mathcal{S}_1\psi|^2+|\mathcal{O}\psi|^2+2\delta_0|\nabla\nabla_T\psi|^2\big)-O(ar^{-6})(\mathfrak{d}^{\leq 2}\psi)^2\\
&\quad+\partial_t\big(r^{-6}M(\psi)\big)+|q|^2\dot{\mathbf{D}}_\beta(O(r^{-6})|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot(\mathcal{S}_1+\mathcal{S}_3)\psi).
\end{aligned}$$

By combining Proposition 8.2.7 and the above for the term $(1-\delta)\widetilde{P}$ in $\widetilde{P}=\delta\widetilde{P}+(1-\delta)\widetilde{P}$, we obtain the following

Proposition 8.3.1. *Upon applying the Poincaré inequality to the principal term, the generalized current verifies the identity, for any sphere $S=S(t,r)$,*

$$\int_S|q|^2\widetilde{\mathcal{E}}[\mathbf{X},\mathbf{w},\mathbf{M}]\ \geq\ \int_S\Big(\delta\widetilde{P}+\widetilde{Qr}_{\mathcal{S}_1,\mathcal{O},\nabla\nabla_T}+\widetilde{\mathcal{E}}_{lot}+Bdr\Big)$$

where the quadratic form $\widetilde{Qr}_{\mathcal{S}_1,\mathcal{O},\nabla\nabla_T}$ is given by

$$\begin{aligned}
\widetilde{Qr}_{\mathcal{S}_1,\mathcal{O},\nabla\nabla_T} &:= Qr_{\mathcal{S}_1,\mathcal{O},\nabla\nabla_T}+(1-\delta)hf^2\big(\delta_0^2|\mathcal{S}_1\psi|^2+|\mathcal{O}\psi|^2+2\delta_0|\nabla\nabla_T\psi|^2\big)\\
&= \mathcal{A}(1+O(r^{-1}\delta_0))\Big(\delta_0^2|\nabla_r\mathcal{S}_1\psi|^2+|\nabla_r\mathcal{O}\psi|^2+2\delta_0|q|^2|\nabla\nabla_T\nabla_r\psi|^2\Big)\\
&\quad+\frac{|q|^2}{2}v\big(\delta_0^2\nabla_r\mathcal{S}_1\psi\cdot\mathcal{S}_1\psi+\nabla_r\mathcal{O}\psi\cdot\mathcal{O}\psi+2\delta_0|q|^2\nabla\nabla_T\nabla_r\psi\cdot\nabla\nabla_T\psi\big)\\
&\quad+\left(\mathcal{V}+\frac{|q|^2}{4}v'+(1-\delta)hf^2+O(\delta_0 r^{-3})\right)\Big(\delta_0^2|\mathcal{S}_1\psi|^2+|\mathcal{O}\psi|^2+2\delta_0|q|^2|\nabla\nabla_T\psi|^2\Big)
\end{aligned}$$

and the lower order term $\widetilde{\mathcal{E}}_{lot}$ is given by

$$\widetilde{\mathcal{E}}_{lot}\ :=\ \mathcal{E}_{lot}-O(ar^7)|\nabla\Psi_z|^2-O(ar^{-1})|\mathfrak{d}^{\leq 2}\psi|^2.$$

We can now apply the Hardy inequality where v is given by Proposition 7.2.5, to show that for $|a|/m \ll 1$ and a universal constant c_1 we have

$$
\begin{aligned}
\widetilde{\text{Qr}}_{\mathcal{S}_1,\mathcal{O},\nabla\nabla_T} &\geq c_1\Big(m\big(|\nabla_{\widehat{R}}\mathcal{S}_1\psi|^2+|\nabla_{\widehat{R}}\mathcal{O}\psi|^2+|q|^2|\nabla\nabla_T\nabla_r\psi|^2\big)\\
&+r^{-1}\big(|\mathcal{S}_1\psi|^2+|\mathcal{O}\psi|^2+|q|^2|\nabla\nabla_T\psi|^2\big)\Big)\\
&-O(ar^{-2})\big(|\nabla_{\widehat{R}}\mathfrak{d}^{\leq 2}\psi|^2+|\mathfrak{d}^{\leq 2}\psi|^2\big).
\end{aligned}
$$

Indeed, we can separate the quadratic form $\widetilde{\text{Qr}}_{\mathcal{S}_1,\mathcal{O},\nabla\nabla_T}$ in the terms involving $\mathcal{S}_1$, $\mathcal{O}$ and $\nabla\nabla_T$, and obtain by Lemma 7.2.4 that

$$
\begin{aligned}
\widetilde{\text{Qr}}_{\mathcal{S}_1} &:= \delta_0^2\left(\mathcal{A}\big|\nabla_r\mathcal{S}_1\psi\big|^2+\frac{|q|^2}{2}v\nabla_r\mathcal{S}_1\psi\cdot\mathcal{S}_1\psi+\left(\mathcal{V}+\frac{|q|^2}{4}v'+(1-\delta)f^2h+O(\delta_0r^{-3})\right)|\mathcal{S}_1\psi|^2\right)\\
&\geq c_0\delta_0^2\Big(\frac{m\Delta^2}{r^4}|\nabla_r\mathcal{S}_1\psi|^2+r^{-1}|\mathcal{S}_1\psi|^2\Big).
\end{aligned}
$$

Similarly, by applying again Lemma 7.2.4 we have

$$
\begin{aligned}
\widetilde{\text{Qr}}_{\mathcal{O}} &:= \mathcal{A}\big|\nabla_r\mathcal{O}\psi\big|^2+\frac{|q|^2}{2}v\nabla_r\mathcal{O}\psi\cdot\mathcal{O}\psi+\left(\mathcal{V}+\frac{|q|^2}{4}v'+(1-\delta)f^2h+O(\delta_0r^{-3})\right)|\mathcal{O}\psi|^2\\
&\geq c_0\Big(\frac{m\Delta^2}{r^4}|\nabla_r\mathcal{O}\psi|^2+r^{-1}|\mathcal{O}\psi|^2\Big)
\end{aligned}
$$

and, recalling $[\nabla_r,|q|\nabla]=O(ar^{-2})\psi$ and $[\nabla_r,\nabla_T]=O(mar^{-4})\mathfrak{d}^{\leq 1}$,

$$
\begin{aligned}
\widetilde{\text{Qr}}_{\nabla\nabla_T} &:= \delta_0|q|^2\left(\mathcal{A}\big|\nabla\nabla_T\nabla_r\psi\big|^2+\frac{|q|^2}{2}v\nabla\nabla_T\nabla_r\psi\cdot\nabla\nabla_T\psi\right.\\
&\left.+\left(\mathcal{V}+\frac{|q|^2}{4}v'+(1-\delta)f^2h+O(\delta_0r^{-3})\right)|\nabla\nabla_T\psi|^2\right)\\
&= \delta_0\left(\mathcal{A}\big|\nabla_r(|q|\nabla\nabla_T\psi)\big|^2+\frac{|q|^2}{2}v\nabla_r(|q|\nabla\nabla_T\psi)\cdot(|q|\nabla\nabla_T\psi)\right.\\
&\left.+\left(\mathcal{V}+\frac{|q|^2}{4}v'+(1-\delta)f^2h+O((\delta_0+a)r^{-3})\right)||q|\nabla\nabla_T\psi|^2\right)\\
&-O(ar^{-2})\big(|\nabla_{\widehat{R}}\mathfrak{d}^{\leq 2}\psi|^2+|\mathfrak{d}^{\leq 2}\psi|^2\big)\\
&\geq c_0\delta_0|q|^2\Big(\frac{m\Delta^2}{r^4}|q|^{-2}\big|\nabla_r(|q|\nabla\nabla_T\psi)|^2+r^{-1}|\nabla\nabla_T\psi|^2\Big)\\
&-O(ar^{-2})\big(|\nabla_{\widehat{R}}\mathfrak{d}^{\leq 2}\psi|^2+|\mathfrak{d}^{\leq 2}\psi|^2\big)\\
&\geq c_0\delta_0|q|^2\Big(\frac{m\Delta^2}{r^4}\big|\nabla\nabla_T\nabla_r\psi)|^2+r^{-1}|\nabla\nabla_T\psi|^2\Big)-O(ar^{-2})\big(|\nabla_{\widehat{R}}\mathfrak{d}^{\leq 2}\psi|^2+|\mathfrak{d}^{\leq 2}\psi|^2\big).
\end{aligned}
$$

This gives for the generalized current, for any sphere $S = S(t,r)$,

$$
\begin{aligned}
\int_S |q|^2 \widetilde{\mathcal{E}}[\mathbf{X}, \mathbf{w}, \mathbf{M}] \geq & \delta \int_S \widetilde{P} \\
& + c_1 \int_S \Big(m\big(|\nabla_{\widehat{R}} \mathcal{S}_1 \psi|^2 + |\nabla_{\widehat{R}} \mathcal{O}\psi|^2 + |q|^2 |\nabla\nabla_T \nabla_{\widehat{R}} \psi|^2\big) \\
& + r^{-1}\big(|\mathcal{S}_1\psi|^2 + |\mathcal{O}\psi|^2 + |q|^2|\nabla\nabla_T\psi|^2\big)\Big) \\
& + \int_S \Big(\widetilde{\mathcal{E}}_{lot} - O(ar^{-2})\big(|\nabla_{\widehat{R}} \mathfrak{d}^{\leq 2}\psi|^2 + |\mathfrak{d}^{\leq 2}\psi|^2\big) + \mathrm{Bdr}\Big).
\end{aligned}
\tag{8.3.2}
$$

8.3.2 Control of the effective generalized current

We have, recall (8.2.13),

$$
\begin{aligned}
\widetilde{\mathcal{E}}_{lot} &= \mathcal{E}_{lot} - O(ar^7)|\nabla\Psi_z|^2 - O(ar^{-1})|\mathfrak{d}^{\leq 2}\psi| \\
&\geq -O(a)\big(|\nabla_{\widehat{R}}\mathfrak{d}^{\leq 2}\psi|^2 + r^{-1}|\mathfrak{d}^{\leq 2}\psi|^2\big) - O(ar^7)|\nabla\Psi_z|^2.
\end{aligned}
$$

Plugging in (8.3.2), we infer

$$
\begin{aligned}
\int_S |q|^2 \widetilde{\mathcal{E}}[\mathbf{X}, \mathbf{w}, \mathbf{M}] \geq & \delta \int_S \widetilde{P} + c_1 \int_S \Big(m\big(|\nabla_{\widehat{R}} \mathcal{S}_1 \psi|^2 + |\nabla_{\widehat{R}} \mathcal{O}\psi|^2\big) + r^{-1}\big(|\mathcal{S}_1\psi|^2 + |\mathcal{O}\psi|^2\big)\Big) \\
& - O(a)\int_S \big(|\nabla_{\widehat{R}}\mathfrak{d}^{\leq 2}\psi|^2 + r^{-1}|\mathfrak{d}^{\leq 2}\psi|^2\big) + \int_S \mathrm{Bdr}.
\end{aligned}
$$

Since $\mathcal{S}_2 = O(a)\mathfrak{d}^{\leq 2}$ and $\mathcal{S}_3 = O(a^2)\mathfrak{d}^{\leq 2}$, we infer

$$
\begin{aligned}
\int_S |q|^2 \widetilde{\mathcal{E}}[\mathbf{X}, \mathbf{w}, \mathbf{M}] \geq & \delta \int_S \widetilde{P} + c_1 \int_S \Big(m\big(|\nabla_{\widehat{R}} \mathcal{S}_1 \psi|^2 + |\nabla_{\widehat{R}} \mathcal{S}_2 \psi|^2 + |\nabla_{\widehat{R}} \mathcal{S}_3 \psi|^2 + |\nabla_{\widehat{R}} \mathcal{O}\psi|^2\big) \\
& + r^{-1}\big(|\mathcal{S}_1\psi|^2 + |\mathcal{S}_2\psi|^2 + |\mathcal{S}_3\psi|^2 + |\mathcal{O}\psi|^2\big)\Big) \\
& - O(a)\int_S \big(|\nabla_{\widehat{R}}\mathfrak{d}^{\leq 2}\psi|^2 + r^{-1}|\mathfrak{d}^{\leq 2}\psi|^2\big) + \int_S \mathrm{Bdr}
\end{aligned}
$$

and hence

$$
\begin{aligned}
\int_S |q|^2 \widetilde{\mathcal{E}}[\mathbf{X}, \mathbf{w}, \mathbf{M}] \geq & \delta \int_S \widetilde{P} + c_1 \int_S \Big(m|\nabla_{\widehat{R}}\psi|^2_{\mathcal{S}} + r^{-1}|\psi|^2_{\mathcal{S}}\Big) \\
& - O(a)\int_S \big(|\nabla_{\widehat{R}}\mathfrak{d}^{\leq 2}\psi|^2 + r^{-1}|\mathfrak{d}^{\leq 2}\psi|^2\big) + \int_S \mathrm{Bdr}.
\end{aligned}
$$

Since

$$
\begin{aligned}
\widetilde{P} &= \frac{1}{2}h\Big(\delta_0\big|\nabla_T\Psi_z\big|^2+a^2|\nabla_Z\Psi_z|^2+O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\Psi_z\dot{\mathbf{D}}_\beta\Psi_z\Big)\\
&\geq c_0r^5\Big(\big|\nabla_T\Psi_z\big|^2+|q|^2|\nabla\Psi_z|^2\Big)
\end{aligned}
$$

we infer

$$
\begin{aligned}
\int_S|q|^2\widetilde{\mathcal{E}}[\mathbf{X},\mathbf{w},\mathbf{M}] \geq& \delta c_0\int_S r^5\Big(\big|\nabla_T\Psi_z\big|^2+|q|^2|\nabla\Psi_z|^2\Big)+c_1\int_S\Big(m|\nabla_{\widehat{R}}\psi|^2_{\mathcal{S}}+r^{-1}|\psi|^2_{\mathcal{S}}\Big)\\
&-O(a)\int_S\big(|\nabla_{\widehat{R}}\mathfrak{d}^{\leq 2}\psi|^2+r^{-1}|\mathfrak{d}^{\leq 2}\psi|^2\big)+\int_S\mathrm{Bdr}.
\end{aligned}
$$

We finally obtain, for $|a|/m \ll 1$,

$$
\begin{aligned}
\int_S\widetilde{\mathcal{E}}[\mathbf{X},\mathbf{w},\mathbf{M}] \gtrsim& \int_S\frac{m}{r^2}|\nabla_{\widehat{R}}\psi|^2_{\mathcal{S}}+r^{-3}|\psi|^2_{\mathcal{S}}+r^3\Big(\big|\nabla_{\widehat{T}}\Psi_z\big|^2+r^2\big|\nabla\Psi_z\big|^2\Big)+\mathrm{Bdr}\\
&-O(a)\int_S\big(r^{-2}|\nabla_{\widehat{R}}\mathfrak{d}^{\leq 2}\psi|^2+r^{-3}|\mathfrak{d}^{\leq 2}\psi|^2\big)
\end{aligned}
\tag{8.3.3}
$$

on any sphere $S=S(t,r)$.

8.3.3 $\mathcal{S}$-derivative version of Lemma 7.2.2

In view of (8.1.3) and (8.1.19), we have

$$
\begin{aligned}
\mathbf{D}^\mu\mathcal{P}_\mu[\mathbf{X},\mathbf{w},\mathbf{M}] =& \widetilde{\mathcal{E}}[\mathbf{X},\mathbf{w},\mathbf{M}]+\mathbf{D}^\mu\mathcal{B}_\mu+\left(\nabla_{X^{\underline{ab}}}\psi_{\underline{a}}+\frac{1}{2}w^{\underline{ab}}\psi_{\underline{a}}\right)\cdot\big(\dot{\square}_2\psi_{\underline{b}}-V\psi_{\underline{b}}\big)\\
&-\big({}^*\rho+\underline{\eta}\wedge\eta\big)\nabla_{(X^{\underline{ab}})^4e_4-(X^{\underline{ab}})^3e_3}\psi_{\underline{a}}\cdot{}^*\psi_{\underline{b}}\\
&-\frac{1}{2}\Im\Big(\mathrm{tr}\underline{X}H(X^{\underline{ab}})^3+\mathrm{tr}X\,\underline{H}(X^{\underline{ab}})^4\Big)\cdot\nabla\psi_{\underline{a}}\cdot{}^*\psi_{\underline{b}}.
\end{aligned}
\tag{8.3.4}
$$

Also, arguing as for the proof of (7.1.9), we have

$$
\nabla_{(X^{\underline{ab}})^4e_4-(X^{\underline{ab}})^3e_3}\psi_{\underline{a}}=\mathcal{F}^{\underline{ab}}\frac{r^2+a^2}{\Delta}\nabla_{\widehat{T}}\psi_{\underline{a}}
$$

and

$$
\Im\Big(\mathrm{tr}\underline{X}H(X^{\underline{ab}})^3+\mathrm{tr}X\,\underline{H}(X^{\underline{ab}})^4\Big) = \frac{4a^2r\cos\theta\mathcal{F}^{\underline{ab}}(r)}{(r^2+a^2)|q|^4}\partial_\phi+\frac{4a^3r\cos\theta(\sin\theta)^2\mathcal{F}^{\underline{ab}}(r)}{|q|^6}\widehat{T}.
$$

Thus we infer

$$\begin{aligned}\mathbf{D}^\mu\mathcal{P}_\mu[\mathbf{X},\mathbf{w},\mathbf{M}] =& \widetilde{\mathcal{E}}[\mathbf{X},\mathbf{w},\mathbf{M}] + \mathbf{D}^\mu\mathcal{B}_\mu + \left(\nabla_{X^{\underline{ab}}}\psi_{\underline{a}} + \frac{1}{2}w^{\underline{ab}}\psi_{\underline{a}}\right)\cdot\left(\dot{\square}_2\psi_{\underline{b}} - V\psi_{\underline{b}}\right)\\ & - \frac{2a^2 r\cos\theta\mathcal{F}^{\underline{ab}}(r)}{(r^2+a^2)|q|^4}\nabla_\phi\psi_{\underline{a}}\cdot {}^*\psi_{\underline{b}} \\ & - \left(\left({}^*\rho + \underline{\eta}\wedge\eta\right)\mathcal{F}^{\underline{ab}}\frac{r^2+a^2}{\Delta} + \frac{2a^3 r\cos\theta(\sin\theta)^2\mathcal{F}^{\underline{ab}}(r)}{|q|^6}\right)\nabla_{\widehat{T}}\psi_{\underline{a}}\cdot {}^*\psi_{\underline{b}}.\end{aligned} \tag{8.3.5}$$

The goal of this section is to control the last three terms on the RHS of (8.3.5). To this end, we derive the $\mathcal{S}$-derivative version of Lemma 7.2.2, being careful to obtain terms involving Ψ_z.

Lemma 8.3.2. *We have, for sufficiently small positive constants δ_2, δ_3:*

$$\begin{aligned}&\left(\nabla_{X^{\underline{ab}}}\psi_{\underline{a}} + \frac{1}{2}w^{\underline{ab}}\psi_{\underline{a}}\right)\cdot\left(\dot{\square}_2\psi_{\underline{b}} - V\psi_{\underline{b}}\right)\\ \geq & -\delta_2 r^{-2}h|\nabla_{\widehat{T}}\Psi_z|^2 - \delta_2 a^2 r^{-6}h|\nabla_Z\Psi_z|^2 + O(ar^{-3})|\psi|^2_{\mathcal{S}}\\ & + O(1)\Big(|\nabla_{\widehat{R}}\psi|_{\mathcal{S}} + r^{-1}|\psi|_{\mathcal{S}}\Big)\sum_{\underline{a}=1}^4|N_{\underline{a}}| + \mathbf{D}_\mu\left(\frac{2a\cos\theta}{|q|^2}(\partial_r)^\mu zhf^{\underline{ab}}\psi_{\underline{a}}\cdot\nabla_T {}^*\psi_{\underline{b}}\right)\\ & - \nabla_T\left(\frac{2a\cos\theta}{|q|^2}\left(zhf^{\underline{ab}}\psi_{\underline{a}}\cdot\nabla_r {}^*\psi_{\underline{b}} + \frac{1}{2}(\partial_r z)h\Psi_z\cdot {}^*\mathcal{L}^{\underline{a}}\psi_{\underline{a}}\right)\right),\end{aligned} \tag{8.3.6}$$

and

$$\begin{aligned}&\left(\left({}^*\rho + \underline{\eta}\wedge\eta\right)\mathcal{F}^{\underline{ab}}\frac{r^2+a^2}{\Delta} + \frac{2a^3 r\cos\theta(\sin\theta)^2\mathcal{F}^{\underline{ab}}(r)}{|q|^6}\right)\nabla_{\widehat{T}}\psi_{\underline{a}}\cdot {}^*\psi_{\underline{b}}\\ & + \frac{2a^2 r\cos\theta\mathcal{F}^{\underline{ab}}(r)}{(r^2+a^2)|q|^4}\nabla_\phi\psi_{\underline{a}}\cdot {}^*\psi_{\underline{b}}\\ \leq & \delta_3 r^3\Big(|\nabla_{\widehat{T}}\Psi_z|^2 + r^2|\nabla\Psi_z|^2\Big) + O(ar^{-3})|\psi|^2_{\mathcal{S}} - \frac{1}{2}\nabla_\phi\left(zh\frac{2a^2 r\cos\theta}{(r^2+a^2)|q|^4}\Psi_z\cdot {}^*(\mathcal{L}^{\underline{b}}\psi_{\underline{b}})\right)\\ & - \frac{1}{2}\nabla_{\widehat{T}}\left(zh\left(\left({}^*\rho + \underline{\eta}\wedge\eta\right)\frac{r^2+a^2}{\Delta} + \frac{2a^3 r\cos\theta(\sin\theta)^2}{|q|^6}\right)\Psi_z\cdot {}^*(\mathcal{L}^{\underline{b}}\psi_{\underline{b}})\right).\end{aligned} \tag{8.3.7}$$

Proof. According to equation (6.3.2), we have

$$\begin{aligned}&\left(\nabla_{X^{\underline{ab}}}(\psi_{\underline{a}}) + \frac{1}{2}w^{\underline{ab}}\psi_{\underline{a}}\right)\cdot\left(\dot{\square}_2\psi_{\underline{b}} - V\psi_{\underline{b}}\right)\\ =&\left(\nabla_{X^{\underline{ab}}}(\psi_{\underline{a}}) + \frac{1}{2}w^{\underline{ab}}\psi_{\underline{a}}\right)\cdot\left(-\frac{4a\cos\theta}{|q|^2}{}^*\nabla_T\psi_{\underline{b}} + N_{\underline{b}}\right).\end{aligned}$$

We consider the first order term, and we write, as in Lemma 7.1.6,

$$\begin{aligned}\Big(\nabla_{X^{\underline{ab}}}(\psi_{\underline{a}})+\frac{1}{2}w^{\underline{ab}}\psi_{\underline{a}}\Big)\cdot\ {}^*\nabla_T\psi_{\underline{b}} &= \Big(-zhf^{\underline{ab}}\nabla_r\psi_{\underline{a}}-\frac{1}{2}z\partial_r\big(hf^{\underline{ab}}\big)\psi_{\underline{a}}\Big)\cdot\ {}^*\nabla_T\psi_{\underline{b}}\\ &= \frac{1}{2}(\partial_r z)hf^{\underline{ab}}\psi_{\underline{a}}\cdot\nabla_T\ {}^*\psi_{\underline{b}}+zhf^{\underline{ab}}\ {}^*\rho\frac{|q|^2}{\Delta}\psi_{\underline{a}}\psi_{\underline{b}}\\ &\quad -\frac{1}{2}\nabla_r\Big(zhf^{\underline{ab}}\psi_{\underline{a}}\cdot\nabla_T\ {}^*\psi_{\underline{b}}\Big)+\frac{1}{2}\nabla_T\Big(zhf^{\underline{ab}}\psi_{\underline{a}}\cdot\nabla_r\ {}^*\psi_{\underline{b}}\Big).\end{aligned}$$

Using that $f^{\underline{ab}}=\tilde{\mathcal{R}}'^{(\underline{a}}\mathcal{L}^{\underline{b})}$ and $\Psi_z=\tilde{\mathcal{R}}'^{\underline{a}}\psi_{\underline{a}}$, we obtain

$$\begin{aligned}&\Big(\nabla_{X^{\underline{ab}}}(\psi_{\underline{a}})+\frac{1}{2}w^{\underline{ab}}\psi_{\underline{a}}\Big)\cdot\ {}^*\nabla_T\psi_{\underline{b}}\\ &= \frac{1}{4}(\partial_r z)h\Big(\mathcal{L}^{\underline{a}}\psi_{\underline{a}}\cdot\nabla_T\ {}^*\Psi_z+\Psi_z\cdot\nabla_T\ {}^*\mathcal{L}^{\underline{a}}\psi_{\underline{a}}\Big)+zh\ {}^*\rho\frac{|q|^2}{\Delta}\Psi_z\mathcal{L}^{\underline{a}}\psi_{\underline{a}}\\ &\quad -\frac{1}{2}\nabla_r\Big(zh\mathcal{L}^{\underline{a}}\psi_{\underline{a}}\cdot\nabla_T\ {}^*\Psi_z\Big)+\frac{1}{2}\nabla_T\Big(zh\Psi_z\cdot\mathcal{L}^{\underline{b}}\nabla_r\ {}^*\psi_{\underline{b}}\Big)\\ &= \frac{1}{2}(\partial_r z)h\mathcal{L}^{\underline{a}}\psi_{\underline{a}}\cdot\nabla_T\ {}^*\Psi_z+zh\ {}^*\rho\frac{|q|^2}{\Delta}\Psi_z\mathcal{L}^{\underline{a}}\psi_{\underline{a}}\\ &\quad -\frac{1}{2}\nabla_r\Big(zhf^{\underline{ab}}\psi_{\underline{a}}\cdot\nabla_T\ {}^*\psi_{\underline{b}}\Big)+\frac{1}{2}\nabla_T\left(zhf^{\underline{ab}}\psi_{\underline{a}}\cdot\nabla_r\ {}^*\psi_{\underline{b}}+\frac{1}{2}(\partial_r z)h\Psi_z\cdot\ {}^*\mathcal{L}^{\underline{a}}\psi_{\underline{a}}\right).\end{aligned}$$

Also, since $\mathbf{X}=\mathbf{O(1)}\widehat{\mathbf{R}}$ and $w^{\underline{ab}}=O(r^{-1})$, we can bound the second product by

$$\left|\Big(\nabla_{X^{\underline{ab}}}(\psi_{\underline{a}})+\frac{1}{2}w^{\underline{ab}}\psi_{\underline{a}}\Big)\cdot N_{\underline{b}}\right| \lesssim \Big(|\nabla_{\widehat{R}}\psi|_{\mathcal{S}}+r^{-1}|\psi|_{\mathcal{S}}\Big)\sum_{\underline{a}=1}^4|N_{\underline{a}}|.$$

By putting together with the previous bounds we obtain

$$
\begin{aligned}
& \left(\nabla_{X^{\underline{ab}}}(\psi_{\underline{a}}) + \frac{1}{2} w^{\underline{ab}} \psi_{\underline{a}}\right) \cdot \left(\dot{\square}_2 \psi_{\underline{b}} - V \psi_{\underline{b}}\right) \\
= \; & \left(\nabla_{X^{\underline{ab}}}(\psi_{\underline{a}}) + \frac{1}{2} w^{\underline{ab}} \psi_{\underline{a}}\right) \cdot \left(-\frac{4a\cos\theta}{|q|^2} \, {}^*\nabla_T \psi_{\underline{b}} + N_{\underline{b}}\right) \\
\geq \; & -\delta_2 r^{-2} h |\nabla_T \Psi_z|^2 + O(ar^{-3})|\psi|^2_{\mathcal{S}} + O(1)\Big(|\nabla_{\widehat{R}}\psi|_{\mathcal{S}} + r^{-1}|\psi|_{\mathcal{S}}\Big) \sum_{\underline{a}=1}^{4} |N_{\underline{a}}| \\
& + \frac{2a\cos\theta}{|q|^2} \nabla_r \left(zhf^{\underline{ab}} \psi_{\underline{a}} \cdot \nabla_T \, {}^*\psi_{\underline{b}}\right) \\
& - \nabla_T \left(\frac{2a\cos\theta}{|q|^2}\left(zhf^{\underline{ab}} \psi_{\underline{a}} \cdot \nabla_r \, {}^*\psi_{\underline{b}} + \frac{1}{2}(\partial_r z) h \Psi_z \cdot \, {}^*\mathcal{L}^{\underline{a}} \psi_{\underline{a}}\right)\right) \\
\geq \; & -\delta_2 r^{-2} h |\nabla_{\widehat{T}} \Psi_z|^2 - \delta_2 a^2 r^{-6} h |\nabla_Z \Psi_z|^2 + O(ar^{-3})|\psi|^2_{\mathcal{S}} \\
& + O(1)\Big(|\nabla_{\widehat{R}}\psi|_{\mathcal{S}} + r^{-1}|\psi|_{\mathcal{S}}\Big) \sum_{\underline{a}=1}^{4} |N_{\underline{a}}| + \frac{2a\cos\theta}{|q|^2} \nabla_r \left(zhf^{\underline{ab}} \psi_{\underline{a}} \cdot \nabla_T \, {}^*\psi_{\underline{b}}\right) \\
& - \nabla_T \left(\frac{2a\cos\theta}{|q|^2}\left(zhf^{\underline{ab}} \psi_{\underline{a}} \cdot \nabla_r \, {}^*\psi_{\underline{b}} + \frac{1}{2}(\partial_r z) h \Psi_z \cdot \, {}^*\mathcal{L}^{\underline{a}} \psi_{\underline{a}}\right)\right).
\end{aligned}
$$

Since we have $\mathbf{D}_\mu(\cos\theta |q|^{-2} (\partial_r)^\mu) = 0$, we infer

$$
\begin{aligned}
& \left(\nabla_{X^{\underline{ab}}}(\psi_{\underline{a}}) + \frac{1}{2} w^{\underline{ab}} \psi_{\underline{a}}\right) \cdot \left(\dot{\square}_2 \psi_{\underline{b}} - V \psi_{\underline{b}}\right) \\
\geq \; & -\delta_2 r^{-2} h |\nabla_{\widehat{T}} \Psi_z|^2 - \delta_2 a^2 r^{-6} h |\nabla_Z \Psi_z|^2 + O(ar^{-3})|\psi|^2_{\mathcal{S}} \\
& + O(1)\Big(|\nabla_{\widehat{R}}\psi|_{\mathcal{S}} + r^{-1}|\psi|_{\mathcal{S}}\Big) \sum_{\underline{a}=1}^{4} |N_{\underline{a}}| + \mathbf{D}_\mu \left(\frac{2a\cos\theta}{|q|^2} (\partial_r)^\mu zhf^{\underline{ab}} \psi_{\underline{a}} \cdot \nabla_T \, {}^*\psi_{\underline{b}}\right) \\
& - \nabla_T \left(\frac{2a\cos\theta}{|q|^2}\left(zhf^{\underline{ab}} \psi_{\underline{a}} \cdot \nabla_r \, {}^*\psi_{\underline{b}} + \frac{1}{2}(\partial_r z) h \Psi_z \cdot \, {}^*\mathcal{L}^{\underline{a}} \psi_{\underline{a}}\right)\right).
\end{aligned}
$$

as desired.

For the second estimate, we write

$$
\begin{aligned}
& \left(\left({}^*\rho + \underline{\eta} \wedge \eta \right) \mathcal{F}^{\underline{ab}} \frac{r^2+a^2}{\Delta} + \frac{2a^3 r\cos\theta(\sin\theta)^2 \mathcal{F}^{\underline{ab}}(r)}{|q|^6} \right) \nabla_{\widehat{T}} \psi_{\underline{a}} \cdot {}^*\psi_{\underline{b}} \\
& + \frac{2a^2 r\cos\theta \mathcal{F}^{\underline{ab}}(r)}{(r^2+a^2)|q|^4} \nabla_\phi \psi_{\underline{a}} \cdot {}^*\psi_{\underline{b}} \\
= \;& zh \tilde{\mathcal{R}}'^{(\underline{a}} \mathcal{L}^{\underline{b})} \left(\left({}^*\rho + \underline{\eta} \wedge \eta \right) \frac{r^2+a^2}{\Delta} + \frac{2a^3 r\cos\theta(\sin\theta)^2}{|q|^6} \right) \nabla_{\widehat{T}} \psi_{\underline{a}} \cdot {}^*\psi_{\underline{b}} \\
& + zh \tilde{\mathcal{R}}'^{(\underline{a}} \mathcal{L}^{\underline{b})} \frac{2a^2 r\cos\theta}{(r^2+a^2)|q|^4} \nabla_\phi \psi_{\underline{a}} \cdot {}^*\psi_{\underline{b}} \\
= \;& zh \tilde{\mathcal{R}}'^{(\underline{a}} \mathcal{L}^{\underline{b})} \left(\left({}^*\rho + \underline{\eta} \wedge \eta \right) \frac{r^2+a^2}{\Delta} + \frac{2a^3 r\cos\theta(\sin\theta)^2}{|q|^6} \right) \nabla_{\widehat{T}} \psi_{\underline{a}} \cdot {}^*\psi_{\underline{b}} \\
& + zh \tilde{\mathcal{R}}'^{(\underline{a}} \mathcal{L}^{\underline{b})} \frac{2a^2 r\cos\theta}{(r^2+a^2)|q|^4} \nabla_\phi \psi \cdot {}^*\psi
\end{aligned}
$$

and hence

$$
\begin{aligned}
& \left(\left({}^*\rho + \underline{\eta} \wedge \eta \right) \mathcal{F}^{\underline{ab}} \frac{r^2+a^2}{\Delta} + \frac{2a^3 r\cos\theta(\sin\theta)^2 \mathcal{F}^{\underline{ab}}(r)}{|q|^6} \right) \nabla_{\widehat{T}} \psi_{\underline{a}} \cdot {}^*\psi_{\underline{b}} \\
& + \frac{2a^2 r\cos\theta \mathcal{F}^{\underline{ab}}(r)}{(r^2+a^2)|q|^4} \nabla_\phi \psi_{\underline{a}} \cdot {}^*\psi_{\underline{b}} \\
= \;& zh \left(\left({}^*\rho + \underline{\eta} \wedge \eta \right) \frac{r^2+a^2}{\Delta} + \frac{2a^3 r\cos\theta(\sin\theta)^2}{|q|^6} \right) \nabla_{\widehat{T}} \Psi_z \cdot {}^*(\mathcal{L}^{\underline{b}} \psi_{\underline{b}}) \\
& - \frac{1}{2} \nabla_{\widehat{T}} \left(zh \left(\left({}^*\rho + \underline{\eta} \wedge \eta \right) \frac{r^2+a^2}{\Delta} + \frac{2a^3 r\cos\theta(\sin\theta)^2}{|q|^6} \right) \Psi_z \cdot {}^*(\mathcal{L}^{\underline{b}} \psi_{\underline{b}}) \right) \\
& + zh \frac{2a^2 r\cos\theta}{(r^2+a^2)|q|^4} \nabla_\phi \Psi_z \cdot {}^*(\mathcal{L}^{\underline{b}} \psi_{\underline{b}}) - \frac{1}{2} \nabla_\phi \left(zh \frac{2a^2 r\cos\theta}{(r^2+a^2)|q|^4} \Psi_z \cdot {}^*(\mathcal{L}^{\underline{b}} \psi_{\underline{b}}) \right) \\
= \;& O(ar^{-1}) \nabla_{\widehat{T}} \Psi_z \cdot {}^*(\mathcal{L}^{\underline{b}} \psi_{\underline{b}}) + O(a^2 r^{-2}) \nabla_\phi \Psi_z \cdot {}^*(\mathcal{L}^{\underline{b}} \psi_{\underline{b}}) \\
& - \frac{1}{2} \nabla_{\widehat{T}} \left(zh \left(\left({}^*\rho + \underline{\eta} \wedge \eta \right) \frac{r^2+a^2}{\Delta} + \frac{2a^3 r\cos\theta(\sin\theta)^2}{|q|^6} \right) \Psi_z \cdot {}^*(\mathcal{L}^{\underline{b}} \psi_{\underline{b}}) \right) \\
& - \frac{1}{2} \nabla_\phi \left(zh \frac{2a^2 r\cos\theta}{(r^2+a^2)|q|^4} \Psi_z \cdot {}^*(\mathcal{L}^{\underline{b}} \psi_{\underline{b}}) \right) \\
\leq \;& \delta_3 r^3 \Big(|\nabla_{\widehat{T}} \Psi_z|^2 + r^2 |\nabla \Psi_z|^2 \Big) + O(ar^{-3}) |\psi|^2_{\mathcal{S}} - \frac{1}{2} \nabla_\phi \left(zh \frac{2a^2 r\cos\theta}{(r^2+a^2)|q|^4} \Psi_z \cdot {}^*(\mathcal{L}^{\underline{b}} \psi_{\underline{b}}) \right) \\
& - \frac{1}{2} \nabla_{\widehat{T}} \left(zh \left(\left({}^*\rho + \underline{\eta} \wedge \eta \right) \frac{r^2+a^2}{\Delta} + \frac{2a^3 r\cos\theta(\sin\theta)^2}{|q|^6} \right) \Psi_z \cdot {}^*(\mathcal{L}^{\underline{b}} \psi_{\underline{b}}) \right)
\end{aligned}
$$

as stated. $\square$

8.3.4 Proof of Proposition 6.3.10

We are now ready to prove Proposition 6.3.10. Recall (8.3.5), i.e.

$$
\begin{aligned}
\mathbf{D}^\mu \mathcal{P}_\mu[\mathbf{X},\mathbf{w},\mathbf{M}] =& \widetilde{\mathcal{E}}[\mathbf{X},\mathbf{w},\mathbf{M}] + \mathbf{D}^\mu \mathcal{B}_\mu + \left(\nabla_{X^{\underline{ab}}}\psi_{\underline{a}} + \frac{1}{2} w^{\underline{ab}}\psi_{\underline{a}}\right)\cdot \left(\dot{\square}_2\psi_{\underline{b}} - V\psi_{\underline{b}}\right)\\
&- \frac{2a^2 r\cos\theta \mathcal{F}^{\underline{ab}}(r)}{(r^2+a^2)|q|^4}\nabla_\phi \psi_{\underline{a}}\cdot {}^*\psi_{\underline{b}}\\
&- \left(\left({}^*\rho + \underline{\eta}\wedge\eta\right)\mathcal{F}^{\underline{ab}}\frac{r^2+a^2}{\Delta} + \frac{2a^3 r\cos\theta(\sin\theta)^2\mathcal{F}^{\underline{ab}}(r)}{|q|^6}\right)\nabla_{\widehat{T}}\psi_{\underline{a}}\cdot {}^*\psi_{\underline{b}}.
\end{aligned}
$$

We apply the divergence theorem to the above on $\mathcal{M}(\tau_1,\tau_2)$, which yields

$$
\begin{aligned}
&\int_{\mathcal{M}(\tau_1,\tau_2)}\Bigg[\widetilde{\mathcal{E}}[\mathbf{X},\mathbf{w},\mathbf{M}] + \left(\nabla_{X^{\underline{ab}}}\psi_{\underline{a}} + \frac{1}{2}w^{\underline{ab}}\psi_{\underline{a}}\right)\cdot\left(\dot{\square}_2\psi_{\underline{b}} - V\psi_{\underline{b}}\right)\\
&-\left(\left({}^*\rho + \underline{\eta}\wedge\eta\right)\mathcal{F}^{\underline{ab}}\frac{r^2+a^2}{\Delta} + \frac{2a^3 r\cos\theta(\sin\theta)^2\mathcal{F}^{\underline{ab}}(r)}{|q|^6}\right)\nabla_{\widehat{T}}\psi_{\underline{a}}\cdot {}^*\psi_{\underline{b}}\\
&-\frac{2a^2 r\cos\theta\mathcal{F}^{\underline{ab}}(r)}{(r^2+a^2)|q|^4}\nabla_\phi\psi_{\underline{a}}\cdot {}^*\psi_{\underline{b}}\Bigg]\\
\leq\ & \int_{\partial\mathcal{M}(\tau_1,\tau_2)}\left(|\mathcal{P}_\mu[\mathbf{X},\mathbf{w},\mathbf{M}]\mathbf{N}^\mu| + |\mathcal{B}_\mu\mathbf{N}^\mu|\right).
\end{aligned}
$$

Using the lower bound (8.3.3) for $\widetilde{\mathcal{E}}[\mathbf{X},\mathbf{w},\mathbf{M}]$, and estimating the three other terms on the LHS thanks to Lemma 8.3.2, we obtain, for sufficiently small δ_2 and δ_3,

$$
\begin{aligned}
\mathrm{Mor}_{\mathcal{S},z,deg}[\psi](\tau_1,\tau_2) \lesssim& \int_{\partial\mathcal{M}(\tau_1,\tau_2)}|M_{\mathcal{S}}(\psi)| + |a|\int_{\mathcal{M}(\tau_1,\tau_2)}\left(r^{-2}|\nabla_{\widehat{R}}\mathfrak{d}^{\leq 2}\psi|^2 + r^{-3}|\mathfrak{d}^{\leq 2}\psi|^2\right)\\
&+\sum_{\underline{a}=1}^{4}\int_{\mathcal{M}(\tau_1,\tau_2)}\left(|\nabla_{\widehat{R}}\psi_{\underline{a}}| + r^{-1}|\psi_{\underline{a}}|\right)|N_{\underline{a}}|
\end{aligned}
$$

where we recall that

$$
\mathrm{Mor}_{\mathcal{S},z,deg}[\psi](\tau_1,\tau_2) = \int_{\mathcal{M}(\tau_1,\tau_2)}\frac{m}{r^2}|\nabla_{\widehat{R}}\psi|^2_{\mathcal{S}} + r^{-3}|\psi|^2_{\mathcal{S}} + r^3\left(\left|\nabla_{\widehat{T}}\Psi_z\right|^2 + r^2\left|\nabla\Psi_z\right|^2\right),
$$

and where $M_{\mathcal{S}}(\psi)$ denotes an expression in ψ for which we have a bound of the form

$$\begin{aligned}
&\int_{\partial\mathcal{M}(\tau_1,\tau_2)}|M_{\mathcal{S}}(\psi)|\\
\lesssim\;& \sum_{\underline{a}=1}^4\left(\sup_{[\tau_1,\tau_2]}E_{deg}[\psi_{\underline{a}}](\tau)+\delta_{\mathcal{H}}F_{\mathcal{A}}[\psi_{\underline{a}}](\tau_1,\tau_2)+F_{\Sigma_*}[\psi_{\underline{a}}](\tau_1,\tau_2)\right)\\
+\;&\left(\sup_{[\tau_1,\tau_2]}E_{deg}[(\nabla_T,\,\not{\mathfrak{d}})^{\leq 1}\psi](\tau)+\delta_{\mathcal{H}}F_{\mathcal{A}}[(\nabla_T,\,\not{\mathfrak{d}})^{\leq 1}\psi](\tau_1,\tau_2)+F_{\Sigma_*}[(\nabla_T,\,\not{\mathfrak{d}})^{\leq 1}\psi](\tau_1,\tau_2)\right)^{\frac{1}{2}}\\
\times\;&\left(\sup_{[\tau_1,\tau_2]}E_{deg}[(\nabla_T,\,\not{\mathfrak{d}})^{\leq 2}\psi](\tau)+\delta_{\mathcal{H}}F_{\mathcal{A}}[(\nabla_T,\,\not{\mathfrak{d}})^{\leq 2}\psi](\tau_1,\tau_2)+F_{\Sigma_*}[(\nabla_T,\,\not{\mathfrak{d}})^{\leq 2}\psi](\tau_1,\tau_2)\right)^{\frac{1}{2}}.
\end{aligned}$$

Finally, since we have

$$\begin{aligned}
&\int_{\mathcal{M}(\tau_1,\tau_2)}\left(r^{-2}|\nabla_{\widehat{R}}\mathfrak{d}^{\leq 2}\psi|^2+r^{-3}|\mathfrak{d}^{\leq 2}\psi|^2\right)\\
\lesssim\;&\int_{\mathcal{M}_{trap}(\tau_1,\tau_2)}\left(r^{-2}|\nabla_{\widehat{R}}\mathfrak{d}^{\leq 2}\psi|^2+r^{-3}|\mathfrak{d}^{\leq 2}\psi|^2\right)+\int_{\mathcal{M}_{\not{trap}}(\tau_1,\tau_2)}\left(r^{-2}|\nabla_3\mathfrak{d}^{\leq 2}\psi|^2+r^{-3}|\mathfrak{d}^{\leq 3}\psi|^2\right)\\
\lesssim\;& B^2_\delta[\psi](\tau_1,\tau_2)
\end{aligned}$$

for any $\delta>0$ in view of the definition of $B^2_\delta[\psi](\tau_1,\tau_2)$, we infer

$$\begin{aligned}
\mathrm{Mor}_{\mathcal{S},z,deg}[\psi](\tau_1,\tau_2)\lesssim&\int_{\partial\mathcal{M}(\tau_1,\tau_2)}|M_{\mathcal{S}}(\psi)|+\frac{|a|}{m}B^2_\delta[\psi](\tau_1,\tau_2)\\
&+\sum_{\underline{a}=1}^4\int_{\mathcal{M}(\tau_1,\tau_2)}\left(|\nabla_{\widehat{R}}\psi_{\underline{a}}|+r^{-1}|\psi_{\underline{a}}|\right)|N_{\underline{a}}|
\end{aligned}$$

which concludes the proof of Proposition 6.3.10.

8.4 Proof of Lemma 6.3.11

In this section, we prove Lemma 6.3.11 on the lower bound for Ψ_z on $\mathcal{M}_{\not{trap}}$. Recall the definition (8.2.11) of Ψ_z

$$\Psi_z=-\frac{2\mathcal{T}}{(r^2+a^2)^3}\big(\delta_0\mathcal{S}_1\psi+(1+O(r^{-2}\delta_0))\mathcal{O}\psi\big)+\frac{4ar}{(r^2+a^2)^2}\nabla_{\widehat{T}}\nabla_Z\psi\big(1+O(r^{-2}\delta_0)\big),$$

as well as the definition of $\mathcal{M}_{tr\not{\phi}}$, see (9.1.2),

$$\mathcal{M}_{tr\not{\phi}} = \left\{\frac{|\mathcal{T}|}{r^3} \geq \frac{1}{10}\right\}.$$

In particular, we have on $\mathcal{M}_{tr\not{\phi}}$

$$\begin{aligned}
|\nabla_T\Psi_z| + r|\nabla\Psi_z| \;\; \geq \;\; & \frac{1}{5}\frac{r^3}{(r^2+a^2)^3}\Big(\big|\nabla_T\big(\delta_0\mathcal{S}_1\psi + \mathcal{O}\psi\big)\big| + |q|\,\big|\nabla\big(\delta_0\mathcal{S}_1\psi + \mathcal{O}\psi\big)\big|\Big)\\
& - O(\delta_0 r^{-3})\Big(|\nabla_T\mathcal{O}\psi| + r|\nabla\mathcal{O}\psi|\Big)\\
& - O(ar^{-3})\Big(|\nabla_T\nabla_{\widehat{T}}\nabla_Z\psi| + r|\nabla\nabla_{\widehat{T}}\nabla_Z\psi|\Big) - O((\delta_0+|a|)r^{-3})|\psi|_{\mathcal{S}}
\end{aligned}$$

and hence for a universal constant $c_0 > 0$

$$\begin{aligned}
r^3\Big(|\nabla_T\Psi_z|^2 + r^2|\nabla\Psi_z|^2\Big) \geq & c_0 r^{-3}\Big(\big|\nabla_T\big(\delta_0\mathcal{S}_1\psi + \mathcal{O}\psi\big)\big|^2 + |q|^2\,\big|\nabla\big(\delta_0\mathcal{S}_1\psi + \mathcal{O}\psi\big)\big|^2\Big)\\
& - O(a^2 r^{-3})\Big(|\nabla_T\nabla_{\widehat{T}}\nabla_Z\psi|^2 + r^2|\nabla\nabla_{\widehat{T}}\nabla_Z\psi|^2\Big)\\
& - O(\delta_0^2 r^{-3})\Big(|\nabla_T\mathcal{O}\psi|^2 + r^2|\nabla\mathcal{O}\psi|^2\Big) - O(r^{-3})|\psi|^2_{\mathcal{S}}.
\end{aligned}$$

which we rewrite

$$\begin{aligned}
r^3\Big(|\nabla_T\Psi_z|^2 + r^2|\nabla\Psi_z|^2\Big) \geq & c_0 r^{-3}\Big(\big|\nabla_T\big(\delta_0\mathcal{S}_1\psi + \mathcal{O}\psi\big)\big|^2 + |q|^2\,\big|\nabla\big(\delta_0\mathcal{S}_1\psi + \mathcal{O}\psi\big)\big|^2\Big)\\
& - O(\delta_0^2 r^{-3})\Big(|\nabla_T\mathcal{O}\psi|^2 + r^2|\nabla\mathcal{O}\psi|^2\Big) \qquad (8.4.1)\\
& - O(ar^{-3})|(\nabla_T, \not{\mathfrak{d}})\mathfrak{d}^{\leq 2}\psi|^2 - O(r^{-3})|\psi|^2_{\mathcal{S}}.
\end{aligned}$$

Next, we focus on the first term on the RHS of (8.4.1). We start with the following computation

$$\begin{aligned}
& \big|\nabla_T\big(\delta_0\mathcal{S}_1\psi + \mathcal{O}\psi\big)\big|^2 + |q|^2\,\big|\nabla\big(\delta_0\mathcal{S}_1\psi + \mathcal{O}\psi\big)\big|^2\\
= \;\; & \delta_0^2\Big(|\nabla_T\mathcal{S}_1\psi|^2 + |q|^2|\nabla\mathcal{S}_1\psi|^2\Big) + 2\delta_0\Big(\nabla_T\mathcal{S}_1\psi\cdot\nabla_T\mathcal{O}\psi + |q|^2\nabla\mathcal{S}_1\psi\cdot\nabla\mathcal{O}\psi\Big)\\
& + |\nabla_T\mathcal{O}\psi|^2 + |q|^2|\nabla\mathcal{O}\psi|^2.
\end{aligned}$$

Using the structure of commutators[3], we write the second term on the RHS as follows

$$
\begin{aligned}
& \nabla_T\mathcal{S}_1\psi\cdot\nabla_T\mathcal{O}\psi+|q|^2\nabla\mathcal{S}_1\psi\cdot\nabla\mathcal{O}\psi\\
= \ & \mathcal{S}_1\nabla_T\psi\cdot\mathcal{O}\nabla_T\psi+|q|^2\mathcal{S}_1\nabla\psi\cdot\mathcal{O}\nabla\psi\\
& +\mathcal{S}_1\nabla_T\psi\cdot[\nabla_T,\mathcal{O}]\psi+|q|^2\nabla\mathcal{S}_1\psi\cdot[\nabla,\mathcal{O}]\psi+|q|^2[\nabla,\mathcal{S}_1]\psi\cdot\mathcal{O}\nabla\psi\\
= \ & \mathcal{S}_1\nabla_T\psi\cdot\mathcal{O}\nabla_T\psi+|q|^2\mathcal{S}_1\nabla\psi\cdot\mathcal{O}\nabla\psi-|q|^2\nabla\mathcal{S}_1\psi\cdot\nabla\psi+O(a)\big((\nabla_T,\,\not{\mathfrak{d}})\mathfrak{d}^{\leq 2}\psi\big)^2\\
= \ & \mathcal{S}_1\nabla_T\psi\cdot\mathcal{O}\nabla_T\psi+|q|^2\mathcal{S}_1\nabla\psi\cdot\mathcal{O}\nabla\psi+|q|^2|\nabla\nabla_T\psi|^2+O(a)\big((\nabla_T,\,\not{\mathfrak{d}})\mathfrak{d}^{\leq 2}\psi\big)^2\\
& -\partial_t(|q|^2\nabla\nabla_T\psi\cdot\nabla\psi).
\end{aligned}
$$

Next, we rely on Lemma 8.2.3 and obtain

$$
\begin{aligned}
& \nabla_T\mathcal{S}_1\psi\cdot\nabla_T\mathcal{O}\psi+|q|^2\nabla\mathcal{S}_1\psi\cdot\nabla\mathcal{O}\psi\\
= \ & |q|^2|\nabla\nabla_T^2\psi|^2+|q|^2\dot{\mathbf{D}}_\beta(|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\nabla_T\psi\cdot\mathcal{S}_1\nabla_T\psi)+\partial_t(M(\nabla_T\psi))\\
& +|q|^4|\nabla^2\nabla_T\psi|^2+|q|^4\dot{\mathbf{D}}_\beta(|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\nabla\psi\cdot\mathcal{S}_1\nabla\psi)+\partial_t(r^2M(\nabla\psi))\\
& +|q|^2|\nabla\nabla_T\psi|^2+O(a)\big((\nabla_T,\,\not{\mathfrak{d}})^{\leq 1}\mathfrak{d}^{\leq 2}\psi\big)^2-\partial_t(|q|^2\nabla\nabla_T\psi\cdot\nabla\psi).
\end{aligned}
$$

Rearranging, this yields

$$
\begin{aligned}
& \nabla_T\mathcal{S}_1\psi\cdot\nabla_T\mathcal{O}\psi+|q|^2\nabla\mathcal{S}_1\psi\cdot\nabla\mathcal{O}\psi\\
= \ & |q|^2|\nabla\nabla_T^2\psi|^2+|q|^4|\nabla^2\nabla_T\psi|^2+|q|^2|\nabla\nabla_T\psi|^2-O(a)\big|(\nabla_T,\,\not{\mathfrak{d}})^{\leq 1}\mathfrak{d}^{\leq 2}\psi\big|^2\\
& +|q|^2\dot{\mathbf{D}}_\beta(|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\nabla_T\psi\cdot\mathcal{S}_1\nabla_T\psi)+|q|^4\dot{\mathbf{D}}_\beta(|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\nabla\psi\cdot\mathcal{S}_1\nabla\psi)\\
& +\partial_t(M(\nabla_T\psi))+\partial_t(M(|q|\nabla\psi))-\partial_t(|q|^2\nabla\nabla_T\psi\cdot\nabla\psi)
\end{aligned}
$$

and hence

$$
\begin{aligned}
& \big|\nabla_T\big(\delta_0\mathcal{S}_1\psi+\mathcal{O}\psi\big)\big|^2+|q|^2\,\big|\nabla\big(\delta_0\mathcal{S}_1\psi+\mathcal{O}\psi\big)\big|^2\\
= \ & \delta_0^2\Big(|\nabla_T\mathcal{S}_1\psi|^2+|q|^2|\nabla\mathcal{S}_1\psi|^2\Big)+2\delta_0\Big(|q|^2|\nabla\nabla_T^2\psi|^2+|q|^4|\nabla^2\nabla_T\psi|^2+|q|^2|\nabla\nabla_T\psi|^2\Big)\\
& +|\nabla_T\mathcal{O}\psi|^2+|q|^2|\nabla\mathcal{O}\psi|^2-O(a)\big|(\nabla_T,\,\not{\mathfrak{d}})^{\leq 1}\mathfrak{d}^{\leq 2}\psi\big|^2\\
& +\delta_0|q|^2\dot{\mathbf{D}}_\beta(|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\nabla_T\psi\cdot\mathcal{S}_1\nabla_T\psi)+\delta_0|q|^4\dot{\mathbf{D}}_\beta(|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\nabla\psi\cdot\mathcal{S}_1\nabla\psi)\\
& +\partial_t(M(\nabla_T\psi))+\partial_t(M(|q|\nabla\psi))-\partial_t(2\delta_0|q|^2\nabla\nabla_T\psi\cdot\nabla\psi). \qquad (8.4.2)
\end{aligned}
$$

Next, notice that $\mathcal{S}_2=O(a)\mathfrak{d}^{\leq 2}$ and $\mathcal{S}_3=O(a^2)\mathfrak{d}^{\leq 2}$ so that

$$
\begin{aligned}
|\nabla_T\mathcal{S}_1\psi|^2+|q|^2|\nabla\mathcal{S}_1\psi|^2+|\nabla_T\mathcal{O}\psi|^2+|q|^2|\nabla\mathcal{O}\psi|^2 \ \geq \ & |\nabla_T\psi|_{\mathcal{S}}+r^2|\nabla\psi|_{\mathcal{S}}\\
& -O(a^2)\big|(\nabla_T,\,\not{\mathfrak{d}})^{\leq 1}\mathfrak{d}^{\leq 2}\psi\big|^2.
\end{aligned}
$$

[3]Note in particular that $[\nabla,\mathcal{O}]=-4\nabla\psi+O(ar^{-1})(\nabla_{\mathbf{T}},\,\not{\mathfrak{d}})^{\leq 2}\psi$.

Together with (8.4.2), we infer

$$\begin{aligned}
&\big|\nabla_T\big(\delta_0\mathcal{S}_1\psi+\mathcal{O}\psi\big)\big|^2+|q|^2\,\big|\nabla\big(\delta_0\mathcal{S}_1\psi+\mathcal{O}\psi\big)\big|^2\\
=\;&\delta_0^2\Big(|\nabla_T\psi|_{\mathcal{S}}+r^2|\nabla\psi|_{\mathcal{S}}\Big)+(1-\delta_0^2)\Big(|\nabla_T\mathcal{O}\psi|^2+|q|^2|\nabla\mathcal{O}\psi|^2\Big)-O(a)\big|(\nabla_T,\,\not{\mathfrak{d}})^{\leq 1}\mathfrak{d}^{\leq 2}\psi\big|^2\\
&+\delta_0|q|^2\dot{\mathbf{D}}_\beta(|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\nabla_T\psi\cdot\mathcal{S}_1\nabla_T\psi)+\delta_0|q|^4\dot{\mathbf{D}}_\beta(|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\nabla\psi\cdot\mathcal{S}_1\nabla\psi)\\
&+\partial_t(M(\nabla_T\psi))+\partial_t(M(|q|\nabla\psi))-\partial_t(2\delta_0|q|^2\nabla\nabla_T\psi\cdot\nabla\psi).
\end{aligned}$$

Next, plugging in (8.4.1), we deduce

$$\begin{aligned}
&r^3\Big(|\nabla_T\Psi_z|^2+r^2|\nabla\Psi_z|^2\Big)\\
\geq\;&c_0r^{-3}\Big(\,\big|\nabla_T\big(\delta_0\mathcal{S}_1\psi+\mathcal{O}\psi\big)\big|^2+|q|^2\,\big|\nabla\big(\delta_0\mathcal{S}_1\psi+\mathcal{O}\psi\big)\big|^2\,\Big)\\
&-O(\delta_0^2r^{-3})\Big(|\nabla_T\mathcal{O}\psi|^2+r^2|\nabla\mathcal{O}\psi|^2\Big)-O(ar^{-3})|(\nabla_T,\,\not{\mathfrak{d}})\mathfrak{d}^{\leq 2}\psi|^2-O(r^{-3})|\psi|^2_{\mathcal{S}}\\
\geq\;&c_0\delta_0^2r^{-3}\Big(|\nabla_T\psi|_{\mathcal{S}}+r^2|\nabla\psi|_{\mathcal{S}}\Big)+c_0r^{-3}(1-O(\delta_0^2))\Big(|\nabla_T\mathcal{O}\psi|^2+|q|^2|\nabla\mathcal{O}\psi|^2\Big)\\
&-O(ar^{-3})\big|(\nabla_T,\,\not{\mathfrak{d}})^{\leq 1}\mathfrak{d}^{\leq 2}\psi\big|^2-O(r^{-3})|\psi|^2_{\mathcal{S}}\\
&+\delta_0r^{-3}|q|^2\dot{\mathbf{D}}_\beta(|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\nabla_T\psi\cdot\mathcal{S}_1\nabla_T\psi)+\delta_0r^{-3}|q|^4\dot{\mathbf{D}}_\beta(|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\nabla\psi\cdot\mathcal{S}_1\nabla\psi)\\
&+\partial_t(r^{-3}M(\nabla_T\psi))+\partial_t(r^{-3}M(|q|\nabla\psi))-\partial_t(2\delta_0r^{-3}|q|^2\nabla\nabla_T\psi\cdot\nabla\psi).
\end{aligned}$$

Hence, fixing $\delta_0>0$ small enough, we infer

$$\begin{aligned}
&r^3\Big(|\nabla_T\Psi_z|^2+r^2|\nabla\Psi_z|^2\Big)\\
\geq\;&c_0\delta_0^2r^{-3}\Big(|\nabla_T\psi|_{\mathcal{S}}+r^2|\nabla\psi|_{\mathcal{S}}\Big)-O(ar^{-3})\big|(\nabla_T,\,\not{\mathfrak{d}})^{\leq 1}\mathfrak{d}^{\leq 2}\psi\big|^2-O(r^{-3})|\psi|^2_{\mathcal{S}}\\
&+\dot{\mathbf{D}}_\beta(\delta_0O(r^{-3})O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\nabla_T\psi\cdot\mathcal{S}_1\nabla_T\psi)+\dot{\mathbf{D}}_\beta(\delta_0O(r^{-1})O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\nabla\psi\cdot\mathcal{S}_1\nabla\psi)\\
&+\partial_t(r^{-3}M(\nabla_T\psi))+\partial_t(r^{-3}M(|q|\nabla\psi))-\partial_t(2\delta_0r^{-3}|q|^2\nabla\nabla_T\psi\cdot\nabla\psi).
\end{aligned}$$

In particular, we deduce the existence of a universal constant $c_0>0$ such that the following holds on $\mathcal{M}_{tr\not{p}}$

$$\begin{aligned}
r^3\Big(|\nabla_T\Psi_z|^2+r^2|\nabla\Psi_z|^2\Big)+r^{-3}|\psi|^2_{\mathcal{S}}\;\;\geq\;\;&c_0r^{-3}\Big(|\nabla_T\psi|^2_{\mathcal{S}}+|\nabla_Z\psi|^2_{\mathcal{S}}+r^2|\nabla\psi|^2_{\mathcal{S}}\Big)\\
&-O(ar^{-3})\big|(\nabla_T,\,\not{\mathfrak{d}})^{\leq 1}\mathfrak{d}^{\leq 2}\psi\big|^2+\mathrm{Bdr}
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{Bdr}\;=\;&\dot{\mathbf{D}}_\beta(O(r^{-3})O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\nabla_T\psi\cdot\mathcal{S}_1\nabla_T\psi)+\dot{\mathbf{D}}_\beta(O(r^{-1})O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\nabla\psi\cdot\mathcal{S}_1\nabla\psi)\\
&+\partial_t(r^{-3}M(\nabla_T\psi))+\partial_t(r^{-3}M(|q|\nabla\psi))-\partial_t(O(r^{-1})\nabla\nabla_T\psi\cdot\nabla\psi).
\end{aligned}$$

Finally, note that the boundary terms verify

$$\text{Bdr} \;=\; \dot{\mathbf{D}}_\alpha F^\alpha, \qquad |F^\mu N_\mu| \lesssim r^{-2}|(\nabla_{\widehat{R}}, \nabla_T, \not{\mathfrak{d}})^{\leq 1}(\nabla_T, \not{\mathfrak{d}})^{\leq 1}\psi||(\nabla_{\widehat{R}}, \nabla_T, \not{\mathfrak{d}})^{\leq 1}(\nabla_T, \not{\mathfrak{d}})^{\leq 2}\psi|,$$

where N denotes the normal to either $\Sigma(\tau)$, $\mathcal{A}$ or Σ_*. This concludes the proof of Lemma 6.3.11.

Chapter 9

Energy-Morawetz in perturbations of Kerr

In this chapter, we prove Theorem 6.3.1, i.e. we establish Energy-Morawetz estimates for solutions to the model gRW equation (6.1.1)

$$\dot{\square}_2\psi - V\psi = -\frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_T\psi + N, \qquad V = \frac{4\Delta}{(r^2+a^2)|q|^2},$$

in perturbations of Kerr. To this end, we proceed as follows: Theorem 6.3.1 is proved in Chapter 9 according to the following steps:

1. First, we revisit the proof of Propositions 6.3.7, 6.3.9 and 6.3.10 by exhibiting the extra terms in perturbations of Kerr, and prove that the conclusions of Propositions 6.3.7, 6.3.9 and 6.3.10 also hold in perturbations of Kerr up to the addition of suitable error terms see section 9.2.

2. Next, we prove redshift estimates to remove the degeneracy on the horizon, see section 9.4.

3. Then, we derive the conclusion of Theorem 6.3.1 for $s = 2$, see section 9.5.1.

4. Finally, we argue by iteration from $s = 2$ to recover higher order derivatives which and conclude the proof of Theorem 6.3.1, see section 9.5.2.

9.1 Preliminaries

In this section, we recall the basic set up and state the analog of Propositions 6.3.7, 6.3.9 and 6.3.10 in perturbations of Kerr.

9.1.1 Admissible perturbations of Kerr

In this chapter, we prove Theorem 6.3.1 for solution of the model gRW equation (6.1.1) on the spacetime $\mathcal{M}$, where $\mathcal{M}$ is an admissible perturbation of Kerr which satisfies the assumptions of section 6.1. We briefly recall them below for the convenience of the reader.

The spacetime $\mathcal{M}$

We consider a given vacuum spacetime $\mathcal{M}$ satisfying the following properties:

- $\mathcal{M}$ comes together with a null pair (e_4, e_3) and its corresponding horizontal structure as in section 2.1.1.
- $\mathcal{M}$ is endowed with a pair of constants (a, m).
- $\mathcal{M}$ is endowed with a pair of scalar functions (r, θ).
- The complex valued scalar function q is defined as

$$q := r + i\cos\theta.$$

- $\mathcal{M}$ is endowed with a complex horizontal 1-form $\mathfrak{J}$.
- $\mathcal{M}$ is endowed with a scalar function τ whose level sets $\Sigma(\tau)$ are spacelike. Also:
  - $\tau \in [1, \tau_*]$ on $\mathcal{M}$ for some arbitrary large constant τ_*.
  - Given a level hypersurface $\Sigma = \Sigma(\tau)$, we denote

$$N_\Sigma := -\mathbf{g}^{\alpha\beta}\partial_\beta\tau\partial_\alpha.$$

  - τ satisfies the properties of Definition 6.1.5.

- The boundary of $\mathcal{M}$ is given by

$$\partial\mathcal{M} = \mathcal{A} \cup \Sigma_* \cup \Sigma(1) \cup \Sigma(\tau_*)$$

where

$$\mathcal{A} := \Big\{ r = r_+ - \delta_{\mathcal{H}},\, 1 \leq \tau \leq \tau_* \Big\},$$

and Σ_* is a spacelike hypersurface on which τ takes the values $[1, \tau_*]$ and $r \geq r_*$ with $r_* \gg \tau_*$.

- Let r_0 a large enough fixed constant. We decompose $\mathcal{M}$ as follows

$${}^{(int)}\mathcal{M} := \mathcal{M} \cap \{r \leq r_0\}, \qquad {}^{(ext)}\mathcal{M} := \mathcal{M} \cap \{r \geq r_0\}.$$

Admissible perturbations of Kerr

Recall that $\mathcal{M}$ comes together three scalar functions (r, θ, τ), and with a null pair (e_4, e_3) and its corresponding horizontal structure as in section 2.1.1. Then:

- We use the complexified Ricci and curvature coefficients of Definition 2.4.8.
- We define the linearized quantities corresponding to these complexified coefficients as in Definition 4.1.1 and 4.1.3, i.e. we consider that the normalization of (e_3, e_4) is ingoing.
- With respect to these linearized quantities, we the notations Γ_g and Γ_b for error terms are given by Definition 4.1.5.

We this definition of Γ_g and Γ_b, we can now state our main assumptions on $\mathcal{M}$ allowing to prove the Energy-Morawetz estimates of Theorem 6.3.1. Let k_L a large enough integer, and let the scalar function τ_{trap} defined by

$$\tau_{trap} := \begin{cases} 1 + \tau & \text{on} \quad \mathcal{M}_{trap}, \\ 1 & \text{on} \quad \mathcal{M}_{\not{trap}}. \end{cases}$$

Then, we assume that (Γ_g, Γ_b) satisfy the following estimates on $\mathcal{M}$

$$\begin{aligned} r^3|\mathfrak{d}^{\leq k}\xi| + r^2|\mathfrak{d}^{\leq k}\Gamma_g| + r|\mathfrak{d}^{\leq k}\Gamma_b| &\leq \frac{\epsilon}{\tau_{trap}^{1+\delta_{dec}}}, \quad 0 \leq k \leq \frac{k_L}{2}, \\ r^3|\mathfrak{d}^{\leq k}\xi| + r^2|\mathfrak{d}^{\leq k}\Gamma_g| + r|\mathfrak{d}^{\leq k}\Gamma_b| &\leq \epsilon, \quad k \leq k_L. \end{aligned} \tag{9.1.1}$$

Remark 9.1.1. *Note that the assumptions for ξ in (9.1.1) are consistent with $\xi \in r^{-1}\Gamma_g$, see Remark 6.1.4 for the justification of these stronger assumptions. We may thus assume in this chapter that $\xi \in r^{-1}\Gamma_g$. This will be needed to deal with various commutators with the wave operator.*

9.1.2 Regions of integration and vectorfields

Regions of integration

Recall the time function τ introduced in Definition 6.1.5. We denote by Σ_τ the level sets of the function τ.

Definition 9.1.2. *We define the following regions of $\mathcal{M}$.*

1. *We define the trapping region of $\mathcal{M}$ to be the set*

$$\mathcal{M}_{trap}(\delta_{trap}) = \left\{ \frac{|\mathcal{T}|}{r^3} \leq \delta_{trap} \right\}, \qquad \delta_{trap} = \frac{1}{10}, \tag{9.1.2}$$

   where $\mathcal{T}$ is the polynomial in r defined in (3.8.5), i.e.

$$\mathcal{T} = r^3 - 3mr^2 + a^2 r + ma^2.$$

2. *We denote $\mathcal{M}_{\not{trap}}$ the complement to the trapping region $\mathcal{M}_{trap}$.*

3. *We define the domain $\mathcal{M}(\tau_1, \tau_2)$ to be the region of $\mathcal{M}$ where $\tau_1 \leq \tau \leq \tau_2$, where τ is the time function defined in Definition 6.1.5.*

Basic vectorfields

$\mathbf{T}$ and $\mathbf{Z}$ are defined in $\mathcal{M}$ as follows

$$\begin{aligned}
\mathbf{T} &= \frac{1}{2}\left(e_4 + \frac{\Delta}{|q|^2} e_3 - 2a\Re(\mathfrak{J})^b e_b\right), \\
\mathbf{Z} &= \frac{1}{2}\left(2(r^2+a^2)\Re(\mathfrak{J})^b e_b - a(\sin\theta)^2 e_4 - \frac{a(\sin\theta)^2\Delta}{|q|^2} e_3\right).
\end{aligned}$$

Remark 9.1.3. *Recall from Lemma 6.1.12 that for $|a|/m$ sufficiently small and $\delta_{trap} = \frac{1}{10}$, the vectorfield $\mathbf{T}$ is strictly timelike in $\mathcal{M}_{trap}$.*

Also, we define the following vectorfields

$$\widehat{T} = \frac{1}{2}\left(\frac{|q|^2}{r^2+a^2}e_4 + \frac{\Delta}{r^2+a^2}e_3\right), \qquad \widehat{R} = \frac{1}{2}\left(\frac{|q|^2}{r^2+a^2}e_4 - \frac{\Delta}{r^2+a^2}e_3\right).$$

Finally, the vectorfield $\widehat{T}_\delta$ that will be used for energy estimates is given by

$$\widehat{T}_\delta = \mathbf{T} + \frac{a}{r^2+a^2}\chi_0\left(\delta^{-1}\frac{\mathcal{T}}{r^3}\right)\mathbf{Z}$$

with $\delta = \delta_{trap}$ and with χ_0 the smooth bump function

$$\chi_0(x) = \begin{cases} 0 & \text{if} \quad |x| \leq 1, \\ 1 & \text{if} \quad |x| \geq 2. \end{cases}$$

We also write

$$\widehat{T}_\delta = \mathbf{T} + \chi_\delta \mathbf{Z}, \qquad \chi_\delta := \frac{a}{r^2+a^2}\chi_0\left(\delta^{-1}\frac{\mathcal{T}}{r^3}\right).$$

9.1.3 Main norms

We recall below the relevant main norms introduced in section 6.1.5:

1. Reduced basic Morawetz norms.

$$\begin{aligned} \text{Mor}[\psi](\tau_1,\tau_2) &:= \int_{\mathcal{M}(\tau_1,\tau_2)} r^{-2}|\nabla_{\widehat{R}}\psi|^2 + r^{-3}|\psi|^2 \\ &\quad + \int_{\mathcal{M}_{trap\!\!\!/}(\tau_1,\tau_2)} \left(r^{-2}|\nabla_3\psi|^2 + r^{-1}|\nabla\!\!\!/\psi|^2\right), \\ \text{Morr}[\psi](\tau_1,\tau_2) &:= \text{Mor}[\psi](\tau_1,\tau_2) + \int_{\mathcal{M}_{r\geq 4m}(\tau_1,\tau_2)} r^{-1-\delta}|\nabla_3\psi|^2. \end{aligned} \tag{9.1.3}$$

2. Basic Energy norm.

$$E[\psi](\tau) := \int_{\Sigma(\tau)} \Big(|\nabla_4\psi|^2 + r^{-2}|\nabla_3\psi|^2 + |\nabla\!\!\!/\psi|^2 + r^{-2}|\psi|^2\Big). \tag{9.1.4}$$

3. Basic Flux norm.

$$
\begin{aligned}
F[\psi](\tau_1,\tau_2) :=& F_{\mathcal{A}}[\psi](\tau_1,\tau_2) + F_{\Sigma_*}[\psi](\tau_1,\tau_2),\\
F_{\mathcal{A}}[\psi](\tau_1,\tau_2) :=& \int_{\mathcal{A}(\tau_1,\tau_2)} \Big(|\nabla_4\psi|^2 + |\nabla_3\psi|^2 + |\nabla\psi|^2 + r^{-2}|\psi|^2\Big),\\
F_{\Sigma_*}[\psi](\tau_1,\tau_2) :=& \int_{\Sigma_*(\tau_1,\tau_2)} \Big(|\nabla_4\psi|^2 + |\nabla_3\psi|^2 + |\nabla\psi|^2 + r^{-2}|\psi|^2\Big).
\end{aligned}
\tag{9.1.5}
$$

4. Basic N- norm.

$$
\begin{aligned}
\mathcal{N}[\psi,N](\tau_1,\tau_2) :=& \int_{\mathcal{M}(\tau_1,\tau_2)} \big(|\nabla_{\widehat{R}}\psi| + r^{-1}|\psi|\big)|N| + \left|\int_{\mathcal{M}} \nabla_{\widehat{T}_\delta}\psi\cdot N\right|\\
&+ \int_{\mathcal{M}(\tau_1,\tau_2)\cap\{r\leq r_+(1+2\delta_{red})\}} |\mathfrak{d}\psi||N|\\
&+ \int_{\mathcal{M}(\tau_1,\tau_2)} |N|^2 + \sup_{\tau\in[\tau_1,\tau_2]}\int_{\Sigma(\tau)} |N|^2 + \int_{\Sigma_*(\tau_1,\tau_2)} |N|^2.
\end{aligned}
\tag{9.1.6}
$$

5. Weighted bulk norm. For $0 < p < 2$, we define

$$
B_p[\psi](\tau_1,\tau_2) \quad := \quad \mathrm{Morr}[\psi](\tau_1,\tau_2) + \int_{\mathcal{M}_{r\geq 4m}(\tau_1,\tau_2)} r^{p-3}\Big(|\mathfrak{d}\psi|^2 + |\psi|^2\Big). \tag{9.1.7}
$$

6. Higher order norms. We define the higher derivative norms $\mathrm{Mor}^s[\psi]$, $E^s[\psi]$, $F^s[\psi]$, $\mathcal{N}^s[\psi,N]$, $B^s_p[\psi]$, by the general procedure for a norm $Q[\psi]$, i.e.

$$
Q^s[\psi] = \sum_{k\leq s} Q[\mathfrak{d}^k\psi].
$$

9.2 Basic energy-Morawetz in perturbations of Kerr

In this section, we revisit the proofs of Chapters 7 and 8 in Kerr and show that the conclusions of Chapters 7 and 8 also hold in perturbations of Kerr up to the addition suitable error terms, see the statements in section 9.2.6.

9.2.1 Some basic commutations with T and Z

In the following lemma, we establish basic properties of $\not{\mathcal{L}}_{\mathbf{T}}$ and $\not{\mathcal{L}}_{\mathbf{Z}}$ that will be used repeatedly.

Lemma 9.2.1. *For a horizontal covariant k-tensor U, we have*

$$\nabla_{\mathbf{T}} U_{b_1\cdots b_k} = \not{\mathcal{L}}_{\mathbf{T}} U_{b_1\cdots b_k} + \frac{2amr\cos\theta}{|q|^4}\sum_{j=1}^k \in_{b_j c} U_{b_1\cdots c\cdots b_k} + \Gamma_b U,$$

$$\nabla_{\mathbf{Z}} U_{b_1\cdots b_k} = \not{\mathcal{L}}_{\mathbf{Z}} U_{b_1\cdots b_k} - \frac{\cos\theta((r^2+a^2)^2 - a^2(\sin\theta)^2\Delta)}{|q|^4}\sum_{j=1}^k \in_{b_j c} U_{b_1\cdots c\cdots b_k} + r\Gamma_b U,$$

and

$$[\not{\mathcal{L}}_{\mathbf{T}}, \mathfrak{d}]U = \mathfrak{d}(\Gamma_b U), \qquad [\not{\mathcal{L}}_{\mathbf{Z}}, \mathfrak{d}]U = r\mathfrak{d}(\Gamma_b U).$$

Proof. First, we compute

$$\begin{aligned}
2\mathbf{g}(\nabla_b \mathbf{T}, e_c) &= \mathbf{g}\left(\nabla_b\left(e_4 + \frac{\Delta}{|q|^2}e_3 - 2a\Re(\mathfrak{J})_d e_d\right), e_c\right)\\
&= \chi_{bc} + \frac{\Delta}{|q|^2}\underline{\chi}_{bc} - 2a\nabla_b\Re(\mathfrak{J})_c\\
&= \frac{1}{2}\left(\mathrm{tr}\,\chi + \frac{\Delta}{|q|^2}\mathrm{tr}\,\underline{\chi}\right)\delta_{bc} + \frac{1}{2}\left({}^{(a)}\mathrm{tr}\chi + \frac{\Delta}{|q|^2}\,{}^{(a)}\mathrm{tr}\underline{\chi}\right)\in_{bc}\\
&\quad -a\,div\,(\Re(\mathfrak{J}))\delta_{bc} - a\,curl\,(\Re(\mathfrak{J}))\in_{bc} + \Gamma_b\\
&= \left(\frac{2a\cos\theta\Delta}{|q|^4} - \frac{2a(r^2+a^2)\cos\theta}{|q|^4}\right)\in_{bc} + \Gamma_b\\
&= -\frac{4amr\cos\theta}{|q|^4}\in_{bc} + \Gamma_b.
\end{aligned}$$

Since we have

$$\not{\mathcal{L}}_{\mathbf{T}} U_{b_1\cdots b_k} = \nabla_{\mathbf{T}} U_{b_1\cdots b_k} + \mathbf{g}(\mathbf{D}_{b_1}\mathbf{T}, e_c)U_{cb_2\cdots b_k} + \cdots,$$

we infer

$$\nabla_{\mathbf{T}} U_{b_1\cdots b_k} = \not{\mathcal{L}}_{\mathbf{T}} U_{b_1\cdots b_k} + \frac{2amr\cos\theta}{|q|^4}\sum_{j=1}^k \in_{b_j c} U_{b_1\cdots c\cdots b_k} + \Gamma_b U$$

as stated.

Next, we compute

$$
\begin{aligned}
2\mathbf{g}(\nabla_b\mathbf{Z}, e_c) &= \mathbf{g}\left(\nabla_b\left(2(r^2+a^2)\Re(\mathfrak{J})_d e_d - a(\sin\theta)^2 e_4 - \frac{a(\sin\theta)^2\Delta}{|q|^2}e_3\right), e_c\right)\\
&= 2(r^2+a^2)\nabla_b\Re(\mathfrak{J})_c + 4re_b(r)\Re(\mathfrak{J})_c - a(\sin\theta)^2\left(\chi_{bc} + \frac{\Delta}{|q|^2}\underline{\chi}_{bc}\right)\\
&= (r^2+a^2)\mathrm{d}iv\,(\Re(\mathfrak{J}))\delta_{bc} + (r^2+a^2)curl\,(\Re(\mathfrak{J}))\in_{bc}\\
&\quad -\frac{a(\sin\theta)^2}{2}\left(\mathrm{tr}\,\chi + \frac{\Delta}{|q|^2}\mathrm{tr}\,\underline{\chi}\right)\delta_{bc} - \frac{a(\sin\theta)^2}{2}\left({}^{(a)}\mathrm{tr}\chi + \frac{\Delta}{|q|^2}\,{}^{(a)}\mathrm{tr}\underline{\chi}\right)\in_{bc}\\
&\quad +r\Gamma_b\\
&= \left(-\frac{2a^2(\sin\theta)^2\cos\theta\Delta}{|q|^4} + \frac{2(r^2+a^2)^2\cos\theta}{|q|^4}\right)\in_{bc} + r\Gamma_b\\
&= \frac{2\cos\theta((r^2+a^2)^2 - a^2(\sin\theta)^2\Delta)}{|q|^4}\in_{bc} + r\Gamma_b.
\end{aligned}
$$

Since we have

$$
\not{\mathcal{L}}_{\mathbf{Z}}U_{b_1\cdots b_k} = \nabla_{\mathbf{Z}}U_{b_1\cdots b_k} + \mathbf{g}(\mathbf{D}_{b_1}\mathbf{Z}, e_c)U_{cb_2\cdots b_k} + \cdots,
$$

we infer

$$
\nabla_{\mathbf{Z}}U_{b_1\cdots b_k} = \not{\mathcal{L}}_{\mathbf{Z}}U_{b_1\cdots b_k} - \frac{\cos\theta((r^2+a^2)^2 - a^2(\sin\theta)^2\Delta)}{|q|^4}\sum_{j=1}^k \in_{b_j c} U_{b_1\cdots c\cdots b_k} + r\Gamma_b U
$$

as stated.

Next, in view of Lemma 2.2.13, together with the fact that ${}^{(\mathbf{T})}\pi \in \Gamma_b$, see Lemma 4.3.2, we have for a horizontal covariant k-tensor U

$$
\begin{aligned}
\nabla_b(\not{\mathcal{L}}_{\mathbf{T}}U_A) - \not{\mathcal{L}}_{\mathbf{T}}(\nabla_b U_A) &= r^{-1}\mathfrak{d}\Gamma_b U,\\
\nabla_4(\not{\mathcal{L}}_{\mathbf{T}}U_A) - \not{\mathcal{L}}_{\mathbf{T}}(\nabla_4 U_A) + \nabla_{\not{\mathcal{L}}_{\mathbf{T}}e_4}U_A &= r^{-1}\mathfrak{d}\Gamma_b U,\\
\nabla_3(\not{\mathcal{L}}_{\mathbf{T}}U_A) - \not{\mathcal{L}}_{\mathbf{T}}(\nabla_3 U_A) + \nabla_{\not{\mathcal{L}}_{\mathbf{T}}e_3}U_A &= \mathfrak{d}\Gamma_b U.
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
2\mathcal{L}_{\mathbf{T}}e_4 &= 2[\mathbf{T}, e_4] = \left[e_4 + \frac{\Delta}{|q|^2}e_3 - 2a\Re(\mathfrak{J})^d e_d, e_4\right]\\
&= \frac{\Delta}{|q|^2}[e_3, e_4] - e_4\left(\frac{\Delta}{|q|^2}\right)e_3 - 2a[\Re(\mathfrak{J})^d e_d, e_4]
\end{aligned}
$$

and hence

$$2 \not{\mathcal{L}}_{\mathbf{T}} e_4 = \frac{2\Delta}{|q|^2}(\check{\eta}_b - \check{\underline{\eta}}_b)e_b - 2a\Re(\widecheck{\nabla_4 \mathfrak{J}})_b e_b$$

which yields

$$\nabla_{\not{\mathcal{L}}_{\mathbf{T}} e_4} = r^{-1}\Gamma_b \not{\mathfrak{d}}.$$

Similarly, we have

$$\nabla_{\not{\mathcal{L}}_{\mathbf{T}} e_3} = r^{-1}\Gamma_b \not{\mathfrak{d}}$$

and hence

$$\begin{aligned}
&\nabla_b(\not{\mathcal{L}}_{\mathbf{T}} U_A) - \not{\mathcal{L}}_{\mathbf{T}}(\nabla_b U_A) = r^{-1}\mathfrak{d}\Gamma_b U, \\
&\nabla_4(\not{\mathcal{L}}_{\mathbf{T}} U_A) - \not{\mathcal{L}}_{\mathbf{T}}(\nabla_4 U_A) = r^{-1}\mathfrak{d}(\Gamma_b U), \\
&\nabla_3(\not{\mathcal{L}}_{\mathbf{T}} U_A) - \not{\mathcal{L}}_{\mathbf{T}}(\nabla_3 U_A) = \mathfrak{d}(\Gamma_b U).
\end{aligned}$$

Since $\mathbf{T}(r) \in r\Gamma_b$, we deduce

$$[\not{\mathcal{L}}_{\mathbf{T}}, \mathfrak{d}]U = \mathfrak{d}(\Gamma_b U)$$

as stated.

Finally, in view of Lemma 2.2.13, together with the fact that ${}^{(\mathbf{Z})}\pi \in r\Gamma_b$, see Lemma 4.3.2, we have for a horizontal covariant k-tensor U

$$\begin{aligned}
\nabla_b(\not{\mathcal{L}}_{\mathbf{Z}} U_A) - \not{\mathcal{L}}_{\mathbf{Z}}(\nabla_b U_A) &= \mathfrak{d}\Gamma_b U, \\
\nabla_4(\not{\mathcal{L}}_{\mathbf{Z}} U_A) - \not{\mathcal{L}}_{\mathbf{Z}}(\nabla_4 U_A) + \nabla_{\not{\mathcal{L}}_{\mathbf{Z}} e_4} U_A &= \mathfrak{d}\Gamma_b U, \\
\nabla_3(\not{\mathcal{L}}_{\mathbf{Z}} U_A) - \not{\mathcal{L}}_{\mathbf{Z}}(\nabla_3 U_A) + \nabla_{\not{\mathcal{L}}_{\mathbf{Z}} e_3} U_A &= r\mathfrak{d}\Gamma_b U.
\end{aligned}$$

Also, we have

$$\begin{aligned}
2\mathcal{L}_{\mathbf{Z}} e_4 &= 2[\mathbf{Z}, e_4] = \left[2(r^2+a^2)\Re(\mathfrak{J})_d e_d - a(\sin\theta)^2 e_4 - \frac{a(\sin\theta)^2\Delta}{|q|^2}e_3, e_4\right] \\
&= 2(r^2+a^2)[\Re(\mathfrak{J})_d e_d, e_4] - 4re_4(r)\Re(\mathfrak{J})_d e_d + ae_4((\sin\theta)^2)e_4 \\
&\quad - \frac{a(\sin\theta)^2\Delta}{|q|^2}[e_3, e_4] + e_4\left(\frac{a(\sin\theta)^2\Delta}{|q|^2}\right)e_3
\end{aligned}$$

and hence

$$2\not{\mathcal{L}}_{\mathbf{Z}} e_4 = 2(r^2+a^2)\Re(\widecheck{\nabla_4 \mathfrak{J}})_b e_b - 4r\widecheck{e_4(r)}\Re(\mathfrak{J})_d e_d - \frac{2a(\sin\theta)^2\Delta}{|q|^2}(\check{\eta}_b - \check{\underline{\eta}}_b)e_b$$

which yields

$$\nabla_{\not{\mathcal{L}}_{\mathbf{Z}} e_4} = r^{-1}\Gamma_b \not{\mathfrak{d}}.$$

Similarly, we have

$$\nabla_{\not{\mathcal{L}}_{\mathbf{T}} e_3} = \Gamma_b \not{\mathfrak{d}}$$

and hence

$$\begin{aligned}
\nabla_b(\not{\mathcal{L}}_{\mathbf{Z}} U_A) - \not{\mathcal{L}}_{\mathbf{Z}}(\nabla_b U_A) &= \mathfrak{d}\Gamma_b U,\\
\nabla_4(\not{\mathcal{L}}_{\mathbf{Z}} U_A) - \not{\mathcal{L}}_{\mathbf{Z}}(\nabla_4 U_A) &= \mathfrak{d}(\Gamma_b U),\\
\nabla_3(\not{\mathcal{L}}_{\mathbf{Z}} U_A) - \not{\mathcal{L}}_{\mathbf{Z}}(\nabla_3 U_A) &= r\mathfrak{d}(\Gamma_b U).
\end{aligned}$$

Since $\mathbf{Z}(r) \in r^2\Gamma_g$, we deduce

$$[\not{\mathcal{L}}_{\mathbf{Z}}, \mathfrak{d}]U = r\mathfrak{d}(\Gamma_b U)$$

as stated. This concludes the proof of Lemma 9.2.1. □

Corollary 9.2.2. *We have the following commutator identities*

$$\begin{aligned}
[\nabla_{\mathbf{T}}, \nabla_{\mathbf{Z}}]\psi &= \mathfrak{d}(\Gamma_b \cdot \psi),\\
[\nabla_{\mathbf{T}}, \not{\mathfrak{d}}]\psi &= \frac{2amr\cos\theta}{|q|^4} r\, {}^*\nabla_b\psi + O(ar^{-3})\psi + \mathfrak{d}(\Gamma_b \cdot \psi),\\
[\nabla_{\mathbf{Z}}, \not{\mathfrak{d}}]\psi &= -\frac{2\cos\theta((r^2+a^2)^2 - a^2(\sin\theta)^2\Delta)}{|q|^4} r\, {}^*\nabla_b\psi + O(1)\psi + r\mathfrak{d}(\Gamma_b \cdot \psi).
\end{aligned}$$

Proof. In view of Lemma 9.2.1, we have

$$\begin{aligned}
[\nabla_{\mathbf{T}}, \nabla_{\mathbf{Z}}]\psi &= [\not{\mathcal{L}}_{\mathbf{T}}, \mathfrak{d}]\psi + \mathbf{Z}\left(\frac{4amr\cos\theta}{|q|^4}\right) {}^*\psi + \mathfrak{d}(\Gamma_b \cdot \psi)\\
&= O(r^{-3})(r^{-1}\mathbf{Z}(r), \mathbf{Z}(\cos\theta))\psi + \mathfrak{d}(\Gamma_b \cdot \psi)
\end{aligned}$$

and hence

$$[\nabla_{\mathbf{T}}, \nabla_{\mathbf{Z}}]\psi = \mathfrak{d}(\Gamma_b \cdot \psi).$$

Using again Lemma 9.2.1, we have

$$\begin{aligned}
[\nabla_{\mathbf{T}}, \not{\mathfrak{d}}]\psi &= [\not{\mathcal{L}}_{\mathbf{T}}, \mathfrak{d}]\psi + \frac{2amr\cos\theta}{|q|^4} r\, {}^*\nabla\psi + r\nabla\left(\frac{4amr\cos\theta}{|q|^4}\right) {}^*\psi + \mathfrak{d}(\Gamma_b \cdot \psi)\\
&= \frac{2amr\cos\theta}{|q|^4} r\, {}^*\nabla\psi + r\nabla\left(\frac{4amr\cos\theta}{|q|^4}\right) {}^*\psi + \mathfrak{d}(\Gamma_b \cdot \psi)
\end{aligned}$$

and hence, since $\nabla(r)\in r\Gamma_g$ and $\widetilde{\nabla(\cos\theta)}\in\Gamma_b$, we infer

$$[\nabla_{\mathbf{T}}, \not{\mathfrak{d}}]\psi = \frac{2amr\cos\theta}{|q|^4} r\,{}^*\nabla\psi + O(ar^{-3})\psi + \mathfrak{d}(\Gamma_b\cdot\psi).$$

Finally, using again Lemma 9.2.1, we have

$$\begin{aligned}
[\nabla_{\mathbf{Z}}, \not{\mathfrak{d}}]\psi &= [\not{\mathcal{L}}_{\mathbf{Z}}, \mathfrak{d}]\psi - \frac{2\cos\theta((r^2+a^2)^2 - a^2(\sin\theta)^2\Delta)}{|q|^4} r\,{}^*\nabla\psi \\
&\quad + r\nabla\left(-\frac{2\cos\theta((r^2+a^2)^2 - a^2(\sin\theta)^2\Delta)}{|q|^4}\right){}^*\psi + r\mathfrak{d}(\Gamma_b\cdot\psi) \\
&= -\frac{2\cos\theta((r^2+a^2)^2 - a^2(\sin\theta)^2\Delta)}{|q|^4} r\,{}^*\nabla\psi \\
&\quad - r\nabla\left(\frac{2\cos\theta((r^2+a^2)^2 - a^2(\sin\theta)^2\Delta)}{|q|^4}\right){}^*\psi + r\mathfrak{d}(\Gamma_b\cdot\psi)
\end{aligned}$$

and hence, since $\nabla(r)\in r\Gamma_g$ and $\widetilde{\nabla(\cos\theta)}\in\Gamma_b$, we infer

$$[\nabla_{\mathbf{Z}}, \not{\mathfrak{d}}]\psi = -\frac{2\cos\theta((r^2+a^2)^2 - a^2(\sin\theta)^2\Delta)}{|q|^4} r\,{}^*\nabla\psi + O(1)\psi + r\mathfrak{d}(\Gamma_b\cdot\psi).$$

This concludes the proof of Corollary 9.2.2. □

9.2.2 Approximate symmetry operators

Recall the set of second order differential operators $\mathcal{S}_{\underline{a}}$, see Definition 4.6.1,

$$\begin{aligned}
\mathcal{S}_1\psi &= \nabla_T\nabla_T\psi, \\
\mathcal{S}_2\psi &= a\nabla_T\nabla_Z\psi, \\
\mathcal{S}_3\psi &= a^2\nabla_Z\nabla_Z\psi, \\
\mathcal{S}_4\psi &= \mathcal{O}(\psi),
\end{aligned}$$

where $\mathcal{O}$ is given by (4.5.2), i.e.

$$\mathcal{O}(\psi) = |q|^2\left(\triangle_k\psi + \frac{2a^2\cos\theta}{|q|^2}\,{}^*\Re(\mathfrak{J})^b\nabla_b\psi\right).$$

These symmetry operators satisfy the following commutation properties.

Lemma 9.2.3. *We have*

$$[\mathcal{S}_1, \mathcal{S}_2],\ [\mathcal{S}_1, \mathcal{S}_3],\ [\mathcal{S}_2, \mathcal{S}_3] = \mathfrak{d}^2(\Gamma_b \cdot \psi).$$

Also, we have

$$\begin{aligned}
[\nabla_{\mathbf{T}}, \mathcal{O}] &= O(ar^{-3})\, \not{\mathfrak{d}}^{\leq 1}\psi + \mathfrak{d}^{\leq 2}(\Gamma_b\psi),\\
[\nabla_{\mathbf{Z}}, \mathcal{O}] &= O(1)\, \not{\mathfrak{d}}^{\leq 1}\psi + r\mathfrak{d}^{\leq 2}(\Gamma_b\psi),
\end{aligned}$$

and

$$\begin{aligned}
[\mathcal{S}_1, \mathcal{O}] &= O(ar^{-3})\mathfrak{d}^{\leq 2}\psi + \mathfrak{d}^{\leq 3}(\Gamma_b\psi),\\
[\mathcal{S}_2, \mathcal{O}] &= O(a)\, \not{\mathfrak{d}}^{\leq 2}\psi + r\mathfrak{d}^{\leq 3}(\Gamma_b\psi),\\
[\mathcal{S}_3, \mathcal{O}] &= O(a^2)\, \not{\mathfrak{d}}^{\leq 2}\psi + r\mathfrak{d}^{\leq 3}(\Gamma_b\psi).
\end{aligned}$$

Proof. First, since $[\nabla_{\mathbf{T}}, \nabla_{\mathbf{Z}}]\psi = \mathfrak{d}(\Gamma_b \cdot \psi)$ in view of Corollary 9.2.2, we immediately have in view of the definition of $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$,

$$[\mathcal{S}_1, \mathcal{S}_2],\ [\mathcal{S}_1, \mathcal{S}_3],\ [\mathcal{S}_2, \mathcal{S}_3] = \mathfrak{d}^2(\Gamma_b \cdot \psi)$$

as stated.

Next, using Lemma 9.2.1 and the definition of $\mathcal{O}$, we have

$$\begin{aligned}
[\nabla_{\mathbf{T}}, \mathcal{O}] &= [\not{\mathcal{L}}_{\mathbf{T}}, \mathcal{O}] + \left[\frac{4amr\cos\theta}{|q|^4}, \mathcal{O}\right]\ {}^*\psi + \mathfrak{d}^{\leq 2}(\Gamma_b\psi)\\
&= \left[\frac{4amr\cos\theta}{|q|^4}, \mathcal{O}\right]\ {}^*\psi + \mathfrak{d}^{\leq 2}(\Gamma_b\psi)\\
&= O(ar^{-3})\, \not{\mathfrak{d}}^{\leq 1}\psi + \mathfrak{d}^{\leq 2}(\Gamma_b\psi)
\end{aligned}$$

and

$$\begin{aligned}
[\nabla_{\mathbf{Z}}, \mathcal{O}] &= [\not{\mathcal{L}}_{\mathbf{Z}}, \mathcal{O}] - \left[\frac{\cos\theta((r^2+a^2)^2 - a^2(\sin\theta)^2\Delta)}{|q|^4}, \mathcal{O}\right]\ {}^*\psi + r\mathfrak{d}^{\leq 2}(\Gamma_b\psi)\\
&= -\left[\frac{\cos\theta((r^2+a^2)^2 - a^2(\sin\theta)^2\Delta)}{|q|^4}, \mathcal{O}\right]\ {}^*\psi + r\mathfrak{d}^{\leq 2}(\Gamma_b\psi)\\
&= O(1)\, \not{\mathfrak{d}}^{\leq 1}\psi + r\mathfrak{d}^{\leq 2}(\Gamma_b\psi)
\end{aligned}$$

as stated.

Finally, the identities for $[\mathcal{S}_1, \mathcal{O}]$, $[\mathcal{S}_2, \mathcal{O}]$ and $[\mathcal{S}_3, \mathcal{O}]$ following immediately from the definition of $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$, and the above commutator identities for $[\nabla_{\mathbf{T}}, \mathcal{O}]$ and $[\nabla_{\mathbf{Z}}, \mathcal{O}]$. This concludes the proof of the lemma. □

We have the following analog of Lemma 8.1.1 in perturbations of Kerr.

Lemma 9.2.4. *Given a $\mathfrak{s}_2$ tensor ψ solution of the equation* (6.1.1)*. Then the commuted $\mathfrak{s}_2$ tensor $\psi_{\underline{a}} := \mathcal{S}_{\underline{a}}\psi$ satisfies*

$$\dot{\square}_2\psi_{\underline{a}} - V\psi_{\underline{a}} = -\frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_{\mathbf{T}}\psi_{\underline{a}} + N_{\underline{a}} \tag{9.2.1}$$

where $N_{\underline{a}}$ satisfies

$$|N_{\underline{a}}| \lesssim |\mathfrak{d}^{\leq 2}N| + |a|r^{-2}|\mathfrak{d}^{\leq 1}\nabla_3\psi| + |a|r^{-3}|\mathfrak{d}^{\leq 2}\psi| + |\mathfrak{d}^{\leq 3}(\Gamma_g\cdot\psi)|. \tag{9.2.2}$$

Proof. Since ψ satisfies (6.1.1), i.e.

$$\dot{\square}_2\psi - V\psi = -\frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_T\psi + N,$$

we infer

$$\begin{aligned}
\dot{\square}_2\psi_{\underline{a}} - V\psi_{\underline{a}} &= -\frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_T\psi_{\underline{a}} + N_{\underline{a}},\\
N_{\underline{a}} &:= -[\mathcal{S}_{\underline{a}},\dot{\square}_2]\psi + [\mathcal{S}_{\underline{a}},V]\psi - {}^*\left[\mathcal{S}_{\underline{a}},\frac{4a\cos\theta}{|q|^2}\nabla_T\right]\psi + \mathfrak{d}^{\leq 2}N, \qquad \underline{a}=1,2,3,\\
N_4 &:= -\frac{1}{|q|^2}[\mathcal{S}_4,|q|^2\dot{\square}_2]\psi + \frac{1}{|q|^2}[\mathcal{S}_4,|q|^2V]\psi - \frac{1}{|q|^2}\,{}^*[\mathcal{S}_4,4a\cos\theta\nabla_T]\psi + \mathfrak{d}^{\leq 2}N,
\end{aligned}$$

and hence

$$\begin{aligned}
|N_{\underline{a}}| &\lesssim |\mathfrak{d}^{\leq 2}N| + |[\mathcal{S}_{\underline{a}},\dot{\square}_2]\psi| + |[\mathcal{S}_{\underline{a}},V]\psi| + a\left|\left[\mathcal{S}_{\underline{a}},\frac{\cos\theta}{|q|^2}\nabla_T\right]\psi\right|, \qquad \underline{a}=1,2,3,\\
|N_4| &\lesssim |\mathfrak{d}^{\leq 2}N| + r^{-2}|[\mathcal{S}_4,|q|^2\dot{\square}_2]\psi| + r^{-2}|[\mathcal{S}_4,|q|^2V]\psi| + ar^{-2}|[\mathcal{S}_4,\cos\theta\nabla_T]\psi|.
\end{aligned}$$

Next, since

$$V = \frac{4\Delta}{(r^2+a^2)|q|^2}, \qquad |q|^2V = \frac{4\Delta}{(r^2+a^2)},$$

see (6.1.1), and since $\mathbf{T}(r)\in r\Gamma_b$, $\mathbf{T}(\cos\theta)\in\Gamma_b$, $\mathbf{Z}(r)\in r^2\Gamma_g$, $\mathbf{Z}(\cos\theta)\in r\Gamma_b$, and $\nabla(r)\in r\Gamma_g$, we infer

$$\begin{aligned}
|[\mathcal{S}_{\underline{a}},V]\psi| &\lesssim r^{-3}|\mathfrak{d}^{\leq 1}((\mathbf{T},\mathbf{Z})(r,\cos\theta))||\mathfrak{d}^{\leq 1}\psi| \lesssim r^{-1}|\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\psi)|, \qquad \underline{a}=1,2,3,\\
r^{-2}|[\mathcal{S}_4,|q|^2V]\psi| &\lesssim r^{-2}|\mathfrak{d}^{\leq 1}(\nabla(r))||\mathfrak{d}^{\leq 1}\psi| \lesssim r^{-1}|\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\psi)|,
\end{aligned}$$

and hence

$$\begin{aligned}|N_{\underline{a}}| &\lesssim |\mathfrak{d}^{\leq 2}N| + r^{-1}|\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\psi)| + |[\mathcal{S}_{\underline{a}}, \dot{\square}_2]\psi| + a\left|\left[\mathcal{S}_{\underline{a}}, \frac{\cos\theta}{|q|^2}\nabla_T\right]\psi\right|, \qquad \underline{a}=1,2,3,\\ |N_4| &\lesssim |\mathfrak{d}^{\leq 2}N| + r^{-1}|\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\psi)| + r^{-2}|[\mathcal{S}_4, |q|^2\dot{\square}_2]\psi| + ar^{-2}|[\mathcal{S}_4, \cos\theta\nabla_T]\psi|.\end{aligned}$$

Also, we have

$$\begin{aligned}\left|\left[\mathcal{S}_{\underline{a}}, \frac{\cos\theta}{|q|^2}\nabla_T\right]\psi\right| &\lesssim r^{-2}|\mathfrak{d}^{\leq 1}((\mathbf{T},\mathbf{Z})(r,\cos\theta))||\mathfrak{d}^{\leq 2}\psi| + r^{-2}|[\mathcal{S}_{\underline{a}}, \nabla_T]\psi|\\ &\lesssim |\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\psi)| + r^{-2}|[\mathcal{S}_{\underline{a}}, \nabla_T]\psi| \qquad \underline{a}=1,2,3,\end{aligned}$$

and

$$\begin{aligned}r^{-2}|[\mathcal{S}_4, \cos\theta\nabla_T]\psi| &\lesssim r^{-1}|\mathfrak{d}^{\leq 1}\nabla(\cos\theta)||\mathfrak{d}^{\leq 1}\nabla_T\psi| + r^{-2}|[\mathcal{S}_4, \nabla_T]\psi|\\ &\lesssim r^{-1}(r^{-1} + |\mathfrak{d}^{\leq 1}\Gamma_b|)|\mathfrak{d}^{\leq 1}\nabla_T\psi| + r^{-2}|[\mathcal{S}_4, \nabla_T]\psi|\\ &\lesssim r^{-2}|\mathfrak{d}^{\leq 1}\nabla_3\psi| + r^{-3}|\mathfrak{d}^{\leq 2}\psi| + |\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\psi)| + r^{-2}|[\mathcal{S}_4, \nabla_T]\psi|\end{aligned}$$

and hence

$$\begin{aligned}|N_{\underline{a}}| &\lesssim |\mathfrak{d}^{\leq 2}N| + |\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\psi)| + |[\mathcal{S}_{\underline{a}}, \dot{\square}_2]\psi| + ar^{-2}\left|[\mathcal{S}_{\underline{a}}, \nabla_T]\psi\right|, \qquad \underline{a}=1,2,3,\\ |N_4| &\lesssim |\mathfrak{d}^{\leq 2}N| + ar^{-2}|\mathfrak{d}^{\leq 1}\nabla_3\psi| + ar^{-3}|\mathfrak{d}^{\leq 2}\psi| + |\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\psi)|\\ &\quad + r^{-2}|[\mathcal{S}_4, |q|^2\dot{\square}_2]\psi| + ar^{-2}|[\mathcal{S}_4, \nabla_T]\psi|.\end{aligned}$$

Next, we use Lemma 9.2.1 which implies

$$\begin{aligned}|[\mathcal{S}_{\underline{a}}, \nabla_T]\psi| &\lesssim |[\mathcal{S}_{\underline{a}}, \not{\mathcal{L}}_T]\psi| + ar^{-2}|\mathfrak{d}^{\leq 2}\psi| + |\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\psi)|\\ &\lesssim ar^{-2}|\mathfrak{d}^{\leq 2}\psi| + |\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\psi)|, \underline{a}=1,2,3,4,\end{aligned}$$

and hence

$$\begin{aligned}|N_{\underline{a}}| &\lesssim |\mathfrak{d}^{\leq 2}N| + a^2r^{-4}|\mathfrak{d}^{\leq 2}\psi| + |\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\psi)| + |[\mathcal{S}_{\underline{a}}, \dot{\square}_2]\psi|, \qquad \underline{a}=1,2,3,\\ |N_4| &\lesssim |\mathfrak{d}^{\leq 2}N| + ar^{-2}|\mathfrak{d}^{\leq 1}\nabla_3\psi| + ar^{-3}|\mathfrak{d}^{\leq 2}\psi| + |\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\psi)| + r^{-2}|[\mathcal{S}_4, |q|^2\dot{\square}_2]\psi|.\end{aligned}$$

Finally, recall from Proposition 4.6.3 that the following commutation formulas hold true for $\psi\in\mathfrak{s}_2$:

$$\begin{aligned}[\mathcal{S}_1, \dot{\square}_2]\psi &= O(ar^{-2})\mathfrak{d}^{\leq 2}\psi + \mathfrak{d}^2\big(\Gamma_g\cdot\mathfrak{d}\psi\big) + \mathfrak{d}\big(\Gamma_b\cdot\dot{\square}_2\psi\big),\\ [\mathcal{S}_2, \dot{\square}_2]\psi &= O(ar^{-2})\mathfrak{d}^{\leq 2}\psi + \mathfrak{d}^2\big(\Gamma_g\cdot\mathfrak{d}\psi\big) + r\mathfrak{d}\big(\Gamma_b\cdot\dot{\square}_2\psi\big),\\ [\mathcal{S}_3, \dot{\square}_2]\psi &= O(ar^{-2})\mathfrak{d}^{\leq 2}\psi + \mathfrak{d}^2\big(\Gamma_g\cdot\mathfrak{d}\psi\big) + r\mathfrak{d}\big(\Gamma_b\cdot\dot{\square}_2\psi\big),\end{aligned}$$

and

$$[\mathcal{S}_4, |q|^2\dot{\square}_2]\psi = |q|^2\Big[O(ar^{-2})\,\mathfrak{d}^{\leq 2}\psi + \mathfrak{d}^2\big(\Gamma_g\cdot\mathfrak{d}\psi\big) + \dot{\mathbf{D}}_3\mathfrak{d}\big(|q|^2\xi\cdot\dot{\mathbf{D}}_a\psi\big)\Big].$$

We deduce

$$\begin{aligned}|N_{\underline{a}}| &\lesssim |\mathfrak{d}^{\leq 2}N| + |a|r^{-2}|\mathfrak{d}^{\leq 1}\nabla_3\psi| + |a|r^{-3}|\mathfrak{d}^{\leq 2}\psi| + |\mathfrak{d}^{\leq 3}(\Gamma_g\cdot\psi)| + r|\mathfrak{d}\big(\Gamma_b\cdot\dot{\square}_2\psi\big)|\\ &\quad + r|\mathfrak{d}^{\leq 3}(\xi\cdot\psi)|.\end{aligned}$$

Next, recall from Remark 9.1.1 that we assume $\xi\in r^{-1}\Gamma_g$ in this chapter which implies

$$|N_{\underline{a}}| \lesssim |\mathfrak{d}^{\leq 2}N| + |a|r^{-2}|\mathfrak{d}^{\leq 1}\nabla_3\psi| + |a|r^{-3}|\mathfrak{d}^{\leq 2}\psi| + |\mathfrak{d}^{\leq 3}(\Gamma_g\cdot\psi)| + r|\mathfrak{d}\big(\Gamma_b\cdot\dot{\square}_2\psi\big)|.$$

Plugging the model RW equation (6.1.1) for ψ in the last term, we have

$$\begin{aligned}r|\mathfrak{d}\big(\Gamma_b\cdot\dot{\square}_2\psi\big)| &\lesssim r\left|\mathfrak{d}\left(\Gamma_b\cdot\left(-V\psi - \frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_T\psi + N\right)\right)\right|\\ &\lesssim r^{-1}|\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\psi)| + |\mathfrak{d}^{\leq 1}N|\end{aligned}$$

and hence

$$|N_{\underline{a}}| \lesssim |\mathfrak{d}^{\leq 2}N| + |a|r^{-2}|\mathfrak{d}^{\leq 1}\nabla_3\psi| + |a|r^{-3}|\mathfrak{d}^{\leq 2}\psi| + |\mathfrak{d}^{\leq 3}(\Gamma_g\cdot\psi)|$$

as stated. This concludes the proof of Lemma 9.2.4. □

9.2.3 Commutation with $\nabla_{\widehat{R}}$

Recall that $\widehat{R}$ is given by

$$\widehat{R} = \frac{1}{2}\left(\frac{|q|^2}{r^2+a^2}e_4 - \frac{\Delta}{r^2+a^2}e_3\right) = \frac{1}{2}\frac{|q|^2}{r^2+a^2}X, \qquad X = e_4 - \frac{\Delta}{|q|^2}e_3.$$

We use the following commutation lemma for vectorfields X spanned by e_3 and e_4.

Lemma 9.2.5. *Let X such that $X = X^4e_4 + X^3e_3$. Then, we have*

$$\begin{aligned}[\nabla_b,\nabla_X]\psi &= \frac{1}{2}(X^4 tr\,\chi + X^3 tr\,\underline{\chi})\nabla_b\psi + \frac{1}{2}(X^{4\,(a)}tr\chi + X^{3\,(a)}tr\underline{\chi})\,{}^*\nabla_b\psi\\ &\quad + (e_b(X^4) - X^4(\underline{\eta}_b + \eta_b))\nabla_4\psi + e_b(X^3)\nabla_3\psi\\ &\quad + O(ar^{-3})(X^4, X^3)\psi + (|X^3| + |X^4|)\big(\Gamma_b\nabla_3\psi + r^{-1}\Gamma_b\mathfrak{d}^{\leq 1}\psi\big).\end{aligned}$$

Proof. In view of Corollary A.1.1 and our definition of Γ_b, Γ_g, we have

$$[\nabla_3, \nabla_b]\psi = -\frac{1}{2}\big(\mathrm{tr}\,\underline{\chi}\nabla_b\psi + {}^{(a)}\mathrm{tr}\underline{\chi}\;{}^*\nabla_b\psi\big) + O(ar^{-3})\psi + \Gamma_b\nabla_3\psi + r^{-1}\Gamma_b\mathfrak{d}^{\leq 1}\psi$$

and

$$\begin{aligned}[\nabla_4, \nabla_b]\psi &= -\frac{1}{2}\big(\mathrm{tr}\,\chi\nabla_b\psi + {}^{(a)}\mathrm{tr}\chi\;{}^*\nabla_b\psi\big) + (\underline{\eta}_b + \zeta_b)\nabla_4\psi\\ &\quad + O(ar^{-3})\psi + \Gamma_g\nabla_3\psi + r^{-1}\Gamma_g\mathfrak{d}^{\leq 1}\psi.\end{aligned}$$

We infer

$$\begin{aligned}(\nabla_b\nabla_X - \nabla_X\nabla_b)\psi &= e_b(X^4)\nabla_4\psi + e_b(X^3)\nabla_3\psi + X^4[\nabla_b, \nabla_4]\psi + X^3[\nabla_b, \nabla_3]\psi\\ &= X^4\left(\frac{1}{2}\mathrm{tr}\,\chi\nabla_b + \frac{1}{2}{}^{(a)}\mathrm{tr}\chi\;{}^*\nabla_b - (\underline{\eta}_b + \eta_b)\nabla_4 + O(ar^{-3})\right)\psi\\ &\quad + X^3\left(\frac{1}{2}\mathrm{tr}\,\underline{\chi}\nabla_b + \frac{1}{2}{}^{(a)}\mathrm{tr}\underline{\chi}\;{}^*\nabla_b + O(ar^{-3})\right)\psi\\ &\quad + e_b(X^4)\nabla_4\psi + e_b(X^3)\nabla_3\psi + (|X^3| + |X^4|)\big(\Gamma_b\nabla_3\psi + r^{-1}\Gamma_b\mathfrak{d}^{\leq 1}\psi\big)\\ &= \frac{1}{2}(X^4\mathrm{tr}\,\chi + X^3\mathrm{tr}\,\underline{\chi})\nabla_b\psi + \frac{1}{2}(X^{4\,(a)}\mathrm{tr}\chi + X^{3\,(a)}\mathrm{tr}\underline{\chi})\;{}^*\nabla_b\psi\\ &\quad + (e_b(X^4) - X^4(\underline{\eta}_b + \eta_b))\nabla_4\psi + e_b(X^3)\nabla_3\psi\\ &\quad + O(ar^{-3})(X^4, X^3)\psi + (|X^3| + |X^4|)\big(\Gamma_b\nabla_3\psi + r^{-1}\Gamma_b\mathfrak{d}^{\leq 1}\psi\big)\end{aligned}$$

as stated. □

Corollary 9.2.6. *We have*

$$[|q|\nabla_b, \nabla_{\widehat{R}}]\psi = O(ar^{-2})\psi + r\Gamma_b\mathfrak{d}^{\leq 1}\psi.$$

In particular, we have

$$[\mathcal{O}, \nabla_{\widehat{R}}]\psi = O(ar^{-2})\mathfrak{d}^{\leq 1}\psi + r\Gamma_b\mathfrak{d}^{\leq 2}\psi.$$

Proof. We apply Lemma 9.2.5 with $X^4 = 1$ and $X^3 = -\frac{\Delta}{|q|^2}$. This yields

$$\begin{aligned}(\nabla_b\nabla_X - \nabla_X\nabla_b)\psi &= \frac{1}{2}\left(\mathrm{tr}\,\chi - \frac{\Delta}{|q|^2}\mathrm{tr}\,\underline{\chi}\right)\nabla_b\psi + \frac{1}{2}\left({}^{(a)}\mathrm{tr}\chi - \frac{\Delta}{|q|^2}{}^{(a)}\mathrm{tr}\underline{\chi}\right){}^*\nabla_b\psi\\ &\quad - (\underline{\eta}_b + \eta_b)\nabla_4\psi - e_b\left(\frac{\Delta}{|q|^2}\right)\nabla_3\psi + O(ar^{-3})\psi\\ &\quad + \Gamma_b\nabla_3\psi + r^{-1}\Gamma_b\mathfrak{d}^{\leq 1}\psi.\end{aligned}$$

Since

$$\mathrm{tr}X - \frac{\Delta}{|q|^2}\mathrm{tr}\underline{X} = \frac{\Delta}{|q|^2}\left(\frac{2}{q}+\frac{2}{\overline{q}}\right) + \Gamma_g = \frac{4r\Delta}{|q|^4} + \Gamma_g,$$

$$\begin{aligned} e_b\left(\frac{\Delta}{|q|^2}\right) &= \partial_r\left(\frac{\Delta}{|q|^2}\right)e_b(r) + \partial_{\cos\theta}\left(\frac{\Delta}{|q|^2}\right)e_b(\cos\theta) \\ &= O(r^{-2})e_b(r) - \frac{2a^2\cos\theta\Delta}{|q|^4}e_b(\cos\theta) = -\frac{2a^2\cos\theta\Delta}{|q|^4}\Re(i\mathfrak{J})_b + r^{-1}\Gamma_g, \end{aligned}$$

and

$$H + \underline{H} = \frac{a(q-\overline{q})}{|q|^2}\mathfrak{J} + \Gamma_b = \frac{2a^2\cos\theta}{|q|^2}i\mathfrak{J} + \Gamma_b,$$

we infer, using also ${}^*\mathfrak{J} = -i\mathfrak{J}$,

$$(\nabla_b\nabla_X - \nabla_X\nabla_b)\psi = \frac{2r\Delta}{|q|^4}\nabla_b\psi + \frac{2a^2\cos\theta}{|q|^2}\,{}^*\Re(\mathfrak{J})_b\nabla_X + O(ar^{-3})\psi + \Gamma_b\nabla_3\psi + r^{-1}\Gamma_b\mathfrak{d}^{\leq 1}\psi.$$

Also, note that

$$\begin{aligned} X(|q|) &= \frac{1}{2}\frac{X(|q|^2)}{|q|} = \frac{rX(r)}{|q|} + O(r^{-1})X(\cos\theta) = \frac{r}{|q|}\left(e_4(r) - \frac{\Delta}{|q|^2}e_3(r)\right) + r^{-1}\Gamma_b \\ &= \frac{2r\Delta}{|q|^3} + r\Gamma_b \end{aligned}$$

and hence

$$\begin{aligned} [|q|\nabla_b, \nabla_X]\psi &= |q|[\nabla_b, \nabla_X]\psi - \nabla_X(|q|)\nabla_b \\ &= \frac{2a^2\cos\theta}{|q|}\,{}^*\Re(\mathfrak{J})_b\nabla_X + O(ar^{-2})\psi + r\Gamma_b\nabla_3\psi + \Gamma_b\mathfrak{d}^{\leq 1}\psi. \end{aligned}$$

Now, notice that

$$\widehat{R} = \frac{|q|^2}{r^2+a^2}X$$

and

$$\begin{aligned} \nabla_b\left(\frac{|q|^2}{r^2+a^2}\right) &= \partial_r\left(\frac{|q|^2}{r^2+a^2}\right)e_b(r) + \partial_{\cos\theta}\left(\frac{|q|^2}{r^2+a^2}\right)e_b(\cos\theta) \\ &= -\frac{2a^2\cos\theta}{r^2+a^2}\,{}^*\Re(\mathfrak{J})_b + r^{-2}\Gamma_b. \end{aligned}$$

We infer

$$
\begin{aligned}
[|q|\nabla_b, \nabla_{\widehat{R}}]\psi &= \left[|q|\nabla_b, \frac{|q|^2}{r^2+a^2}X\right]\psi \\
&= \frac{|q|^2}{r^2+a^2}[|q|\nabla_b, \nabla_X]\psi + |q|\nabla_b\left(\frac{|q|^2}{r^2+a^2}\right)\nabla_X\psi \\
&= O(ar^{-2})\psi + r\Gamma_b\nabla_3\psi + \Gamma_b\mathfrak{d}^{\leq 1}\psi
\end{aligned}
$$

as stated. □

Next, we compute the commutators $[\nabla_{\widehat{R}}, \nabla_{\mathbf{T}}]$ and $[\nabla_{\widehat{R}}, \nabla_{\mathbf{Z}}]$.

Lemma 9.2.7. *We have*

$$
\begin{aligned}
[\nabla_{\mathbf{T}}, \nabla_{\widehat{R}}]\psi &= O(amr^{-4})\psi + \mathfrak{d}(\Gamma_b\cdot\psi), \\
[\nabla_{\mathbf{Z}}, \nabla_{\widehat{R}}]\psi &= O(a^2r^{-3})\psi + r\mathfrak{d}(\Gamma_b\cdot\psi),
\end{aligned}
$$

and

$$
[\nabla_{\widehat{T}}, \nabla_{\widehat{R}}]\psi = O(ar^{-3})\nabla_{\mathbf{Z}}\psi + O(amr^{-4})\psi + \mathfrak{d}(\Gamma_b\cdot\psi).
$$

In particular, we have

$$
\begin{aligned}
[\nabla_{\widehat{R}}, \mathcal{S}_1] &= O(amr^{-4})\mathfrak{d}^{\leq 1}\psi + \mathfrak{d}^{\leq 2}(\Gamma_b\cdot\psi), \\
[\nabla_{\widehat{R}}, \mathcal{S}_2] &= O(a^2r^{-3})\mathfrak{d}^{\leq 1}\psi + r\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\psi), \\
[\nabla_{\widehat{R}}, \mathcal{S}_3] &= O(a^4r^{-3})\mathfrak{d}^{\leq 1}\psi + r\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\psi).
\end{aligned}
$$

Proof. In view of Lemma 9.2.1, we have

$$
\begin{aligned}
[\nabla_{\mathbf{T}}, \nabla_{\widehat{R}}]\psi &= [\not{\mathcal{L}}_{\mathbf{T}}, \mathfrak{d}]\psi + \widehat{R}\left(\frac{4amr\cos\theta}{|q|^4}\right)\,{}^*\psi + \mathfrak{d}(\Gamma_b\cdot\psi) \\
&= \widehat{R}\left(\frac{4amr\cos\theta}{|q|^4}\right)\,{}^*\psi + \mathfrak{d}(\Gamma_b\cdot\psi) \\
&= O(amr^{-4})\psi + \mathfrak{d}(\Gamma_b\cdot\psi)
\end{aligned}
$$

and

$$
\begin{aligned}
[\nabla_{\mathbf{Z}}, \nabla_{\widehat{R}}]\psi &= [\not{\mathcal{L}}_{\mathbf{Z}}, \mathfrak{d}]\psi + \widehat{R}\left(-\frac{2\cos\theta((r^2+a^2)^2 - a^2(\sin\theta)^2\Delta)}{|q|^4}\right)\,{}^*\psi + r\mathfrak{d}(\Gamma_b\cdot\psi) \\
&= \widehat{R}\left(-\frac{2\cos\theta((r^2+a^2)^2 - a^2(\sin\theta)^2\Delta)}{|q|^4}\right)\,{}^*\psi + r\mathfrak{d}(\Gamma_b\cdot\psi) \\
&= O(1)\widehat{R}(\cos\theta)\psi + O(r^{-3})\widehat{R}(r)\psi + r\mathfrak{d}(\Gamma_b\cdot\psi) \\
&= O(a^2r^{-3})\psi + r\mathfrak{d}(\Gamma_b\cdot\psi)
\end{aligned}
$$

as stated.

Also, we have

$$
\begin{aligned}
[\nabla_{\widehat{T}}, \nabla_{\widehat{R}}]\psi &= \left[\nabla_{\mathbf{T}} + \frac{a}{r^2+a^2}\nabla_{\mathbf{Z}}, \nabla_{\widehat{R}}\right]\psi \\
&= [\nabla_{\mathbf{T}}, \nabla_{\widehat{R}}]\psi + O(ar^{-2})[\nabla_{\mathbf{Z}}, \nabla_{\widehat{R}}]\psi - \widehat{R}\left(\frac{a}{r^2+a^2}\right)\nabla_{\mathbf{Z}}\psi \\
&= O(ar^{-3})\nabla_{\mathbf{Z}}\psi + O(amr^{-4})\psi + \mathfrak{d}(\Gamma_b\cdot\psi)
\end{aligned}
$$

as stated. □

9.2.4 The modified $\widetilde{\mathcal{O}}$ operator

The commutation properties of the operator $\mathcal{O}$ with $\dot{\square}_2$, see Proposition 4.5.3, are not good enough to derive energy estimates for $\mathcal{O}\psi$ with $\psi\in\mathfrak{s}_2$. In the lemma below, we introduce a modified operator $\widetilde{\mathcal{O}}$ which enjoys better properties.

Lemma 9.2.8. *Let*

$$
\widetilde{\mathcal{O}}\psi := \mathcal{O}\psi + \frac{4a(r^2+a^2+|q|^2)\cos\theta}{|q|^2}\nabla_{\mathbf{T}}{}^*\psi + \frac{4a^2\cos\theta}{|q|^2}\nabla_{\mathbf{Z}}{}^*\psi. \tag{9.2.3}
$$

Then, we have

$$
\begin{aligned}
\frac{1}{|q|^2}\left[|q|^2\left(\dot{\square}_2 - V + \frac{4a\cos\theta}{|q|^2}{}^*\nabla_{\mathbf{T}}\right), \widetilde{\mathcal{O}}\right]\psi &= O(ar^{-2})\nabla_{\widehat{R}}\mathfrak{d}^{\leq 1}\psi + O(ar^{-2})\mathfrak{d}^{\leq 1}\psi \\
&\quad + \mathfrak{d}^{\leq 3}(\Gamma_g\cdot\psi) + \Gamma_b\cdot\dot{\square}_2\psi.
\end{aligned} \tag{9.2.4}
$$

Proof. Recall from Proposition 4.5.3 that we have

$$
\begin{aligned}
\frac{1}{|q|^2}[|q|^2\dot{\square}_2, \mathcal{O}]\psi &= -\nabla\left(\frac{8a(r^2+a^2)\cos\theta}{|q|^2}\right)\cdot\nabla\nabla_{\widehat{T}}{}^*\psi + O(ar^{-2})\nabla_{\widehat{R}}^{\leq 1}\mathfrak{d}^{\leq 1}\psi \\
&\quad + \mathfrak{d}^2\big(\Gamma_g\cdot\mathfrak{d}\psi\big) + \dot{\mathbf{D}}_3\mathfrak{d}\big(|q|^2\xi\cdot\dot{\mathbf{D}}_a\psi\big).
\end{aligned}
$$

We infer, using the fact that $|q|^2V$ only depends on r and satisfies $|q|^2V = 4 + O(r^{-1})$, as well as the computation of the commutator $[\nabla_{\mathbf{T}}, \mathcal{O}]\psi$ in Lemma 9.2.3,

$$
\begin{aligned}
& \frac{1}{|q|^2}\left[|q|^2\left(\dot{\square}_2 - V + \frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_{\mathbf{T}}\right), \mathcal{O}\right]\psi \\
= \; & \frac{1}{|q|^2}[|q|^2\dot{\square}_2, \mathcal{O}]\psi + \frac{4}{|q|^2}[a\cos\theta\,{}^*\nabla_{\mathbf{T}}, \mathcal{O}]\psi + O(r^{-1})\,\not{\mathfrak{d}}^{\leq 1}\nabla(r)\cdot\not{\mathfrak{d}}^{\leq 1}\psi \\
= \; & -\nabla\left(\frac{8a(r^2+a^2)\cos\theta}{|q|^2}\right)\cdot\nabla\nabla_{\widehat{T}}\,{}^*\psi - 8a\nabla(\cos\theta)\cdot\nabla\nabla_{\mathbf{T}}\,{}^*\psi \\
& + O(a^2r^{-2})\nabla_{\widehat{R}}\mathfrak{d}^{\leq 1}\psi + O(ar^{-2})\mathfrak{d}^{\leq 1}\psi \\
& + \mathfrak{d}^2\big(\Gamma_g\cdot\mathfrak{d}\psi\big) + \dot{\mathbf{D}}_3\mathfrak{d}\big(|q|^2\xi\cdot\dot{\mathbf{D}}_a\psi\big) + r^{-2}\mathfrak{d}^{\leq 2}(\Gamma_b\psi) + \mathfrak{d}^{\leq 1}\Gamma_g\mathfrak{d}^{\leq 1}\psi.
\end{aligned}
$$

Since $\xi \in r^{-1}\Gamma_g$, see Remark 9.1.1, we deduce

$$
\begin{aligned}
& \frac{1}{|q|^2}\left[|q|^2\left(\dot{\square}_2 - V + \frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_{\mathbf{T}}\right), \mathcal{O}\right]\psi \\
= \; & -\nabla\left(\frac{8a(r^2+a^2)\cos\theta}{|q|^2}\right)\cdot\nabla\nabla_{\widehat{T}}\,{}^*\psi - 8a\nabla(\cos\theta)\cdot\nabla\nabla_{\mathbf{T}}\,{}^*\psi \\
& + O(a^2r^{-2})\nabla_{\widehat{R}}\mathfrak{d}^{\leq 1}\psi + O(ar^{-2})\mathfrak{d}^{\leq 1}\psi + \mathfrak{d}^{\leq 3}(\Gamma_g\cdot\psi)
\end{aligned}
$$

and hence, since $\widehat{T} = \mathbf{T} + \frac{a}{r^2+a^2}\mathbf{Z}$,

$$
\begin{aligned}
& \frac{1}{|q|^2}\left[|q|^2\left(\dot{\square}_2 - V + \frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_{\mathbf{T}}\right), \mathcal{O}\right]\psi \\
= \; & -\nabla\left(\frac{8a(r^2+a^2+|q|^2)\cos\theta}{|q|^2}\right)\cdot\nabla\nabla_{\mathbf{T}}\,{}^*\psi - \nabla\left(\frac{8a^2\cos\theta}{|q|^2}\right)\cdot\nabla\nabla_{\mathbf{Z}}\,{}^*\psi \\
& + O(ar^{-2})\nabla_{\widehat{R}}\mathfrak{d}^{\leq 1}\psi + O(ar^{-2})\mathfrak{d}^{\leq 1}\psi + \mathfrak{d}^{\leq 3}(\Gamma_g\cdot\psi).
\end{aligned}
$$

Next, recall from Proposition 4.3.3 that we have

$$
[\nabla_{\mathbf{T}}, \dot{\square}_2]\psi \;=\; O(ar^{-4})\mathfrak{d}^{\leq 1}\psi + \mathfrak{d}\big(\Gamma_g\cdot\mathfrak{d}\psi\big) + \Gamma_b\cdot\dot{\square}_2\psi.
$$

Given a smooth function $f_1(r,\cos\theta)$ such that $f_1 = O(a)$, $\partial_r f_1 = O(ar^{-3})$ and $\partial_{\cos\theta} f_1 =$

$O(a)$, we deduce

$$\begin{aligned}
[f_1(r,\cos\theta)\nabla_{\mathbf{T}}, \dot{\square}_2]\psi &= f_1(r,\cos\theta)\,[\nabla_{\mathbf{T}}, \dot{\square}_2]\psi + [f_1(r,\cos\theta), \dot{\square}_2]\nabla_{\mathbf{T}}\psi\\
&= -2\mathbf{g}^{\alpha\beta}\nabla_\alpha(f_1)\nabla_\beta\nabla_{\mathbf{T}}\psi + O\Big(|\square_{\mathbf{g}}(f_1)| + ar^{-4}\Big)\mathfrak{d}^{\leq 1}\psi\\
&\quad +\mathfrak{d}\big(\Gamma_g\cdot\mathfrak{d}\psi\big) + \Gamma_b\cdot\dot{\square}_2\psi\\
&= -2\partial_r f_1(r,\cos\theta)\mathbf{g}^{\alpha\beta}\nabla_\alpha(r)\nabla_\beta\nabla_{\mathbf{T}}\psi\\
&\quad -2\partial_{\cos\theta}f_1(r,\cos\theta)\mathbf{g}^{\alpha\beta}\nabla_\alpha(\cos\theta)\nabla_\beta\nabla_{\mathbf{T}}\psi\\
&\quad +O(ar^{-2})\mathfrak{d}^{\leq 1}\psi + \mathfrak{d}\big(\Gamma_g\cdot\mathfrak{d}\psi\big) + \Gamma_b\cdot\dot{\square}_2\psi\\
&= -2\partial_{\cos\theta}f_1(r,\cos\theta)\nabla(\cos\theta)\cdot\nabla\nabla_{\mathbf{T}}\psi\\
&\quad +O(\partial_r f_1)\nabla_{\widehat{R}}\nabla_{\mathbf{T}}\psi + O(ar^{-2})\mathfrak{d}^{\leq 1}\psi + \mathfrak{d}\big(\Gamma_g\cdot\mathfrak{d}\psi\big) + \Gamma_b\cdot\dot{\square}_2\psi\\
&= -2\nabla(f_1(r,\cos\theta))\cdot\nabla\nabla_{\mathbf{T}}\psi\\
&\quad +O(ar^{-3})\nabla_{\widehat{R}}\mathfrak{d}^{\leq 1}\psi + O(ar^{-2})\mathfrak{d}^{\leq 1}\psi + \mathfrak{d}\big(\Gamma_g\cdot\mathfrak{d}\psi\big) + \Gamma_b\cdot\dot{\square}_2\psi.
\end{aligned}$$

Also, recall from Proposition 4.3.3 that we have

$$[\nabla_{\mathbf{Z}}, \dot{\square}_2]\psi = O(r^{-2})\mathfrak{d}^{\leq 1}\psi + \mathfrak{d}\big(\Gamma_g\cdot\mathfrak{d}\psi\big) + r\Gamma_b\cdot\dot{\square}_2\psi.$$

Given a smooth function $f_2(r,\cos\theta)$ such that $f_2 = O(ar^{-2})$, $\partial_r f_2 = O(ar^{-3})$ and $\partial_{\cos\theta}f_2 = O(ar^{-2})$, we deduce similarly

$$\begin{aligned}
[f_2(r,\cos\theta)\nabla_{\mathbf{Z}}, \dot{\square}_2]\psi &= f_2(r,\cos\theta)\,[\nabla_{\mathbf{Z}}, \dot{\square}_2]\psi + [f_2(r,\cos\theta), \dot{\square}_2]\nabla_{\mathbf{Z}}\psi\\
&= -2\nabla(f_2(r,\cos\theta))\cdot\nabla\nabla_{\mathbf{Z}}\psi\\
&\quad +O(ar^{-3})\nabla_{\widehat{R}}\mathfrak{d}^{\leq 1}\psi + O(ar^{-2})\mathfrak{d}^{\leq 1}\psi + \mathfrak{d}\big(\Gamma_g\cdot\mathfrak{d}\psi\big) + \Gamma_b\cdot\dot{\square}_2\psi.
\end{aligned}$$

Setting

$$f_1(r,\cos\theta) = \frac{4a(r^2+a^2+|q|^2)\cos\theta}{|q|^2}, \qquad f_2(r,\cos\theta) = \frac{4a^2\cos\theta}{|q|^2},$$

we infer

$$\begin{aligned}
&\left[\dot{\square}_2, \frac{4a(r^2+a^2+|q|^2)\cos\theta}{|q|^2}\nabla_{\mathbf{T}} + \frac{4a^2\cos\theta}{|q|^2}\nabla_{\mathbf{Z}}\right]\psi\\
&= \nabla\left(\frac{8a(r^2+a^2+|q|^2)\cos\theta}{|q|^2}\right)\cdot\nabla\nabla_{\mathbf{T}}\psi + \nabla\left(\frac{8a^2\cos\theta}{|q|^2}\right)\cdot\nabla\nabla_{\mathbf{Z}}\psi\\
&\quad +O(ar^{-3})\nabla_{\widehat{R}}\mathfrak{d}^{\leq 1}\psi + O(ar^{-2})\mathfrak{d}^{\leq 1}\psi + \mathfrak{d}\big(\Gamma_g\cdot\mathfrak{d}\psi\big) + \Gamma_b\cdot\dot{\square}_2\psi
\end{aligned}$$

and hence

$$
\begin{aligned}
& \frac{1}{|q|^2}\left[|q|^2\left(\dot{\square}_2 - V + \frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_{\mathbf{T}}\right), \frac{4a(r^2+a^2+|q|^2)\cos\theta}{|q|^2}\,{}^*\nabla_{\mathbf{T}} + \frac{4a^2\cos\theta}{|q|^2}\,{}^*\nabla_{\mathbf{Z}}\right]\psi \\
= \;& -\nabla\left(\frac{8a^3(\sin\theta)^2\cos\theta}{|q|^2}\right)\cdot\nabla\nabla_{\mathbf{T}}\,{}^*\psi - \nabla\left(\frac{8a^2\cos\theta}{|q|^2}\right)\cdot\nabla\nabla_{\mathbf{Z}}\,{}^*\psi \\
& + O(ar^{-3})\nabla_{\widehat{R}}\mathfrak{d}^{\leq 1}\psi + O(ar^{-2})\mathfrak{d}^{\leq 1}\psi + \mathfrak{d}\big(\Gamma_g\cdot\mathfrak{d}\psi\big) + \Gamma_b\cdot\dot{\square}_2\psi.
\end{aligned}
$$

Setting

$$
\widetilde{\mathcal{O}}\psi = \mathcal{O}\psi + \frac{4a(r^2+a^2+|q|^2)\cos\theta}{|q|^2}\nabla_{\mathbf{T}}\,{}^*\psi + \frac{4a^2\cos\theta}{|q|^2}\nabla_{\mathbf{Z}}\,{}^*\psi,
$$

we infer

$$
\begin{aligned}
\frac{1}{|q|^2}\left[|q|^2\left(\dot{\square}_2 - V + \frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_{\mathbf{T}}\right), \widetilde{\mathcal{O}}\right]\psi \;&=\; O(ar^{-2})\nabla_{\widehat{R}}\mathfrak{d}^{\leq 1}\psi + O(ar^{-2})\mathfrak{d}^{\leq 1}\psi \\
& \quad + \mathfrak{d}^{\leq 3}(\Gamma_g\cdot\psi) + \Gamma_b\cdot\dot{\square}_2\psi
\end{aligned}
$$

as stated. This concludes the proof of Lemma 9.2.3. □

9.2.5 Additional energy flux and bulk quantities

We recall below the relevant main norms introduced in section 6.3.1:

Pointwise notation

Definition 9.2.9. *We introduce the following pointwise notation for $\psi \in \mathfrak{s}_2$.*

1. *We denote*

$$
|\psi|^2_{\mathcal{S}} \;:=\; \sum_{\underline{a}=1}^{4} |\psi_{\underline{a}}|^2.
$$

2. *Given a vectorfield Y we denote*

$$
|\nabla_Y\psi|^2_{\mathcal{S}} \;:=\; \sum_{\underline{a}=1}^{4} |\nabla_Y\psi_{\underline{a}}|^2.
$$

Degenerate energy norm

Definition 9.2.10 (Degenerate energy norm). *We define the following degenerate energy for $\psi \in \mathfrak{s}_2$ along $\Sigma(\tau)$:*

$$E_{deg}[\psi](\tau) := \int_{\Sigma(\tau)} \left(|\nabla_4\psi|^2 + \frac{|\Delta|}{r^4}|\nabla_3\psi|^2 + |\nabla\psi|^2 + r^{-2}|\psi|^2\right).$$

Refined Morawetz norms

Definition 9.2.11 (Refined Morawetz norms). *We define the following Morawetz norms for $\psi \in \mathfrak{s}_2$.*

1. *The degenerate axially symmetric Morawetz norms in $\mathcal{M} = \mathcal{M}(\tau_1, \tau_2)$:*

$$\begin{aligned} \dot{Mor}^{ax}_{deg}[\psi](\tau_1,\tau_2) &:= \int_{\mathcal{M}(\tau_1,\tau_2)} \frac{m}{r^2}|\nabla_{\widehat{R}}\psi|^2 + \frac{\mathcal{T}^2}{r^6}\left(\frac{m}{r^2}|\nabla_{\widehat{T}}\psi|^2 + r^{-1}|\nabla\psi|^2\right), \\ Mor^{ax}_{deg}[\psi](\tau_1,\tau_2) &:= \int_{\mathcal{M}(\tau_1,\tau_2)} \frac{m}{r^2}|\nabla_{\widehat{R}}\psi|^2 + r^{-3}|\psi|^2 + \frac{\mathcal{T}^2}{r^6}\left(\frac{m}{r^2}|\nabla_{\widehat{T}}\psi|^2 + r^{-1}|\nabla\psi|^2\right). \end{aligned}$$

2. *We also define the higher degenerate and non-degenerate Morawetz norms in $\mathcal{M} = \mathcal{M}(\tau_1, \tau_2)$, for a scalar function z:*

$$\begin{aligned} \dot{Mor}_{\mathcal{S},z,deg}[\psi](\tau_1,\tau_2) &:= \int_{\mathcal{M}} \frac{m}{r^2}|\nabla_{\widehat{R}}\psi|^2_{\mathcal{S}} + r^3\Big(\big|\nabla_{\widehat{T}}\Psi_z\big|^2 + r^2\big|\nabla\Psi_z\big|^2\Big), \\ Mor_{\mathcal{S},z,deg}[\psi](\tau_1,\tau_2) &:= \int_{\mathcal{M}} \frac{m}{r^2}|\nabla_{\widehat{R}}\psi|^2_{\mathcal{S}} + r^{-3}|\psi|^2_{\mathcal{S}} + r^3\Big(\big|\nabla_{\widehat{T}}\Psi_z\big|^2 + r^2\big|\nabla\Psi_z\big|^2\Big), \end{aligned}$$

where[1]

$$\Psi_z = \Psi_z[\psi] \;:=\; \tilde{\mathcal{R}}'^{\underline{a}}[z]\psi_{\underline{a}}, \qquad \tilde{\mathcal{R}}'^{\underline{a}}[z] := \partial_r\left(\frac{z}{\Delta}\mathcal{R}^{\underline{a}}\right),$$

with z a suitable function of r to be chosen later, and with the scalar functions $\mathcal{R}^{\underline{a}}$ given by (3.5.8), *i.e.*

$$\mathcal{R}^1 \;=\; -(r^2+a^2)^2, \qquad \mathcal{R}^2 = -2(r^2+a^2), \qquad \mathcal{R}^3 = -1, \qquad \mathcal{R}^4 = \Delta.$$

[1]Note that z, $\mathcal{R}^{\underline{a}}$ and Δ are functions depending only on r so that ∂_r in the formula for $\tilde{\mathcal{R}}'^{\underline{a}}[z]$ simply denotes differentiation w.r.t. r.

9.2.6 Basic energy and Morawetz in perturbations of Kerr

Let $\mathcal{M}$ satisfying the properties in section 9.1.1. In particular, we assume the estimates (9.1.1) for (Γ_b, Γ_b). We now state the main results of section 9.2 whose goal is to show that the results proved in Kerr in Chapters 7 and 8 also hold on $\mathcal{M}$.

First, we have the following analog of Proposition 6.3.7.

Proposition 9.2.12. *For $|a|/m \ll 1$ sufficiently small, the solution ψ to the model gRW equation* (6.1.1) *satisfies in $\mathcal{M}$*

$$\begin{aligned} Mor^{ax}_{deg}[\psi](\tau_1,\tau_2) \lesssim & \sup_{[\tau_1,\tau_2]} E_{deg}[\psi](\tau) + \delta_{\mathcal{H}} F_{\mathcal{A}}[\psi](\tau_1,\tau_2) + F_{\Sigma_*}[\psi](\tau_1,\tau_2) \\ & + \int_{\mathcal{M}(\tau_1,\tau_2)} a r^{-2}\big(|\nabla\psi|^2 + |\nabla_{\mathbf{T}}\psi|^2\big) \\ & + \int_{\mathcal{M}(\tau_1,\tau_2)} \big(|\nabla_{\widehat{R}}\psi| + r^{-1}|\psi|\big)|N| \\ & + \epsilon\left(\sup_{[\tau_1,\tau_2]} E[\psi](\tau) + B^1_\delta[\psi](\tau_1,\tau_2)\right). \end{aligned} \tag{9.2.5}$$

Also, we have the following analog of Proposition 6.3.9 for perturbations of Kerr.

Proposition 9.2.13. *For $|a|/m \ll 1$ sufficiently small, the solution ψ to the model gRW equation* (6.1.1) *satisfies in $\mathcal{M}$*

$$\begin{aligned} E_{deg}[\psi](\tau_2) + F_{\Sigma_*}[\psi](\tau_1,\tau_2) \lesssim & E_{deg}[\psi](\tau_1) + \delta_{\mathcal{H}}\Big(E_{r\le r_+(1+\delta_{\mathcal{H}})}[\psi](\tau_2) + F_{\mathcal{A}}[\psi](\tau_1,\tau_2)\Big) \\ & + \frac{|a|}{m} Mor^{ax}_{deg}[\psi](\tau_1,\tau_2) + \left|\int_{\mathcal{M}(\tau_1,\tau_2)} \nabla_{\widehat{T}_\delta}\psi\cdot N\right| \\ & + \int_{\mathcal{M}(\tau_1,\tau_2)} |N|^2 + \epsilon\left(\sup_{[\tau_1,\tau_2]} E[\psi](\tau) + B_\delta[\psi](\tau_1,\tau_2)\right). \end{aligned} \tag{9.2.6}$$

Next, we have the following analog of Proposition 7.4.1 for perturbations of Kerr.

Proposition 9.2.14. *Let ψ a solution to the gRW equation* (6.1.1)*. Also, recall the norms $E[\psi]$ and $Mor[\psi]$ introduced in section 6.1.5.*

$$\begin{aligned} Mor_{r\ge 2r_1}[\psi](\tau_1,\tau_2) \lesssim & \sup_{\tau\in[\tau_1,\tau_2]} E_{r\ge r_1}[\psi](\tau) + r_1 Mor_{r_1\le r\le 2r_1}[\psi](\tau_1,\tau_2) \\ & + \mathcal{N}_{r\ge r_1}[\psi, N](\tau_1,\tau_2) + \epsilon B_\delta[\psi](\tau_1,\tau_2), \end{aligned} \tag{9.2.7}$$

and

$$\begin{aligned}\sup_{\tau\in[\tau_1,\tau_2]} E_{r\geq 2r_1}[\psi](\tau) \lesssim & E_{r\geq r_1}[\psi](\tau_1) + \mathcal{N}_{r\geq r_1}[\psi, N](\tau_1,\tau_2) + r_1 Mor_{r_1\leq r\leq 2r_1}[\psi](\tau_1,\tau_2)\\ & + \frac{|a|}{r_1} Mor_{r\geq 2r_1}[\psi](\tau_1,\tau_2) + \epsilon B_\delta[\psi](\tau_1,\tau_2).\end{aligned} \tag{9.2.8}$$

Next, we have the following analog of Proposition 6.3.10 for perturbations of Kerr.

Proposition 9.2.15 ($\mathcal{S}$-derivatives Morawetz estimates)**.** *Let the scalar function z given by*

$$z = z_0 - \delta_0 z_0^2, \qquad z_0 = \frac{\Delta}{(r^2+a^2)^2}.$$

Then, for $|a|/m \ll 1$ sufficiently small, the solution ψ to the model gRW equation (6.1.1) *satisfies in $\mathcal{M}$*

$$\begin{aligned}& Mor_{\mathcal{S},z,deg}[\psi](\tau_1,\tau_2)\\ \lesssim\ & \sum_{\underline{a}=1}^{4}\left(\sup_{[\tau_1,\tau_2]} E_{deg}[\psi_{\underline{a}}](\tau) + \delta_{\mathcal{H}} F_{\mathcal{A}}[\psi_{\underline{a}}](\tau_1,\tau_2) + F_{\Sigma_*}[\psi_{\underline{a}}](\tau_1,\tau_2)\right)\\ & + \left(\sup_{[\tau_1,\tau_2]} E_{deg}[(\nabla_T, \not{\mathfrak{d}})^{\leq 1}\psi](\tau) + \delta_{\mathcal{H}} F_{\mathcal{A}}[(\nabla_T, \not{\mathfrak{d}})^{\leq 1}\psi](\tau_1,\tau_2) + F_{\Sigma_*}[(\nabla_T, \not{\mathfrak{d}})^{\leq 1}\psi](\tau_1,\tau_2)\right)^{\frac{1}{2}}\\ & \times\left(\sup_{[\tau_1,\tau_2]} E_{deg}[(\nabla_T, \not{\mathfrak{d}})^{\leq 2}\psi](\tau) + \delta_{\mathcal{H}} F_{\mathcal{A}}[(\nabla_T, \not{\mathfrak{d}})^{\leq 2}\psi](\tau_1,\tau_2) + F_{\Sigma_*}[(\nabla_T, \not{\mathfrak{d}})^{\leq 2}\psi](\tau_1,\tau_2)\right)^{\frac{1}{2}}\\ & + \sum_{\underline{a}=1}^{4}\int_{\mathcal{M}(\tau_1,\tau_2)} \big(|\nabla_{\widehat{R}}\psi_{\underline{a}}| + r^{-1}|\psi_{\underline{a}}|\big)|N_{\underline{a}}| + \epsilon\left(\sup_{[\tau_1,\tau_2]} E^2[\psi](\tau) + B^2_\delta[\psi](\tau_1,\tau_2)\right).\end{aligned} \tag{9.2.9}$$

Finally, to show that $\mathrm{Mor}_{\mathcal{S},z}$ controls ψ in $\mathcal{M}_{\not{trap}}$, we will rely on the following lemma which is the analog of Lemma 6.3.11 for perturbations of Kerr.

Lemma 9.2.16. *For $\delta_0 > 0$ small enough*[2] *and $|a|/m \ll \delta_0$, there exists a universal constant $c_0 > 0$ such that the following holds on $\mathcal{M}_{\not{trap}}$:*

$$\begin{aligned} r^3\Big(|\nabla_T\Psi_z|^2 + r^2|\nabla\Psi_z|^2\Big) + r^{-3}|\psi|^2_{\mathcal{S}} \ \geq\ & c_0 r^{-3}\Big(|\nabla_T\psi|^2_{\mathcal{S}} + |\nabla_Z\psi|^2_{\mathcal{S}} + r^2|\nabla\psi|^2_{\mathcal{S}}\Big)\\ & - O(ar^{-3})\big|(\nabla_T, \not{\mathfrak{d}})^{\leq 1}\mathfrak{d}^{\leq 2}\psi\big|^2 + \dot{\mathbf{D}}_\alpha F^\alpha + Err_\epsilon\end{aligned}$$

[2]Recall that the constant $\delta_0 > 0$ is involved in the definition of $z = z_0 - \delta_0 z_0^2$.

where the 1-form F denotes an expression in ψ for which we have a bound of the form

$$\begin{aligned}
&\left|\int_{\partial\mathcal{M}(\tau_1,\tau_2)} F^\mu N_\mu\right| \\
\lesssim\ & \left(\sup_{[\tau_1,\tau_2]} E_{deg}[(\nabla_T,\, \not{\mathfrak{d}})^{\leq 1}\psi](\tau) + \delta_{\mathcal{H}} F_{\mathcal{A}}[(\nabla_T,\, \not{\mathfrak{d}})^{\leq 1}\psi](\tau_1,\tau_2) + F_{\Sigma_*}[(\nabla_T,\, \not{\mathfrak{d}})^{\leq 1}\psi](\tau_1,\tau_2)\right)^{\frac{1}{2}} \\
\times\ & \left(\sup_{[\tau_1,\tau_2]} E_{deg}[(\nabla_T,\, \not{\mathfrak{d}})^{\leq 2}\psi](\tau) + \delta_{\mathcal{H}} F_{\mathcal{A}}[(\nabla_T,\, \not{\mathfrak{d}})^{\leq 2}\psi](\tau_1,\tau_2) + F_{\Sigma_*}[(\nabla_T,\, \not{\mathfrak{d}})^{\leq 2}\psi](\tau_1,\tau_2)\right)^{\frac{1}{2}},
\end{aligned}$$

and the scalar function Err_ϵ satisfies

$$\int_{\mathcal{M}} |Err_\epsilon| \ \lesssim\ \epsilon\left(\sup_{[\tau_1,\tau_2]} E^2[\psi](\tau) + B^2_\delta[\psi](\tau_1,\tau_2)\right).$$

The rest of section 9.2 is devoted to the proof of the above results. Propositions 9.2.12, 9.2.13 and 9.2.14 are proved in section 9.2.10 and Proposition 9.2.15 and Lemma 9.2.16 are proved in section 9.2.11.

9.2.7 Acceptable error terms

Recall the definition of the energy-momentum tensor associated to the model gRW equation (6.1.1), see (4.7.2),

$$\mathcal{Q}_{\mu\nu} := \dot{\mathbf{D}}_\mu\psi\cdot\dot{\mathbf{D}}_\nu\psi - \frac{1}{2}\mathbf{g}_{\mu\nu}\left(\dot{\mathbf{D}}_\lambda\psi\cdot\dot{\mathbf{D}}^\lambda\psi + V\psi\cdot\psi\right) = \dot{\mathbf{D}}_\mu\psi\cdot\dot{\mathbf{D}}_\nu\psi - \frac{1}{2}\mathbf{g}_{\mu\nu}\mathcal{L}[\psi]. \quad (9.2.10)$$

When revisiting the proofs of Chapters 7 and 8 in the context of admissible perturbations $\mathcal{M}$ of Kerr, we generate additional terms. We introduce below acceptable terms.

Definition 9.2.17 (Acceptable error terms)**.** *The following quantity*

$$F^{\mu\nu}\mathcal{Q}_{\mu\nu}[\psi] + G^\mu\psi\cdot\dot{\mathbf{D}}_\mu\psi + H|\psi|^2$$

is said to be of the acceptable type, and denoted Good, if:

- $F_{44}\in\Gamma_g$*, and all other components of* $F_{\mu\nu}$ *belong to* Γ_b.
- *All components of* G_μ *belong to* Γ_g.

- $H \in r^{-1}\Gamma_g$.

The justification for Definition 9.2.17 is provided by the following lemma.

Lemma 9.2.18. *Assume that the quantity*

$$F^{\mu\nu}\mathcal{Q}_{\mu\nu}[\psi] + G^\mu\psi \cdot \dot{\mathbf{D}}_\mu\psi + H|\psi|^2$$

is of the acceptable type in the sense of Definition 9.2.17. Then, it satisfies the following estimate

$$\int_{\mathcal{M}_{trap}(\tau_1,\tau_2)} \Big|F^{\mu\nu}\mathcal{Q}_{\mu\nu}[\psi] + G^\mu\psi \cdot \dot{\mathbf{D}}_\mu\psi + H|\psi|^2\Big|^2 \lesssim \epsilon \sup_{[\tau_1,\tau_2]} E[\psi](\tau)$$

and

$$\int_{{}^{(trap)}\!\!\!\!\!/\mathcal{M}(\tau_1,\tau_2)} \Big|F^{\mu\nu}\mathcal{Q}_{\mu\nu}[\psi] + G^\mu\psi \cdot \dot{\mathbf{D}}_\mu\psi + H|\psi|^2\Big|^2 \lesssim \epsilon B_\delta[\psi](\tau_1,\tau_2).$$

Remark 9.2.19. *Recall from Definition 9.2.17 that terms of the acceptable type are denoted Good. In view of Lemma 9.2.18, we infer that such term satisfy the following estimate*

$$\int_{\mathcal{M}} |Good| \quad\lesssim\quad \epsilon\left(\sup_{[\tau_1,\tau_2]} E[\psi](\tau) + B_\delta[\psi](\tau_1,\tau_2)\right).$$

Proof. We start with the control on $\mathcal{M}_{trap}$. We have, using the control of Γ_g and Γ_b,

$$\begin{aligned}
&\int_{\mathcal{M}_{trap}(\tau_1,\tau_2)} \Big|F^{\mu\nu}\mathcal{Q}_{\mu\nu}[\psi] + G^\mu\psi \cdot \dot{\mathbf{D}}_\mu\psi + H|\psi|^2\Big|^2 \\
\lesssim\ & \epsilon\int_{\mathcal{M}_{trap}(\tau_1,\tau_2)} \frac{1}{\tau_{trap}^{1+\delta_{dec}}}|\mathfrak{d}^{\leq 1}\psi|^2 \\
\lesssim\ & \epsilon\left(\int_{\tau_1}^{\tau_2}\frac{1}{\tau_{trap}^{1+\delta_{dec}}}\right)E[\psi] \\
\lesssim\ & \epsilon\sup_{\tau\in[\tau_1,\tau_2]} E[\psi]
\end{aligned}$$

as stated.

Concerning the control on $\mathcal{M}_{tr\not{p}}$, we have

$$\begin{aligned}
&\int_{\mathcal{M}_{tr\not{p}}(\tau_1,\tau_2)} \Big|F^{\mu\nu}\mathcal{Q}_{\mu\nu}[\psi] + G^\mu\psi\cdot\dot{\mathbf{D}}_\mu\psi + H|\psi|^2\Big|^2 \\
\lesssim\ & \epsilon\int_{\mathcal{M}_{tr\not{p}}(\tau_1,\tau_2)} \Big(|\Gamma_g||\nabla_3\psi||\mathfrak{d}^{\leq 1}\psi| + r^{-1}|\Gamma_g||\mathfrak{d}^{\leq 1}\psi|^2\Big) \\
\lesssim\ & \epsilon\int_{\mathcal{M}_{tr\not{p}}(\tau_1,\tau_2)} r^{-2}\Big(|\nabla_3\psi||\mathfrak{d}^{\leq 1}\psi| + r^{-1}|\mathfrak{d}^{\leq 1}\psi|^2\Big) \\
\lesssim\ & \epsilon\int_{\mathcal{M}_{tr\not{p}}(\tau_1,\tau_2)} \Big(r^{-1-\delta}|\nabla_3\psi|^2 + r^{\delta-3}|\mathfrak{d}^{\leq 1}\psi|^2\Big) \\
\lesssim\ & \epsilon B^s_\delta[\psi](\tau_1,\tau_2)
\end{aligned}$$

as stated. This concludes the proof of the lemma. □

Next, we introduce the linearization of the quantities $F_{\mu\nu}$, G_μ and H.

Definition 9.2.20. *Let $F_{\mu\nu}$, G_μ and H. We define their linearization as follows*

$$F_{\mu\nu} = (F_{\mu\nu})_K + \widecheck{F_{\mu\nu}}, \qquad G_\mu = (G_\mu)_K + \widecheck{G}_\mu, \qquad H = H_K + \check{H},$$

where[3]*:*

1. *the quantities*

$$(F_{44})_K, \qquad (F_{34})_K, \qquad (F_{33})_K, \qquad (G_4)_K, \qquad (G_3)_K, \qquad H_K,$$

are given as explicit functions of $(r, \cos\theta)$ coinciding with the corresponding expressions in Kerr,

2. *the quantities*

$$(F_{4a})_K, \qquad (F_{3a})_K, \qquad (G_a)_K,$$

are given as the 1-form $\Re(\mathfrak{J})_a$ multiplied by explicit functions of $(r, \cos\theta)$ coinciding with the corresponding expressions in Kerr,

[3]This reflects the fact that in the principal null frame of Kerr, the only nontrivial 1-form is $\Re(\mathfrak{J})$, and that there are no nontrivial symmetric traceless 2-tensors.

3. *the quantity* $(F_{ab})_K$ *is given by the the symmetric 2-tensor* γ_{ab} *multiplied by explicit functions of* $(r, \cos\theta)$ *coinciding with the corresponding expressions in Kerr.*

In view of Definition 9.2.20, we can decompose the above expressions in their main part and error terms as follows

$$\begin{aligned} F^{\mu\nu}\mathcal{Q}_{\mu\nu}[\psi] + G^{\mu}\psi\cdot\dot{\mathbf{D}}_\mu\psi + H|\psi|^2 &= \Big[F^{\mu\nu}\mathcal{Q}_{\mu\nu}[\psi] + G^{\mu}\psi\cdot\dot{\mathbf{D}}_\mu\psi + H|\psi|^2\Big]_K + \mathrm{Err}, \\ \Big[F^{\mu\nu}\mathcal{Q}_{\mu\nu}[\psi] + G^{\mu}\psi\cdot\dot{\mathbf{D}}_\mu\psi + H|\psi|^2\Big]_K &= (F^{\mu\nu})_K\mathcal{Q}_{\mu\nu}[\psi] + (G^{\mu})_K\psi\cdot\dot{\mathbf{D}}_\mu\psi + H_K|\psi|^2, \\ \mathrm{Err} &= \widecheck{F^{\mu\nu}}\mathcal{Q}_{\mu\nu}[\psi] + \widecheck{G^{\mu}}\psi\cdot\dot{\mathbf{D}}_\mu\psi + \check{H}|\psi|^2. \end{aligned} \tag{9.2.11}$$

The proof of the results of section 9.2.6 will rely in particular on showing that the extra terms in perturbations of Kerr appearing in the various divergence identities involved in energy and Morawetz estimates are of the acceptable type in the sense of Definition 9.2.17, i.e, that Err $\in$ Good in (9.2.11).

9.2.8 Some deformation tensors

We star with computing the deformation tensor of e_3 and e_4.

Lemma 9.2.21. *We have*

$$\begin{aligned} {}^{(e_3)}\pi_{44} &= 4\partial_r\left(\frac{\Delta}{|q|^2}\right) + \Gamma_g, \quad {}^{(e_3)}\pi_{34} = \Gamma_b, \quad {}^{(e_3)}\pi_{33} = 0, \\ {}^{(e_3)}\pi_{4a} &= -\frac{2ar}{|q|^2}\Re(\mathfrak{J})_a + \Gamma_g, \quad {}^{(e_3)}\pi_{3a} = \Gamma_b, \quad {}^{(e_3)}\pi_{ab} = -\frac{2r}{|q|^2}\delta_{ab} + \Gamma_b, \end{aligned} \tag{9.2.12}$$

and

$$\begin{aligned} {}^{(e_4)}\pi_{33} &= \Gamma_b, \quad {}^{(e_4)}\pi_{34} = -2\partial_r\left(\frac{\Delta}{|q|^2}\right) + \Gamma_g, \quad {}^{(e_4)}\pi_{44} = 0, \\ {}^{(e_4)}\pi_{3a} &= \Gamma_b \quad {}^{(e_4)}\pi_{4a} = \Gamma_g, \quad {}^{(e_4)}\pi_{ab} = \frac{2r\Delta}{|q|^4}\delta_{ab} + \Gamma_g. \end{aligned} \tag{9.2.13}$$

Proof. We have

$$
\begin{aligned}
{}^{(e_3)}\pi_{44} &= 2\mathbf{g}(\mathbf{D}_4 e_3, e_4) = -8\omega = 4\partial_r\left(\frac{\Delta}{|q|^2}\right) + \Gamma_g,\\
{}^{(e_3)}\pi_{34} &= \mathbf{g}(\mathbf{D}_3 e_3, e_4) = 4\underline{\omega} = \Gamma_b,\\
{}^{(e_3)}\pi_{33} &= 2\mathbf{g}(\mathbf{D}_3 e_3, e_3) = 0,\\
{}^{(e_3)}\pi_{4a} &= \mathbf{g}(\mathbf{D}_4 e_3, e_a) + \mathbf{g}(\mathbf{D}_a e_3, e_4) = 2(\underline{\eta}_a - \zeta_a)\\
&= -\Re\left(\left(\frac{aq}{|q|^2} + \frac{a\overline{q}}{|q|^2}\right)\mathfrak{J}_a\right) + \Gamma_g = -\frac{2ar}{|q|^2}\Re(\mathfrak{J})_a + \Gamma_g,\\
{}^{(e_3)}\pi_{3a} &= \mathbf{g}(\mathbf{D}_3 e_3, e_a) + \mathbf{g}(\mathbf{D}_a e_3, e_3) = 2\underline{\xi}_a = \Gamma_b,\\
{}^{(e_3)}\pi_{ab} &= \mathbf{g}(\mathbf{D}_a e_3, e_b) + \mathbf{g}(\mathbf{D}_b e_3, e_a) = \underline{\chi}_{ab} + \underline{\chi}_{ab} = \mathrm{tr}\,\underline{\chi}\delta_{ab} + 2\widehat{\underline{\chi}}_{ab}\\
&= -\frac{2r}{|q|^2}\delta_{ab} + \Gamma_b,
\end{aligned}
$$

and

$$
\begin{aligned}
{}^{(e_4)}\pi_{33} &= 2\mathbf{g}(\mathbf{D}_3 e_4, e_3) = -8\underline{\omega} = \Gamma_b,\\
{}^{(e_4)}\pi_{34} &= \mathbf{g}(\mathbf{D}_4 e_4, e_3) = 4\omega = -2\partial_r\left(\frac{\Delta}{|q|^2}\right) + \Gamma_g,\\
{}^{(e_4)}\pi_{44} &= 2\mathbf{g}(\mathbf{D}_4 e_4, e_4) = 0,\\
{}^{(e_4)}\pi_{3a} &= \mathbf{g}(\mathbf{D}_3 e_4, e_a) + \mathbf{g}(\mathbf{D}_a e_4, e_3) = 2(\eta_a + \zeta_a) = \Gamma_b\\
{}^{(e_4)}\pi_{4a} &= \mathbf{g}(\mathbf{D}_4 e_4, e_a) + \mathbf{g}(\mathbf{D}_a e_4, e_4) = 2\xi_a = \Gamma_g,\\
{}^{(e_4)}\pi_{ab} &= \mathbf{g}(\mathbf{D}_a e_4, e_b) + \mathbf{g}(\mathbf{D}_b e_4, e_a) = \chi_{ab} + \chi_{ab} = \mathrm{tr}\,\chi\delta_{ab} + 2\widehat{\chi}_{ab}\\
&= \frac{2r\Delta}{|q|^4}\delta_{ab} + \Gamma_g,
\end{aligned}
$$

as desired. □

Recall that we have

$$
\begin{aligned}
\widehat{T} &= \frac{1}{2}\left(\frac{|q|^2}{r^2+a^2}e_4 + \frac{\Delta}{r^2+a^2}e_3\right),\\
\widehat{R} &= \frac{1}{2}\left(\frac{|q|^2}{r^2+a^2}e_4 - \frac{\Delta}{r^2+a^2}e_3\right).
\end{aligned}
$$

Recall also that $\widehat{T}_\delta = \widehat{T}$ on $\mathcal{M}_{tr\not{\phi}}$ by construction. We now show that particular quantities $\widetilde{F^{\mu\nu}}\mathcal{Q}_{\mu\nu}[\psi]$ are acceptable error terms.

Lemma 9.2.22. *We have*

$$\widetilde{{}^{(\widehat{R})}\pi^{\mu\nu}\mathcal{Q}_{\mu\nu}[\psi]} = Good, \qquad \widetilde{{}^{(\widehat{T})}\pi^{\mu\nu}\mathcal{Q}_{\mu\nu}[\psi]} = Good, \qquad \widetilde{{}^{(\widehat{T}_\delta)}\pi^{\mu\nu}\mathcal{Q}_{\mu\nu}[\psi]} = Good.$$

Proof. Since

$$\begin{aligned} \widehat{T} &= \frac{1}{2}\left(\frac{|q|^2}{r^2+a^2}e_4 + \frac{\Delta}{r^2+a^2}e_3\right), \\ \widehat{R} &= \frac{1}{2}\left(\frac{|q|^2}{r^2+a^2}e_4 - \frac{\Delta}{r^2+a^2}e_3\right), \end{aligned}$$

we infer

$$\begin{aligned} \widetilde{{}^{(\widehat{R})}\pi_{\mu\nu}}, \widetilde{{}^{(\widehat{T})}\pi_{\mu\nu}} &= O(1)\widetilde{{}^{(e_4)}\pi_{\mu\nu}} + O(1)\widetilde{{}^{(e_3)}\pi_{\mu\nu}} + O(r^{-1})\widetilde{e_4(r)} + O(r^{-1})\widetilde{e_3(r)} \\ &\quad + O(r^{-1})e_4(\cos\theta) + O(r^{-1})e_3(\cos\theta). \end{aligned}$$

In view of Lemma 9.2.21 and the fact that $\widetilde{e_3(r)} \in r\Gamma_b$, while the other components behave better, we infer $\widetilde{{}^{(\widehat{R})}\pi_{\mu\nu}}, \widetilde{{}^{(\widehat{T})}\pi_{\mu\nu}} \in \Gamma_b$.

Also, we have for the particular case $\mu = \nu = 4$

$$\widetilde{{}^{(\widehat{R})}\pi_{44}}, \widetilde{{}^{(\widehat{T})}\pi_{44}} = O(1)\widetilde{{}^{(e_4)}\pi_{44}} + O(1)\widetilde{{}^{(e_3)}\pi_{44}} + O(r^{-1})\widetilde{e_4(r)} + O(r^{-1})e_4(\cos\theta).$$

In view of Lemma 9.2.21 and the fact that $\widetilde{e_4(r)} \in r\Gamma_g$, while $e_4(\cos\theta)$ behaves better, we infer $\widetilde{{}^{(\widehat{R})}\pi_{44}}, \widetilde{{}^{(\widehat{T})}\pi_{44}} \in \Gamma_g$.

Since we have shown that $\widetilde{{}^{(\widehat{R})}\pi_{\mu\nu}}, \widetilde{{}^{(\widehat{T})}\pi_{\mu\nu}} \in \Gamma_b$ and $\widetilde{{}^{(\widehat{R})}\pi_{44}}, \widetilde{{}^{(\widehat{T})}\pi_{44}} \in \Gamma_g$, we have, in view of Definition 9.2.17

$$\widetilde{{}^{(\widehat{R})}\pi^{\mu\nu}\mathcal{Q}_{\mu\nu}[\psi]} = \text{Good}, \qquad \widetilde{{}^{(\widehat{T})}\pi^{\mu\nu}\mathcal{Q}_{\mu\nu}[\psi]} = \text{Good}.$$

Also, since $\widehat{T}_\delta = \widehat{T}$ on $\mathcal{M}_{tr\not{p}}$, we also have $\widetilde{{}^{(\widehat{T}_\delta)}\pi_{\mu\nu}} \in \Gamma_b$ and $\widetilde{{}^{(\widehat{T}_\delta)}\pi_{44}} \in \Gamma_g$, and thus

$$\widetilde{{}^{(\widehat{T}_\delta)}\pi^{\mu\nu}\mathcal{Q}_{\mu\nu}[\psi]} = \text{Good}.$$

This concludes the proof of the Lemma 9.2.22. □

9.2.9 Poincaré inequality

Recall from Definition 6.1.7 that the spheres $S(\tau, r)$ are covered by three coordinates systems (x^1_S, x^2_S), (x^1_E, x^2_E) and (x^1_N, x^2_N) so that we have the following control on each corresponding coordinate chart

$$\max_{b,c=1,2} |g_{bc} - (g_{a,m})_{bc}| \lesssim r^2\epsilon,$$

where g_{bc} denotes the induced metric coefficients in these coordinates systems, and $(g_{a,m})_{bc}$ the corresponding expression in Kerr.

The following Poincaré inequality is the analog of Lemma 7.2.3 in perturbations of Kerr.

Lemma 9.2.23. *For $\psi \in \mathfrak{s}_2$, we have*

$$\begin{aligned}\int_S |\nabla\psi|^2 \geq & \frac{2\big(1+O(\epsilon+a^2r^{-2})\big)}{r^2}\int_S |\psi|^2 - O(a+\epsilon)\int_S \big(|\nabla\psi|^2 + r^{-2}|\nabla_{\mathbf{T}}\psi|^2\big)\\ & -O(\epsilon r^{-2})\int_S |\nabla_3\psi|^2.\end{aligned}$$

Proof. We denote by e_b, $b = 1, 2$, an orthonormal basis for the horizontal structure associated to (e_3, e_4), and we look for vectorfields X^S_b, $b = 1, 2$, tangent to S as

$$X^S_b = e_b - a\Re(\mathfrak{J})_b\mathbf{T} + \lambda_b e_3 + \underline{\lambda}_b e_4, \quad b = 1, 2,$$

for some 1-forms λ and $\underline{\lambda}$. The vectorfields X^S_b, $b = 1, 2$, are tangent to $S(\tau, r)$ if and only if $X^S_b(r) = X^S_b(\tau) = 0$, and hence, if and only if

$$\begin{aligned}0 &= \nabla(r) - a\Re(\mathfrak{J})\mathbf{T}(r) + \lambda e_3(r) + \underline{\lambda} e_4(r),\\ 0 &= \nabla(\tau) - a\Re(\mathfrak{J})\mathbf{T}(\tau) + \lambda e_3(\tau) + \underline{\lambda} e_4(\tau),\end{aligned}$$

or

$$\begin{aligned}\lambda e_3(r) + \underline{\lambda} e_4(r) &= -\nabla(r) + a\Re(\mathfrak{J})\mathbf{T}(r),\\ \lambda e_3(\tau) + \underline{\lambda} e_4(\tau) &= -\nabla(\tau) + a\Re(\mathfrak{J})\mathbf{T}(\tau).\end{aligned}$$

Since we have

$$e_4(r)e_3(\tau) - e_3(r)e_4(\tau) = \frac{\Delta}{|q|^2}e_3(\tau) + e_4(\tau) + O(\epsilon) \gtrsim 1 + O(\epsilon + \delta_{\mathcal{H}}) > 0,$$

we infer the existence and uniqueness of $(\lambda, \underline{\lambda})$ with

$$|\lambda| + |\underline{\lambda}| \lesssim \big|\nabla(r) - a\Re(\mathfrak{J})\mathbf{T}(r)\big| + \big|\nabla(\tau) - a\Re(\mathfrak{J})\mathbf{T}(\tau)\big|.$$

Together with

$$|\nabla(r)| \lesssim r^{-1}\epsilon, \qquad |\mathbf{T}(r)| \lesssim \epsilon, \qquad |\mathbf{T}(\tau) - 1| \lesssim \epsilon, \qquad |\nabla(\tau) - a\Re(\mathfrak{J})| \lesssim r^{-1}\epsilon,$$

we deduce

$$|\lambda| + |\underline{\lambda}| \lesssim \frac{\epsilon}{r}.$$

Coming back to the tangent vectorfields

$$X_b^S = e_b - a\Re(\mathfrak{J})_b\mathbf{T} + \lambda_b e_3 + \underline{\lambda}_b e_4, \quad b = 1, 2,$$

we infer

$$\begin{aligned}\mathbf{g}(X_b^S, X_c^S) &= \delta_{bc} - 2a^2\Re(\mathfrak{J})_b\Re(\mathfrak{J})_c + a^4|\Re(\mathfrak{J})|^2\Re(\mathfrak{J})_b\Re(\mathfrak{J})_c + a\lambda_b\Re(\mathfrak{J})_c + a\lambda_c\Re(\mathfrak{J})_b\\ &\quad +a\frac{\Delta}{|q|^2}\underline{\lambda}_b\Re(\mathfrak{J})_c + a\frac{\Delta}{|q|^2}\underline{\lambda}_c\Re(\mathfrak{J})_b - \lambda_b\underline{\lambda}_c - \lambda_c\underline{\lambda}_b, \quad b = 1, 2,\end{aligned}$$

and hence, since $|\lambda| = O(r^{-1}\epsilon)$, $|\underline{\lambda}| = O(r^{-1}\epsilon)$, $|\Re(\mathfrak{J})| = O(r^{-1})$,

$$\mathbf{g}(X_b^S, X_c^S) = \delta_{bc} + O((a^2 + \epsilon^2)r^{-2}), \quad b = 1, 2.$$

We deduce

$$\begin{aligned}\nabla^S &= \Big(1 + O((a^2 + \epsilon^2)r^{-2})\Big)\Big(\nabla + O(ar^{-1})\nabla_T + O(r^{-1}\epsilon)e_3 + O(r^{-1}\epsilon)e_4\Big)\\ &= \Big(1 + O((a^2 + \epsilon^2)r^{-2})\Big)\Big(\nabla + O((a + \epsilon)r^{-1})\nabla_T + O(r^{-1}\epsilon)e_3\Big).\end{aligned}$$

This yields

$$|\nabla\psi|^2 = |\nabla^S\psi|^2 - O(a + \epsilon)\big(|\nabla\psi|^2 + r^{-2}|\nabla_{\mathbf{T}}\psi|^2\big) - O(\epsilon r^{-2})|\nabla_3\psi|^2$$

so that

$$\int_S |\nabla\psi|^2 = \int_S |\nabla^S\psi|^2 - O(a + \epsilon)\int_S \big(|\nabla\psi|^2 + r^{-2}|\nabla_{\mathbf{T}}\psi|^2\big) - O(\epsilon r^{-2})\int_S |\nabla_3\psi|^2.$$

Next, recall from the proof of Lemma 7.2.3 that in the three coordinates systems of Remark 6.1.8, we have

$$\max_{b,c=1,2}\Big|\partial^{\leq 2}((g_{a,m})_{x^a x^b} - r^2(\gamma_{\mathbb{S}^2})_{x^a x^b})\Big| \lesssim a^2$$

where $\partial^{\leq 2}$ denotes at most 2 coordinates derivatives, and $(\gamma_{\mathbb{S}^2})_{x^a x^b}$ the metric coefficients on $\mathbb{S}^2$ in the corresponding coordinates system. Together with Definition 6.1.7, we infer, for the induced metric on $S(\tau, r)$,

$$\max_{b,c=1,2} \left|\partial^{\leq 2}((g_{x^a x^b} - r^2(\gamma_{\mathbb{S}^2})_{x^a x^b})\right| \lesssim a^2 + r^2\epsilon.$$

We deduce in particular, for the Gauss curvature K_S of S and the area radius r_S of S,

$$K_S = \frac{1}{r^2}\big(1 + O(\epsilon + a^2 r^{-2})\big), \qquad r_S = r\big(1 + O(\epsilon + a^2 r^{-2})\big).$$

Applying the effective uniformization result of Corollary 3.8 in [52], we obtain a map $\Phi : \mathbb{S}^2 \to S$ and a scalar function u on $\mathbb{S}^2$ such that

$$\Phi^\# g = (r_S)^2 e^{2u}\gamma_{\mathbb{S}^2}, \qquad \|\not\partial^{\leq 2}(u \circ \Phi^{-1})\|_{L^2(S)} \lesssim (a^2 r^{-2}) r_S.$$

We infer, relying on the well known Poincaré inequality for $\psi \in \mathfrak{s}_2(\mathbb{S}^2)$, see for example [36],

$$\begin{aligned}
\int_S |\nabla^S \psi|^2 &= \frac{1}{r^2}\big(1 + O(\epsilon + a^2 r^{-2})\big)\int_{\mathbb{S}^2} |\nabla_{\mathbb{S}^2}\Phi^\#\psi|^2 + \frac{1}{r^2}O(\epsilon + a^2 r^{-2})\int_{\mathbb{S}^2} |\Phi^\#\psi|^2\\
&\geq \frac{2}{r^2}\big(1 + O(\epsilon + a^2 r^{-2})\big)\int_{\mathbb{S}^2} |\Phi^\#\psi|^2\\
&\geq \frac{2}{r^2}\int_S \big(1 + O(\epsilon + a^2 r^{-2})\big)|\psi|^2
\end{aligned}$$

and hence

$$\begin{aligned}
\int_S |\nabla\psi|^2 &= \int_S |\nabla^S\psi|^2 - O(a+\epsilon)\int_S \big(|\nabla\psi|^2 + r^{-2}|\nabla_{\mathbf{T}}\psi|^2\big) - O(\epsilon r^{-2})\int_S |\nabla_3\psi|^2\\
&\geq \frac{2\big(1 + O(\epsilon + a^2 r^{-2})\big)}{r^2}\int_S |\psi|^2\\
&\quad - O(a+\epsilon)\int_S \big(|\nabla\psi|^2 + r^{-2}|\nabla_{\mathbf{T}}\psi|^2\big) - O(\epsilon r^{-2})\int_S |\nabla_3\psi|^2
\end{aligned}$$

as stated. This concludes the proof of Lemma 9.2.23. □

9.2.10 Proof of Propositions 9.2.12, 9.2.13 and 9.2.14

Preliminaries for the Morawetz estimate

We start with some preliminaries for the Morawetz estimate. Let $\psi \in \mathfrak{s}_k(\mathcal{M})$ be a solution of (6.1.1), X a vectorfield of the form

$$X = X^3 e_3 + X^4 e_4,$$

w a scalar and M a one form. Define

$$\mathcal{P}_\mu[X, w, M] \;:=\; \mathcal{Q}_{\mu\nu}X^\nu + \frac{1}{2}w\psi \cdot \dot{\mathbf{D}}_\mu\psi - \frac{1}{4}|\psi|^2\partial_\mu w + \frac{1}{4}|\psi|^2 M_\mu.$$

Then, we have in view of Proposition 4.7.3

$$\begin{aligned}
\mathbf{D}^\mu\mathcal{P}_\mu[X, w, M] \;=\;& \frac{1}{2}\mathcal{Q}\cdot {}^{(X)}\pi - \frac{1}{2}X(V)|\psi|^2 + \frac{1}{2}w\mathcal{L}[\psi] - \frac{1}{4}|\psi|^2\square_{\mathbf{g}}w\\
&- \big({}^*\rho + \underline{\eta}\wedge\eta\big)\nabla_{X^4e_4 - X^3e_3}\psi\cdot {}^*\psi\\
&- \frac{1}{2}\Im\Big(\mathrm{tr}\underline{X}HX^3 + \mathrm{tr}X\,\underline{H}X^4\Big)\cdot\nabla\psi\cdot {}^*\psi\\
&+ \frac{1}{4}\mathrm{Div}(|\psi|^2M) + \left(\nabla_X\psi + \frac{1}{2}w\psi\right)\cdot\big(\dot{\square}_2 - V\psi\big)\\
&+ r^{-2}\big(X^3\Gamma_b + X^4\Gamma_g\big)\mathfrak{d}\psi\cdot\psi.
\end{aligned}$$

We make the following choices for (X, w, M), consistent with the ones in Chapter 7:

1. X of the type[4]

$$X = \mathcal{F}\frac{(r^2+a^2)}{\Delta}\widehat{R}, \qquad \widehat{R} = \frac{|q|^2}{2(r^2+a^2)}e_4 - \frac{\Delta}{2(r^2+a^2)}e_3,$$

   where $\mathcal{F} = \mathcal{F}(r)$ is such that $\mathcal{F}/\Delta$ is a smooth function of r, and $(r\partial_r)^k\mathcal{F}(r)$ is uniformly bounded on $\mathcal{M}$. In particular, there holds $X = X^3e_3 + X^4e_4$ as anticipated, with X^3 and X^4 smooth functions of r such that $(r\partial_r)^kX^3$ and $(r\partial_r)^kX^4$ are uniformly bounded on $\mathcal{M}$.

2. The scalar function $w(r)$ is such that w/Δ is a smooth function of r and $(r\partial_r)^k(rw(r))$ is uniformly bounded on $\mathcal{M}$.

3. M is of the type

$$M = v(r)\frac{(r^2+a^2)}{\Delta}\widehat{R}$$

   where the scalar function $v(r)$ is such that v/Δ is a smooth function of r and $(r\partial_r)^k(r^{\frac{5}{2}}v(r))$ is uniformly bounded on $\mathcal{M}$.

---

[4]Recall that we have in Kerr

$$\widehat{R} = \frac{\Delta}{r^2+a^2}\partial_r.$$

We have the following lemma.

Lemma 9.2.24. *Using the decomposition* (9.2.11), *we have*

$$\begin{aligned}\mathbf{D}^\mu\mathcal{P}_\mu[X,w,M] &= \left[\frac{1}{2}\mathcal{Q}\cdot{}^{(X)}\pi-\frac{1}{2}X(V)|\psi|^2+\frac{1}{2}w\mathcal{L}[\psi]-\frac{1}{4}|\psi|^2\square_{\mathbf{g}}w+\frac{1}{4}Div(|\psi|^2M)\right]_K\\ &\quad -\big({}^*\rho+\underline{\eta}\wedge\eta\big)\nabla_{X^4e_4-X^3e_3}\psi\cdot{}^*\psi\\ &\quad -\frac{1}{2}\Im\Big(tr\underline{X}HX^3+trX\,\underline{H}X^4\Big)\cdot\nabla\psi\cdot{}^*\psi\\ &\quad +\left(\nabla_X\psi+\frac{1}{2}w\psi\right)\cdot\big(\dot{\square}_2-V\psi\big)+Good\end{aligned}$$

where Good is given by Definition 9.2.17.

Proof. First, since $(r\partial_r)^kX^3$ and $(r\partial_r)^kX^4$ are uniformly bounded on $\mathcal{M}$, we easily check that

$$r^{-2}\big(X^3\Gamma_b+X^4\Gamma_g\big)\mathfrak{d}\psi\cdot\psi \;=\; \text{Good}$$

in the sense of Definition 9.2.17. Together with the above, we deduce

$$\begin{aligned}\mathbf{D}^\mu\mathcal{P}_\mu[X,w,M] &= \frac{1}{2}\mathcal{Q}\cdot{}^{(X)}\pi-\frac{1}{2}X(V)|\psi|^2+\frac{1}{2}w\mathcal{L}[\psi]-\frac{1}{4}|\psi|^2\square_{\mathbf{g}}w\\ &\quad -\big({}^*\rho+\underline{\eta}\wedge\eta\big)\nabla_{X^4e_4-X^3e_3}\psi\cdot{}^*\psi\\ &\quad -\frac{1}{2}\Im\Big(\mathrm{tr}\underline{X}HX^3+\mathrm{tr}X\,\underline{H}X^4\Big)\cdot\nabla\psi\cdot{}^*\psi\\ &\quad +\frac{1}{4}\mathrm{Div}(|\psi|^2M)+\left(\nabla_X\psi+\frac{1}{2}w\psi\right)\cdot\big(\dot{\square}_2-V\psi\big)+\text{Good}.\end{aligned}$$

Next, applying the decomposition (9.2.11), and using the fact that X^4, X^3, v and w are functions of r, we obtain

$$\begin{aligned}&\frac{1}{2}\mathcal{Q}\cdot{}^{(X)}\pi-\frac{1}{2}X(V)|\psi|^2+\frac{1}{2}w\mathcal{L}[\psi]-\frac{1}{4}|\psi|^2\square_{\mathbf{g}}w+\frac{1}{4}\mathrm{Div}(|\psi|^2M)\\ =\;&\left[\frac{1}{2}\mathcal{Q}\cdot{}^{(X)}\pi-\frac{1}{2}X(V)|\psi|^2+\frac{1}{2}w\mathcal{L}[\psi]-\frac{1}{4}|\psi|^2\square_{\mathbf{g}}w+\frac{1}{4}\mathrm{Div}(|\psi|^2M)\right]_K+\text{Err}\end{aligned}$$

where

$$\text{Err} \;=\; \frac{1}{2}\mathcal{Q}\cdot\widetilde{{}^{(X)}\pi}-\frac{1}{2}\widetilde{X(V)}|\psi|^2-\frac{1}{4}|\psi|^2\widetilde{\square_{\mathbf{g}}w}+\frac{1}{4}\widetilde{\mathrm{Div}(M)}|\psi|^2.$$

Since

$$
\begin{aligned}
\widecheck{X(V)} &= O(r^{-3})\Big(\widecheck{e_3(r)}, \widecheck{e_4(r)}, e_3(\cos\theta), e_4(\cos\theta)\Big) = r^{-2}\Gamma_b = r^{-1}\Gamma_g,\\
\widecheck{\square_{\mathbf{g}} w} &= r^{-3}\mathfrak{d}^{\leq 1}\big(\widecheck{e_3(r)}, \nabla(r)\big) + r^{-2}\mathfrak{d}^{\leq 1}\widecheck{e_4(r)} + r^{-2}\Gamma_g = r^{-2}\Gamma_b = r^{-1}\Gamma_g,\\
\widecheck{\mathrm{Div}(M)} &= O\left(r^{-7/2}\right)\big(\widecheck{e_3(r)}, \widecheck{e_4(r)}, \nabla(r)\big) + O\left(r^{-5/2}\right)\widecheck{\mathrm{Div}(\widehat{R})}\\
&= r^{-\frac{5}{2}}\Gamma_b + O\left(r^{-5/2}\right)\widecheck{\mathrm{Div}(\widehat{R})} = r^{-\frac{3}{2}}\Gamma_g + O\left(r^{-5/2}\right)\widecheck{\mathrm{Div}(\widehat{R})}
\end{aligned}
$$

we infer in view of Definition 9.2.17

$$
\mathrm{Err} = \frac{1}{2}\mathcal{Q}\cdot\widecheck{{}^{(X)}\pi} + O\left(r^{-5/2}\right)\widecheck{\mathrm{Div}(\widehat{R})}|\psi|^2 + \mathrm{Good}.
$$

Also, recall from the proof of Lemma 9.2.22 that $\widecheck{{}^{(\widehat{R})}\pi} \in \Gamma_b$ so that $\widecheck{\mathrm{Div}(\widehat{R})} \in \Gamma_b$ and hence

$$
O\left(r^{-5/2}\right)\widecheck{\mathrm{Div}(\widehat{R})}|\psi|^2 = O\left(r^{-5/2}\right)\Gamma_b|\psi|^2 = O\left(r^{-3/2}\right)\Gamma_g|\psi|^2 = \mathrm{Good}
$$

so that

$$
\mathrm{Err} = \frac{1}{2}\mathcal{Q}\cdot\widecheck{{}^{(X)}\pi} + \mathrm{Good}.
$$

Also, in view of the definition of X, we have

$$
\begin{aligned}
\widecheck{{}^{(X)}\pi_{\mu\nu}} &= O(1)\widecheck{{}^{(\widehat{R})}\pi_{\mu\nu}} + O(r^{-1})\big(\widecheck{e_3(r)}, \widecheck{e_4(r)}, \nabla r\big) = O(1)\widecheck{{}^{(\widehat{R})}\pi_{\mu\nu}} + \Gamma_b,\\
\widecheck{{}^{(X)}\pi_{44}} &= O(1)\widecheck{{}^{(\widehat{R})}\pi_{44}} + O(r^{-1})\widecheck{e_4(r)} = O(1)\widecheck{{}^{(\widehat{R})}\pi_{44}} + \Gamma_g,
\end{aligned}
$$

which together with Lemma 9.2.22 implies

$$
\mathcal{Q}\cdot\widecheck{{}^{(X)}\pi} = \mathrm{Good}
$$

and hence

$$
\mathrm{Err} = \mathrm{Good}.
$$

This yields

$$
\begin{aligned}
&\frac{1}{2}\mathcal{Q}\cdot{}^{(X)}\pi - \frac{1}{2}X(V)|\psi|^2 + \frac{1}{2}w\mathcal{L}[\psi] - \frac{1}{4}|\psi|^2\square_{\mathbf{g}}w + \frac{1}{4}\mathrm{Div}(|\psi|^2M)\\
=\ &\left[\frac{1}{2}\mathcal{Q}\cdot{}^{(X)}\pi - \frac{1}{2}X(V)|\psi|^2 + \frac{1}{2}w\mathcal{L}[\psi] - \frac{1}{4}|\psi|^2\square_{\mathbf{g}}w + \frac{1}{4}\mathrm{Div}(|\psi|^2M)\right]_K + \mathrm{Good}
\end{aligned}
$$

and thus

$$\begin{aligned}\mathbf{D}^\mu\mathcal{P}_\mu[X,w,M] &= \left[\frac{1}{2}\mathcal{Q}\cdot {}^{(X)}\pi - \frac{1}{2}X(V)|\psi|^2 + \frac{1}{2}w\mathcal{L}[\psi] - \frac{1}{4}|\psi|^2\square_{\mathbf{g}}w + \frac{1}{4}\mathrm{Div}(|\psi|^2 M)\right]_K\\ &\quad -\left({}^*\rho + \underline{\eta}\wedge\eta\right)\nabla_{X^4 e_4 - X^3 e_3}\psi\cdot {}^*\psi\\ &\quad -\frac{1}{2}\Im\left(\mathrm{tr}\underline{X}HX^3 + \mathrm{tr}X\,\underline{H}X^4\right)\cdot\nabla\psi\cdot {}^*\psi\\ &\quad +\left(\nabla_X\psi + \frac{1}{2}w\psi\right)\cdot\left(\dot{\square}_2 - V\psi\right) + \mathrm{Good}\end{aligned}$$

as stated. This concludes the proof of Lemma 9.2.24. □

Proof of Proposition 9.2.12

We choose (X, w, M) as in Chapter 7:

1. X is given by

$$X = \mathcal{F}\frac{(r^2+a^2)}{\Delta}\widehat{R}, \qquad \mathcal{F} = -zhf,$$
$$z = \frac{\Delta}{(r^2+a^2)^2}, \qquad f = -\frac{2\mathcal{T}}{(r^2+a^2)^3}, \qquad h = \frac{(r^2+a^2)^4}{r(r^2-a^2)},$$

   i.e. z, f, h correspond to the choices made in Proposition 7.1.8 and X is given by

$$X = \frac{2\mathcal{T}}{r(r^2-a^2)}\widehat{R}.$$

2. The scalar function $w(r)$ is given, as in Proposition 7.1.8, by

$$w = -z\partial_r(hf).$$

3. M is given by

$$M = v(r)\frac{(r^2+a^2)}{\Delta}\widehat{R}$$

   where the scalar function $v(r)$ is the one of Lemma 7.2.4 and satisfies in particular $v(r) = O(m^{1/2}\Delta r^{-9/2})$.

Next, we introduce the expression

$$\begin{aligned}\mathcal{E}_K[X,w,M] := \Bigg[&\frac{1}{2}\mathcal{Q}\cdot {}^{(X)}\pi - \frac{1}{2}X(V)|\psi|^2 + \frac{1}{2}w\mathcal{L}[\psi] - \frac{1}{4}|\psi|^2\square_{\mathbf{g}}w \\ &+ \frac{1}{4}\mathrm{Div}(|\psi|^2 M)\Bigg]_K.\end{aligned} \tag{9.2.14}$$

Notice that $\mathcal{E}_K[X,w,M]$ coincides in fact with the quantity $\mathcal{E}[X,w,M]$ in (7.1.8). Thus, we have according to Proposition 7.1.5

$$|q|^2\mathcal{E}_K[X,w,M] \;=\; \mathcal{A}|\nabla_r\psi|^2 + P + \mathcal{V}|\psi|^2 + \frac{1}{4}|q|^2\Big[\mathbf{D}^\mu(|\psi|^2 M_\mu)\Big]_K,$$

where the coefficients $\mathcal{A}$ and $\mathcal{V}$ are given by (7.1.27) and (7.1.28) respectively, and the principal term P given by (7.1.26), i.e.

$$P \;=\; \frac{\mathcal{T}}{r}\frac{r^2+a^2}{r^2-a^2}\left(\frac{2\mathcal{T}}{(r^2+a^2)^3}|q|^2|\nabla\psi|^2 - \frac{4ar}{(r^2+a^2)^2}\nabla_{\widehat{T}}\psi\cdot\nabla_{\mathbf{Z}}\psi\right).$$

Next, as in (7.2.7), we define, for $\delta > 0$ sufficiently small chosen later, the following quadratic form

$$\begin{aligned}\mathrm{Qr}_\delta[\psi] =& (1-\delta)\mathcal{A}|\nabla_r\psi|^2 + \left(\mathcal{V} + (1-\delta)\frac{2\mathcal{T}^2}{(r^2+a^2)^2(r^2-a^2)}\right)|\psi|^2 \\ &+ \frac{1}{4}|q|^2\Big[\mathbf{D}^\mu(|\psi|^2 M_\mu)\Big]_K\end{aligned} \tag{9.2.15}$$

so that, in view of the above, we have

$$|q|^2\mathcal{E}_K[X,w,M] \;=\; \delta\mathcal{A}|\nabla_r\psi|^2 + \delta P + (1-\delta)\left(P - \frac{2\mathcal{T}^2}{r(r^2+a^2)^2(r^2-a^2)}|\psi|^2\right) + \mathrm{Qr}_\delta[\psi].$$

Together with Lemma 9.2.23, we obtain the following analog of (7.2.8)

$$\begin{aligned}\int_S \mathcal{E}_K[X,w,M] \;\geq\;& \delta\int_S\left(\frac{m}{r^2}|\nabla_{\widehat{R}}\psi|^2 + \frac{\mathcal{T}^2}{r^7}|\nabla\psi|^2\right) + \int_S\frac{1}{|q|^2}\mathrm{Qr}_\delta[\psi] \\ &-O((a+\epsilon)r^{-1})\int_S\left(|\nabla\psi|^2 + r^{-2}|\nabla_{\mathbf{T}}\psi|^2\right) - O(\epsilon r^{-3})\int_S|\nabla_3\psi|^2 \\ &-O(\epsilon r^{-1} + ar^{-2})\int_S\left(r^{-2}|\psi|^2\right).\end{aligned} \tag{9.2.16}$$

Also, in view of the definition of $\mathrm{Qr}_\delta[\psi]$ in (9.2.15), Lemma 7.2.4 applies to $\mathrm{Qr}_\delta[\psi]$, and hence, for $|a|/m \ll 1$ and $\delta > 0$ sufficiently small, we have

$$\mathrm{Qr}_\delta[\psi] \;\geq\; O(\delta)\left(m\big|\nabla_{\widehat{R}}\Phi|^2 + r^{-1}|\Phi|^2\right).$$

Together with (9.2.16), we deduce the following analog of Proposition 7.2.5.

Proposition 9.2.25. *There exists a small universal constant $c_0 > 0$ such that, for $|a|/m \ll 1$ and $\epsilon > 0$ small enough, we have on $\mathcal{M}$*

$$\begin{aligned}\int_S \mathcal{E}_K[X,w,M] \quad &\geq \quad c_0 \int_S \left(\frac{m}{r^2}|\nabla_{\widehat{R}}\psi|^2 + \frac{\mathcal{T}^2}{r^7}|\nabla\psi|^2 + r^{-3}|\psi|^2\right) \\ &\quad -O((a+\epsilon)r^{-1})\int_S \left(|\nabla\psi|^2 + r^{-2}|\nabla_{\mathbf{T}}\psi|^2\right) - O(\epsilon r^{-3})\int_S |\nabla_3\psi|^2.\end{aligned}$$

In view of Lemma 9.2.24 and (9.2.14), we have

$$\begin{aligned}\mathbf{D}^\mu\mathcal{P}_\mu[X,w,M] \quad &= \quad \mathcal{E}_K[X,w,M] - \left({}^*\rho + \underline{\eta}\wedge\eta\right)\nabla_{X^4e_4 - X^3e_3}\psi\cdot{}^*\psi \\ &\quad -\frac{1}{2}\Im\Big(\mathrm{tr}\underline{X}HX^3 + \mathrm{tr}X\,\underline{H}X^4\Big)\cdot\nabla\psi\cdot{}^*\psi \\ &\quad +\left(\nabla_X\psi + \frac{1}{2}w\psi\right)\cdot\left(\dot{\square}_2 - V\psi\right) + \mathrm{Good}\end{aligned}$$

with $\mathcal{E}_K[X,w,M]$ satisfying the estimate of Proposition 9.2.25. Also, following the computation leading to (7.1.9), we have

$$\nabla_{X^4e_4 - X^3e_3} = \mathcal{F}\frac{r^2+a^2}{\Delta}\nabla_{\widehat{T}}\psi$$

and

$$\begin{aligned}\Im\Big(\mathrm{tr}\underline{X}HX^3 + \mathrm{tr}X\,\underline{H}X^4\Big)\cdot\nabla \quad &= \quad \frac{4a^2r\cos\theta\mathcal{F}(r)}{|q|^4}\Re(\mathfrak{J})\cdot\nabla + r^{-1}\Gamma_b\nabla \\ &= \quad \frac{4a^2r\cos\theta\mathcal{F}(r)}{(r^2+a^2)|q|^4}\left(\nabla_{\mathbf{Z}} + a(\sin\theta)^2\frac{r^2+a^2}{|q|^2}\nabla_{\widehat{T}}\right) + r^{-1}\Gamma_b\nabla \\ &= \quad \frac{4a^2r\cos\theta\mathcal{F}(r)}{(r^2+a^2)|q|^4}\nabla_Z + \frac{4a^3r\cos\theta(\sin\theta)^2\mathcal{F}(r)}{|q|^6}\nabla_{\widehat{T}} + r^{-1}\Gamma_b\nabla\end{aligned}$$

which yields

$$\begin{aligned}\mathbf{D}^\mu\mathcal{P}_\mu[X,w,M] \quad &= \quad \mathcal{E}_K[X,w,M] - \left({}^*\rho + \underline{\eta}\wedge\eta\right)\nabla_{X^4e_4 - X^3e_3}\psi\cdot{}^*\psi \\ &\quad -\left(\left({}^*\rho + \underline{\eta}\wedge\eta\right)\frac{r^2+a^2}{\Delta} + \frac{2a^3r\cos\theta(\sin\theta)^2}{|q|^6}\right)\mathcal{F}\nabla_{\widehat{T}}\psi\cdot{}^*\psi \\ &\quad -\frac{2a^2r\cos\theta\mathcal{F}(r)}{(r^2+a^2)|q|^4}\nabla_{\mathbf{Z}}\psi\cdot{}^*\psi + \left(\nabla_X\psi + \frac{1}{2}w\psi\right)\cdot\left(\dot{\square}_2 - V\psi\right) + \mathrm{Good}\end{aligned}$$

where we used the fact that $r^{-1}\Gamma_b\psi\nabla\psi$ is of type Good.

Next, following Section 7.2.2, we consider

$$\begin{aligned}\mathcal{P}_\mu[0,w',0] &= \frac{1}{2}w'\psi\cdot\dot{\mathbf{D}}_\mu\psi - \frac{1}{4}|\psi|^2\partial_\mu w',\\ w' &= -\delta_1\frac{4m\Delta\mathcal{T}^2}{r^2(r^2+a^2)^4},\end{aligned}$$

for some $\delta_1>0$ small enouh. Lemma 9.2.24 applies, and yields

$$\mathbf{D}^\mu\mathcal{P}_\mu[0,w',0] = \left[\frac{1}{2}w'\mathcal{L}[\psi] - \frac{1}{4}|\psi|^2\square_{\mathbf{g}}w'\right]_K + \frac{1}{2}w'\psi\cdot\big(\dot{\square}_2 - V\psi\big) + \text{Good}.$$

We introduce the expression

$$\mathcal{E}_K[0,w',0] := \left[\frac{1}{2}w'\mathcal{L}[\psi] - \frac{1}{4}|\psi|^2\square_{\mathbf{g}}w'\right]_K \tag{9.2.17}$$

which yields

$$\begin{aligned}\mathbf{D}^\mu\mathcal{P}_\mu[X,w+w',M] &= \mathcal{E}_K[X,w,M] + \mathcal{E}_K[0,w',0]\\ &\quad -\left(\big({}^*\rho + \underline{\eta}\wedge\eta\big)\frac{r^2+a^2}{\Delta} + \frac{2a^3r\cos\theta(\sin\theta)^2}{|q|^6}\right)\mathcal{F}\nabla_{\widehat{T}}\psi\cdot{}^*\psi\\ &\quad -\frac{2a^2r\cos\theta\mathcal{F}(r)}{(r^2+a^2)|q|^4}\nabla_{\mathbf{Z}}\psi\cdot{}^*\psi\\ &\quad +\left(\nabla_X\psi + \frac{1}{2}(w+w')\psi\right)\cdot\big(\dot{\square}_2 - V\psi\big) + \text{Good}.\end{aligned} \tag{9.2.18}$$

Also, in view of Section 7.2.2, we have

$$\begin{aligned}|q|^2\mathcal{E}_K[0,w',0] &= \frac{1}{2}\frac{w'}{\Delta}(r^2+a^2)^2|\nabla_{\widehat{R}}\psi|^2 - \frac{w'(r^2+a^2)^2}{2\Delta}|\nabla_{\widehat{T}}\psi|^2 + \frac{1}{2}w'|q|^2|\nabla\psi|^2\\ &\quad -\frac{1}{2}\left(\frac{1}{2}|q|^2\big[\square_{\mathbf{g}}w'\big]_K - |q|^2w'V\right)|\psi|^2.\end{aligned}$$

Proposition 9.2.25 together with the choice of w' easily yields, for $\delta_1>0$ small enough,

$$\begin{aligned}&\int_S\Big(\mathcal{E}_K[X,w,M] + \mathcal{E}_K[0,w',0]\Big)\\ \geq\ & c_0\int_S\left(\frac{m}{r^2}|\nabla_{\widehat{R}}\psi|^2 + \frac{\mathcal{T}^2}{r^6}\left(\frac{m}{r^2}|\nabla_{\widehat{T}}\psi|^2 + r^{-1}|\nabla\psi|^2\right) + r^{-3}|\psi|^2\right)\\ & -O((a+\epsilon)r^{-1})\int_S\big(|\nabla\psi|^2 + r^{-2}|\nabla_{\mathbf{T}}\psi|^2\big) - O(\epsilon r^{-3})\int_S|\nabla_3\psi|^2.\end{aligned} \tag{9.2.19}$$

Finally, we derive the following analog of Lemma 7.2.2.

Lemma 9.2.26. *We have, for arbitrarily small positive constants δ_2, δ_3, to be fixed later:*

$$
\begin{aligned}
\left(\nabla_X\psi+\frac{1}{2}w\psi\right)\cdot\left(\dot{\square}_2\psi-V\psi\right) \geq& -\delta_2\frac{m}{r^2}|\nabla_{\widehat{R}}\psi|^2-\delta_2\frac{\mathcal{T}^2}{r^6}\frac{m}{r^2}|\nabla_{\widehat{T}}\psi|^2\\
&+O(1)(|\nabla_{\widehat{R}}\psi|+r^{-1}|\psi|)|N|\\
&+O(a^3r^{-6})|\nabla_{\mathbf{Z}}\psi|^2+O(ar^{-4})|\psi|^2+Good\\
&+\mathbf{D}_\mu\left(\frac{2a\cos\theta}{|q|^2}(\widehat{R})^\mu(r^2+a^2)\frac{z}{\Delta}hf\psi\cdot\nabla_T\,{}^*\psi\right)\\
&-\mathbf{D}_\mu\left(\mathbf{T}^\mu\frac{2a\cos\theta}{|q|^2}\frac{z}{\Delta}(r^2+a^2)hf\psi\cdot\nabla_{\widehat{R}}\,{}^*\psi\right)
\end{aligned}
\tag{9.2.20}
$$

and

$$
\begin{aligned}
&\left|\left(\left({}^*\rho+\underline{\eta}\wedge\eta\right)\frac{r^2+a^2}{\Delta}+\frac{2a^3r\cos\theta(\sin\theta)^2}{|q|^6}\right)\mathcal{F}\nabla_{\widehat{T}}\psi\cdot{}^*\psi\right|\\
&+\left|\frac{2a^2r\cos\theta\mathcal{F}(r)}{(r^2+a^2)|q|^4}\nabla_{\mathbf{Z}}\psi\cdot{}^*\psi\right|\\
\leq\ & \delta_3\frac{\mathcal{T}^2}{r^6}\left(\frac{m}{r^2}|\nabla_{\widehat{T}}\psi|^2+\frac{1}{r^4}|\nabla_{\mathbf{Z}}\psi|^2\right)+O(a^2r^{-6})|\psi|^2+Good.
\end{aligned}
\tag{9.2.21}
$$

Proof. According to equation (6.1.1), we have

$$
\left(\nabla_X\psi+\frac{1}{2}w\psi\right)\cdot\left(\dot{\square}_2\psi-V\psi\right) = \left(\nabla_X\psi+\frac{1}{2}w\psi\right)\cdot\left(-\frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_T\psi+N\right).
$$

We have

$$
\begin{aligned}
&-\frac{4a\cos\theta}{|q|^2}\left(\nabla_X\psi+\frac{1}{2}w\psi\right)\cdot\left({}^*\nabla_T\psi\right)\\
=\ & -\left[\frac{4a\cos\theta}{|q|^2}\left(\nabla_X\psi+\frac{1}{2}w\psi\right)\cdot\left({}^*\nabla_T\psi\right)\right]_K\\
&+O(r^{-2})\,{}^{(\widehat{R})}\pi_{\mu\nu}\psi\cdot\nabla_T\,{}^*\psi+O(r^{-2})\,{}^{(\mathbf{T})}\pi_{\mu\nu}\psi\cdot\nabla_{\widehat{R}}\,{}^*\psi+O(r^{-2})\psi\cdot{}^*[\nabla_{\widehat{R}},\nabla_{\mathbf{T}}]\psi\\
&+O(r^{-3})\left(\widehat{R}(r),\widehat{R}(\cos\theta),\mathbf{T}(r),\mathbf{T}(\cos\theta)\right)|\psi|^2
\end{aligned}
$$

where the first term in the RHS is the one computed in the proof of Lemma 7.2.2, i.e it verifies

$$
\begin{aligned}
\left[\left(\nabla_X\psi+\frac{1}{2}w\psi\right)\cdot\left(-\frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_T\psi\right)\right]_K \geq\ & -\delta_2\frac{\mathcal{T}^2}{r^6}\frac{m}{r^2}|\nabla_{\widehat{T}}\psi|^2+O(a^3r^{-6})|\nabla_{\mathbf{Z}}\psi|^2+O(ar^{-4})|\psi|^2\\
&+\mathbf{D}_\mu\left(\frac{2a\cos\theta}{|q|^2}(\widehat{R})^\mu(r^2+a^2)\frac{z}{\Delta}hf\psi\cdot\nabla_T\,{}^*\psi\right)\\
&-\mathbf{D}_\mu\left(\mathbf{T}^\mu\frac{2a\cos\theta}{|q|^2}\frac{z}{\Delta}(r^2+a^2)hf\psi\cdot\nabla_{\widehat{R}}\,{}^*\psi\right).
\end{aligned}
$$

Also, since $\widetilde{{}^{(\widehat{R})}\pi_{\mu\nu}}, {}^{(\mathbf{T})}\pi_{\mu\nu} \in \Gamma_b$, $\widetilde{\widehat{R}(r)}, \mathbf{T}(r) \in r\Gamma_b$, and $\widehat{R}(\cos\theta), \mathbf{T}(\cos\theta) \in \Gamma_b$, we have

$$
\begin{aligned}
& -\frac{4a\cos\theta}{|q|^2}\left(\nabla_X\psi + \frac{1}{2}w\psi\right)\cdot\left({}^*\nabla_T\psi\right) \\
= & -\Big[\frac{4a\cos\theta}{|q|^2}\left(\nabla_X\psi + \frac{1}{2}w\psi\right)\cdot\left({}^*\nabla_T\psi\right)\Big]_K + r^{-2}\Gamma_b\cdot\psi\cdot\nabla_T{}^*\psi + r^{-2}\Gamma_b\cdot\psi\cdot\nabla_{\widehat{R}}{}^*\psi \\
& + O(r^{-2})\psi\cdot{}^*\widetilde{[\nabla_{\widehat{R}}, \nabla_{\mathbf{T}}]}\psi + r^{-2}\Gamma_b|\psi|^2.
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
[\nabla_{\widehat{R}}, \nabla_{\mathbf{T}}]\psi &= \nabla_{[\widehat{R},\mathbf{T}]}\psi + \mathbf{R}^{bc}{}_{\mu\nu}\psi_{cd}\widehat{R}^\mu\mathbf{T}^\nu \\
&= \Gamma_b\cdot(\nabla_3, \nabla_4, \nabla)\psi + \widehat{R}^3\mathbf{R}^{bc}{}_{3\nu}\psi_{cd}\mathbf{T}^\nu + \widehat{R}^4\mathbf{R}^{bc}{}_{4\nu}\psi_{cd}\mathbf{T}^\nu
\end{aligned}
$$

and hence

$$
\begin{aligned}
\widetilde{[\nabla_{\widehat{R}}, \nabla_{\mathbf{T}}]}\psi &= \Gamma_b\cdot(\nabla_3, \nabla_4, \nabla)\psi + \Big(O(1)\,\widecheck{{}^*\rho} + O(ar^{-1})(\underline{\beta}, \beta)\Big)\psi \\
&= \Gamma_b\cdot(\nabla_3, \nabla_4, \nabla)\psi + r^{-1}\Gamma_g\cdot\psi
\end{aligned}
$$

which yields

$$
\begin{aligned}
& -\frac{4a\cos\theta}{|q|^2}\left(\nabla_X\psi + \frac{1}{2}w\psi\right)\cdot\left({}^*\nabla_T\psi\right) \\
= & -\Big[\frac{4a\cos\theta}{|q|^2}\left(\nabla_X\psi + \frac{1}{2}w\psi\right)\cdot\left({}^*\nabla_T\psi\right)\Big]_K + r^{-2}\Gamma_b\cdot\psi\cdot(\nabla_3, \nabla_4, \nabla)\psi + r^{-2}\Gamma_b|\psi|^2.
\end{aligned}
$$

In view of Definition 9.2.17 for Good, we infer

$$
-\frac{4a\cos\theta}{|q|^2}\left(\nabla_X\psi + \frac{1}{2}w\psi\right)\cdot\left({}^*\nabla_T\psi\right) = -\Big[\frac{4a\cos\theta}{|q|^2}\left(\nabla_X\psi + \frac{1}{2}w\psi\right)\cdot\left({}^*\nabla_T\psi\right)\Big]_K + \text{Good}.
$$

Together with the above, we infer

$$
\begin{aligned}
\left(\nabla_X\psi + \frac{1}{2}w\psi\right)\cdot\left(-\frac{4a\cos\theta}{|q|^2}{}^*\nabla_T\psi\right) \geq & -\delta_2\frac{\mathcal{T}^2}{r^6}\frac{m}{r^2}|\nabla_{\widehat{T}}\psi|^2 + O(a^3r^{-6})|\nabla_{\mathbf{Z}}\psi|^2 + O(ar^{-4})|\psi|^2 \\
& + \mathbf{D}_\mu\left(\frac{2a\cos\theta}{|q|^2}(\widehat{R})^\mu(r^2+a^2)\frac{z}{\Delta}hf\psi\cdot\nabla_T{}^*\psi\right) \\
& - \mathbf{D}_\mu\left(\mathbf{T}^\mu\frac{2a\cos\theta}{|q|^2}\frac{z}{\Delta}(r^2+a^2)hf\psi\cdot\nabla_{\widehat{R}}{}^*\psi\right) + \text{Good}.
\end{aligned}
$$

As in the proof of Lemma 7.2.2, we bound the second product by

$$
\Big|\left(\nabla_X\psi + \frac{1}{2}w\psi\right)\cdot N\Big| \lesssim \Big(|\nabla_{\widehat{R}}\psi| + r^{-1}|\psi|\Big)|N|.
$$

By putting together with the previous bound we obtain the first desired estimate.

Next, since $\widecheck{{}^*\rho} \in r^{-1}\Gamma_g$, $\widecheck{\eta} \in \Gamma_b$, and $\widecheck{\underline{\eta}} \in \Gamma_g$, and since $\frac{r^2+a^2}{\Delta}\mathcal{F}$ is bounded, we have

$$({}^*\rho + \underline{\eta} \wedge \eta)\frac{r^2+a^2}{\Delta}\mathcal{F}\nabla_{\widehat{T}}\psi \cdot {}^*\psi = \left[({}^*\rho + \underline{\eta} \wedge \eta)\frac{r^2+a^2}{\Delta}\mathcal{F}\nabla_{\widehat{T}}\psi \cdot {}^*\psi\right]_K + \text{Good}.$$

Also, proceeding as in Lemma 7.2.2, we have

$$\begin{aligned} &\left|\left[({}^*\rho + \underline{\eta} \wedge \eta)\frac{r^2+a^2}{\Delta}\mathcal{F}\nabla_{\widehat{T}}\psi \cdot {}^*\psi\right]_K\right| + \left|\frac{2a^3 r\cos\theta(\sin\theta)^2}{|q|^6}\mathcal{F}\nabla_{\widehat{T}}\psi \cdot {}^*\psi\right| \\ &+\left|\frac{2a^2 r\cos\theta\mathcal{F}(r)}{(r^2+a^2)|q|^4}\nabla_{\mathbf{Z}}\psi \cdot {}^*\psi\right| \\ \leq\ & \delta_3\frac{\mathcal{T}^2}{r^6}\left(\frac{m}{r^2}|\nabla_{\widehat{T}}\psi|^2 + \frac{1}{r^4}|\nabla_{\mathbf{Z}}\psi|^2\right) + O(a^2 r^{-6})|\psi|^2 \end{aligned}$$

and hence

$$\begin{aligned} &\left|\left(({}^*\rho + \underline{\eta} \wedge \eta)\frac{r^2+a^2}{\Delta} + \frac{2a^3 r\cos\theta(\sin\theta)^2}{|q|^6}\right)\mathcal{F}\nabla_{\widehat{T}}\psi \cdot {}^*\psi\right| \\ &+\left|\frac{2a^2 r\cos\theta\mathcal{F}(r)}{(r^2+a^2)|q|^4}\nabla_{\mathbf{Z}}\psi \cdot {}^*\psi\right| \\ \leq\ & \delta_3\frac{\mathcal{T}^2}{r^6}\left(\frac{m}{r^2}|\nabla_{\widehat{T}}\psi|^2 + \frac{1}{r^4}|\nabla_{\mathbf{Z}}\psi|^2\right) + O(a^2 r^{-6})|\psi|^2 + \text{Good} \end{aligned}$$

as stated. □

By applying the divergence theorem to (9.2.18), relying on the lower bound (9.2.19), and using Lemma 7.2.2, we then obtain the following estimate

$$\begin{aligned} &\int_{\mathcal{M}(\tau_1,\tau_2)} \frac{m}{r^2}|\nabla_{\widehat{R}}\psi|^2 + r^{-3}|\psi|^2 + \frac{\mathcal{T}^2}{r^6}\left(\frac{m}{r^2}|\nabla_{\widehat{T}}\psi|^2 + r^{-1}|\nabla\psi|^2\right) \\ \lesssim\ & \int_{\partial\mathcal{M}(\tau_1,\tau_2)} |\mathcal{P}\cdot N_\Sigma| + \int_{\mathcal{M}(\tau_1,\tau_2)} \Big((a+\epsilon)r^{-1}\big(|\nabla\psi|^2 + r^{-2}|\nabla_{\mathbf{T}}\psi|^2\big) + \epsilon r^{-3}|\nabla_3\psi|^2 + \text{Good}\Big) \\ &+ \int_{\mathcal{M}(\tau_1,\tau_2)} \Big(|\nabla_{\widehat{R}}\psi| + r^{-1}|\psi|\Big)|N|. \end{aligned}$$

Since the extra terms in perturbations of Kerr satisfy

$$\begin{aligned}
&\int_{\mathcal{M}(\tau_1,\tau_2)} \Big(\epsilon r^{-1}\big(|\nabla\psi|^2 + r^{-2}|\nabla_{\mathbf{T}}\psi|^2 + r^{-2}|\nabla_3\psi|^2\big) + \text{Good}\Big)\\
\lesssim\ & \epsilon\int_{\mathcal{M}(\tau_1,\tau_2)} \Big(r^{-3}|\nabla_3\psi|^2 + r^{-4}|\mathfrak{d}\psi|^2\Big) + \epsilon\left(\sup_{[\tau_1,\tau_2]} E[\psi](\tau) + B_\delta[\psi](\tau_1,\tau_2)\right)\\
\lesssim\ & \epsilon\left(\sup_{[\tau_1,\tau_2]} E[\psi](\tau) + B^1_\delta[\psi](\tau_1,\tau_2)\right),
\end{aligned}$$

where we used in particular Remark 9.2.19, this concludes the proof of Proposition 9.2.12.

Proof of Proposition 9.2.13

We start with the following analog of Lemma 7.3.1.

Lemma 9.2.27. *The following hold true with a sufficiently small $c_0 > 0$, for any $|a| \ll m$,*

$$\begin{aligned}
\int_{\Sigma(\tau)} \mathcal{Q}(\widehat{T}, N_\Sigma) &\geq c_0 E_{deg}[\psi](\tau) - O(\delta_{\mathcal{H}}) E_{r\leq r_+(1+\delta_{\mathcal{H}})}[\psi](\tau),\\
\int_{\Sigma_*(\tau_1,\tau_2)} \mathcal{Q}(\widehat{T}, N_{\Sigma_*}) &\geq c_0 F_{\Sigma_*}[\psi](\tau_1,\tau_2), \qquad (9.2.22)\\
\int_{\mathcal{A}(\tau_1,\tau_2)} \mathcal{Q}(\widehat{T}, N_{\mathcal{A}}) &\gtrsim -\delta_{\mathcal{H}} F_{\mathcal{A}}[\psi](\tau_1,\tau_2).
\end{aligned}$$

Proof. For $\epsilon > 0$ sufficiently small, satisfying in particular $\epsilon \ll \delta_{\mathcal{H}}$, proceeding as in the proof of Lemma 7.3.1, we obtain

$$\begin{aligned}
\int_{\Sigma(\tau)} \mathcal{Q}(\widehat{T}, N_\Sigma) &\geq c_0 \int_{\Sigma(\tau)} \left(|\nabla_4\psi|^2 + \frac{|\Delta|}{r^4}|\nabla_3\psi|^2 + |\nabla\psi|^2\right) - O(\delta_{\mathcal{H}}) E_{r\leq r_+}[\psi](\tau),\\
\int_{\Sigma_*(\tau_1,\tau_2)} \mathcal{Q}(\widehat{T}, N_{\Sigma_*}) &\geq c_0 F_{\Sigma_*}[\psi](\tau_1,\tau_2),\\
\int_{\mathcal{A}(\tau_1,\tau_2)} \mathcal{Q}(\widehat{T}, N_{\mathcal{A}}) &\gtrsim -\delta_{\mathcal{H}} F_{\mathcal{A}}[\psi](\tau_1,\tau_2).
\end{aligned}$$

In particular, we have obtained the desired estimates on Σ_* and $\mathcal{A}$.

Also, we have in view of Lemma 9.2.23, for $\psi \in \mathfrak{s}_2$ and $|a| \ll m$,

$$\begin{aligned}\frac{1}{r^2}\int_S |\psi|^2 &\lesssim \int_S \left(|\nabla\psi|^2 + r^{-2}|\nabla_{\mathbf{T}}\psi|^2 + \epsilon r^{-2}|\nabla_3\psi|^2\right)\\ &\lesssim \int_S \left(|\nabla_4\psi|^2 + \frac{|\Delta|}{r^4}|\nabla_3\psi|^2 + |\nabla\psi|^2 + \epsilon r^{-2}|\nabla_3\psi|^2\right).\end{aligned}$$

Together with the above, we infer, for $\epsilon \ll \delta_{\mathcal{H}}$,

$$\begin{aligned}\int_{\Sigma(\tau)} \mathcal{Q}(\widehat{T}, N_\Sigma) \;\geq\; & c_0 \int_{\Sigma(\tau)} \left(|\nabla_4\psi|^2 + \frac{|\Delta|}{r^4}|\nabla_3\psi|^2 + |\nabla\psi|^2 + r^{-2}|\psi|^2\right)\\ & -O(\delta_{\mathcal{H}})E_{r\leq r_+(1+\delta_{\mathcal{H}})}[\psi](\tau),\end{aligned}$$

and hence

$$\int_{\Sigma(\tau)} \mathcal{Q}(\widehat{T}, N_\Sigma) \;\geq c_0 E_{deg}[\psi](\tau) - O(\delta_{\mathcal{H}})E_{r\leq r_+(1+\delta_{\mathcal{H}})}[\psi](\tau)$$

as stated. This concludes the proof of Lemma 9.2.27. □

We consider the energy current associated to the modified timelike vectorfield $\widehat{T}_\delta$ introduced in Definition 6.1.13 by

$$\widehat{T}_\delta = \mathbf{T} + \chi_\delta \mathbf{Z}, \qquad \chi_\delta = \frac{a}{r^2+a^2}\chi_0\left(\delta^{-1}\frac{\mathcal{T}}{r^3}\right),$$

with $\delta = \frac{1}{10}$ and $|a|/m \ll 1$ small enough, where $\chi_\delta = \frac{a}{r^2+a^2}\chi_0\Big(\delta^{-1}\frac{\mathcal{T}}{r^3}\Big)$ with χ_0 given by (6.1.11) satisfying in particular $\chi_0 = 0$ in $\mathcal{M}_{trap}$.

From Proposition 4.7.2, we have for the current associated to $\widehat{T}_\delta$:

$$\mathbf{D}^\mu \mathcal{P}_\mu[\widehat{T}_\delta, 0, 0] = \frac{1}{2}\mathcal{Q}\cdot {}^{(\widehat{T}_\delta)}\pi - \frac{1}{2}\widehat{T}_\delta(V)|\psi|^2 + \widehat{T}_\delta^\mu \dot{\mathbf{D}}^\nu\psi^a \mathbf{R}_{ab\nu\mu}\psi^b + \nabla_{\widehat{T}_\delta}\psi\cdot\big(\dot{\square}_2\psi - V\psi\big).$$

Since

$$\begin{aligned}\widehat{T}_\delta(V) &= O(r^{-3})\Big(\widehat{T}_\delta(r),\, \widehat{T}_\delta(\cos\theta)\Big) = O(r^{-3})\Big(\mathbf{T}(r),\, \mathbf{T}(\cos\theta),\, \nabla(r),\, \widetilde{\nabla(\cos\theta)}\Big)\\ &= r^{-2}\Gamma_b\end{aligned}$$

and hence $\widehat{T}_\delta(V)|\psi|^2 = \text{Good}$ which yields

$$\mathbf{D}^\mu \mathcal{P}_\mu[\widehat{T}_\delta, 0, 0] = \frac{1}{2}\mathcal{Q}\cdot {}^{(\widehat{T}_\delta)}\pi + \widehat{T}_\delta^\mu \dot{\mathbf{D}}^\nu\psi^a \mathbf{R}_{ab\nu\mu}\psi^b + \nabla_{\widehat{T}_\delta}\psi\cdot\big(\dot{\square}_2\psi - V\psi\big) + \text{Good}$$

or

$$\begin{aligned}\mathbf{D}^\mu \mathcal{P}_\mu[\widehat{T}_\delta, 0, 0] &= \frac{1}{2}\mathcal{Q}\cdot {}^{(\widehat{T}_\delta)}\pi + \mathbf{T}^\mu \dot{\mathbf{D}}^\nu \psi^a \dot{\mathbf{R}}_{ab\nu\mu}\psi^b + (\widehat{T}_\delta - \mathbf{T})^\mu \dot{\mathbf{D}}^\nu \psi^a \dot{\mathbf{R}}_{ab\nu\mu}\psi^b \\ &\quad + \nabla_{\widehat{T}_\delta}\psi\cdot\big(\dot{\square}_2\psi - V\psi\big) + \text{Good}.\end{aligned}$$

Since $\dot{\mathbf{R}}_{ab\nu\mu}$ is antisymmetric with respect to (a,b), we rewrite

$$\begin{aligned}\mathbf{D}^\mu \mathcal{P}_\mu[\widehat{T}_\delta, 0, 0] &= \frac{1}{2}\mathcal{Q}\cdot {}^{(\widehat{T}_\delta)}\pi + \frac{1}{2}\mathbf{T}^\mu \in^{ab} \dot{\mathbf{R}}_{ab\nu\mu}\ {}^*\psi\cdot\dot{\mathbf{D}}^\nu\psi + (\widehat{T}_\delta - \mathbf{T})^\mu \dot{\mathbf{D}}^\nu \psi^a \dot{\mathbf{R}}_{ab\nu\mu}\psi^b \\ &\quad + \nabla_{\widehat{T}_\delta}\psi\cdot\big(\dot{\square}_2\psi - V\psi\big) + \text{Good}.\end{aligned}$$

Introducing the following spacetime 1-form

$$A_\mu \ := \ \in^{bc} \dot{\mathbf{R}}_{bc\mu\nu}\mathbf{T}^\nu, \tag{9.2.23}$$

we infer the following analog of (7.3.3)

$$\begin{aligned}\mathbf{D}^\mu \mathcal{P}_\mu[\widehat{T}_\delta, 0, 0] &= \frac{1}{2}\mathcal{Q}\cdot {}^{(\widehat{T}_\delta)}\pi + \frac{1}{2}A_\nu\ {}^*\psi\cdot\dot{\mathbf{D}}^\nu\psi + (\widehat{T}_\delta - \mathbf{T})^\mu \dot{\mathbf{D}}^\nu \psi^a \dot{\mathbf{R}}_{ab\nu\mu}\psi^b \\ &\quad + \nabla_{\widehat{T}_\delta}\psi\cdot\big(\dot{\square}_2\psi - V\psi\big) + \text{Good}.\end{aligned} \tag{9.2.24}$$

Next, we compute the components of A.

Lemma 9.2.28. *Let A the spacetime 1-form given by* (9.2.23). *Then, we have*

$$\begin{aligned}A_4 &= -4\ {}^*\rho\mathbf{T}^3 - 4(\underline{\eta}\wedge\eta)\mathbf{T}^3 + tr\,\chi\big({}^{(h)}\mathbf{T}\wedge\underline{\eta}\big) - {}^{(a)}tr\chi\big(\underline{\eta}\cdot{}^{(h)}\mathbf{T}\big) + r^{-1}\Gamma_g,\\ A_3 &= 4\ {}^*\rho\mathbf{T}^4 + 4(\underline{\eta}\wedge\eta)\mathbf{T}^4 + tr\,\underline{\chi}\big({}^{(h)}\mathbf{T}\wedge\eta\big) - {}^{(a)}tr\underline{\chi}\big(\eta\cdot{}^{(h)}\mathbf{T}\big) + r^{-2}\Gamma_b,\\ A_e &= \Big(-tr\,\underline{\chi}\ {}^*\eta_e + {}^{(a)}tr\underline{\chi}\eta_e\Big)\mathbf{T}^3 + \Big(-tr\,\chi\ {}^*\underline{\eta}_e + {}^{(a)}tr\chi\underline{\eta}_e\Big)\mathbf{T}^4\\ &\quad -\frac{1}{2}\Big(4\rho + tr\,\chi tr\underline{\chi} + {}^{(a)}tr\chi\,{}^{(a)}tr\underline{\chi}\Big)\ {}^*({}^{(h)}\mathbf{T})_e + r^{-1}\Gamma_b.\end{aligned}$$

Proof. First, note that we have in view of Proposition 2.2.4 and the definition of Γ_b and Γ_g

$$\begin{aligned}&\mathbf{B}_{abc3} = -\mathbf{B}_{ab3c} = -\mathrm{tr}\,\underline{\chi}\big(\delta_{ca}\eta_b - \delta_{cb}\eta_a\big) - {}^{(a)}\mathrm{tr}\underline{\chi}\big(\in_{ca}\eta_b - \in_{cb}\eta_a\big) + r^{-1}\Gamma_b,\\ &\mathbf{B}_{abc4} = -\mathbf{B}_{abc4c} = -\mathrm{tr}\,\chi\big(\delta_{ca}\underline{\eta}_b - \delta_{cb}\underline{\eta}_a\big) - {}^{(a)}\mathrm{tr}\chi\big(\in_{ca}\underline{\eta}_b - \in_{cb}\underline{\eta}_a\big) + r^{-1}\Gamma_g,\\ &\mathbf{B}_{ab34} = -\mathbf{B}_{ab43} = 4(\underline{\eta}_a\eta_b - \eta_a\underline{\eta}_b) + \Gamma_b\cdot\Gamma_g,\\ &\mathbf{B}_{abcd} = -\mathbf{B}_{abdc} = -\frac{1}{2}\big(\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} + {}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi}\big)\in_{ab}\in_{cd} + r^{-1}\Gamma_b.\end{aligned} \tag{9.2.25}$$

Next, we rewrite A_μ as

$$A_\mu \;=\; \in^{bc} \dot{\mathbf{R}}_{bc\mu 3}\mathbf{T}^3 + \in^{bc} \dot{\mathbf{R}}_{bc\mu 4}\mathbf{T}^4 + \in^{bc} \dot{\mathbf{R}}_{bc\mu d}\mathbf{T}^d.$$

and compute the various components of A_μ. We have, using the horizontal tensor ${}^{(h)}\mathbf{T}$ defined by ${}^{(h)}\mathbf{T}_b = \mathbf{T}_b$, the definition (2.1.13) of $\dot{\mathbf{R}}$, (9.2.25), and the computations of the main terms in the proof of Lemma 7.3.2,

$$\begin{aligned}
A_4 &= \in^{bc} \dot{\mathbf{R}}_{bc43}\mathbf{T}^3 + \in^{bc} \dot{\mathbf{R}}_{bc4d}\mathbf{T}^d,\\
&= -4\ {}^*\rho\mathbf{T}^3 - 4(\underline{\eta}\wedge\eta)\mathbf{T}^3 + \mathrm{tr}\,\chi\big({}^{(h)}\mathbf{T}\wedge\underline{\eta}\big) - {}^{(a)}\mathrm{tr}\chi\big(\underline{\eta}\cdot{}^{(h)}\mathbf{T}\big)\\
&\quad +O(\mathbf{T}^3)\Gamma_b\cdot\Gamma_g + O(|{}^{(h)}\mathbf{T}|)\big(\beta, r^{-1}\Gamma_g\big)\\
&= -4\ {}^*\rho\mathbf{T}^3 - 4(\underline{\eta}\wedge\eta)\mathbf{T}^3 + \mathrm{tr}\,\chi\big({}^{(h)}\mathbf{T}\wedge\underline{\eta}\big) - {}^{(a)}\mathrm{tr}\chi\big(\underline{\eta}\cdot{}^{(h)}\mathbf{T}\big) + r^{-1}\Gamma_g,
\end{aligned}$$

$$\begin{aligned}
A_3 &= \in^{bc} \dot{\mathbf{R}}_{bc34}\mathbf{T}^4 + \in^{bc} \dot{\mathbf{R}}_{bc3d}\mathbf{T}^d\\
&= 4\ {}^*\rho\mathbf{T}^4 + 4(\underline{\eta}\wedge\eta)\mathbf{T}^4 + \mathrm{tr}\,\underline{\chi}\big({}^{(h)}\mathbf{T}\wedge\eta\big) - {}^{(a)}\mathrm{tr}\underline{\chi}\big(\eta\cdot{}^{(h)}\mathbf{T}\big)\\
&\quad +O(\mathbf{T}^4)\Gamma_b\cdot\Gamma_g + O(|{}^{(h)}\mathbf{T}|)\big(\underline{\beta}, r^{-1}\Gamma_b\big)\\
&= 4\ {}^*\rho\mathbf{T}^4 + 4(\underline{\eta}\wedge\eta)\mathbf{T}^4 + \mathrm{tr}\,\underline{\chi}\big({}^{(h)}\mathbf{T}\wedge\eta\big) - {}^{(a)}\mathrm{tr}\underline{\chi}\big(\eta\cdot{}^{(h)}\mathbf{T}\big) + r^{-2}\Gamma_b,
\end{aligned}$$

and

$$\begin{aligned}
A_e &= \in^{bc} \dot{\mathbf{R}}_{bce3}\mathbf{T}^3 + \in^{bc} \dot{\mathbf{R}}_{bce4}\mathbf{T}^4 + \in^{bc} \dot{\mathbf{R}}_{bced}\mathbf{T}^d\\
&= \Big(-\mathrm{tr}\,\underline{\chi}\ {}^*\eta_e + {}^{(a)}\mathrm{tr}\underline{\chi}\eta_e\Big)\mathbf{T}^3 + \Big(-\mathrm{tr}\,\chi\ {}^*\underline{\eta}_e + {}^{(a)}\mathrm{tr}\chi\underline{\eta}_e\Big)\mathbf{T}^4\\
&\quad -\frac{1}{2}\Big(4\rho + \mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} + {}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi}\Big)\ {}^*({}^{(h)}\mathbf{T})_e\\
&\quad +O(\mathbf{T}^3)\big(\underline{\beta}, r^{-1}\Gamma_b\big) + O(\mathbf{T}^4)\big(\beta, r^{-1}\Gamma_g\big) + O(|{}^{(h)}\mathbf{T}|)r^{-1}\Gamma_b\\
&= \Big(-\mathrm{tr}\,\underline{\chi}\ {}^*\eta_e + {}^{(a)}\mathrm{tr}\underline{\chi}\eta_e\Big)\mathbf{T}^3 + \Big(-\mathrm{tr}\,\chi\ {}^*\underline{\eta}_e + {}^{(a)}\mathrm{tr}\chi\underline{\eta}_e\Big)\mathbf{T}^4\\
&\quad -\frac{1}{2}\Big(4\rho + \mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} + {}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi}\Big)\ {}^*({}^{(h)}\mathbf{T})_e + r^{-1}\Gamma_b
\end{aligned}$$

as stated. This concludes the proof of Lemma 9.2.28. □

We infer the following corollary.

Corollary 9.2.29. *We have*

$$A_\mu = -\mathbf{D}_\mu\left(\Im\left(\frac{2m}{q^2}\right)\right) + r^{-1}\Gamma_b. \tag{9.2.26}$$

Proof. We have

$$\mathbf{T}^4=\frac{1}{2},\qquad \mathbf{T}^3=\frac{1}{2}\frac{\Delta}{|q|^2},\qquad \mathbf{T}_b={}^{(h)}\mathbf{T}_b=-a\Re(\mathfrak{J})_b.$$

Plugging in the identities of Lemma 9.2.28, we infer

$$\begin{aligned}
A_4 &= -2\ {}^*\rho\frac{\Delta}{|q|^2}-2(\underline{\eta}\wedge\eta)\frac{\Delta}{|q|^2}-a\mathrm{tr}\,\chi\big(\Re(\mathfrak{J})\wedge\underline{\eta}\big)+a\,{}^{(a)}\mathrm{tr}\chi\big(\underline{\eta}\cdot\Re(\mathfrak{J})\big)+r^{-1}\Gamma_g,\\
A_3 &= 2\ {}^*\rho+2(\underline{\eta}\wedge\eta)-a\mathrm{tr}\,\underline{\chi}\big(\Re(\mathfrak{J})\wedge\eta\big)+a\,{}^{(a)}\mathrm{tr}\underline{\chi}\big(\eta\cdot\Re(\mathfrak{J})\big)+r^{-2}\Gamma_b,\\
A_e &= \frac{1}{2}\Big(-\mathrm{tr}\,\underline{\chi}\ {}^*\eta_e+{}^{(a)}\mathrm{tr}\underline{\chi}\eta_e\Big)\frac{\Delta}{|q|^2}+\frac{1}{2}\Big(-\mathrm{tr}\,\chi\ {}^*\underline{\eta}_e+{}^{(a)}\mathrm{tr}\chi\underline{\eta}_e\Big)\\
&\quad+\frac{a}{2}\Big(4\rho+\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}+{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi}\Big)\ {}^*\Re(\mathfrak{J})_e+r^{-1}\Gamma_b.
\end{aligned}$$

Next, since $\widecheck{\mathrm{tr}X},\widecheck{\mathrm{tr}\underline{X}}$, $\widecheck{\underline{H}}\in\Gamma_g$, $\check{\rho}$, $\widecheck{{}^*\rho}\in r^{-1}\Gamma_g$ and $\widecheck{H}\in\Gamma_b$, and using the proof of the main terms in the proof of Corollary 7.3.3, we infer

$$\begin{aligned}
A_e &= -a\Re\left(\frac{4m}{q^3}\right)\ {}^*\Re(\mathfrak{J})_e+r^{-1}\Gamma_b,\\
A_4 &= 4\Im\left(\frac{m}{q^3}\right)\frac{\Delta}{|q|^2}+r^{-1}\Gamma_g,\\
A_3 &= -4\Im\left(\frac{m}{q^3}\right)+r^{-1}\Gamma_g.
\end{aligned}$$

Using again the proof of the main terms in the proof of Corollary 7.3.3, and since $\widecheck{\nabla(q)}\in r\Gamma_g$, $\widecheck{e_3(q)}\in r\Gamma_b$ and $\widecheck{e_4(q)}\in r^{-1}\Gamma_g$, we infer

$$\begin{aligned}
A_e &= -\nabla_e\left(\Im\left(\frac{2m}{q^2}\right)\right)+r^{-1}\Gamma_b,\\
A_4 &= -e_4\left(\Im\left(\frac{2m}{q^2}\right)\right)+r^{-1}\Gamma_g,\\
A_3 &= -e_3\left(\Im\left(\frac{2m}{q^2}\right)\right)+r^{-1}\Gamma_g,
\end{aligned}$$

and hence

$$A_\mu=-\mathbf{D}_\mu\left(\Im\left(\frac{2m}{q^2}\right)\right)+r^{-1}\Gamma_b$$

as stated. This concludes the proof of Corollary 9.2.29. □

Recall from Definition 9.2.17 that the terms $G_\mu \psi \cdot \dot{\mathbf{D}}^\mu \psi$ with $G_\mu \in \Gamma_g$ belong to Good. Given that the terms $G_\mu \,{}^*\psi \cdot \dot{\mathbf{D}}^\mu \psi$ with $G_\mu \in \Gamma_g$ satisfy the same estimates, i.e the ones of Lemma 9.2.18, we will from now on also assume that they are part of Good. In particular, in view of Corollary 9.2.29, this implies

$$\frac{1}{2} A_\nu \,{}^*\psi \cdot \dot{\mathbf{D}}^\nu \psi = -\mathbf{D}_\nu \left(\Im \left(\frac{m}{q^2} \right) \right) {}^*\psi \cdot \dot{\mathbf{D}}^\nu \psi + \text{Good}$$

which together with (9.2.24) implies

$$\begin{aligned} \mathbf{D}^\mu \mathcal{P}_\mu[\widehat{T}_\delta, 0, 0] &= \frac{1}{2} \mathcal{Q} \cdot {}^{(\widehat{T}_\delta)}\pi - \mathbf{D}_\nu \left(\Im \left(\frac{m}{q^2} \right) \right) {}^*\psi \cdot \dot{\mathbf{D}}^\nu \psi + (\widehat{T}_\delta - \mathbf{T})^\mu \dot{\mathbf{D}}^\nu \psi^a \dot{\mathbf{R}}_{ab\nu\mu} \psi^b \\ &\quad + \nabla_{\widehat{T}_\delta} \psi \cdot \big(\dot{\square}_2 \psi - V\psi \big) + \text{Good}. \end{aligned} \tag{9.2.27}$$

Next, we modify the identity (9.2.27) to cancel the second term on the RHS. To this end, we consider the following modified current

$$\widetilde{\mathcal{P}}_\mu := \mathcal{P}_\mu[\widehat{T}_\delta, 0, 0] + \tilde{w} \,{}^*\psi \cdot \dot{\mathbf{D}}_\mu \psi, \tag{9.2.28}$$

for a scalar function $\tilde{w} = \tilde{w}(r, \cos\theta)$ to be chosen below and satisfying $w = O(mar^{-3})$. We have

$$\begin{aligned} \mathbf{D}^\mu \Big[\tilde{w} \,{}^*\psi \cdot \dot{\mathbf{D}}_\mu \psi \Big] &= \tilde{w} \,{}^*\psi \cdot \dot{\mathbf{D}}^\mu \dot{\mathbf{D}}_\mu \psi + \tilde{w} \,{}^*\dot{\mathbf{D}}^\mu \psi \cdot \dot{\mathbf{D}}_\mu \psi + \mathbf{D}^\mu(\tilde{w}) \,{}^*\psi \cdot \dot{\mathbf{D}}_\mu \psi \\ &= \tilde{w} \,{}^*\psi \cdot \dot{\square}_2 \psi + \mathbf{D}^\mu(\tilde{w}) \,{}^*\psi \cdot \dot{\mathbf{D}}_\mu \psi \\ &= \tilde{w} \,{}^*\psi \cdot \left(V\psi - \frac{4a\cos\theta}{|q|^2} \,{}^*\nabla_{\mathbf{T}} \psi + N \right) + \mathbf{D}^\mu(\tilde{w}) \,{}^*\psi \cdot \dot{\mathbf{D}}_\mu \psi \\ &= -\tilde{w} \frac{4a\cos\theta}{|q|^2} \nabla_{\mathbf{T}} (|\,{}^*\psi|^2) + \tilde{w} \,{}^*\psi \cdot N + \mathbf{D}^\mu(\tilde{w}) \,{}^*\psi \cdot \dot{\mathbf{D}}_\mu \psi. \end{aligned}$$

Since $\tilde{w} = \tilde{w}(r, \cos\theta)$ and $w = O(mar^{-3})$, we have

$$\begin{aligned} -\tilde{w} \frac{4a\cos\theta}{|q|^2} \nabla_{\mathbf{T}} (|\,{}^*\psi|^2) &= -\mathbf{D}_\mu \left(\mathbf{T}^\mu \tilde{w} \frac{4a\cos\theta}{|q|^2} |\psi|^2 \right) + O(a^2 r^{-6}) \Big(\mathbf{T}(r), \mathbf{T}(\cos\theta) \Big) |\psi|^2 \\ &\quad + O(a^2 r^{-5}) \text{Div}(\mathbf{T}) |\psi|^2 \\ &= -\mathbf{D}_\mu \left(\mathbf{T}^\mu \tilde{w} \frac{4a\cos\theta}{|q|^2} |\psi|^2 \right) + r^{-5} \Gamma_b |\psi|^2 \\ &= -\mathbf{D}_\mu \left(\mathbf{T}^\mu \tilde{w} \frac{4a\cos\theta}{|q|^2} |\psi|^2 \right) + \text{Good} \end{aligned}$$

and hence

$$\begin{aligned} \mathcal{D}^\mu \left(\widetilde{\mathcal{P}}_\mu + \mathbf{T}^\mu \tilde{w} \frac{4a\cos\theta}{|q|^2} |\psi|^2 \right) &= \frac{1}{2} \mathcal{Q} \cdot {}^{(\widehat{T}_\delta)}\pi + \mathbf{D}_\nu \left(-\Im \left(\frac{m}{q^2} \right) + \tilde{w} \right) {}^*\psi \cdot \dot{\mathbf{D}}^\nu \psi \\ &\quad + (\widehat{T}_\delta - \mathbf{T})^\mu \dot{\mathbf{D}}^\nu \psi^a \dot{\mathbf{R}}_{ab\nu\mu} \psi^b \\ &\quad + \nabla_{\widehat{T}_\delta} \psi \cdot \big(\dot{\square}_2 \psi - V\psi \big) + \tilde{w} \,{}^*\psi \cdot N + \text{Good}. \end{aligned}$$

Next, we make the following choice for $\tilde{w}$

$$\tilde{w} := \Im\left(\frac{m}{q^2}\right) = -\frac{2amr\cos\theta}{|q|^4} \tag{9.2.29}$$

which yields

$$\begin{aligned}\mathcal{D}^\mu\left(\widetilde{\mathcal{P}}_\mu + \mathbf{T}^\mu\tilde{w}\frac{4a\cos\theta}{|q|^2}|\psi|^2\right) =& \frac{1}{2}\mathcal{Q}\cdot {}^{(\widehat{T}_\delta)}\pi + (\widehat{T}_\delta - \mathbf{T})^\mu\dot{\mathbf{D}}^\nu\psi^a\dot{\mathbf{R}}_{ab\nu\mu}\psi^b\\ &+\nabla_{\widehat{T}_\delta}\psi\cdot\left(\dot{\square}_2\psi - V\psi\right) + \tilde{w}\,{}^*\psi\cdot N + \text{Good}.\end{aligned} \tag{9.2.30}$$

Next, recalling that $\widehat{T}_\delta = \mathbf{T} + \chi_\delta\mathbf{Z}$, we have

$$\begin{aligned}{}^{(\widehat{T}_\delta)}\pi_{\mu\nu} &= {}^{(\mathbf{T})}\pi_{\mu\nu} + \chi_\delta\,{}^{(\mathbf{Z})}\pi_{\mu\nu} + \chi_\delta'(r)\mathbf{D}_\mu(r)\mathbf{Z}_\nu + \chi_\delta'(r)\mathbf{D}_\nu(r)\mathbf{Z}_\mu\\ &= {}^{(\mathbf{T})}\pi_{\mu\nu} + O(r^{-2})\,{}^{(\mathbf{Z})}\pi_{\mu\nu} + \chi_\delta'(r)\mathbf{D}_\mu(r)\mathbf{Z}_\nu + \chi_\delta'(r)\mathbf{D}_\nu(r)\mathbf{Z}_\mu.\end{aligned}$$

Together with the control of ${}^{(\mathbf{T})}\pi$ and ${}^{(\mathbf{Z})}\pi$ provided by Lemma 4.3.2, and the fact that $\widetilde{e_3(r)} \in r\Gamma_b$, $\widetilde{e_4(r)} \in r\Gamma_g$, $\nabla(r) \in r\Gamma_g$, and $\chi_\delta'(r) = O(r^{-3})$, we infer

$$\mathcal{Q}\cdot {}^{(\widehat{T}_\delta)}\pi = \left[\mathcal{Q}\cdot {}^{(\widehat{T}_\delta)}\pi\right]_K + \text{Good}.$$

In view of the estimate for $[\mathcal{Q}\cdot {}^{(\widehat{T}_\delta)}\pi]_K$ in section 7.3, we infer

$$\left|\mathcal{Q}\cdot {}^{(\widehat{T}_\delta)}\pi\right| \lesssim \mathbb{1}_{\mathcal{M}_{trap\!\!\!/}}\delta^{-1}\frac{|a|}{r^3}|\nabla_{\mathbf{Z}}\psi||\nabla_{\widehat{R}}\psi| + \text{Good}.$$

Also, we have $\widehat{T}_\delta - \mathbf{T} = \chi_\delta(r)\mathbf{Z}$ and hence

$$\begin{aligned}&(\widehat{T}_\delta - \mathbf{T})^\mu\dot{\mathbf{D}}^\nu\psi^a\dot{\mathbf{R}}_{ab\nu\mu}\psi^b\\ =& \frac{1}{2}\chi_\delta(r)\mathbf{Z}^\mu \in^{ab}\dot{\mathbf{R}}_{ab\nu\mu}\,{}^*\psi\cdot\dot{\mathbf{D}}^\nu\psi\\ =& \frac{1}{2}\chi_\delta(r)\left(-\frac{1}{2}\mathbf{Z}^\mu\in^{ab}\dot{\mathbf{R}}_{ab3\mu}\,{}^*\psi\cdot\nabla_4\psi - \frac{1}{2}\mathbf{Z}^\mu\in^{ab}\dot{\mathbf{R}}_{ab4\mu}\,{}^*\psi\cdot\nabla_3\psi + \mathbf{Z}^\mu\in^{ab}\dot{\mathbf{R}}_{abc\mu}\,{}^*\psi\cdot\nabla^c\psi\right).\end{aligned}$$

Since we have

$$\mathbf{Z}^3 = O(ar^{-2}\Delta), \qquad \mathbf{Z}^4 = O(a), \qquad \mathbf{Z}^c = O(r^2)\Re(\mathfrak{J})^c,$$

we infer, together with the definition (2.1.13) of $\dot{\mathbf{R}}$, and (9.2.25),

$$\begin{aligned}\mathbf{Z}^\mu\in^{ab}\dot{\mathbf{R}}_{ab3\mu} &= \mathbf{Z}^4\in^{ab}\dot{\mathbf{R}}_{ab34} + \mathbf{Z}^c\in^{ab}\dot{\mathbf{R}}_{ab3c} = O(ar^{-2}) + \Gamma_b,\\ \mathbf{Z}^\mu\in^{ab}\dot{\mathbf{R}}_{ab4\mu} &= \mathbf{Z}^3\in^{ab}\dot{\mathbf{R}}_{ab43} + \mathbf{Z}^c\in^{ab}\dot{\mathbf{R}}_{ab4c} = O(ar^{-4}\Delta) + \Gamma_g,\\ \mathbf{Z}^\mu\in^{ab}\dot{\mathbf{R}}_{abc\mu} &= \mathbf{Z}^4\in^{ab}\dot{\mathbf{R}}_{abc4} + \mathbf{Z}^3\in^{ab}\dot{\mathbf{R}}_{abc3} + \mathbf{Z}^d\in^{ab}\dot{\mathbf{R}}_{abcd}\\ &= O(r^{-1}) + \Gamma_b,\end{aligned}$$

which implies, since $\chi_\delta = \frac{a}{r^2+a^2}\chi_0\Big(\delta^{-1}\frac{\mathcal{T}}{r^3}\Big)$,

$$\Big|(\widehat{T}_\delta - \mathbf{T})^\mu \dot{\mathbf{D}}^\nu \psi^a \dot{\mathbf{R}}_{ab\nu\mu}\psi^b\Big| \lesssim \frac{|a|}{r^3}\mathbb{1}_{\mathcal{M}_{trap}}\Big(|\nabla_{\widehat{R}}\psi| + |\nabla_{\mathbf{T}}\psi| + |\nabla\psi|\Big)|\psi| + |\text{Good}|.$$

Finally, using equation (6.1.1), we have

$$\nabla_{\widehat{T}_\delta}\psi \cdot \big(\dot{\square}_2\psi - V\psi\big) = -\frac{4a\cos\theta}{|q|^2}\chi_\delta \nabla_{\mathbf{Z}}\psi \cdot {}^*\nabla_{\mathbf{T}}\psi + \nabla_{\widehat{T}_\delta}\psi \cdot N$$

where we have the crucial cancellation $\nabla_{\mathbf{T}}\psi \cdot {}^*\nabla_{\mathbf{T}}\psi = 0$.

We summarize the result in the following.

Lemma 9.2.30. *Consider the modified current*

$$\widetilde{\mathcal{P}}_\mu = \mathcal{P}_\mu[\widehat{T}_\delta, 0, 0] + \tilde{w}\,{}^*\psi \cdot \dot{\mathbf{D}}_\mu\psi, \qquad \tilde{w} = \Im\left(\frac{m}{q^2}\right),$$

where the vectorfield $\widehat{T}_\delta$ is given by $\widehat{T}_\delta = \mathbf{T} + \chi_\delta(r)\mathbf{Z}$ for $\delta = \frac{1}{10}$ and $|a|/m \ll 1$ small enough. Then, we have on $\mathcal{M}$,

$$\begin{aligned}
&\left|\mathbf{D}^\mu\left(\widetilde{\mathcal{P}}_\mu + \mathbf{T}^\mu \tilde{w}\frac{4a\cos\theta}{|q|^2}|\psi|^2\right) - \Big(\nabla_{\widehat{T}_\delta}\psi + \tilde{w}\,{}^*\psi\Big)\cdot N\right| \\
\lesssim\ & \mathbb{1}_{\mathcal{M}_{trap}}\left(\delta^{-1}\frac{|a|}{r^3}|\nabla_{\widehat{R}}\psi||\nabla_{\mathbf{Z}}\psi| + \frac{|a|}{r^4}|\nabla_{\mathbf{T}}\psi||\nabla_{\mathbf{Z}}\psi| + \frac{|a|m}{r^4}\Big[|\nabla_{\widehat{R}}\psi| + |\nabla_{\mathbf{T}}\psi| + |\nabla\psi|\Big]|\psi|\right) \\
&+|\textit{Good}|.
\end{aligned}$$

Integrating the above inequality on $\mathcal{M} = \mathcal{M}(\tau_1,\tau_2)$ and applying the divergence theorem we deduce, in view of the definition of $\text{Mor}^{ax}_{deg}[\psi](\tau_1,\tau_2)$,

$$\begin{aligned}
&\int_{\Sigma(\tau_2)}\left(\widetilde{\mathcal{P}}_\mu + \mathbf{T}^\mu\tilde{w}\frac{4a\cos\theta}{|q|^2}|\psi|^2\right)\cdot N_\Sigma + \int_{\Sigma_*(\tau_1,\tau_2)}\left(\widetilde{\mathcal{P}}_\mu + \mathbf{T}^\mu\tilde{w}\frac{4a\cos\theta}{|q|^2}|\psi|^2\right)\cdot N_{\Sigma_*} \\
&+\int_{\mathcal{A}(\tau_1,\tau_2)}\left(\widetilde{\mathcal{P}}_\mu + \mathbf{T}^\mu\tilde{w}\frac{4a\cos\theta}{|q|^2}|\psi|^2\right)\cdot N_{\mathcal{A}} \\
\lesssim\ & \int_{\Sigma(\tau_1)}\left(\widetilde{\mathcal{P}}_\mu + \mathbf{T}^\mu\tilde{w}\frac{4a\cos\theta}{|q|^2}|\psi|^2\right)\cdot N_\Sigma + \frac{|a|}{m}\text{Mor}^{ax}_{deg}[\psi](\tau_1,\tau_2) + \left|\int_{\mathcal{M}(\tau_1,\tau_2)}\nabla_{\widehat{T}_\delta}\psi\cdot N\right| \\
&+\int_{\mathcal{M}(\tau_1,\tau_2)}|N|^2 + \int_{\mathcal{M}(\tau_1,\tau_2)}|\text{Good}|.
\end{aligned}$$

The rest of the proof is a simple adaptation of the one of Proposition 6.3.9, using Lemma 9.2.27 for the sign of boundary terms, and Remark 9.2.19 to control the term Good. This concludes the proof of Proposition 9.2.13.

Proof of Proposition 9.2.14

The proof of Proposition 9.2.14 is a simple adaptation of the one of Proposition 7.4.1, where the extra error terms are tracked as in the proof of Propositions 9.2.12 and 9.2.13.

9.2.11 Proof of Proposition 9.2.15 and Lemma 9.2.16

Let the second order differential operators $\mathcal{S}_{\underline{a}}$, $\underline{a} = 1, 2, 3, 4$, introduced in Definition 4.6.1. Given a $\mathfrak{s}_2$ tensor ψ solution of the equation (6.1.1), we consider in this section the commuted $\mathfrak{s}_2$ tensors

$$\psi_{\underline{a}} := \mathcal{S}_{\underline{a}}\psi, \qquad \underline{a} = 1, 2, 3, 4, \tag{9.2.31}$$

that satisfy according to Lemma 9.2.4 the equation (9.2.1), i.e.

$$\dot{\square}_2\psi_{\underline{a}} - V\psi_{\underline{a}} \quad = \quad -\frac{4a\cos\theta}{|q|^2}\, {}^*\nabla_{\mathbf{T}}\psi_{\underline{a}} + N_{\underline{a}} \tag{9.2.32}$$

where $N_{\underline{a}}$ satisfies (9.2.2).

Acceptable error terms for $\mathcal{S}$-Morawetz

To (9.2.1), we associate the following generalized energy-momentum tensor

$$\begin{aligned}
\mathcal{Q}_{\mu\nu}(\psi_{\underline{a}}, \psi_{\underline{b}}) &= \dot{\mathbf{D}}_\mu\psi_{\underline{a}} \cdot \dot{\mathbf{D}}_\nu\psi_{\underline{b}} - \frac{1}{2}\mathbf{g}_{\mu\nu}\left(\mathbf{g}^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi_{\underline{a}} \cdot \dot{\mathbf{D}}_\beta\psi_{\underline{b}} + V\psi_{\underline{a}} \cdot \psi_{\underline{b}}\right)\\
&= \dot{\mathbf{D}}_\mu\psi_{\underline{a}} \cdot \dot{\mathbf{D}}_\nu\psi_{\underline{b}} - \frac{1}{2}\mathbf{g}_{\mu\nu}\mathcal{L}[\psi_{\underline{a}}, \psi_{\underline{b}}],\\
\mathcal{L}[\psi_{\underline{a}}, \psi_{\underline{b}}] &= \mathbf{g}^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi_{\underline{a}} \cdot \dot{\mathbf{D}}_\beta\psi_{\underline{b}} + V\psi_{\underline{a}} \cdot \psi_{\underline{b}}.
\end{aligned}$$

When revisiting the proofs of Chapter 8 in the context of admissible perturbations $\mathcal{M}$ of Kerr, we generate additional terms. We introduce below acceptable terms.

Definition 9.2.31 (Acceptable error terms)**.** *The following quantity*

$$F^{\underline{ab}\,\mu\nu}\mathcal{Q}_{\mu\nu}(\psi_{\underline{a}}, \psi_{\underline{b}}) + G^{\underline{ab}\,\mu}\psi_{\underline{a}} \cdot \dot{\mathbf{D}}_\mu\psi_{\underline{b}} + H^{\underline{ab}}\psi_{\underline{a}} \cdot \psi_{\underline{b}} + I[\psi]$$

is said to be of the acceptable type, and denoted $Good_{\mathcal{S}}$, if:

- $F_{44}^{\underline{ab}} \in \Gamma_g$, *and all other components of* $F_{\mu\nu}^{\underline{ab}}$ *belong to* Γ_b.
- *All components of* $G_{\mu}^{\underline{ab}}$ *belong to* Γ_g.
- $H^{\underline{ab}} \in r^{-1}\Gamma_g$.
- $I[\psi] = \mathfrak{d}^{\leq 3}(\Gamma_g \cdot \psi) \cdot \nabla_3(\mathfrak{d}^{\leq 2}\psi) + r^{-1}\mathfrak{d}^{\leq 3}(\Gamma_g \cdot \psi) \cdot \mathfrak{d}^{\leq 3}\psi$.

The justification for Definition 9.2.17 is provided by the following lemma.

Lemma 9.2.32. *Assume that the quantity*

$$F^{\underline{ab}\,\mu\nu}\mathcal{Q}_{\mu\nu}(\psi_{\underline{a}}, \psi_{\underline{b}}) + G^{\underline{ab}\,\mu}\psi_{\underline{a}} \cdot \dot{\mathbf{D}}_\mu\psi_{\underline{b}} + H^{\underline{ab}}\psi_{\underline{a}} \cdot \psi_{\underline{b}} + I[\psi]$$

is of the acceptable type in the sense of Definition 9.2.31. Then, it satisfies the following estimate

$$\int_{\mathcal{M}_{trap}(\tau_1,\tau_2)} \Big|F^{\underline{ab}\,\mu\nu}\mathcal{Q}_{\mu\nu}(\psi_{\underline{a}}, \psi_{\underline{b}}) + G^{\underline{ab}\,\mu}\psi_{\underline{a}} \cdot \dot{\mathbf{D}}_\mu\psi_{\underline{b}} + H^{\underline{ab}}\psi_{\underline{a}} \cdot \psi_{\underline{b}} + I[\psi]\Big|^2 \lesssim \epsilon \sup_{[\tau_1,\tau_2]} E^2[\psi](\tau)$$

and

$$\int_{\mathcal{M}_{\not{trap}}(\tau_1,\tau_2)} \Big|F^{\underline{ab}\,\mu\nu}\mathcal{Q}_{\mu\nu}(\psi_{\underline{a}}, \psi_{\underline{b}}) + G^{\underline{ab}\,\mu}\psi_{\underline{a}} \cdot \dot{\mathbf{D}}_\mu\psi_{\underline{b}} + H^{\underline{ab}}\psi_{\underline{a}} \cdot \psi_{\underline{b}} + I[\psi]\Big|^2 \lesssim \epsilon B_\delta^2[\psi](\tau_1, \tau_2).$$

Remark 9.2.33. *Recall from Definition 9.2.31 that terms of the acceptable type are denoted* $Good_{\mathcal{S}}$. *In view of Lemma 9.2.32, we infer that such term satisfy the following estimate*

$$\int_{\mathcal{M}} |Good_{\mathcal{S}}| \lesssim \epsilon \left(\sup_{[\tau_1,\tau_2]} E^2[\psi](\tau) + B_\delta^2[\psi](\tau_1, \tau_2) \right).$$

Proof. The proof is analogous to the one of Lemma 9.2.18 noticing that $\psi_{\underline{a}}$ is schematically of the type $\mathfrak{d}^{\leq 2}\psi$. □

Next, we introduce the linearization of the quantities $F_{\mu\nu}^{\underline{ab}}$, $G_{\mu}^{\underline{ab}}$ and $H^{\underline{ab}}$.

Definition 9.2.34. *Let* $F_{\mu\nu}^{\underline{ab}}$, $G_{\mu}^{\underline{ab}}$ *and* $H^{\underline{ab}}$. *We define their linearization as follows*

$$F_{\mu\nu}^{\underline{ab}} = (F_{\mu\nu}^{\underline{ab}})_K + \widecheck{F_{\mu\nu}^{\underline{ab}}}, \qquad G_{\mu}^{\underline{ab}} = (G_{\mu}^{\underline{ab}})_K + \widecheck{G_{\mu}^{\underline{ab}}}, \qquad H^{\underline{ab}} = H_K^{\underline{ab}} + \widecheck{H^{\underline{ab}}},$$

where:

1. *the quantities*

$$(F^{\underline{ab}}_{44})_K, \qquad (F^{\underline{ab}}_{34})_K, \qquad (F^{\underline{ab}}_{33})_K, \qquad (G^{\underline{ab}}_{4})_K, \qquad (G^{\underline{ab}}_{3})_K, \qquad H^{\underline{ab}}_{K},$$

   are given as explicit functions of $(r, \cos\theta)$ *coinciding with the corresponding expressions in Kerr,*

2. *the quantities*

$$(F^{\underline{ab}}_{4a})_K, \qquad (F^{\underline{ab}}_{3a})_K, \qquad (G^{\underline{ab}}_{a})_K,$$

   are given as the 1-form $\Re(\mathfrak{J})_a$ *multiplied by explicit functions of* $(r, \cos\theta)$ *coinciding with the corresponding expressions in Kerr,*

3. *the quantity* $(F^{\underline{ab}}_{ab})_K$ *is given by the the symmetric 2-tensor* γ_{ab} *multiplied by explicit functions of* $(r, \cos\theta)$ *coinciding with the corresponding expressions in Kerr.*

In view of Definition 9.2.34, we can decompose the above expressions in their main part and error terms as follows

$$\begin{aligned}
&F^{\underline{ab}\,\mu\nu}\mathcal{Q}_{\mu\nu}(\psi_{\underline{a}}, \psi_{\underline{b}}) + G^{\underline{ab}\,\mu}\psi_{\underline{a}}\cdot\dot{\mathbf{D}}_\mu\psi_{\underline{b}} + H^{\underline{ab}}\psi_{\underline{a}}\cdot\psi_{\underline{b}} + I[\psi]\\
=&\Big[F^{\underline{ab}\,\mu\nu}\mathcal{Q}_{\mu\nu}(\psi_{\underline{a}}, \psi_{\underline{b}}) + G^{\underline{ab}\,\mu}\psi_{\underline{a}}\cdot\dot{\mathbf{D}}_\mu\psi_{\underline{b}} + H^{\underline{ab}}\psi_{\underline{a}}\cdot\psi_{\underline{b}}\Big]_K + \mathrm{Err},\\
&\Big[F^{\underline{ab}\,\mu\nu}\mathcal{Q}_{\mu\nu}(\psi_{\underline{a}}, \psi_{\underline{b}}) + G^{\underline{ab}\,\mu}\psi_{\underline{a}}\cdot\dot{\mathbf{D}}_\mu\psi_{\underline{b}} + H^{\underline{ab}}\psi_{\underline{a}}\cdot\psi_{\underline{b}}\Big]_K\\
=&(F^{\underline{ab}\,\mu\nu})_K\mathcal{Q}_{\mu\nu}[\psi] + (G^{\underline{ab}\,\mu})_K\psi\cdot\dot{\mathbf{D}}_\mu\psi + H^{\underline{ab}}_K|\psi|^2,\\
\mathrm{Err} =&\widetilde{F^{\underline{ab}\,\mu\nu}}\mathcal{Q}_{\mu\nu}[\psi] + \widetilde{G^{\underline{ab}\,\mu}}\psi\cdot\dot{\mathbf{D}}_\mu\psi + \widetilde{H^{\underline{ab}}}|\psi|^2 + I[\psi].
\end{aligned} \tag{9.2.33}$$

The proof of section 9.2.11 will rely in particular on showing that the extra terms in perturbations of Kerr appearing in the various divergence identities involved in energy and Morawetz estimates are of the acceptable type in the sense of Definition 9.2.31, i.e, that $\mathrm{Err} \in \mathrm{Good}_{\mathcal{S}}$ in (9.2.33).

Preliminaries for $\mathcal{S}$-Morawetz

We start with some preliminaries for the $\mathcal{S}$-Morawetz estimate. Let $\mathbf{X}$ be a double-indexed collection of vector fields $\mathbf{X} = \{\mathbf{X}^{\underline{\mathbf{ab}}}\}$, $\mathbf{w}$ be a double-indexed collection of functions $\mathbf{w} = \{w^{\underline{ab}}\}$, and $\mathbf{M} = \{M^{\underline{ab}}\}$ a double-indexed collection of 1-forms, all symmetric in the indices $\underline{a}, \underline{b}$. Assume furthermore that $\mathbf{X}$ are of the form

$$X^{\underline{ab}} = (X^{\underline{ab}})^3 e_3 + (X^{\underline{ab}})^4 e_4.$$

Define

$$\mathcal{P}_\mu[\mathbf{X},\mathbf{w},\mathbf{M}] := \mathcal{Q}_{\mu\nu}(\psi_{\underline{a}},\psi_{\underline{b}})X^{\underline{ab}\nu}+\frac{1}{2}w^{\underline{ab}}\dot{\mathbf{D}}_\mu\psi_{\underline{a}}\cdot\psi_{\underline{b}}-\frac{1}{4}(\partial_\mu w^{\underline{ab}})\psi_{\underline{a}}\cdot\psi_{\underline{b}}+\frac{1}{4}M^{\underline{ab}}_\mu\psi_{\underline{a}}\cdot\psi_{\underline{b}}.$$

We make the following choices for $(\mathbf{X},\mathbf{w},\mathbf{M})$, consistent with the ones in Chapter 8:

1. $\mathbf{X}$ of the type
$$X^{\underline{ab}} = \mathcal{F}^{\underline{ab}}\frac{(r^2+a^2)}{\Delta}\widehat{R},$$
where $\mathcal{F}^{\underline{ab}}=\mathcal{F}^{\underline{ab}}(r)$ are such that $\mathcal{F}^{\underline{ab}}/\Delta$ is a smooth function of r, and $(r\partial_r)^k\mathcal{F}^{\underline{ab}}(r)$ is uniformly bounded on $\mathcal{M}$. In particular, there holds $X^{\underline{ab}}=(X^{\underline{ab}})^3e_3+(X^{\underline{ab}})^4e_4$ as anticipated, with $(X^{\underline{ab}})^3$ and $(X^{\underline{ab}})^4$ smooth functions of r such that $(r\partial_r)^k(X^{\underline{ab}})^3$ and $(r\partial_r)^k(X^{\underline{ab}})^4$ are uniformly bounded on $\mathcal{M}$.

2. The scalar functions $w^{\underline{ab}}(r)$ are such that $w^{\underline{ab}}/\Delta$ is a smooth function of r and $(r\partial_r)^k(rw^{\underline{ab}}(r))$ is uniformly bounded on $\mathcal{M}$.

3. $M^{\underline{ab}}$ are of the type
$$M^{\underline{ab}} = v^{\underline{ab}}(r)\frac{(r^2+a^2)}{\Delta}\widehat{R}$$
where the scalar functions $v^{\underline{ab}}(r)$ are such that $v^{\underline{ab}}/\Delta$ is a smooth function of r and $(r\partial_r)^k(r^{\frac{5}{2}}v^{\underline{ab}}(r))$ is uniformly bounded on $\mathcal{M}$.

We have the following lemma.

Lemma 9.2.35. *Using the decomposition* (9.2.33), *we have*

$$\begin{aligned}\mathbf{D}^\mu\mathcal{P}_\mu[\mathbf{X},\mathbf{w},\mathbf{M}] &= \Bigg[\frac{1}{2}\mathcal{Q}_{\underline{ab}}\cdot{}^{(X^{\underline{ab}})}\pi-\frac{1}{2}X^{\underline{ab}}(V)\psi_{\underline{a}}\cdot\psi_{\underline{b}}+\frac{1}{2}w^{\underline{ab}}\mathcal{L}[\psi_{\underline{a}},\psi_{\underline{b}}]\\ &\quad -\frac{1}{4}\psi_{\underline{a}}\cdot\psi_{\underline{b}}\square_{\mathbf{g}}w^{\underline{ab}}+\frac{1}{4}Div(\psi_{\underline{a}}\cdot\psi_{\underline{b}}M^{\underline{ab}})\Bigg]_K\\ &\quad -\big({}^*\rho+\underline{\eta}\wedge\eta\big)\nabla_{(X^{\underline{ab}})^4e_4-(X^{\underline{ab}})^3e_3}\psi_{\underline{a}}\cdot{}^*\psi_{\underline{b}}\\ &\quad -\frac{1}{2}\Im\Big(tr\underline{X}H(X^{\underline{ab}})^3+trX\,\underline{H}(X^{\underline{ab}})^4\Big)\cdot\nabla\psi_{\underline{a}}\cdot{}^*\psi_{\underline{b}}\\ &\quad +\left(\nabla_{X^{\underline{ab}}}\psi_{\underline{a}}+\frac{1}{2}w^{\underline{ab}}\psi_{\underline{a}}\right)\cdot\big(\dot{\square}_2\psi_{\underline{b}}-V\psi_{\underline{b}}\big)+Good_{\mathcal{S}}\end{aligned}$$

where $Good_{\mathcal{S}}$ *is given by Definition 9.2.31.*

Proof. The proof is analogous to the one of Lemma 9.2.24. □

Integration by parts identities

We now generalize the results of section 8.2.2 to admissible perturbations of Kerr. We start with the following analog of Definition 8.2.1 that will be useful to take care of boundary terms in the integrations by parts.

Definition 9.2.36. *We denote by $M(\psi)$ quadratic terms of the following type*

$$\nabla_T\psi\cdot\mathcal{S}_{\underline{a}}\psi, \qquad |q|^2\nabla\psi\cdot\nabla\nabla_T\psi, \qquad |q|^2\nabla\psi\cdot\nabla\nabla_Z\psi,$$

Also, we denote by $M(\nabla_{\widehat{R}}\psi)$ denote quadratic terms of the following type

$$\nabla_T\nabla_{\widehat{R}}\psi\cdot\mathcal{S}_{\underline{a}}\nabla_{\widehat{R}}\psi, \qquad |q|^2\nabla\nabla_{\widehat{R}}\psi\cdot\nabla\nabla_T\nabla_{\widehat{R}}\psi, \qquad |q|^2\nabla\nabla_{\widehat{R}}\psi\cdot\nabla\nabla_Z\nabla_{\widehat{R}}\psi.$$

Remark 9.2.37. *Notice the following pointwise estimates for $M(\psi)$ and $M(\nabla_{\widehat{R}}\psi)$:*

$$\begin{aligned}|M(\psi)| &\lesssim |(\nabla_T,\, \not{\mathfrak{d}})^{\leq 1}\psi||(\nabla_T,\, \not{\mathfrak{d}})^{\leq 2}\psi|,\\ |M(\nabla_{\widehat{R}}\psi)| &\lesssim |\nabla_{\widehat{R}}^{\leq 1}(\nabla_T,\, \not{\mathfrak{d}})^{\leq 1}\psi||\nabla_{\widehat{R}}^{\leq 1}(\nabla_T,\, \not{\mathfrak{d}})^{\leq 2}\psi|.\end{aligned} \tag{9.2.34}$$

Lemma 9.2.38. *For any function $H=H(r)$, the following identities hold true:*

$$\begin{aligned}
H\mathcal{O}(\psi)\cdot\mathcal{S}_1\psi &= H|q|^2|\nabla\nabla_{\mathbf{T}}\psi|^2 - O(ar^{-2})H(\mathfrak{d}^{\leq 2}\psi)^2\\
&\quad +|q|^2\dot{\mathbf{D}}_\beta(H|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot\mathcal{S}_1\psi) + \dot{\mathbf{D}}_\mu(HM(\psi)\mathbf{T}^\mu)\\
&\quad +r^2(|H|+|H'|)Good_{\mathcal{S}},\\
H\nabla_{\widehat{R}}\mathcal{O}\psi\cdot\nabla_{\widehat{R}}\mathcal{S}_1\psi &= H|q|^2|\nabla\nabla_T\nabla_{\widehat{R}}\psi|^2 - O(ar^{-2})H(\nabla_{\widehat{R}}\mathfrak{d}^{\leq 2}\psi)^2 - O(ar^{-2})H(\mathfrak{d}^{\leq 2}\psi)^2\\
&\quad +|q|^2\dot{\mathbf{D}}_\beta(H|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\nabla_{\widehat{R}}\psi\cdot\mathcal{S}_1\nabla_{\widehat{R}}\psi) + \dot{\mathbf{D}}_\mu(HM(\nabla_{\widehat{R}}\psi)\mathbf{T}^\mu)\\
&\quad +r^2(|H|+|H'|)Good_{\mathcal{S}},\\
H\nabla_{\widehat{R}}\mathcal{O}(\psi)\cdot\mathcal{S}_1\psi &= H|q|^2\nabla\nabla_{\mathbf{T}}\nabla_{\widehat{R}}\psi\cdot\nabla\nabla_{\mathbf{T}}\psi - O(ar^{-2})H(\nabla_{\widehat{R}}\mathfrak{d}^{\leq 2}\psi)^2\\
&\quad -O(ar^{-2})H(\mathfrak{d}^{\leq 2}\psi)^2 + |q|^2\dot{\mathbf{D}}_\beta(H|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\nabla_{\widehat{R}}\psi\cdot\mathcal{S}_1\psi)\\
&\quad +\dot{\mathbf{D}}_\mu(HM(\nabla_{\widehat{R}}\psi)\mathbf{T}^\mu) + r^2(|H|+|H'|)Good_{\mathcal{S}},\\
H\mathcal{O}(\psi)\cdot\nabla_{\widehat{R}}\mathcal{S}_1\psi &= H|q|^2\nabla\nabla_{\mathbf{T}}\nabla_{\widehat{R}}\psi\cdot\nabla\nabla_{\mathbf{T}}\psi - O(ar^{-2})H(\nabla_{\widehat{R}}\mathfrak{d}^{\leq 2}\psi)^2\\
&\quad -O(ar^{-2})H(\mathfrak{d}^{\leq 2}\psi)^2 + |q|^2\dot{\mathbf{D}}_\beta(H|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot\mathcal{S}_1\nabla_{\widehat{R}}\psi)\\
&\quad +\dot{\mathbf{D}}_\mu(HM(\nabla_{\widehat{R}}\psi)\mathbf{T}^\mu) + r^2(|H|+|H'|)Good_{\mathcal{S}}.
\end{aligned}$$

In all the above, $M(\psi)$ and $M(\nabla_{\widehat{R}}\psi)$ denote the quadratic expressions in ψ and its derivatives of Definition 8.2.1.

Proof. First, we have in view of Lemma 4.6.2

$$|q|^2\dot{\mathbf{D}}_\beta(|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi) = \mathcal{O}(\psi) + r\Gamma_b\mathfrak{d}\psi.$$

We obtain for any Φ,

$$
\begin{aligned}
|q|^{-2}\mathcal{O}(\psi)\cdot\Phi &= \dot{\mathbf{D}}_\beta(|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi)\cdot\Phi + r^{-1}\Gamma_b\Phi\cdot\mathfrak{d}\psi\\
&= \dot{\mathbf{D}}_\beta(|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot\Phi) - |q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot\dot{\mathbf{D}}_\beta\Phi\\
&\quad + r^{-1}\Gamma_b\Phi\cdot\mathfrak{d}\psi.
\end{aligned}
\tag{9.2.35}
$$

Applying the above to $\Phi = \mathcal{S}_1\psi = \nabla_T\nabla_T\psi$, and using that

$$[\nabla, \nabla_T]\psi = O(ar^{-4})\,\not{\mathfrak{d}}^{\leq 1}\psi + r^{-1}\mathfrak{d}(\Gamma_b\cdot\psi)$$

from Corollary 9.2.2, $\mathrm{tr}^{(\mathbf{T})}\pi\in\Gamma_g$ from Lemma 4.3.2, and $\nabla(r)\in r\Gamma_g$ and $\mathbf{T}(r)\in r\Gamma_b$, we infer, proceeding as in Lemma 8.2.3,

$$
\begin{aligned}
H\mathcal{O}(\psi)\cdot\mathcal{S}_1\psi &= -H|q|^2\nabla\psi\cdot\nabla\nabla_{\mathbf{T}}\nabla_{\mathbf{T}}\psi + H|q|^2\dot{\mathbf{D}}_\beta(|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot\mathcal{S}_1\psi)\\
&\quad + Hr\mathfrak{d}(\Gamma_b\cdot\psi)\nabla_{\mathbf{T}}\mathfrak{d}^{\leq 1}\psi\\
&= H|q|^2|\nabla\nabla_{\mathbf{T}}\psi|^2 - O(ar^{-2})H(\mathfrak{d}^{\leq 2}\psi)^2 + |q|^2\dot{\mathbf{D}}_\beta(H|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot\mathcal{S}_1\psi)\\
&\quad + \dot{\mathbf{D}}_\mu(HM(\psi)\mathbf{T}^\mu) + H|q|^2\nabla\psi\cdot[\nabla_{\mathbf{T}},\nabla]\nabla_{\mathbf{T}}\psi + H|q|^2[\nabla_{\mathbf{T}},\nabla]\psi\cdot\nabla\nabla_{\mathbf{T}}\psi\\
&\quad + H|q|^2\nabla\psi\cdot\nabla\nabla_{\mathbf{T}}\psi\mathbf{D}_\mu\mathbf{T}^\mu + H'(r)\big(\mathbf{T}(r), r\nabla(r)\big)\mathfrak{d}^{\leq 1}\psi\nabla_{\mathbf{T}}\mathfrak{d}^{\leq 1}\psi\\
&\quad + Hr\mathfrak{d}(\Gamma_b\cdot\psi)\nabla_{\mathbf{T}}\mathfrak{d}^{\leq 1}\psi\\
&= |q|^2|\nabla\nabla_{\mathbf{T}}\psi|^2 - O(ar^{-2})(\mathfrak{d}^{\leq 2}\psi)^2 + |q|^2\dot{\mathbf{D}}_\beta(|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot\mathcal{S}_1\psi)\\
&\quad + \dot{\mathbf{D}}_\mu(HM(\psi)\mathbf{T}^\mu) + (|H|+|H'|)r^2\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\psi)\nabla_{\mathbf{T}}\mathfrak{d}^{\leq 1}\psi\\
&= H|q|^2|\nabla\nabla_{\mathbf{T}}\psi|^2 - O(ar^{-2})H(\mathfrak{d}^{\leq 2}\psi)^2 + |q|^2\dot{\mathbf{D}}_\beta(H|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot\mathcal{S}_1\psi)\\
&\quad + \dot{\mathbf{D}}_\mu(HM(\psi)\mathbf{T}^\mu) + r^2(|H|+|H'|)\mathrm{Good}_{\mathcal{S}}
\end{aligned}
$$

as stated, where we have used the fact that

$$\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\psi)\nabla_{\mathbf{T}}\mathfrak{d}^{\leq 1}\psi = \mathfrak{d}^{\leq 2}(\Gamma_g\cdot\psi)\nabla_3\mathfrak{d}^{\leq 1}\psi + r^{-1}\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\psi)\mathfrak{d}^{\leq 2}\psi = \mathrm{Good}_{\mathcal{S}}.$$

Similarly, using in addition $[\mathcal{O},\nabla_{\widehat{R}}]\psi = O(ar^{-2})\mathfrak{d}^{\leq 1}\psi + r\Gamma_b\mathfrak{d}^{\leq 2}\psi$ from Corollary 9.2.6, and $[\nabla_{\widehat{R}},\mathcal{S}_1] = O(amr^{-4})\mathfrak{d}^{\leq 1}\psi + \mathfrak{d}^{\leq 2}(\Gamma_b\cdot\psi)$ from Lemma 9.2.7, we have

$$
\begin{aligned}
& H\nabla_{\widehat{R}}\mathcal{O}\psi\cdot\nabla_{\widehat{R}}\mathcal{S}_1\psi\\
={}& H\mathcal{O}\nabla_{\widehat{R}}\psi\cdot\mathcal{S}_1\nabla_{\widehat{R}}\psi + HO(ar^{-2})\,\not{\mathfrak{d}}^{\leq 1}\psi\cdot\nabla_{\widehat{R}}\mathcal{S}_1\psi + H\mathcal{O}\nabla_{\widehat{R}}\psi\cdot O(ar^{-4})\mathfrak{d}^{\leq 1}\psi\\
& + Hr\Gamma_b\mathfrak{d}^{\leq 2}\psi\mathfrak{d}^{\leq 3}\psi\\
={}& H|q|^2|\nabla\nabla_{\mathbf{T}}\nabla_{\widehat{R}}\psi|^2 - O(ar^{-2})H\nabla\nabla_{\widehat{R}}\psi\cdot\not{\mathfrak{d}}^{\leq 1}\nabla_{\mathbf{T}}\nabla_{\widehat{R}}\psi\\
& + |q|^2\dot{\mathbf{D}}_\beta(H|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\nabla_{\widehat{R}}\psi\cdot\mathcal{S}_1\nabla_{\widehat{R}}\psi) - \dot{\mathbf{D}}_\mu(|q|^2H\nabla\nabla_{\widehat{R}}\psi\cdot\nabla\nabla_{\mathbf{T}}\nabla_{\widehat{R}}\psi\mathbf{T}^\mu)\\
& + HO(ar^{-2})\,\not{\mathfrak{d}}^{\leq 1}\psi\cdot\nabla_{\widehat{R}}\mathcal{S}_1\psi + H\mathcal{O}\nabla_{\widehat{R}}\psi\cdot O(ar^{-4})\mathfrak{d}^{\leq 1}\psi + (|H|+|H'|)r^2\Gamma_g\mathfrak{d}^{\leq 2}\psi\mathfrak{d}^{\leq 3}\psi
\end{aligned}
$$

which can be written as

$$
\begin{aligned}
H\nabla_{\widehat{R}}\mathcal{O}\psi\cdot\nabla_{\widehat{R}}\mathcal{S}_1\psi &= H|q|^2|\nabla\nabla_{\mathbf{T}}\nabla_{\widehat{R}}\psi|^2+|q|^2\dot{\mathbf{D}}_\beta(H|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\nabla_{\widehat{R}}\psi\cdot\mathcal{S}_1\nabla_{\widehat{R}}\psi)\\
&+\dot{\mathbf{D}}_\mu(HM(\nabla_{\widehat{R}}\psi)\mathbf{T}^\mu)-O(ar^{-2})H(\nabla_{\widehat{R}}\mathfrak{d}^{\leq 2}\psi)^2-O(ar^{-2})H(\mathfrak{d}^{\leq 2}\psi)^2\\
&+r^2\mathrm{Good}_{\mathcal{S}},
\end{aligned}
$$

as stated.

The mixed products $H\nabla_{\widehat{R}}\mathcal{O}\psi\cdot\mathcal{S}_1\psi$ and $H\mathcal{O}\psi\cdot\nabla_{\widehat{R}}\mathcal{S}_1\psi$ are treated similarly. This concludes the proof of Lemma 9.2.38. □

We use the above lemma to derive the following analog of Lemma 8.2.4.

Lemma 9.2.39. *Let $\Phi_{\underline{a}}=\mathcal{S}_{\underline{a}}\Phi$ for some*[5] *Φ, and let $\mathcal{Y}^{\underline{a}}$ be some coefficients only depending on r, such that*

$$
\mathcal{Y}^1=\delta_0\mathcal{Y},\qquad \mathcal{Y}^4=\mathcal{Y}.
$$

Then for $\mathcal{L}^{\underline{a}}$ given by (8.2.2), *i.e.*

$$
\mathcal{L}^1=\delta_0,\qquad \mathcal{L}^2=0,\qquad \mathcal{L}^3=\mathcal{L}^4=1,
$$

we have

$$
\begin{aligned}
\big(\mathcal{Y}^{\underline{a}}\Phi_{\underline{a}}\big)\cdot\big(\mathcal{L}^{\underline{a}}\Phi_{\underline{a}}\big) &= \mathcal{Y}\Big(\delta_0^2|\mathcal{S}_1\Phi|^2+|\mathcal{O}\Phi|^2+2\delta_0|q|^2|\nabla\nabla_T\Phi|^2\Big)\\
&-O(a)(|\mathcal{Y}|+|\mathcal{Y}^2|+|\mathcal{Y}^3|)(\mathfrak{d}^{\leq 2}\psi)^2+Bdr[\Phi]+r^2(|\mathcal{Y}|+|\mathcal{Y}'|)\,Good_{\mathcal{S}},
\end{aligned}
$$

where the boundary term is given by

$$
Bdr[\Phi] = \dot{\mathbf{D}}_\mu\Big(\delta_0\mathcal{Y}r^2M(\Phi)\mathbf{T}^\mu\Big)+|q|^2\dot{\mathbf{D}}_\beta\Big(2\delta_0|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\Phi\cdot\mathcal{Y}\mathcal{S}_1\Phi\Big).
$$

Proof. As in the proof of Lemma 8.2.4, we have

$$
\begin{aligned}
\big(\mathcal{Y}^{\underline{a}}\Phi_{\underline{a}}\big)\cdot\big(\mathcal{L}^{\underline{a}}\Phi_{\underline{a}}\big) &= \delta_0^2\mathcal{Y}|\mathcal{S}_1\Phi|^2+\mathcal{Y}|\mathcal{O}\Phi|^2+2\delta_0\mathcal{Y}(\mathcal{S}_1\Phi\cdot\mathcal{O}\Phi)\\
&+O(a)(|\mathcal{Y}|+|\mathcal{Y}^2|+|\mathcal{Y}^3|)(\mathfrak{d}^{\leq 2}\psi)^2.
\end{aligned}
$$

Using Lemma 9.2.38, we obtain the desired identity. □

[5] We will apply it to $\Phi=\psi$ and $\Phi=\nabla_r\psi$.

Proof of Proposition 9.2.15

We choose $(\mathbf{X}, \mathbf{w}, \mathbf{M})$ as in Chapter 8:

1. $\mathbf{X}$ is given by

$$\begin{aligned}
&X^{\underline{ab}} = \mathcal{F}^{\underline{ab}} \frac{(r^2+a^2)}{\Delta} \widehat{R}, \qquad \mathcal{F}^{\underline{ab}} = -zhf^{\underline{ab}},\\
&z = z_0 - \delta_0 z_0^2. \qquad z_0 = \frac{\Delta}{(r^2+a^2)^2}, \qquad \delta_0 > 0, \qquad h = \frac{(r^2+a^2)^4}{r(r^2-a^2)},\\
&f^{\underline{ab}} = \frac{1}{2}\big(\tilde{\mathcal{R}}'^{\underline{a}}\mathcal{L}^{\underline{b}} + \tilde{\mathcal{R}}'^{\underline{b}}\mathcal{L}^{\underline{a}}\big), \qquad \tilde{\mathcal{R}}'^{\underline{a}} = \partial_r\Big(\frac{z}{\Delta}\mathcal{R}^{\underline{a}}\Big),\\
&\mathcal{L}^1 = \delta_0, \qquad \mathcal{L}^2 = 0, \qquad \mathcal{L}^3 = \mathcal{L}^4 = 1,\\
&\mathcal{R}^1 = -(r^2+a^2)^2, \qquad \mathcal{R}^2 = -2(r^2+a^2), \qquad \mathcal{R}^3 = -1, \qquad \mathcal{R}^4 = \Delta.
\end{aligned}$$

2. $\mathbf{w}$ is given by

$$w^{\underline{ab}} = -z\partial_r(hf^{\underline{ab}}).$$

3. $\mathbf{M}$ is given by

$$M^{\underline{ab}} = v^{\underline{a}}\mathcal{L}^{\underline{b}}\frac{(r^2+a^2)}{\Delta}\widehat{R}, \qquad v^1 = \delta_0 v, \qquad v^2 = v^3 = 0, \qquad v^4 = v,$$

where the scalar function $v(r)$ is the one of Lemma 7.2.4 and satisfies in particular $v(r) = O(m^{1/2}\Delta r^{-9/2})$.

Next, we introduce the expression

$$\begin{aligned}
\mathcal{E}_K[\mathbf{X}, \mathbf{w}, \mathbf{M}] := \Bigg[&\frac{1}{2}\mathcal{Q}_{\underline{ab}} \cdot {}^{(X^{\underline{ab}})}\pi - \frac{1}{2}X^{\underline{ab}}(V)\psi_{\underline{a}} \cdot \psi_{\underline{b}} + \frac{1}{2}w^{\underline{ab}}\mathcal{L}[\psi_{\underline{a}}, \psi_{\underline{b}}]\\
&- \frac{1}{4}\psi_{\underline{a}} \cdot \psi_{\underline{b}}\square_{\mathbf{g}}w^{\underline{ab}} + \frac{1}{4}\mathrm{Div}(\psi_{\underline{a}} \cdot \psi_{\underline{b}}M^{\underline{ab}})\Bigg]_K.
\end{aligned} \tag{9.2.36}$$

Notice that $\mathcal{E}_K[\mathbf{X}, \mathbf{w}, \mathbf{M}]$ coincides in fact with the quantity $\mathcal{E}[\mathbf{X}, \mathbf{w}, \mathbf{M}]$ in (8.1.3). Thus, we have according to (8.1.7)

$$|q|^2\mathcal{E}_K[\mathbf{X}, \mathbf{w}, \mathbf{M}] = P + I + J + K \tag{9.2.37}$$

where

$$\begin{aligned} P &:= \mathcal{U}^{\alpha\beta\underline{a}\underline{b}}\dot{\mathbf{D}}_\alpha\psi_{\underline{a}}\cdot\dot{\mathbf{D}}_\beta\psi_{\underline{b}}, \\ I &:= \mathcal{A}^{\underline{a}\underline{b}}\nabla_r\psi_{\underline{a}}\cdot\nabla_r\psi_{\underline{b}}, \\ J &:= \mathcal{V}^{\underline{a}\underline{b}}\psi_{\underline{a}}\cdot\psi_{\underline{b}}, \\ K &:= \frac{1}{4}|q|^2\mathbf{D}^\mu\Big(M_\mu^{\underline{a}\underline{b}}\psi_{\underline{a}}\cdot\psi_{\underline{b}}\Big), \end{aligned}$$

and where the coefficients $\mathcal{U}^{\alpha\beta\underline{a}\underline{b}}$, $\mathcal{A}^{\underline{a}\underline{b}}$ and $\mathcal{V}^{\underline{a}\underline{b}}$ are provided by Proposition 8.1.3.

We now derive the analog of Lemma 8.1.4.

Lemma 9.2.40. *Let P the principal term defined as above. We then have the identity*

$$P = \frac{1}{2}hL^{\alpha\beta}\dot{\mathbf{D}}_\alpha\Psi\cdot\dot{\mathbf{D}}_\beta\Psi - \frac{1}{2}h\Psi\cdot(\tilde{\mathcal{R}}'^{\underline{c}}\mathcal{L}^{\underline{b}}[\mathcal{S}_{\underline{c}},\mathcal{S}_{\underline{b}}]\psi) + |q|^2\dot{\mathbf{D}}_\alpha\mathcal{B}^\alpha + r^2\,Good_{\mathcal{S}},$$

where Ψ is defined as

$$\Psi := \tilde{\mathcal{R}}'^{\underline{a}}\psi_{\underline{a}}, \tag{9.2.38}$$

and the boundary term $\mathcal{B}$ is given by

$$\mathcal{B}^\alpha := |q|^{-2}\frac{1}{2}h\Psi\tilde{\mathcal{R}}'^{\underline{c}}\mathcal{L}^{\underline{b}}\cdot\Big(S_{\underline{c}}^{\alpha\beta}\,\dot{\mathbf{D}}_\beta\psi_{\underline{b}} - S_{\underline{b}}^{\alpha\beta}\dot{\mathbf{D}}_\beta\psi_{\underline{c}}\Big). \tag{9.2.39}$$

Proof. As in the proof of Lemma 8.1.4, we have

$$\begin{aligned} |q|^{-2}P &= -\mathcal{U}^{\underline{a}\underline{c}}\mathcal{L}^{\underline{b}}\psi_{\underline{a}}\cdot\dot{\mathbf{D}}_\alpha(|q|^{-2}S_{\underline{c}}^{\alpha\beta}\,\dot{\mathbf{D}}_\beta\psi_{\underline{b}}) + \dot{\mathbf{D}}_\alpha(|q|^{-2}\mathcal{U}^{\underline{a}\underline{c}}\mathcal{L}^{\underline{b}}S_{\underline{c}}^{\alpha\beta}\psi_{\underline{a}}\cdot\dot{\mathbf{D}}_\beta\psi_{\underline{b}}) \\ &\quad -\dot{\mathbf{D}}_\alpha(\mathcal{U}^{\underline{a}\underline{c}}\mathcal{L}^{\underline{b}})|q|^{-2}S_{\underline{c}}^{\alpha\beta}\psi_{\underline{a}}\cdot\dot{\mathbf{D}}_\beta\psi_{\underline{b}}. \end{aligned}$$

Since $\mathcal{L}^{\underline{a}}$ and $\mathcal{U}^{\underline{a}\underline{b}}$ only depend on r, and in view of the form of the 2-tensors $S_{\underline{c}}^{\alpha\beta}$, we have

$$\begin{aligned} \dot{\mathbf{D}}_\alpha\big(\mathcal{U}^{\underline{a}\underline{c}}\mathcal{L}^{\underline{b}}\big)|q|^{-2}S_{\underline{c}}^{\alpha\beta}\psi_{\underline{a}}\cdot\dot{\mathbf{D}}_\beta\psi_{\underline{b}} &= \partial_r\big(\mathcal{U}^{\underline{a}\underline{c}}\mathcal{L}^{\underline{b}}\big)e_\alpha(r)|q|^{-2}S_{\underline{c}}^{\alpha\beta}\psi_{\underline{a}}\cdot\dot{\mathbf{D}}_\beta\psi_{\underline{b}} \\ &= O(r^{-2})\partial_r\big(\mathcal{U}^{\underline{a}\underline{c}}\mathcal{L}^{\underline{b}}\big)\Big(\mathbf{T}(r),\mathbf{Z}(r),r\nabla(r)\Big)\psi\mathfrak{d}^{\leq 1}\psi \\ &= \partial_r\big(\mathcal{U}^{\underline{a}\underline{c}}\mathcal{L}^{\underline{b}}\big)\Gamma_g\psi\mathfrak{d}^{\leq 1}\psi \\ &= \partial_r\big(\mathcal{U}^{\underline{a}\underline{c}}\mathcal{L}^{\underline{b}}\big)r\mathrm{Good}_{\mathcal{S}}. \end{aligned}$$

Now, note that the choices of Chapter 8 imply $\mathcal{U}^{\underline{a}\underline{c}}\mathcal{L}^{\underline{b}} = O(r^{-1})$ and $\partial_r(\mathcal{U}^{\underline{a}\underline{c}}\mathcal{L}^{\underline{b}}) = O(r^{-2})$ and hence

$$\dot{\mathbf{D}}_\alpha\big(\mathcal{U}^{\underline{a}\underline{c}}\mathcal{L}^{\underline{b}}\big)|q|^{-2}S_{\underline{c}}^{\alpha\beta}\psi_{\underline{a}}\cdot\dot{\mathbf{D}}_\beta\psi_{\underline{b}} = r^{-1}\mathrm{Good}_{\mathcal{S}}.$$

Also, we have in view of Lemma 4.6.2

$$\widetilde{\mathcal{S}}_1 = \mathcal{S}_1 + \Gamma_b \cdot \mathfrak{d}, \qquad \widetilde{\mathcal{S}}_{\underline{a}} \;=\; \mathcal{S}_{\underline{a}} + r\Gamma_b \cdot \mathfrak{d}, \quad \underline{a} = 2, 3, 4,$$

where

$$\widetilde{\mathcal{S}}_{\underline{a}}\psi \;=\; |q|^2 \dot{\mathbf{D}}_\alpha (|q|^{-2} S_{\underline{a}}^{\alpha\beta} \dot{\mathbf{D}}_\beta \psi), \qquad \text{for } \underline{a} = 1, 2, 3, 4.$$

Therefore we write

$$\begin{aligned}
\dot{\mathbf{D}}_\alpha(|q|^{-2} S_{\underline{c}}^{\alpha\beta}\, \dot{\mathbf{D}}_\beta \psi_{\underline{b}}) &= |q|^{-2} \widetilde{\mathcal{S}}_{\underline{c}}(\mathcal{S}_{\underline{b}}\psi) = |q|^{-2}\mathcal{S}_{\underline{c}}\mathcal{S}_{\underline{b}}\psi + r^{-1}\Gamma_b \mathfrak{d}\mathcal{S}_{\underline{b}}\psi \\
&= |q|^{-2}\mathcal{S}_{\underline{b}}\mathcal{S}_{\underline{c}}\psi + |q|^{-2}[\mathcal{S}_{\underline{c}}, \mathcal{S}_{\underline{b}}]\psi + r^{-1}\Gamma_b \mathfrak{d}^{\leq 3}\psi \\
&= \dot{\mathbf{D}}_\alpha(|q|^{-2} S_{\underline{b}}^{\alpha\beta}\, \dot{\mathbf{D}}_\beta \psi_{\underline{c}}) + |q|^{-2}[\mathcal{S}_{\underline{c}}, \mathcal{S}_{\underline{b}}]\psi + r^{-1}\Gamma_b \mathfrak{d}^{\leq 3}\psi.
\end{aligned}$$

Thus, repeating the integration by parts procedure and noting, as above, that $\mathcal{U}^{\underline{ac}}\mathcal{L}^{\underline{b}} = O(r^{-1})$ and that the last term is of the type $r^{-1}\mathrm{Good}_{\mathcal{S}}$, we obtain

$$\begin{aligned}
&\mathcal{U}^{\underline{ac}}\mathcal{L}^{\underline{b}}\psi_{\underline{a}} \cdot \dot{\mathbf{D}}_\alpha\big(|q|^{-2} S_{\underline{c}}^{\alpha\beta} \dot{\mathbf{D}}_\beta \psi_{\underline{b}}\big) \\
=\;& \mathcal{U}^{\underline{ac}}\mathcal{L}^{\underline{b}}\psi_{\underline{a}} \cdot \dot{\mathbf{D}}_\alpha(|q|^{-2} S_{\underline{b}}^{\alpha\beta}\, \dot{\mathbf{D}}_\beta \psi_{\underline{c}}) + |q|^{-2}\mathcal{U}^{\underline{ac}}\mathcal{L}^{\underline{b}}\psi_{\underline{a}} \cdot [\mathcal{S}_{\underline{c}}, \mathcal{S}_{\underline{b}}]\psi + r^{-1}\mathcal{U}^{\underline{ac}}\mathcal{L}^{\underline{b}}\psi_{\underline{a}} \cdot \Gamma_b \mathfrak{d}^{\leq 3}\psi \\
=\;& -|q|^{-2}\mathcal{U}^{\underline{ac}}\mathcal{L}^{\underline{b}} S_{\underline{b}}^{\alpha\beta} \dot{\mathbf{D}}_\alpha \psi_{\underline{a}} \cdot \dot{\mathbf{D}}_\beta \psi_{\underline{c}} + \dot{\mathbf{D}}_\alpha\Big(|q|^{-2}\mathcal{U}^{\underline{ac}}\mathcal{L}^{\underline{b}} S_{\underline{b}}^{\alpha\beta}\psi_{\underline{a}} \cdot \dot{\mathbf{D}}_\beta\psi_{\underline{c}}\Big) \\
&+|q|^{-2}\mathcal{U}^{\underline{ac}}\mathcal{L}^{\underline{b}}\psi_{\underline{a}} \cdot [\mathcal{S}_{\underline{c}}, \mathcal{S}_{\underline{b}}]\psi + r^{-1}\mathrm{Good}_{\mathcal{S}} + r^{-2}\Gamma_b \mathfrak{d}^{\leq 2}\psi \cdot \mathfrak{d}^{\leq 3}\psi \\
=\;& -|q|^{-2}\mathcal{U}^{\underline{ac}}\mathcal{L}^{\underline{b}} S_{\underline{b}}^{\alpha\beta} \dot{\mathbf{D}}_\alpha \psi_{\underline{a}} \cdot \dot{\mathbf{D}}_\beta \psi_{\underline{c}} + \dot{\mathbf{D}}_\alpha\Big(|q|^{-2}\mathcal{U}^{\underline{ac}}\mathcal{L}^{\underline{b}} S_{\underline{b}}^{\alpha\beta}\psi_{\underline{a}} \cdot \dot{\mathbf{D}}_\beta\psi_{\underline{c}}\Big) \\
&+|q|^{-2}\mathcal{U}^{\underline{ac}}\mathcal{L}^{\underline{b}}\psi_{\underline{a}} \cdot [\mathcal{S}_{\underline{c}}, \mathcal{S}_{\underline{b}}]\psi + \mathrm{Good}_{\mathcal{S}}.
\end{aligned}$$

Therefore, recalling that $\mathcal{L}^{\underline{b}} S_{\underline{b}}^{\alpha\beta} = L^{\alpha\beta}$,

$$\begin{aligned}
P \;=\;& \mathcal{U}^{\underline{ac}} L^{\alpha\beta} \dot{\mathbf{D}}_\alpha \psi_{\underline{a}} \cdot \dot{\mathbf{D}}_\beta \psi_{\underline{c}} + |q|^2 \dot{\mathbf{D}}_\alpha \Big(|q|^{-2}\mathcal{U}^{\underline{ac}}\mathcal{L}^{\underline{b}}\psi_{\underline{a}} \cdot \Big(S_{\underline{c}}^{\alpha\beta}\, \dot{\mathbf{D}}_\beta\psi_{\underline{b}} - S_{\underline{b}}^{\alpha\beta}\dot{\mathbf{D}}_\beta\psi_{\underline{c}}\Big)\Big) \\
&-\mathcal{U}^{\underline{ac}}\mathcal{L}^{\underline{b}}\psi_{\underline{a}} \cdot [\mathcal{S}_{\underline{c}}, \mathcal{S}_{\underline{b}}]\psi + r^2\mathrm{Good}_{\mathcal{S}}.
\end{aligned}$$

Finally, since we have $\tilde{\mathcal{R}}'^{\underline{a}} = O(r^{-3})$ and $\partial_r \tilde{\mathcal{R}}'^{\underline{a}} = O(r^{-4})$, and since $\tilde{\mathcal{R}}'^{\underline{a}}$ only depend on r, we obtain

$$\begin{aligned}
&L^{\alpha\beta}\dot{\mathbf{D}}_\alpha(\tilde{\mathcal{R}}'^{\underline{a}}\psi_{\underline{a}}) \cdot \dot{\mathbf{D}}_\beta(\tilde{\mathcal{R}}'^{\underline{c}}\psi_{\underline{c}}) \\
=\;& L^{\alpha\beta}\tilde{\mathcal{R}}'^{\underline{a}}\tilde{\mathcal{R}}'^{\underline{c}}\dot{\mathbf{D}}_\alpha\psi_{\underline{a}} \cdot \dot{\mathbf{D}}_\beta\psi_{\underline{c}} + \big(\mathbf{T}(r), \mathbf{Z}(r), r\nabla(r)\big)O(r^{-7})\mathfrak{d}^{\leq 2}\psi\mathfrak{d}^{\leq 3}\psi \\
=\;& L^{\alpha\beta}\tilde{\mathcal{R}}'^{\underline{a}}\tilde{\mathcal{R}}'^{\underline{c}}\dot{\mathbf{D}}_\alpha\psi_{\underline{a}} \cdot \dot{\mathbf{D}}_\beta\psi_{\underline{c}} + r^{-5}\Gamma_g\mathfrak{d}^{\leq 2}\psi\mathfrak{d}^{\leq 3}\psi \\
=\;& L^{\alpha\beta}\tilde{\mathcal{R}}'^{\underline{a}}\tilde{\mathcal{R}}'^{\underline{c}}\dot{\mathbf{D}}_\alpha\psi_{\underline{a}} \cdot \dot{\mathbf{D}}_\beta\psi_{\underline{c}} + r^{-4}\mathrm{Good}_{\mathcal{S}}
\end{aligned}$$

which together with (8.1.10) implies

$$\begin{aligned} P &= \frac{1}{2}h\tilde{\mathcal{R}}'^{\underline{a}}\tilde{\mathcal{R}}'^{\underline{c}}L^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi_{\underline{a}}\cdot\dot{\mathbf{D}}_\beta\psi_{\underline{c}}-\frac{1}{2}h\tilde{\mathcal{R}}'^{\underline{a}}\tilde{\mathcal{R}}'^{\underline{c}}\mathcal{L}^{\underline{b}}\psi_{\underline{a}}\cdot[\mathcal{S}_{\underline{c}},\mathcal{S}_{\underline{b}}]\psi\\ &+|q|^2\dot{\mathbf{D}}_\alpha\left(|q|^{-2}\frac{1}{2}h\tilde{\mathcal{R}}'^{\underline{a}}\tilde{\mathcal{R}}'^{\underline{c}}\mathcal{L}^{\underline{b}}\psi_{\underline{a}}\cdot\Big(S_{\underline{c}}^{\alpha\beta}\,\dot{\mathbf{D}}_\beta\psi_{\underline{b}}-S_{\underline{b}}^{\alpha\beta}\dot{\mathbf{D}}_\beta\psi_{\underline{c}}\Big)\right)+r^2\mathrm{Good}_{\mathcal{S}}\\ &= \frac{1}{2}hL^{\alpha\beta}\dot{\mathbf{D}}_\alpha(\tilde{\mathcal{R}}'^{\underline{a}}\psi_{\underline{a}})\cdot\dot{\mathbf{D}}_\beta(\tilde{\mathcal{R}}'^{\underline{c}}\psi_{\underline{c}})-\frac{1}{2}h\tilde{\mathcal{R}}'^{\underline{c}}\mathcal{L}^{\underline{b}}(\tilde{\mathcal{R}}'^{\underline{a}}\psi_{\underline{a}})\cdot[\mathcal{S}_{\underline{c}},\mathcal{S}_{\underline{b}}]\psi\\ &+|q|^2\dot{\mathbf{D}}_\alpha\left(|q|^{-2}\frac{1}{2}h\tilde{\mathcal{R}}'^{\underline{c}}\mathcal{L}^{\underline{b}}(\tilde{\mathcal{R}}'^{\underline{a}}\psi_{\underline{a}})\cdot\Big(S_{\underline{c}}^{\alpha\beta}\,\dot{\mathbf{D}}_\beta\psi_{\underline{b}}-S_{\underline{b}}^{\alpha\beta}\dot{\mathbf{D}}_\beta\psi_{\underline{c}}\Big)\right)+r^2\mathrm{Good}_{\mathcal{S}}. \end{aligned}$$

By denoting $\Psi=\tilde{\mathcal{R}}'^{\underline{a}}\psi_{\underline{a}}$ we obtain the stated expression. □

In view of (9.2.37) and Lemma 9.2.40, we obtain the following analog of (8.1.13)

$$|q|^2\mathcal{E}_K[\mathbf{X},\mathbf{w},\mathbf{M}]-|q|^2\mathbf{D}^\mu\mathcal{B}_\mu=\widetilde{P}+P_{lot}+I+J+K+r^2\mathrm{Good}_{\mathcal{S}} \tag{9.2.40}$$

where the quadratic forms I, J and K are defined below (9.2.37) and

- $\mathcal{B}$ is the boundary term defined in (9.2.39),
- The principal term $\widetilde{P}$ is positive definite and given by

$$\widetilde{P} := \frac{1}{2}hL^{\alpha\beta}\dot{\mathbf{D}}_\alpha\Psi\cdot\dot{\mathbf{D}}_\beta\Psi,\qquad \Psi=\tilde{\mathcal{R}}'^{\underline{a}}\psi_{\underline{a}}, \tag{9.2.41}$$

  with $L^{\alpha\beta}$ as in (8.1.9),
- The lower order term P_{lot} is given by

$$P_{lot} = -\frac{1}{2}h\Psi\cdot(\tilde{\mathcal{R}}'^{\underline{c}}\mathcal{L}^{\underline{b}}[\mathcal{S}_{\underline{c}},\mathcal{S}_{\underline{b}}]\psi). \tag{9.2.42}$$

As in (8.1.19), we denote the effective generalized current

$$|q|^2\widetilde{\mathcal{E}}_K[\mathbf{X},\mathbf{w},\mathbf{M}]:=|q|^2\mathcal{E}_K[\mathbf{X},\mathbf{w},\mathbf{M}]-|q|^2\mathbf{D}^\mu\mathcal{B}_\mu. \tag{9.2.43}$$

Next, we evaluate each term on the RHS of (9.2.40):

1. For $\widetilde{P}$, we immediately have the analog of (8.2.3) of, i.e.

$$\widetilde{P}=\frac{1}{2}h\Big(\delta_0\big|\nabla_T\Psi_z\big|^2+a^2\big|\nabla_Z\Psi_z\big|^2+O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\Psi_z\dot{\mathbf{D}}_\beta\Psi_z\Big). \tag{9.2.44}$$

2. Concerning P_{lot}, we make use of Lemma 9.2.3 according to which we have

$$[\mathcal{S}_1,\mathcal{S}_2],\ [\mathcal{S}_1,\mathcal{S}_3],\ [\mathcal{S}_2,\mathcal{S}_3] = \mathfrak{d}^2(\Gamma_b\cdot\psi)$$

and

$$\begin{aligned}
[\mathcal{S}_1,\mathcal{O}] &= O(ar^{-3})\mathfrak{d}^{\leq 2}\psi + \mathfrak{d}^{\leq 3}(\Gamma_b\psi),\\
[\mathcal{S}_2,\mathcal{O}] &= O(a)\,\not{\mathfrak{d}}^{\leq 2}\psi + r\mathfrak{d}^{\leq 3}(\Gamma_b\psi),\\
[\mathcal{S}_3,\mathcal{O}] &= O(a^2)\,\not{\mathfrak{d}}^{\leq 2}\psi + r\mathfrak{d}^{\leq 3}(\Gamma_b\psi).
\end{aligned}$$

In view of the definition of P_{lot} in (9.2.42), we infer, since $h = O(r^5)$, $\tilde{\mathcal{R}}'^{\underline{a}} = O(r^{-3})$, and $\Psi = O(r^{-3})\mathfrak{d}^{\leq 2}\psi$,

$$\begin{aligned}
P_{lot} &= -\frac{1}{2}h\Psi\cdot(\tilde{\mathcal{R}}'^{\underline{c}}\mathcal{L}^{\underline{b}}[\mathcal{S}_{\underline{c}},\mathcal{S}_{\underline{b}}]\psi)\\
&= [P_{lot}]_K + h\Psi\cdot\tilde{\mathcal{R}}'^{\underline{c}}\mathcal{L}^{\underline{b}}r\mathfrak{d}^{\leq 3}(\Gamma_b\psi) = [P_{lot}]_K + \mathfrak{d}^{\leq 2}\psi\cdot\mathfrak{d}^{\leq 3}(\Gamma_b\psi)\\
&= [P_{lot}]_K + r^2\mathrm{Good}_{\mathcal{S}}
\end{aligned}$$

where $[P_{lot}]_K$ denotes the corresponding expression in Kerr, i.e the one of (8.1.15). In particular, $[P_{lot}]_K$ satisfies (8.2.4) which implies the following analog of (8.1.15) for P_{lot}

$$P_{lot} = O(ar^{-1})(\mathfrak{d}^{\leq 2}\psi)^2 + r^2\mathrm{Good}_{\mathcal{S}}. \tag{9.2.45}$$

3. I is estimated as in section 8.2.3, where Lemma 8.2.4 is replaced with Lemma 9.2.39 which yields the following analog of (8.2.7)

$$\begin{aligned}
I =&\mathcal{A}(1+O(r^{-1}\delta_0))\Big(\delta_0^2|\nabla_r\mathcal{S}_1\psi|^2 + |\nabla_r\mathcal{O}\psi|^2 + 2\delta_0|q|^2|\nabla\nabla_T\nabla_r\psi|^2\Big)\\
&- O(a)(|\mathcal{A}|+|\widetilde{\mathcal{A}}|)(1+O(r^{-1}\delta_0))\Big((\nabla_r\mathfrak{d}^{\leq 2}\psi)^2 + r^{-2}(\mathfrak{d}^{\leq 2}\psi)^2\Big)\\
&+ \mathrm{Bdr}[\psi]_I + r^2\mathrm{Good}_{\mathcal{S}},
\end{aligned} \tag{9.2.46}$$

where the boundary term is given by

$$\mathrm{Bdr}[\psi]_I = \dot{\mathbf{D}}_\mu\Big(\delta_0\mathcal{A}M(\nabla_r\psi)\mathbf{T}^\mu\Big) + |q|^2\dot{\mathbf{D}}_\beta\Big(2\delta_0|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\nabla_r\psi\cdot\mathcal{A}\mathcal{S}_1\nabla_r\psi\Big).$$

4. J is estimated as in Lemma 8.2.6 where Lemma 8.2.4 is replaced with Lemma 9.2.39 which yields the following analog of Lemma 8.2.6

$$\begin{aligned}
J =&\big(\mathcal{V}+O(\delta_0 r^{-3})\big)\Big(\delta_0^2|\mathcal{S}_1\psi|^2 + |\mathcal{O}\psi|^2 + 2\delta_0|q|^2|\nabla\nabla_T\psi|^2\Big)\\
&- O(ar^{-1})(\mathfrak{d}^{\leq 2}\psi)^2 + \mathrm{Bdr}[\psi]_J + r\mathrm{Good}_{\mathcal{S}},
\end{aligned} \tag{9.2.47}$$

with boundary term

$$\begin{aligned}\mathrm{Bdr}_J[\psi] &= \dot{\mathbf{D}}_\mu\Big(\delta_0(\mathcal{V}+O(r^{-3}))r^2M(\psi)\mathbf{T}^\mu\Big)\\ &\quad +|q|^2\dot{\mathbf{D}}_\beta\Big(2\delta_0|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot(\mathcal{V}+O(a^2r^{-3}))\mathcal{S}_1\psi\Big).\end{aligned}$$

5. For K, we assume as in Lemma 8.2.6 that

$$M^{\underline{a}} := v^{\underline{a}}\frac{r^2+a^2}{\Delta}\widehat{R}, \qquad v^1=\delta_0 v, \qquad v^2=v^3=0, \qquad v^4=v,$$

for some given function $v=v(r)$. Furthermore, under the assumption $v=O(m^{1/2}\Delta r^{-9/2})$, we have

$$\begin{aligned}\mathbf{D}_\mu((M^{\underline{a}})^\mu) &= \mathbf{D}_\mu\left(\frac{v^{\underline{a}}}{\Delta}(r^2+a^2)\widehat{R}^\mu\right)\\ &= \left[\mathbf{D}_\mu\left(\frac{v^{\underline{a}}}{\Delta}(r^2+a^2)\widehat{R}^\mu\right)\right]_K + O(r^{-\frac{7}{2}})\widecheck{\widehat{R}(r)} + O(r^{-\frac{5}{2}})\widecheck{\mathbf{D}_\mu\widehat{R}^\mu}\\ &= \left[\mathbf{D}_\mu\left(\frac{v^{\underline{a}}}{\Delta}(r^2+a^2)\widehat{R}^\mu\right)\right]_K + r^{-\frac{5}{2}}\Gamma_b\end{aligned}$$

which together with the proof of Lemma 8.2.6 yields

$$\begin{aligned}\frac{4}{|q|^2}K &= (\mathbf{D}^\mu M^{\underline{a}}_\mu)\psi_{\underline{a}}\cdot(\mathcal{L}^{\underline{b}}\psi_{\underline{b}}) + (M^{\underline{a}}_\mu\mathbf{D}^\mu\psi_{\underline{a}})\cdot(\mathcal{L}^{\underline{b}}\psi_{\underline{b}}) + (M^{\underline{a}}_\mu\psi_{\underline{a}})\cdot\mathbf{D}^\mu(\mathcal{L}^{\underline{b}}\psi_{\underline{b}})\\ &= (v'^{\underline{a}}\psi_{\underline{a}})\cdot(\mathcal{L}^{\underline{b}}\psi_{\underline{b}}) + v^{\underline{a}}\nabla_r\psi_{\underline{a}}\cdot(\mathcal{L}^{\underline{b}}\psi_{\underline{b}}) + (v^{\underline{a}}\psi_{\underline{a}})\cdot\mathcal{L}^{\underline{b}}\nabla_r\psi_{\underline{b}} + r^{-\frac{5}{2}}\Gamma_b\psi_{\underline{a}}\cdot(\mathcal{L}^{\underline{b}}\psi_{\underline{b}})\\ &= (v'^{\underline{a}}\psi_{\underline{a}})\cdot(\mathcal{L}^{\underline{b}}\psi_{\underline{b}}) + v^{\underline{a}}\nabla_r\psi_{\underline{a}}\cdot(\mathcal{L}^{\underline{b}}\psi_{\underline{b}}) + (v^{\underline{a}}\psi_{\underline{a}})\cdot\mathcal{L}^{\underline{b}}\nabla_r\psi_{\underline{b}} + r^{-\frac{5}{2}}\Gamma_b(\mathfrak{d}^{\leq 2}\psi)^2\\ &= (v'^{\underline{a}}\psi_{\underline{a}})\cdot(\mathcal{L}^{\underline{b}}\psi_{\underline{b}}) + v^{\underline{a}}\nabla_r\psi_{\underline{a}}\cdot(\mathcal{L}^{\underline{b}}\psi_{\underline{b}}) + (v^{\underline{a}}\psi_{\underline{a}})\cdot\mathcal{L}^{\underline{b}}\nabla_r\psi_{\underline{b}} + r^{-\frac{1}{2}}\mathrm{Good}_{\mathcal{S}}\end{aligned}$$

where we wrote $v'^{\underline{a}} := \partial_r v^{\underline{a}} + \frac{2r}{|q|^2}v^{\underline{a}}$. Then, we follow the rest of the proof of Lemma 8.2.6 where Lemma 8.2.4 is replaced with Lemma 9.2.39 and Lemma 8.2.3 is replaced by Lemma 9.2.38 which yields the following analog of Lemma 8.2.6

$$\begin{aligned}K &= \frac{|q|^2}{2}v\Big(\delta_0^2\nabla_r\mathcal{S}_1\psi\cdot\mathcal{S}_1\psi + \nabla_r\mathcal{O}\psi\cdot\mathcal{O}\psi + 2\delta_0|q|^2\nabla\nabla_T\nabla_r\psi\cdot\nabla\nabla_T\psi\Big)\\ &\quad +\frac{|q|^2}{4}v'\Big(\delta_0^2|\mathcal{S}_1\psi|^2 + |\mathcal{O}\psi|^2 + 2\delta_0|q|^2|\nabla\nabla_T\psi|^2\Big) - vO(ar^{\frac{5}{2}})(\nabla_r\mathfrak{d}^{\leq 2}\psi)^2\\ &\quad -O(ar^{\frac{3}{2}})v(\mathfrak{d}^{\leq 2}\psi)^2 - O(ar^2)v'(\mathfrak{d}^{\leq 2}\psi)^2 + \mathrm{Bdr}[\psi]_K + r^{\frac{3}{2}}\mathrm{Good}_{\mathcal{S}}, \qquad (9.2.48)\end{aligned}$$

where we denoted

$$v'^{\underline{a}} := \partial_r v^{\underline{a}} + \frac{2r}{|q|^2}v^{\underline{a}},$$

and with boundary term

$$\begin{aligned}\mathrm{Bdr}[\psi]_K &= \dot{\mathbf{D}}_\mu\Big(\big(vr^4M(\nabla_r\psi)\big)\mathbf{T}^\mu\Big)+\frac{|q|^4}{4}\dot{\mathbf{D}}_\beta(2\delta_0 v|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot\mathcal{S}_1\nabla_r\psi)\\ &+\frac{|q|^4}{4}\dot{\mathbf{D}}_\beta(2\delta_0 v|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\nabla_r\psi\cdot\mathcal{S}_1\psi)+\dot{\mathbf{D}}_\mu\Big(\delta_0 v' r^4 M(\psi)\mathbf{T}^\mu\Big)\\ &+\frac{|q|^4}{4}\dot{\mathbf{D}}_\beta\Big(2\delta_0 v'|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot\mathcal{S}_1\psi\Big).\end{aligned}$$

The above computations of $\widetilde{P}$, P_{lot}, I, J and K immediately yields the following analog of Proposition 8.2.7.

Proposition 9.2.41. *The effective generalized current is given by*

$$|q|^2\widetilde{\mathcal{E}}_K[\mathbf{X},\mathbf{w},\mathbf{M}] = \widetilde{P}+Qr_{\mathcal{S}_1,\mathcal{O},\nabla\nabla_T}+\mathcal{E}_{lot}+Bdr+r^2Good_{\mathcal{S}} \tag{9.2.49}$$

with the following terms.

1. *The principal trapping term $\widetilde{P}$ is given by, see* (9.2.44),

$$\widetilde{P} = \frac{1}{2}h\Big(\delta_0\big|\nabla_T\Psi_z\big|^2+a^2|\nabla_Z\Psi_z|^2+O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\Psi_z\dot{\mathbf{D}}_\beta\Psi_z\Big), \tag{9.2.50}$$

   where

$$\begin{aligned}\Psi_z = &-\frac{2\mathcal{T}}{(r^2+a^2)^3}\big(\delta_0\mathcal{S}_1\psi+(1+O(r^{-2}\delta_0))\mathcal{O}\psi\big)\\ &+\frac{4ar}{(r^2+a^2)^2}\nabla_{\widehat{T}}\nabla_Z\psi\big(1+O(r^{-2}\delta_0)\big).\end{aligned} \tag{9.2.51}$$

2. *The quadratic form $Qr_{\mathcal{S}_1,\mathcal{O},\nabla\nabla_T}$ is given by*

$$\begin{aligned}Qr_{\mathcal{S}_1,\mathcal{O},\nabla\nabla_T} := &\ \mathcal{A}(1+O(r^{-1}\delta_0))\Big(\delta_0^2|\nabla_r\mathcal{S}_1\psi|^2+|\nabla_r\mathcal{O}\psi|^2+2\delta_0|q|^2|\nabla\nabla_T\nabla_r\psi|^2\Big)\\ &+\frac{|q|^2}{2}v\Big(\delta_0^2\nabla_r\mathcal{S}_1\psi\cdot\mathcal{S}_1\psi+\nabla_r\mathcal{O}\psi\cdot\mathcal{O}\psi+2\delta_0|q|^2\nabla\nabla_T\nabla_r\psi\cdot\nabla\nabla_T\psi\Big)\\ &+\left(\mathcal{V}+\frac{|q|^2}{4}v'+O(\delta_0 r^{-3})\right)\Big(\delta_0^2|\mathcal{S}_1\psi|^2+|\mathcal{O}\psi|^2+2\delta_0|q|^2|\nabla\nabla_T\psi|^2\Big).\end{aligned}$$

3. *The terms $\mathcal{E}_{lot}$ are lower order terms in a satisfying*

$$\mathcal{E}_{lot}\geq -O(a)\big(|\nabla_{\widehat{R}}\mathfrak{d}^{\leq 2}\psi|^2+r^{-1}|\mathfrak{d}^{\leq 2}\psi|^2\big). \tag{9.2.52}$$

4. *The boundary terms are given by*

$$Bdr = \dot{\mathbf{D}}_\mu\Big(\big(M(\nabla_{\widehat{R}}\psi) + M(\psi)\big)\mathbf{T}^\mu\Big) + |q|^2\mathbf{D}_\beta\widehat{\mathcal{B}}^\beta$$

with

$$\begin{aligned}\widehat{\mathcal{B}}^\beta := {} & 2\delta_0|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\nabla_r\psi\cdot\mathcal{A}\mathcal{S}_1\nabla_r\psi + 2\delta_0|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot(\mathcal{V}+O(a^2r^{-3}))\mathcal{S}_1\psi\\ & +\frac{1}{4}\delta_0 vO^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot\mathcal{S}_1\nabla_r\psi + \frac{1}{4}\delta_0 vO^{\alpha\beta}\dot{\mathbf{D}}_\alpha\nabla_r\psi\cdot\mathcal{S}_1\psi + \frac{1}{2}\delta_0O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi\cdot v'\mathcal{S}_1\psi,\end{aligned}$$

where $M(\psi)$ denotes the quadratic expressions in ψ and $M(\nabla_{\widehat{R}}\psi)$ denotes the quadratic expressions in ψ and its derivatives of Definition 9.2.36.

Next, we proceed as in section 8.3.1 in order to control the quadratic form $\mathrm{Qr}_{\mathcal{S}_1,\mathcal{O},\nabla\nabla_T}$ appearing in Proposition 9.2.41. The proof is analogous provided we use:

- the Poincaré inequality of Lemma 9.2.23 which is the analog of the Poincaré inequality of Lemma 7.2.3,
- the integration by parts identities of Lemma 9.2.38 which is the analog of the ones of Lemma 8.2.3.

This leads to the following analog of (8.3.2), for any sphere $S = S(\tau, r)$,

$$\begin{aligned}\int_S|q|^2\widetilde{\mathcal{E}}_K[\mathbf{X},\mathbf{w},\mathbf{M}] \geq{} & \delta\int_S\widetilde{P}\\ & + c_1\int_S\Big(m\big(|\nabla_{\widehat{R}}\mathcal{S}_1\psi|^2 + |\nabla_{\widehat{R}}\mathcal{O}\psi|^2 + |q|^2|\nabla\nabla_T\nabla_{\widehat{R}}\psi|^2\big)\\ & + r^{-1}\big(|\mathcal{S}_1\psi|^2 + |\mathcal{O}\psi|^2 + |q|^2|\nabla\nabla_T\psi|^2\big)\Big)\\ & + \int_S\Big(\widetilde{\mathcal{E}}_{lot} - O(ar^{-2})\big(|\nabla_{\widehat{R}}\mathfrak{d}^{\leq 2}\psi|^2 + |\mathfrak{d}^{\leq 2}\psi|^2\big) + \mathrm{Bdr} + r^2\mathrm{Good}_{\mathcal{S}}\Big).\end{aligned} \tag{9.2.53}$$

Next, we proceed exactly as in section 8.3.2 and obtain the following analog of (8.3.3)

$$\begin{aligned}\int_S\widetilde{\mathcal{E}}_K[\mathbf{X},\mathbf{w},\mathbf{M}] \gtrsim{} & \int_S\frac{m}{r^2}|\nabla_{\widehat{R}}\psi|^2_{\mathcal{S}} + r^{-3}|\psi|^2_{\mathcal{S}} + r^3\Big(\big|\nabla_{\widehat{T}}\Psi_z\big|^2 + r^2\big|\nabla\Psi_z\big|^2\Big) + \mathrm{Bdr}\\ & - O(a)\int_S\big(r^{-2}|\nabla_{\widehat{R}}\mathfrak{d}^{\leq 2}\psi|^2 + r^{-3}|\mathfrak{d}^{\leq 2}\psi|^2\big) + \int_S\mathrm{Good}_{\mathcal{S}}\end{aligned} \tag{9.2.54}$$

on any sphere $S = S(\tau, r)$.

Next, proceeding as in the beginning of section 8.3.3, relying on Lemma 9.2.35, (9.2.36) and (9.2.43), we derive the following analog of (8.3.5)

$$
\begin{aligned}
\mathbf{D}^\mu \mathcal{P}_\mu[\mathbf{X},\mathbf{w},\mathbf{M}] =& \widetilde{\mathcal{E}}_K[\mathbf{X},\mathbf{w},\mathbf{M}] + \mathbf{D}^\mu \mathcal{B}_\mu + \left(\nabla_{X^{\underline{ab}}}\psi_{\underline{a}} + \frac{1}{2} w^{\underline{ab}}\psi_{\underline{a}}\right)\cdot \left(\dot{\square}_2\psi_{\underline{b}} - V\psi_{\underline{b}}\right)\\
&- \frac{2a^2 r\cos\theta \mathcal{F}^{\underline{ab}}(r)}{(r^2+a^2)|q|^4}\nabla_{\mathbf{Z}}\psi_{\underline{a}}\cdot {}^*\psi_{\underline{b}}\\
&- \left(\left({}^*\rho + \underline{\eta}\wedge\eta\right)\mathcal{F}^{\underline{ab}}\frac{r^2+a^2}{\Delta} + \frac{2a^3 r\cos\theta(\sin\theta)^2\mathcal{F}^{\underline{ab}}(r)}{|q|^6}\right)\nabla_{\widehat{T}}\psi_{\underline{a}}\cdot {}^*\psi_{\underline{b}}\\
&+ r^{-1}\Gamma_b\Big(|(X^{\underline{ab}})^3| + (X^{\underline{ab}})^4|\Big)\nabla\psi_{\underline{a}}\cdot {}^*\psi_{\underline{b}}
\end{aligned}
$$

where we used the fact that $\check{H}\in\Gamma_b$ and $\widetilde{\mathrm{tr}\underline{X}}\in\Gamma_g$. Since $|(X^{\underline{ab}})^3| + (X^{\underline{ab}})^4| \lesssim 1$, we have

$$
r^{-1}\Gamma_b\Big(|(X^{\underline{ab}})^3| + (X^{\underline{ab}})^4|\Big)\nabla\psi_{\underline{a}}\cdot {}^*\psi_{\underline{b}} \;=\; r^{-2}\Gamma_b\mathfrak{d}^{\leq 1}\psi\cdot {}^*\psi = \mathrm{Good}_{\mathcal{S}}
$$

and we deduce

$$
\begin{aligned}
&\mathbf{D}^\mu \mathcal{P}_\mu[\mathbf{X},\mathbf{w},\mathbf{M}]\\
=&\widetilde{\mathcal{E}}_K[\mathbf{X},\mathbf{w},\mathbf{M}] + \mathbf{D}^\mu \mathcal{B}_\mu + \left(\nabla_{X^{\underline{ab}}}\psi_{\underline{a}} + \frac{1}{2} w^{\underline{ab}}\psi_{\underline{a}}\right)\cdot \left(\dot{\square}_2\psi_{\underline{b}} - V\psi_{\underline{b}}\right)\\
&- \left(\left({}^*\rho + \underline{\eta}\wedge\eta\right)\mathcal{F}^{\underline{ab}}\frac{r^2+a^2}{\Delta} + \frac{2a^3 r\cos\theta(\sin\theta)^2\mathcal{F}^{\underline{ab}}(r)}{|q|^6}\right)\nabla_{\widehat{T}}\psi_{\underline{a}}\cdot {}^*\psi_{\underline{b}}\\
&- \frac{2a^2 r\cos\theta \mathcal{F}^{\underline{ab}}(r)}{(r^2+a^2)|q|^4}\nabla_{\mathbf{Z}}\psi_{\underline{a}}\cdot {}^*\psi_{\underline{b}} + \mathrm{Good}_{\mathcal{S}}.
\end{aligned}
\tag{9.2.55}
$$

We now derive the following analog of Lemma 8.3.2.

Lemma 9.2.42. *We have, for sufficiently small positive constants δ_2, δ_3:*

$$
\begin{aligned}
&\left(\nabla_{X^{\underline{ab}}}\psi_{\underline{a}} + \frac{1}{2} w^{\underline{ab}}\psi_{\underline{a}}\right)\cdot \left(\dot{\square}_2\psi_{\underline{b}} - V\psi_{\underline{b}}\right)\\
\geq& -\delta_2 r^{-2}h|\nabla_{\widehat{T}}\Psi_z|^2 - \delta_2 a^2 r^{-6}h|\nabla_{\mathbf{Z}}\Psi_z|^2 + O(ar^{-3})|\psi|^2_{\mathcal{S}}\\
&+ O(1)\Big(|\nabla_{\widehat{R}}\psi|_{\mathcal{S}} + r^{-1}|\psi|_{\mathcal{S}}\Big)\sum_{\underline{a}=1}^{4}|N_{\underline{a}}|\\
&+ \dot{\mathbf{D}}_\mu\left(\frac{2a\cos\theta}{|q|^2}\frac{r^2+a^2}{\Delta}\widehat{R}^\mu z h f^{\underline{ab}}\psi_{\underline{a}}\cdot\nabla_{\mathbf{T}}\,{}^*\psi_{\underline{b}}\right)\\
&- \dot{\mathbf{D}}_\mu\left(\frac{2a\cos\theta}{|q|^2}\left(zhf^{\underline{ab}}\psi_{\underline{a}}\cdot\frac{r^2+a^2}{\Delta}\nabla_{\widehat{R}}\,{}^*\psi_{\underline{b}} + \frac{1}{2}(\partial_r z)h\Psi_z\cdot {}^*\mathcal{L}^{\underline{a}}\psi_{\underline{a}}\right)\mathbf{T}^\mu\right) + \mathit{Good}_{\mathcal{S}},
\end{aligned}
\tag{9.2.56}
$$

and

$$
\begin{aligned}
&\left(\left({}^*\rho+\underline{\eta}\wedge\eta\right)\mathcal{F}^{\underline{ab}}\frac{r^2+a^2}{\Delta}+\frac{2a^3r\cos\theta(\sin\theta)^2\mathcal{F}^{\underline{ab}}(r)}{|q|^6}\right)\nabla_{\widehat{T}}\psi_{\underline{a}}\cdot {}^*\psi_{\underline{b}}\\
&+\frac{2a^2r\cos\theta\mathcal{F}^{\underline{ab}}(r)}{(r^2+a^2)|q|^4}\nabla_{\mathbf{Z}}\psi_{\underline{a}}\cdot {}^*\psi_{\underline{b}}\\
\leq&\delta_3r^3\Big(|\nabla_{\widehat{T}}\Psi_z|^2+r^2|\nabla\Psi_z|^2\Big)+O(ar^{-3})|\psi|^2_{\mathcal{S}}-\frac{1}{2}\dot{\mathbf{D}}_\mu\left(zh\frac{2a^2r\cos\theta}{(r^2+a^2)|q|^4}\Psi_z\cdot {}^*(\mathcal{L}^{\underline{b}}\psi_{\underline{b}})\mathbf{Z}^\mu\right)\\
&-\frac{1}{2}\dot{\mathbf{D}}_\mu\left(zh\left(\left({}^*\rho+\underline{\eta}\wedge\eta\right)\frac{r^2+a^2}{\Delta}+\frac{2a^3r\cos\theta(\sin\theta)^2}{|q|^6}\right)\Psi_z\cdot {}^*(\mathcal{L}^{\underline{b}}\psi_{\underline{b}})\widehat{T}^\mu\right)+\mathit{Good}_{\mathcal{S}}.
\end{aligned}
\tag{9.2.57}
$$

Proof. We proceed as in the proof of Lemma 8.3.2. It then suffices to check that the extra error terms are all of the type $\text{Good}_{\mathcal{S}}$. For the first estimate, the extra error terms are of the type

$$
\begin{aligned}
&O(r^{-2})\Big(r^{-1}\widecheck{\widehat{R}(r)},r^{-1}\mathbf{T}(r),(\mathbf{D}_\mu\mathbf{T}^\mu),\widecheck{\mathbf{D}_\mu\widehat{R}^\mu},\mathbf{T}(\cos\theta),\widehat{R}(\cos\theta)\Big)\mathfrak{d}^{\leq 2}\psi\cdot\mathfrak{d}^{\leq 3}\psi\\
=\;& r^{-2}\Gamma_b\mathfrak{d}^{\leq 2}\psi\cdot\mathfrak{d}^{\leq 3}\psi=\text{Good}_{\mathcal{S}}
\end{aligned}
$$

as desired.

For the second estimate, the extra error terms are of the type

$$
O(1)\Big(\widecheck{{}^*\rho},O(r^{-2})\check{H},O(r^{-2})\widecheck{\underline{H}}\Big)\mathfrak{d}^{\leq 2}\psi\cdot\mathfrak{d}^{\leq 3}\psi=r^{-1}\Gamma_g\mathfrak{d}^{\leq 2}\psi\cdot\mathfrak{d}^{\leq 3}\psi=\text{Good}_{\mathcal{S}}
$$

and

$$
\begin{aligned}
&O(r^{-4})\Big(r^{-1}\mathbf{T}(r),r^{-1}\mathbf{Z}(r),(\mathbf{D}_\mu\mathbf{T}^\mu),(\mathbf{D}_\mu\mathbf{Z}^\mu),\mathbf{T}(\cos\theta),\mathbf{Z}(\cos\theta)\Big)\mathfrak{d}^{\leq 2}\psi\cdot\mathfrak{d}^{\leq 3}\psi\\
=\;& r^{-2}\Gamma_b\mathfrak{d}^{\leq 2}\psi\cdot\mathfrak{d}^{\leq 3}\psi=\text{Good}_{\mathcal{S}}
\end{aligned}
$$

as desired. This concludes the proof of Lemma 9.2.42. □

We are now ready to prove Proposition 9.2.15. Recall (9.2.55), i.e.

$$
\begin{aligned}
&\mathbf{D}^\mu\mathcal{P}_\mu[\mathbf{X},\mathbf{w},\mathbf{M}]\\
=&\widetilde{\mathcal{E}}_K[\mathbf{X},\mathbf{w},\mathbf{M}]+\mathbf{D}^\mu\mathcal{B}_\mu+\left(\nabla_{X^{\underline{ab}}}\psi_{\underline{a}}+\frac{1}{2}w^{\underline{ab}}\psi_{\underline{a}}\right)\cdot\left(\dot{\square}_2\psi_{\underline{b}}-V\psi_{\underline{b}}\right)\\
&-\left(\left({}^*\rho+\underline{\eta}\wedge\eta\right)\mathcal{F}^{\underline{ab}}\frac{r^2+a^2}{\Delta}+\frac{2a^3r\cos\theta(\sin\theta)^2\mathcal{F}^{\underline{ab}}(r)}{|q|^6}\right)\nabla_{\widehat{T}}\psi_{\underline{a}}\cdot {}^*\psi_{\underline{b}}\\
&-\frac{2a^2r\cos\theta\mathcal{F}^{\underline{ab}}(r)}{(r^2+a^2)|q|^4}\nabla_{\mathbf{Z}}\psi_{\underline{a}}\cdot {}^*\psi_{\underline{b}}+\text{Good}_{\mathcal{S}}.
\end{aligned}
$$

We apply the divergence theorem to the above on $\mathcal{M}(\tau_1,\tau_2)$, which yields

$$\begin{aligned}
&\int_{\mathcal{M}(\tau_1,\tau_2)}\Bigg[\widetilde{\mathcal{E}}_K[\mathbf{X},\mathbf{w},\mathbf{M}]+\left(\nabla_{X^{\underline{ab}}}\psi_{\underline{a}}+\frac{1}{2}w^{\underline{ab}}\psi_{\underline{a}}\right)\cdot(\dot{\square}_2\psi_{\underline{b}}-V\psi_{\underline{b}})\\
&-\left(\left({}^*\rho+\underline{\eta}\wedge\eta\right)\mathcal{F}^{\underline{ab}}\frac{r^2+a^2}{\Delta}+\frac{2a^3r\cos\theta(\sin\theta)^2\mathcal{F}^{\underline{ab}}(r)}{|q|^6}\right)\nabla_{\widehat{T}}\psi_{\underline{a}}\cdot\ {}^*\psi_{\underline{b}}\\
&-\frac{2a^2r\cos\theta\mathcal{F}^{\underline{ab}}(r)}{(r^2+a^2)|q|^4}\nabla_{\mathbf{Z}}\psi_{\underline{a}}\cdot\ {}^*\psi_{\underline{b}}\Bigg]\\
\leq\ &\int_{\partial\mathcal{M}(\tau_1,\tau_2)}\big(|\mathcal{P}_\mu[\mathbf{X},\mathbf{w},\mathbf{M}]\mathbf{N}^\mu|+|\mathcal{B}_\mu\mathbf{N}^\mu|+\mathrm{Good}_{\mathcal{S}}\big).
\end{aligned}$$

Then, we proceed as in section 8.3.4 relying on the lower bound (9.2.54) for $\widetilde{\mathcal{E}}_K[\mathbf{X},\mathbf{w},\mathbf{M}]$ and Lemma 9.2.42 which leads to

$$\begin{aligned}
\mathrm{Mor}_{\mathcal{S},z,deg}[\psi](\tau_1,\tau_2)\lesssim&\int_{\partial\mathcal{M}(\tau_1,\tau_2)}|M_{\mathcal{S}}(\psi)|+\frac{|a|}{m}B^2_\delta[\psi](\tau_1,\tau_2)\\
&+\sum_{\underline{a}=1}^{4}\int_{\mathcal{M}(\tau_1,\tau_2)}\big(|\nabla_{\widehat{R}}\psi_{\underline{a}}|+r^{-1}|\psi_{\underline{a}}|\big)|N_{\underline{a}}|+\int_{\mathcal{M}(\tau_1,\tau_2)}|\mathrm{Good}_{\mathcal{S}}|.
\end{aligned}$$

where $M_{\mathcal{S}}(\psi)$ denotes an expression in ψ for which we have a bound of the form

$$\begin{aligned}
&\int_{\partial\mathcal{M}(\tau_1,\tau_2)}|M_{\mathcal{S}}(\psi)|\\
\lesssim\ &\sum_{\underline{a}=1}^{4}\left(\sup_{[\tau_1,\tau_2]}E_{deg}[\psi_{\underline{a}}](\tau)+\delta_{\mathcal{H}}F_{\mathcal{A}}[\psi_{\underline{a}}](\tau_1,\tau_2)+F_{\Sigma_*}[\psi_{\underline{a}}](\tau_1,\tau_2)\right)\\
+\ &\left(\sup_{[\tau_1,\tau_2]}E_{deg}[(\nabla_T,\,\not{\mathfrak{d}})^{\leq1}\psi](\tau)+\delta_{\mathcal{H}}F_{\mathcal{A}}[(\nabla_T,\,\not{\mathfrak{d}})^{\leq1}\psi](\tau_1,\tau_2)+F_{\Sigma_*}[(\nabla_T,\,\not{\mathfrak{d}})^{\leq1}\psi](\tau_1,\tau_2)\right)^{\frac{1}{2}}\\
\times\ &\left(\sup_{[\tau_1,\tau_2]}E_{deg}[(\nabla_T,\,\not{\mathfrak{d}})^{\leq2}\psi](\tau)+\delta_{\mathcal{H}}F_{\mathcal{A}}[(\nabla_T,\,\not{\mathfrak{d}})^{\leq2}\psi](\tau_1,\tau_2)+F_{\Sigma_*}[(\nabla_T,\,\not{\mathfrak{d}})^{\leq2}\psi](\tau_1,\tau_2)\right)^{\frac{1}{2}}.
\end{aligned}$$

Together with the control of $\mathrm{Good}_{\mathcal{S}}$ provided by Remark 9.2.33, this concludes the proof of Proposition 9.2.15.

Proof of Lemma 9.2.16

Proceeding exactly as for the proof of Lemma 6.3.11 in section 8.4, we obtain on $\mathcal{M}_{tr\not{p}}$ the following analog of (8.4.1)

$$\begin{aligned} r^3\Big(|\nabla_{\mathbf{T}}\Psi_z|^2+r^2|\nabla\Psi_z|^2\Big) \geq& c_0 r^{-3}\Big(\big|\nabla_{\mathbf{T}}\big(\delta_0\mathcal{S}_1\psi+\mathcal{O}\psi\big)\big|^2+|q|^2\big|\nabla\big(\delta_0\mathcal{S}_1\psi+\mathcal{O}\psi\big)\big|^2\Big)\\ &-O(\delta_0^2r^{-3})\Big(|\nabla_{\mathbf{T}}\mathcal{O}\psi|^2+r^2|\nabla\mathcal{O}\psi|^2\Big)\\ &-O(ar^{-3})|(\nabla_{\mathbf{T}},\,\not{\mathfrak{d}})\mathfrak{d}^{\leq 2}\psi|^2-O(r^{-3})|\psi|^2_{\mathcal{S}}. \end{aligned} \tag{9.2.58}$$

as well a the following computation

$$\begin{aligned} &\big|\nabla_{\mathbf{T}}\big(\delta_0\mathcal{S}_1\psi+\mathcal{O}\psi\big)\big|^2+|q|^2\big|\nabla\big(\delta_0\mathcal{S}_1\psi+\mathcal{O}\psi\big)\big|^2\\ =\;& \delta_0^2\Big(|\nabla_{\mathbf{T}}\mathcal{S}_1\psi|^2+|q|^2|\nabla\mathcal{S}_1\psi|^2\Big)+2\delta_0\Big(\nabla_{\mathbf{T}}\mathcal{S}_1\psi\cdot\nabla_{\mathbf{T}}\mathcal{O}\psi+|q|^2\nabla\mathcal{S}_1\psi\cdot\nabla\mathcal{O}\psi\Big)\\ &+|\nabla_{\mathbf{T}}\mathcal{O}\psi|^2+|q|^2|\nabla\mathcal{O}\psi|^2. \end{aligned}$$

Next, recalling the definition of $\mathcal{O}$ in (4.5.2), we have

$$\begin{aligned} [\nabla,\mathcal{O}] &= \left[\nabla,|q|^2\left(\triangle_2\psi+\frac{2a^2\cos\theta}{|q|^2}\;{}^*\Re(\mathfrak{J})^b\nabla_b\psi\right)\right]\\ &= |q|^2[\nabla,\Delta_2]\psi+O(ar^{-3})\,\not{\mathfrak{d}}^{\leq 2}\psi+\Big(r^{-1}\nabla(r),r^{-2}\nabla\widecheck{(\cos\theta)},r^{-3}\nabla\widecheck{\Re(\mathfrak{J})}\Big)\,\not{\mathfrak{d}}^{\leq 2}\psi\\ &= |q|^2[\nabla,\Delta_2]\psi+O(ar^{-3})\,\not{\mathfrak{d}}^{\leq 2}\psi+\Gamma_g\,\not{\mathfrak{d}}^{\leq 2}\psi. \end{aligned}$$

Now, in view of Proposition 2.1.43, we have

$$\begin{aligned} [\nabla_a,\nabla_b]\psi &= \Big(\frac{1}{2}({}^{(a)}\mathrm{tr}\chi\nabla_3+{}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4)\psi+2\;{}^{(h)}K\;{}^*\psi\Big)\in_{ab}\\ &= \frac{2}{r^2}\;{}^*\psi\in_{ab}+O(ar^{-2})(\nabla_{\mathbf{T}},\,\not{\mathfrak{d}})^{\leq 1}\psi+\Gamma_g\mathfrak{d}^{\leq 1}\psi \end{aligned}$$

and hence

$$\begin{aligned} [\nabla_c,\Delta_2]\psi &= \mathbf{g}^{ab}[\nabla_c,\nabla_a]\nabla_b+\mathbf{g}^{ab}\nabla_a[\nabla_c,\nabla_b]\\ &= -\frac{4}{r^2}\nabla_c\psi+O(ar^{-3})(\nabla_{\mathbf{T}},\,\not{\mathfrak{d}})^{\leq 2}\psi+r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_g\mathfrak{d}^{\leq 1}\psi) \end{aligned}$$

which implies

$$[\nabla,\mathcal{O}] = -4\nabla\psi+O(ar^{-1})(\nabla_{\mathbf{T}},\,\not{\mathfrak{d}})^{\leq 2}\psi+\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\psi).$$

Together with the commutators in Lemma 9.2.3 and Corollary 9.2.2, we infer

$$\begin{aligned}
&\nabla_{\mathbf{T}}\mathcal{S}_1\psi\cdot\nabla_{\mathbf{T}}\mathcal{O}\psi+|q|^2\nabla\mathcal{S}_1\psi\cdot\nabla\mathcal{O}\psi\\
=\;&\mathcal{S}_1\nabla_{\mathbf{T}}\psi\cdot\mathcal{O}\nabla_{\mathbf{T}}\psi+|q|^2\mathcal{S}_1\nabla\psi\cdot\mathcal{O}\nabla\psi\\
&+\mathcal{S}_1\nabla_{\mathbf{T}}\psi\cdot[\nabla_{\mathbf{T}},\mathcal{O}]\psi+|q|^2\nabla\mathcal{S}_1\psi\cdot[\nabla,\mathcal{O}]\psi+|q|^2[\nabla,\mathcal{S}_1]\psi\cdot\mathcal{O}\nabla\psi\\
=\;&\mathcal{S}_1\nabla_{\mathbf{T}}\psi\cdot\mathcal{O}\nabla_{\mathbf{T}}\psi+|q|^2\mathcal{S}_1\nabla\psi\cdot\mathcal{O}\nabla\psi-4|q|^2\nabla\mathcal{S}_1\psi\cdot\nabla\psi+O(a)\big((\nabla_{\mathbf{T}},\,\not{\mathfrak{d}})\mathfrak{d}^{\leq 2}\psi\big)^2\\
&+r\mathfrak{d}^{\leq 3}\psi\cdot\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\psi)\\
=\;&\mathcal{S}_1\nabla_{\mathbf{T}}\psi\cdot\mathcal{O}\nabla_{\mathbf{T}}\psi+|q|^2\mathcal{S}_1\nabla\psi\cdot\mathcal{O}\nabla\psi+|q|^2|\nabla\nabla_{\mathbf{T}}\psi|^2+O(ar^{-3})\big((\nabla_{\mathbf{T}},\,\not{\mathfrak{d}})\mathfrak{d}^{\leq 2}\psi\big)^2\\
&-\mathbf{D}_\mu(|q|^2\nabla\nabla_{\mathbf{T}}\psi\cdot\nabla\psi\mathbf{T}^\mu)+r\mathfrak{d}^{\leq 3}\psi\cdot\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\psi).
\end{aligned}$$

Next, we rely on Lemma 9.2.38 and obtain

$$\begin{aligned}
&\nabla_{\mathbf{T}}\mathcal{S}_1\psi\cdot\nabla_{\mathbf{T}}\mathcal{O}\psi+|q|^2\nabla\mathcal{S}_1\psi\cdot\nabla\mathcal{O}\psi\\
=\;&|q|^2|\nabla\nabla_{\mathbf{T}}^2\psi|^2+|q|^2\dot{\mathbf{D}}_\beta(|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\nabla_{\mathbf{T}}\psi\cdot\mathcal{S}_1\nabla_{\mathbf{T}}\psi)+\dot{\mathbf{D}}_\mu(M(\nabla_{\mathbf{T}}\psi)\mathbf{T}^\mu)\\
&+|q|^4|\nabla^2\nabla_{\mathbf{T}}\psi|^2+|q|^4\dot{\mathbf{D}}_\beta(|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\nabla\psi\cdot\mathcal{S}_1\nabla\psi)+\dot{\mathbf{D}}_\mu(r^2M(\nabla\psi)\mathbf{T}^\mu)\\
&+|q|^2|\nabla\nabla_{\mathbf{T}}\psi|^2+O(a)\big((\nabla_{\mathbf{T}},\,\not{\mathfrak{d}})^{\leq 1}\mathfrak{d}^{\leq 2}\psi\big)^2\\
&-\mathbf{D}_\mu(|q|^2\nabla\nabla_{\mathbf{T}}\psi\cdot\nabla\psi\mathbf{T}^\mu)+r\mathfrak{d}^{\leq 3}\psi\cdot\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\psi)+r^2\mathrm{Good}_{\mathcal{S}}.
\end{aligned}$$

Rearranging, and using the definition of $\mathrm{Good}_{\mathcal{S}}$, this yields

$$\begin{aligned}
&\nabla_{\mathbf{T}}\mathcal{S}_1\psi\cdot\nabla_{\mathbf{T}}\mathcal{O}\psi+|q|^2\nabla\mathcal{S}_1\psi\cdot\nabla\mathcal{O}\psi\\
=\;&|q|^2|\nabla\nabla_{\mathbf{T}}^2\psi|^2+|q|^4|\nabla^2\nabla_{\mathbf{T}}\psi|^2+|q|^2|\nabla\nabla_{\mathbf{T}}\psi|^2-O(a)\big|(\nabla_{\mathbf{T}},\,\not{\mathfrak{d}})^{\leq 1}\mathfrak{d}^{\leq 2}\psi\big|^2\\
&+|q|^2\dot{\mathbf{D}}_\beta(|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\nabla_{\mathbf{T}}\psi\cdot\mathcal{S}_1\nabla_{\mathbf{T}}\psi)+|q|^4\dot{\mathbf{D}}_\beta(|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\nabla\psi\cdot\mathcal{S}_1\nabla\psi)\\
&+\dot{\mathbf{D}}_\mu(M(\nabla_{\mathbf{T}}\psi)\mathbf{T}^\mu)+\dot{\mathbf{D}}_\mu(M(|q|\nabla\psi)\mathbf{T}^\mu)-\dot{\mathbf{D}}_\mu(|q|^2\nabla\nabla_{\mathbf{T}}\psi\cdot\nabla\psi\mathbf{T}^\mu)+r^2\mathrm{Good}_{\mathcal{S}}.
\end{aligned}$$

The remainder of the proceeds exactly as for the proof of Lemma 6.3.11 in section 8.4, and leads to a universal constant $c_0>0$ such that the following holds on $\mathcal{M}_{tr\not{p}}$

$$\begin{aligned}
r^3\Big(|\nabla_{\mathbf{T}}\Psi_z|^2+r^2|\nabla\Psi_z|^2\Big)+r^{-3}|\psi|^2_{\mathcal{S}}\;\;\geq\;\;& c_0r^{-3}\Big(|\nabla_{\mathbf{T}}\psi|^2_{\mathcal{S}}+|\nabla_Z\psi|^2_{\mathcal{S}}+r^2|\nabla\psi|^2_{\mathcal{S}}\Big)\\
&-O(ar^{-3})\big|(\nabla_{\mathbf{T}},\,\not{\mathfrak{d}})^{\leq 1}\mathfrak{d}^{\leq 2}\psi\big|^2+\dot{\mathbf{D}}_\alpha F^\alpha+\mathrm{Err}_\epsilon,
\end{aligned}$$

where the additional error term Err_ϵ is given in view of the above by

$$\mathrm{Err}_\epsilon=\mathrm{Good}_{\mathcal{S}}$$

and thus satisfies in view of the control of $\mathrm{Good}_{\mathcal{S}}$ provided by Remark 9.2.33

$$\int_{\mathcal{M}}|\mathrm{Err}_\epsilon|\;\;\lesssim\;\;\epsilon\left(\sup_{[\tau_1,\tau_2]}E^2[\psi](\tau)+B^2_\delta[\psi](\tau_1,\tau_2)\right)$$

as stated. This concludes the proof of Lemma 9.2.16.

9.3 Non-integrable Hodge estimates

In this section, we derive elliptic type estimates by projecting the basic horizontal Hodge operators on surfaces S of constant (τ, r) in $\mathcal{M}$ and by relying on the standard elliptic estimates on spheres, see Proposition 2.1.35. We start with the following lemma.

Lemma 9.3.1. *Let ${}^S e_b$ denote the projections of e_b to the spheres S of fixed (τ, r). Then*

$$ {}^S e_b = \big(1 + O(ar^{-2})\big)e_b + O(ar^{-1})\mathbf{T} + r\Gamma_g(e_3, e_4). \tag{9.3.1} $$

Proof. Note that

$$ \mathbf{T}(r) = r\Gamma_b, \qquad \mathbf{T}(\tau) = 1 + r\Gamma_b, \qquad \nabla(r) = r\Gamma_g, \qquad \nabla(\tau) = a\Re(\mathfrak{J}) + \Gamma_b. $$

Thus, setting $e'_b = e_b + \Lambda_b \mathbf{T}$ for a 1-form Λ, and assuming $\Lambda = O(r^{-1})$, we have

$$ e'_b(r) = r\Gamma_g, \qquad e'_b(\tau) = a\Re(\mathfrak{J})_b + \Lambda_b + \Gamma_b. $$

We may thus choose $\Lambda = -a\Re(\mathfrak{J})$, which indeed satisfies $\Lambda = O(r^{-1})$, and deduce $e'_b(r), e'_b(\tau) = r\Gamma_g$.

We then look for a pair of vectorfields $\widetilde{e}^S_b$, $b = 1, 2$, under the form $\widetilde{e}^S_b = e'_b + \lambda e_3 + \underline{\lambda} e_4$ and choose λ and $\underline{\lambda}$ to enforce $\widetilde{e}^S_b(\tau) = 0$ and $\widetilde{e}^S_b(r) = 0$ so that $\widetilde{e}^S_b$, $b = 1, 2$ are tangent to S. We infer

$$ \begin{aligned} 0 &= \widetilde{e}^S_b(r) = e'_b(r) + \lambda e_3(r) + \underline{\lambda} e_4(r) = (-1 + r\Gamma_b)\,\lambda + \left(\frac{\Delta}{|q|^2} + r\Gamma_b\right)\underline{\lambda} + r\Gamma_g, \\ 0 &= \widetilde{e}^S_b(\tau) = e'_b(\tau) + \lambda e_3(\tau) + \underline{\lambda} e_4(\tau) = \lambda e_3(\tau) + \underline{\lambda} e_4(\tau) + r\Gamma_g. \end{aligned} $$

Since $e_3(\tau) > 0$ and $e_4(\tau) > 0$, see Definition 6.1.5, we infer that $\lambda, \underline{\lambda}$ exist and satisfy $\lambda, \underline{\lambda} \in r\Gamma_g$. Hence

$$ \begin{aligned} \widetilde{e}^S_b &= e'_b + \lambda e_3 + \underline{\lambda} e_4 = e_b - a\Re(\mathfrak{J})_b \mathbf{T} + r\Gamma_g(e_3, e_4) \\ &= e_b + O(ar^{-1})\mathbf{T} + r\Gamma_g(e_3, e_4). \end{aligned} $$

We then apply Gram-Schmidt to $\widetilde{e}^S_b$, $b = 1, 2$ so obtain an orthonormal frame e^S_b, $b = 1, 2$ of S satisfying the desired properties. This concludes the proof of Lemma 9.3.1. □

Proposition 9.3.2. *Consider a sphere $S \subset \mathcal{M}$ of the type $S(\tau, r)$. The following estimates hold true for a small enough:*

1. *For* $f \in \mathfrak{s}_p$*,* $p = 1, 2$*, we have*

$$\int_S \big(r^2|\nabla f|^2 + |f|^2\big) \lesssim \int_S r^2|\,{\not{\mathcal{D}}}_p f|^2 + O(a^2)\int_S |\nabla_{\mathbf{T}} f|^2 + O(\epsilon^2)\int_S |(\nabla_3, \nabla_4) f|^2. \tag{9.3.2}$$

Moreover for higher derivatives, in a simplified form, we have

$$\int_S \big(r^2|\nabla \mathfrak{d}^{\le k} f|^2 + |\mathfrak{d}^{\le k} f|^2\big) \lesssim \int_S r^2|\mathfrak{d}^{\le k}\,{\not{\mathcal{D}}}_p f|^2 + O(a^2, \epsilon^2)\int_S |\mathfrak{d}^{\le k+1} f|^2. \tag{9.3.3}$$

2. *For* $f \in \mathfrak{s}_p$*,* $p = 0, 1$*, we have*

$$\int_S r^2|\nabla f|^2 \lesssim \int_S \big(r^2|\,{\not{\mathcal{D}}}_p^* f|^2 + |f|^2\big) + O(a^2)\int_S |\nabla_{\mathbf{T}} f|^2 + O(\epsilon^2)\int_S |(\nabla_3, \nabla_4) f|^2. \tag{9.3.4}$$

Moreover for higher derivatives, in a simplified form, we have

$$\int_S r^2|\nabla \mathfrak{d}^{\le k} f|^2 \lesssim \int_S \big(r^2|\mathfrak{d}^{\le k}\,{\not{\mathcal{D}}}_p f|^2 + |\mathfrak{d}^{\le k}\,{\not{\mathcal{D}}}_p f|^2\big) + O(a^2, \epsilon^2)\int_S |\mathfrak{d}^{\le k+1} f|^2. \tag{9.3.5}$$

Proof. We start with the proof of (9.3.2), and consider the case $p = 2$, i.e. $f \in \mathfrak{s}_2$. Recalling from Lemma 9.3.1 that ${}^S e_b = \big(1 + O(ar^{-2})\big)e_b + O(ar^{-1})\mathbf{T} + r\Gamma_g(e_3, e_4)$, we infer

$$\nabla^b f_{ab} = {}^S\nabla^b f_{ab} + O(ar^{-2})\nabla f + O(ar^{-1})\nabla_{\mathbf{T}} f + r\Gamma_g \nabla_3 f.$$

Hence

$${\not{\mathcal{D}}}_2 f = {}^S{\not{\mathcal{D}}}_2 f + O(ar^{-2})\nabla f + O(ar^{-1})\nabla_{\mathbf{T}} f + r\Gamma_g(\nabla_3, \nabla_4) f.$$

Integrating over S and using the standard elliptic estimates of Proposition 2.1.35 we have

$$\int_S |\,{}^S\nabla f|^2 + 2\,K^S|f|^2 = 2\int_S |\,{}^S{\not{\mathcal{D}}}_2 f|^2$$

where K^S is the Gauss curvature of S. We deduce

$$\int_S \big(|\,{}^S\nabla f|^2 + 2\,K^S|f|^2\big) \lesssim 2\int_S |\,{\not{\mathcal{D}}}_2 f|^2 + a^2\int_S r^{-4}|\nabla f|^2 + a^2\int_S r^{-2}|\nabla_{\mathbf{T}} f|^2 + \int_S |r\Gamma_g(\nabla_3, \nabla_4) f|^2.$$

Also, recall from the proof of Lemma 9.2.23 that the Gauss curvature K_S verifies

$$K_S = \frac{1}{r^2}\big(1+O(\epsilon+a^2r^{-2})\big).$$

Hence

$$\begin{aligned}\int_S\left(|{}^S\nabla f|^2+\frac{2}{r^2}|f|^2\right) &\lesssim 2\int_S|\mathcal{D}\!\!\!\!/_2 f|^2+a^2\int_S r^{-4}|\nabla f|^2+a^2\int_S r^{-4}|f|^2\\ &+a^2\int_S r^{-2}|\nabla_{\mathbf{T}}f|^2+\int_S|r\Gamma_g(\nabla_3,\nabla_4)f|^2.\end{aligned}$$

Also

$$\int_S|{}^S\nabla f|^2 = \int_S\big(1+O(a^2r^{-4})\big)|\nabla f|^2+O(a^2)\int_S r^{-2}|\nabla_{\mathbf{T}}f|^2+\int_S|r\Gamma_g(\nabla_3,\nabla_4)f|^2$$

and thus, absorbing terms on the RHS for a small enough, we infer

$$\int_S\left(|\nabla f|^2+\frac{2}{r^2}|f|^2\right) \lesssim \int_S|\mathcal{D}\!\!\!\!/_2 f|^2+a^2\int_S r^{-2}|\nabla_{\mathbf{T}}f|^2+\epsilon^2\int_S r^{-2}|(\nabla_3,\nabla_4)f|^2.$$

This ends the proof of (9.3.2) for $p=2$. The case $p=1$ is derived in the same manner.

Next, we consider the higher derivative estimates (9.3.3). For $p=1,2$ and $k\geq 1$, we have the following non sharp commutator estimate

$$\begin{aligned}\mathcal{D}_p\mathfrak{d}^k f &= \mathfrak{d}^k\mathcal{D}_p f+[\mathcal{D}_p,\mathfrak{d}^k]f\\ &= \mathfrak{d}^k\mathcal{D}_p f+O(r^{-1})\nabla\mathfrak{d}^{\leq k-1}f+O(ar^{-2})\mathfrak{d}^{\leq k}f+O(r^{-1})\mathfrak{d}^{\leq k-1}f+O(\epsilon r^{-1})\mathfrak{d}^{\leq k}f.\end{aligned}$$

Hence, applying (9.3.2), we deduce

$$\begin{aligned}\int_S r^2|\nabla\mathfrak{d}^k f|^2+|\mathfrak{d}^k f|^2 &\lesssim \int_S|\mathfrak{d}^k\mathcal{D}_p f|^2+\int_S r^2|\nabla\mathfrak{d}^{\leq k-1}f|^2+|\mathfrak{d}^{\leq k-1}f|^2\\ &+O(a^2,\epsilon^2)\int_S|\mathfrak{d}^{\leq k+1}f|^2\end{aligned}$$

and (9.3.3) follows immediately by iteration starting from the case $k=0$ provided by (9.3.2).

The proof of (9.3.4) is similar to the one of (9.3.2). Then, the proof of (9.3.5) is derived by iteration starting from the case $k=0$ provided by (9.3.4) as above. This concludes the proof of Proposition 9.3.2. □

9.4 Redshift estimates

To remove degeneracies of the energy in the neighborhood of the horizon, we make use of the Dafermos-Rodnianski redshift vectorfield. The goal of this section is to prove the following proposition.

Proposition 9.4.1 (Redshift estimates). *Let ψ a solution to* (6.1.1). *Then, for $|a| < m$, there exists a small enough constant $\delta_{red} > 0$ such that $\delta_{red} = \delta_{red}(m - |a|)$ with $\delta_{red} \geq \delta_{\mathcal{H}}$, and a small constant $c_0 > 0$ with $c_0 = c_0(m - |a|)$, such that the following estimate holds true in $\mathcal{M}(\tau_1, \tau_2)$:*

$$\begin{aligned} & c_0 E_{r \leq r_+(1+\delta_{red})}[\psi](\tau_2) + c_0 Mor_{r \leq r_+(1+\delta_{red})}[\psi](\tau_1, \tau_2) + c_0 F_{\mathcal{A}}[\psi](\tau_1, \tau_2) \\ \leq & E_{r \leq r_+(1+2\delta_{red})}[\psi](\tau_1) + \delta_{red}^{-1} Mor_{r_+(1+\delta_{red}) \leq r \leq r_+(1+2\delta_{red})}[\psi](\tau_1, \tau_2) \\ & + \int_{\mathcal{M}(\tau_1,\tau_2) \cap \left\{ \frac{r}{r_+} \leq 1+2\delta_{red} \right\}} |N|^2. \end{aligned} \tag{9.4.1}$$

The proof of Proposition 9.4.1 is postponed to the end of this section. We start with the following lemma.

Lemma 9.4.2. *Given the vectorfield*

$$Y = \underline{d}(r) e_3 + d(r) e_4,$$

and assuming

$$\sup_{r \leq 4m} \Big(|d(r)| + |d'(r)| + |\underline{d}(r)| + |\underline{d}'(r)| \Big) \quad \lesssim \quad 1,$$

we have for $r \leq 4m$

$$\begin{aligned} \mathcal{Q} \cdot {}^{(Y)}\pi \quad = \quad & \left(\partial_r \left(\frac{\Delta}{|q|^2} \right) \underline{d}(r) - \frac{\Delta}{|q|^2} \underline{d}'(r) \right) |\nabla_3 \psi|^2 + d'(r) |\nabla_4 \psi|^2 \\ & + \left(\underline{d}'(r) - \partial_r \left(\frac{\Delta}{|q|^2} \right) d(r) - \frac{\Delta}{|q|^2} d'(r) \right) \Big(|\nabla \psi|^2 + V |\psi|^2 \Big) \\ & + \frac{2ar}{|q|^2} \Re(\mathfrak{J})^b \underline{d}(r) \nabla_b \Psi \cdot \nabla_3 \Psi - \frac{2r}{|q|^2} \left(\underline{d}(r) - \frac{\Delta}{|q|^2} d(r) \right) \Big(\nabla_3 \psi \cdot \nabla_4 \psi - V |\psi|^2 \Big) \\ & + O(\epsilon) \Big(|\nabla_3 \psi|^2 + |\nabla_4 \psi|^2 + |\nabla \psi|^2 + r^{-2} |\psi|^2 \Big) \end{aligned}$$

and

$$\begin{aligned} \mathbf{D}^\mu \mathcal{P}_\mu[Y, 0, 0] \quad = \quad & \frac{1}{2} \mathcal{Q} \cdot {}^{(Y)}\pi + 2 \left(\underline{d}(r) - \frac{\Delta}{|q|^2} d(r) \right) \partial_r \left(\frac{\Delta}{(r^2 + a^2)|q|^2} \right) |\psi|^2 \\ & - \; {}^*\rho \in_{AB} \Big(d(r) \nabla_4 \psi^A \psi^B - \underline{d}(r) \nabla_3 \psi^A \psi^B \Big) + Y(\psi) \cdot \big(\dot{\square}_2 \psi - V \psi \big) \\ & + O(\epsilon) \Big(|\nabla_3 \psi|^2 + |\nabla_4 \psi|^2 + |\nabla \psi|^2 + r^{-2} |\psi|^2 \Big). \end{aligned}$$

Also, we have for $r \le 4m$

$$\begin{aligned}\mathbf{D}^\mu \mathcal{P}_\mu[\mathbf{T}, 0, 0] &= -\frac{1}{2}\ {}^*\rho \in_{AB} \left(\frac{\Delta}{|q|^2}\nabla_4\psi^A\psi^B - \nabla_3\psi^A\psi^B\right) - a\ {}^*\Re(\mathfrak{J})^d\nabla_d\psi_A\rho\ {}^*\psi^A \\ &+\mathbf{T}(\psi)\cdot\big(\dot{\square}_2\psi - V\psi\big) + O(\epsilon)\Big(|\nabla_3\psi|^2 + |\nabla_4\psi|^2 + |\nabla\psi|^2 + r^{-2}|\psi|^2\Big).\end{aligned}$$

Proof. We have

$$\begin{aligned}{}^{(Y)}\pi_{\mu\nu} &= \underline{d}(r)^{(e_3)}\pi_{\mu\nu} + e_\mu(r)\underline{d}'(r)\mathbf{g}(e_3, e_\nu) + e_\nu(r)\underline{d}'(r)\mathbf{g}(e_3, e_\mu) \\ &+d(r)^{(e_4)}\pi_{\mu\nu} + e_\mu(r)d'(r)\mathbf{g}(e_4, e_\nu) + e_\nu(r)d'(r)\mathbf{g}(e_4, e_\mu).\end{aligned}$$

Together with the fact, in the region $r \le 4m$,

$$e_3(r) = -1 + O(\epsilon), \qquad e_4(r) = \frac{\Delta}{|q|^2} + O(\epsilon), \qquad \nabla(r) = O(\epsilon),$$

and using Lemma 9.2.21, we infer

$$\begin{aligned}{}^{(Y)}\pi_{44} &= \underline{d}(r)^{(e_3)}\pi_{44} + 2e_4(r)\underline{d}'(r)\mathbf{g}(e_3, e_4) + d(r)^{(e_4)}\pi_{44} \\ &= 4\partial_r\left(\frac{\Delta}{|q|^2}\right)\underline{d}(r) - \frac{4\Delta}{|q|^2}\underline{d}'(r) + O(\epsilon), \\ {}^{(Y)}\pi_{34} &= \underline{d}(r)^{(e_3)}\pi_{34} + e_3(r)\underline{d}'(r)\mathbf{g}(e_3, e_4) + d(r)^{(e_4)}\pi_{34} + e_4(r)d'(r)\mathbf{g}(e_4, e_3), \\ &= 2\underline{d}'(r) - 2\partial_r\left(\frac{\Delta}{|q|^2}\right)d(r) - \frac{2\Delta}{|q|^2}d'(r) + O(\epsilon), \\ {}^{(Y)}\pi_{33} &= \underline{d}(r)^{(e_3)}\pi_{33} + d(r)^{(e_4)}\pi_{33} + 2e_3(r)d'(r)\mathbf{g}(e_4, e_3) = 4d'(r) + O(\epsilon), \\ {}^{(Y)}\pi_{4b} &= \underline{d}(r)^{(e_3)}\pi_{4b} + d(r)^{(e_4)}\pi_{4b} = -\frac{2ar}{|q|^2}\Re(\mathfrak{J})_b\underline{d}(r) + O(\epsilon), \\ {}^{(Y)}\pi_{3b} &= \underline{d}(r)^{(e_3)}\pi_{3b} + d(r)^{(e_4)}\pi_{3b} = O(\epsilon), \\ {}^{(Y)}\pi_{bc} &= \underline{d}(r)^{(e_3)}\pi_{bc} + d(r)^{(e_4)}\pi_{bc} = -\frac{2r}{|q|^2}\left(\underline{d}(r) - \frac{\Delta}{|q|^2}d(r)\right)\delta_{bc} + O(\epsilon).\end{aligned}$$

This yields, in the region $r \le 4m$,

$$\begin{aligned}\mathcal{Q}\cdot{}^{(Y)}\pi &= \left(\partial_r\left(\frac{\Delta}{|q|^2}\right)\underline{d}(r) - \frac{\Delta}{|q|^2}\underline{d}'(r)\right)\mathcal{Q}_{33} + d'(r)\mathcal{Q}_{44} \\ &+\left(\underline{d}'(r) - \partial_r\left(\frac{\Delta}{|q|^2}\right)d(r) - \frac{\Delta}{|q|^2}d'(r)\right)\mathcal{Q}_{34} \\ &+\frac{2ar}{|q|^2}\Re(\mathfrak{J})^b\underline{d}(r)\mathcal{Q}_{3b} - \frac{2r}{|q|^2}\left(\underline{d}(r) - \frac{\Delta}{|q|^2}d(r)\right)\delta^{bc}\mathcal{Q}_{bc} \\ &+O(\epsilon)\Big(|\nabla_3\psi|^2 + |\nabla_4\psi|^2 + |\nabla\psi|^2 + r^{-2}|\psi|^2\Big).\end{aligned}$$

Since we have in view of (4.7.2)

$$\mathcal{Q}_{33} = |\nabla_3\psi|^2, \qquad \mathcal{Q}_{44} = |\nabla_4\psi|^2, \qquad \mathcal{Q}_{34} = |\nabla\psi|^2 + V|\psi|^2,$$
$$\mathcal{Q}_{4a} = \nabla_4\Psi\cdot\nabla_a\Psi, \qquad \mathcal{Q}_{3a} = \nabla_3\Psi\cdot\nabla_a\Psi, \qquad \delta^{bc}\mathcal{Q}_{bc} = \nabla_3\psi\cdot\nabla_4\psi - V|\psi|^2,$$

we deduce

$$\begin{aligned}\mathcal{Q}\cdot{}^{(Y)}\pi &= \left(\partial_r\left(\frac{\Delta}{|q|^2}\right)\underline{d}(r) - \frac{\Delta}{|q|^2}\underline{d}'(r)\right)|\nabla_3\psi|^2 + d'(r)|\nabla_4\psi|^2\\ &+\left(\underline{d}'(r) - \partial_r\left(\frac{\Delta}{|q|^2}\right)d(r) - \frac{\Delta}{|q|^2}d'(r)\right)\Big(|\nabla\psi|^2 + V|\psi|^2\Big)\\ &+\frac{2ar}{|q|^2}\Re(\mathfrak{J})^b\underline{d}(r)\nabla_b\Psi\cdot\nabla_3\Psi - \frac{2r}{|q|^2}\left(\underline{d}(r) - \frac{\Delta}{|q|^2}d(r)\right)\Big(\nabla_3\psi\cdot\nabla_4\psi - V|\psi|^2\Big)\\ &+O(\epsilon)\Big(|\nabla_3\psi|^2 + |\nabla_4\psi|^2 + |\nabla\psi|^2 + r^{-2}|\psi|^2\Big).\end{aligned}$$

as stated.

Also, in view of Proposition 4.7.3, we have, in the region $r\leq 4m$,

$$\begin{aligned}\mathbf{D}^\mu\mathcal{P}_\mu[Y,0,0] &= \frac{1}{2}\mathcal{Q}\cdot{}^{(Y)}\pi - \frac{1}{2}Y(V)|\psi|^2 - {}^*\rho\in_{AB}\Big(d(r)\nabla_4\psi^A\psi^B - \underline{d}(r)\nabla_3\psi^A\psi^B\Big)\\ &+Y(\psi)\cdot\big(\dot{\square}_2\psi - V\psi\big) + O(\epsilon)\Big(|\nabla_3\psi|^2 + |\nabla_4\psi|^2 + |\nabla\psi|^2 + r^{-2}|\psi|^2\Big).\end{aligned}$$

Since $V = \frac{4\Delta}{(r^2+a^2)|q|^2}$ in view of (6.1.1), and since, in the region $r\leq 4m$,

$$e_3(r) = -1 + O(\epsilon), \qquad e_4(r) = \frac{\Delta}{|q|^2} + O(\epsilon), \qquad e_4(\cos\theta) = O(\epsilon), \quad e_3(\cos\theta) = O(\epsilon),$$

we have

$$Y(V) = -4\left(\underline{d}(r) - \frac{\Delta}{|q|^2}d(r) + O(\epsilon)\right)\partial_r\left(\frac{\Delta}{(r^2+a^2)|q|^2}\right).$$

We deduce

$$\begin{aligned}\mathbf{D}^\mu\mathcal{P}_\mu[Y,0,0] &= \frac{1}{2}\mathcal{Q}\cdot{}^{(Y)}\pi + 2\left(\underline{d}(r) - \frac{\Delta}{|q|^2}d(r)\right)\partial_r\left(\frac{\Delta}{(r^2+a^2)|q|^2}\right)|\psi|^2\\ &- {}^*\rho\in_{AB}\Big(d(r)\nabla_4\psi^A\psi^B - \underline{d}(r)\nabla_3\psi^A\psi^B\Big) + Y(\psi)\cdot\big(\dot{\square}_2\psi - V\psi\big)\\ &+O(\epsilon)\Big(|\nabla_3\psi|^2 + |\nabla_4\psi|^2 + |\nabla\psi|^2 + r^{-2}|\psi|^2\Big)\end{aligned}$$

as stated.

Finally, we have ${}^{(\mathbf{T})}\pi_{\mu\nu} = O(\epsilon)$ in the region $r \le 4m$ so that

$$\mathcal{Q} \cdot {}^{(\mathbf{T})}\pi = O(\epsilon)\Big(|\nabla_3\psi|^2 + |\nabla_4\psi|^2 + |\nabla\psi|^2 + r^{-2}|\psi|^2\Big).$$

Together with Proposition 4.7.2, and using that $\mathbf{T}(V) = O(\epsilon)$ in the region $r \le 4m$, this implies

$$\begin{aligned}
\mathbf{D}^\mu \mathcal{P}_\mu[\mathbf{T}, 0, 0] &= \mathbf{T}^\mu \dot{\mathbf{D}}^\nu \psi^a \mathbf{R}_{ab\nu\mu}\psi^b + \mathbf{T}(\psi) \cdot \big(\dot{\square}_2\psi - V\psi\big) \\
&\quad + O(\epsilon)\Big(|\nabla_3\psi|^2 + |\nabla_4\psi|^2 + |\nabla\psi|^2 + r^{-2}|\psi|^2\Big).
\end{aligned}$$

Also, using the computations in the proof of Proposition 4.7.3, we have, in the region $r \le 4m$,

$$\begin{aligned}
& \mathbf{T}^\mu \dot{\mathbf{D}}^\nu \psi^A \mathbf{R}_{AB\nu\mu}\psi^B \\
={}& - {}^*\rho \in_{AB} \Big(\mathbf{T}^4 \dot{\mathbf{D}}_4 \psi^A \psi^B - \mathbf{T}^3 \dot{\mathbf{D}}_3 \psi^A \psi^B\Big) \\
& + \mathbf{T}^c \left(-\frac{1}{2}\dot{\mathbf{D}}_3 \psi^A \mathbf{R}_{AB4c}\psi^B - \frac{1}{2}\dot{\mathbf{D}}_4 \psi^A \mathbf{R}_{AB3c}\psi^B + \dot{\mathbf{D}}^d \psi^A \mathbf{R}_{ABdc}\psi^B\right) \\
& + O(\epsilon)\Big(|\nabla_3\psi|^2 + |\nabla_4\psi|^2 + |\nabla\psi|^2 + r^{-2}|\psi|^2\Big) \\
={}& -\frac{1}{2}\ {}^*\rho \in_{AB} \left(\frac{\Delta}{|q|^2}\nabla_4\psi^A\psi^B - \nabla_3\psi^A\psi^B\right) \\
& - a\Re(\mathfrak{J})^c \left(-\frac{1}{2}\nabla_3\psi^A \in_{AB}\ {}^*\beta_c \psi^B + \frac{1}{2}\nabla_4\psi^A \in_{AB}\ {}^*\underline{\beta}_c\psi^B + \nabla^d\psi^A \in_{AB}\in_{cd} \rho\psi^B\right) \\
& + O(\epsilon)\Big(|\nabla_3\psi|^2 + |\nabla_4\psi|^2 + |\nabla\psi|^2 + r^{-2}|\psi|^2\Big) \\
={}& -\frac{1}{2}\ {}^*\rho \in_{AB} \left(\frac{\Delta}{|q|^2}\nabla_4\psi^A\psi^B - \nabla_3\psi^A\psi^B\right) - a\ {}^*\Re(\mathfrak{J})^d \nabla_d\psi_A \rho\ {}^*\psi^A \\
& + O(\epsilon)\Big(|\nabla_3\psi|^2 + |\nabla_4\psi|^2 + |\nabla\psi|^2 + r^{-2}|\psi|^2\Big)
\end{aligned}$$

and hence

$$\begin{aligned}
\mathbf{D}^\mu \mathcal{P}_\mu[\mathbf{T}, 0, 0] &= -\frac{1}{2}\ {}^*\rho \in_{AB} \left(\frac{\Delta}{|q|^2}\nabla_4\psi^A\psi^B - \nabla_3\psi^A\psi^B\right) - a\ {}^*\Re(\mathfrak{J})^d \nabla_d\psi_A \rho\ {}^*\psi^A \\
&\quad + \mathbf{T}(\psi) \cdot \big(\dot{\square}_2\psi - V\psi\big) + O(\epsilon)\Big(|\nabla_3\psi|^2 + |\nabla_4\psi|^2 + |\nabla\psi|^2 + r^{-2}|\psi|^2\Big)
\end{aligned}$$

as stated. This concludes the proof of Lemma 9.4.2. □

Corollary 9.4.3. *Let*

$$Y_{(0)} := Y + 2T, \qquad Y = \underline{d}(r)e_3 + d(r)e_4.$$

If we choose

$$d(r_+) = 0, \qquad \underline{d}(r_+) = 1, \qquad d'(r_+) \geq \frac{c_1}{r_+ - m}, \qquad \underline{d}'(r_+) \geq \frac{c_1}{r_+ - m},$$

for some large enough universal constant c_1, then at $r = r_+$, we have, for any sphere $S = S(\tau, r_+)$,

$$\begin{aligned}\int_S \mathbf{D}^\mu \mathcal{P}_\mu[Y_{(0)}, 0, 0] \;\; &\geq \;\; \frac{(r_+ - m)}{32 r_+^2} \int_S \Big(|\nabla_3 \psi|^2 + |\nabla_4 \psi|^2 + |\nabla \psi|^2 + r^{-2} |\psi|^2 \Big) \\ &+ \int_S Y_{(0)}(\psi) \cdot \left(\dot{\square}_2 \psi - V\psi + \frac{4a\cos\theta}{|q|^2} \,{}^*\nabla_T \psi \right). \qquad (9.4.2)\end{aligned}$$

Proof. At $r = r+$, assuming that $\underline{d}(r_+) = 1$ and $d(r_+) = 0$, and using $\Delta(r_+) = 0$, $\partial_r \left(\frac{\Delta}{|q|^2}\right)_{|r=r_+} = \frac{2(r_+ - m)}{|q|^2}$ and $V(r_+) = 0$, we obtain from Lemma 9.4.2

$$\begin{aligned}\mathcal{Q} \cdot {}^{(Y)}\pi \;\; &= \;\; \frac{2(r_+ - m)}{|q|^2} |\nabla_3 \psi|^2 + d'(r_+) |\nabla_4 \psi|^2 + \underline{d}'(r_+) |\nabla \psi|^2 \\ &+ \frac{2ar}{|q|^2} \Re(\mathfrak{J})^b \nabla_b \Psi \cdot \nabla_3 \Psi - \frac{2r}{|q|^2} \nabla_3 \psi \cdot \nabla_4 \psi \\ &+ O(\epsilon) \Big(|\nabla_3 \psi|^2 + |\nabla_4 \psi|^2 + |\nabla \psi|^2 + r^{-2} |\psi|^2 \Big).\end{aligned}$$

Also, since $\partial_r \left(\frac{\Delta}{(r^2+a^2)|q|^2}\right)_{|r=r_+} = \frac{2(r_+ - m)}{(r^2+a^2)|q|^2}$, we obtain from Lemma 9.4.2, at $r = r+$,

$$\begin{aligned}\mathbf{D}^\mu \mathcal{P}_\mu[Y, 0, 0] \;\; &= \;\; \frac{1}{2} \mathcal{Q} \cdot {}^{(Y)}\pi + \frac{4(r_+ - m)}{(r^2 + a^2)|q|^2} |\psi|^2 + \; {}^*\rho \in_{AB} \nabla_3 \psi^A \psi^B \\ &+ Y(\psi) \cdot \big(\dot{\square}_2 \psi - V\psi \big) + O(\epsilon) \Big(|\nabla_3 \psi|^2 + |\nabla_4 \psi|^2 + |\nabla \psi|^2 + r^{-2} |\psi|^2 \Big).\end{aligned}$$

Together with Lemma 9.4.2, and since $Y_{(0)} = Y + 2T$, we infer, at $r = r+$,

$$\begin{aligned}\mathbf{D}^\mu \mathcal{P}_\mu[Y_{(0)}, 0, 0] \;\; &= \;\; \mathbf{D}^\mu \mathcal{P}_\mu[Y, 0, 0] + 2\mathbf{D}^\mu \mathcal{P}_\mu[\mathbf{T}, 0, 0] \\ &= \;\; \frac{(r_+ - m)}{|q|^2} |\nabla_3 \psi|^2 + \frac{d'(r_+)}{2} |\nabla_4 \psi|^2 + \frac{\underline{d}'(r_+)}{2} |\nabla \psi|^2 + \frac{4(r_+ - m)}{(r^2 + a^2)|q|^2} |\psi|^2 \\ &+ \frac{ar}{|q|^2} \Re(\mathfrak{J})^b \nabla_b \Psi \cdot \nabla_3 \Psi - \frac{r}{|q|^2} \nabla_3 \psi \cdot \nabla_4 \psi + 2 \; {}^*\rho \in_{AB} \nabla_3 \psi^A \psi^B \\ &- 2a \; {}^*\Re(\mathfrak{J})^d \nabla_d \psi_A \rho \; {}^*\psi^A + Y_{(0)}(\psi) \cdot \big(\dot{\square}_2 \psi - V\psi \big) \\ &+ O(\epsilon) \Big(|\nabla_3 \psi|^2 + |\nabla_4 \psi|^2 + |\nabla \psi|^2 + r^{-2} |\psi|^2 \Big).\end{aligned}$$

We deduce, at $r = r_+$,

$$\begin{aligned}
\mathbf{D}^\mu \mathcal{P}_\mu[Y_{(0)}, 0, 0] \geq & \frac{(r_+ - m)}{4|q|^2}|\nabla_3\psi|^2 + \left(\frac{d'(r_+)}{2} - \frac{r_+^2}{(r_+ - m)|q|^2}\right)|\nabla_4\psi|^2 \\
& + \left(\frac{\underline{d}'(r_+)}{2} - \frac{a^2 r_+^2}{(r_+ - m)|q|^2}|\Re(\mathfrak{J})|^2 - a|\Re(\mathfrak{J})||\rho| r_+\right)|\nabla\psi|^2 \\
& + \left(\frac{4(r_+ - m)}{(r^2 + a^2)|q|^2} - \frac{4|q|^2 \ {}^*\rho^2}{(r_+ - m)} - \frac{a|\Re(\mathfrak{J})||\rho|}{r_+}\right)|\psi|^2 \\
& + Y_{(0)}(\psi) \cdot \left(\dot{\square}_2\psi - V\psi\right) + O(\epsilon)\Big(|\nabla_3\psi|^2 + |\nabla_4\psi|^2 + |\nabla\psi|^2 + r^{-2}|\psi|^2\Big).
\end{aligned}$$

Also, since $Y_{(0)} = Y + 2T$, we have

$$\begin{aligned}
-\frac{4a\cos\theta}{|q|^2} Y_{(0)}(\psi) \cdot \ {}^*\nabla_T\psi &= -\frac{4a\cos\theta}{|q|^2}\nabla_Y\psi \cdot \ {}^*\nabla_T\psi - \frac{8a\cos\theta}{|q|^2}\nabla_T\psi \cdot \ {}^*\nabla_T\psi \\
&= -\frac{4a\cos\theta}{|q|^2}\nabla_Y\psi \cdot \ {}^*\nabla_T\psi.
\end{aligned}$$

At $r = r_+$, we obtain

$$\begin{aligned}
-\frac{4a\cos\theta}{|q|^2} Y_{(0)}(\psi) \cdot \ {}^*\nabla_T\psi &= -\frac{2a\cos\theta}{|q|^2}\nabla_3\psi \cdot \nabla_{e_4 - 2a\Re(\mathfrak{J})^b e_b} \ {}^*\psi \\
&\geq -\frac{2|a|}{|q|^2}|\nabla_3\psi||\nabla_4\psi| - \frac{4a^2}{|q|^2}|\Re(\mathfrak{J})||\nabla_3\psi||\nabla\psi| \\
&\geq -\frac{(r_+ - m)}{8|q|^2}|\nabla_3\psi|^2 - \frac{16a^2}{(r_+ - m)|q|^2}|\nabla_4\psi|^2 \\
&\quad -\frac{64a^4}{(r_+ - m)|q|^2}|\Re(\mathfrak{J})|^2|\nabla\psi|^2
\end{aligned}$$

and hence

$$\begin{aligned}
\mathbf{D}^\mu \mathcal{P}_\mu[Y_{(0)}, 0, 0] \geq & \frac{(r_+ - m)}{8|q|^2}|\nabla_3\psi|^2 + \left(\frac{d'(r_+)}{2} - \frac{r_+^2}{(r_+ - m)|q|^2} - \frac{16a^2}{(r_+ - m)|q|^2}\right)|\nabla_4\psi|^2 \\
& + \left(\frac{\underline{d}'(r_+)}{2} - \frac{a^2 r_+^2}{(r_+ - m)|q|^2}|\Re(\mathfrak{J})|^2 - a|\Re(\mathfrak{J})||\rho| r_+ - \frac{64a^4}{(r_+ - m)|q|^2}|\Re(\mathfrak{J})|^2\right)|\nabla\psi|^2 \\
& + \left(\frac{4(r_+ - m)}{(r^2 + a^2)|q|^2} - \frac{4|q|^2 \ {}^*\rho^2}{(r_+ - m)} - \frac{a|\Re(\mathfrak{J})||\rho|}{r_+}\right)|\psi|^2 \\
& + Y_{(0)}(\psi) \cdot \left(\dot{\square}_2\psi - V\psi + \frac{4a\cos\theta}{|q|^2} \ {}^*\nabla_T\psi\right) \\
& + O(\epsilon)\Big(|\nabla_3\psi|^2 + |\nabla_4\psi|^2 + |\nabla\psi|^2 + r^{-2}|\psi|^2\Big).
\end{aligned}$$

Since we have, for $r\leq 4m$,

$$r_+>m,\quad r\leq |q|\leq 2r,\quad |\rho|\leq \frac{2m}{r^3}+O(\epsilon),\quad |{}^*\rho|\leq \frac{6|a|m}{r^4}+O(\epsilon),\quad |\Re(\mathfrak{J})|\leq \frac{1}{r}+O(\epsilon),$$

we infer, at $r=r_+$,

$$\begin{aligned}
\mathbf{D}^\mu\mathcal{P}_\mu[Y_{(0)},0,0] \;\geq\;& \frac{(r_+-m)}{16r_+^2}|\nabla_3\psi|^2+\left(\frac{d'(r_+)}{2}-\frac{17}{r_+-m}\right)|\nabla_4\psi|^2\\
&+\left(\frac{\underline{d}'(r_+)}{2}-\frac{67m|a|}{r_+^2(r_+-m)}\right)|\nabla\psi|^2+\left(\frac{(r_+-m)}{4r_+^4}-\frac{578|a|m}{(r_+-m)r_+^4}\right)|\psi|^2\\
&+Y_{(0)}(\psi)\cdot\left(\dot{\square}_2\psi-V\psi+\frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_T\psi\right)\\
&+O(\epsilon)\Big(|\nabla_3\psi|^2+|\nabla_4\psi|^2+|\nabla\psi|^2+r^{-2}|\psi|^2\Big).
\end{aligned}$$

Next, using the Poincaré inequality of Lemma 9.2.23, we have for some universal constant $c_0>0$, and for any sphere $S=S(\tau,r)$,

$$\int_S\left(|\nabla_4\psi|^2+\frac{\Delta}{r^4}|\nabla_3\psi|^2+|\nabla\psi|^2\right)\geq \frac{c_0}{r^2}\int_S|\psi|^2.$$

In particular, we deduce at $r=r_+$

$$\int_S\left(|\nabla_4\psi|^2+|\nabla\psi|^2\right)\geq\frac{c_0}{r^2}\int_S|\psi|^2,$$

and hence, for any sphere $S=S(\tau,r_+)$

$$\begin{aligned}
\int_S\mathbf{D}^\mu\mathcal{P}_\mu[Y_{(0)},0,0] \;\geq\;& \frac{(r_+-m)}{16r_+^2}\int_S|\nabla_3\psi|^2+\frac{(r_+-m)}{4r_+^4}\int_S|\psi|^2\\
&+\left(\frac{d'(r_+)}{2}-\frac{17}{r_+-m}-\frac{1}{c_0}\frac{578m|a|}{(r_+-m)r_+^2}\right)\int_S|\nabla_4\psi|^2\\
&+\left(\frac{\underline{d}'(r_+)}{2}-\frac{67m|a|}{r_+^2(r_+-m)}-\frac{1}{c_0}\frac{578m|a|}{(r_+-m)r_+^2}\right)\int_S|\nabla\psi|^2\\
&+\int_S Y_{(0)}(\psi)\cdot\left(\dot{\square}_2\psi-V\psi+\frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_T\psi\right)\\
&+O(\epsilon)\int_S\Big(|\nabla_3\psi|^2+|\nabla_4\psi|^2+|\nabla\psi|^2+r^{-2}|\psi|^2\Big).
\end{aligned}$$

Thus, if we choose

$$d(r_+)=0,\qquad \underline{d}(r_+)=1,\qquad d'(r_+)\geq\frac{c_1}{r_+-m},\qquad \underline{d}'(r_+)\geq\frac{c_1}{r_+-m},$$

for some large enough universal constant c_1, then at $r = r_+$, we have, for any sphere $S = S(\tau, r_+)$,

$$\int_S \mathbf{D}^\mu \mathcal{P}_\mu[Y_{(0)}, 0, 0] \geq \frac{(r_+ - m)}{32 r_+^2} \int_S \Big(|\nabla_3 \psi|^2 + |\nabla_4 \psi|^2 + |\nabla \psi|^2 + r^{-2} |\psi|^2 \Big) + \int_S Y_{(0)}(\psi) \cdot \left(\dot{\square}_2 \psi - V\psi + \frac{4a\cos\theta}{|q|^2} {}^*\nabla_T \psi \right)$$

as stated. This concludes the proof of Corollary 9.4.3. □

Proposition 9.4.4. *Let $\kappa(r)$ a positive bump function supported in $[2, 2]$ and equal to 1 on $[-1, 1]$. Also, for $|a| < m$, let a small enough constant $\delta_{red} > 0$ such that $\delta_{red} = \delta_{red}(m - |a|)$ with $\delta_{red} \geq \delta_{\mathcal{H}}$. Let $Y_{\mathcal{H}}$ the vectorfield given by*

$$Y_{\mathcal{H}} := \kappa_{\mathcal{H}} Y_{(0)}, \qquad \kappa_{\mathcal{H}} = \kappa \left(\frac{\frac{r}{r_+} - 1}{\delta_{red}} \right), \tag{9.4.3}$$

where $Y_{(0)}$ is the vectorfield of Corollary 9.4.3. Then, the following estimate holds, for any sphere $S = S(\tau, r)$,

$$\begin{aligned} \int_S \mathbf{D}^\mu \mathcal{P}_\mu[Y_{\mathcal{H}}, 0, 0] \geq\ & \frac{(r_+ - m)}{64 r_+^2} \int_S \Big(|\nabla_3 \psi|^2 + |\nabla_4 \psi|^2 + |\nabla \psi|^2 + r^{-2} |\psi|^2 \Big) \mathbb{1}_{\left|\frac{r}{r_+} - 1\right| \leq \delta_{red}} \\ & - O(r_+^{-1} \delta_{red}^{-1}) \int_S \Big(|\nabla_3 \psi|^2 + |\nabla_4 \psi|^2 + |\nabla \psi|^2 + r^{-2} |\psi|^2 \Big) \mathbb{1}_{\delta_{red} \leq \left|\frac{r}{r_+} - 1\right| \leq 2\delta_{red}} \\ & + \int_S Y_{\mathcal{H}}(\psi) \cdot \left(\dot{\square}_2 \psi - V\psi + \frac{4a\cos\theta}{|q|^2} {}^*\nabla_T \psi \right). \end{aligned} \tag{9.4.4}$$

Also, recallling that the vectorfield N_Σ is given by $N_\Sigma = -\mathbf{g}^{\alpha\beta} \partial_\alpha(\tau) \partial_\beta$, there holds on $\mathcal{M}$

$$\mathcal{P}_\mu[Y_{\mathcal{H}}, 0, 0] \cdot N_\Sigma \geq 0, \tag{9.4.5}$$

and, there exists, for $|a| < m$, a constant $c_0 > 0$, with $c_0 = c_0(m - |a|)$, such that we have for any sphere $S = S(\tau, r)$ with $|\frac{r}{r_+} - 1| \leq \delta_{red}$,

$$\int_S \mathcal{P}_\mu[Y_{\mathcal{H}}, 0, 0] \cdot N_\Sigma \geq c_0 \int_S \Big(|\nabla_3 \psi|^2 + |\nabla_4 \psi|^2 + |\nabla \psi|^2 + r^{-2} |\psi|^2 \Big). \tag{9.4.6}$$

Finally, there exists, for $|a| < m$, a constant $c_0 > 0$, with $c_0 = c_0(m - |a|)$, such that we have for any sphere $S = S(\tau, r)$ with $r = (1 - \delta_{\mathcal{H}}) r_+$, i.e. any sphere in $\mathcal{A}$,

$$\int_S \mathcal{P}_\mu[Y_{\mathcal{H}}, 0, 0] \cdot N_{\mathcal{A}} \geq c_0 \int_S \Big(\delta_{\mathcal{H}} |\nabla_3 \psi|^2 + |\nabla_4 \psi|^2 + |\nabla \psi|^2 + r^{-2} |\psi|^2 \Big). \tag{9.4.7}$$

Proof. In view of the definition of $Y_{\mathcal{H}}$ and the properties of $\kappa_{\mathcal{H}}$, we have

$$
\begin{aligned}
\mathbf{D}^\mu\mathcal{P}_\mu[Y_{\mathcal{H}},0,0] &= \frac{1}{2}\mathcal{Q}\cdot{}^{(Y_{\mathcal{H}})}\pi + Y_{\mathcal{H}}(\psi)\cdot\big(\dot{\square}_2\psi - V\psi\big) - \frac{1}{2}Y_{\mathcal{H}}(V)|\psi|^2\\
&\quad + Y_{\mathcal{H}}^\mu\dot{\mathbf{D}}^\nu\psi^a\mathbf{R}_{ab\nu\mu}\psi^b\\
&= \kappa_{\mathcal{H}}\mathbf{D}^\mu\mathcal{P}_\mu[Y_{(0)},0,0] + \mathcal{Q}(Y_{(0)},d\kappa_{\mathcal{H}})\\
&= \kappa_{\mathcal{H}}\mathbf{D}^\mu\mathcal{P}_\mu[Y_{(0)},0,0]\\
&\quad -O(r_+^{-1}\delta_{red}^{-1})\mathbb{1}_{\delta_{red}\leq\left|\frac{r}{r_+}-1\right|\leq 2\delta_{red}}\Big(|\nabla_3\psi|^2+|\nabla_4\psi|^2+|\nabla\psi|^2+r^{-2}|\psi|^2\Big).
\end{aligned}
$$

Together with the lower bound for $\mathbf{D}^\mu\mathcal{P}_\mu[Y_{(0)},0,0]$ of Corollary 9.4.3 and the properties of $\kappa_{\mathcal{H}}$, we infer, provided δ_{red} is chosen sufficiently small, for any sphere $S=S(\tau,r)$,

$$
\begin{aligned}
\int_S\mathbf{D}^\mu\mathcal{P}_\mu[Y_{\mathcal{H}},0,0] &\geq \frac{(r_+-m)}{64r_+^2}\int_S\Big(|\nabla_3\psi|^2+|\nabla_4\psi|^2+|\nabla\psi|^2+r^{-2}|\psi|^2\Big)\mathbb{1}_{\left|\frac{r}{r_+}-1\right|\leq\delta_{red}}\\
&\quad -O(r_+^{-1}\delta_{red}^{-1})\int_S\Big(|\nabla_3\psi|^2+|\nabla_4\psi|^2+|\nabla\psi|^2+r^{-2}|\psi|^2\Big)\mathbb{1}_{\delta_{red}\leq\left|\frac{r}{r_+}-1\right|\leq 2\delta_{red}}\\
&\quad +\int_S Y_{\mathcal{H}}(\psi)\cdot\left(\dot{\square}_2\psi - V\psi + \frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_T\psi\right)
\end{aligned}
$$

as stated.

Next, we have

$$
\begin{aligned}
\mathcal{P}_\mu[Y_{\mathcal{H}},0,0]\cdot N_\Sigma &= \mathcal{Q}(Y_{\mathcal{H}},N_\Sigma) = \kappa_{\mathcal{H}}\mathcal{Q}(Y_{(0)},N_\Sigma)\\
&= \kappa_{\mathcal{H}}\mathcal{Q}\left(\underline{d}(r)e_3 + d(r)e_4 + e_4 + \frac{\Delta}{|q|^2}e_3 - 2a\Re(\mathfrak{J})^be_b, N_\Sigma\right)\\
&= \kappa_{\mathcal{H}}\mathcal{Q}\Big(e_3+e_4+O(r-r_+)e_3+O(r-r_+)e_4-2a\Re(\mathfrak{J})^be_b,N_\Sigma\Big).
\end{aligned}
$$

In view of the support of $\kappa_{\mathcal{H}}$, we infer

$$
\begin{aligned}
\mathcal{P}_\mu[Y_{\mathcal{H}},0,0]\cdot N_\Sigma &= \kappa_{\mathcal{H}}\Bigg[\mathcal{Q}\Big(e_3+e_4-2a\Re(\mathfrak{J})^be_b,N_\Sigma\Big)\\
&\qquad +O(\delta_{red})\Big(|\nabla_3\psi|^2+|\nabla_4\psi|^2+|\nabla\psi|^2+r^{-2}|\psi|^2\Big)\Bigg].
\end{aligned}
$$

Next, in view of (4.7.2), and since $V(r_+)=0$ in view of its explicit formula in (6.1.1), we have on the support of $\kappa_{\mathcal{H}}$

$$
\begin{aligned}
&\mathcal{Q}_{33}=|\nabla_3\psi|^2,\qquad \mathcal{Q}_{44}=|\nabla_4\psi|^2,\qquad \mathcal{Q}_{34}=|\nabla\psi|^2+O(\delta_{red})|\psi|^2,\\
&\mathcal{Q}_{4a}=\nabla_4\Psi\cdot\nabla_a\Psi,\qquad \mathcal{Q}_{3a}=\nabla_3\Psi\cdot\nabla_a\Psi,\\
&\mathrm{tr}\mathcal{Q}=\nabla_4\psi\cdot\nabla_3\psi+O(\delta_{red})|\psi|^2,\qquad \widehat{\mathcal{Q}}_{ab}=\frac{1}{2}(\nabla\psi\widehat{\otimes}\nabla\psi)_{ab}+O(\delta_{red})|\psi|^2,
\end{aligned}
$$

and hence

$$\mathcal{Q}\Big(e_3+e_4-2a\Re(\mathfrak{J})^be_b, N_\Sigma\Big) = \widetilde{\mathcal{Q}}\Big(e_3+e_4-2a\Re(\mathfrak{J})^be_b, N_\Sigma\Big) + O(\delta_{red})|\psi|^2$$

where the symmetric 2-tensor $\widetilde{\mathcal{Q}}$ is given by

$$\widetilde{\mathcal{Q}}_{\mu\nu} := \dot{\mathbf{D}}_\mu\psi\cdot\dot{\mathbf{D}}_\nu\psi - \frac{1}{2}\mathbf{g}_{\mu\nu}\dot{\mathbf{D}}_\lambda\psi\cdot\dot{\mathbf{D}}^\lambda\psi.$$

We deduce, on the support of $\kappa_{\mathcal{H}}$,

$$\begin{aligned}\mathcal{P}_\mu[Y_{\mathcal{H}},0,0]\cdot N_\Sigma &= \kappa_{\mathcal{H}}\Bigg[\widetilde{\mathcal{Q}}\Big(e_3+e_4-2a\Re(\mathfrak{J})^be_b, N_\Sigma\Big)\\ &\quad +O(\delta_{red})\Big(|\nabla_3\psi|^2+|\nabla_4\psi|^2+|\nabla\psi|^2+r^{-2}|\psi|^2\Big)\Bigg].\end{aligned}$$

Next, recall that the choice of τ in Definition 6.1.5 is such that

$$\mathbf{g}(N_\Sigma, N_\Sigma) \le -\frac{m^2}{8r^2} < 0.$$

Also, we have

$$\mathbf{g}\Big(e_3+e_4-2a\Re(\mathfrak{J})^be_b, e_3+e_4-2a\Re(\mathfrak{J})^be_b\Big) = -4\Big(1-a^2|\Re(\mathfrak{J})|^2\Big)$$

and hence, on the support of $\kappa_{\mathcal{H}}$,

$$\begin{aligned}\mathbf{g}\Big(e_3+e_4-2a\Re(\mathfrak{J})^be_b, e_3+e_4-2a\Re(\mathfrak{J})^be_b\Big) &= -4\left(1-\frac{a^2(\sin\theta)^2}{|q|^2}+O(\epsilon)\right)\\ &\le -4\left(1-\frac{a^2}{r_+^2}+O(\delta_{red}+\epsilon)\right) < 0\end{aligned}$$

for $|a|<m$, and δ_{red} and ϵ small enough. Since both vectorfields are uniformly timelike on the support of $\kappa_{\mathcal{H}}$, we deduce from the above, together with the Poincaré inequality of Lemma 9.2.23, the existence of a constant $c_0>0$ such that, for δ_{red} small enough, for any sphere $S=S(\tau,r)$,

$$\int_S \mathcal{P}_\mu[Y_{\mathcal{H}},0,0]\cdot N_\Sigma \ \ge\ c_0\kappa_{\mathcal{H}}\int_S\Big(|\nabla_3\psi|^2+|\nabla_4\psi|^2+|\nabla\psi|^2+r^{-2}|\psi|^2\Big).$$

In particular, since $\kappa_{\mathcal{H}}\ge 0$, we have on $\mathcal{M}$

$$\int_S \mathcal{P}_\mu[Y_{\mathcal{H}},0,0]\cdot N_\Sigma \ \ge\ 0,$$

and moreover, in view of the definition of $\kappa_{\mathcal{H}}$, we have, for any sphere $S(\tau, r)$ with $|\frac{r}{r_+} - 1| \leq \delta_{red}$

$$\int_S \mathcal{P}_\mu[Y_{\mathcal{H}}, 0, 0] \cdot N_\Sigma \;\; \geq \;\; c_0 \int_S \Big(|\nabla_3 \psi|^2 + |\nabla_4 \psi|^2 + |\nabla \psi|^2 + r^{-2} |\psi|^2 \Big)$$

as stated.

Finally, we have on $\mathcal{A} = \{r = (1 - \delta_{\mathcal{H}}) r_+\}$

$$\begin{aligned} N_{\mathcal{A}} &= \mathbf{g}^{\alpha\beta} \partial_\alpha(r) \partial_\beta = -\frac{1}{2} e_3(r) e_4 - \frac{1}{2} e_4(r) e_3 + \nabla(r) \\ &= \frac{1}{2} \big(1 + O(\epsilon)\big) e_4 - \frac{1}{2} \left(\frac{\Delta}{|q|^2} + O(\epsilon) \right) e_3 + O(\epsilon) \nabla \\ &= \frac{1}{2} \big(1 + O(\epsilon)\big) e_4 + \frac{1}{2} \left(\frac{|\Delta|}{|q|^2} + O(\epsilon) \right) e_3 + O(\epsilon) \nabla \end{aligned}$$

and

$$\begin{aligned} Y_{(0)} &= \underline{d}(r) e_3 + d(r) e_4 + e_4 + \frac{\Delta}{|q|^2} e_3 - 2a \Re(\mathfrak{J})^b e_b \\ &= \big(1 + O(\delta_{\mathcal{H}})\big) e_3 + \big(1 + O(\delta_{\mathcal{H}})\big) e_3 - 2a \Re(\mathfrak{J})^b e_b. \end{aligned}$$

In view of the definition of $\kappa_{\mathcal{H}}$ and the fact that $\delta_{red} \geq \delta_{\mathcal{H}}$, we have $\kappa_{\mathcal{H}} = 1$ on $\mathcal{A}$. We infer

$$\begin{aligned} \mathcal{P}_\mu[Y_{\mathcal{H}}, 0, 0] \cdot N_{\mathcal{A}} &= \mathcal{Q}(Y_{\mathcal{H}}, N_{\mathcal{A}}) = \kappa_{\mathcal{H}} \mathcal{Q}(Y_{(0)}, N_{\mathcal{A}}) = \mathcal{Q}(Y_{(0)}, N_{\mathcal{A}}) \\ &= \frac{1}{2} \big(1 + O(\delta_{\mathcal{H}} + \epsilon)\big) \mathcal{Q}_{43} + \frac{1}{2} \big(1 + O(\delta_{\mathcal{H}} + \epsilon)\big) \mathcal{Q}_{44} - a \big(1 + O(\epsilon)\big) \Re(\mathfrak{J})^b \mathcal{Q}_{4b} \\ & \quad + \frac{1}{2} \left(\frac{|\Delta|}{|q|^2} + O(\epsilon) \right) \Big(\big(1 + O(\delta_{\mathcal{H}} + \epsilon)\big) \mathcal{Q}_{33} - 2a \Re(\mathfrak{J})^b \mathcal{Q}_{3b} \Big) \\ & \quad + O(\epsilon) \Big(\mathcal{Q}_{b4} + \mathcal{Q}_{b3} - 2a \Re(\mathfrak{J})^c \mathcal{Q}_{bc} \Big). \end{aligned}$$

Next, in view of (4.7.2), and since $V(r_+) = 0$ in view of its explicit formula in (6.1.1), we have on $\mathcal{A} = \{r = (1 - \delta_{\mathcal{H}}) r_+\}$

$$\begin{aligned} & \mathcal{Q}_{33} = |\nabla_3 \psi|^2, \qquad \mathcal{Q}_{44} = |\nabla_4 \psi|^2, \qquad \mathcal{Q}_{34} = |\nabla \psi|^2 + O(\delta_{\mathcal{H}}) |\psi|^2, \\ & \mathcal{Q}_{4a} = \nabla_4 \Psi \cdot \nabla_a \Psi, \qquad \mathcal{Q}_{3a} = \nabla_3 \Psi \cdot \nabla_a \Psi, \\ & \mathrm{tr} \mathcal{Q} = \nabla_4 \psi \cdot \nabla_3 \psi + O(\delta_{\mathcal{H}}) |\psi|^2, \qquad \widehat{\mathcal{Q}}_{ab} = \frac{1}{2} (\nabla \psi \widehat{\otimes} \nabla \psi)_{ab} + O(\delta_{\mathcal{H}}) |\psi|^2. \end{aligned}$$

We infer on $\mathcal{A}$

$$
\begin{aligned}
\mathcal{P}_\mu[Y_{\mathcal{H}},0,0]\cdot N_{\mathcal{A}} &= \frac{1}{2}|\nabla\psi|^2+\frac{1}{2}|\nabla_4\psi|^2-a\Re(\mathfrak{J})^b\nabla_4\Psi\cdot\nabla_b\Psi\\
&+\frac{1}{2}\frac{|\Delta|}{|q|^2}\Big(|\nabla_3\psi|^2-2a\Re(\mathfrak{J})^b\nabla_3\Psi\cdot\nabla_b\Psi\Big)\\
&+O(\delta_{\mathcal{H}}^2+\epsilon)|\nabla_3\psi|^2+O(\delta_{\mathcal{H}}+\epsilon)\Big(|\nabla_4\psi|^2+|\nabla\psi|^2+r^{-2}|\psi|^2\Big).
\end{aligned}
$$

Arguing as above, for $|a|<m$, and $\delta_{\mathcal{H}}$ and ϵ small enough, this yields the existence of a constant $c_0>0$ depending on a and m such that

$$
\mathcal{P}_\mu[Y_{\mathcal{H}},0,0]\cdot N_{\mathcal{A}} \geq c_0\Big(\delta_{\mathcal{H}}|\nabla_3\psi|^2+|\nabla_4\psi|^2+|\nabla\psi|^2\Big)+O(\delta_{\mathcal{H}})|\psi|^2.
$$

Using again the above Poincaré inequality, we infer, for a possibly smaller $c_0>0$,

$$
\int_S\mathcal{P}_\mu[Y_{\mathcal{H}},0,0]\cdot N_{\mathcal{A}} \geq c_0\int_S\Big(\delta_{\mathcal{H}}|\nabla_3\psi|^2+|\nabla_4\psi|^2+|\nabla\psi|^2+r^{-2}|\psi|^2\Big)
$$

for any sphere $S(\tau,r)$ with $r=(1-\delta_{\mathcal{H}})r_+$ as stated. This concludes the proof of Proposition 9.4.4 $\Box$

We are now ready to prove the redshift estimates of Proposition 9.4.1.

Proof of Proposition 9.4.1. Let $Y_{\mathcal{H}}$ the vectorfield of Proposition 9.4.4. We integrate $\mathbf{D}^\mu\mathcal{P}_\mu[Y_{\mathcal{H}},0,0]$ on $\mathcal{M}(\tau_1,\tau_2)$ and apply the divergence theorem. The proof of Proposition 9.4.1 follows then immediately from the lower bounds for $\mathbf{D}^\mu\mathcal{P}_\mu[Y_{\mathcal{H}},0,0]$, $\mathcal{P}_\mu[Y_{\mathcal{H}},0,0]\cdot N_\Sigma$ and $\mathcal{P}_\mu[Y_{\mathcal{H}},0,0]\cdot N_{\mathcal{A}}$ derived in Proposition 9.4.4. $\Box$

The following lemma shows that e_3 commute well with $\dot{\Box}_k$ in the redshift region.

Lemma 9.4.5. *We have, for $r\leq 4m$,*

$$
\begin{aligned}
[\nabla_3,\dot{\Box}_k] &= -\partial_r\left(\frac{\Delta}{|q|^2}\right)\nabla_3^2\psi+O(1)\nabla\nabla_3\psi+O(1)\nabla_4\nabla_3\psi\\
&+O(1)\dot{\Box}_k\psi+O(1)\mathfrak{d}^{\leq 1}\psi+O(\epsilon)\mathfrak{d}^{\leq 2}\psi
\end{aligned}
$$

Proof. Recall the following decomposition, see (4.7.5),

$$
\begin{aligned}
\dot{\Box}_k\psi &= -\nabla_4\nabla_3\psi-\frac{1}{2}\mathrm{tr}\,\underline{\chi}\nabla_4\psi+\left(2\omega-\frac{1}{2}\mathrm{tr}\,\chi\right)\nabla_3\psi+\triangle_k\psi+2\underline{\eta}\cdot\nabla\psi\\
&+2i\left({}^*\rho-\eta\wedge\underline{\eta}\right)\psi+(\Gamma_b\cdot\Gamma_g)\cdot\psi.
\end{aligned}
$$

We infer, for $r \le 4m$,

$$
\begin{aligned}
[\nabla_3, \dot{\square}_k] &= \Bigg[\nabla_3, -\nabla_4\nabla_3\psi - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\nabla_4 + \left(2\omega - \frac{1}{2}\mathrm{tr}\,\chi\right)\nabla_3 + \triangle_k + 2\underline{\eta}\cdot\nabla \\
&\quad +2i\left({}^*\rho - \eta\wedge\underline{\eta}\right)\psi + (\Gamma_b\cdot\Gamma_g)\cdot\Bigg]\psi \\
&= -[\nabla_3, \nabla_4]\nabla_3\psi + [\nabla_3, \Delta_k]\psi + O(1)\mathfrak{d}^{\le 1}\psi \\
&= 2\omega\nabla_3^2\psi + 2(\eta - \underline{\eta})\cdot\nabla\nabla_3\psi + 2\underline{\omega}\nabla_4\nabla\psi - 2\underline{\chi}_{bc}\nabla_c\nabla_b\psi \\
&\quad +2(\eta-\zeta)\cdot\nabla\nabla_3\psi + 2\underline{\xi}\cdot\nabla_4\nabla\psi + O(1)\mathfrak{d}^{\le 1}\psi \\
&= 2\omega\nabla_3^2\psi + 2(\eta - \underline{\eta})\cdot\nabla\nabla_3\psi - \mathrm{tr}\,\underline{\chi}\Delta_k\psi + O(1)\mathfrak{d}^{\le 1}\psi + O(\epsilon)\mathfrak{d}^{\le 2}\psi.
\end{aligned}
$$

Using again (4.7.5), we infer

$$
\begin{aligned}
[\nabla_3, \dot{\square}_k] &= 2\omega\nabla_3^2\psi + 2(\eta - \underline{\eta})\cdot\nabla\nabla_3\psi - \mathrm{tr}\,\underline{\chi}\nabla_4\nabla_3\psi \\
&\quad -\mathrm{tr}\,\underline{\chi}\dot{\square}_k\psi + O(1)\mathfrak{d}^{\le 1}\psi + O(\epsilon)\mathfrak{d}^{\le 2}\psi
\end{aligned}
$$

and hence

$$
\begin{aligned}
[\nabla_3, \dot{\square}_k] &= -\partial_r\left(\frac{\Delta}{|q|^2}\right)\nabla_3^2\psi + O(1)\nabla\nabla_3\psi + O(1)\nabla_4\nabla_3\psi \\
&\quad +O(1)\dot{\square}_k\psi + O(1)\mathfrak{d}^{\le 1}\psi + O(\epsilon)\mathfrak{d}^{\le 2}\psi
\end{aligned}
$$

as stated. □

In view of Lemma 9.4.5, the following corollary of Proposition 9.4.1 will be useful when commuting the wave equation with ∇_3.

Corollary 9.4.6. *Let ψ a solution, in $\mathcal{M}\cap\{r \le 4m\}$, to*

$$
\dot{\square}_2\psi - V\psi = -\frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_T\psi + N + \partial_r\left(\frac{\Delta}{|q|^2}\right)\nabla_3\psi + O(1)\nabla\psi + O(1)\nabla_4\psi.
$$

Then, for $|a| < m$, there exists a small enough constant $\delta_{red} > 0$ such that $\delta_{red} = \delta_{red}(m - |a|)$ with $\delta_{red} \ge \delta_{\mathcal{H}}$, and a small constant $c_0 > 0$ with $c_0 = c_0(m - |a|)$, such that the following estimate holds true in $\mathcal{M}(\tau_1, \tau_2)$:

$$
\begin{aligned}
& c_0 E_{r\le r_+(1+\delta_{red})}[\psi](\tau_2) + c_0 Mor_{r\le r_+(1+\delta_{red})}[\psi](\tau_1,\tau_2) + c_0 F_{\mathcal{A}}[\psi](\tau_1,\tau_2) \\
\le & E_{r\le r_+(1+2\delta_{red})}[\psi](\tau_1) + \delta_{red}^{-1} Mor_{r_+(1+\delta_{red})\le r\le r_+(1+2\delta_{red})}[\psi](\tau_1,\tau_2) \\
& + \int_{\mathcal{M}(\tau_1,\tau_2)\cap\left\{\frac{r}{r_+}\le 1+2\delta_{red}\right\}} |N|^2.
\end{aligned}
\tag{9.4.8}
$$

Proof. Let $Y_{\mathcal{H}}$ the redshift vectorfield of Proposition 9.4.4. As in the proof of that proposition, we have

$$\begin{aligned}\mathbf{D}^\mu\mathcal{P}_\mu[Y_{\mathcal{H}},0,0] &= \kappa_{\mathcal{H}}\mathbf{D}^\mu\mathcal{P}_\mu[Y_{(0)},0,0]\\ &\quad -O(r_+^{-1}\delta_{red}^{-1})\mathbb{1}_{\delta_{red}\leq\left|\frac{r}{r_+}-1\right|\leq 2\delta_{red}}\Big(|\nabla_3\psi|^2+|\nabla_4\psi|^2+|\nabla\psi|^2+r^{-2}|\psi|^2\Big).\end{aligned}$$

Together with the lower bound for $\mathbf{D}^\mu\mathcal{P}_\mu[Y_{(0)},0,0]$ of Corollary 9.4.3 and the properties of $\kappa_{\mathcal{H}}$, we infer, provided δ_{red} is chosen sufficiently small, for any sphere $S=S(\tau,r)$, choosing $d'(r_+)$ and $\underline{d}'(r_+)$ even larger than in Corollary 9.4.3,

$$\begin{aligned}\int_S\mathbf{D}^\mu\mathcal{P}_\mu[Y_{\mathcal{H}},0,0] &\geq \frac{(r_+-m)}{64r_+^2}\int_S\Big(|\nabla_3\psi|^2+|\nabla_4\psi|^2+|\nabla\psi|^2+r^{-2}|\psi|^2\Big)\mathbb{1}_{\left|\frac{r}{r_+}-1\right|\leq\delta_{red}}\\ &\quad -O(r_+^{-1}\delta_{red}^{-1})\int_S\Big(|\nabla_3\psi|^2+|\nabla_4\psi|^2+|\nabla\psi|^2+r^{-2}|\psi|^2\Big)\mathbb{1}_{\delta_{red}\leq\left|\frac{r}{r_+}-1\right|\leq 2\delta_{red}}\\ &\quad +\int_S Y_{\mathcal{H}}(\psi)\cdot\left(\dot{\square}_2\psi-V\psi+\frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_T\psi\right)\\ &\quad +\frac{1}{2}\left(d'(r_+)-\frac{c_1}{r_+-m}\right)\int_S|\nabla_4\psi|^2\mathbb{1}_{\left|\frac{r}{r_+}-1\right|\leq\delta_{red}}\\ &\quad +\frac{1}{2}\left(\underline{d}'(r_+)-\frac{c_1}{r_+-m}\right)\int_S|\nabla\psi|^2\mathbb{1}_{\left|\frac{r}{r_+}-1\right|\leq\delta_{red}}.\end{aligned}$$

Plugging the equation for ψ, we infer

$$\begin{aligned}\int_S\mathbf{D}^\mu\mathcal{P}_\mu[Y_{\mathcal{H}},0,0] &\geq \frac{(r_+-m)}{64r_+^2}\int_S\Big(|\nabla_3\psi|^2+|\nabla_4\psi|^2+|\nabla\psi|^2+r^{-2}|\psi|^2\Big)\mathbb{1}_{\left|\frac{r}{r_+}-1\right|\leq\delta_{red}}\\ &\quad -O(r_+^{-1}\delta_{red}^{-1})\int_S\Big(|\nabla_3\psi|^2+|\nabla_4\psi|^2+|\nabla\psi|^2+r^{-2}|\psi|^2\Big)\mathbb{1}_{\delta_{red}\leq\left|\frac{r}{r_+}-1\right|\leq 2\delta_{red}}\\ &\quad +\int_S Y_{\mathcal{H}}(\psi)\cdot\left(N+\partial_r\left(\frac{\Delta}{|q|^2}\right)\nabla_3\psi+O(1)\nabla\psi+O(1)\nabla_4\psi\right)\\ &\quad +\frac{1}{2}\left(d'(r_+)-\frac{c_1}{r_+-m}\right)\int_S|\nabla_4\psi|^2\mathbb{1}_{\left|\frac{r}{r_+}-1\right|\leq\delta_{red}}\\ &\quad +\frac{1}{2}\left(\underline{d}'(r_+)-\frac{c_1}{r_+-m}\right)\int_S|\nabla\psi|^2\mathbb{1}_{\left|\frac{r}{r_+}-1\right|\leq\delta_{red}}.\end{aligned}$$

Now, in view of the choice of $Y_{\mathcal{H}}$, we have

$$\begin{aligned}&\int_S Y_{\mathcal{H}}(\psi)\cdot\left(\partial_r\left(\frac{\Delta}{|q|^2}\right)\nabla_3\psi+O(1)\nabla\psi+O(1)\nabla_4\psi\right)\\ =&\int_S\kappa_{\mathcal{H}}\Big(\big(1+O(\delta_{red})\big)\nabla_3+O(1)\nabla_4+O(1)\nabla\Big)\psi\cdot\left(\partial_r\left(\frac{\Delta}{|q|^2}\right)\nabla_3\psi+O(1)\nabla\psi+O(1)\nabla_4\psi\right).\end{aligned}$$

Since $\partial_r\left(\frac{\Delta}{|q|^2}\right) \geq 0$ on the support of $\kappa_{\mathcal{H}}$, we infer

$$\begin{aligned}
&\int_S Y_{\mathcal{H}}(\psi)\cdot\left(\partial_r\left(\frac{\Delta}{|q|^2}\right)\nabla_3\psi + O(1)\nabla\psi + O(1)\nabla_4\psi\right)\\
\geq\ & -O(1)\int_S \kappa_{\mathcal{H}}\Big(|\nabla_4\psi|^2 + |\nabla\psi|^2 + |\nabla_3\psi|\big(|\nabla_4\psi| + |\nabla\psi|\big)\Big)
\end{aligned}$$

and hence, for $d'(r_+)$ and $\underline{d}'(r_+)$ large enough,

$$\begin{aligned}
\int_S \mathbf{D}^\mu\mathcal{P}_\mu[Y_{\mathcal{H}},0,0] \ \geq\ & \frac{(r_+-m)}{128r_+^2}\int_S\Big(|\nabla_3\psi|^2 + |\nabla_4\psi|^2 + |\nabla\psi|^2 + r^{-2}|\psi|^2\Big)\mathbb{1}_{\left|\frac{r}{r_+}-1\right|\leq\delta_{red}}\\
& -O(r_+^{-1}\delta_{red}^{-1})\int_S\Big(|\nabla_3\psi|^2 + |\nabla_4\psi|^2 + |\nabla\psi|^2 + r^{-2}|\psi|^2\Big)\mathbb{1}_{\delta_{red}\leq\left|\frac{r}{r_+}-1\right|\leq 2\delta_{red}}\\
& -O(1)\int_S |N|^2\mathbb{1}_{\left|\frac{r}{r_+}-1\right|\leq 2\delta_{red}}.
\end{aligned}$$

We integrate $\mathbf{D}^\mu\mathcal{P}_\mu[Y_{\mathcal{H}},0,0]$ on $\mathcal{M}(\tau_1,\tau_2)$ and apply the divergence theorem. The proof of Corollary 9.4.6 follows then immediately from the above lower bounds for $\mathbf{D}^\mu\mathcal{P}_\mu[Y_{\mathcal{H}},0,0]$, and from the lower bound for $\mathcal{P}_\mu[Y_{\mathcal{H}},0,0]\cdot N_\Sigma$ and $\mathcal{P}_\mu[Y_{\mathcal{H}},0,0]\cdot N_{\mathcal{A}}$ derived in Proposition 9.4.4. □

9.5 Proof of Theorem 6.3.1

In this section, we finally prove our main result for Energy-Morawetz estimates in perturbations of Kerr. We first start with the particular case $s = 2$, and then treat the general case.

9.5.1 Proof of Theorem 6.3.1 in the case $s = 2$

Commutating the model RW equation (6.1.1) satisfied by ψ with various vectorfields will generate in particular error terms of the type $\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\psi)$. We start with the following simple lemma which will allow us to control the contribution of these error terms.

Lemma 9.5.1. *We have*

$$\int_{\mathcal{M}_{trap}(\tau_1,\tau_2)} |\mathfrak{d}^{\leq s+1}(\Gamma_g\cdot\psi)||\mathfrak{d}^{\leq s+1}\psi| \lesssim \epsilon \sup_{\tau\in[\tau_1,\tau_2]} E^s[\psi]$$

and

$$\int_{\mathcal{M}_{\not{trap}}(\tau_1,\tau_2)} |\mathfrak{d}^{\leq s+1}(\Gamma_g\cdot\psi)|\Big(|\nabla_{\widehat{T}_\delta}\mathfrak{d}^{\leq s}\psi| + |\nabla_{\widehat{R}}\mathfrak{d}^{\leq s}\psi| + r^{-1}|\mathfrak{d}^{\leq s}\psi|\Big) \lesssim \epsilon B^s_\delta[\psi](\tau_1,\tau_2).$$

Proof. We start with the control on $\mathcal{M}_{trap}$. We have

$$\begin{aligned}
& \int_{\mathcal{M}_{trap}(\tau_1,\tau_2)} |\mathfrak{d}^{\leq s+1}(\Gamma_g\cdot\psi)||\nabla_{\widehat{T}_\delta}\mathfrak{d}^{\leq s}\psi| \\
\lesssim\ & \epsilon\int_{\mathcal{M}_{trap}(\tau_1,\tau_2)} \frac{1}{\tau_{trap}^{1+\delta_{dec}}}|\mathfrak{d}^{\leq s+1}\psi|^2 \\
\lesssim\ & \epsilon\left(\int_{\tau_1}^{\tau_2}\frac{1}{\tau_{trap}^{1+\delta_{dec}}}\right)E^s[\psi] \\
\lesssim\ & \epsilon\sup_{\tau\in[\tau_1,\tau_2]}E^s[\psi]
\end{aligned}$$

as stated.

Concerning the control on $\mathcal{M}_{\not{trap}}$, we have

$$\begin{aligned}
& \int_{\mathcal{M}_{\not{trap}}(\tau_1,\tau_2)} |\mathfrak{d}^{\leq s+1}(\Gamma_g\cdot\psi)|\Big(|\nabla_{\widehat{T}_\delta}\mathfrak{d}^{\leq s}\psi| + |\nabla_{\widehat{R}}\mathfrak{d}^{\leq s}\psi| + r^{-1}|\mathfrak{d}^{\leq s}\psi|\Big) \\
\lesssim\ & \epsilon\int_{\mathcal{M}_{\not{trap}}(\tau_1,\tau_2)} r^{-2}|\mathfrak{d}^{\leq s+1}\psi|\Big(|\nabla_{\widehat{T}_\delta}\mathfrak{d}^{\leq s}\psi| + |\nabla_{\widehat{R}}\mathfrak{d}^{\leq s}\psi| + r^{-1}|\mathfrak{d}^{\leq s}\psi|\Big) \\
\lesssim\ & \epsilon\int_{\mathcal{M}_{\not{trap}}(\tau_1,\tau_2)} r^{-2}|\mathfrak{d}^{\leq s+1}\psi|\Big(|\nabla_3\mathfrak{d}^{\leq s}\psi| + r^{-1}|\mathfrak{d}^{\leq s+1}\psi|\Big) \\
\lesssim\ & \epsilon\int_{\mathcal{M}_{\not{trap}}(\tau_1,\tau_2)} \Big(r^{-1-\delta}|\nabla_3\mathfrak{d}^{\leq s}\psi|^2 + r^{\delta-3}|\mathfrak{d}^{\leq s+1}\psi|^2\Big) \\
\lesssim\ & \epsilon B^s_\delta[\psi](\tau_1,\tau_2)
\end{aligned}$$

as stated. This concludes the proof of the lemma. □

First, we derive the control of energy for at most one derivative of ψ and Morawetz for at most one $(\nabla_{\mathbf{T}}, \nabla_{\mathbf{Z}})$ derivative of ψ.

Proposition 9.5.2. *The solution ψ of of the model RW equation* (6.1.1) *satisfies the following energy estimate*

$$\begin{aligned}
& E_{deg}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq 1}\psi](\tau_2) + F_{\Sigma_*}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq 1}\psi](\tau_1,\tau_2) + Mor[(\nabla_{\mathbf{T}}, \nabla_{\mathbf{Z}})^{\leq 1}\psi](\tau_1,\tau_2) \\
\lesssim\ & \delta_{\mathcal{H}}\Big(E_{r\leq r_+}[\not{\mathfrak{d}}\psi](\tau_2) + F_{\mathcal{A}}[\not{\mathfrak{d}}\psi](\tau_1,\tau_2)\Big) + E^1[\psi](\tau_1) + \mathcal{N}^1[\psi, N](\tau_1,\tau_2) \\
& + \left(\frac{|a|}{m} + \epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^2[\psi] + B^2_\delta[\psi](\tau_1,\tau_2) + F^2[\psi](\tau_1,\tau_2)\right). \qquad (9.5.1)
\end{aligned}$$

Proof. We proceed in several steps.

Step 1. From Proposition 9.2.12, we have

$$\begin{aligned}
\text{Mor}^{ax}_{deg}[\psi](\tau_1,\tau_2) \ \lesssim\ & \sup_{\tau\in[\tau_1,\tau_2]} E_{deg}[\psi] + \int_{\mathcal{M}(\tau_1,\tau_2)} ar^{-2}\big(|\nabla\psi|^2 + |\nabla_{\mathbf{T}}\psi|^2\big) \\
& + \int_{\mathcal{M}(\tau_1,\tau_2)} (|\nabla_{\widehat{R}}\psi| + r^{-1}|\psi|)|N| + \epsilon\left(\sup_{[\tau_1,\tau_2]} E[\psi](\tau) + B^1_\delta[\psi]\right).
\end{aligned}$$

Also, from Proposition 9.2.13, we have

$$\begin{aligned}
& E_{deg}[\psi](\tau_2) + F_{\Sigma_*}[\psi](\tau_1,\tau_2) \\
\lesssim\ & \delta_{\mathcal{H}}\left(E_{r\leq r_+(1+\delta_{\mathcal{H}})}(\tau_2) + F_{\mathcal{A}}[\psi](\tau_1,\tau_2)\right) + E_{deg}[\psi](\tau_1) + \frac{|a|}{m}\text{Mor}^{ax}_{deg}[\psi](\tau_1,\tau_2) \\
& + \left|\int_{\mathcal{M}} \nabla_{\widehat{T}_\delta}\psi\cdot N\right| + \epsilon\left(\sup_{[\tau_1,\tau_2]} E[\psi](\tau) + B_\delta[\psi]\right).
\end{aligned}$$

Also, we have the redshift estimate of Proposition 9.4.1

$$\begin{aligned}
& \text{Mor}_{r\leq r_+(1+\delta_{red})}[\psi](\tau_1,\tau_2) + E_{r\leq r_+(1+\delta_{red})}[\psi](\tau_2) + F_{\mathcal{A}}[\psi](\tau_1,\tau_2) \\
\lesssim\ & E_{r\leq r_+(1+2\delta_{red})}[\psi](\tau_1) + \delta_{red}^{-3}\text{Mor}_{deg}[\psi](\tau_1,\tau_2) + \int_{\mathcal{M}(\tau_1,\tau_2)\cap\left\{\frac{r}{r_+}\leq 1+2\delta_{red}\right\}} |N|^2.
\end{aligned}$$

Combining the three estimates, where the second is multiplied by the large constant Λ

and the third by δ_{red}^4, we infer, for a constant c_0 only depending on $m-|a|$,

$$
\begin{aligned}
&c_0\Big(\delta_{red}^4\big(E_{red}[\psi](\tau_2)+\mathrm{Mor}_{red}[\psi](\tau_1,\tau_2)+F_{\mathcal{A}}[\psi](\tau_1,\tau_2)\big)+\Lambda\big(E_{deg}(\tau_2)+F_{\Sigma_*}[\psi](\tau_1,\tau_2)\big)\\
&+\mathrm{Mor}_{deg}^{ax}[\psi](\tau_1,\tau_2)\Big)\\
\leq\ & \sup_{\tau\in[\tau_1,\tau_2]}E_{deg}[\psi]+\int_{\mathcal{M}(\tau_1,\tau_2)}ar^{-2}\big(|\nabla\psi|^2+|\nabla_{\mathbf{T}}\psi|^2\big)+\int_{\mathcal{M}(\tau_1,\tau_2)}(|\nabla_{\widehat{R}}\psi|+r^{-1}|\psi|)|N|\\
&+\int_{\mathcal{M}(\tau_1,\tau_2)}|\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\psi)||\mathfrak{d}^{\leq 1}\psi|\\
&+\Lambda\Bigg(\delta_{\mathcal{H}}\left(E_{r\leq r_+(1+\delta_{\mathcal{H}})}(\tau_2)+F_{\mathcal{A}}[\psi](\tau_1,\tau_2)\right)+E_{deg}[\psi](\tau_1)+\frac{|a|}{m}\mathrm{Mor}_{deg}^{ax}[\psi](\tau_1,\tau_2)\\
&+\left|\int_{\mathcal{M}}\nabla_{\widehat{T}_\delta}\psi\cdot N\right|+\epsilon\left(\sup_{[\tau_1,\tau_2]}E[\psi](\tau)+B_\delta^1[\psi]\right)\Bigg)\\
&+\delta_{red}^4E_{r\leq r_+(1+2\delta_{red})}[\psi](\tau_1)+\delta_{red}\mathrm{Mor}_{deg}[\psi](\tau_1,\tau_2)+\delta_{red}^4\int_{\mathcal{M}(\tau_1,\tau_2)\cap\left\{\frac{r}{r_+}\leq 1+2\delta_{red}\right\}}|N|^2.
\end{aligned}
$$

We now choose Λ and δ_{red}, depending on $m-|a|$, such that

$$
c_0\Lambda\geq 2,\qquad c_0\delta_{red}^4\geq 2\Lambda\delta_{\mathcal{H}},\qquad c_0\geq 2\delta_{red}
$$

which is possible provided $\delta_{\mathcal{H}}$ is sufficiently small. We infer

$$
\begin{aligned}
&\frac{c_0}{2}\Big(\delta_{red}^4\big(E_{red}[\psi](\tau_2)+\mathrm{Mor}_{red}[\psi](\tau_1,\tau_2)+F_{\mathcal{A}}[\psi](\tau_1,\tau_2)\big)+\Lambda\big(E_{deg}(\tau_2)+F_{\Sigma_*}[\psi](\tau_1,\tau_2)\big)\\
&+\mathrm{Mor}_{deg}^{ax}[\psi](\tau_1,\tau_2)\Big)\\
\leq\ & \int_{\mathcal{M}(\tau_1,\tau_2)}ar^{-2}\big(|\nabla\psi|^2+|\nabla_{\mathbf{T}}\psi|^2\big)+\int_{\mathcal{M}(\tau_1,\tau_2)}(|\nabla_{\widehat{R}}\psi|+r^{-1}|\psi|)|N|+\int_{\mathcal{M}(\tau_1,\tau_2)}|\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\psi)||\mathfrak{d}^{\leq 1}\psi|\\
&+\Lambda\Bigg(E_{deg}[\psi](\tau_1)+\frac{|a|}{m}\mathrm{Mor}_{deg}^{ax}[\psi](\tau_1,\tau_2)+\left|\int_{\mathcal{M}}\nabla_{\widehat{T}_\delta}\psi\cdot N\right|+\epsilon\left(\sup_{[\tau_1,\tau_2]}E[\psi](\tau)+B_\delta^1[\psi]\right)\Bigg)\\
&+\delta_{red}^4E_{r\leq r_+(1+2\delta_{red})}[\psi](\tau_1)+\delta_{red}^4\int_{\mathcal{M}(\tau_1,\tau_2)\cap\left\{\frac{r}{r_+}\leq 1+2\delta_{red}\right\}}|N|^2.
\end{aligned}
$$

We deduce

$$
\begin{aligned}
&E[\psi](\tau_2)+F[\psi](\tau_1,\tau_2)+\mathrm{Mor}[\psi](\tau_1,\tau_2)\\
\lesssim\ & E[\psi](\tau_1)+\mathcal{N}[\psi,N](\tau_1,\tau_2)+\left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]}E^1[\psi]+B_\delta^1[\psi](\tau_1,\tau_2)+F^1[\psi](\tau_1,\tau_2)\right).
\end{aligned}
$$

Step 2. We commute with $(\dot{\mathcal{L}}_{\mathbf{T}}, \dot{\mathcal{L}}_{\mathbf{Z}})$ the RW model equation satisfied by ψ and obtain

$$\begin{aligned}\dot{\square}_2(\not{\mathcal{L}}_{\mathbf{T}}\psi) &= -[\dot{\mathcal{L}}_{\mathbf{T}}, \dot{\square}_2]\psi + \not{\mathcal{L}}_{\mathbf{T}}\left(V\psi - \frac{4a\cos\theta}{|q|^2}\nabla_{\mathbf{T}}\ {}^*\psi + N\right)\\ &= -[\dot{\mathcal{L}}_{\mathbf{T}}, \dot{\square}_2]\psi + V\not{\mathcal{L}}_{\mathbf{T}}\psi - \frac{4a\cos\theta}{|q|^2}\nabla_{\mathbf{T}}\ {}^*\not{\mathcal{L}}_{\mathbf{T}}\psi + \mathfrak{d}^{\leq 1}N + \Gamma_g\mathfrak{d}^{\leq 1}\psi\end{aligned}$$

and

$$\begin{aligned}\dot{\square}_2(\not{\mathcal{L}}_{\mathbf{Z}}\psi) &= -[\dot{\mathcal{L}}_{\mathbf{Z}}, \dot{\square}_2]\psi + \not{\mathcal{L}}_{\mathbf{Z}}\left(V\psi - \frac{4a\cos\theta}{|q|^2}\nabla_{\mathbf{T}}\ {}^*\psi + N\right)\\ &= -[\dot{\mathcal{L}}_{\mathbf{Z}}, \dot{\square}_2]\psi + V\not{\mathcal{L}}_{\mathbf{Z}}\psi - \frac{4a\cos\theta}{|q|^2}\nabla_{\mathbf{T}}\ {}^*\not{\mathcal{L}}_{\mathbf{Z}}\psi + \mathfrak{d}^{\leq 1}N + \Gamma_g\mathfrak{d}^{\leq 1}\psi.\end{aligned}$$

Recalling Corollary 4.3.4, i.e.

$$\begin{aligned}[\dot{\mathcal{L}}_{\mathbf{T}}, \dot{\square}_2]\psi &= \mathfrak{d}\big(\Gamma_g\cdot\mathfrak{d}\psi\big) + \Gamma_b\cdot\square_{\mathbf{g}}\psi,\\ [\dot{\mathcal{L}}_{\mathbf{Z}}, \dot{\square}_2]\psi &= \mathfrak{d}\big(\Gamma_g\cdot\mathfrak{d}\psi\big) + r\Gamma_b\cdot\square_{\mathbf{g}}\psi,\end{aligned}$$

we infer, using again the RW model equation for ψ to simplify the RHS,

$$\begin{aligned}\dot{\square}_2(\not{\mathcal{L}}_{\mathbf{T}}\psi) - V\not{\mathcal{L}}_{\mathbf{T}}\psi &= -\frac{4a\cos\theta}{|q|^2}\ {}^*\nabla_{\mathbf{T}}\not{\mathcal{L}}_{\mathbf{T}}\psi + N_{\not{\mathcal{L}}_{\mathbf{T}}\psi},\\ N_{\not{\mathcal{L}}_{\mathbf{T}}\psi} &= \mathfrak{d}^{\leq 1}N + \mathfrak{d}^{\leq 1}(\Gamma_g\mathfrak{d}^{\leq 1}\psi) + \Gamma_b\dot{\square}_2\psi\\ &= \mathfrak{d}^{\leq 1}N + \mathfrak{d}^{\leq 1}(\Gamma_g\mathfrak{d}^{\leq 1}\psi)\end{aligned}$$

and

$$\begin{aligned}\dot{\square}_2(\not{\mathcal{L}}_{\mathbf{Z}}\psi) - V\not{\mathcal{L}}_{\mathbf{Z}}\psi &= -\frac{4a\cos\theta}{|q|^2}\ {}^*\nabla_{\mathbf{T}}\not{\mathcal{L}}_{\mathbf{Z}}\psi + N_{\not{\mathcal{L}}_{\mathbf{Z}}\psi},\\ N_{\not{\mathcal{L}}_{\mathbf{Z}}\psi} &= \mathfrak{d}^{\leq 1}N + \mathfrak{d}^{\leq 1}(\Gamma_g\mathfrak{d}^{\leq 1}\psi) + r\Gamma_b\dot{\square}_2\psi\\ &= \mathfrak{d}^{\leq 1}N + \mathfrak{d}^{\leq 1}(\Gamma_g\mathfrak{d}^{\leq 1}\psi).\end{aligned}$$

We may apply the control derived in Step 1 to these RW model equations which yields, using Lemma 9.5.1 to control the $\mathfrak{d}^{\leq 1}(\Gamma_g\mathfrak{d}^{\leq 1}\psi)$ error terms,

$$\begin{aligned}&E[(\not{\mathcal{L}}_{\mathbf{T}}, \not{\mathcal{L}}_{\mathbf{Z}})\psi](\tau_2) + F[(\not{\mathcal{L}}_{\mathbf{T}}, \not{\mathcal{L}}_{\mathbf{Z}})\psi](\tau_1,\tau_2) + \mathrm{Mor}[(\not{\mathcal{L}}_{\mathbf{T}}, \not{\mathcal{L}}_{\mathbf{Z}})\psi](\tau_1,\tau_2)\\ \lesssim\ & E^1[\psi](\tau_1) + \mathcal{N}^1[\psi, N](\tau_1,\tau_2) + \left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]}E^2[\psi] + B^2_\delta[\psi](\tau_1,\tau_2) + F^2[\psi](\tau_1,\tau_2)\right).\end{aligned}$$

Given the relation between $\not{\mathcal{L}}_{\mathbf{T}}$ and $\nabla_{\mathbf{T}}$, and the one between $\not{\mathcal{L}}_{\mathbf{Z}}$ and $\nabla_{\mathbf{Z}}$, see Lemma 9.2.1, we infer, together with the estimate of Step 1,

$$\begin{aligned}&E[(\nabla_{\mathbf{T}}, \nabla_{\mathbf{Z}})^{\leq 1}\psi](\tau_2) + F[(\nabla_{\mathbf{T}}, \nabla_{\mathbf{Z}})^{\leq 1}\psi](\tau_1,\tau_2) + \mathrm{Mor}[(\nabla_{\mathbf{T}}, \nabla_{\mathbf{Z}})^{\leq 1}\psi](\tau_1,\tau_2)\\ \lesssim\ & E^1[\psi](\tau_1) + \mathcal{N}^1[\psi, N](\tau_1,\tau_2) + \left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]}E^2[\psi] + B^2_\delta[\psi](\tau_1,\tau_2) + F^2[\psi](\tau_1,\tau_2)\right).\end{aligned}$$

Step 3. Next, we commute the wave equation (6.1.1) satisfied by ψ by $|q|\mathcal{D}_2$ using the commutation formula of Lemma 4.7.13

$$\begin{aligned}\dot{\square}_1(|q|\mathcal{D}_2\psi) - V|q|\mathcal{D}_2\psi &= -\frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_T|q|\mathcal{D}_2\psi + N_{|q|\mathcal{D}_2},\\ N_{\not{\mathfrak{d}}_2} &:= \mathfrak{d}^{\leq 1}N - \frac{3}{r^2}|q|\mathcal{D}_2\psi + O(ar^{-2})\mathfrak{d}^{\leq 2}\psi + \mathfrak{d}^{\leq 2}(\Gamma_g\cdot\psi).\end{aligned}$$

From Proposition 9.2.13, we infer

$$\begin{aligned}&E_{deg}[|q|\mathcal{D}_2\psi](\tau_2) + F_{\Sigma_*}[|q|\mathcal{D}_2\psi](\tau_1,\tau_2)\\ \lesssim\;& \delta_{\mathcal{H}}\Big(E_{r\leq r_+}[\not{\mathfrak{d}}\psi](\tau_2) + F_{\mathcal{A}}[\not{\mathfrak{d}}\psi](\tau_1,\tau_2)\Big) + E^1[\psi](\tau_1) + \left|\int_{\mathcal{M}(\tau_1,\tau_2)}\nabla_{\widehat{T}_\delta}(|q|\mathcal{D}_2\psi)\cdot N_{|q|\mathcal{D}_2}\right|\\ &+\left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]}E^2[\psi] + B^2_\delta[\psi](\tau_1,\tau_2) + F^2[\psi](\tau_1,\tau_2)\right).\end{aligned}$$

Now, from the definition of $N_{|q|\mathcal{D}_2}$, we have, integrating by parts the second term in $N_{|q|\mathcal{D}_2}$,

$$\begin{aligned}&\left|\int_{\mathcal{M}(\tau_1,\tau_2)}\nabla_{\widehat{T}_\delta}(|q|\mathcal{D}_2\psi)\cdot N_{|q|\mathcal{D}_2}\right|\\ \lesssim\;& \left|\int_{\mathcal{M}(\tau_1,\tau_2)}\frac{1}{r^2}\nabla_{\widehat{T}_\delta}(|q|\mathcal{D}_2\psi)\cdot|q|\mathcal{D}_2\psi\right| + \mathcal{N}^1[\psi,N](\tau_1,\tau_2)\\ &+\left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]}E^2[\psi] + B^2_\delta[\psi](\tau_1,\tau_2) + F^2[\psi](\tau_1,\tau_2)\right)\\ \lesssim\;& \sup_{\tau\in[\tau_1,\tau_2]}E[\psi] + \mathcal{N}^1[\psi,N](\tau_1,\tau_2)\\ &+\left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]}E^2[\psi] + B^2_\delta[\psi](\tau_1,\tau_2) + F^2[\psi](\tau_1,\tau_2)\right),\end{aligned}$$

where we used the fact that $\widehat{T}_\delta(r)\in r\Gamma_b$, the control of $\mathbf{D}_\mu\widehat{T}^\mu$ induced from the one of ${}^{(\widehat{T}_\delta)}\pi$ in the proof of Proposition 9.2.13 in section 9.2.10, and Lemma 9.5.1 to control the various error terms.

In view of the above, and in particular the estimate for $E[\psi]$ of Step 1, we infer

$$\begin{aligned}&E_{deg}[|q|\mathcal{D}_2\psi](\tau_2) + F_{\Sigma_*}[|q|\mathcal{D}_2\psi](\tau_1,\tau_2)\\ \lesssim\;& \delta_{\mathcal{H}}\Big(E_{r\leq r_+}[\not{\mathfrak{d}}\psi](\tau_2) + F_{\mathcal{A}}[\not{\mathfrak{d}}\psi](\tau_1,\tau_2)\Big) + E^1[\psi](\tau_1)\\ &+\mathcal{N}^1[\psi,N](\tau_1,\tau_2) + \left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]}E^2[\psi] + B^2_\delta[\psi](\tau_1,\tau_2) + F^2[\psi](\tau_1,\tau_2)\right).\end{aligned}$$

Together with Step 2, and using the Hodge estimates of Proposition 9.3.2 to control $\not{\mathfrak{d}}$ from $|q|\mathcal{D}_2$ and $\nabla_{\mathbf{T}}$, we deduce

$$
\begin{aligned}
& E_{deg}[(\not{\mathfrak{d}}, \nabla_T)^{\leq 1}\psi](\tau_2) + F_{\Sigma_*}[(\not{\mathfrak{d}}, \nabla_T)^{\leq 1}\psi](\tau_1,\tau_2) + \mathrm{Mor}[(\nabla_{\mathbf{T}}, \nabla_{\mathbf{Z}})^{\leq 1}\psi](\tau_1,\tau_2)\\
\lesssim\ & \delta_{\mathcal{H}}\Big(E_{r\leq r_+}[\not{\mathfrak{d}}\psi](\tau_2) + F_{\mathcal{A}}[\not{\mathfrak{d}}\psi](\tau_1,\tau_2)\Big) + E^1[\psi](\tau_1)\\
& +\mathcal{N}^1[\psi,N](\tau_1,\tau_2) + \left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^2[\psi] + B^2_\delta[\psi](\tau_1,\tau_2) + F^2[\psi](\tau_1,\tau_2)\right).
\end{aligned}
$$

Step 4. Using the representation of the wave operator provided by (4.7.7), i.e.

$$
\begin{aligned}
|q|^2\dot{\Box}_2\psi = {}& \frac{(r^2+a^2)^2}{\Delta}\big(-\nabla_{\widehat{T}}\nabla_{\widehat{T}}\psi + \nabla_{\widehat{R}}\nabla_{\widehat{R}}\psi\big) + 2r\nabla_{\widehat{R}}\psi\\
& + |q|^2\triangle_2\psi + |q|^2(\eta+\underline{\eta})\cdot\nabla\psi + r^2\Gamma_g\cdot\mathfrak{d}\psi,
\end{aligned}
$$

together with the wave equation (6.1.1) satisfied by ψ, we have

$$
\begin{aligned}
\int_{\Sigma(\tau_2)}|\nabla^2_{\widehat{R}}\psi|^2 + \int_{\Sigma_*(\tau_1,\tau_2)}|\nabla^2_{\widehat{R}}\psi|^2 \ \lesssim\ & E_{deg}[(\nabla_T,\nabla_Z)^{\leq 1}\psi](\tau) + F_{\Sigma_*}[(\nabla_T,\nabla_Z)^{\leq 1}\psi]\\
& + \int_{\Sigma(\tau_2)}|N|^2 + \int_{\Sigma_*(\tau_1,\tau_2)}|N|^2.
\end{aligned}
$$

Since we have

$$
\begin{aligned}
& E_{deg}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq 1}\psi](\tau_2) + F_{\Sigma_*}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq 1}\psi](\tau_1,\tau_2)\\
\lesssim\ & E_{deg}[(\not{\mathfrak{d}}, \nabla_T)^{\leq 1}\psi](\tau_2) + F_{\Sigma_*}[(\not{\mathfrak{d}}, \nabla_T)^{\leq 1}\psi](\tau_1,\tau_2) + \int_{\Sigma(\tau_2)}|\nabla^2_{\widehat{R}}\psi|^2 + \int_{\Sigma_*(\tau_1,\tau_2)}|\nabla^2_{\widehat{R}}\psi|^2,
\end{aligned}
$$

we infer

$$
\begin{aligned}
& E_{deg}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq 1}\psi](\tau_2) + F_{\Sigma_*}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq 1}\psi](\tau_1,\tau_2)\\
\lesssim\ & E_{deg}[(\not{\mathfrak{d}}, \nabla_T)^{\leq 1}\psi](\tau_2) + F_{\Sigma_*}[(\not{\mathfrak{d}}, \nabla_T)^{\leq 1}\psi](\tau_1,\tau_2) + \int_{\Sigma(\tau_2)}|N|^2 + \int_{\Sigma_*(\tau_1,\tau_2)}|N|^2.
\end{aligned}
$$

Together with Step 3, we deduce

$$
\begin{aligned}
& E_{deg}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq 1}\psi](\tau_2) + F_{\Sigma_*}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq 1}\psi](\tau_1,\tau_2) + \mathrm{Mor}[(\nabla_{\mathbf{T}}, \nabla_{\mathbf{Z}})^{\leq 1}\psi](\tau_1,\tau_2)\\
\lesssim\ & \delta_{\mathcal{H}}\Big(E_{r\leq r_+}[\not{\mathfrak{d}}\psi](\tau_2) + F_{\mathcal{A}}[\not{\mathfrak{d}}\psi](\tau_1,\tau_2)\Big) + E^1[\psi](\tau_1)\\
& +\mathcal{N}^1[\psi,N](\tau_1,\tau_2) + \left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^2[\psi] + B^2_\delta[\psi](\tau_1,\tau_2) + F^2[\psi](\tau_1,\tau_2)\right)
\end{aligned}
$$

as stated. This concludes the proof of Proposition 9.5.2. □

Next, we control the energy of $\psi_{\underline{a}} = \mathcal{S}_{\underline{a}}\psi$ for $\underline{a} = 1, 2, 3, 4$.

Lemma 9.5.3. *We have*

$$\begin{aligned}
&\sum_{\underline{a}=1}^{4}\Big(E_{deg}[\psi_{\underline{a}}](\tau_2) + F_{\Sigma_*}[\psi_{\underline{a}}](\tau_1,\tau_2)\Big)\\
\lesssim\; & \delta_{\mathcal{H}}\Big(E^2_{r\leq r_+}[\psi](\tau_2) + F^2_{\mathcal{A}}[\psi](\tau_1,\tau_2)\Big) + E^2[\psi](\tau_1) + \mathcal{N}^2[\psi, N](\tau_1,\tau_2)\\
& + \left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^2[\psi] + B^2_\delta[\psi](\tau_1,\tau_2) + F^2[\psi](\tau_1,\tau_2)\right).
\end{aligned} \tag{9.5.2}$$

Proof. We first consider $\psi_{\underline{a}}$ for $\underline{a} = 1, 2, 3$. We commute with $(\dot{\mathcal{L}}_{\mathbf{T}}, \dot{\mathcal{L}}_{\mathbf{Z}})^2$ the RW model equation satisfied by ψ and obtain, proceeding as in Step 2 of the proof of Proposition 9.5.2,

$$\begin{aligned}
\dot{\square}_2((\not{\mathcal{L}}_{\mathbf{T}}, \not{\mathcal{L}}_{\mathbf{Z}})^2\psi) - V(\not{\mathcal{L}}_{\mathbf{T}}, \not{\mathcal{L}}_{\mathbf{Z}})^2\psi &= -\frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_{\mathbf{T}}(\not{\mathcal{L}}_{\mathbf{T}}, \not{\mathcal{L}}_{\mathbf{Z}})^2\psi + N_{(\not{\mathcal{L}}_{\mathbf{T}}, \not{\mathcal{L}}_{\mathbf{Z}})^2},\\
N_{(\not{\mathcal{L}}_{\mathbf{T}}, \not{\mathcal{L}}_{\mathbf{Z}})^2} &= \mathfrak{d}^{\leq 2}N + \mathfrak{d}^{\leq 2}(\Gamma_g\mathfrak{d}^{\leq 1}\psi).
\end{aligned}$$

From Proposition 9.2.13, we infer

$$\begin{aligned}
& E_{deg}[(\not{\mathcal{L}}_{\mathbf{T}}, \not{\mathcal{L}}_{\mathbf{Z}})^2\psi](\tau_2) + F_{\Sigma_*}[(\not{\mathcal{L}}_{\mathbf{T}}, \not{\mathcal{L}}_{\mathbf{Z}})^2\psi](\tau_1,\tau_2)\\
\lesssim\; & \delta_{\mathcal{H}}\Big(E^2_{r\leq r_+}[\psi](\tau_2) + F^2_{\mathcal{A}}[\psi](\tau_1,\tau_2)\Big) + E^2[\psi](\tau_1)\\
& + \left|\int_{\mathcal{M}(\tau_1,\tau_2)} \nabla_{\widehat{T}_\delta}((\not{\mathcal{L}}_{\mathbf{T}}, \not{\mathcal{L}}_{\mathbf{Z}})^2\psi)\cdot N_{(\not{\mathcal{L}}_{\mathbf{T}}, \not{\mathcal{L}}_{\mathbf{Z}})^2}\right|\\
& + \left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^2[\psi] + B^2_\delta[\psi](\tau_1,\tau_2) + F^2[\psi](\tau_1,\tau_2)\right).
\end{aligned}$$

Together with Lemma 9.5.1 and the structure of $N_{(\not{\mathcal{L}}_{\mathbf{T}}, \not{\mathcal{L}}_{\mathbf{Z}})^2}$, we deduce

$$\begin{aligned}
& E_{deg}[(\not{\mathcal{L}}_{\mathbf{T}}, \not{\mathcal{L}}_{\mathbf{Z}})^2\psi](\tau_2) + F_{\Sigma_*}[(\not{\mathcal{L}}_{\mathbf{T}}, \not{\mathcal{L}}_{\mathbf{Z}})^2\psi](\tau_1,\tau_2)\\
\lesssim\; & \delta_{\mathcal{H}}\Big(E^2_{r\leq r_+}[\psi](\tau_2) + F^2_{\mathcal{A}}[\psi](\tau_1,\tau_2)\Big) + E^2[\psi](\tau_1) + \mathcal{N}^2[\psi, N](\tau_1,\tau_2)\\
& + \left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^2[\psi] + B^2_\delta[\psi](\tau_1,\tau_2) + F^2[\psi](\tau_1,\tau_2)\right).
\end{aligned}$$

Given the relation between $\not{\mathcal{L}}_{\mathbf{T}}$ and $\nabla_{\mathbf{T}}$, and the one between $\not{\mathcal{L}}_{\mathbf{Z}}$ and $\nabla_{\mathbf{Z}}$, see Lemma 9.2.1, and using the control of $E_{deg}[(\nabla_{\mathbf{T}}, \nabla_{\mathbf{Z}})^{\leq 1}\psi]$ and $F_{\Sigma_*}[(\nabla_{\mathbf{T}}, \nabla_{\mathbf{Z}})^{\leq 1}\psi]$ provided by

Proposition 9.5.2, we obtain

$$\begin{aligned}
& E_{deg}[(\nabla_{\mathbf{T}}, \nabla_{\mathbf{Z}})^2\psi](\tau_2) + F_{\Sigma_*}[(\nabla_{\mathbf{T}}, \nabla_{\mathbf{Z}})^2\psi](\tau_1, \tau_2) \\
\lesssim\ & \delta_{\mathcal{H}}\Big(E^2_{r\leq r_+}[\psi](\tau_2) + F^2_{\mathcal{A}}[\psi](\tau_1, \tau_2)\Big) + E^2[\psi](\tau_1) + \mathcal{N}^2[\psi, N](\tau_1, \tau_2) \\
& + \left(\frac{|a|}{m} + \epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^2[\psi] + B^2_\delta[\psi](\tau_1, \tau_2) + F^2[\psi](\tau_1, \tau_2)\right),
\end{aligned}$$

and hence, in view of the definition of $\psi_{\underline{a}}$ for $\underline{a} = 1, 2, 3$,

$$\begin{aligned}
& \sum_{\underline{a}=1}^{3}\Big(E_{deg}[\psi_{\underline{a}}](\tau_2) + F_{\Sigma_*}[\psi_{\underline{a}}](\tau_1, \tau_2)\Big) \\
\lesssim\ & \delta_{\mathcal{H}}\Big(E^2_{r\leq r_+}[\psi](\tau_2) + F^2_{\mathcal{A}}[\psi](\tau_1, \tau_2)\Big) + E^2[\psi](\tau_1) + \mathcal{N}^2[\psi, N](\tau_1, \tau_2) \\
& + \left(\frac{|a|}{m} + \epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^2[\psi] + B^2_\delta[\psi](\tau_1, \tau_2) + F^2[\psi](\tau_1, \tau_2)\right).
\end{aligned}$$

It remains to control the energy of $\psi_4 = \mathcal{O}\psi$. To this end, we rely on the modified operator $\widetilde{\mathcal{O}}$ defined by (9.2.3), i.e.

$$\widetilde{\mathcal{O}}\psi = \mathcal{O}\psi + \frac{4a(r^2 + a^2 + |q|^2)\cos\theta}{|q|^2}\nabla_{\mathbf{T}}\,{}^*\psi + \frac{4a^2\cos\theta}{|q|^2}\nabla_{\mathbf{Z}}\,{}^*\psi$$

and derive the model RW equation satisfied by $\widetilde{\mathcal{O}}\psi$. In view of the model RW equation for ψ, we have

$$\begin{aligned}
\dot{\square}_2(\widetilde{\mathcal{O}}\psi) - V\widetilde{\mathcal{O}}\psi \ =\ & -\frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_{\mathbf{T}}\widetilde{\mathcal{O}}\psi + \frac{1}{|q|^2}\widetilde{\mathcal{O}}(|q|^2 N) \\
& + \frac{1}{|q|^2}\left[|q|^2\left(\dot{\square}_2 - V + \frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_{\mathbf{T}}\right), \widetilde{\mathcal{O}}\right]\psi.
\end{aligned}$$

Using Lemma 9.2.8 to estimate the last term, we infer

$$\dot{\square}_2(\widetilde{\mathcal{O}}\psi) - V\widetilde{\mathcal{O}}\psi \ =\ -\frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_{\mathbf{T}}\widetilde{\mathcal{O}}\psi + N_{\widetilde{\mathcal{O}}},$$

with

$$\begin{aligned}
N_{\widetilde{\mathcal{O}}} \ &=\ \mathfrak{d}^{\leq 2}N + O(ar^{-2})\nabla_{\widehat{R}}\mathfrak{d}^{\leq 1}\psi + O(ar^{-2})\mathfrak{d}^{\leq 1}\psi + \mathfrak{d}^{\leq 3}(\Gamma_g\cdot\psi) + \Gamma_b\cdot\dot{\square}_2\psi \\
&=\ \mathfrak{d}^{\leq 2}N + O(ar^{-2})\nabla_{\widehat{R}}\mathfrak{d}^{\leq 1}\psi + O(ar^{-2})\mathfrak{d}^{\leq 1}\psi + \mathfrak{d}^{\leq 3}(\Gamma_g\cdot\psi)
\end{aligned}$$

where we used again the model RW equation for ψ. In view of Proposition 9.2.13, we deduce

$$\begin{aligned}
& E_{deg}[\widetilde{\mathcal{O}}\psi](\tau_2) + F_{\Sigma_*}[\widetilde{\mathcal{O}}\psi](\tau_1, \tau_2) \\
\lesssim \;& \delta_{\mathcal{H}}\Big(E^2_{r\leq r_+}[\psi](\tau_2) + F^2_{\mathcal{A}}[\psi](\tau_1, \tau_2)\Big) + E^2[\psi](\tau_1) + \left|\int_{\mathcal{M}(\tau_1,\tau_2)} \nabla_{\widehat{T}_\delta}(\widetilde{\mathcal{O}}\psi)\cdot N_{\widetilde{\mathcal{O}}}\right| \\
& + \left(\frac{|a|}{m} + \epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^2[\psi] + B^2_\delta[\psi](\tau_1, \tau_2) + F^2[\psi](\tau_1, \tau_2)\right).
\end{aligned}$$

Together with Lemma 9.5.1 and the structure of $N_{(\not{\mathcal{L}}_{\mathbf{T}},\not{\mathcal{L}}_{\mathbf{Z}})^2}$, we deduce

$$\begin{aligned}
& E_{deg}[\widetilde{\mathcal{O}}\psi](\tau_2) + F_{\Sigma_*}[\widetilde{\mathcal{O}}\psi](\tau_1, \tau_2) \\
\lesssim \;& \delta_{\mathcal{H}}\Big(E^2_{r\leq r_+}[\psi](\tau_2) + F^2_{\mathcal{A}}[\psi](\tau_1, \tau_2)\Big) + E^2[\psi](\tau_1) + \mathcal{N}^2[\psi, N](\tau_1, \tau_2) \\
& + \left|\int_{\mathcal{M}(\tau_1,\tau_2)} \nabla_{\widehat{T}_\delta}(\widetilde{\mathcal{O}}\psi)\cdot\Big(O(ar^{-2})\nabla_{\widehat{R}}\mathfrak{d}^{\leq 1}\psi + O(ar^{-2})\mathfrak{d}^{\leq 1}\psi\Big)\right| \\
& + \left(\frac{|a|}{m} + \epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^2[\psi] + B^2_\delta[\psi](\tau_1, \tau_2) + F^2[\psi](\tau_1, \tau_2)\right).
\end{aligned}$$

Integrating by parts the $\widehat{T}_\delta$ derivative, we obtain

$$\begin{aligned}
& E_{deg}[\widetilde{\mathcal{O}}\psi](\tau_2) + F_{\Sigma_*}[\widetilde{\mathcal{O}}\psi](\tau_1, \tau_2) \\
\lesssim \;& \delta_{\mathcal{H}}\Big(E^2_{r\leq r_+}[\psi](\tau_2) + F^2_{\mathcal{A}}[\psi](\tau_1, \tau_2)\Big) + E^2[\psi](\tau_1) + \mathcal{N}^2[\psi, N](\tau_1, \tau_2) \\
& + |a|\int_{\mathcal{M}(\tau_1,\tau_2)} r^{-2}|\mathfrak{d}^{\leq 2}\psi|\Big(|\nabla_{\widehat{R}}\mathfrak{d}^{\leq 2}\psi| + |\nabla_{\widehat{T}_\delta}\mathfrak{d}^{\leq 1}\psi|\Big) \\
& + \left(\frac{|a|}{m} + \epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^2[\psi] + B^2_\delta[\psi](\tau_1, \tau_2) + F^2[\psi](\tau_1, \tau_2)\right).
\end{aligned}$$

where we used the fact that $\widehat{T}_\delta(r) \in r\Gamma_b$ and $\widehat{T}_\delta(\cos\theta) \in \Gamma_b$, the structure of the commutator $[\nabla_{\mathbf{T}}, \nabla_{\widehat{R}}]$ provided by Lemma 9.2.7, the control of $\mathbf{D}_\mu \widehat{T}^\mu$ induced from the one of ${}^{(\widehat{T}_\delta)}\pi$ in the proof of Proposition 9.2.13 in section 9.2.10, and Lemma 9.5.1 to control the various error terms. Since

$$\begin{aligned}
& \int_{\mathcal{M}(\tau_1,\tau_2)} r^{-2}|\mathfrak{d}^{\leq 2}\psi|\Big(|\nabla_{\widehat{R}}\mathfrak{d}^{\leq 2}\psi| + |\nabla_{\widehat{T}_\delta}\mathfrak{d}^{\leq 1}\psi|\Big) \\
\lesssim \;& \int_{\mathcal{M}_{trap}\tau_1,\tau_2)}\Big(|\nabla_{\widehat{R}}\mathfrak{d}^{\leq 2}\psi|^2 + |\mathfrak{d}^{\leq 2}\psi|^2\Big) + \int_{\mathcal{M}_{\not{trap}}\tau_1,\tau_2)} r^{-2}\Big(|\nabla_3\mathfrak{d}^{\leq 2}\psi||\mathfrak{d}^{\leq 3}\psi| + r^{-1}|\mathfrak{d}^{\leq 3}\psi|^2\Big) \\
\lesssim \;& B^2_\delta[\psi](\tau_1, \tau_2),
\end{aligned}$$

we infer

$$
\begin{aligned}
& E_{deg}[\widetilde{\mathcal{O}}\psi](\tau_2) + F_{\Sigma_*}[\widetilde{\mathcal{O}}\psi](\tau_1,\tau_2) \\
\lesssim\;& \delta_{\mathcal{H}}\Big(E^2_{r\le r_+}[\psi](\tau_2) + F^2_{\mathcal{A}}[\psi](\tau_1,\tau_2)\Big) + E^2[\psi](\tau_1) + \mathcal{N}^2[\psi,N](\tau_1,\tau_2) \\
& + \left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^2[\psi] + B^2_\delta[\psi](\tau_1,\tau_2) + F^2[\psi](\tau_1,\tau_2)\right).
\end{aligned}
$$

Together with the definition of $\widetilde{\mathcal{O}}$ and the control of $E_{deg}[(\nabla_{\mathbf{T}},\nabla_{\mathbf{Z}})^{\le 1}\psi]$ and $F_{\Sigma_*}[(\nabla_{\mathbf{T}},\nabla_{\mathbf{Z}})^{\le 1}\psi]$ provided by Proposition 9.5.2, we obtain

$$
\begin{aligned}
& E_{deg}[\mathcal{O}\psi](\tau_2) + F_{\Sigma_*}[\mathcal{O}\psi](\tau_1,\tau_2) \\
\lesssim\;& \delta_{\mathcal{H}}\Big(E^2_{r\le r_+}[\psi](\tau_2) + F^2_{\mathcal{A}}[\psi](\tau_1,\tau_2)\Big) + E^2[\psi](\tau_1) + \mathcal{N}^2[\psi,N](\tau_1,\tau_2) \\
& + \left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^2[\psi] + B^2_\delta[\psi](\tau_1,\tau_2) + F^2[\psi](\tau_1,\tau_2)\right).
\end{aligned}
$$

The is the stated energy estimate for $\psi_4 = \mathcal{O}\psi$. Together with the above energy estimates for $\psi_{\underline{a}}$, $\underline{a} = 1,2,3$, this concludes the proof of Lemma 9.5.3. □

The following proposition provides the control of Morawetz from the energy.

Proposition 9.5.4. *The solution ψ of of the model RW equation* (6.1.1) *satisfies the following Morawetz estimate*

$$
\begin{aligned}
& \delta^7_{red}\Big(E^2_{r\le r_+(1+\delta_{red})}[\psi](\tau_2) + Mor[(\not{\mathfrak{d}},\nabla_3,\nabla_4)^{\le 2}\psi](\tau_1,\tau_2) + F^2_{\mathcal{A}}[\psi](\tau_1,\tau_2)\Big) \\
\lesssim\;& \delta_{\mathcal{H}}\left(\sup_{\tau\in[\tau_1,\tau_2]} E_{deg}[(\not{\mathfrak{d}},\nabla_T,\nabla_{\widehat{R}})^{\le 2}\psi] + F_{\Sigma_*}[(\not{\mathfrak{d}},\nabla_T,\nabla_{\widehat{R}})^{\le 2}\psi]\right) \\
& + \Bigg(E^1[\psi](\tau_1) + \mathcal{N}^1[\psi,N](\tau_1,\tau_2) \\
& + \left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^2[\psi] + B^2_\delta[\psi](\tau_1,\tau_2) + F^2[\psi](\tau_1,\tau_2)\right)\Bigg)^{\frac{1}{2}} \\
& \times\left(\sup_{\tau\in[\tau_1,\tau_2]} E_{deg}[(\not{\mathfrak{d}},\nabla_T,\nabla_{\widehat{R}})^{\le 2}\psi] + F_{\Sigma_*}[(\not{\mathfrak{d}},\nabla_T,\nabla_{\widehat{R}})^{\le 2}\psi]\right)^{\frac{1}{2}} \\
& + \mathcal{N}^2[\psi,N](\tau_1,\tau_2) + \left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^2[\psi] + B^2_\delta[\psi](\tau_1,\tau_2) + F^2[\psi](\tau_1,\tau_2)\right) \\
& + E^2[\psi](\tau_1) \qquad (9.5.3)
\end{aligned}
$$

Proof. We proceed in several steps.

Step 1. From Proposition 9.2.15, together with Lemma 9.2.16, we infer

$$
\begin{aligned}
& \mathrm{Mor}_{deg}[\nabla_T^2\psi](\tau_1,\tau_2) + \mathrm{Mor}_{deg}[\mathcal{O}\psi](\tau_1,\tau_2) + \mathrm{Mor}_{deg}[\not{\mathfrak{d}}\nabla_T\psi](\tau_1,\tau_2) \\
\lesssim\ & \sum_{\underline{a}=1}^{4}\left(\sup_{[\tau_1,\tau_2]} E_{deg}[\psi_{\underline{a}}](\tau) + \delta_{\mathcal{H}} F_{\mathcal{A}}[\psi_{\underline{a}}](\tau_1,\tau_2) + F_{\Sigma_*}[\psi_{\underline{a}}](\tau_1,\tau_2)\right) \\
& + \left(\sup_{\tau\in[\tau_1,\tau_2]} E_{deg}[(\not{\mathfrak{d}},\nabla_T)^{\leq 2}\psi] + F_{\Sigma_*}[(\not{\mathfrak{d}},\nabla_T)^{\leq 2}\psi] + \delta_{\mathcal{H}} F_{\mathcal{A}}[(\not{\mathfrak{d}},\nabla_T)^{\leq 2}\psi]\right)^{\frac{1}{2}} \\
& \times \left(\sup_{\tau\in[\tau_1,\tau_2]} E_{deg}[(\not{\mathfrak{d}},\nabla_T)^{\leq 1}\psi] + F_{\Sigma_*}[(\not{\mathfrak{d}},\nabla_T)^{\leq 1}\psi] + \delta_{\mathcal{H}} F_{\mathcal{A}}[(\not{\mathfrak{d}},\nabla_T)^{\leq 1}\psi]\right)^{\frac{1}{2}} \\
& + \int_{\mathcal{M}(\tau_1,\tau_2)} (|\nabla_{\widehat{R}}\psi_{\underline{a}}| + r^{-1}|\psi_{\underline{a}}|)|N_{\underline{a}}| + \epsilon\left(\sup_{[\tau_1,\tau_2]} E^2[\psi](\tau) + B^2_\delta[\psi]\right).
\end{aligned}
$$

Plugging the control of the energy provided by Proposition 9.5.2 and Lemma 9.5.3 in the RHS, we infer

$$
\begin{aligned}
& \mathrm{Mor}_{deg}[\nabla_T^2\psi](\tau_1,\tau_2) + \mathrm{Mor}_{deg}[\mathcal{O}\psi](\tau_1,\tau_2) + \mathrm{Mor}_{deg}[\not{\mathfrak{d}}\nabla_T\psi](\tau_1,\tau_2) \\
\lesssim\ & \delta_{\mathcal{H}}\Big(E^2_{r\leq r_+}[\psi](\tau_2) + F^2_{\mathcal{A}}[\psi](\tau_1,\tau_2)\Big) + \mathcal{I}_{RHS} + \int_{\mathcal{M}(\tau_1,\tau_2)} (|\nabla_{\widehat{R}}\psi_{\underline{a}}| + r^{-1}|\psi_{\underline{a}}|)|N_{\underline{a}}|
\end{aligned}
$$

where we have introduce the following notation

$$
\begin{aligned}
\mathcal{I}_{RHS} \ :=\ & \delta_{\mathcal{H}}\left(\sup_{\tau\in[\tau_1,\tau_2]} E_{deg}[(\not{\mathfrak{d}},\nabla_T,\nabla_{\widehat{R}})^{\leq 2}\psi] + F_{\Sigma_*}[(\not{\mathfrak{d}},\nabla_T,\nabla_{\widehat{R}})^{\leq 2}\psi]\right) \\
& + \Bigg(E^1[\psi](\tau_1) + \mathcal{N}^1[\psi,N](\tau_1,\tau_2) \\
& + \left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^2[\psi] + B^2_\delta[\psi](\tau_1,\tau_2) + F^2[\psi](\tau_1,\tau_2)\right)\Bigg)^{\frac{1}{2}} \\
& \times \left(\sup_{\tau\in[\tau_1,\tau_2]} E_{deg}[(\not{\mathfrak{d}},\nabla_T,\nabla_{\widehat{R}})^{\leq 2}\psi] + F_{\Sigma_*}[(\not{\mathfrak{d}},\nabla_T,\nabla_{\widehat{R}})^{\leq 2}\psi]\right)^{\frac{1}{2}} \\
& + \mathcal{N}^2[\psi,N](\tau_1,\tau_2) + \left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^2[\psi] + B^2_\delta[\psi](\tau_1,\tau_2) + F^2[\psi](\tau_1,\tau_2)\right) \\
& + E^2[\psi](\tau_1).
\end{aligned}
$$

Also, recall from (9.2.2) that we have

$$|N_{\underline{a}}| \lesssim |\mathfrak{d}^{\leq 2}N| + \frac{|a|}{m}r^{-2}|\mathfrak{d}^{\leq 1}\nabla_3\psi| + \frac{|a|}{m}r^{-3}|\mathfrak{d}^{\leq 2}\psi| + |\mathfrak{d}^{\leq 3}(\Gamma_g \cdot \psi)|$$

so that, using Lemma 9.5.1,

$$\begin{aligned}
&\int_{\mathcal{M}(\tau_1,\tau_2)} (|\nabla_{\widehat{R}}\psi_{\underline{a}}| + r^{-1}|\psi_{\underline{a}}|)|N_{\underline{a}}| \\
\lesssim\ & \mathcal{N}^2[\psi, N](\tau_1,\tau_2) + \left(\frac{|a|}{m} + \epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^2[\psi] + B^2_\delta[\psi](\tau_1,\tau_2) + F^2[\psi](\tau_1,\tau_2)\right) \\
\lesssim\ & \mathcal{I}_{RHS}.
\end{aligned}$$

We infer

$$\begin{aligned}
&\mathrm{Mor}_{deg}[\nabla_T^2\psi](\tau_1,\tau_2) + \mathrm{Mor}_{deg}[\mathcal{O}\psi](\tau_1,\tau_2) + \mathrm{Mor}_{deg}[\not{\mathfrak{d}}\nabla_T\psi](\tau_1,\tau_2) \\
\lesssim\ & \mathcal{I}_{RHS} + \delta_{\mathcal{H}}\Big(E^2_{r\leq r_+}[\psi](\tau_2) + F^2_{\mathcal{A}}[\psi](\tau_1,\tau_2)\Big).
\end{aligned}$$

Moreover, using the definition of $\mathcal{O}$ and the Hodge estimates of Proposition 9.3.2, we have

$$\begin{aligned}
\mathrm{Mor}_{deg}[\not{\mathfrak{d}}^{\leq 2}\psi](\tau_1,\tau_2) &\lesssim \mathrm{Mor}_{deg}[r^2\Delta\psi](\tau_1,\tau_2) + \mathrm{Mor}_{deg}[\nabla_{\widetilde{T}}^{\leq 2}\psi](\tau_1,\tau_2) \\
&\lesssim \mathrm{Mor}_{deg}[\mathcal{O}\psi](\tau_1,\tau_2) + \mathrm{Mor}_{deg}[\nabla_{\widetilde{T}}^{\leq 2}\psi](\tau_1,\tau_2) + \frac{|a|}{m}B^1_\delta[\psi](\tau_1,\tau_2)
\end{aligned}$$

and hence

$$\begin{aligned}
&\mathrm{Mor}_{deg}[\nabla_T^2\psi](\tau_1,\tau_2) + \mathrm{Mor}_{deg}[\not{\mathfrak{d}}^{\leq 2}\psi](\tau_1,\tau_2) + \mathrm{Mor}_{deg}[\not{\mathfrak{d}}\nabla_T\psi](\tau_1,\tau_2) \\
\lesssim\ & \delta_{\mathcal{H}}\Big(E^2_{r\leq r_+}[\psi](\tau_2) + F^2_{\mathcal{A}}[\psi](\tau_1,\tau_2)\Big) + \mathcal{I}_{RHS} + \mathrm{Mor}_{deg}[\nabla_{\widetilde{T}}^{\leq 1}\psi](\tau_1,\tau_2) + \frac{|a|}{m}B^1_\delta[\psi](\tau_1,\tau_2) \\
\lesssim\ & \delta_{\mathcal{H}}\Big(E^2_{r\leq r_+}[\psi](\tau_2) + F^2_{\mathcal{A}}[\psi](\tau_1,\tau_2)\Big) + \mathcal{I}_{RHS} + \mathrm{Mor}_{deg}[\nabla_{\widetilde{T}}^{\leq 1}\psi](\tau_1,\tau_2).
\end{aligned}$$

Together with the control of $\mathrm{Mor}_{deg}[\nabla_{\widetilde{T}}^{\leq 1}\psi]$ provided by Proposition 9.5.2, we deduce

$$\mathrm{Mor}_{deg}[(\nabla_T, \not{\mathfrak{d}})^{\leq 2}\psi](\tau_1,\tau_2) \lesssim \delta_{\mathcal{H}}\Big(E^2_{r\leq r_+}[\psi](\tau_2) + F^2_{\mathcal{A}}[\psi](\tau_1,\tau_2)\Big) + \mathcal{I}_{RHS}.$$

Step 2. We now introduce the operators $\not{\mathfrak{d}}_2^j$ acting on $\mathfrak{s}_2$ as follows, for $j \geq 0$,

$$\not{\mathfrak{d}}_2^{2j} := (|q|^2\Delta_2)^j, \qquad \not{\mathfrak{d}}_2^{2j+1} := |q|\mathcal{D}_2(|q|^2\Delta_2)^j. \tag{9.5.4}$$

Then, we commute the equation for ψ by $\not{\mathfrak{d}}_2^{j_1}\not{\mathcal{L}}_{\mathbf{T}}^{j_2}$ with $j_1+j_2\leq 2$, and obtain in $r\leq r_+(1+2\delta_{red})$, using the commutation formulas of Corollary 4.3.4, Lemma 4.5.4 and Lemma 4.7.13

$$\dot{\square}_{k_{j_1}}(\not{\mathfrak{d}}_2^{j_1}\not{\mathcal{L}}_{\mathbf{T}}^{j_2}\psi)-V\not{\mathfrak{d}}_2^{j_1}\not{\mathcal{L}}_{\mathbf{T}}^{j_2}\psi \;=\; -\frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_T(\not{\mathfrak{d}}_2^{j_1}\not{\mathcal{L}}_{\mathbf{T}}^{j_2}\psi)+N_{\not{\mathfrak{d}}_2^{j_1}\not{\mathcal{L}}_{\mathbf{T}}^{j_2}},$$

where $k_0=k_2=2$ and $k_1=1$, and, we have, for $r\leq 4m$,

$$N_{\not{\mathfrak{d}}_2^{j_1}\not{\mathcal{L}}_{\mathbf{T}}^{j_2}} := \mathfrak{d}^{\leq 2}N+\delta_{j_1 1}O(1)\not{\mathfrak{d}}_2^{j_1}\not{\mathcal{L}}_{\mathbf{T}}^{j_2}\psi+O\left(\frac{|a|}{m}+\epsilon\right)\mathfrak{d}^{\leq 3}\psi.$$

Using the redshift estimates of Proposition 9.4.1, we deduce, for $|j|=1,2$,

$$\begin{aligned}
&c_0E_{r\leq r_+(1+\delta_{red})}[\not{\mathfrak{d}}_2^{j_1}\not{\mathcal{L}}_{\mathbf{T}}^{j_2}\psi](\tau_2)+c_0\mathrm{Mor}_{r\leq r_+(1+\delta_{red})}[\not{\mathfrak{d}}_2^{j_1}\not{\mathcal{L}}_{\mathbf{T}}^{j_2}\psi](\tau_1,\tau_2)+c_0F_{\mathcal{A}}[\not{\mathfrak{d}}_2^{j_1}\not{\mathcal{L}}_{\mathbf{T}}^{j_2}\psi](\tau_1,\tau_2)\\
\leq& E^2_{r\leq r_+(1+2\delta_{red})}[\psi](\tau_1)+\delta_{red}^{-1}\mathrm{Mor}_{r_+(1+\delta_{red})\leq r\leq r_+(1+2\delta_{red})}[(\not{\mathfrak{d}},\not{\mathcal{L}}_{\mathbf{T}})^{\leq 2}\psi](\tau_1,\tau_2)\\
&+\int_{\mathcal{M}(\tau_1,\tau_2)\cap\left\{\frac{r}{r_+}\leq 1+2\delta_{red}\right\}}|N_{\not{\mathfrak{d}}_2^{j_1}\not{\mathcal{L}}_{\mathbf{T}}^{j_2}}|^2.
\end{aligned}$$

In view of the form of $N_{\not{\mathfrak{d}}_2^{j_1}\not{\mathcal{L}}_{\mathbf{T}}^{j_2}}$, we infer

$$\begin{aligned}
&c_0E_{r\leq r_+(1+\delta_{red})}[\not{\mathfrak{d}}_2^{j_1}\not{\mathcal{L}}_{\mathbf{T}}^{j_2}\psi](\tau_2)+c_0\mathrm{Mor}_{r\leq r_+(1+\delta_{red})}[\not{\mathfrak{d}}_2^{j_1}\not{\mathcal{L}}_{\mathbf{T}}^{j_2}\psi](\tau_1,\tau_2)+c_0F_{\mathcal{A}}[\not{\mathfrak{d}}_2^{j_1}\not{\mathcal{L}}_{\mathbf{T}}^{j_2}\psi](\tau_1,\tau_2)\\
\leq& E^2_{r\leq r_+(1+2\delta_{red})}[\psi](\tau_1)+(\delta_{j_1 1}+\delta_{j_1 2})\mathrm{Mor}_{r\leq r_+(1+\delta_{red})}[\not{\mathfrak{d}}_2^{j_1-1}\not{\mathcal{L}}_{\mathbf{T}}^{j_2}\psi](\tau_1,\tau_2)\\
&+\delta_{red}^{-1}\mathrm{Mor}_{r_+(1+\delta_{red})\leq r\leq r_+(1+2\delta_{red})}[(\not{\mathfrak{d}},\not{\mathcal{L}}_{\mathbf{T}})^{\leq 2}\psi](\tau_1,\tau_2)\\
&+\mathcal{N}^2[\psi,N](\tau_1,\tau_2)+\left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]}E^2[\psi]+B^2_\delta[\psi](\tau_1,\tau_2)+F^2[\psi](\tau_1,\tau_2)\right).
\end{aligned}$$

Arguing by iteration on j_1, using also the comparison of $\not{\mathcal{L}}_{\mathbf{T}}$ and $\nabla_{\mathbf{T}}$ of Lemma 9.2.1, and the Hodge estimates of Proposition 9.3.2, we obtain

$$\begin{aligned}
&c_0E_{r\leq r_+(1+\delta_{red})}[(\not{\mathfrak{d}},\nabla_{\mathbf{T}})^{\leq 2}\psi](\tau_2)+c_0\mathrm{Mor}_{r\leq r_+(1+\delta_{red})}[(\not{\mathfrak{d}},\nabla_{\mathbf{T}})^{\leq 2}\psi](\tau_1,\tau_2)\\
&+c_0F_{\mathcal{A}}[(\not{\mathfrak{d}},\nabla_{\mathbf{T}})^{\leq 2}\psi](\tau_1,\tau_2)\\
\leq& E^2_{r\leq r_+(1+2\delta_{red})}[\psi](\tau_1)+\delta_{red}^{-1}\mathrm{Mor}_{r_+(1+\delta_{red})\leq r\leq r_+(1+2\delta_{red})}[(\not{\mathfrak{d}},\nabla_{\mathbf{T}})^{\leq 2}\psi](\tau_1,\tau_2)\\
&+\mathcal{N}^2[\psi,N](\tau_1,\tau_2)+\left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]}E^2[\psi]+B^2_\delta[\psi](\tau_1,\tau_2)+F^2[\psi](\tau_1,\tau_2)\right).
\end{aligned}$$

Multiplying this estimate by δ_{red}^6, and summing it with the above control obtained in Step

1 for $\mathrm{Mor}_{deg}[(\not{\mathfrak{d}}, \nabla_T)^{\leq 2}\psi](\tau_1, \tau_2)$, we infer

$$\begin{aligned}
& \delta_{red}^6\Big(E_{r\leq r_+(1+\delta_{red})}[(\not{\mathfrak{d}}, \nabla_{\mathbf{T}})^{\leq 2}\psi](\tau_2) + \mathrm{Mor}_{r\leq r_+(1+\delta_{red})}[(\not{\mathfrak{d}}, \nabla_{\mathbf{T}})^{\leq 2}\psi](\tau_1, \tau_2) \\
& +F_{\mathcal{A}}[(\not{\mathfrak{d}}, \nabla_{\mathbf{T}})^{\leq 2}\psi](\tau_1, \tau_2)\Big) + \mathrm{Mor}_{deg}[(\not{\mathfrak{d}}, \nabla_T)^{\leq 2}\psi](\tau_1, \tau_2) \\
\lesssim\ & \delta_{\mathcal{H}}\Big(E^2_{r\leq r_+}[\psi](\tau_2) + F^2_{\mathcal{A}}[\psi](\tau_1, \tau_2)\Big) + \mathcal{I}_{RHS} \\
& +\delta_{red}^5 \mathrm{Mor}_{r_+(1+\delta_{red})\leq r\leq r_+(1+2\delta_{red})}[(\not{\mathfrak{d}}, \nabla_{\mathbf{T}})^{\leq 2}\psi](\tau_1, \tau_2).
\end{aligned}$$

Since

$$\delta_{red}^5 \mathrm{Mor}_{r_+(1+\delta_{red})\leq r\leq r_+(1+2\delta_{red})}[(\not{\mathfrak{d}}, \nabla_{\mathbf{T}})^{\leq 2}\psi](\tau_1, \tau_2) \lesssim \delta_{red}\mathrm{Mor}_{deg}[(\not{\mathfrak{d}}, \nabla_T)^{\leq 2}\psi](\tau_1, \tau_2)$$

we may absorb the last term of the RHS by the LHS for δ_{red} small enough, and hence

$$\begin{aligned}
& \delta_{red}^6\Big(E_{r\leq r_+(1+\delta_{red})}[(\not{\mathfrak{d}}, \nabla_{\mathbf{T}})^{\leq 2}\psi](\tau_2) + \mathrm{Mor}_{r\leq r_+(1+\delta_{red})}[(\not{\mathfrak{d}}, \nabla_{\mathbf{T}})^{\leq 2}\psi](\tau_1, \tau_2) \\
& +F_{\mathcal{A}}[(\not{\mathfrak{d}}, \nabla_{\mathbf{T}})^{\leq 2}\psi](\tau_1, \tau_2)\Big) + \mathrm{Mor}_{deg}[(\not{\mathfrak{d}}, \nabla_T)^{\leq 2}\psi](\tau_1, \tau_2) \\
\lesssim\ & \delta_{\mathcal{H}}\Big(E^2_{r\leq r_+}[\psi](\tau_2) + F^2_{\mathcal{A}}[\psi](\tau_1, \tau_2)\Big) + \mathcal{I}_{RHS}.
\end{aligned}$$

Step 3. Next, recalling the representation of the wave operator provided by (4.7.7), i.e.

$$\begin{aligned}
|q|^2\dot{\square}_2\psi = \frac{(r^2+a^2)^2}{\Delta}\big(-\nabla_{\widehat{T}}\nabla_{\widehat{T}}\psi + \nabla_{\widehat{R}}\nabla_{\widehat{R}}\psi\big) + 2r\nabla_{\widehat{R}}\psi \\
+ |q|^2\triangle_2\psi + |q|^2(\eta+\underline{\eta})\cdot\nabla\psi + r^2\Gamma_g\cdot\mathfrak{d}\psi,
\end{aligned}$$

we infer

$$\begin{aligned}
\int_{\mathcal{M}}\Big(|\nabla^3_{\widehat{R}}\psi|^2 + |\nabla^2_{\widehat{R}}\psi|^2\Big) \lesssim\ & \mathrm{Mor}_{deg}[(\not{\mathfrak{d}}, \nabla_T)^{\leq 2}\psi](\tau_1, \tau_2) \\
& + \int_{\mathcal{M}}|\mathfrak{d}^{\leq 1}N|^2 + \int_{\mathcal{M}}|\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\psi)|^2.
\end{aligned}$$

Then, we have by integration by parts

$$\begin{aligned}
& \int_{\mathcal{M}}\Big(|(\not{\mathfrak{d}}, \nabla_T)\nabla^2_{\widehat{R}}\psi|^2 + |(\not{\mathfrak{d}}, \nabla_T)\nabla_{\widehat{R}}\psi|^2\Big) \\
\lesssim\ & \int_{\mathcal{M}}\Big(|\nabla_{\widehat{R}}(\not{\mathfrak{d}}, \nabla_T)^2\psi||\nabla^3_{\widehat{R}}\psi| + |(\not{\mathfrak{d}}, \nabla_T)^2\psi||\nabla^2_{\widehat{R}}\psi|\Big) \\
& + \left(\sup_{\tau\in[\tau_1,\tau_2]} E_{deg}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq 1}\psi] + F_{\Sigma_*}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq 1}\psi] + \delta_{\mathcal{H}}F_{\mathcal{A}}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq 1}\psi]\right)^{\frac{1}{2}} \\
& \times\left(\sup_{\tau\in[\tau_1,\tau_2]} E_{deg}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq 2}\psi] + F_{\Sigma_*}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq 2}\psi] + \delta_{\mathcal{H}}F_{\mathcal{A}}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq 2}\psi]\right)^{\frac{1}{2}},
\end{aligned}$$

where the last term is used to control the boundary terms. Since we have

$$\begin{aligned}M_{deg}[(\nabla_T, \nabla_{\widehat{R}}, \not{\mathfrak{d}})^{\leq 2}\psi] \lesssim & M_{deg}[(\nabla_T, \not{\mathfrak{d}})^{\leq 2}\psi] + \int_{\mathcal{M}} \Big(|\nabla^3_{\widehat{R}}\psi|^2 + |\nabla^2_{\widehat{R}}\psi|^2\Big)\\ & + \int_{\mathcal{M}} \Big(|(\not{\mathfrak{d}}, \nabla_T)\nabla^2_{\widehat{R}}\psi|^2 + |(\not{\mathfrak{d}}, \nabla_T)\nabla_{\widehat{R}}\psi|^2\Big),\end{aligned}$$

we infer from the above

$$\begin{aligned}& M_{deg}[(\nabla_T, \nabla_{\widehat{R}}, \not{\mathfrak{d}})^{\leq 2}\psi]\\ \lesssim\ & M_{deg}[(\nabla_T, \not{\mathfrak{d}})^{\leq 2}\psi] + \int_{\mathcal{M}} |\mathfrak{d}^{\leq 1}N|^2 + \int_{\mathcal{M}} |\mathfrak{d}^{\leq 1}(\Gamma_g \cdot \psi)|^2\\ & + \left(\sup_{\tau\in[\tau_1,\tau_2]} E_{deg}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq 1}\psi] + F_{\Sigma_*}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq 1}\psi] + \delta_{\mathcal{H}} F_{\mathcal{A}}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq 1}\psi]\right)^{\frac{1}{2}}\\ & \times \left(\sup_{\tau\in[\tau_1,\tau_2]} E_{deg}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq 2}\psi] + F_{\Sigma_*}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq 2}\psi] + \delta_{\mathcal{H}} F_{\mathcal{A}}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq 2}\psi]\right)^{\frac{1}{2}}\end{aligned}$$

and hence

$$M_{deg}[(\nabla_T, \nabla_{\widehat{R}}, \not{\mathfrak{d}})^{\leq 2}\psi] \lesssim M_{deg}[(\nabla_T, \not{\mathfrak{d}})^{\leq 2}\psi] + \mathcal{I}_{RHS}$$

where we used the control of the energy in Proposition 9.5.2 and the definition of $\mathcal{I}_{RHS}$. Together with the above estimate of Step 2, we infer

$$\begin{aligned}& \delta^6_{red}\Big(E_{r\leq r_+(1+\delta_{red})}[(\not{\mathfrak{d}}, \nabla_{\mathbf{T}})^{\leq 2}\psi](\tau_2) + \mathrm{Mor}_{r\leq r_+(1+\delta_{red})}[(\not{\mathfrak{d}}, \nabla_{\mathbf{T}})^{\leq 2}\psi](\tau_1,\tau_2)\\ & + F_{\mathcal{A}}[(\not{\mathfrak{d}}, \nabla_{\mathbf{T}})^{\leq 2}\psi](\tau_1,\tau_2)\Big) + \mathrm{Mor}_{deg}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq 2}\psi](\tau_1,\tau_2)\\ \lesssim\ & \delta_{\mathcal{H}}\Big(E^2_{r\leq r_+}[\psi](\tau_2) + F^2_{\mathcal{A}}[\psi](\tau_1,\tau_2)\Big) + \mathcal{I}_{RHS}.\end{aligned}$$

Step 4. Next, we commute the wave equation (6.1.1) for ψ in $r \leq 4m$ by $(\nabla_3, \not{\mathfrak{d}}_2, \not{\mathcal{L}}_{\mathbf{T}})^{\leq 2}$, using the commutation formulas of Corollary 4.3.4, Lemma 4.5.4 and Lemma 4.7.13, as well as Lemma 9.4.5 for the commutations w.r.t to ∇_3. Together with Corollary 9.4.6, we infer

$$\begin{aligned}& c_0 E_{r\leq r_+(1+\delta_{red})}[(\nabla_3, \not{\mathfrak{d}}_2, \not{\mathcal{L}}_{\mathbf{T}})^{\leq 2}\psi](\tau_2) + c_0\mathrm{Mor}_{r\leq r_+(1+\delta_{red})}[(\nabla_3, \not{\mathfrak{d}}_2, \not{\mathcal{L}}_{\mathbf{T}})^{\leq 2}\psi](\tau_1,\tau_2)\\ & + c_0 F_{\mathcal{A}}[(\nabla_3, \not{\mathfrak{d}}_2, \not{\mathcal{L}}_{\mathbf{T}})^{\leq 2}\psi](\tau_1,\tau_2)\\ \leq & E^2_{r\leq r_+(1+2\delta_{red})}[\psi](\tau_1) + \delta^{-1}_{red}\mathrm{Mor}_{r_+(1+\delta_{red})\leq r\leq r_+(1+2\delta_{red})}[(\nabla_3, \not{\mathfrak{d}}, \not{\mathcal{L}}_{\mathbf{T}})^{\leq 2}\psi](\tau_1,\tau_2)\\ & + \mathcal{N}^2[\psi, N](\tau_1,\tau_2) + \left(\frac{|a|}{m} + \epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^2[\psi] + B^2_\delta[\psi](\tau_1,\tau_2) + F^2[\psi](\tau_1,\tau_2)\right).\end{aligned}$$

Using the comparison of $\not{\mathcal{L}}_{\mathbf{T}}$ and $\nabla_{\mathbf{T}}$ of Lemma 9.2.1, and the Hodge estimates of Proposition 9.3.2, and using

$$\begin{aligned}
&\delta_{red}^{-1}\text{Mor}_{r_+(1+\delta_{red})\leq r\leq r_+(1+2\delta_{red})}[(\nabla_3,\,\not{\mathfrak{d}},\not{\mathcal{L}}_{\mathbf{T}})^{\leq 2}\psi](\tau_1,\tau_2)\\
\lesssim\;&\delta_{red}^{-7}\text{Mor}_{r_+(1+\delta_{red})\leq r\leq r_+(1+2\delta_{red})}[(\not{\mathfrak{d}},\nabla_{\mathbf{T}},\nabla_{\widehat{R}})^{\leq 2}\psi](\tau_1,\tau_2),
\end{aligned}$$

we infer

$$\begin{aligned}
&c_0E_{r\leq r_+(1+\delta_{red})}[(\nabla_3,\,\not{\mathfrak{d}},\nabla_4)^{\leq 2}\psi](\tau_2)+c_0\text{Mor}_{r\leq r_+(1+\delta_{red})}[(\nabla_3,\,\not{\mathfrak{d}},\nabla_4)^{\leq 2}\psi](\tau_1,\tau_2)\\
&+c_0F_{\mathcal{A}}[(\nabla_3,\,\not{\mathfrak{d}},\nabla_4)^{\leq 2}\psi](\tau_1,\tau_2)\\
\leq&E^2_{r\leq r_+(1+2\delta_{red})}[\psi](\tau_1)\\
&+\delta_{red}^{-7}\text{Mor}_{r_+(1+\delta_{red})\leq r\leq r_+(1+2\delta_{red})}[(\not{\mathfrak{d}},\nabla_{\mathbf{T}},\nabla_{\widehat{R}})^{\leq 2}\psi](\tau_1,\tau_2)\\
&+\mathcal{N}^2[\psi,N](\tau_1,\tau_2)+\left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]}E^2[\psi]+B^2_\delta[\psi](\tau_1,\tau_2)+F^2[\psi](\tau_1,\tau_2)\right).
\end{aligned}$$

Together with the control of Step 3, we deduce

$$\begin{aligned}
&\delta_{red}^7\Big(E_{r\leq r_+(1+\delta_{red})}[\mathfrak{d}^{\leq 2}\psi](\tau_2)+\text{Mor}[(\not{\mathfrak{d}},\nabla_3,\nabla_4)^{\leq 2}\psi](\tau_1,\tau_2)+F_{\mathcal{A}}[\mathfrak{d}^{\leq 2}\psi](\tau_1,\tau_2)\Big)\\
\lesssim\;&\delta_{\mathcal{H}}\Big(E^2_{r\leq r_+}[\psi](\tau_2)+F^2_{\mathcal{A}}[\psi](\tau_1,\tau_2)\Big)+\mathcal{I}_{RHS}.
\end{aligned}$$

Choosing δ_{red} such that $\delta_{\mathcal{H}}\ll\delta_{red}^7$, we finally obtain

$$\delta_{red}^7\Big(E^2_{r\leq r_+(1+\delta_{red})}[\psi](\tau_2)+\text{Mor}[(\not{\mathfrak{d}},\nabla_3,\nabla_4)^{\leq 2}\psi](\tau_1,\tau_2)+F^2_{\mathcal{A}}[\psi](\tau_1,\tau_2)\Big)\lesssim\mathcal{I}_{RHS}.$$

In view of the definition of $\mathcal{I}_{RHS}$, this yields

$$\begin{aligned}
&\delta_{red}^7\Big(E^2_{r\leq r_+(1+\delta_{red})}[\psi](\tau_2)+\text{Mor}[(\not{\mathfrak{d}},\nabla_3,\nabla_4)^{\leq 2}\psi](\tau_1,\tau_2)+F^2_{\mathcal{A}}[\psi](\tau_1,\tau_2)\Big)\\
\lesssim\;&\delta_{\mathcal{H}}\left(\sup_{\tau\in[\tau_1,\tau_2]}E_{deg}[(\not{\mathfrak{d}},\nabla_T,\nabla_{\widehat{R}})^{\leq 2}\psi]+F_{\Sigma_*}[(\not{\mathfrak{d}},\nabla_T,\nabla_{\widehat{R}})^{\leq 2}\psi]\right)\\
&+\Bigg(E^1[\psi](\tau_1)+\mathcal{N}^1[\psi,N](\tau_1,\tau_2)\\
&+\left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]}E^2[\psi]+B^2_\delta[\psi](\tau_1,\tau_2)+F^2[\psi](\tau_1,\tau_2)\right)\Bigg)^{\frac{1}{2}}\\
&\times\left(\sup_{\tau\in[\tau_1,\tau_2]}E_{deg}[(\not{\mathfrak{d}},\nabla_T,\nabla_{\widehat{R}})^{\leq 2}\psi]+F_{\Sigma_*}[(\not{\mathfrak{d}},\nabla_T,\nabla_{\widehat{R}})^{\leq 2}\psi]\right)^{\frac{1}{2}}\\
&+\mathcal{N}^2[\psi,N](\tau_1,\tau_2)+\left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]}E^2[\psi]+B^2_\delta[\psi](\tau_1,\tau_2)+F^2[\psi](\tau_1,\tau_2)\right)\\
&+E^2[\psi](\tau_1)
\end{aligned}$$

as stated. This concludes the proof of Proposition 9.5.4. □

The following proposition provides the control of the energy from Morawetz.

Proposition 9.5.5. *The solution ψ of of the model RW equation* (6.1.1) *satisfies the following energy estimate*

$$\begin{aligned}
&\delta_{red}^8\Big(F_{\mathcal{A}}^2[\psi](\tau_1,\tau_2) + E[(\nabla_3,\nabla_4,\,\not{\mathfrak{d}})^{\leq 2}\psi](\tau_2) + F_{\Sigma_*}[(\nabla_3,\nabla_4,\,\not{\mathfrak{d}})^{\leq 2}\psi](\tau_1,\tau_2)\Big)\\
\lesssim\;& E^2[\psi](\tau_1) + Mor[(\nabla_3,\nabla_4,\,\not{\mathfrak{d}})^{\leq 2}\psi](\tau_1,\tau_2) + \mathcal{N}^2[\psi,N](\tau_1,\tau_2)\\
&+\left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]}E^2[\psi] + B_\delta^2[\psi](\tau_1,\tau_2) + F^2[\psi](\tau_1,\tau_2)\right). \qquad (9.5.5)
\end{aligned}$$

Proof. The proof proceeds in several steps.

Step 1. First, note from the control of Proposition 9.5.2 and Lemma 9.5.3 that we have in particular

$$\begin{aligned}
&E_{deg}[(\,\not{\mathfrak{d}},\nabla_T,\nabla_{\widehat{R}})^{\leq 1}\psi](\tau_2) + F_{\Sigma_*}[(\,\not{\mathfrak{d}},\nabla_T,\nabla_{\widehat{R}})^{\leq 1}\psi](\tau_1,\tau_2)\\
&+\sup_{\tau\in[\tau_1,\tau_2]}E_{deg}[(\nabla_T,\nabla_Z)^2\psi] + F_{\Sigma_*}[(\nabla_T,\nabla_Z)^2\psi](\tau_1,\tau_2)\\
\lesssim\;& \delta_{\mathcal{H}}\Big(E^2_{r\leq r_+}[\psi](\tau_2) + F^2_{\mathcal{A}}[\psi](\tau_1,\tau_2)\Big) + E^2[\psi](\tau_1) + \mathcal{N}^2[\psi,N](\tau_1,\tau_2)\\
&+\left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]}E^2[\psi] + B_\delta^2[\psi](\tau_1,\tau_2) + F^2[\psi](\tau_1,\tau_2)\right).
\end{aligned}$$

Step 2. Commuting the wave equation (6.1.1) satisfied by ψ with $\not{\mathfrak{d}}_2^{j_1}\not{\mathcal{L}}_{\mathbf{T}}^{j_2}$ with $j_1+j_2\leq 2$, and obtain, using the commutation formulas of using the commutation formulas of Corollary 4.3.4, Lemma 4.5.4 and Lemma 4.7.13

$$\dot{\square}_{k_{j_1}}(\,\not{\mathfrak{d}}_2^{j_1}\not{\mathcal{L}}_{\mathbf{T}}^{j_2}\psi) - V\,\not{\mathfrak{d}}_2^{j_1}\not{\mathcal{L}}_{\mathbf{T}}^{j_2}\psi \;=\; -\frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_T(\,\not{\mathfrak{d}}_2^{j_1}\not{\mathcal{L}}_{\mathbf{T}}^{j_2}\psi) + N_{\not{\mathfrak{d}}_2^{j_1}\not{\mathcal{L}}_{\mathbf{T}}^{j_2}},$$

where $k_0=k_2=2$ and $k_1=1$, and

$$N_{\not{\mathfrak{d}}_2^{j_1}\not{\mathcal{L}}_{\mathbf{T}}^{j_2}} := \mathfrak{d}^{\leq 2}N - \delta_{j_1 1}\frac{3}{r^2}\not{\mathfrak{d}}_2^{j_1}\not{\mathcal{L}}_{\mathbf{T}}^{j_2}\psi + O(ar^{-2})\mathfrak{d}^{\leq 3}\psi + \mathfrak{d}^{\leq 3}(\Gamma_g\cdot\psi).$$

Next, we rely on Proposition 9.2.14. We infer, for $r\geq r_1$ with $r_1=r_1(m)$ sufficiently large, together with the Hodge estimates of Proposition 9.3.2

$$\begin{aligned}
&E_{r\geq 2r_1}[(\nabla_T, \mathfrak{d}\!\!/)^{\leq 2}\psi](\tau_2) + F_{\Sigma_*}[(\nabla_T, \mathfrak{d}\!\!/)^{\leq 2}\psi](\tau_1,\tau_2)\\
\lesssim\ & E^2[\psi](\tau_1) + \frac{r_1}{m}\mathrm{Mor}_{r_1\leq r\leq 2r_1}[(\nabla_T, \mathfrak{d}\!\!/)^{\leq 2}\psi](\tau_1,\tau_2)\\
&+\mathcal{N}^2[\psi, N](\tau_1,\tau_2) + \left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^2[\psi] + B^2_\delta[\psi](\tau_1,\tau_2) + F^2[\psi](\tau_1,\tau_2)\right).
\end{aligned}$$

Also, using again the representation of the wave operator provided by (4.7.7), we have

$$\begin{aligned}
&E_{r\geq 2r_1}[\nabla^2_{\widehat{R}}\psi] + F_{\Sigma_*}[\nabla^2_{\widehat{R}}\psi]\\
\lesssim\ & E_{r\geq 2r_1}[(\nabla_T, \mathfrak{d}\!\!/)^{\leq 2}\psi](\tau_2) + F_{\Sigma_*}[(\nabla_T, \mathfrak{d}\!\!/)^{\leq 2}\psi](\tau_1,\tau_2) + \int_{\Sigma(\tau_2)}|\mathfrak{d}^{\leq 1}N|^2\\
&+\int_{\Sigma(\tau_2)}|\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\mathfrak{d}\psi)|^2 + \int_{\Sigma_*(\tau_1,\tau_2)}|\mathfrak{d}^{\leq 1}N|^2 + \int_{\Sigma_*(\tau_1,\tau_2)}|\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\mathfrak{d}\psi)|^2.
\end{aligned}$$

Since

$$\begin{aligned}
&E_{r\geq 2r_1}[(\nabla_T, \nabla_{\widehat{R}}, \mathfrak{d}\!\!/)^{\leq 2}\psi](\tau_2) + F_{\Sigma_*}[(\nabla_T, \nabla_{\widehat{R}}, \mathfrak{d}\!\!/)^{\leq 2}\psi](\tau_2)\\
\lesssim\ & E_{r\geq 2r_1}[(\nabla_T, \mathfrak{d}\!\!/)^{\leq 2}\psi](\tau_2) + F_{\Sigma_*}[(\nabla_T, \mathfrak{d}\!\!/)^{\leq 2}\psi](\tau_1,\tau_2) + E_{r\geq 2r_1}[\nabla^2_{\widehat{R}}\psi] + F_{\Sigma_*}[\nabla^2_{\widehat{R}}\psi],
\end{aligned}$$

we infer from the above

$$\begin{aligned}
&E_{r\geq 2r_1}[(\nabla_T, \mathfrak{d}\!\!/, \nabla_{\widehat{R}})^{\leq 2}\psi](\tau_2) + F_{\Sigma_*}[(\nabla_T, \mathfrak{d}\!\!/, \nabla_{\widehat{R}})^{\leq 2}\psi](\tau_1,\tau_2)\\
\lesssim\ & E^2[\psi](\tau_1) + \frac{r_1}{m}\mathrm{Mor}_{r_1\leq r\leq 2r_1}[(\nabla_T, \mathfrak{d}\!\!/)^{\leq 2}\psi](\tau_1,\tau_2)\\
&+\mathcal{N}^2[\psi, N](\tau_1,\tau_2) + \left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^2[\psi] + B^2_\delta[\psi](\tau_1,\tau_2) + F^2[\psi](\tau_1,\tau_2)\right).
\end{aligned}$$

Step 3. Next, we commute the wave equation (6.1.1) for ψ in $\leq 4m$ by $(\nabla_3, |q|\nabla, \mathcal{L}\!\!/_{\mathbf{T}})^{\leq 2}$, using the commutation formulas of Corollary 4.3.4, Lemma 4.5.4 and Lemma 4.7.13, as well as Lemma 9.4.5 for the commutations w.r.t to ∇_3. Together with Corollary 9.4.6, we infer, as in Step 4 of the proof of Proposition 9.5.4,

$$\begin{aligned}
&c_0 E_{r\leq r_+(1+\delta_{red})}[(\nabla_3, \mathfrak{d}\!\!/_2, \mathcal{L}\!\!/_{\mathbf{T}})^{\leq 2}\psi](\tau_2) + c_0\mathrm{Mor}_{r\leq r_+(1+\delta_{red})}[(\nabla_3, \mathfrak{d}\!\!/_2, \mathcal{L}\!\!/_{\mathbf{T}})^{\leq 2}\psi](\tau_1,\tau_2)\\
&+ c_0 F_{\mathcal{A}}[(\nabla_3, \mathfrak{d}\!\!/_2, \mathcal{L}\!\!/_{\mathbf{T}})^{\leq 2}\psi](\tau_1,\tau_2)\\
\leq\ & E^2_{r\leq r_+(1+2\delta_{red})}[\psi](\tau_1) + \delta_{red}^{-1}\mathrm{Mor}_{r_+(1+\delta_{red})\leq r\leq r_+(1+2\delta_{red})}[(\nabla_3, \mathfrak{d}\!\!/, \mathcal{L}\!\!/_{\mathbf{T}})^{\leq 2}\psi](\tau_1,\tau_2)\\
&+\mathcal{N}^2[\psi, N](\tau_1,\tau_2) + \left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^2[\psi] + B^2_\delta[\psi](\tau_1,\tau_2) + F^2[\psi](\tau_1,\tau_2)\right).
\end{aligned}$$

from which we deduce, dropping the Morawetz term on the LHS, and using the Hodge estimates of Proposition 9.3.2

$$
\begin{aligned}
& c_0 E_{r\leq r_+(1+\delta_{red})}[\mathfrak{d}^{\leq 2}\psi](\tau_2) + c_0 F_{\mathcal{A}}[\mathfrak{d}^{\leq 2}\psi](\tau_1,\tau_2)\\
\leq & E^2[\psi](\tau_1) + \delta_{red}^{-1}\mathrm{Mor}_{r_+(1+\delta_{red})\leq r\leq r_+(1+2\delta_{red})}[(\nabla_3,\, \not{\mathfrak{d}},\nabla_{\mathbf{T}})^{\leq 2}\psi](\tau_1,\tau_2)\\
& + \mathcal{N}^2[\psi,N](\tau_1,\tau_2) + \left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]}E^2[\psi] + B^2_\delta[\psi](\tau_1,\tau_2) + F^2[\psi](\tau_1,\tau_2)\right).
\end{aligned}
$$

Step 4. Using again the representation of the wave operator provided by (4.7.7), i.e.

$$
\begin{aligned}
|q|^2\dot{\square}_2\psi = & \frac{(r^2+a^2)^2}{\Delta}\big(-\nabla_{\widehat{T}}\nabla_{\widehat{T}}\psi + \nabla_{\widehat{R}}\nabla_{\widehat{R}}\psi\big) + 2r\nabla_{\widehat{R}}\psi\\
& + |q|^2\triangle_2\psi + |q|^2(\eta+\underline{\eta})\cdot\nabla\psi + r^2\Gamma_g\cdot\mathfrak{d}\psi,
\end{aligned}
$$

together with the wave equation (6.1.1) satisfied by ψ, we infer

$$
E_{deg}[U] \lesssim E_{deg}[(\nabla_T,\nabla_Z)^{\leq 2}\psi](\tau) + E_{deg}[(\nabla_T,\nabla,\nabla_{\widehat{R}})^{\leq 1}\psi](\tau) + \int_{\Sigma(\tau)}|\mathfrak{d}^{\leq 1}N|^2,
$$

where the constant in $\lesssim$ is independent of τ, and where the tensor U is given by

$$
U := \nabla_{\widehat{R}}\nabla_{\widehat{R}}\psi + \frac{|q|^2\Delta}{(r^2+a^2)^2}\triangle_2\psi.
$$

Next, we proceed as follows:

Step 4a. We consider a cut-off $\kappa_{\delta_{\mathcal{H}}}(r)$ such that $0\leq\kappa_{\delta_{\mathcal{H}}}\leq 1$, $\kappa_{\delta_{\mathcal{H}}}=1$ on $r\geq r_+$, and $\kappa_{\delta_{\mathcal{H}}}$ vanishes on $\mathcal{A}$. Also, for any integer $2\leq n\leq\tau_*-2$, we consider a cut-off $\kappa_n(\tau)$ such that $0\leq\kappa_n\leq 1$, $\kappa_n(\tau)=1$ on $\{n\leq\tau\leq n+1\}$, and κ_n vanishes on $\tau\leq n-1$ and $\tau\geq n+2$. We have, by integrating by parts, and using the support properties of $\kappa_n(\tau)$

and the above control of U,

$$\begin{aligned}
&\int_{\mathcal{M}} \kappa_{\delta_{\mathcal{H}}}(r)\kappa_n(\tau) \left(|\nabla_4 \nabla_{\widehat{R}}^2 \psi|^2 + \frac{|\Delta|}{r^2} |\nabla_4 \nabla \nabla_{\widehat{R}} \psi|^2 + \frac{|\Delta|^2}{r^4} |\nabla_4 \nabla^2 \psi|^2 \right) \\
&+ \int_{\mathcal{M}} \kappa_{\delta_{\mathcal{H}}}(r)\kappa_n(\tau) r^{-2} \left(|\nabla_{\widehat{T}} \nabla_{\widehat{R}}^2 \psi|^2 + \frac{|\Delta|}{r^2} |\nabla_{\widehat{T}} \nabla \nabla_{\widehat{R}} \psi|^2 + \frac{|\Delta|^2}{r^4} |\nabla_{\widehat{T}} \nabla^2 \psi|^2 \right) \\
&+ \int_{\mathcal{M}} \kappa_{\delta_{\mathcal{H}}}(r)\kappa_n(\tau) \left(|\nabla \nabla_{\widehat{R}}^2 \psi|^2 + \frac{|\Delta|}{r^2} |\nabla \nabla_{\widehat{R}} \nabla \psi|^2 + \frac{|\Delta|^2}{r^4} |\nabla^3 \psi|^2 \right) \\
\lesssim\ & \sup_{\tau \in [n-1, n+2]} E_{deg}[U] \\
&+ \left(\sup_{\tau \in [n-1, n+2]} E_{deg}[(\nabla_T, \nabla, \nabla_{\widehat{R}})^{\leq 1} \psi](\tau) + F_{\Sigma_*}[(\nabla_T, \nabla_{\widehat{R}}, \nabla)^{\leq 1} \psi](n-1, n+2) \right)^{\frac{1}{2}} \\
&\times \left(\sup_{\tau \in [n-1, n+2]} E_{deg}[(\nabla_T, \nabla, \nabla_{\widehat{R}})^{\leq 2} \psi](\tau) + F_{\Sigma_*}[(\nabla_T, \nabla_{\widehat{R}}, \nabla)^{\leq 2} \psi](n-1, n+2) \right)^{\frac{1}{2}}.
\end{aligned}$$

Together with the above estimates, we deduce

$$\begin{aligned}
&\int_{\mathcal{M}} \kappa_{\delta_{\mathcal{H}}}(r)\kappa_n(\tau) \left(|\nabla_4 \nabla_{\widehat{R}}^2 \psi|^2 + \frac{|\Delta|}{r^2} |\nabla_4 \nabla \nabla_{\widehat{R}} \psi|^2 + \frac{|\Delta|^2}{r^4} |\nabla_4 \nabla^2 \psi|^2 \right) \\
&+ \int_{\mathcal{M}} \kappa_{\delta_{\mathcal{H}}}(r)\kappa_n(\tau) r^{-2} \left(|\nabla_{\widehat{T}} \nabla_{\widehat{R}}^2 \psi|^2 + \frac{|\Delta|}{r^2} |\nabla_{\widehat{T}} \nabla \nabla_{\widehat{R}} \psi|^2 + \frac{|\Delta|^2}{r^4} |\nabla_{\widehat{T}} \nabla^2 \psi|^2 \right) \\
&+ \int_{\mathcal{M}} \kappa_{\delta_{\mathcal{H}}}(r)\kappa_n(\tau) \left(|\nabla \nabla_{\widehat{R}}^2 \psi|^2 + \frac{|\Delta|}{r^2} |\nabla \nabla_{\widehat{R}} \nabla \psi|^2 + \frac{|\Delta|^2}{r^4} |\nabla^3 \psi|^2 \right) \\
\lesssim\ & \sup_{\tau \in [n-1, n+2]} E_{deg}[(\nabla_T, \nabla_Z)^{\leq 2} \psi](\tau) + \sup_{\tau \in [n-1, n+2]} E_{deg}[(\nabla_T, \nabla, \nabla_{\widehat{R}})^{\leq 1} \psi](\tau) \\
&+ \left(\sup_{\tau \in [n-1, n+2]} E_{deg}[(\nabla_T, \nabla, \nabla_{\widehat{R}})^{\leq 1} \psi](\tau) + F_{\Sigma_*}[(\nabla_T, \nabla_{\widehat{R}}, \nabla)^{\leq 1} \psi](n-1, n+2) \right)^{\frac{1}{2}} \\
&\times \left(\sup_{\tau \in [n-1, n+2]} E_{deg}[(\nabla_T, \nabla, \nabla_{\widehat{R}})^{\leq 2} \psi](\tau) + F_{\Sigma_*}[(\nabla_T, \nabla_{\widehat{R}}, \nabla)^{\leq 2} \psi](n-1, n+2) \right)^{\frac{1}{2}} \\
&+ \sup_{\tau \in [n-1, n+2]} \int_{\Sigma(\tau)} |\mathfrak{d}^{\leq 1} N|^2.
\end{aligned}$$

Step 4b. In view of the properties of the cut-offs, the above estimates imply

$$\begin{aligned}
&\int_{\mathcal{M}\cap\{r\geq r_+\}\cap\{n\leq\tau\leq n+1\}}\Bigg[|\nabla_4\nabla_{\widehat{R}}^2\psi|^2+\frac{|\Delta|}{r^2}|\nabla_4\nabla\nabla_{\widehat{R}}\psi|^2+\frac{|\Delta|^2}{r^4}|\nabla_4\nabla^2\psi|^2\\
&+r^{-2}\left(|\nabla_{\widehat{T}}\nabla_{\widehat{R}}^2\psi|^2+\frac{|\Delta|}{r^2}|\nabla_{\widehat{T}}\nabla\nabla_{\widehat{R}}\psi|^2+\frac{|\Delta|^2}{r^4}|\nabla_{\widehat{T}}\nabla^2\psi|^2\right)+|\nabla\nabla_{\widehat{R}}^2\psi|^2\\
&+\frac{|\Delta|}{r^2}|\nabla\nabla_{\widehat{R}}\nabla\psi|^2+\frac{|\Delta|^2}{r^4}|\nabla^3\psi|^2\Bigg]\\
\lesssim\;& \sup_{\tau\in[n-1,n+2]}E_{deg}[(\nabla_T,\nabla_Z)^{\leq 2}\psi](\tau)+\sup_{\tau\in[n-1,n+2]}E_{deg}[(\nabla_T,\nabla,\nabla_{\widehat{R}})^{\leq 1}\psi](\tau)\\
&+\left(\sup_{\tau\in[n-1,n+2]}E_{deg}[(\nabla_T,\nabla,\nabla_{\widehat{R}})^{\leq 1}\psi](\tau)+F_{\Sigma_*}[(\nabla_T,\nabla_{\widehat{R}},\nabla)^{\leq 1}\psi](n-1,n+2)\right)^{\frac{1}{2}}\\
&\times\left(\sup_{\tau\in[n-1,n+2]}E_{deg}[(\nabla_T,\nabla,\nabla_{\widehat{R}})^{\leq 2}\psi](\tau)+F_{\Sigma_*}[(\nabla_T,\nabla_{\widehat{R}},\nabla)^{\leq 2}\psi](n-1,n+2)\right)^{\frac{1}{2}}\\
&+\sup_{\tau\in[n-1,n+2]}\int_{\Sigma(\tau)}|\mathfrak{d}^{\leq 1}N|^2.
\end{aligned}$$

By the mean value theorem, we infer the existence of $\tau_{(n)}\in[n,n+1]$ such that

$$\begin{aligned}
&\int_{\Sigma(\tau_{(n)})\cap\{r\geq r_+\}}\Bigg[|\nabla_4\nabla_{\widehat{R}}^2\psi|^2+\frac{|\Delta|}{r^2}|\nabla_4\nabla\nabla_{\widehat{R}}\psi|^2+\frac{|\Delta|^2}{r^4}|\nabla_4\nabla^2\psi|^2\\
&+r^{-2}\left(|\nabla_{\widehat{T}}\nabla_{\widehat{R}}^2\psi|^2+\frac{|\Delta|}{r^2}|\nabla_{\widehat{T}}\nabla\nabla_{\widehat{R}}\psi|^2+\frac{|\Delta|^2}{r^4}|\nabla_{\widehat{T}}\nabla^2\psi|^2\right)+|\nabla\nabla_{\widehat{R}}^2\psi|^2\\
&+\frac{|\Delta|}{r^2}|\nabla\nabla_{\widehat{R}}\nabla\psi|^2+\frac{|\Delta|^2}{r^4}|\nabla^3\psi|^2\Bigg]\\
\lesssim\;& \sup_{\tau\in[n-1,n+2]}E_{deg}[(\nabla_T,\nabla_Z)^{\leq 2}\psi](\tau)+\sup_{\tau\in[n-1,n+2]}E_{deg}[(\nabla_T,\nabla,\nabla_{\widehat{R}})^{\leq 1}\psi](\tau)\\
&+\left(\sup_{\tau\in[n-1,n+2]}E_{deg}[(\nabla_T,\nabla,\nabla_{\widehat{R}})^{\leq 1}\psi](\tau)+F_{\Sigma_*}[(\nabla_T,\nabla_{\widehat{R}},\nabla)^{\leq 1}\psi](n-1,n+2)\right)^{\frac{1}{2}}\\
&\times\left(\sup_{\tau\in[n-1,n+2]}E_{deg}[(\nabla_T,\nabla,\nabla_{\widehat{R}})^{\leq 2}\psi](\tau)+F_{\Sigma_*}[(\nabla_T,\nabla_{\widehat{R}},\nabla)^{\leq 2}\psi](n-1,n+2)\right)^{\frac{1}{2}}\\
&+\sup_{\tau\in[n-1,n+2]}\int_{\Sigma(\tau)}|\mathfrak{d}^{\leq 1}N|^2.
\end{aligned}$$

Step 4c. Together with the above control of $E_{deg}[(\nabla_T,\nabla_Z)^{\leq 2}\psi](\tau)$, the above control of $E_{r\geq 2r_1}[(\nabla_T, \mathfrak{d}\!\!/, \nabla_{\widehat{R}})^{\leq 2}\psi](\tau_2)$ and $F_{\Sigma_*}[(\nabla_T, \mathfrak{d}\!\!/, \nabla_{\widehat{R}})^{\leq 2}\psi](\tau_1,\tau_2)$, and the above control of $E_{r\leq r_+(1+\delta_{red})}[\mathfrak{d}^{\leq 2}\psi](\tau_2)$ and $F_{\mathcal{A}}[\mathfrak{d}^{\leq 2}\psi](\tau_1,\tau_2)$, and fixing the value of $r_1 = r_1(m)$, we infer, the following non sharp estimate, for any n such that $\tau_{(n)} \geq \tau_1$,

$$
\begin{aligned}
&\delta_{red}^4\Big(F_{\mathcal{A}}[\mathfrak{d}^{\leq 2}\psi](\tau_1,\tau_{(n)}) + E[(\nabla_3,\nabla_4,\mathfrak{d}\!\!/)^{\leq 2}\psi](\tau_{(n)}) + F_{\Sigma_*}[(\nabla_3,\nabla_4,\mathfrak{d}\!\!/)^{\leq 2}\psi](\tau_1,\tau_{(n)})\Big)\\
\lesssim\ & \sup_{\tau\in[n-1,n+2]} E_{deg}[(\nabla_T,\nabla,\nabla_{\widehat{R}})^{\leq 1}\psi](\tau) + \delta_{\mathcal{H}}\sup_{\tau\in[\tau_1,n+2]} E_{r\leq r_+}[(\nabla_T,\nabla_Z)^{\leq 2}\psi]\\
&+\delta_{\mathcal{H}}F_{\mathcal{A}}[(\nabla_T,\nabla_Z)^{\leq 2}\psi](\tau_1,n+2)\\
&+\left(\sup_{\tau\in[n-1,n+2]} E_{deg}[(\nabla_T,\nabla,\nabla_{\widehat{R}})^{\leq 1}\psi](\tau) + F_{\Sigma_*}[(\nabla_T,\nabla_{\widehat{R}},\nabla)^{\leq 1}\psi](n-1,n+2)\right)^{\frac{1}{2}}\\
&\times\left(\sup_{\tau\in[n-1,n+2]} E_{deg}[(\nabla_T,\nabla,\nabla_{\widehat{R}})^{\leq 2}\psi](\tau) + F_{\Sigma_*}[(\nabla_T,\nabla_{\widehat{R}},\nabla)^{\leq 2}\psi](n-1,n+2)\right)^{\frac{1}{2}}\\
&+E^2[\psi](\tau_1) + \mathrm{Mor}[(\nabla_3,\nabla_4,\mathfrak{d}\!\!/)^{\leq 2}\psi](\tau_1,n+2)\\
&+\mathcal{N}^2[\psi,N](\tau_1,n+2) + \left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,n+2]} E^2[\psi] + B^2_\delta[\psi](\tau_1,n+2) + F^2[\psi](\tau_1,n+2)\right).
\end{aligned}
$$

Step 4d. Let $\tau_2 \geq \tau_1$. By local energy estimates, it suffices to consider the case $\tau_2 \geq \tau_1+5$. We then choose n such that $\tau_1 \leq n-1 \leq n+2 \leq \tau_2 < n+3$. In particular, we have $\tau_{(n)}+1 \leq \tau_2 \leq \tau_{(n)}+3$, and hence, using local energy estimates between $\tau_{(n)}$ and τ_2, we infer from the previous estimate, choosing also $\delta_{\mathcal{H}} \ll \delta_{red}^4$ to absorb some terms on the RHS from the LHS,

$$
\begin{aligned}
&\delta_{red}^4\Big(F_{\mathcal{A}}[\mathfrak{d}^{\leq 2}\psi](\tau_1,\tau_2) + E[(\nabla_3,\nabla_4,\mathfrak{d}\!\!/)^{\leq 2}\psi](\tau_2) + F_{\Sigma_*}[(\nabla_3,\nabla_4,\mathfrak{d}\!\!/)^{\leq 2}\psi](\tau_1,\tau_2)\Big)\\
\lesssim\ & \sup_{\tau\in[\tau_1,\tau_2]} E_{deg}[(\nabla_T,\nabla,\nabla_{\widehat{R}})^{\leq 1}\psi](\tau)\\
&+\left(\sup_{\tau\in[\tau_1,\tau_2]} E_{deg}[(\nabla_T,\nabla,\nabla_{\widehat{R}})^{\leq 1}\psi](\tau) + F_{\Sigma_*}[(\nabla_T,\nabla_{\widehat{R}},\nabla)^{\leq 1}\psi](\tau_1,\tau_2)\right)^{\frac{1}{2}}\\
&\times\left(\sup_{\tau\in[\tau_1,\tau_2]} E_{deg}[(\nabla_T,\nabla,\nabla_{\widehat{R}})^{\leq 2}\psi](\tau) + F_{\Sigma_*}[(\nabla_T,\nabla_{\widehat{R}},\nabla)^{\leq 2}\psi](\tau_1,\tau_2)\right)^{\frac{1}{2}}\\
&+E^2[\psi](\tau_1) + \mathrm{Mor}[(\nabla_3,\nabla_4,\mathfrak{d}\!\!/)^{\leq 2}\psi](\tau_1,\tau_2)\\
&+\mathcal{N}^2[\psi,N](\tau_1,\tau_2) + \left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^2[\psi] + B^2_\delta[\psi](\tau_1,\tau_2) + F^2[\psi](\tau_1,\tau_2)\right)
\end{aligned}
$$

and thus

$$
\begin{aligned}
&\delta_{red}^8\Big(F_{\mathcal{A}}[\mathfrak{d}^{\leq 2}\psi](\tau_1,\tau_2)+E[(\nabla_3,\nabla_4,\not\mathfrak{d})^{\leq 2}\psi](\tau_2)+F_{\Sigma_*}[(\nabla_3,\nabla_4,\not\mathfrak{d})^{\leq 2}\psi](\tau_1,\tau_2)\Big)\\
\lesssim\; &\sup_{\tau\in[\tau_1,\tau_2]}E_{deg}[(\nabla_T,\nabla,\nabla_{\widehat{R}})^{\leq 1}\psi](\tau)+F_{\Sigma_*}[(\nabla_T,\nabla_{\widehat{R}},\nabla)^{\leq 1}\psi](\tau_1,\tau_2)\\
&+E^2[\psi](\tau_1)+\mathrm{Mor}[(\nabla_3,\nabla_4,\not\mathfrak{d})^{\leq 2}\psi](\tau_1,\tau_2)\\
&+\mathcal{N}^2[\psi,N](\tau_1,\tau_2)+\left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]}E^2[\psi]+B^2_\delta[\psi](\tau_1,\tau_2)+F^2[\psi](\tau_1,\tau_2)\right).
\end{aligned}
$$

Step 5. Plugging the control of $E_{deg}[(\nabla_T,\nabla,\nabla_{\widehat{R}})^{\leq 1}\psi](\tau_2)$ and $F_{\Sigma_*}[(\not\mathfrak{d},\nabla_T,\nabla_{\widehat{R}})^{\leq 1}\psi](\tau_1,\tau_2)$ derived in Proposition 9.5.2, we infer, choosing also $\delta_{\mathcal{H}}$ such that $\delta_{\mathcal{H}}\ll\delta_{red}^8$ to absorb terms from the RHS by the LHS,

$$
\begin{aligned}
&\delta_{red}^8\Big(F_{\mathcal{A}}[\mathfrak{d}^{\leq 2}\psi](\tau_1,\tau_2)+E[(\nabla_3,\nabla_4,\not\mathfrak{d})^{\leq 2}\psi](\tau_2)+F_{\Sigma_*}[(\nabla_3,\nabla_4,\not\mathfrak{d})^{\leq 2}\psi](\tau_1,\tau_2)\Big)\\
\lesssim\; &E^2[\psi](\tau_1)+\mathrm{Mor}[(\nabla_3,\nabla_4,\not\mathfrak{d})^{\leq 2}\psi](\tau_1,\tau_2)\\
&+\mathcal{N}^2[\psi,N](\tau_1,\tau_2)+\left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]}E^2[\psi]+B^2_\delta[\psi](\tau_1,\tau_2)+F^2[\psi](\tau_1,\tau_2)\right)
\end{aligned}
$$

as stated. This concludes the proof of Proposition 9.5.5. □

We are now ready to prove Theorem 6.3.1 in the particular case $s=2$.

Proof of Theorem 6.3.1 in the case $s=2$. Plugging the energy estimate of Proposition

9.5.5 in the RHS of the Morawetz estimate of Proposition 9.5.4, we infer

$$
\begin{aligned}
&\delta_{red}^7\Big(E^2_{r\leq r_+(1+\delta_{red})}[\psi](\tau_2)+\mathrm{Mor}[(\not{\mathfrak{d}},\nabla_3,\nabla_4)^{\leq 2}\psi](\tau_1,\tau_2)+F^2_{\mathcal{A}}[\psi](\tau_1,\tau_2)\Big)\\
\lesssim\;& \delta_{red}^{-8}\delta_{\mathcal{H}}\Bigg(E^2[\psi](\tau_1)+\mathrm{Mor}[(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{\leq 2}\psi](\tau_1,\tau_2)\\
&+\mathcal{N}^2[\psi,N](\tau_1,\tau_2)+\left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]}E^2[\psi]+B^2_\delta[\psi](\tau_1,\tau_2)+F^2[\psi](\tau_1,\tau_2)\right)\Bigg)\\
&+\delta_{red}^{-4}\Bigg(E^1[\psi](\tau_1)+\mathcal{N}^1[\psi,N](\tau_1,\tau_2)\\
&+\left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]}E^2[\psi]+B^2_\delta[\psi](\tau_1,\tau_2)+F^2[\psi](\tau_1,\tau_2)\right)\Bigg)^{\frac{1}{2}}\\
&\times\Bigg(E^2[\psi](\tau_1)+\mathrm{Mor}[(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{\leq 2}\psi](\tau_1,\tau_2)\\
&+\mathcal{N}^2[\psi,N](\tau_1,\tau_2)+\left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]}E^2[\psi]+B^2_\delta[\psi](\tau_1,\tau_2)+F^2[\psi](\tau_1,\tau_2)\right)\Bigg)^{\frac{1}{2}}\\
&+\mathcal{N}^2[\psi,N](\tau_1,\tau_2)+\left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]}E^2[\psi]+B^2_\delta[\psi](\tau_1,\tau_2)+F^2[\psi](\tau_1,\tau_2)\right)\\
&+E^2[\psi](\tau_1).
\end{aligned}
$$

Choosing $\delta_{\mathcal{H}}$ such that $\delta_{\mathcal{H}}\ll\delta_{red}^{15}$, we may absorb the Morawetz terms on the LHS from the one on the RHS. We deduce

$$
\begin{aligned}
&\delta_{red}^{22}\Big(E_{r\leq r_+(1+\delta_{red})}[\mathfrak{d}^{\leq 2}\psi](\tau_2)+\mathrm{Mor}[(\not{\mathfrak{d}},\nabla_3,\nabla_4)^{\leq 2}\psi](\tau_1,\tau_2)+F_{\mathcal{A}}[\mathfrak{d}^{\leq 2}\psi](\tau_1,\tau_2)\Big)\\
\lesssim\;& E^2[\psi](\tau_1)+\mathcal{N}^2[\psi,N](\tau_1,\tau_2)\\
&+\left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]}E^2[\psi]+B^2_\delta[\psi](\tau_1,\tau_2)+F^2[\psi](\tau_1,\tau_2)\right).
\end{aligned}
$$

Together with the energy estimate of Proposition 9.5.5, and fixing the value of δ_{red}, we obtain

$$
\begin{aligned}
&E[(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{\leq 2}\psi](\tau_2)+\mathrm{Mor}[(\not{\mathfrak{d}},\nabla_3,\nabla_4)^{\leq 2}\psi](\tau_1,\tau_2)+F[(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{\leq 2}\psi](\tau_1,\tau_2)\\
\lesssim\;& E^2[\psi](\tau_1)+\mathcal{N}^2[\psi,N](\tau_1,\tau_2)\\
&+\left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]}E^2[\psi]+B^2_\delta[\psi](\tau_1,\tau_2)+F^2[\psi](\tau_1,\tau_2)\right)
\end{aligned}
\tag{9.5.6}
$$

which concludes the proof of Theorem 6.3.1 in the particular case $s=2$. $\square$

9.5.2 Proof of Theorem 6.3.1

We are in position to concludes the proof of Theorem 6.3.1. We consider the following iteration assumption for $2 \leq j \leq s-1$:

$$\begin{aligned}
& E[(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{\leq j}\psi](\tau_2) + \mathrm{Mor}[(\not{\mathfrak{d}}, \nabla_3, \nabla_4)^{\leq j}\psi](\tau_1,\tau_2) + F[(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{\leq j}\psi](\tau_1,\tau_2) \\
\lesssim\ & E^j[\psi](\tau_1) + \mathcal{N}^j[\psi, N](\tau_1,\tau_2) \\
& + \left(\frac{|a|}{m} + \epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^j[\psi] + B^j_\delta[\psi](\tau_1,\tau_2) + F^j[\psi](\tau_1,\tau_2)\right). \qquad (9.5.7)
\end{aligned}$$

Note that (9.5.7) holds for $j=2$ in view of (9.5.6). We may thus assume that (9.5.7) holds for some $2 \leq j \leq s-1$, and our goal is to prove that this estimate also holds for j replaced by $j+1$.

We start with the following Morawetz estimate.

Proposition 9.5.6. *Let $2 \leq j \leq s-1$. Assume that the solution ψ of of the model RW equation* (6.1.1) *satisfies the iteration assumption* (9.5.7)*. Then, the following Morawetz estimate holds:*

$$\begin{aligned}
& E_{r\leq r_+(1+\delta_{red})}[\mathfrak{d}^{\leq j+1}\psi](\tau_2) + Mor[(\not{\mathfrak{d}}, \nabla_3, \nabla_4)^{\leq j+1}\psi](\tau_1,\tau_2) + F_{\mathcal{A}}[\mathfrak{d}^{\leq j+1}\psi](\tau_1,\tau_2) \\
\lesssim\ & \Bigg(E^j[\psi](\tau_1) + \mathcal{N}^j[\psi, N](\tau_1,\tau_2) \\
& + \left(\frac{|a|}{m} + \epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^j[\psi] + B^j_\delta[\psi](\tau_1,\tau_2) + F^j[\psi](\tau_1,\tau_2)\right)\Bigg)^{\frac{1}{2}} \\
& \times\Bigg(\sup_{\tau\in[\tau_1,\tau_2]} E_{deg}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq j+1}\psi] \\
& + F_{\Sigma_*}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq j+1}\psi](\tau_1,\tau_2) + \delta_{\mathcal{H}} F_{\mathcal{A}}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq j+1}\psi](\tau_1,\tau_2)\Bigg)^{\frac{1}{2}} \\
& + E^{j+1}[\psi](\tau_1) + \mathcal{N}^{j+1}[\psi, N](\tau_1,\tau_2) \\
& + \left(\frac{|a|}{m} + \epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^{j+1}[\psi] + B^{j+1}_\delta[\psi](\tau_1,\tau_2) + F^{j+1}[\psi](\tau_1,\tau_2)\right). \qquad (9.5.8)
\end{aligned}$$

Proof. We proceed in several steps.

Step 1. This step is analogous to Step 2 of the proof of Proposition 9.5.2, so we only sketch it. We commute the wave equation (6.1.1) satisfied by ψ with $(\dot{\mathcal{L}}_{\mathbf{T}}, \dot{\mathcal{L}}_{\mathbf{Z}})$ and apply the iteration assumption (9.5.7) to the commuted wave equation. This yields, comparing also (∇_T, ∇_Z) and $(\not{\mathcal{L}}_{\mathbf{T}}, \not{\mathcal{L}}_{\mathbf{Z}})$, and using the iteration assumption (9.5.7) to absorb lower order terms,

$$\begin{aligned}
& E[(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{\leq j}(\nabla_T, \nabla_Z)\psi](\tau_2) + \text{Mor}[(\not{\mathfrak{d}}, \nabla_3, \nabla_4)^{\leq j}(\nabla_T, \nabla_Z)\psi](\tau_1, \tau_2) \\
& + F[(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{\leq j}(\nabla_T, \nabla_Z)\psi](\tau_1, \tau_2) \\
\lesssim \; & E^{j+1}[\psi](\tau_1) + \mathcal{N}^{j+1}[\psi, N](\tau_1, \tau_2) \\
& + \left(\frac{|a|}{m} + \epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^{j+1}[\psi] + B_\delta^{j+1}[\psi](\tau_1, \tau_2) + F^{j+1}[\psi](\tau_1, \tau_2)\right).
\end{aligned}$$

Step 2. Using again the representation of the wave operator provided by (4.7.7), i.e.

$$\begin{aligned}
|q|^2\dot{\square}_2\psi = \; & \frac{(r^2+a^2)^2}{\Delta}\big(- \nabla_{\widehat{T}}\nabla_{\widehat{T}}\psi + \nabla_{\widehat{R}}\nabla_{\widehat{R}}\psi\big) + 2r\nabla_{\widehat{R}}\psi \\
& + |q|^2\triangle_2\psi + |q|^2(\eta + \underline{\eta})\cdot\nabla\psi + r^2\Gamma_g\cdot\mathfrak{d}\psi,
\end{aligned}$$

together with the wave equation (6.1.1) satisfied by ψ, we infer

$$\begin{aligned}
\text{Mor}[(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{\leq j-1}U] \; \lesssim \; & \text{Mor}[(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{\leq j-1}(\nabla_T, \nabla_Z)^{\leq 2}\psi](\tau) \\
& + \text{Mor}[(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{\leq j}\psi](\tau) + \int_{\Sigma(\tau)} |\mathfrak{d}^{\leq j-1}N|^2,
\end{aligned}$$

where the constant in $\lesssim$ is independent of τ, and where the tensor U is given by

$$U := \nabla_{\widehat{R}}\nabla_{\widehat{R}}\psi + \frac{|q|^2\Delta}{(r^2+a^2)^2}\triangle_2\psi.$$

Together with Step 1 and the iteration assumption (9.5.7), we deduce

$$\begin{aligned}
& \text{Mor}[(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{\leq j-1}U] \\
\lesssim \; & E^{j+1}[\psi](\tau_1) + \mathcal{N}^{j+1}[\psi, N](\tau_1, \tau_2) \\
& + \left(\frac{|a|}{m} + \epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^{j+1}[\psi] + B_\delta^{j+1}[\psi](\tau_1, \tau_2) + F^{j+1}[\psi](\tau_1, \tau_2)\right).
\end{aligned}$$

Next, using integration by parts and the iteration assumption (9.5.7) to absorb lower order terms, we have

$$\begin{aligned}
&\int_{\mathcal{M}(\tau_1,\tau_2)} r^{-2}\Big(|\nabla_{\widehat{R}}^3(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{j-1}\psi|^2 + \frac{|\Delta|}{r^2}|\nabla_{\widehat{R}}^2\nabla(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{j-1}\psi|^2\\
&+\frac{\Delta^2}{r^4}|\Delta_2\nabla_{\widehat{R}}(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{j-1}\psi|^2\Big)\\
\lesssim\ & \mathrm{Mor}[(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{\leq j-1}U]\\
&+\Bigg(\sup_{\tau\in[\tau_1,\tau_2]} E_{deg}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq j}\psi] + F_{\Sigma_*}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq j}\psi](\tau_1, \tau_2)\\
&+\delta_{\mathcal{H}}F_{\mathcal{A}}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq j}\psi](\tau_1, \tau_2)\Bigg)^{\frac{1}{2}}\Bigg(\sup_{\tau\in[\tau_1,\tau_2]} E_{deg}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq j+1}\psi]\\
&+F_{\Sigma_*}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq j+1}\psi](\tau_1, \tau_2) + \delta_{\mathcal{H}}F_{\mathcal{A}}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq j+1}\psi](\tau_1, \tau_2)\Bigg)^{\frac{1}{2}}.
\end{aligned}$$

Using again the iteration assumption (9.5.7), we infer

$$\begin{aligned}
&\int_{\mathcal{M}(\tau_1,\tau_2)} r^{-2}\Big(|\nabla_{\widehat{R}}^3(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{j-1}\psi|^2 + \frac{|\Delta|}{r^2}|\nabla_{\widehat{R}}^2\nabla(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{j-1}\psi|^2\\
&+\frac{\Delta^2}{r^4}|\Delta_2\nabla_{\widehat{R}}(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{j-1}\psi|^2\Big)\\
\lesssim\ & \mathrm{Mor}[(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{\leq j-1}U]\\
&+\Bigg(E^j[\psi](\tau_1) + \mathcal{N}^j[\psi, N](\tau_1, \tau_2)\\
&+\left(\frac{|a|}{m} + \epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^j[\psi] + B_\delta^j[\psi](\tau_1, \tau_2) + F^j[\psi](\tau_1, \tau_2)\right)\Bigg)^{\frac{1}{2}}\\
&\times\Bigg(\sup_{\tau\in[\tau_1,\tau_2]} E_{deg}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq j+1}\psi]\\
&+F_{\Sigma_*}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq j+1}\psi](\tau_1, \tau_2) + \delta_{\mathcal{H}}F_{\mathcal{A}}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq j+1}\psi](\tau_1, \tau_2)\Bigg)^{\frac{1}{2}}.
\end{aligned}$$

Together with the above control of $\mathrm{Mor}[(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{\leq j-1}U]$, we infer

$$\begin{aligned}
\int_{\mathcal{M}(\tau_1,\tau_2)} r^{-2}\Big(|\nabla_{\widehat{R}}^3(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{j-1}\psi|^2 + \frac{|\Delta|}{r^2}|\nabla_{\widehat{R}}^2\nabla(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{j-1}\psi|^2&\\
+\frac{\Delta^2}{r^4}|\Delta_2\nabla_{\widehat{R}}(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{j-1}\psi|^2\Big)\ \lesssim\ \mathcal{J}_{RHS}&
\end{aligned}$$

where we have introduced the following notation

$$\begin{aligned}\mathcal{J}_{RHS} &:= \Bigg(E^j[\psi](\tau_1) + \mathcal{N}^j[\psi, N](\tau_1, \tau_2)\\ &+ \left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^j[\psi] + B^j_\delta[\psi](\tau_1,\tau_2) + F^j[\psi](\tau_1,\tau_2)\right)\Bigg)^{\frac{1}{2}}\\ &\times\Bigg(\sup_{\tau\in[\tau_1,\tau_2]} E_{deg}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq j+1}\psi]\\ &+F_{\Sigma_*}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq j+1}\psi](\tau_1,\tau_2) + \delta_{\mathcal{H}} F_{\mathcal{A}}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq j+1}\psi](\tau_1,\tau_2)\Bigg)^{\frac{1}{2}}\\ &+E^{j+1}[\psi](\tau_1) + \mathcal{N}^{j+1}[\psi, N](\tau_1,\tau_2)\\ &+\left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^{j+1}[\psi] + B^{j+1}_\delta[\psi](\tau_1,\tau_2) + F^{j+1}[\psi](\tau_1,\tau_2)\right).\end{aligned}$$

Using the Hodge estimates of Proposition 9.3.2, as well as the estimate of Step 1 and the iteration assumption (9.5.7), we deduce

$$\begin{aligned}\int_{\mathcal{M}(\tau_1,\tau_2)} r^{-2}\Big(|\nabla^3_{\widehat{R}}(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{j-1}\psi|^2 + \frac{|\Delta|}{r^2}|\nabla^2_{\widehat{R}}\nabla(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{j-1}\psi|^2\\ +\frac{\Delta^2}{r^4}|\nabla_{\widehat{R}}\nabla^2(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{j-1}\psi|^2\Big) \lesssim \mathcal{J}_{RHS}.\end{aligned}$$

Together with Step 1, we finally obtain

$$\int_{\mathcal{M}(\tau_1,\tau_2)} r^{-6}\frac{\Delta^2}{r^4}|\nabla_{\widehat{R}}(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{j+1}\psi|^2 \lesssim \mathcal{J}_{RHS}.$$

Step 3. Using again the representation of the wave operator provided by (4.7.7), we have, together with the wave equation (6.1.1) satisfied by ψ,

$$\begin{aligned}&\int_{\mathcal{M}_{\not{trap}}(\tau_1,\tau_2)} r^{-6}\frac{\Delta^4}{r^8}|\Delta_2(\nabla_3,\nabla_4,\not{\mathfrak{d}})^j\psi|^2 + \int_{\mathcal{M}(\tau_1,\tau_2)} r^{-6}\frac{\Delta^4}{r^8}|\Delta_2(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{j-1}\psi|^2\\ \lesssim\ &\int_{\mathcal{M}_{\not{trap}}(\tau_1,\tau_2)} r^{-6}\frac{\Delta^2}{r^4}|\nabla^2_{\widehat{R}}(\nabla_3,\nabla_4,\not{\mathfrak{d}})^j\psi|^2 + \mathrm{Mor}[(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{\leq j-1}(\nabla_T,\nabla_Z)^{\leq 2}\psi](\tau)\\ &+\mathrm{Mor}[(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{\leq j}\psi](\tau) + \int_{\Sigma(\tau)}|\mathfrak{d}^{\leq j-1}N|^2.\end{aligned}$$

In view of Step 1 and Step 2, we infer

$$\int_{\mathcal{M}_{tr\not{p}}(\tau_1,\tau_2)} r^{-6}\frac{\Delta^4}{r^8}|\Delta_2(\nabla_3,\nabla_4,\not{\mathfrak{d}})^j\psi|^2+\int_{\mathcal{M}(\tau_1,\tau_2)} r^{-6}\frac{\Delta^4}{r^8}|\Delta_2(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{j-1}\psi|^2 \lesssim \mathcal{J}_{RHS}.$$

Using the Hodge estimates of Proposition 9.3.2, as well as the estimate of Step 1 and the iteration assumption (9.5.7), we deduce

$$\int_{\mathcal{M}_{tr\not{p}}(\tau_1,\tau_2)} r^{-6}\frac{\Delta^4}{r^8}|\nabla^2(\nabla_3,\nabla_4,\not{\mathfrak{d}})^j\psi|^2+\int_{\mathcal{M}(\tau_1,\tau_2)} r^{-6}\frac{\Delta^4}{r^8}|\nabla^2(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{j-1}\psi|^2 \lesssim \mathcal{J}_{RHS}.$$

Using again Step 1 and Step 2, we obtain

$$\begin{aligned}
&\int_{\mathcal{M}(\tau_1,\tau_2)} r^{-6}\frac{\Delta^2}{r^4}|\nabla_{\widehat{R}}(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{j+1}\psi|^2\\
&+\int_{\mathcal{M}_{tr\not{p}}(\tau_1,\tau_2)} r^{-8}\frac{\Delta^4}{r^8}|(\nabla,\nabla_T)(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{j+1}\psi|^2\\
&+\int_{\mathcal{M}(\tau_1,\tau_2)} r^{-10}\frac{\Delta^4}{r^8}|(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{j+1}\psi|^2 \lesssim \mathcal{J}_{RHS}.
\end{aligned}$$

Step 4. This step is analogous to Step 2 of the proof of Proposition 9.5.5, so we only sketch it. Commuting the wave equation (6.1.1) satisfied by ψ with $\not{\mathfrak{d}}_2^{j_1}\not{\mathcal{L}}_{\mathbf{T}}^{j_2}$ with $j_1+j_2\leq j+1$, and relying on Proposition 9.2.14, we infer, for $r\geq r_1$ with $r_1=r_1(m)$ sufficiently large, together with the Hodge estimates of Proposition 9.3.2

$$\mathrm{Mor}_{r\geq 2r_1}[(\nabla_T,\not{\mathfrak{d}})^{\leq j+1}\psi](\tau_1,\tau_2) \lesssim \frac{r_1}{m}\mathrm{Mor}_{r_1\leq r\leq 2r_1}[(\nabla_T,\not{\mathfrak{d}})^{\leq 2}\psi](\tau_1,\tau_2)+\mathcal{J}_{RHS},$$

where we used also the iteration assumption (9.5.7) to absorb lower order terms coming from commutations with $\not{\mathfrak{d}}_2$. Noticing that $\mathrm{Mor}_{r_1\leq r\leq 2r_1}[(\nabla_T,\not{\mathfrak{d}})^{\leq 2}\psi](\tau_1,\tau_2)$ is controlled by the LHS of the final estimate of Step 3, we infer

$$\mathrm{Mor}_{r\geq 2r_1}[(\nabla_T,\not{\mathfrak{d}})^{\leq j+1}\psi](\tau_1,\tau_2) \lesssim \mathcal{J}_{RHS}.$$

Next, using again the representation of the wave operator provided by (4.7.7), and arguing by iteration, we infer from the above estimate, for any $0\leq 2l\leq j+1$,

$$\int_{\mathcal{M}_{r\geq 2r_1}(\tau_1,\tau_2)} r^{-2}|\nabla_{\widehat{R}}^{1+2l}(\nabla_T,\not{\mathfrak{d}})^{j+1-2l}\psi|^2 \lesssim \mathcal{J}_{RHS},$$

and for any $2 \leq 2l \leq j+2$,

$$\int_{\mathcal{M}_{r\geq 2r_1}(\tau_1,\tau_2)} r^{-2}|\nabla^{2l}_{\widehat{R}}(\nabla_T, \not{\mathfrak{d}})^{j+2-2l}\psi|^2 \lesssim \mathcal{J}_{RHS},$$

and hence, together with the above estimate

$$\text{Mor}_{r\geq 2r_1}[(\nabla_T, \not{\mathfrak{d}}, \nabla_{\widehat{R}})^{\leq j+1}\psi](\tau_1,\tau_2) \lesssim \mathcal{J}_{RHS}.$$

Together with Step 3, we deduce

$$\begin{aligned}
&\int_{\mathcal{M}_{\not{trap}}(\tau_1,\tau_2)} \left(r^{-1}\frac{\Delta^4}{r^8}|\nabla(\nabla_3,\nabla_4, \not{\mathfrak{d}})^{j+1}\psi|^2 + r^{-2}\frac{\Delta^4}{r^8}|\nabla_T(\nabla_3,\nabla_4, \not{\mathfrak{d}})^{j+1}\psi|^2\right)\\
&+\int_{\mathcal{M}(\tau_1,\tau_2)} r^{-2}\frac{\Delta^2}{r^4}|\nabla_{\widehat{R}}(\nabla_3,\nabla_4, \not{\mathfrak{d}})^{j+1}\psi|^2 + \int_{\mathcal{M}(\tau_1,\tau_2)} r^{-3}\frac{\Delta^4}{r^8}|(\nabla_3,\nabla_4, \not{\mathfrak{d}})^{j+1}\psi|^2 \lesssim \mathcal{J}_{RHS}.
\end{aligned}$$

Step 5. Next, we commute the wave equation (6.1.1) for ψ in $r \leq 4m$ by $(\nabla_3, \not{\mathfrak{d}}_2, \not{\mathcal{L}}_{\mathbf{T}})^{\leq j+1}$, using the commutation formulas of Corollary 4.3.4, Lemma 4.5.4 and Lemma 4.7.13, as well as Lemma 9.4.5 for the commutations w.r.t to ∇_3. Together with Corollary 9.4.6, we infer

$$\begin{aligned}
&c_0 E_{r\leq r_+(1+\delta_{red})}[(\nabla_3, \not{\mathfrak{d}}_2, \not{\mathcal{L}}_{\mathbf{T}})^{\leq j+1}\psi](\tau_2) + c_0\text{Mor}_{r\leq r_+(1+\delta_{red})}[(\nabla_3, \not{\mathfrak{d}}_2, \not{\mathcal{L}}_{\mathbf{T}})^{\leq j+1}\psi](\tau_1,\tau_2)\\
&\quad + c_0 F_{\mathcal{A}}[(\nabla_3, \not{\mathfrak{d}}_2, \not{\mathcal{L}}_{\mathbf{T}})^{\leq j+1}\psi](\tau_1,\tau_2)\\
&\leq \delta_{red}^{-1}\text{Mor}_{r_+(1+\delta_{red})\leq r\leq r_+(1+2\delta_{red})}[(\nabla_3, \not{\mathfrak{d}}, \not{\mathcal{L}}_{\mathbf{T}})^{\leq j+1}\psi](\tau_1,\tau_2) + \mathcal{J}_{RHS}.
\end{aligned}$$

Using the comparison of $\not{\mathcal{L}}_{\mathbf{T}}$ and $\nabla_{\mathbf{T}}$ of Lemma 9.2.1, and the Hodge estimates of Proposition 9.3.2, and using

$$\begin{aligned}
&\delta_{red}^{-1}\text{Mor}_{r_+(1+\delta_{red})\leq r\leq r_+(1+2\delta_{red})}[(\nabla_3, \not{\mathfrak{d}}, \not{\mathcal{L}}_{\mathbf{T}})^{\leq j+1}\psi](\tau_1,\tau_2)\\
\lesssim\ &\delta_{red}^{-7}\Bigg(\int_{\mathcal{M}_{\not{trap}}(\tau_1,\tau_2)} \left(r^{-1}\frac{\Delta^4}{r^8}|\nabla(\nabla_3,\nabla_4, \not{\mathfrak{d}})^{j+1}\psi|^2 + r^{-2}\frac{\Delta^4}{r^8}|\nabla_T(\nabla_3,\nabla_4, \not{\mathfrak{d}})^{j+1}\psi|^2\right)\\
&+\int_{\mathcal{M}(\tau_1,\tau_2)} r^{-2}\frac{\Delta^2}{r^4}|\nabla_{\widehat{R}}(\nabla_3,\nabla_4, \not{\mathfrak{d}})^{j+1}\psi|^2 + \int_{\mathcal{M}(\tau_1,\tau_2)} r^{-3}\frac{\Delta^4}{r^8}|(\nabla_3,\nabla_4, \not{\mathfrak{d}})^{j+1}\psi|^2\Bigg),
\end{aligned}$$

we infer from Step 4

$$\begin{aligned}
&E_{r\leq r_+(1+\delta_{red})}[(\nabla_3, \not{\mathfrak{d}}, \nabla_4)^{\leq j+1}\psi](\tau_2) + \text{Mor}_{r\leq r_+(1+\delta_{red})}[(\nabla_3, \not{\mathfrak{d}}, \nabla_4)^{\leq j+1}\psi](\tau_1,\tau_2)\\
&\qquad\qquad + F_{\mathcal{A}}[(\nabla_3, \not{\mathfrak{d}}, \nabla_4)^{\leq j+1}\psi](\tau_1,\tau_2) \lesssim \mathcal{J}_{RHS}.
\end{aligned}$$

Together with the control of Step 4, we deduce

$$\begin{aligned}
& E_{r\leq r_+(1+\delta_{red})}[\mathfrak{d}^{\leq j+1}\psi](\tau_2) + \mathrm{Mor}[(\not{\mathfrak{d}}, \nabla_3, \nabla_4)^{\leq j+1}\psi](\tau_1,\tau_2) + F_{\mathcal{A}}[\mathfrak{d}^{\leq j+1}\psi](\tau_1,\tau_2)\\
\lesssim\ & \mathcal{J}_{RHS}
\end{aligned}$$

which in view of the definition of $\mathcal{J}_{RHS}$ yields

$$\begin{aligned}
& E_{r\leq r_+(1+\delta_{red})}[\mathfrak{d}^{\leq j+1}\psi](\tau_2) + \mathrm{Mor}[(\not{\mathfrak{d}}, \nabla_3, \nabla_4)^{\leq j+1}\psi](\tau_1,\tau_2) + F_{\mathcal{A}}[\mathfrak{d}^{\leq j+1}\psi](\tau_1,\tau_2)\\
\lesssim\ & \Bigg(E^j[\psi](\tau_1) + \mathcal{N}^j[\psi, N](\tau_1,\tau_2)\\
& + \left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^j[\psi] + B^j_\delta[\psi](\tau_1,\tau_2) + F^j[\psi](\tau_1,\tau_2)\right)\Bigg)^{\frac{1}{2}}\\
& \times\Bigg(\sup_{\tau\in[\tau_1,\tau_2]} E_{deg}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq j+1}\psi]\\
& + F_{\Sigma_*}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq j+1}\psi](\tau_1,\tau_2) + \delta_{\mathcal{H}} F_{\mathcal{A}}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq j+1}\psi](\tau_1,\tau_2)\Bigg)^{\frac{1}{2}}\\
& + E^{j+1}[\psi](\tau_1) + \mathcal{N}^{j+1}[\psi, N](\tau_1,\tau_2)\\
& + \left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^{j+1}[\psi] + B^{j+1}_\delta[\psi](\tau_1,\tau_2) + F^{j+1}[\psi](\tau_1,\tau_2)\right)
\end{aligned}$$

as stated. This concludes the proof of Proposition 9.5.6. □

Next, we consider the following combined energy-Morawetz estimate.

Proposition 9.5.7. *Let $2\leq j\leq s-1$. Assume that the solution ψ of of the model RW equation* (6.1.1) *satisfies the iteration assumption* (9.5.7)*. Then, the following combined energy-Morawetz estimate holds:*

$$\begin{aligned}
& E[(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{\leq j+1}\psi](\tau_2) + Mor[(\not{\mathfrak{d}}, \nabla_3, \nabla_4)^{\leq j+1}\psi](\tau_1,\tau_2) + F[(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{\leq j+1}\psi](\tau_1,\tau_2)\\
\lesssim\ & E^{j+1}[\psi](\tau_1) + \mathcal{N}^{j+1}[\psi, N](\tau_1,\tau_2)\\
& + \left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^{j+1}[\psi] + B^{j+1}_\delta[\psi](\tau_1,\tau_2) + F^{j+1}[\psi](\tau_1,\tau_2)\right). \qquad (9.5.9)
\end{aligned}$$

Proof. Since the proof is similar, and in fact easier, than the one of (9.5.6), we only sketch it:

1. First, we commute the wave equation (6.1.1) satisfied by ψ with $(\mathcal{L}_{\mathbf{T}}, \mathcal{L}_{\mathbf{Z}})$ and apply the iteration assumption 9.5.7 to the commuted wave equation. This yields, comparing also (∇_T, ∇_Z) and $(\mathcal{L}_{\mathbf{T}}, \mathcal{L}_{\mathbf{Z}})$, and using the iteration assumption (9.5.7) to absorb lower order terms,

$$\begin{aligned}
& E[(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{\leq j}(\nabla_T, \nabla_Z)\psi](\tau_2) + \mathrm{Mor}[(\not{\mathfrak{d}}, \nabla_3, \nabla_4)^{\leq j}(\nabla_T, \nabla_Z)\psi](\tau_1, \tau_2) \\
& + F[(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{\leq j}(\nabla_T, \nabla_Z)\psi](\tau_1, \tau_2) \\
\lesssim \; & E^{j+1}[\psi](\tau_1) + \mathcal{N}^{j+1}[\psi, N](\tau_1, \tau_2) \\
& + \left(\frac{|a|}{m} + \epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^{j+1}[\psi] + B_\delta^{j+1}[\psi](\tau_1, \tau_2) + F^{j+1}[\psi](\tau_1, \tau_2)\right).
\end{aligned}$$

2. Next, we commute the wave equation (6.1.1) satisfied by ψ with $(\mathcal{L}_{\mathbf{T}}, \mathcal{L}_{\mathbf{Z}})^{j+1-j}\not{\mathfrak{d}}_2^k$ for $1 \leq k \leq j+1$, where we recall that

$$\not{\mathfrak{d}}_2^{2k} = (|q|^2\Delta_2)^k, \qquad \not{\mathfrak{d}}_2^{2k+1} = |q|\mathcal{D}_2(|q|^2\Delta_2)^k.$$

Denoting

$$\psi_{k,j} = \mathcal{L}_{\mathbf{T}}^{j+1-k}\not{\mathfrak{d}}_2^k\psi,$$

and using the commutation formulas of using the commutation formulas of Corollary 4.3.4, Lemma 4.5.4 and Lemma 4.7.13, we obtain

$$\begin{aligned}
\square_2\psi_{2k,j} &= N_{2k,j}, & N_{2k,j} &= \mathfrak{d}^{\leq j+1}N + O(ar^{-2})\mathfrak{d}^{j+2}\psi + \mathfrak{d}^{\leq j+2}(\Gamma_g\cdot\psi), \\
\square_1\psi_{2k+1,j} &= N_{2k+1,j}, & N_{2k+1,j} &= \mathfrak{d}^{\leq j+1}N - \frac{3}{r^2}\psi_{2k+1,j} + O(ar^{-2})\mathfrak{d}^{j+2}\psi + \mathfrak{d}^{\leq j+2}(\Gamma_g\cdot\psi).
\end{aligned}$$

Relying on Proposition 9.2.14, we infer, for $r \geq r_1$ with $r_1 = r_1(m)$ sufficiently large,

$$\begin{aligned}
& E_{r\geq 2r_1}[\psi_{k,j}](\tau_2) + F_{\Sigma_*}[\psi_{k,j}](\tau_1, \tau_2) \\
\lesssim \; & \mathrm{Mor}_{r_1\leq r\leq 2r_1}[(\nabla_{\mathbf{T}}, \not{\mathfrak{d}})^{\leq j+1}\psi](\tau_1, \tau_2) + E^{j+1}[\psi](\tau_1) + \mathcal{N}^{j+1}[\psi, N](\tau_1, \tau_2) \\
& + \left(\frac{|a|}{m} + \epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^{j+1}[\psi] + B_\delta^{j+1}[\psi](\tau_1, \tau_2) + F^{j+1}[\psi](\tau_1, \tau_2)\right).
\end{aligned}$$

In view of the definition of $\psi_{k,j}$, together with the Hodge estimates of Proposition 9.3.2 and the iteration assumption 9.5.7, we obtain

$$\begin{aligned}
& E_{r\geq 2r_1}[(\nabla_T, \not{\mathfrak{d}})^{\leq j+1}](\tau_2) + F_{\Sigma_*}[(\nabla_T, \not{\mathfrak{d}})^{\leq j+1}](\tau_1, \tau_2) \\
\lesssim \; & \mathrm{Mor}_{r_1\leq r\leq 2r_1}[(\nabla_{\mathbf{T}}, \not{\mathfrak{d}})^{\leq j+1}\psi](\tau_1, \tau_2) + E^{j+1}[\psi](\tau_1) + \mathcal{N}^{j+1}[\psi, N](\tau_1, \tau_2) \\
& + \left(\frac{|a|}{m} + \epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^{j+1}[\psi] + B_\delta^{j+1}[\psi](\tau_1, \tau_2) + F^{j+1}[\psi](\tau_1, \tau_2)\right).
\end{aligned}$$

Also, using again the representation of the wave operator provided by (4.7.7), we have, for $2k \leq j+1$,

$$\begin{aligned}
& E_{r\geq 2r_1}[(\nabla_T, \not{\mathfrak{d}})^{\leq j+1-2k}\nabla_{\widehat{R}}^{2k}\psi] + F_{\Sigma_*}[(\nabla_T, \not{\mathfrak{d}})^{\leq j+1-2k}\nabla_{\widehat{R}}^{2k}\psi] \\
\lesssim\ & E_{r\geq 2r_1}[(\nabla_T, \not{\mathfrak{d}})^{\leq j+1}\psi](\tau_2) + F_{\Sigma_*}[(\nabla_T, \not{\mathfrak{d}})^{\leq j+1}\psi](\tau_1,\tau_2) \\
& +\mathcal{N}^{j+1}[\psi, N](\tau_1,\tau_2) + \epsilon\left(\sup_{\tau\in[\tau_1,\tau_2]} E^{j+1}[\psi] + F^{j+1}[\psi](\tau_1,\tau_2)\right).
\end{aligned}$$

and for $2k \leq j+2$,

$$\begin{aligned}
& \int_{\Sigma_{r\geq 2r_1(\tau_2)}} |\nabla_{\widehat{R}}^{2k}\psi|^2 + \int_{\Sigma_*(\tau_1,\tau_2)} |\nabla_{\widehat{R}}^{2k}\psi|^2 \\
\lesssim\ & E_{r\geq 2r_1}[(\nabla_T, \not{\mathfrak{d}})^{\leq j+1}\psi](\tau_2) + F_{\Sigma_*}[(\nabla_T, \not{\mathfrak{d}})^{\leq j+1}\psi](\tau_1,\tau_2) \\
& +\mathcal{N}^{j+1}[\psi, N](\tau_1,\tau_2) + \epsilon\left(\sup_{\tau\in[\tau_1,\tau_2]} E^{j+1}[\psi] + F^{j+1}[\psi](\tau_1,\tau_2)\right).
\end{aligned}$$

We infer from the above

$$\begin{aligned}
& E_{r\geq 2r_1}[(\nabla_T, \not{\mathfrak{d}}, \nabla_{\widehat{R}})^{\leq j+1}\psi](\tau_2) + F_{\Sigma_*}[(\nabla_T, \not{\mathfrak{d}}, \nabla_{\widehat{R}})^{\leq j+1}\psi](\tau_1,\tau_2) \\
\lesssim\ & \mathrm{Mor}_{r_1\leq r\leq 2r_1}[(\nabla_{\mathbf{T}}, \not{\mathfrak{d}})^{\leq j+1}\psi](\tau_1,\tau_2) + E^{j+1}[\psi](\tau_1) + \mathcal{N}^{j+1}[\psi, N](\tau_1,\tau_2) \\
& + \left(\frac{|a|}{m} + \epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^{j+1}[\psi] + B_\delta^{j+1}[\psi](\tau_1,\tau_2) + F^{j+1}[\psi](\tau_1,\tau_2)\right).
\end{aligned}$$

3. Using again the representation of the wave operator provided by (4.7.7), i.e.

$$\begin{aligned}
|q|^2\dot{\square}_2\psi = \frac{(r^2+a^2)^2}{\Delta}\big(-\nabla_{\widehat{T}}\nabla_{\widehat{T}}\psi + \nabla_{\widehat{R}}\nabla_{\widehat{R}}\psi\big) + 2r\nabla_{\widehat{R}}\psi \\
+ |q|^2\triangle_2\psi + |q|^2(\eta + \underline{\eta})\cdot\nabla\psi + r^2\Gamma_g\cdot\mathfrak{d}\psi,
\end{aligned}$$

together with the wave equation (6.1.1) satisfied by ψ, we infer

$$\begin{aligned}
E[(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{\leq j-1}U] \ \lesssim\ & E[(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{\leq j-1}(\nabla_T, \nabla_Z)^{\leq 2}\psi](\tau) \\
& +E[(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{\leq j}\psi](\tau) + \int_{\Sigma(\tau)} |\mathfrak{d}^{\leq j-1}N|^2,
\end{aligned}$$

where the constant in $\lesssim$ is independent of τ, and where the tensor U is given by

$$U := \nabla_{\widehat{R}}\nabla_{\widehat{R}}\psi + \frac{|q|^2\Delta}{(r^2+a^2)^2}\triangle_2\psi.$$

Next, we proceed as follows:

(a) We consider a cut-off $\kappa_{\delta_{\mathcal{H}}}(r)$ such that $0\leq\kappa_{\delta_{\mathcal{H}}}\leq 1$, $\kappa_{\delta_{\mathcal{H}}}=1$ on $r\geq r_+$, and $\kappa_{\delta_{\mathcal{H}}}$ vanishes on $\mathcal{A}$. Also, for any integer $2\leq n\leq\tau_*-2$, we consider a cut-off $\kappa_n(\tau)$ such that $0\leq\kappa_n\leq 1$, $\kappa_n(\tau)=1$ on $\{n\leq\tau\leq n+1\}$, and κ_n vanishes on $\tau\leq n-1$ and $\tau\geq n+2$. We have, by integrating by parts, and using the support properties of $\kappa_n(\tau)$ and the above control of U,

$$
\begin{aligned}
&\int_{\mathcal{M}}\kappa_{\delta_{\mathcal{H}}}(r)\kappa_n(\tau)|(\nabla_T,\nabla,\nabla_{\widehat{R}})(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{\leq j-1}\nabla_{\widehat{R}}^2\psi|^2\\
&+\int_{\mathcal{M}}\kappa_{\delta_{\mathcal{H}}}(r)\kappa_n(\tau)\frac{|\Delta|}{r^2}|(\nabla_T,\nabla,\nabla_{\widehat{R}})(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{\leq j-1}\nabla\nabla_{\widehat{R}}\psi|^2\\
&+\int_{\mathcal{M}}\kappa_{\delta_{\mathcal{H}}}(r)\kappa_n(\tau)\frac{|\Delta|^2}{r^4}|(\nabla_T,\nabla,\nabla_{\widehat{R}})(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{\leq j-1}\nabla^2\psi|^2\\
\lesssim\ &\sup_{\tau\in[n-1,n+2]}E[(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{\leq j-1}U]\\
&+\left(\sup_{\tau\in[n-1,n+2]}E[(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{\leq j}\psi](\tau)+F_{\Sigma_*}[(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{\leq j}\psi](n-1,n+2)\right)^{\frac{1}{2}}\\
&\times\left(\sup_{\tau\in[n-1,n+2]}E[(\nabla_T,\nabla,\nabla_{\widehat{R}})^{\leq j+1}\psi](\tau)+F_{\Sigma_*}[(\nabla_T,\nabla_{\widehat{R}},\nabla)^{\leq j+1}\psi](n-1,n+2)\right)^{\frac{1}{2}}.
\end{aligned}
$$

Together with the above estimate for U, we deduce

$$
\begin{aligned}
&\int_{\mathcal{M}}\kappa_{\delta_{\mathcal{H}}}(r)\kappa_n(\tau)|(\nabla_T,\nabla,\nabla_{\widehat{R}})(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{\leq j-1}\nabla_{\widehat{R}}^2\psi|^2\\
&+\int_{\mathcal{M}}\kappa_{\delta_{\mathcal{H}}}(r)\kappa_n(\tau)\frac{|\Delta|}{r^2}|(\nabla_T,\nabla,\nabla_{\widehat{R}})(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{\leq j-1}\nabla\nabla_{\widehat{R}}\psi|^2\\
&+\int_{\mathcal{M}}\kappa_{\delta_{\mathcal{H}}}(r)\kappa_n(\tau)\frac{|\Delta|^2}{r^4}|(\nabla_T,\nabla,\nabla_{\widehat{R}})(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{\leq j-1}\nabla^2\psi|^2\\
\lesssim\ &\sup_{\tau\in[n-1,n+2]}\Big[E[(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{\leq j-1}(\nabla_T,\nabla_Z)^{\leq 2}\psi](\tau)+E[(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{\leq j}\psi](\tau)\Big]\\
&+\left(\sup_{\tau\in[n-1,n+2]}E[(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{\leq j}\psi](\tau)+F_{\Sigma_*}[(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{\leq j}\psi](n-1,n+2)\right)^{\frac{1}{2}}\\
&\times\left(\sup_{\tau\in[n-1,n+2]}E[(\nabla_T,\nabla,\nabla_{\widehat{R}})^{\leq j+1}\psi](\tau)+F_{\Sigma_*}[(\nabla_T,\nabla_{\widehat{R}},\nabla)^{\leq j+1}\psi](n-1,n+2)\right)^{\frac{1}{2}}\\
&+\mathcal{N}^{j+1}[\psi,N](n-1,n+2).
\end{aligned}
$$

(b) In view of the properties of the cut-offs, the above estimates imply

$$\begin{aligned}
&\int_{\mathcal{M}\cap\{r\geq r_+\}\cap\{n\leq\tau\leq n+1\}}\Bigg[|(\nabla_T,\nabla,\nabla_{\widehat{R}})(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{\leq j-1}\nabla_{\widehat{R}}^2\psi|^2\\
&+\frac{|\Delta|}{r^2}|(\nabla_T,\nabla,\nabla_{\widehat{R}})(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{\leq j-1}\nabla\nabla_{\widehat{R}}\psi|^2\\
&+\frac{|\Delta|^2}{r^4}|(\nabla_T,\nabla,\nabla_{\widehat{R}})(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{\leq j-1}\nabla^2\psi|^2\Bigg]\\
\lesssim\;& \sup_{\tau\in[n-1,n+2]}\Big[E[(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{\leq j-1}(\nabla_T,\nabla_Z)^{\leq 2}\psi](\tau)+E[(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{\leq j}\psi](\tau)\Big]\\
&+\left(\sup_{\tau\in[n-1,n+2]}E[(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{\leq j}\psi](\tau)+F_{\Sigma_*}[(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{\leq j}\psi](n-1,n+2)\right)^{\frac{1}{2}}\\
&\times\left(\sup_{\tau\in[n-1,n+2]}E[(\nabla_T,\nabla,\nabla_{\widehat{R}})^{\leq j+1}\psi](\tau)+F_{\Sigma_*}[(\nabla_T,\nabla_{\widehat{R}},\nabla)^{\leq j+1}\psi](n-1,n+2)\right)^{\frac{1}{2}}\\
&+\mathcal{N}^{j+1}[\psi,N](n-1,n+2).
\end{aligned}$$

By the mean value theorem, we infer the existence of $\tau_{(n)}\in[n,n+1]$ such that

$$\begin{aligned}
&\int_{\Sigma(\tau_{(n)})\cap\{r\geq r_+\}}\Bigg[|\nabla_4\nabla_{\widehat{R}}^2\psi|^2+\frac{|\Delta|}{r^2}|\nabla_4\nabla\nabla_{\widehat{R}}\psi|^2+\frac{|\Delta|^2}{r^4}|\nabla_4\nabla^2\psi|^2\\
&+r^{-2}\left(|\nabla_{\widehat{T}}\nabla_{\widehat{R}}^2\psi|^2+\frac{|\Delta|}{r^2}|\nabla_{\widehat{T}}\nabla\nabla_{\widehat{R}}\psi|^2+\frac{|\Delta|^2}{r^4}|\nabla_{\widehat{T}}\nabla^2\psi|^2\right)+|\nabla\nabla_{\widehat{R}}^2\psi|^2\\
&+\frac{|\Delta|}{r^2}|\nabla\nabla_{\widehat{R}}\nabla\psi|^2+\frac{|\Delta|^2}{r^4}|\nabla^3\psi|^2\Bigg]\\
\lesssim\;& \sup_{\tau\in[n-1,n+2]}\Big[E[(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{\leq j-1}(\nabla_T,\nabla_Z)^{\leq 2}\psi](\tau)+E[(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{\leq j}\psi](\tau)\Big]\\
&+\left(\sup_{\tau\in[n-1,n+2]}E[(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{\leq j}\psi](\tau)+F_{\Sigma_*}[(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{\leq j}\psi](n-1,n+2)\right)^{\frac{1}{2}}\\
&\times\left(\sup_{\tau\in[n-1,n+2]}E[(\nabla_T,\nabla,\nabla_{\widehat{R}})^{\leq j+1}\psi](\tau)+F_{\Sigma_*}[(\nabla_T,\nabla_{\widehat{R}},\nabla)^{\leq j+1}\psi](n-1,n+2)\right)^{\frac{1}{2}}\\
&+\mathcal{N}^{j+1}[\psi,N](n-1,n+2).
\end{aligned}$$

(c) Together with the above control of $E[(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{\leq j}(\nabla_T,\nabla_Z)\psi](\tau)$, the above control of $E_{r\geq 2r_1}[(\nabla_T,\not{\mathfrak{d}},\nabla_{\widehat{R}})^{\leq j+1}\psi](\tau_2)$ and $F_{\Sigma_*}[(\nabla_T,\not{\mathfrak{d}},\nabla_{\widehat{R}})^{\leq j+1}\psi](\tau_1,\tau_2)$, the control of $E_{r\leq r_+(1+\delta_{red})}[\mathfrak{d}^{\leq j+1}\psi](\tau_2)$ and $F_{\mathcal{A}}[\mathfrak{d}^{\leq j+1}\psi](\tau_1,\tau_2)$ provided by Proposition 9.5.6, and the iteration assumption 9.5.7, and fixing the value of $r_1 = r_1(m)$, we infer, the following estimate, for any n such that $\tau_{(n)}\geq\tau_1$,

$$\begin{aligned}
& F_{\mathcal{A}}[\mathfrak{d}^{\leq j+1}\psi](\tau_1,\tau_{(n)}) + E[(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{\leq j+1}\psi](\tau_{(n)}) + F_{\Sigma_*}[(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{\leq j+1}\psi](\tau_1,\tau_{(n)})\\
\lesssim\ & \Bigg(E^j[\psi](\tau_1)+\mathcal{N}^j[\psi,N](\tau_1,n+2)\\
&+\left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,n+2]}E^j[\psi]+B^j_\delta[\psi](\tau_1,n+2)+F^j[\psi](\tau_1,n+2)\right)\Bigg)^{\frac{1}{2}}\\
&\times\left(\sup_{\tau\in[n-1,n+2]}E[(\nabla_T,\nabla,\nabla_{\widehat{R}})^{\leq j+1}\psi](\tau)+F_{\Sigma_*}[(\nabla_T,\nabla_{\widehat{R}},\nabla)^{\leq j+1}\psi](n-1,n+2)\right)^{\frac{1}{2}}\\
&+\mathrm{Mor}[(\nabla_{\mathbf{T}},\not{\mathfrak{d}})^{\leq j+1}\psi](\tau_1,n+2)+E^{j+1}[\psi](\tau_1)+\mathcal{N}^{j+1}[\psi,N](\tau_1,n+2)\\
&+\left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,n+2]}E^{j+1}[\psi]+B^{j+1}_\delta[\psi](\tau_1,n+2)+F^{j+1}[\psi](\tau_1,n+2)\right).
\end{aligned}$$

(d) Let $\tau_2\geq\tau_1$. By local energy estimates, it suffices to consider the case $\tau_2\geq\tau_1+5$. We then choose n such that $\tau_1\leq n-1\leq n+2\leq\tau_2<n+3$. In particular, we have $\tau_{(n)}+1\leq\tau_2\leq\tau_{(n)}+3$, and hence, using local energy estimates between $\tau_{(n)}$ and τ_2, we infer from the previous estimate

$$\begin{aligned}
& F_{\mathcal{A}}[\mathfrak{d}^{\leq j+1}\psi](\tau_1,\tau_2) + E[(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{\leq j+1}\psi](\tau_2) + F_{\Sigma_*}[(\nabla_3,\nabla_4,\not{\mathfrak{d}})^{\leq j+1}\psi](\tau_1,\tau_2)\\
\lesssim\ & \Bigg(E^j[\psi](\tau_1)+\mathcal{N}^j[\psi,N](\tau_1,\tau_2)\\
&+\left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]}E^j[\psi]+B^j_\delta[\psi](\tau_1,\tau_2)+F^j[\psi](\tau_1,\tau_2)\right)\Bigg)^{\frac{1}{2}}\\
&\times\left(\sup_{\tau\in[\tau_1,\tau_2]}E[(\nabla_T,\nabla,\nabla_{\widehat{R}})^{\leq j+1}\psi](\tau)+F_{\Sigma_*}[(\nabla_T,\nabla_{\widehat{R}},\nabla)^{\leq j+1}\psi](\tau_1,\tau_2)\right)^{\frac{1}{2}}\\
&+\mathrm{Mor}[(\nabla_{\mathbf{T}},\not{\mathfrak{d}})^{\leq j+1}\psi](\tau_1,\tau_2)+E^{j+1}[\psi](\tau_1)+\mathcal{N}^{j+1}[\psi,N](\tau_1,\tau_2)\\
&+\left(\frac{|a|}{m}+\epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]}E^{j+1}[\psi]+B^{j+1}_\delta[\psi](\tau_1,\tau_2)+F^{j+1}[\psi](\tau_1,\tau_2)\right).
\end{aligned}$$

and thus

$$
\begin{aligned}
& F_{\mathcal{A}}[\mathfrak{d}^{\leq j+1}\psi](\tau_1,\tau_2) + E[(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{\leq j+1}\psi](\tau_2) + F_{\Sigma_*}[(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{\leq j+1}\psi](\tau_1,\tau_2) \\
\lesssim\ & \mathrm{Mor}[(\nabla_{\mathbf{T}}, \not{\mathfrak{d}})^{\leq j+1}\psi](\tau_1,\tau_2) + E^{j+1}[\psi](\tau_1) + \mathcal{N}^{j+1}[\psi, N](\tau_1,\tau_2) \\
& + \left(\frac{|a|}{m} + \epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^{j+1}[\psi] + B^{j+1}_{\delta}[\psi](\tau_1,\tau_2) + F^{j+1}[\psi](\tau_1,\tau_2)\right).
\end{aligned}
$$

4. Together with the control for $\mathrm{Mor}[(\not{\mathfrak{d}}, \nabla_3, \nabla_4)^{\leq j+1}\psi]$ obtained in Proposition 9.5.6, we infer

$$
\begin{aligned}
& E[(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{\leq j+1}\psi](\tau_2) + \mathrm{Mor}[(\not{\mathfrak{d}}, \nabla_3, \nabla_4)^{\leq j+1}\psi](\tau_1,\tau_2) + F[(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{\leq j+1}\psi](\tau_1,\tau_2) \\
\lesssim\ & \left(E^{j}[\psi](\tau_1) + \mathcal{N}^{j}[\psi, N](\tau_1,\tau_2) + \left(\frac{|a|}{m} + \epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^{j}[\psi] + B^{j}_{\delta}[\psi](\tau_1,\tau_2)\right)\right)^{\frac{1}{2}} \\
& \times\left(\sup_{\tau\in[\tau_1,\tau_2]} E_{deg}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq j+1}\psi] + F_{\Sigma_*}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq j+1}\psi](\tau_1,\tau_2)\right. \\
& \left.+\delta_{\mathcal{H}} F_{\mathcal{A}}[(\not{\mathfrak{d}}, \nabla_T, \nabla_{\widehat{R}})^{\leq j+1}\psi](\tau_1,\tau_2)\right)^{\frac{1}{2}} + E^{j+1}[\psi](\tau_1) + \mathcal{N}^{j+1}[\psi, N](\tau_1,\tau_2) \\
& + \left(\frac{|a|}{m} + \epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^{j+1}[\psi] + B^{j+1}_{\delta}[\psi](\tau_1,\tau_2) + F^{j+1}[\psi](\tau_1,\tau_2)\right).
\end{aligned}
$$

We deduce

$$
\begin{aligned}
& E[(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{\leq j+1}\psi](\tau_2) + \mathrm{Mor}[(\not{\mathfrak{d}}, \nabla_3, \nabla_4)^{\leq j+1}\psi](\tau_1,\tau_2) + F[(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{\leq j+1}\psi](\tau_1,\tau_2) \\
\lesssim\ & E^{j+1}[\psi](\tau_1) + \mathcal{N}^{j+1}[\psi, N](\tau_1,\tau_2) \\
& + \left(\frac{|a|}{m} + \epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^{j+1}[\psi] + B^{j}_{\delta}[\psi](\tau_1,\tau_2) + F^{j+1}[\psi](\tau_1,\tau_2)\right)
\end{aligned}
$$

as stated.

This concludes the proof of Proposition 9.5.7. □

We are now ready to prove Theorem 6.3.1.

Proof of Theorem 6.3.1. Recall that (9.5.7) holds for $j = 2$ in view of (9.5.6). Also, if the iteration assumption (9.5.7) holds for some $2 \leq j \leq s-1$, then, it also holds for j

replaced by $j+1$ in view of Proposition 9.5.7. Thus, we infer that (9.5.7) holds for all $2 \leq j \leq s$. Thus, we have, for all $2 \leq s \leq k_L$, and for any $\delta > 0$,

$$\begin{aligned} & \mathrm{Mor}[(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{\leq s}\psi](\tau_1,\tau_2) + E[(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{\leq s}\psi](\tau_2) + F[(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{\leq s}\psi](\tau_1,\tau_2) \\ \lesssim\; & E^s[\psi](\tau_1) + \mathcal{N}^s[\psi, N](\tau_1,\tau_2) + \left(\frac{|a|}{m} + \epsilon\right)\left(\sup_{\tau\in[\tau_1,\tau_2]} E^s[\psi] + B^s_\delta[\psi](\tau_1,\tau_2)\right). \end{aligned}$$

This concludes the proof of Theorem 6.3.1. □

9.6 Conditional estimate for the scalar wave

Proposition 9.6.1. *Let ψ be a solution to the following scalar wave equation*

$$\square\psi + V\psi \;=\; N \tag{9.6.1}$$

where V is real and satisfies $V \sim r^{-3}$ for r large. Then the following estimates hold true for all $2 \leq s \leq k_L$ and some small $\delta > 0$,

$$\begin{aligned} Mor[(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{\leq s}\psi](\tau_1,\tau_2) \lesssim & \int_{\mathcal{M}} r^{-3}|(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{\leq s}\psi|^2 + Mor^{s-1}[\psi](\tau_1,\tau_2) \\ & + F^{s-1}[\psi](\tau_1,\tau_2) + \sup_{\tau\in[\tau_1,\tau_2]} E^s[\psi](\tau) \\ & + {}^{(mor)}\mathcal{N}^s[\psi, N](\tau_1,\tau_2) + {}^{(mor)}\mathcal{N}^s[\psi, N](\tau_1,\tau_2) + \epsilon B^s_\delta[\psi] \end{aligned} \tag{9.6.2}$$

and

$$\begin{aligned} & E[(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{\leq s}\psi](\tau_2) + Mor[(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{\leq s}\psi](\tau_1,\tau_2) + F[(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{\leq s}\psi](\tau_1,\tau_2) \\ & \lesssim \int_{\mathcal{M}} r^{-3}|(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{\leq s}\psi|^2 + Mor^{s-1}[\psi](\tau_1,\tau_2) \\ & + F^{s-1}[\psi](\tau_1,\tau_2) + \sup_{\tau\in[\tau_1,\tau_2]} E^{s-1}[\psi](\tau) + \mathcal{N}^s[\psi, N](\tau_1,\tau_2) + \epsilon B^s_\delta[\psi] \end{aligned} \tag{9.6.3}$$

where

$$\mathcal{N}^s[\psi, N](\tau_1,\tau_2) := \int_{\mathcal{M}} \left(|\nabla_{\widehat{R}}\mathfrak{d}^{\leq s}\psi| + r^{-1}|\mathfrak{d}^{\leq s}\psi|\right)|\mathfrak{d}^{\leq s}N|.$$

Remark 9.6.2. *Note that both estimates are conditional on the control of the quantities $\int_{\mathcal{M}} r^{-3}|(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{\leq s}\psi|^2$, $Mor^{s-1}[\psi](\tau_1,\tau_2)$ and $F^{s-1}[\psi](\tau_1,\tau_2)$. Proposition 9.6.1 will be extended to a conditional r^p-weighted version in Proposition 10.5.1, see section 10.5. These estimates will be used to control $\check{P}$ in Chapter 14.*

The proof of Proposition 9.6.1 is similar to the one of Theorem 6.3.1. Given that the estimates are only conditional, and in view of the the strong decay in r for the potential V, and the fact that extending the Andersson-Blue method to perturbations of Kerr is more straightforward for scalar waves, the proof of Proposition 9.6.1 is in fact simpler than one of Theorem 6.3.1.

Chapter 10

Proof of Theorems 6.2.1 and 6.2.2

In this chapter, we derive the r^p-weighted estimates for the reduced gRW equation (6.1.1)

$$\dot{\square}_2\psi - V\psi = -\frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_T\psi + N, \qquad V = \frac{4\Delta}{(r^2+a^2)|q|^2}, \tag{10.0.1}$$

on a spacetime $\mathcal{M}$ which is an admissible perturbation of Kerr in the sense that (6.1.6) holds. Together with the Energy-Morawetz estimates of Theorem 6.3.1, this will conclude the proof of Theorems 6.2.1 and 6.2.2. These r^p weighted estimates concern only the region $r \geq R$ for a sufficiently large R. In such a region the equation (6.1.1) closely resembles the equation

$$\dot{\square}_2\psi - V\psi = N, \qquad V = \left(1-\frac{2m}{r}\right)\frac{4}{r^2},$$

which was studied in Chapter 10 of [50]. The estimates in this chapter are thus similar to the r^p estimates in sections 10.2-10.5 of [50]. There is however an important difference in that the hypersurfaces $\Sigma(\tau)$ are not null, as in [50], but spacelike asymptotically null. This leads to some significant differences in the proof of Theorem 6.2.1.

10.1 Proof of Theorem 6.2.1

In order to prove Theorem 6.2.1, we need to show that the following estimates hold true for solutions $\psi \in \mathfrak{s}_2$ of (10.0.1) on an admissible $\mathcal{M}$, for all $\delta \leq p \leq 2-\delta$ and $2 \leq s \leq k_L$,

$$BEF^s_p[\psi](\tau_1,\tau_2) \lesssim E^s_p[\psi](\tau_1) + \mathcal{N}^s_p[\psi, N](\tau_1,\tau_2), \tag{10.1.1}$$

where the BEF_p^s norms have been introduced in section 6.1.5.

We proceed in steps as follows.

Step 0. Recall that we have proved in section 9.5 the global energy-Morawetz estimates of Theorem 6.3.1, i.e. the fact that the following estimates hold true for solutions $\psi \in \mathfrak{s}_2$ of (10.0.1) on an admissible $\mathcal{M}$, for $|a|/m \ll 1$ sufficiently small, for $2 \leq s \leq k_L$, and for any $\delta > 0$,

$$\begin{aligned}&\mathrm{Mor}[(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{\leq s}\psi](\tau_1, \tau_2) + E[(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{\leq s}\psi](\tau_2) + F[(\nabla_3, \nabla_4, \not{\mathfrak{d}})^{\leq s}\psi](\tau_1, \tau_2)\\ &\lesssim E^s[\psi](\tau_1) + \mathcal{N}^s[\psi, N](\tau_1, \tau_2) + \left(\frac{|a|}{m} + \epsilon\right) BEF_\delta^s[\psi](\tau_1, \tau_2).\end{aligned} \tag{10.1.2}$$

In view of (10.1.2), to complete the proof of Theorem 6.2.1, i.e. to prove (10.1.1), it remains to derive r-weighted estimates in the region $r \geq R$ for R large enough.

Step 1. We first derive a basic r-weighted estimate for $\nabla_3\psi$ in the region $r \geq R$.

Proposition 10.1.1. *Let $R \gg m$ large enough. We have*

$$\begin{aligned}\int_{\mathcal{M}_{\geq R}(\tau_1,\tau_2)} r^{-1-\delta}|\nabla_3\psi|^2 \lesssim& \int_{\mathcal{M}_{\geq R}(\tau_1,\tau_2)} r^{\delta-1}\Big(|\nabla_4\psi|^2 + |\nabla\psi|^2 + r^{-2}|\psi|^2\Big)\\ &+ E_{\geq \frac{R}{2}}[\psi](\tau_1) + Mor_{\frac{R}{2}\leq r\leq R}[\psi](\tau_1, \tau_2) + \mathcal{N}[\psi, N](\tau_1, \tau_2).\end{aligned} \tag{10.1.3}$$

Proposition 10.1.1 is proved in section 10.2.3.

Step 2. Next, we derive r-weighted estimates in the region $r \geq R$.

Proposition 10.1.2. *Let $R \gg m$ large enough. We have, for $\delta \leq p \leq 2-\delta$, $0 \leq s \leq k_L$,*

$$BEF^s_{p,\geq R}[\psi](\tau_1, \tau_2) \lesssim E^s_{p,\geq \frac{R}{2}}[\psi](\tau_1) + \mathcal{N}^s_{p,\geq R/2}[\psi, N] + Mor^s_{\frac{R}{2}\leq r\leq R}[\psi](\tau_1, \tau_2). \tag{10.1.4}$$

The proof of Proposition 10.1.2 is obtained in section 10.3.2.

Step 3. Combining the energy-Morawetz estimates (10.1.2) with the r-weighted estimates of Proposition 10.1.2, we infer, for $\delta \leq p \leq 2-\delta$, $0 \leq s \leq k_L$,

$$BEF^s_p[\psi](\tau_1, \tau_2) \lesssim E^s_p[\psi](\tau_1) + \mathcal{N}^s_p[\psi, N](\tau_1, \tau_2) + \left(\frac{|a|}{m} + \epsilon\right) BEF^s_\delta[\psi](\tau_1, \tau_2).$$

For a and ϵ small enough, may absorb the last term on the RHS from the LHS and obtain (10.1.1) which concludes the proof of Theorem 6.2.1.

10.2 Basic setup and control of $\nabla_3\psi$

10.2.1 Renormalization of the horizontal structure

As in sections 10.2-10.5 of [50] (see in particular formula (10.2.6) in [50]), it is convenient to work with the renormalized frame

$$e_4' = \lambda e_4, \qquad e_3' = \lambda^{-1} e_3, \qquad e_a' = e_a, \qquad \lambda := \frac{|q|^2}{\Delta}. \tag{10.2.1}$$

Note that for r large we have $\lambda = \Upsilon^{-1}\big(1 + O(a^2 r^{-2})\big)$, with $\Upsilon = (1 - \frac{2m}{r})$ as in Schwarzschild.

The corresponding renormalized quantities verify the following.

Lemma 10.2.1. *Let an unprimed frame and a primed frame related by* (10.2.1). *Using the ingoing normalization for the linearized quantities associated to the unprimed horizontal structure, and the outgoing normalization for the linearized quantities associated to the primed horizontal structure, see section 4.1.1 for the definition of the linearized quantities, we have*

$$\begin{aligned}
&\widecheck{trX}' = \lambda\widecheck{trX}, \quad \widehat{X}' = \lambda\widehat{X}, \quad \widecheck{tr\underline{X}}' = \lambda^{-1}\widecheck{tr\underline{X}}, \quad \widehat{\underline{X}}' = \lambda^{-1}\widehat{\underline{X}},\\
&\widecheck{H}' = \widecheck{H}, \quad \widecheck{\underline{H}}' = \widecheck{\underline{H}}, \quad \Xi' = \lambda^2\Xi, \quad \underline{\Xi}' = \lambda^{-2}\underline{\Xi},
\end{aligned}$$

$$\begin{aligned}
\widecheck{Z}' &= \widecheck{Z} - \frac{2a^2\cos\theta}{|q|^2}\widecheck{\mathcal{D}(\cos\theta)} - \left(\frac{2r}{|q|^2} - \frac{2r-2m}{\Delta}\right)\mathcal{D}(r),\\
\omega' &= \lambda\left(\widecheck{\omega} + \frac{1}{2}\lambda\partial_r\left(\frac{\Delta}{|q|^2}\right)\widecheck{e_4(r)} - \frac{a^2\cos\theta}{|q|^2}e_4(\cos\theta)\right),\\
\widecheck{\underline{\omega}}' &= \lambda^{-1}\left(\underline{\omega} - \frac{1}{2}\lambda\partial_r\left(\frac{\Delta}{|q|^2}\right)\widecheck{e_3(r)} + \frac{a^2\cos\theta}{|q|^2}e_3(\cos\theta)\right),
\end{aligned}$$

$$A' = \lambda^2 A, \quad B' = \lambda B, \quad \widecheck{P}' = \widecheck{P}, \quad \underline{B}' = \lambda^{-1}\underline{B}, \quad \underline{A}' = \lambda^{-2}\underline{A},$$

and

$$\begin{aligned}
&\widecheck{e_4'(r)} = \lambda\widecheck{e_4(r)}, \quad e_4'(\cos\theta) = \lambda e_4(\cos\theta), \quad \widecheck{\nabla_4'\mathfrak{J}} = \lambda\widecheck{\nabla_4\mathfrak{J}},\\
&\widecheck{e_3'(r)} = \lambda^{-1}\widecheck{e_3(r)}, \quad \widecheck{e_3'(\cos\theta)} = \lambda^{-1}\widecheck{e_3(\cos\theta)}, \quad \widecheck{\nabla_3'\mathfrak{J}} = \lambda^{-1}\widecheck{\nabla_3\mathfrak{J}},\\
&\mathcal{D}'(r) = \mathcal{D}(r), \quad \widecheck{\mathcal{D}'(\cos\theta)} = \widecheck{\mathcal{D}(\cos\theta)}, \quad \mathcal{D}'\widehat{\otimes}\mathfrak{J} = \mathcal{D}\widehat{\otimes}\mathfrak{J}, \quad \widecheck{\overline{\mathcal{D}'}\cdot\mathfrak{J}} = \widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}}.
\end{aligned}$$

Proof. Under the conformal transformation (10.2.1), the complexified Ricci coefficients transform as follows:

$$
\begin{aligned}
&\mathrm{tr}X' = \lambda \mathrm{tr}X, \quad \widehat{X}' = \lambda\widehat{X}, \quad \mathrm{tr}\underline{X}' = \lambda^{-1}\mathrm{tr}\underline{X}, \quad \widehat{\underline{X}}' = \lambda^{-1}\widehat{\underline{X}},\\
&Z' = Z - \mathcal{D}'(\log\lambda), \quad H' = H, \quad \underline{H}' = \underline{H}, \quad \Xi' = \lambda^2\Xi, \quad \underline{\Xi}' = \lambda^{-2}\underline{\Xi},\\
&\omega' = \lambda\left(\omega - \frac{1}{2}\lambda^{-1}e_4'(\log\lambda)\right), \quad \underline{\omega}' = \lambda^{-1}\left(\underline{\omega} + \frac{1}{2}\lambda e_3'(\log\lambda)\right).
\end{aligned}
$$

Also, the curvature components transform as follows

$$
A' = \lambda^2 A, \quad B' = \lambda B, \quad P' = P, \quad \underline{B}' = \lambda^{-1}\underline{B}, \quad \underline{A}' = \lambda^{-2}\underline{A}.
$$

Using the choice $\lambda = \frac{|q|^2}{\Delta}$ in (10.2.1), the ingoing normalization for the linearized quantities associated to the unprimed horizontal structure, and the outgoing normalization for the linearized quantities associated to the primed horizontal structure, we easily infer the stated identities for the linearized Ricci and curvature coefficients.

Also, we have

$$
\begin{aligned}
&e_4'(r) = \lambda e_4(r), \quad e_4'(\cos\theta) = \lambda e_4(\cos\theta), \quad \nabla_4'\mathfrak{J} = \lambda\nabla_4\mathfrak{J},\\
&e_3'(r) = \lambda^{-1}e_3(r), \quad e_3'(\cos\theta) = \lambda^{-1}e_3(\cos\theta), \quad \nabla_3'\mathfrak{J} = \lambda^{-1}\nabla_3\mathfrak{J},\\
&\mathcal{D}'(r) = \mathcal{D}(r), \quad \mathcal{D}'(\cos\theta) = \mathcal{D}(\cos\theta), \quad \mathcal{D}'\mathfrak{J} = \mathcal{D}\mathfrak{J},
\end{aligned}
$$

which immediately yields the remaining statements. This concludes the proof of Lemma 10.2.1. □

Lemma 10.2.2. *Assume that the primed horizontal structure is related to the unprimed one by the conformal transformation* (10.2.1). *Also, assume that* (Γ_b, Γ_g) *associated to the unprimed horizontal structure satisfies* (6.1.6). *Then,* (Γ_b', Γ_g') *associated to the primed horizontal structure satisfies in the region* $r \geq 4m_0$

$$
\begin{aligned}
&r^3|\mathfrak{d}^{\leq k}\xi'| + r^2|\mathfrak{d}^{\leq k}\Gamma_g'| + r|\mathfrak{d}^{\leq k}\Gamma_b'| \lesssim \epsilon, && k \leq k_L,\\
&r^3|\mathfrak{d}^{\leq k}\xi'| + r^2|\mathfrak{d}^{\leq k}\Gamma_g'| + r|\mathfrak{d}^{\leq k}\Gamma_b'| \lesssim \frac{\epsilon}{\tau_{trap}^{1+\delta_{dec}}}, && k \leq \frac{k_L}{2}.
\end{aligned}
\tag{10.2.2}
$$

Proof. The proof is immediate in view of Lemma 10.2.1 and the fact that $\lambda = \frac{|q|^2}{\Delta}$ is smooth in the region $r \geq 4m_0$. □

Remark 10.2.3. *In view of Lemma 10.2.2, in the remainder of the chapter, we make all the calculations in the renormalized frame and, since there is no danger of confusion, we* *drop the primes. In particular, in the frame we shall use throughout the chapter, we thus have*

$$e_4(r) = 1 + \Gamma_g, \quad e_3(r) = -\frac{\Delta}{|q|^2} + r\Gamma_b, \quad tr\,\chi = \frac{2r}{|q|^2} + \Gamma_g, \quad tr\,\underline{\chi} = -\frac{2r\Delta}{|q|^4} + \Gamma_g,$$
$$\omega \in \Gamma_g, \quad \underline{\omega} = \frac{1}{2}\partial_r\left(\frac{\Delta}{|q|^2}\right) + \Gamma_b, \quad Z = \frac{a\overline{q}}{|q|^2}\mathfrak{J} + \Gamma_g.$$

10.2.2 Boundaries and integral identities

Spacelike, asymptotically null hypersurface $\Sigma = \Sigma(\tau)$

In view of Definition 6.1.5, and normalizing the normal N_Σ such that $\mathbf{g}(N_\Sigma, e_3) = -2$, we have

$$\begin{aligned} N_\Sigma &= e_4 + \frac{1}{2}r^{-2}\lambda e_3 + Y^b e_b, \qquad |Y| = O(ar^{-1}), \\ g(N_\Sigma, N_\Sigma) &= -2r^{-2}\lambda + |Y|^2 \lesssim -\frac{m^2}{r^2}, \end{aligned} \tag{10.2.3}$$

with λ satisfying

$$2m^2 \lesssim \lambda \lesssim 2m^2, \qquad D\lambda = O(\epsilon + R^{-1}). \tag{10.2.4}$$

Also, define the vectorfield orthogonal to N_Σ,

$$\nu_\Sigma := e_4 - \frac{1}{2}r^{-2}\lambda e_3 \tag{10.2.5}$$

and note that ν_Σ is tangent to Σ.

Spacelike hypersurface Σ_*

Let N_* be the vectorfield normal to Σ_* of the form[1]

$$N_* \;=\; e_4 + Ue_3 + Y_*, \tag{10.2.6}$$

with U a scalar function and Y_* horizontal vectorfield verifying

$$|U| = 1 + O(\epsilon), \qquad |Y_*| = O(ar^{-1}).$$

Note also that the vectorfield

$$\nu_* = \nu_{\Sigma_*} = e_4 - Ue_3 \tag{10.2.7}$$

is perpendicular to N_* and thus tangent to Σ_*.

[1]See for comparison Lemma 10.44 of [50].

Basic pointwise identity of Proposition 4.7.3

According to Proposition 4.7.3, if $\psi \in \mathfrak{s}_2(\mathcal{M})$ is a solution of $\dot{\square}_2\psi - V\psi = N'$, with $N' := -\frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_T\psi + N$, and X be a vectorfield of the form $X = X^3e_3 + X^4e_4$, w a scalar, M a one form and $\mathcal{P}_\mu[X, w, M]$ the current defined by

$$\mathcal{P}_\mu[X, w, M] \;:=\; \mathcal{Q}_{\mu\nu}X^\nu + \frac{1}{2}w\psi \cdot \dot{\mathbf{D}}_\mu\psi - \frac{1}{4}|\psi|^2\partial_\mu w + \frac{1}{4}|\psi|^2 M_\mu, \tag{10.2.8}$$

then,

$$\begin{aligned}\mathbf{D}^\mu\mathcal{P}_\mu[X, w, M] &= \frac{1}{2}\mathcal{Q}\cdot {}^{(X)}\pi - \frac{1}{2}X(V)|\psi|^2 + \frac{1}{2}w\mathcal{L}[\psi] - \frac{1}{4}|\psi|^2\square_{\mathbf{g}}w + \frac{1}{4}\mathrm{Div}(|\psi|^2 M)\\ &+ \left(X(\psi) + \frac{1}{2}w\psi\right)\cdot N' - \left({}^*\rho + \underline{\eta}\wedge\eta\right)\nabla_{X^4e_4 - X^3e_3}\psi\cdot {}^*\psi \\ &- \frac{1}{2}\Im\Big(\mathrm{tr}\underline{X}HX^3 + \mathrm{tr}X\,\underline{H}X^4\Big)\cdot\nabla\psi\cdot {}^*\psi + r^{-2}\big(X^3\Gamma_b + X^4\Gamma_g\big)\mathfrak{d}\psi\cdot\psi.\end{aligned} \tag{10.2.9}$$

Recalling the definition of the expression $\mathcal{E}[X, w, M]$ introduced in (7.1.8), i.e.

$$\begin{aligned}\mathcal{E}[X, w, M] &= \frac{1}{2}\mathcal{Q}\cdot {}^{(X)}\pi - \frac{1}{2}X(V)|\psi|^2 + \frac{1}{2}w\mathcal{L}[\psi] - \frac{1}{4}|\psi|^2\square_{\mathbf{g}}w\\ &+ \frac{1}{4}\mathrm{Div}(|\psi|^2 M),\end{aligned} \tag{10.2.10}$$

we deduce, with $\mathcal{P}_\mu = \mathcal{P}_\mu[X, w, M]$, $\mathcal{E} = \mathcal{E}[X, w, M]$,

$$\begin{aligned}\mathbf{D}^\mu\mathcal{P}_\mu &= \mathcal{E} + \big(\nabla_X\psi + \frac{1}{2}w\psi\big)\cdot\big(\dot{\square}_k\psi - V\psi\big) + \left({}^*\rho + \underline{\eta}\wedge\eta\right)\nabla_{X^4e_4 - X^3e_3}\psi\cdot {}^*\psi\\ &+ \frac{1}{2}\Im\Big(\mathrm{tr}\underline{X}HX^3 + \mathrm{tr}X\,\underline{H}X^4\Big)\cdot\nabla\psi\cdot {}^*\psi + r^{-2}\big(X^3\Gamma_b + X^4\Gamma_g\big)\mathfrak{d}\psi\cdot\psi.\end{aligned} \tag{10.2.11}$$

10.2.3 Proof of Proposition 10.1.1

The proof of Proposition 10.1.1 follows easily by integration from the following lemma[2].

Lemma 10.2.4. *With the notation in* (10.2.10) *the following identity holds true in the region* $r \geq R$

$$\mathcal{E}[f_{-\delta}T, 0, 0] = \frac{1}{4}\delta r^{-1-\delta}|\nabla_3\psi|^2 - \frac{1}{4}\frac{\Delta^2}{|q|^4}\delta r^{-1-\delta}|\nabla_4\psi|^2 + O(\epsilon + R^{-1})r^{-1-\delta}\big(|D\psi|^2 + r^{-2}|\psi|^2\big).$$

[2]This is the precise analogue of Proposition 10.36 in [50]. Note that the identity for $\mathcal{E}[f_{-\delta}T, 0, 0]$ is used together with Proposition 4.7.2 which generalizes (10.2.9) to the case of vectorfields which are not spanned by (e_3, e_4).

Moreover

$$\mathcal{P}_\mu[f_{-\delta}T,0,0]\cdot e_4 = f_{-\delta}\mathcal{Q}(T,e_4)\geq 0, \qquad \mathcal{P}_\mu[f_{-\delta}T,0,0]\cdot e_3 = f_{-\delta}\mathcal{Q}(T,e_3)\geq 0.$$

Proof. The first part of the lemma can be derived by using the identity (10.2.10) with the vectorfield $X = f_{-\delta}\mathbf{T}$, $w = 0$, $M = 0$, with $f_{-\delta} = r^{-\delta}$ for $r \geq R$ and supported for $r \geq R/2$ with R sufficiently large. We refer the reader to the proof of Proposition 10.36 in [50]. □

10.3 r^p weighted estimates

10.3.1 Basic pointwise identities

In what follows we apply formulas (10.2.9), (10.2.10) with the choice

$$X = f(r)\left(e_4 + \frac{1}{2}r^{-2}\lambda e_3\right), \qquad w = \frac{2r}{|q|^2}f, \qquad M = \frac{2r}{|q|^2}f'e_4. \tag{10.3.1}$$

In view of (10.2.3), we have $X = f(N_\Sigma - Y^b e_b)$. We choose $f = f_p$ non-negative defined as $f_p = r^p$ for $r \geq R$ and $f_p = 0$ for $r \leq R/2$, where R is a fixed sufficiently large constant.

Remark 10.3.1. *In view of the intended choice $f = f_p$ we can write schematically*

$$f' = O(R^{-1})f, \qquad f'' = O(R^{-2})f, \qquad (r\partial_r)^{\leq 2}(f) = O(1+R^{-1})f.$$

We will use this to simplify various error terms in the identities that follow.

Based on the remark above we rewrite formula (10.2.11) for X as in (10.3.1) in the form

$$\begin{aligned}\mathbf{D}^\mu\mathcal{P}_\mu[X,w,M] &= \mathcal{E}[X,w,M] + \left(\nabla_X\psi + \frac{1}{2}w\psi\right)\cdot\left(\dot{\square}_k\psi - V\psi\right)\\ &+ \left(\epsilon + R^{-1}\right)r^{-1}f\Big(|\nabla_4\psi|^2 + |\nabla\psi|^2 + r^{-2}|\nabla_3\psi|^2 + r^{-2}|\psi|^2\Big).\end{aligned} \tag{10.3.2}$$

We write $X = {}^{(1)}X + {}^{(2)}X$ with ${}^{(1)}X = fe_4$ and ${}^{(2)}X = \frac{1}{2}r^{-2}\lambda f e_3$. The following lemma is similar to Lemma 10.40 in [50].

Lemma 10.3.2. *The following hold true.*

1. *The deformation tensor of the vectorfield* ${}^{(1)}X = f(r)e_4$ *is given by*

$$ {}^{(1)}\pi = \frac{2r}{|q|^2} f\mathbf{g} + \widetilde{{}^{(1)}\pi}, $$

with symmetric tensor $\widetilde{{}^{(1)}\pi}$ *which verifies*

$$ \begin{aligned} \widetilde{{}^{(1)}\pi}_{43} &= -2f' + \frac{4}{r}f + O(\epsilon + R^{-1})r^{-1}f, \\ \widetilde{{}^{(1)}\pi}_{33} &= 4f' + O(\epsilon + R^{-1})r^{-1}f, \\ \widetilde{{}^{(1)}\pi}_{44} &= 0, \quad \widetilde{{}^{(1)}\pi}_{3a} = O(\epsilon + R^{-1})r^{-1}f, \quad \widetilde{{}^{(1)}\pi}_{4a}, \ \widetilde{{}^{(1)}\pi}_{ab} = O(\epsilon)r^{-2}f. \end{aligned} \tag{10.3.3} $$

2. *The deformation tensor of the vectorfield* ${}^{(2)}X = \frac{1}{2}r^{-2}\lambda f e_3$ *is given by*

$$ \begin{aligned} &{}^{(2)}\pi_{33} = 0, \\ &{}^{(2)}\pi_{44} = O(\epsilon + R^{-1})r^{-2}f, \qquad {}^{(2)}\pi_{34} = O(\epsilon + R^{-1})r^{-2}f, \\ &{}^{(2)}\pi_{ab} = O(\epsilon + R^{-1})r^{-2}f, \qquad {}^{(2)}\pi_{3a} = O(\epsilon)r^{-3}f, \\ &{}^{(2)}\pi_{4a} = O(\epsilon + R^{-1})r^{-3}f. \end{aligned} \tag{10.3.4} $$

3. *For* $w = \frac{2r}{|q|^2}f$

$$ \square_{\mathbf{g}} w = \frac{2r}{|q|^2}f'' + O(\epsilon + R^{-1})r^{-2}f. \tag{10.3.5} $$

4. *For* $M = \frac{2r}{|q|^2}f'e_4$ *we have*

$$ \begin{aligned} Div\big(|\psi|^2 M\big) &= 4r^{-1}f'\nabla_4\psi \cdot \psi \\ &\quad + \left(\frac{2f'}{r^2} + \frac{2f''}{r} + O(\epsilon + R^{-1})r^{-3}f\right)|\psi|^2. \end{aligned} \tag{10.3.6} $$

Proof. Since ${}^{(X)}\pi_{\mu\nu} = \mathbf{g}(\mathbf{D}_\mu X, e_\nu) + \mathbf{g}(\mathbf{D}_\nu X, e_\mu)$ we deduce

$$ \begin{aligned} &{}^{(1)}\pi_{44} = 0, \qquad {}^{(1)}\pi_{43} = (e_4 f)\mathbf{g}_{34} + 4f\omega, \qquad {}^{(1)}\pi_{33} = -8f\underline{\omega} - 4e_3 f, \\ &{}^{(1)}\pi_{4a} = 2f\xi_a, \qquad {}^{(1)}\pi_{3a} = 2f(\eta + \zeta)_a, \qquad {}^{(1)}\pi_{ab} = 2f\left(\widehat{\chi}_{ab} + \frac{1}{2}\frac{2r}{|q|^2}\mathbf{g}_{ab}\right), \end{aligned} $$

from which (10.3.3) easily follows.

Similarly

$$\begin{aligned} {}^{(2)}\pi_{33} &= 0, & {}^{(2)}\pi_{44} = -4r^{-2}\lambda f\omega - 2e_4(r^{-2}\lambda f)\\ {}^{(2)}\pi_{ab} &= \frac{1}{2}r^{-2}\lambda f\underline{\chi}_{ab}, & {}^{(2)}\pi_{34} = -2r^{-2}\lambda f\underline{\omega} - 2e_3(r^{-2}\lambda f),\\ {}^{(2)}\pi_{3a} &= r^{-2}f\lambda\underline{\xi}_a, & {}^{(2)}\pi_{4a} = r^{-2}f\lambda(\underline{\eta}_a - \zeta_a), \end{aligned}$$

from which, since $\lambda = (O(1)$ and $D\lambda = O(\epsilon + R^{-1})$, (10.3.4) follows.

Using formula (7.1.15) for $H = w = \frac{2r}{|q|^2}f$,

$$\begin{aligned} \square_{\mathbf{g}} H &= \frac{1}{\sqrt{|\mathbf{g}|}}\partial_\alpha\big(\sqrt{|\mathbf{g}|}\mathbf{g}^{\alpha\beta}\partial_\beta\big)H = \frac{1}{|q|^2}\partial_r\big(\Delta\partial_r H\big) + \not{\mathfrak{d}}^{\leq 1}(\Gamma_g\cdot\mathfrak{d}^{\leq 1}H)\\ &= \frac{\Delta}{|q|^2}H'' + \frac{2(r-m)}{|q|^2}H' + r^{-2}\Big((r\partial_r)^{\leq 2}f\cdot\Gamma_b + (r\partial_r)^{\leq 1}f\cdot\mathfrak{d}^{\leq 1}\Gamma_b\Big)\\ &= \frac{\Delta}{|q|^2}H'' + \frac{2(r-m)}{|q|^2}H' + O(\epsilon, R^{-1})r^{-3}f. \end{aligned}$$

Hence, modulo error terms of the form $O(\epsilon, R^{-1})r^{-3}f$,

$$\begin{aligned} \square_{\mathbf{g}} w &= \frac{\Delta}{|q|^2}\left(\frac{2r}{|q|^2}f\right)'' + \frac{2(r-m)}{|q|^2}\left(\frac{2r}{|q|^2}f\right)'\\ &= \frac{\Delta}{|q|^2}\left(\frac{2r}{|q|^2}f'' + 2\left(\frac{2r}{|q|^2}\right)'f' + \left(\frac{2r}{|q|^2}\right)''f\right) + \frac{2(r-m)}{|q|^2}\left(\frac{2r}{|q|^2}f' + \left(\frac{2r}{|q|^2}\right)'f\right). \end{aligned}$$

Since $\partial_r(\frac{2r}{|q|^2}) = \frac{-2r^2+2a^2\cos^2\theta}{|q|^4} = -\frac{2}{|q|^2} + \frac{4a^2\cos^2\theta}{|q|^4}$,

$$\begin{aligned} \square_{\mathbf{g}} w &= \frac{\Delta}{|q|^2}\left(\frac{2r}{|q|^2}f'' + \left(-\frac{4}{|q|^2} + \frac{8a^2\cos^2\theta}{|q|^4}\right)f' + \left(\frac{4r}{|q|^4} - \frac{16a^2\cos^2\theta r}{|q|^6}\right)f\right)\\ &\quad + \frac{2(r-m)}{|q|^2}\left(\frac{2r}{|q|^2}f' + \left(-\frac{2}{|q|^2} + \frac{4a^2\cos^2\theta}{|q|^4}\right)f\right) + O(R^{-1})r^{-3}f\\ &= \frac{2r\Delta}{|q|^4}f'' - \frac{4(\Delta - r^2 + mr)}{|q|^4}f' + \frac{4(\Delta - r^2 + mr)r}{|q|^6}f + O\left(\frac{a^2}{r^5}\right)(r\partial_r)^{\leq 1}f\\ &\quad + O(R^{-1})r^{-3}f\\ &= \frac{2r}{|q|^2}f'' + O(R^{-1})r^{-3}f. \end{aligned}$$

Thus $\square_{\mathbf{g}} w = \frac{2r}{|q|^2}f'' + O(\epsilon + R^{-1})r^{-3}f$ as stated.

To prove the last part of the lemma we write

$$\begin{aligned} \mathrm{Div}(|\psi|^2 M) &= 4r^{-1}f'\nabla_4\psi\cdot\psi + \mathrm{Div}M|\psi|^2,\\ \mathrm{Div}M &= 2\mathrm{Div}(r^{-1}f'e_4) = \frac{2f'}{r^2} + \frac{2f''}{r} + O(\epsilon + R^{-1})r^{-3}f, \end{aligned}$$

as stated. □

We are now ready to prove the following.

Proposition 10.3.3. *The following hold true:*

1. *We have, for* $\mathcal{E} = \mathcal{E}[X, w, M]$, $X = f(e_4 + \frac{1}{2}r^{-2}\lambda e_3)$ *and* $w = \frac{2r}{|q|^2}f$,

$$\begin{aligned}\mathcal{E} &= f'|\nabla_4\psi|^2 + \frac{1}{2}\left(-f' + \frac{2}{r}f\right)\left(|\nabla\psi|^2 + V|\psi|^2\right) - \frac{1}{2}\frac{r}{|q|^2}f''|\psi|^2 + \frac{1}{4}Div(|\psi|^2 M)\\ &+ O(\epsilon + R^{-1})r^{-1}f\Big(|\nabla_4\psi|^2 + |\nabla\psi|^2 + r^{-2}|\psi|^2 + r^{-2}|\nabla_3\psi|^2\Big).\end{aligned} \tag{10.3.7}$$

2. *If in addition we choose* $M = 2\frac{r}{|q|^2}f'e_4$ *we deduce, with* $\check{\nabla}_4\psi = \nabla_4\psi + r^{-1}\psi$,

$$\begin{aligned}\mathcal{E} &= \frac{1}{2}f'|\check{\nabla}_4\psi|^2 + \frac{1}{2}\left(-f' + \frac{2}{r}f\right)\left(|\nabla\psi|^2 + V|\psi|^2\right)\\ &+ O(\epsilon + R^{-1})r^{-1}f\Big(|\nabla_4\psi|^2 + |\nabla\psi|^2 + r^{-2}|\psi|^2 + r^{-2}|\nabla_3\psi|^2\Big).\end{aligned} \tag{10.3.8}$$

Proof. We calculate the expression

$$\begin{aligned}\mathcal{E} &= \frac{1}{2}\mathcal{Q}\cdot {}^{(X)}\pi + \frac{1}{2}w\mathcal{L}[\psi] - \frac{1}{4}|\psi|^2\square_{\mathbf{g}}w - \frac{1}{2}X(V)|\psi|^2 + \frac{1}{4}\mathrm{Div}\big(|\psi|^2 M\big)\\ &= \mathcal{E}' + \frac{1}{4}\mathrm{Div}\big(|\psi|^2 M\big)\end{aligned}$$

with X, w as in (10.3.1). Recalling that

$$\mathcal{Q}_{\mu\nu} = \dot{\mathbf{D}}_\mu\psi\cdot\dot{\mathbf{D}}_\nu\psi - \frac{1}{2}\mathbf{g}_{\mu\nu}\mathcal{L}, \qquad \mathcal{L} = \dot{\mathbf{D}}^\mu\psi\cdot\dot{\mathbf{D}}_\mu\psi + V|\psi|^2,$$

we calculate, using Lemma 10.3.2,

$$\begin{aligned}
\mathcal{E}' &= \frac{1}{2}\mathcal{Q}\cdot {}^{(X)}\pi + \frac{1}{2}w\mathcal{L}[\psi] - \frac{1}{2}X(V)|\psi|^2 - \frac{1}{4}|\psi|^2\square_{\mathbf{g}}w\\
&= \frac{1}{2}\mathcal{Q}\cdot {}^{(1)}\pi + \frac{1}{2}\mathcal{Q}\cdot {}^{(2)}\pi + \frac{1}{2}w\mathcal{L}[\psi] - \frac{1}{2}X(V)|\psi|^2 - \frac{1}{4}|\psi|^2\square_{\mathbf{g}}w\\
&= \frac{1}{2}\mathcal{Q}\cdot\left(\frac{2r}{|q|^2}f\mathbf{g} + \widetilde{{}^{(1)}\pi}\right) + \frac{1}{2}\frac{2r}{|q|^2}f\dot{\mathbf{D}}^\mu\psi\cdot\dot{\mathbf{D}}_\mu\psi + \frac{1}{2}\frac{2r}{|q|^2}fV|\psi|^2 - \frac{1}{2}X(V)|\psi|^2\\
&\quad +\frac{1}{2}\mathcal{Q}\cdot {}^{(2)}\pi - \frac{1}{4}|\psi|^2\square_{\mathbf{g}}w\\
&= -\frac{2r}{|q|^2}f\mathcal{L}[\psi] + \frac{2r}{|q|^2}f\dot{\mathbf{D}}^\mu\psi\cdot\dot{\mathbf{D}}_\mu\psi + \frac{1}{2}\frac{2r}{|q|^2}fV|\psi|^2 - \frac{1}{2}X(V)|\psi|^2\\
&\quad +\frac{1}{2}\mathcal{Q}\cdot\widetilde{{}^{(1)}\pi} + \frac{1}{2}\mathcal{Q}\cdot {}^{(2)}\pi - \frac{1}{4}|\psi|^2\square_{\mathbf{g}}w\\
&= -\left(\frac{r}{|q|^2}fV + \frac{1}{2}X(V)\right)|\psi|^2 + \frac{1}{2}\mathcal{Q}\cdot\widetilde{{}^{(1)}\pi} + \frac{1}{2}\mathcal{Q}\cdot {}^{(2)}\pi\\
&\quad -\frac{1}{4}|\psi|^2\left(\frac{2r}{|q|^2}f'' + O(\epsilon + R^{-1})r^{-3}f\right).
\end{aligned}$$

We deduce

$$\begin{aligned}
\mathcal{E}' &= \frac{1}{2}\mathcal{Q}\cdot\widetilde{{}^{(1)}\pi} + \frac{1}{2}\mathcal{Q}\cdot {}^{(2)}\pi - \frac{1}{2}\left(\frac{2r}{|q|^2}fV + X(V)\right)|\psi|^2\\
&\quad -\frac{1}{4}|\psi|^2\left(\frac{2r}{|q|^2}f'' + O(\epsilon + R^{-1})r^{-3}f\right).
\end{aligned}$$

Recall that

$$\mathcal{Q}_{33} = |\nabla_3\psi|^2, \qquad \mathcal{Q}_{44} = |\nabla_4\psi|^2, \qquad \mathcal{Q}_{34} = |\nabla\psi|^2 + V|\psi|^2, \qquad \mathcal{Q}_{4a} = \nabla_4\psi\cdot\nabla_a\psi.$$

Therefore in view of Lemma 10.3.2 we have

$$\begin{aligned}
\mathcal{Q}\cdot\widetilde{{}^{(1)}\pi} &= f'|\nabla_4\psi|^2 + \left(-f' + \frac{2}{r}f\right)\left(|\nabla\psi|^2 + V|\psi|^2\right)\\
&\quad +O(\epsilon + R^{-1})r^{-1}f\Big(|\nabla_4\psi|^2 + |\nabla\psi|^2 + V|\psi|^2\Big)\\
&\quad O(\epsilon + R^{-1})r^{-1}f\mathcal{Q}_{4a} + O(\epsilon + R^{-1})r^{-2}f\mathcal{Q}_{3a} + O(\epsilon)r^{-2}f\mathcal{Q}_{ab}.
\end{aligned}$$

Since

$$|\mathcal{Q}_{3a}| \le |\nabla_3\psi||\nabla\psi|, \quad |\mathcal{Q}_{ab}| \le |\nabla_3\psi||\nabla_4\psi| + |\nabla\psi|^2 + |V||\psi|^2, \quad |\mathcal{Q}_{4a}| \le |\nabla_4\psi||\nabla\psi|,$$

we easily deduce

$$\begin{aligned}\mathcal{Q}\cdot \widetilde{^{(1)}\pi} &= f'|\nabla_4\psi|^2+\left(-f'+\frac{2}{r}f\right)\left(|\nabla\psi|^2+V|\psi|^2\right)\\ &\quad +O(\epsilon+R^{-1})r^{-1}f\Big(|\nabla_4\psi|^2+|\nabla\psi|^2+r^{-2}|\psi|^2+r^{-2}|\nabla_3\psi|^2\Big).\end{aligned}$$

Similarly

$$\mathcal{Q}\cdot {}^{(2)}\pi = O(\epsilon+R^{-1})r^{-1}f\Big(|\nabla_4\psi|^2+|\nabla\psi|^2+r^{-2}|\psi|^2+r^{-2}|\nabla_3\psi|^2\Big).$$

Recall also that $V=\frac{4\Delta}{(r^2+a^2)|q|^2}$. Therefore, since [3]

$$\frac{1}{2}e_4(V)+\frac{f}{r}V=O(mr^{-4})+O(a^2r^{-6}),\qquad \frac{r}{|q|^2}fV=\frac{1}{r}V+O(ar^{-4}),$$

we deduce

$$\frac{r}{|q|^2}fV+\frac{1}{2}X(V)=\frac{r}{|q|^2}fV+\frac{1}{2}f\left(e_4(V)+\frac{1}{2}r^{-2}\lambda e_3(V)\right)=O(R^{-1})r^{-3}f.$$

Consequently

$$\begin{aligned}\mathcal{E}' &= \frac{1}{2}f'|\nabla_4\psi|^2+\frac{1}{2}\left(-f'+\frac{2}{r}f\right)\left(|\nabla\psi|^2+V|\psi|^2\right)-\frac{1}{2}\frac{r}{|q|^2}f''|\psi|^2\\ &\quad +O(\epsilon+R^{-1})r^{-1}f\Big(|\nabla_4\psi|^2+|\nabla\psi|^2+r^{-2}|\psi|^2+r^{-2}|\nabla_3\psi|^2\Big),\end{aligned}$$

and thus

$$\begin{aligned}\mathcal{E} &= \mathcal{E}'+\frac{1}{4}\mathrm{Div}(|\psi|^2M)\\ &= \frac{1}{2}f'|\nabla_4\psi|^2+\frac{1}{2}\left(-f'+\frac{2}{r}f\right)\left(|\nabla\psi|^2+V|\psi|^2\right)-\frac{1}{2}\frac{r}{|q|^2}f''|\psi|^2+\frac{1}{4}\mathrm{Div}(|\psi|^2M)\\ &\quad +O(\epsilon+R^{-1})r^{-1}f\Big(|\nabla_4\psi|^2+|\nabla\psi|^2+r^{-2}|\psi|^2+r^{-2}|\nabla_3\psi|^2\Big)\end{aligned}$$

as stated.

To derive the second part of the Proposition we choose $M=2\frac{r}{|q|^2}f'e_4$ and make use of the formula (10.3.6)

$$\mathrm{Div}(|\psi|^2M) = 4r^{-1}f'\nabla_4\psi\cdot\psi+\left(\frac{2f'}{r^2}+\frac{2f''}{r}+O(\epsilon+R^{-1})r^{-3}f\right)|\psi|^2.$$

[3] See for example the proof of Proposition 10.2.5 in [50].

We deduce

$$\begin{aligned}\mathcal{E} &= \frac{1}{2}f'|\nabla_4\psi|^2+\frac{1}{2}\left(-f'+\frac{2}{r}f\right)\left(|\nabla\psi|^2+V|\psi|^2\right)-\frac{1}{2}\frac{r}{|q|^2}f''|\psi|^2\\ &+\left(r^{-1}f'\nabla_4\psi\cdot\psi+\frac{1}{2}\left(\frac{f'}{r^2}+\frac{f''}{r}\right)|\psi|^2\right)\\ &+O(\epsilon+R^{-1})r^{-1}f\Big(|\nabla_4\psi|^2+|\nabla\psi|^2+r^{-2}|\psi|^2+r^{-2}|\nabla_3\psi|^2\Big).\end{aligned}$$

Consider the term

$$\begin{aligned}J &= \frac{1}{2}f'|\nabla_4\psi|^2-\frac{1}{2}\frac{r}{|q|^2}f''|\psi|^2+r^{-1}f'\nabla_4\psi\cdot\psi+\frac{1}{4}\left(\frac{2f'}{r^2}+\frac{2f''}{r}+O(\epsilon+R^{-1})r^{-3}f\right)|\psi|^2\\ &= \frac{1}{2}f'\Big(|\nabla_4\psi|^2+2r^{-1}\nabla_4\psi\cdot\psi+r^{-2}|\psi|^2\Big)+O(\epsilon+R^{-1})r^{-3}f|\psi|^2\\ &= \frac{1}{2}f'\big|\nabla_4\psi+r^{-1}\psi\big|^2+O(\epsilon+R^{-1})r^{-3}f|\psi|^2.\end{aligned}$$

We thus infer that, recalling that $\check{\nabla}_4\psi=\nabla_4\psi+r^{-1}\psi$,

$$\begin{aligned}\mathcal{E} &= \frac{1}{2}f'|\check{\nabla}_4\psi|^2+\frac{1}{2}\left(-f'+\frac{2}{r}f\right)\left(|\nabla\psi|^2+V|\psi|^2\right)\\ &+O(\epsilon+R^{-1})r^{-1}f\Big(|\nabla_4\psi|^2+|\nabla\psi|^2+r^{-2}|\psi|^2+r^{-2}|\nabla_3\psi|^2\Big)\end{aligned}$$

as stated. □

Boundary terms

When we integrate formula (10.3.2) we get, in addition to the bulk terms expressed in Proposition 10.3.3, boundary terms on $\Sigma_{\geq R/2}(\tau)\cup\Sigma_*$. In what follows we deal with these boundary terms.

Lemma 10.3.4. *Given $\mathcal{P}=\mathcal{P}[X,w,M]$ as in (10.2.8) with (X,w,M) as in (10.3.1), we have*

$$\begin{aligned}\mathcal{P}\cdot e_4 &= f\big|\check{\nabla}_4\psi\big|^2-\frac{1}{2}r^{-2}\nabla_4(rf|\psi|^2)+\frac{1}{2}r^{-2}\lambda f\mathcal{Q}_{34}+O(\epsilon+R^{-1})r^{-3}f|\psi|^2,\\ \mathcal{P}\cdot e_3 &= f\mathcal{Q}_{34}+f\frac{1}{2}r^{-2}\lambda\mathcal{Q}_{33}+\frac{1}{2}r^{-2}\nabla_3\big(rf|\psi|^2\big)+(\epsilon+R^{-1})fr^{-3}|\psi|^2.\end{aligned}\tag{10.3.9}$$

Also,

$$\begin{aligned}\big|\mathcal{P}\cdot Y\big| &\lesssim f\big|\check{\nabla}_4\psi\big||Y||\nabla\psi|+O(R^{-\delta})r^{-2}|\nabla\psi|^2+r^{-4+\delta}|\nabla_3\psi|^2\\ &+O(\epsilon+R^{-1})r^{-3}f|\psi|^2.\end{aligned}\tag{10.3.10}$$

Proof. We write, since $X = f(e_4 + \frac{1}{2}r^{-2}\lambda e_3)$

$$\begin{aligned}
\mathcal{P}\cdot e_4 &= \left(\mathcal{Q}_{\mu\nu}X^\nu + \frac{1}{2}w\psi\cdot\dot{\mathbf{D}}_\mu\psi - \frac{1}{4}|\psi|^2\partial_\mu w + \frac{1}{4}|\psi|^2 M_\mu\right)e_4^\mu\\
&= \mathcal{Q}(X, e_4) + f\frac{r}{|q|^2}\psi\cdot\nabla_4\psi - \frac{1}{2}e_4\left(f\frac{r}{|q|^2}\right)|\psi|^2\\
&= f\mathcal{Q}(e_4, e_4) + \frac{1}{2}r^{-2}f\lambda\mathcal{Q}(e_4, e_3) + f\frac{r}{|q|^2}\psi\cdot\nabla_4\psi - \frac{1}{2}e_4\left(f\frac{r}{|q|^2}\right)|\psi|^2\\
&= f\big|\nabla_4\psi|^2 + fr^{-1}\psi\cdot\nabla_4\psi - \frac{1}{2}e_4(fr^{-1})|\psi|^2 + \frac{1}{2}r^{-2}f\lambda\mathcal{Q}_{34} + O(\epsilon + R^{-1})r^{-3}f|\psi|^2.
\end{aligned}$$

We rewrite the expression

$$I = f\big|\nabla_4\psi|^2 + fr^{-1}\psi\cdot\nabla_4\psi - \frac{1}{2}e_4(fr^{-1})|\psi|^2$$

in the form

$$\begin{aligned}
I &= f\left(|\nabla_4\psi|^2 + \frac{1}{r}\psi\cdot\nabla_4\psi\right) - \frac{1}{2}e_4(r^{-1}f)|\psi|^2\\
&= f\left|\nabla_4\psi + \frac{1}{r}\psi\right|^2 - \frac{1}{r}f\psi\cdot\nabla_4\psi - r^{-2}f|\psi|^2 - \frac{1}{2}e_4(r^{-1}f)|\psi|^2\\
&= f\left|\check{\nabla}_4\psi\right|^2 - \frac{1}{2}r^{-2}\nabla_4(rf|\psi|^2) + \frac{1}{2}r^{-2}e_4(rf)|\psi|^2 - r^{-2}f|\psi|^2 - \frac{1}{2}e_4(r^{-1}f)|\psi|^2\\
&= f\left|\check{\nabla}_4\psi\right|^2 - \frac{1}{2}r^{-2}\nabla_4(rf|\psi|^2) + r^{-2}(e_4(r) - 1)f|\psi|^2.
\end{aligned}$$

Hence

$$\mathcal{P}\cdot e_4 = f\big|\check{\nabla}_4\psi\big|^2 - \frac{1}{2}r^{-2}\nabla_4(rf|\psi|^2) + \frac{1}{2}r^{-2}f\lambda\mathcal{Q}_{34} + O(\epsilon + R^{-1})r^{-3}f|\psi|^2$$

as stated.

Also, since $M = 2\frac{r}{|q|^2}f'e_4$,

$$\begin{aligned}
\mathcal{P}\cdot e_3 &= \left(\mathcal{Q}_{\mu\nu}X^\nu + \frac{1}{2}w\psi\cdot\dot{\mathbf{D}}_\mu\psi - \frac{1}{4}|\psi|^2\partial_\mu w + \frac{1}{4}|\psi|^2 M_\mu\right)e_3^\mu\\
&= \mathcal{Q}(X, e_3) + f\frac{r}{|q|^2}\psi\cdot\nabla_3\psi - \frac{1}{2}e_3\left(f\frac{r}{|q|^2}\right)|\psi|^2 - \frac{r}{|q|^2}f'|\psi|^2\\
&= f\mathcal{Q}_{34} + \frac{1}{2}r^{-2}f\lambda\mathcal{Q}_{33} + \frac{1}{2}f\frac{r}{|q|^2}\nabla_3(|\psi|^2) - \frac{1}{2}e_3\left(f\frac{r}{|q|^2}\right)|\psi|^2 - \frac{r}{|q|^2}f'|\psi|^2\\
&= J + f\mathcal{Q}_{34} + \frac{1}{2}r^{-2}f\lambda\mathcal{Q}_{33} + O(\epsilon + R^{-1})r^{-2}|\psi|^2
\end{aligned}$$

with

$$J = \frac{1}{2}r^{-1}f\nabla_3(|\psi|^2) - \frac{1}{2}e_3(r^{-1}f)|\psi|^2 - \frac{1}{r}f'|\psi|^2.$$

We rewrite J in the form

$$\begin{aligned}
2J &= r^{-2}\nabla_3\big(rf|\psi|^2\big) - r^{-2}e_3(rf)|\psi|^2 - e_3(r^{-1}f)|\psi|^2 - 2r^{-1}f'|\psi|^2\\
&= r^{-2}\nabla_3\big(rf|\psi|^2\big) - \Big(2r^{-1}e_3(f) + r^{-2}fe_3(r) + e_3(r^{-1})f + 2r^{-1}f'\Big)|\psi|^2\\
&= r^{-2}\nabla_3\big(rf|\psi|^2\big) - \Big(2r^{-1}e_3(r)f' + r^{-2}fe_3(r) - r^{-2}e_3(r)f + 2r^{-1}f'\Big)|\psi|^2\\
&= r^{-2}\nabla_3\big(rf|\psi|^2\big) - 2f'\Big(r^{-1}e_3(r) + r^{-1}\Big)|\psi|^2.
\end{aligned}$$

Since $e_3(r) = -\frac{\Delta}{|q|^2} + r\Gamma_b = -1 + (\epsilon + R^{-1})$ we deduce

$$2J = r^{-2}\nabla_3\big(rf|\psi|^2\big) + (\epsilon + R^{-1})f'r^{-2}|\psi|^2.$$

Therefore,

$$\mathcal{P}\cdot e_3 = f\mathcal{Q}_{34} + \frac{1}{2}r^{-2}f\lambda\mathcal{Q}_{33} + \frac{1}{2}r^{-2}\nabla_3\big(rf|\psi|^2\big) + (\epsilon + R^{-1})fr^{-3}|\psi|^2$$

as stated.

Finally,

$$\begin{aligned}
\mathcal{P}\cdot Y &= \left(\mathcal{Q}_{\mu\nu}X^\nu + \frac{1}{2}w\psi\cdot\dot{\mathbf{D}}_\mu\psi - \frac{1}{4}|\psi|^2\partial_\mu w + \frac{1}{4}|\psi|^2 M_\mu\right)Y^\mu\\
&= \mathcal{Q}(X,Y) + \frac{r}{|q|^2}f\psi\cdot\nabla_Y\psi - |\psi|^2 Y\left(\frac{r}{2|q|^2}f\right)\\
&= f\mathcal{Q}(e_4,Y) + \frac{1}{2}r^{-2}\lambda f\mathcal{Q}(e_3,Y) + \frac{r}{|q|^2}f\psi\cdot\nabla_Y\psi - |\psi|^2 Y\left(\frac{r}{2|q|^2}f\right)\\
&= f\nabla_4\psi\cdot\nabla_Y\psi + \frac{1}{2}r^{-2}f\lambda\nabla_3\psi\cdot\nabla_Y\psi + \frac{r}{|q|^2}f\psi\cdot\nabla_Y\psi - |\psi|^2 Y\left(\frac{r}{2|q|^2}f\right)\\
&= f\big(\check{\nabla}_4\psi - r^{-1}\psi\big)\cdot\nabla_Y\psi + \frac{1}{2}r^{-2}f\lambda\nabla_3\psi\cdot\nabla_Y\psi + \frac{r}{|q|^2}f\psi\cdot\nabla_Y\psi - |\psi|^2 Y\left(\frac{r}{2|q|^2}f\right).
\end{aligned}$$

We deduce

$$\mathcal{P}\cdot Y = f\check{\nabla}_4\psi\cdot\nabla_Y\psi + \left(\frac{r}{|q|^2} - \frac{1}{r}\right)f\psi\cdot\nabla_Y\psi + \frac{1}{2}r^{-2}f\lambda\nabla_3\psi\cdot\nabla_Y\psi - |\psi|^2 Y\left(\frac{r}{2|q|^2}f\right)$$

and hence

$$\begin{aligned}
\big|\mathcal{P}\cdot Y\big| &\lesssim f\big|\check{\nabla}_4\psi\big||Y||\nabla\psi| + O(r^{-4})f|\psi||\nabla\psi| + O(R^{-\delta/2})|Y||\nabla\psi|r^{-2+\delta/2}|\nabla_3\psi| \\
&\quad +O(\epsilon+R^{-1})r^{-3}f|\psi|^2\\
&\lesssim f\big|\check{\nabla}_4\psi\big||Y||\nabla\psi| + O(R^{-\delta})r^{-2}|\nabla\psi|^2 + r^{-4+\delta}|\nabla_3\psi|^2 + O(\epsilon+R^{-1})r^{-3}f|\psi|^2
\end{aligned}$$

as stated. This concludes the proof of Lemma 10.3.4. □

Proposition 10.3.5. *The following bounds hold true, for $r \geq R$ sufficiently large:*

1. *On $\Sigma = \Sigma(\tau)$, for $0 \leq p \leq 2-\delta$,*

$$\begin{aligned}
\mathcal{P}\cdot N_\Sigma \gtrsim\, & f\left|\check{\nabla}_4\psi\right|^2 + r^{-2}f|\nabla\psi|^2 - \frac{1}{2}div_{\,\Sigma}\big(r^{-1}f|\psi|^2\nu_\Sigma\big)\\
&- O(\epsilon+R^{-1})r^{-3}f|\psi|^2 - r^{-4+\delta}f\Big(|\nabla_3\psi|^2 + |\nabla\psi|^2 + r^{-2}|\psi|^2\Big).
\end{aligned} \tag{10.3.11}$$

2. *On Σ_*, with[4] $N_* = e_4 + Ue_3 + Y_*$, $\nu_* = e_4 - Ue_3$, for $0 \leq p \leq 2-\delta$,*

$$\begin{aligned}
\mathcal{P}\cdot N_* \gtrsim\, & f\big(|\check{\nabla}_4\psi|^2 + |\nabla\psi|^2 + r^{-2}|\nabla_3\psi|^2\big) - \frac{1}{2}div_{\,\Sigma_*}\big(r^{-1}f|\psi|^2\nu_*\big)\\
&- O(R^{-1}+\epsilon)r^{-3}f|\psi|^2.
\end{aligned} \tag{10.3.12}$$

3. *On Σ we have, with $f = r^p$ and for $0 \leq p \leq 1-\delta$,*

$$\begin{aligned}
\mathcal{P}\cdot N_\Sigma \geq\, & \frac{\delta^2}{8}r^{p-2}|\psi|^2 - \frac{p}{2}r^{-2}\nu_\Sigma(r^{p+1}|\psi|^2) + \frac{m}{r^2}r^p\mathcal{Q}_{34}\\
&- O(r^{p-3})\Big(|\nabla_3\psi|^2 + |\psi|^2\Big).
\end{aligned} \tag{10.3.13}$$

4. *On Σ_* we have, with $f = r^p$ and for $0 \leq p \leq 1-\delta$,*

$$\begin{aligned}
\mathcal{P}\cdot N_* \geq\, & \frac{\delta^2}{8}r^{p-2}|\psi|^2 - \frac{p}{2}r^{-2}\nu_*(r^{p+1}|\psi|^2) + r^p|\nabla\psi|^2\\
&- O(r^{p-3})\Big(|\nabla_3\psi|^2 + |\psi|^2\Big).
\end{aligned} \tag{10.3.14}$$

Proof. We calculate using the definition $N_\Sigma = e_4 + \frac{1}{2}r^{-2}\lambda e_3 + Y^b e_b + O(r^{-3})e_3$, see (10.2.3),

[4] Recall (10.2.6), (10.2.7).

and the lemma above

$$\begin{aligned}
&\mathcal{P}\cdot N_\Sigma - \mathcal{P}\cdot Y = \mathcal{P}\cdot\left(e_4+\frac{1}{2}r^{-2}\lambda e_3 + O(r^{-3})e_3\right)\\
=\;& f\left|\check{\nabla}_4\psi\right|^2 - \frac{1}{2}r^{-2}\nabla_4(rf|\psi|^2) + \frac{1}{2}r^{-2}\lambda f\mathcal{Q}_{34} + O(\epsilon+R^{-1})r^{-3}f|\psi|^2\\
&+\frac{1}{2}r^{-2}\lambda\left(f\mathcal{Q}_{34} + f\frac{1}{2}r^{-2}\lambda\mathcal{Q}_{33} + \frac{1}{2}r^{-2}\nabla_3\big(rf|\psi|^2\big) + (\epsilon+R^{-1})fr^{-3}|\psi|^2\right)\\
=\;& f\left|\check{\nabla}_4\psi\right|^2 + \frac{1}{2}r^{-2}\lambda f\mathcal{Q}_{34} + f\frac{1}{4}r^{-4}\lambda|\nabla_3\psi|^2 - \frac{1}{2}r^{-2}\left(\nabla_4 - \frac{1}{2}r^{-2}\lambda\nabla_3\right)(rf|\psi|^2)\\
&+O(\epsilon+R^{-1})r^{-3}f|\psi|^2\\
=\;& f\left|\check{\nabla}_4\psi\right|^2 + \frac{1}{2}r^{-2}\lambda f|\nabla\psi|^2 + f\frac{1}{4}r^{-4}\lambda|\nabla_3\psi|^2 - \frac{1}{2}r^{-2}\nu_\Sigma(rf|\psi|^2) + O(\epsilon+R^{-1})r^{-3}f|\psi|^2,
\end{aligned}$$

where we recall the definition of $\nu_\Sigma = e_4 - \frac{1}{2}r^{-2}\lambda e_3$, see (10.2.5).

We now write

$$\begin{aligned}
\mathrm{d}iv_{\,\Sigma}\big(r^{-1}f|\psi|^2\nu_\Sigma\big) &= \nu_\Sigma\big(r^{-1}f|\psi|^2\big) + r^{-1}f|\psi|^2\mathrm{d}iv_{\,\Sigma}(\nu_\Sigma)\\
&= r^{-2}\nu_\Sigma\big(rf|\psi|^2\big) + \nu_\Sigma(r^{-2})rf|\psi|^2 + r^{-1}f|\psi|^2\mathrm{d}iv_{\,\Sigma}(\nu_\Sigma)\\
&= r^{-2}\nu_\Sigma\big(rf|\psi|^2\big) - 2r^{-2}f|\psi|^2\nu_\Sigma(r) + r^{-1}f|\psi|^2\mathrm{d}iv_{\,\Sigma}(\nu_\Sigma).
\end{aligned}$$

Therefore,

$$\begin{aligned}
r^{-2}\nu_\Sigma\big(rf|\psi|^2\big) &= \mathrm{d}iv_{\,\Sigma}\big(r^{-1}f|\psi|^2\nu_\Sigma\big) - r^{-1}f|\psi|^2\mathrm{d}iv_{\,\Sigma}(\nu_\Sigma) + 2fr^{-2}|\psi|^2\nu_\Sigma(r)\\
&= \mathrm{d}iv_{\,\Sigma}\big(r^{-1}f|\psi|^2\nu_\Sigma\big) + fr^{-1}|\psi|^2\big(2r^{-1}\nu_\Sigma(r) - \mathrm{d}iv_{\,\Sigma}(\nu_\Sigma)\big).
\end{aligned}$$

Note that $\nu_\Sigma(r) = e_4(r) - \frac{1}{2}r^{-2}\lambda e_3(r) = 1 + O(r^{-2}) + r\Gamma_g$ and

$$\begin{aligned}
\mathrm{d}iv_{\,\Sigma}(\nu_\Sigma) &= \mathrm{tr}\,\chi + O(r^{-1}+\epsilon)r^{-1} = \frac{2}{r} + O(r^{-1}+\epsilon)r^{-2},\\
2\nu_\Sigma(r) - r^{-1}\mathrm{d}iv_{\,\Sigma}(\nu_\Sigma) &= O(r^{-1}+\epsilon)r^{-2}.
\end{aligned}$$

Hence

$$r^{-2}\nu_\Sigma\big(rf|\psi|^2\big) = \mathrm{d}iv_{\,\Sigma}\big(r^{-1}f|\psi|^2\nu_\Sigma\big) + O(r^{-1}+\epsilon)fr^{-3}|\psi|^2.$$

We deduce

$$\begin{aligned}
\mathcal{P}\cdot N_\Sigma - \mathcal{P}\cdot Y \;\geq\;& f\left|\check{\nabla}_4\psi\right|^2 + \frac{1}{2}r^{-2}\lambda f|\nabla\psi|^2 - \frac{1}{2}\mathrm{d}iv_{\,\Sigma}\big(r^{-1}f|\psi|^2\nu_\Sigma\big)\\
&+O(\epsilon+R^{-1})r^{-3}f|\psi|^2.
\end{aligned}$$

In view of the estimate for $\big|\mathcal{P}\cdot Y\big|$ of Lemma 10.3.4, we obtain

$$\begin{aligned}\mathcal{P}\cdot N_\Sigma &\geq f\left|\check{\nabla}_4\psi\right|^2+\frac{1}{2}r^{-2}\lambda f|\nabla\psi|^2-\frac{1}{2}\mathrm{d}iv_{\,\Sigma}\big(r^{-1}f|\psi|^2\nu_\Sigma\big)\\&+O(\epsilon+R^{-1})r^{-3}f|\psi|^2-f\big|\check{\nabla}_4\psi\big||Y||\nabla\psi|\\&+O(R^{-\delta})r^{-2}f|\nabla\psi|^2+r^{-4+\delta}f\Big(|\nabla_3\psi|^2+|\nabla\psi|^2+|\psi|^2\Big)\\&+O(\epsilon+R^{-1})r^{-3}f|\psi|^2.\end{aligned}$$

Note that

$$f\left|\check{\nabla}_4\psi\right|^2+\frac{1}{2}r^{-2}\lambda f|\nabla\psi|^2-f\big|\check{\nabla}_4\psi\big||Y||\nabla\psi|\gtrsim f\left|\check{\nabla}_4\psi\right|^2+fr^{-2}|\nabla\psi|^2.$$

Hence, for R large,

$$\begin{aligned}\mathcal{P}\cdot N_\Sigma &\gtrsim f\left|\check{\nabla}_4\psi\right|^2+r^{-2}f|\nabla\psi|^2+O(\epsilon+R^{-1})r^{-3}f|\psi|^2-\frac{1}{2}\mathrm{d}iv_{\,\Sigma}\big(r^{-1}f|\psi|^2\nu_\Sigma\big)\\&-fO(r^{-4+\delta})|\nabla_3\psi|^2,\end{aligned}$$

as stated.

To prove (10.3.12) we write, recalling that $N_*=e_4+Ue_3+Y_*$ and $\nu_*=e_4-Ue_3$,

$$\begin{aligned}\mathcal{P}\cdot N_* &= U\mathcal{P}\cdot e_3+\mathcal{P}\cdot e_4+\mathcal{P}\cdot Y_*\\&= U\left(f\mathcal{Q}_{34}+f\frac{1}{2}r^{-2}\lambda\mathcal{Q}_{33}+\frac{1}{2}r^{-2}\nabla_3\big(rf|\psi|^2\big)+(\epsilon+R^{-1})fr^{-3}|\psi|^2\right)\\&+\left(f\left|\check{\nabla}_4\psi\right|^2-\frac{1}{2}r^{-2}\nabla_4(rf|\psi|^2)+\frac{1}{2}r^{-2}\lambda f\mathcal{Q}_{34}+O(\epsilon+R^{-1})r^{-3}f|\psi|^2\right)\\&= f\left(\big(U+\frac{1}{2}r^{-2}\lambda\big)\mathcal{Q}_{34}+\left|\check{\nabla}_4\psi\right|^2+\frac{1}{2}r^{-2}\lambda|\nabla_3\psi|^2\right)+\frac{1}{2}Uf'r^{-2}|\psi|^2\\&+\frac{1}{2}fr^{-2}\Big(\nabla_{\nu_*}\big(rf|\psi|^2\big)\Big).\end{aligned}$$

As in the proof of part 1, see also the proof of Lemma 10.44 in [50], we have

$$\frac{1}{2}r^{-2}\Big(\nabla_{\nu_*}\big(rf|\psi|^2\big)\Big)=\frac{1}{2}\mathrm{d}iv_{\,\Sigma_*}\big(r^{-1}f|\psi|^2\nu_*\big)+O(\epsilon+R^{-1})r^{-3}f|\psi|^2$$

and therefore,

$$\begin{aligned}\mathcal{P}\cdot N_* &\gtrsim f\big(|\check{\nabla}_4\psi|^2+|\nabla\psi|^2+r^{-2}|\nabla_3\psi|^2\big)-\frac{1}{2}\mathrm{d}iv_{\,\Sigma}\big(r^{-1}f|\psi|^2\nu_\Sigma\big)\\&+O(R^{-1}+\epsilon)r^{-3}|\psi|^2\end{aligned}$$

as stated.

It remains to derive the last two parts. Starting with the identities in Lemma 10.3.4

$$\begin{aligned}\mathcal{P}\cdot e_4 &= f\big|\nabla_4\psi|^2 + fr^{-1}\psi\nabla_4\psi - \frac{1}{2}e_4(fr^{-1})|\psi|^2 + \frac{1}{2}r^{-2}\lambda f\mathcal{Q}_{34} + O(\epsilon+R^{-1})r^{-3}f|\psi|^2,\\ \mathcal{P}\cdot e_3 &= f\mathcal{Q}_{34} + f\frac{1}{2}r^{-2}\lambda\mathcal{Q}_{33} + r^{-1}f\psi\cdot\nabla_3\psi - \frac{1}{2}e_3(r^{-1}f)|\psi|^2 - \frac{r}{|q|^2}f'|\psi|^2\\ &\quad +O(\epsilon+R^{-1})r^{-3}f|\psi|^2,\end{aligned}$$

and combining them along Σ we derive

$$\begin{aligned}\mathcal{P}\cdot N_\Sigma &= f\big|\nabla_4\psi\big|^2 + fr^{-1}\psi\cdot\nu_\Sigma\psi - \frac{1}{2}\nu_\Sigma(fr^{-1})|\psi|^2 + \frac{1}{2}r^{-2}\lambda f\mathcal{Q}_{34} + f\frac{1}{4}r^{-4}\lambda^2|\nabla_3\psi|^2\\ &\quad +O(R^{-1}+\epsilon)r^{-2}f|\psi|^2 + \mathcal{P}\cdot Y.\end{aligned}$$

Note that, since $\nu_\Sigma = e_4 - \frac{1}{2}r^{-2}\lambda$,

$$\big|\nabla_4\psi\big|^2 + \frac{1}{4}r^{-4}\lambda^2|\nabla_3\psi|^2 = \big|\nabla_\Sigma\psi\big|^2 + r^{-2}\lambda\nabla_4\psi\cdot\nabla_3\psi.$$

Hence

$$\begin{aligned}\mathcal{P}\cdot N_\Sigma &= f\big|\nabla_\Sigma\psi\big|^2 + fr^{-1}\psi\cdot\nu_\Sigma\psi - \frac{1}{2}\nu_\Sigma(fr^{-1})|\psi|^2\\ &\quad +\frac{1}{2}r^{-2}\lambda f\mathcal{Q}_{34} + fr^{-2}\lambda\nabla_4\psi\cdot\nabla_3\psi + O(R^{-1}+\epsilon)r^{-2}|\psi|^2 + \mathcal{P}\cdot Y.\end{aligned}$$

Since

$$r^{-2}\lambda\nabla_4\psi\cdot\nabla_3\psi = r^{-2}\lambda\nabla_\nu\psi\nabla_3\psi - O(r^{-4})|\nabla_3\psi|^2 \lesssim r^{-1}|\nabla_\nu\psi|^2 + O(r^{-3})|\nabla_3\psi|^2$$

we deduce[5]

$$\begin{aligned}\mathcal{P}\cdot N_\Sigma &\geq f(1-r^{-1})\big|\nabla_\Sigma\psi\big|^2 + fr^{-1}\psi\cdot\nu_\Sigma\psi - \frac{1}{2}\nu_\Sigma(fr^{-1})|\psi|^2\\ &\quad +\frac{1}{2}r^{-2}\lambda f\mathcal{Q}_{34} + O(r^{-3})f\Big(|\nabla_3\psi|^2 + |\nabla\psi|^2 + |\psi|^2\Big).\end{aligned}$$

Lemma 10.3.6. *We re-express*

$$J = f(1-r^{-1})\big|\nabla_\Sigma\psi\big|^2 + fr^{-1}\psi\cdot\nu_\Sigma\psi - \frac{1}{2}\nu_\Sigma(fr^{-1})|\psi|^2,$$

for $f = r^p$, $0\leq p\leq 1-\delta$, $r\geq R$ sufficiently large, as follows

$$J + \frac{p}{2}r^{-2}\nu_\Sigma(rf|\psi|^2) \geq \frac{\delta^2}{8}r^{p-2}|\psi|^2.$$

[5]Note that we also estimate $\mathcal{P}\cdot Y$ slightly differently than in Lemma 10.3.4.

Proof. We calculate

$$
\begin{aligned}
& J+\frac{\lambda}{2}r^{-2}\nu_\Sigma(rf|\psi|^2)\\
= \;& f\big(1-r^{-1}\big)|\nu_\Sigma\psi|^2+r^{-1}f\psi\cdot\nu_\Sigma(\psi)-\frac{1}{2}\nu_\Sigma(r^{-1}f)|\psi|^2+\frac{\lambda}{2}r^{-2}\nu_\Sigma(rf|\psi|^2)\\
= \;& f\big(1-r^{-1}\big)|\nu_\Sigma\psi|^2+(1+\lambda)r^{-1}f\psi\cdot\nu_\Sigma\psi+\left(\frac{\lambda}{2}r^{-2}\nu_\Sigma(rf)-\frac{1}{2}\nu_\Sigma(r^{-1}f)\right)|\psi|^2\\
= \;& f\left|\big(1-r^{-1}\big)^{1/2}\nu_\Sigma\psi+\frac{1+\lambda}{2\big(1-r^{-1}\big)^{1/2}}r^{-1}\psi\right|^2-\frac{(1+\lambda)^2}{4\big(1-O(r^{-1})\big)}r^{-2}f|\psi|^2\\
& -\frac{1}{2}\nu_\Sigma(r^{-1}f)|\psi|^2+\frac{\lambda}{2}r^{-2}\nu_\Sigma(rf)|\psi|^2\\
\geq \;& L|\psi|^2,\\
L \;:= \;& \left(-\frac{(1+\lambda)^2}{4\big(1-r^{-1}\big)}r^{-2}f-\frac{1}{2}\nu_\Sigma(r^{-1}f)+\frac{\lambda}{2}r^{-2}\nu_\Sigma(rf)\right)|\psi|^2.
\end{aligned}
$$

Using $\nu_\Sigma(r)=1+r\Gamma_g+r^{-1}\Gamma_b=1+r^{-1}\epsilon$ we deduce for $f=r^p$, $r\geq R$,

$$
\begin{aligned}
L &= r^{p-2}\left(-\frac{(1+\lambda)^2}{4}\big(1+O(r^{-1})\big)+\frac{\lambda(p+1)}{2}-\frac{p-1}{2}\right)\\
&= r^{p-2}\left(-\frac{(1+\lambda)^2}{4}+\frac{\lambda(p+1)}{2}-\frac{p-1}{2}\right)+O(r^{p-3})\frac{(1+\lambda)^2}{4}.
\end{aligned}
$$

For $\lambda=p$, $0\leq p\leq 1-\delta$ we derive

$$
L = r^{p-2}\left(\frac{1}{4}(p-1)^2+O(R^{-2})\frac{(p+1)^2}{4}\right)\geq r^{-1-\delta}\left(\frac{\delta^2}{4}-O(R^{-2})\right).
$$

We need $R^{-1}\lesssim\frac{\delta}{2}$ to deduce, for $p\leq 1-\delta$,

$$
J+\frac{p}{2}r^{-2}\nu_\Sigma(rf|\psi|^2)\geq\frac{\delta^2}{8}r^{p-2}|\psi|^2
$$

as stated. □

Using the above lemma we deduce

$$
\mathcal{P}\cdot N_\Sigma \;\geq\; \frac{\delta^2}{8}r^{p-2}f|\psi|^2+fr^{-2}|\nabla\psi|^2-\frac{p}{2}r^{-2}\nu_\Sigma(rf|\psi|^2)+O(r^{p-3})f\Big(|\nabla_3\psi|^2+|\psi|^2\Big),
$$

as stated. The last inequality can be derived in the same manner. This concludes the proof of Proposition 10.3.5. □

10.3.2 Proof of Proposition 10.1.2 in the case $s = 0$

The goal of this section is to prove Proposition 10.1.2 in the case $s = 0$[6] for solutions of the wave equation (10.0.1)

$$\dot{\square}_2\psi - V\psi = -\frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_T\psi + N.$$

Using the estimates already derived in Propositions 10.3.3 and 10.3.5, the proof is very similar to that of Theorem 10.37 in section 10.2.3 of [50].

The proof of Proposition 10.1.2 in the case $s = 0$ proceeds in the following steps.

Step 1. We start by integrating the expression (10.3.2), i.e.

$$\begin{aligned}\mathbf{D}^\mu\mathcal{P}_\mu[X, w, M] &= \mathcal{E}[X, w, M] + \left(\nabla_X\psi + \frac{1}{2}w\psi\right)\cdot\left(\dot{\square}_2\psi - V\psi\right)\\ &+ O(\epsilon + R^{-1})r^{-1}f\Big(|\nabla_4\psi|^2 + |\nabla\psi|^2 + r^{-2}|\nabla_3\psi|^2 + r^{-2}|\psi|^2\Big)\end{aligned}$$

with, see (10.3.8),

$$\begin{aligned}\mathcal{E} &= \frac{1}{2}f'|\check{\nabla}_4\psi|^2 + \frac{1}{2}\left(-f' + \frac{2}{r}f\right)\left(|\nabla\psi|^2 + V|\psi|^2\right)\\ &+ O(\epsilon + R^{-1})r^{-1}f\Big(|\nabla_4\psi|^2 + |\nabla\psi|^2 + r^{-2}|\psi|^2 + r^{-2}|\nabla_3\psi|^2\Big)\end{aligned}$$

for the specific choice $f = f_p = f_{p,R}$, defined as $f_p = r^p$ for $r \geq R$ and $f_p = 0$ for $r \leq R/2$, where R is a fixed sufficiently large constant.

We derive

$$\begin{aligned}&\int_{\Sigma(\tau_2)}\mathcal{P}\cdot N_\Sigma + \int_{\Sigma_*(\tau_1,\tau_2)}\mathcal{P}\cdot N_{\Sigma_*} + \int_{\mathcal{M}(\tau_1,\tau_2)}\mathcal{E} = \int_{\Sigma(\tau_1)}\mathcal{P}\cdot N_\Sigma + \mathrm{Err},\\ &\mathrm{Err}(\tau_1,\tau_2) := -\int_{\mathcal{M}(\tau_1,\tau_2)}f_p\check{\nabla}_4\psi\cdot N - \int_{\mathcal{M}(\tau_1,\tau_2)}f_p\check{\nabla}_4\psi\cdot\frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_T\psi.\end{aligned} \tag{10.3.15}$$

Denoting the boundary terms,

$$\begin{aligned}K_{\geq R}(\tau_1,\tau_2) :&= \int_{\Sigma_{\geq R}(\tau_2)}\mathcal{P}\cdot N_\Sigma + \int_{\Sigma_*(\tau_1,\tau_2)}\mathcal{P}\cdot N_{\Sigma_*} - \int_{\Sigma_{\geq R}(\tau_1)}\mathcal{P}\cdot N_\Sigma,\\ K_{\leq R}(\tau_1,\tau_2) :&= \int_{\Sigma_{\leq R}(\tau_1)}\mathcal{P}\cdot N_\Sigma - \int_{\Sigma_{\leq R}(\tau_2)}\mathcal{P}\cdot N_\Sigma,\end{aligned}$$

[6]The case of higher order derivatives $0 \leq s \leq k_L$ is postponed to section 10.3.3.

we write,

$$K_{\geq R}(\tau_1,\tau_2)+\int_{\mathcal{M}_{\geq R}(\tau_1,\tau_2)}\mathcal{E}=K_{\leq R}(\tau_1,\tau_2)-\int_{\mathcal{M}_{\leq R}(\tau_1,\tau_2)}\mathcal{E}+\text{Err}(\tau_1,\tau_2).$$

We estimate the term $K_{\leq R}(\tau_1,\tau_2)-\int_{\mathcal{M}_{\leq R}(\tau_1,\tau_2)}\mathcal{E}$ on the right hand side and deduce (see also Lemma 10.45 in [50])

$$\begin{aligned}K_{\geq R}(\tau_1,\tau_2)+\int_{\mathcal{M}_{\geq R}(\tau_1,\tau_2)}\mathcal{E} \;\lesssim\;& R^{p+2}\Big(E_{R/2\leq r\leq R}[\psi](\tau_1)+\text{Mor}_{R/2\leq r\leq R}[\psi](\tau_1,\tau_2)\Big)\\ &+\text{Err}(\tau_1,\tau_2).\end{aligned}\tag{10.3.16}$$

Step 2. We make use of the second identity in Proposition 10.3.3 with $f=f_p$ to deduce, exactly as in the Lemma 10.46 in [50], for $\delta\leq p\leq 2-\delta$ and R sufficiently large,

$$\begin{aligned}\int_{\mathcal{M}_{\geq R}(\tau_1,\tau_2)}\mathcal{E} \;\geq\;& \frac{1}{4}\int_{\mathcal{M}_{\geq R}(\tau_1,\tau_2)}r^{p-1}\Big(p|\check{\nabla}_4(\psi)|^2+(2-p)(|\nabla\psi|^2+r^{-2}|\psi|^2)\Big)\\ &-\;O(\epsilon+R^{-1})\int_{\mathcal{M}_{\geq R}(\tau_1,\tau_2)}r^{p-3}|\nabla_3\psi|^2.\end{aligned}\tag{10.3.17}$$

Indeed, according to (10.3.8), since $f=r^p$ for $r\geq R$,

$$\begin{aligned}\mathcal{E}&=\frac{1}{2}r^{p-1}\Big(p|\check{\nabla}_4\psi|^2+(2-p)\big(|\nabla\psi|^2+V|\psi|^2\big)\Big)\\ &\quad+O(\epsilon+R^{-1})r^{p-1}\Big(|\check{\nabla}_4\psi|^2+|\nabla\psi|^2+r^{-2}|\psi|^2+r^{-2}|\nabla_3\psi|^2\Big)\Big)\\ &\geq\frac{1}{4}r^{p-1}\Big(p|\check{\nabla}_4\psi|^2+(2-p)\big(|\nabla\psi|^2+r^{-2}|\psi|^2\big)\Big)+O(\epsilon+R^{-1})r^{p-3}|\nabla_3\psi|^2,\end{aligned}$$

and integrating on $\mathcal{M}(\tau_1,\tau_2)$ immediately yields (10.3.17).

Step 3. According to Proposition 10.1.1, we have

$$\begin{aligned}\int_{\mathcal{M}_{\geq R}(\tau_1,\tau_2)}r^{-1-\delta}|\nabla_3\psi|^2 \lesssim& \int_{\mathcal{M}_{\geq R}(\tau_1,\tau_2)}r^{\delta-1}\Big(|\nabla_4\psi|^2+|\nabla\psi|^2+r^{-2}|\psi|^2\Big)\\ &+E_{\geq\frac{R}{2}}[\psi](\tau_1)+Mor_{\frac{R}{2}\leq r\leq R}[\psi](\tau_1,\tau_2)+\mathcal{N}_{\geq R/2}[\psi,N](\tau_1,\tau_2).\end{aligned}$$

Together with (10.3.17), we deduce, for $\delta\leq p\leq 2-\delta$ and for ϵ and R^{-1} small enough,

$$\begin{aligned}B_{p,\geq R}[\psi](\tau_1,\tau_2)\lesssim& \int_{\mathcal{M}_{\geq R}(\tau_1,\tau_2)}\mathcal{E}+Mor_{\frac{R}{2}\leq r\leq R}[\psi](\tau_1,\tau_2)\\ &+\mathcal{N}_{\geq R/2}[\psi,N](\tau_1,\tau_2)+E_{\geq\frac{R}{2}}[\psi](\tau_1)\end{aligned}\tag{10.3.18}$$

where, see section 6.1.5,

$$B_{p,\geq R}[\psi](\tau_1,\tau_2) \;=\; \int_{\mathcal{M}_{\geq R}(\tau_1,\tau_2)} \Big(r^{p-1}\big(|\nabla_4\psi|^2+|\nabla\psi|^2+r^{-2}|\psi|^2\big)+r^{-1-\delta}|\nabla_3\psi|^2\Big).$$

Step 4. We treat the boundary terms

$$K_{\geq R}(\tau_1,\tau_2)=\int_{\Sigma_{\geq R}(\tau_2)}\mathcal{P}\cdot N_\Sigma+\int_{\Sigma_*(\tau_1,\tau_2)}\mathcal{P}\cdot N_{\Sigma_*}-\int_{\Sigma_{\geq R}(\tau_1)}\mathcal{P}\cdot N_\Sigma,$$

by making use of Proposition 10.3.5 (see also Step 2 in the proof of Theorem 10.37 in [50]). Integrating the inequality (10.3.11) we deduce

$$\begin{aligned}\int_{\Sigma_{\geq R}(\tau_2)}\mathcal{P}\cdot N_\Sigma \;&\gtrsim\; \int_{\Sigma_{\geq R}(\tau_2)} r^p\Big(|\check{\nabla}_4\psi|^2+r^{-2}|\nabla\psi|^2\Big)-\frac12\int_{\Sigma_{\geq R}(\tau_2)}\mathrm{d}iv_{\,\Sigma}\big(r^{-1}f|\psi|^2\nu_\Sigma\big)\\ &-O(1)\int_{\Sigma_{\geq R}(\tau_2)} r^{p-4+\delta}|\nabla_3\psi|^2-O(\epsilon+R^{-1})\int_{\Sigma_{\geq R}(\tau_2)} r^{p-3}|\psi|^2.\end{aligned}$$

Since $p-4+\delta\leq -2$ and, see section 6.1.5, $E_{\geq R}[\psi](\tau)\geq\int_{\Sigma_{\geq R}(\tau)} r^{-2}|\nabla_3\psi|^2$ we deduce

$$\begin{aligned}\int_{\Sigma_{\geq R}(\tau_2)}\mathcal{P}\cdot N_\Sigma \;&\gtrsim\; \int_{\Sigma_{\geq R}(\tau_2)} r^p\Big(|\check{\nabla}_4\psi|^2+r^{-2}|\nabla\psi|^2\Big)+O(R^{-1}+\epsilon)\int_{\Sigma_{\geq R}(\tau_2)} r^{p-3}|\psi|^2\\ &-O(1)E_{\geq R}[\psi](\tau_2)-\frac12\int_{\Sigma(\tau_2)}\mathrm{d}iv_{\,\Sigma}\big(r^{-1}f|\psi|^2\nu_\Sigma\big).\end{aligned}$$

Similarly, integrating (10.3.12) on $\Sigma_*(\tau_1,\tau_2)$, we deduce

$$\begin{aligned}\int_{\Sigma_*(\tau_1,\tau_2)}\mathcal{P}\cdot N_{\Sigma_*}\geq\frac12\int_{\Sigma_*(\tau_1,\tau_2)} r^p\left(|\check{\nabla}_4\psi|^2+\frac{m}{r^2}|\nabla\psi|^2+\frac{m^2}{2r^4}|\nabla_3\psi|^2\right)\\ -\frac12\int_{\Sigma_*(\tau_1,\tau_2)}\mathrm{d}iv_{\,\Sigma}\big(r^{-1}f|\psi|^2\nu_\Sigma\big)+O(R^{-1}+\epsilon)r^{p-3}|\psi|^2.\end{aligned}$$

Applying the divergence theorem on $\Sigma(\tau_2)\cup\Sigma_*(\tau_1,\tau_2)$ to $\mathrm{d}iv_{\,\Sigma}(r^{-1}f|\psi|^2\nu_\Sigma)$, and noticing that the corresponding boundary terms from $\Sigma(\tau)$ and $\Sigma_*(\tau_1,\tau_2)$ cancel each other at $\Sigma(\tau)\cap\Sigma_*$, we deduce, for R sufficiently large, $\delta\leq p\leq 2-\delta$,

$$\begin{aligned}K_{\geq R}(\tau_1,\tau_2)\gtrsim&\int_{\Sigma_{\geq R}(\tau_2)} r^p\Big(|\check{\nabla}_4\psi|^2+r^{-2}|\nabla\psi|^2\Big)+\int_{\Sigma_*(\tau_1,\tau_2)} r^p\Big(|\check{\nabla}_4\psi|^2+r^{-2}|\nabla\psi|^2\Big)\\ &-O(R^{-1}+\epsilon)\int_{\Sigma_{\geq R}(\tau_2)} r^{p-3}|\psi|^2-O(R^{-1}+\epsilon)\int_{\Sigma_*(\tau_1,\tau_2)} r^{p-3}|\psi|^2\\ &-E_{\geq R}[\psi](\tau_2)-E_{p,\geq R}[\psi](\tau_1).\end{aligned}\tag{10.3.19}$$

Step 5. Combining (10.3.19) with (10.3.18) and (10.3.16) we derive

$$\begin{aligned} BEF_{p,\geq R}[\psi](\tau_1,\tau_2) \lesssim & \int_{\Sigma_{\geq R}(\tau_2)} r^{p-3}|\psi|^2 + \int_{\Sigma_*} r^{p-3}|\psi|^2 + E_{p,\geq R}[\psi](\tau_1) \\ & + R^{p+2}\Big(E_{R/2\leq r\leq R}[\psi](\tau_1) + \mathrm{Mor}_{R/2\leq r\leq R}[\psi](\tau_1,\tau_2)\Big) \\ & + E_{\geq R}[\psi](\tau_2) + \mathrm{Err}(\tau_1,\tau_2) + \mathcal{N}_{p,\geq R/2}[\psi,N](\tau_1,\tau_2). \end{aligned} \tag{10.3.20}$$

Step 6. We now estimate the error term $\mathrm{Err}(\tau_1,\tau_2)$ which we decompose as

$$\begin{aligned} \mathrm{Err}(\tau_1,\tau_2) =& \mathrm{Err}_1(\tau_1,\tau_2) + \mathrm{Err}_2(\tau_1,\tau_2), \\ \mathrm{Err}_1(\tau_1,\tau_2) :=& -\int_{\mathcal{M}(\tau_1,\tau_2)} f_p \check{\nabla}_4\psi \cdot N, \\ \mathrm{Err}_2(\tau_1,\tau_2) :=& -\int_{\mathcal{M}(\tau_1,\tau_2)} f_p \check{\nabla}_4\psi \cdot \frac{4a\cos\theta}{|q|^2}\ {}^*\nabla_T\psi. \end{aligned}$$

First, in view of the definition of the $\mathcal{N}_p$ norms in section 6.1.5, we have

$$\big|\mathrm{Err}_1(\tau_1,\tau_2)\big| \ \lesssim\ \mathcal{N}_{p,\geq R/2}[\psi,N](\tau_1,\tau_2).$$

Next, we focus on the control of $\mathrm{Err}_2(\tau_1,\tau_2)$. Recalling that the vectorfield $\mathbf{T}$ is given by[7] $\mathbf{T} = \frac{1}{2}(\frac{\Delta}{|q|^2}e_4 + e_3 - 2a\Re(\mathfrak{J})^b e_b)$, we deduce

$$\begin{aligned} -\frac{4a\cos\theta}{|q|^2} f_p\check{\nabla}_4\psi\cdot\ {}^*\nabla_{\mathbf{T}}\psi \ =\ & -\frac{2a\cos\theta}{|q|^2} f_p\check{\nabla}_4\psi\cdot\ {}^*\nabla_3\psi \\ & + O(r^{-2})|f_p||\check{\nabla}_4\psi|\cdot\big(|\nabla_4\psi| + r^{-1}|\nabla\psi|\big). \end{aligned}$$

Hence

$$\begin{aligned} \mathrm{Err}_2(\tau_1,\tau_2) \ =\ & -\int_{\mathcal{M}(\tau_1,\tau_2)} f_p\check{\nabla}_4\psi\cdot\left(\frac{2a\cos\theta}{|q|^2}\ {}^*\nabla_3\psi\right) \\ & + R^{-1}\int_{\mathcal{M}_{r\geq R}(\tau_1,\tau_2)} r^{p-1}\big(|\check{\nabla}_4\psi|^2 + |\nabla\psi|^2\big). \end{aligned}$$

To estimate the integral

$$I := \left|\int_{\mathcal{M}(\tau_1,\tau_2)} f_p\check{\nabla}_4\psi\cdot\left(\frac{2a\cos\theta}{|q|^2}\ {}^*\nabla_3\psi\right)\right|,$$

[7]The formula for $\mathbf{T}$ takes into account the renormalization of the frame, see Remark 10.2.3.

we write, with $p \le 2-\delta$,

$$\begin{aligned} I &\lesssim \int_{\mathcal{M}_{\ge R/2}(\tau_1,\tau_2)} r^{p-2}|\check{\nabla}_4\psi|\,|\nabla_3\psi| \\ &\lesssim \left(\int_{\mathcal{M}_{\ge R/2}(\tau_1,\tau_2)} r^{p-1}|\check{\nabla}_4\psi|^2\right)^{\frac12}\left(\int_{\mathcal{M}_{\ge R/2}(\tau_1,\tau_2)} r^{p-3}|\nabla_3\psi|^2\right)^{\frac12} \\ &\lesssim r^{-\frac{2-\delta-p}{2}}\left(\int_{\mathcal{M}_{\ge R/2}(\tau_1,\tau_2)} r^{p-1}|\check{\nabla}_4\psi|^2\right)^{\frac12}\left(\int_{\mathcal{M}_{\ge R/2}(\tau_1,\tau_2)} r^{-1-\delta}|\nabla_3\psi|^2\right)^{\frac12}. \end{aligned}$$

Plugging the above estimates for $\mathrm{Err}_1(\tau_1,\tau_2)$ and $\mathrm{Err}_2(\tau_1,\tau_2)$ in (10.3.20), and making use once more of Proposition 10.1.1 we deduce

$$\begin{aligned} BEF_{p,\ge R}[\psi](\tau_1,\tau_2) \lesssim & \int_{\Sigma_{\ge R}(\tau_2)} r^{p-3}|\psi|^2 + \int_{\Sigma_*(\tau_1,\tau_2)} r^{p-3}|\psi|^2 + E_{p,\ge R}[\psi](\tau_1) \\ & + \mathcal{N}_{p,\ge R/2}[\psi,N](\tau_1,\tau_2) + E_{\ge R}[\psi](\tau_2) \\ & + R^{p+2}\Big(E_{R/2\le r\le R}[\psi](\tau_1) + \mathrm{Mor}_{R/2\le r\le R}[\psi](\tau_1,\tau_2)\Big). \end{aligned} \tag{10.3.21}$$

Step 7. Next, we eliminate the term

$$I_p(\tau_1,\tau_2) := \int_{\Sigma_{\ge R}(\tau_2)} r^{p-3}|\psi|^2 + \int_{\Sigma_*(\tau_1,\tau_2)} r^{p-3}|\psi|^2$$

on the right hand side of (10.3.21). Note that for $p \le 1$ we have

$$I_p(\tau_1,\tau_2) \;\lesssim\; E_{\ge R}[\psi](\tau_2) + F[\psi](\tau_1,\tau_2).$$

Hence, for $p \le 1$,

$$\begin{aligned} BEF_{p,\ge R}[\psi](\tau_1,\tau_2) \lesssim & \, EF_{\ge R}[\psi](\tau_1,\tau_2) + E_{p,\ge \frac{R}{2}}[\psi](\tau_1) + \mathcal{N}_{p,\ge R/2}[\psi,N](\tau_1,\tau_2) \\ & + R^{p+2}\Big(E_{R/2\le r\le R}[\psi](\tau_1) + \mathrm{Mor}_{R/2\le r\le R}[\psi](\tau_1,\tau_2)\Big). \end{aligned} \tag{10.3.22}$$

For the remaining range $1 \le p \le 2-\delta$ we have

$$\begin{aligned} I_p(\tau_1,\tau_2) &\lesssim I_{2-\delta}(\tau_1,\tau_2) = \int_{\Sigma_{\ge R}(\tau_2)} r^{-1-\delta}|\psi|^2 + \int_{\Sigma_*} r^{-1-\delta}|\psi|^2 \\ &\lesssim EF_{1-\delta,\ge R}[\psi](\tau_1,\tau_2) \end{aligned}$$

which together with (10.3.22) implies, for $1 \le p \le 2-\delta$,

$$\begin{aligned} I_p(\tau_1,\tau_2) \lesssim & \, EF_{\ge R}[\psi](\tau_1,\tau_2) + E_{p,\ge R}[\psi](\tau_1) + \mathcal{N}_{1-\delta,\ge R/2}[\psi,N](\tau_1,\tau_2) \\ & + R^{3-\delta}\Big(E_{R/2\le r\le R}[\psi](\tau_1) + \mathrm{Mor}_{R/2\le r\le R}[\psi](\tau_1,\tau_2)\Big). \end{aligned}$$

Combining with (10.3.21) we deduce, for $\delta \leq p \leq 2-\delta$,

$$\begin{aligned} BEF_{p,\geq R}[\psi](\tau_1,\tau_2) \lesssim\; & EF_{\geq R}[\psi](\tau_1,\tau_2) + E_{p,\geq R}[\psi](\tau_1) + \mathcal{N}_{p,\geq R/2}[\psi,N](\tau_1,\tau_2) \\ & + R^{p+2}\Big(E_{R/2\leq r\leq R}[\psi](\tau_1) + \mathrm{Mor}_{R/2\leq r\leq R}[\psi](\tau_1,\tau_2)\Big). \end{aligned} \tag{10.3.23}$$

Step 8. It remains to eliminate the term $EF_{\geq R}[\psi](\tau_1,\tau_2)$ on the RHS of (10.3.23). We rely on Proposition 9.2.14 which yields

$$\begin{aligned} EF_{\geq R}[\psi](\tau_1,\tau_2) \lesssim\; & E_{\geq R}[\psi](\tau_1) + \mathcal{N}_{p,\geq R/2}[\psi,N](\tau_1,\tau_2) + \epsilon B_\delta[\psi](\tau_1,\tau_2) \\ & + R\Big(E_{R/2\leq r\leq R}[\psi](\tau_1) + \mathrm{Mor}_{R/2\leq r\leq R}[\psi](\tau_1,\tau_2)\Big). \end{aligned}$$

Together with (10.3.23), we infer

$$\begin{aligned} BEF_{p,\geq R}[\psi](\tau_1,\tau_2) \lesssim\; & E_{p,\geq R}[\psi](\tau_1) + \mathcal{N}_{p,\geq R/2}[\psi,N](\tau_1,\tau_2) \\ & + R^{p+2}\Big(E_{R/2\leq r\leq R}[\psi](\tau_1) + \mathrm{Mor}_{R/2\leq r\leq R}[\psi](\tau_1,\tau_2)\Big) \end{aligned} \tag{10.3.24}$$

which concludes the proof of Proposition 10.1.2 in the case $s=0$.

10.3.3 Proof of Proposition 10.1.2

In order to prove Proposition 10.1.2, we proceed as in section 10.4 in [50], i.e. we extend the estimates derived in the previous section for $s=0$ to $\mathfrak{d}^s$ derivatives of ψ with $0\leq s\leq k_L$ by recovering the derivatives one by one. We indicate below how to go from $s=0$ to $s=1$ and note that the procedure to recover the estimate for $s+1$ from the one for s is completely analogous.

To derive the estimate for $s=1$ we proceed in the following steps, see also section 10.4 in [50].

Step 1. We start with the following result, in the spirit of Lemma 9.5.1, which will be used to deal with terms generated by commutators with $\dot{\square}_2$.

Lemma 10.3.7. *Let $\widetilde{N}$ a tensor with the following schematic structure*

$$\widetilde{N} \;=\; O(r^{-2})\mathfrak{d}F + O(r^{-2})\mathfrak{d}^{\leq 1}\psi + O(r^{-3})\mathfrak{d}^{\leq 2}\psi + \mathfrak{d}^{\leq 2}(\Gamma_g\cdot\psi),$$

where F is a given tensor. Also, let $\psi^{(1)}$ a tensor satisfying $|\psi^{(1)}|\lesssim|\mathfrak{d}^{\leq 1}\psi|$. Then, we have, for $\delta\leq p\leq 2-\delta$,

$$\begin{aligned} & \mathcal{N}_{p,\geq R/2}[\psi^{(1)},\widetilde{N}](\tau_1,\tau_2) \\ \lesssim\; & \sqrt{B_{p,\geq R/2}[\psi^{(1)}](\tau_1,\tau_2)}\left(\sqrt{B_{p,\geq R/2}[F](\tau_1,\tau_2)} + \sqrt{B_{p,\geq R/2}[\psi](\tau_1,\tau_2)}\right) \\ & +(\epsilon+R^{-1})B^1_{p,\geq R/2}[\psi](\tau_1,\tau_2). \end{aligned}$$

Proof. In view of the definition of $\mathcal{N}_p$, we have, for $\delta \leq p \leq 2-\delta$,

$$\begin{aligned}\mathcal{N}_{p,\geq R/2}[\psi^{(1)}, \widetilde{N}](\tau_1,\tau_2) &\lesssim \int_{\mathcal{M}(r\geq R/2)} |\widetilde{N}|\Big(|\nabla_3\psi^{(1)}| + r^{p-1}|\mathfrak{d}^{\leq 1}\psi^{(1)}|\Big)\\ &\lesssim \sqrt{B_{p,\geq R/2}[\psi^{(1)}](\tau_1,\tau_2)}\left(\int_{\mathcal{M}(r\geq R/2)} r^{1+p}|\widetilde{N}|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Now, in view of the structure of $\widetilde{N}$ and the assumptions on Γ_g, we have

$$\begin{aligned}\int_{\mathcal{M}(r\geq R/2)} r^{1+p}|\widetilde{N}|^2 &\lesssim \int_{\mathcal{M}(r\geq R/2)} r^{p-3}\Big(|\mathfrak{d}F|^2 + |\mathfrak{d}^{\leq 1}\psi|^2 + O(\epsilon^2+R^{-2})|\mathfrak{d}^{\leq 2}\psi|^2\Big)\\ &\lesssim B_{p,\geq R/2}[F](\tau_1,\tau_2) + B_{p,\geq R/2}[\psi](\tau_1,\tau_2)\\ &\quad +O(\epsilon^2+R^{-2})B^1_{p,\geq R/2}[\psi](\tau_1,\tau_2).\end{aligned}$$

Plugging the second estimate in the first one implies

$$\begin{aligned}&\mathcal{N}_{p,\geq R/2}[\psi^{(1)}, \widetilde{N}](\tau_1,\tau_2)\\ \lesssim\ & \sqrt{B_{p,\geq R/2}[\psi^{(1)}](\tau_1,\tau_2)}\Big(\sqrt{B_{p,\geq R/2}[F](\tau_1,\tau_2)} + \sqrt{B_{p,\geq R/2}[\psi](\tau_1,\tau_2)}\Big)\\ &+(\epsilon+R^{-1})B^1_{p,\geq R/2}[\psi](\tau_1,\tau_2)\end{aligned}$$

where we used also the fact that $B_{p,\geq R/2}[\psi^{(1)}](\tau_1,\tau_2) \lesssim B_{p,\geq R/2}[\psi^{(1)}](\tau_1,\tau_2)$. This concludes the proof of Lemma 10.3.7. □

Step 2. Next, we derive and estimate for $\not{\mathcal{L}}_{\mathbf{T}}\psi$. To this end, we commute the wave equation (10.0.1) with $\not{\mathcal{L}}_{\mathbf{T}}$ and obtain, using Corollary 4.3.4,

$$\dot{\square}_2\not{\mathcal{L}}_{\mathbf{T}}\psi - V\not{\mathcal{L}}_{\mathbf{T}}\psi = -\frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_T\not{\mathcal{L}}_{\mathbf{T}}\psi + N_{\not{\mathcal{L}}_{\mathbf{T}}},$$

where

$$N_{\not{\mathcal{L}}_{\mathbf{T}}} = \not{\mathcal{L}}_{\mathbf{T}}N + \mathfrak{d}(\Gamma_g\cdot\mathfrak{d}\psi) + \Gamma_b\cdot\dot{\square}_2\psi.$$

In view of (10.0.1), we may rewrite $N_{\not{\mathcal{L}}_{\mathbf{T}}}$ in the following form

$$N_{\not{\mathcal{L}}_{\mathbf{T}}} = \not{\mathcal{L}}_{\mathbf{T}}N + \mathfrak{d}^{\leq 2}(\Gamma_g\cdot\psi) + \Gamma_b\cdot N.$$

Applying (10.3.24) to the above wave equation for $\not{\mathcal{L}}_{\mathbf{T}}\psi$, we obtain

$$\begin{aligned}BEF_{p,\geq R}[\not{\mathcal{L}}_{\mathbf{T}}\psi](\tau_1,\tau_2) \lesssim\ & E^1_{p,\geq R}[\psi](\tau_1) + \mathcal{N}_{p,\geq R/2}[\not{\mathcal{L}}_{\mathbf{T}}\psi, N_{\not{\mathcal{L}}_{\mathbf{T}}}](\tau_1,\tau_2)\\ &+ R^{p+2}\Big(E^1_{R/2\leq r\leq R}[\psi](\tau_1) + \mathrm{Mor}^1_{R/2\leq r\leq R}[\psi](\tau_1,\tau_2)\Big).\end{aligned}$$

Next, we have

$$\begin{aligned}\mathcal{N}_{p,\geq R/2}[\mathcal{L}\!\!\!/_{\mathbf{T}}\psi, N_{\mathcal{L}\!\!\!/_{\mathbf{T}}}](\tau_1,\tau_2) &\lesssim \mathcal{N}_{p,\geq R/2}[\mathcal{L}\!\!\!/_{\mathbf{T}}\psi, \mathfrak{d}^{\leq 2}(\Gamma_g\cdot\psi)](\tau_1,\tau_2) + \mathcal{N}^1_{p,\geq R/2}[\psi, N](\tau_1,\tau_2)\\ &\lesssim \epsilon B^1_{p,\geq R/2}[\psi](\tau_1,\tau_2) + \mathcal{N}^1_{p,\geq R/2}[\psi, N](\tau_1,\tau_2)\end{aligned}$$

where we have applied Lemma 10.3.7 in the particular case where $\widetilde{N} = \mathfrak{d}^{\leq 2}(\Gamma_g\cdot\psi)$. We deduce

$$\begin{aligned}BEF_{p,\geq R}[\mathcal{L}\!\!\!/_{\mathbf{T}}\psi](\tau_1,\tau_2) \lesssim{}& E^1_{p,\geq R}[\psi](\tau_1) + \mathcal{N}^1_{p,\geq R/2}[\psi, N](\tau_1,\tau_2) + \epsilon B^1_{p,\geq R/2}[\psi](\tau_1,\tau_2)\\ &+ R^{p+2}\Big(E^1_{R/2\leq r\leq R}[\psi](\tau_1) + \mathrm{Mor}^1_{R/2\leq r\leq R}[\psi](\tau_1,\tau_2)\Big).\end{aligned} \tag{10.3.25}$$

Step 3. Next, we derive and estimate for $\not{\mathfrak{d}}\psi$. To this end, we commute the wave equation (10.0.1) with $|q|\,\mathcal{D}\!\!\!/_2$ and obtain, using Lemma 4.7.13,

$$\dot{\square}_1(|q|\,\mathcal{D}\!\!\!/_2\psi) - V|q|\,\mathcal{D}\!\!\!/_2\psi = -\frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_T(|q|\,\mathcal{D}\!\!\!/_2\psi) + N_{|q|\mathcal{D}\!\!\!/_2},$$

where

$$\begin{aligned}N_{|q|\mathcal{D}\!\!\!/_2} ={}& O(1)\mathfrak{d}^{\leq 1}N + (O(r^{-1}) + r\Gamma_b)\cdot\dot{\square}_2\psi + O(r^{-2})\mathfrak{d}\mathcal{L}\!\!\!/_{\mathbf{T}}\psi + O(r^{-2})\mathfrak{d}^{\leq 1}\psi\\ &+O(r^{-3})\mathfrak{d}^{\leq 2}\psi + \mathfrak{d}^{\leq 2}(\Gamma_g\cdot\psi).\end{aligned}$$

In view of (10.0.1), we may rewrite $N_{|q|\mathcal{D}\!\!\!/_2}$ in the following form

$$N_{|q|\mathcal{D}\!\!\!/_2} = O(1)\mathfrak{d}^{\leq 1}N + O(r^{-2})\mathfrak{d}\mathcal{L}\!\!\!/_{\mathbf{T}}\psi + O(r^{-2})\mathfrak{d}^{\leq 1}\psi + O(r^{-3})\mathfrak{d}^{\leq 2}\psi + \mathfrak{d}^{\leq 2}(\Gamma_g\cdot\psi).$$

Applying (10.3.24) to the above wave equation for $|q|\,\mathcal{D}\!\!\!/_2\psi$, we obtain[8]

$$\begin{aligned}BEF_{p,\geq R}[|q|\,\mathcal{D}\!\!\!/_2\psi](\tau_1,\tau_2) \lesssim{}& E^1_{p,\geq R}[\psi](\tau_1) + \mathcal{N}_{p,\geq R/2}[|q|\,\mathcal{D}\!\!\!/_2\psi, N_{|q|\mathcal{D}\!\!\!/_2}](\tau_1,\tau_2)\\ &+ R^{p+2}\Big(E^1_{R/2\leq r\leq R}[\psi](\tau_1) + \mathrm{Mor}^1_{R/2\leq r\leq R}[\psi](\tau_1,\tau_2)\Big).\end{aligned}$$

Next, we have

$$\begin{aligned}&\mathcal{N}_{p,\geq R/2}[|q|\,\mathcal{D}\!\!\!/_2\psi, N_{|q|\mathcal{D}\!\!\!/_2}](\tau_1,\tau_2)\\ \lesssim{}& \sqrt{B_{p,\geq R/2}[|q|\,\mathcal{D}\!\!\!/_2\psi](\tau_1,\tau_2)}\Big(\sqrt{B_{p,\geq R/2}[\mathcal{L}\!\!\!/_{\mathbf{T}}\psi](\tau_1,\tau_2)} + \sqrt{B_{p,\geq R/2}[\psi](\tau_1,\tau_2)}\Big)\\ &+(\epsilon + R^{-1})B^1_{p,\geq R/2}[\psi](\tau_1,\tau_2) + \mathcal{N}^1_{p,\geq R/2}[\psi, N](\tau_1,\tau_2)\end{aligned}$$

---

[8] In fact, we apply a variant for tensors in $\mathfrak{s}_1$. Adapting (10.3.24) to this case can be done along the same lines, and is in fact easier as an estimate conditional on $\int_{\mathcal{M}(r\geq R/2)} r^{p-3}|\psi|^2$ suffices for this step. See Theorem 10.61 in [50] for a proof of this variant in the case of perturbations of Schwarzschild.

where we have applied Lemma 10.3.7 in the particular case where $F = \not{\mathcal{L}}_{\mathbf{T}}\psi$. We deduce

$$\begin{aligned}
&BEF_{p,\geq R}[|q|\not{\mathcal{P}}_2\psi](\tau_1,\tau_2)\\
\lesssim & E^1_{p,\geq R}[\psi](\tau_1) + \mathcal{N}^1_{p,\geq R/2}[\psi, N](\tau_1,\tau_2) + (\epsilon + R^{-1})B^1_{p,\geq R/2}[\psi](\tau_1,\tau_2)\\
&+ R^{p+2}\Big(E^1_{R/2\leq r\leq R}[\psi](\tau_1) + \mathrm{Mor}^1_{R/2\leq r\leq R}[\psi](\tau_1,\tau_2)\Big)\\
&\sqrt{B_{p,\geq R/2}[|q|\not{\mathcal{P}}_2\psi](\tau_1,\tau_2)}\left(\sqrt{B_{p,\geq R/2}[\not{\mathcal{L}}_{\mathbf{T}}\psi](\tau_1,\tau_2)} + \sqrt{B_{p,\geq R/2}[\psi](\tau_1,\tau_2)}\right).
\end{aligned}$$

Together with (10.3.24) and (10.3.25), we infer

$$\begin{aligned}
&BEF_{p,\geq R}[|q|\not{\mathcal{P}}_2\psi](\tau_1,\tau_2) + BEF_{p,\geq R}[\not{\mathcal{L}}_{\mathbf{T}}\psi](\tau_1,\tau_2) + BEF_{p,\geq R}[\psi](\tau_1,\tau_2)\\
\lesssim & E^1_{p,\geq R}[\psi](\tau_1) + \mathcal{N}^1_{p,\geq R/2}[\psi, N](\tau_1,\tau_2) + (\epsilon + R^{-1})B^1_{p,\geq R/2}[\psi](\tau_1,\tau_2)\\
&+ R^{p+2}\Big(E^1_{R/2\leq r\leq R}[\psi](\tau_1) + \mathrm{Mor}^1_{R/2\leq r\leq R}[\psi](\tau_1,\tau_2)\Big).
\end{aligned}$$

Using the Hodge estimates of Proposition 9.3.2, and comparing $\nabla_{\mathbf{T}}$ and $\not{\mathcal{L}}_{\mathbf{T}}$, we deduce

$$\begin{aligned}
&BEF_{p,\geq R}[(\nabla_{\mathbf{T}}, \not{\mathfrak{d}})\psi](\tau_1,\tau_2) + BEF_{p,\geq R}[\psi](\tau_1,\tau_2)\\
\lesssim & E^1_{p,\geq R}[\psi](\tau_1) + \mathcal{N}^1_{p,\geq R/2}[\psi, N](\tau_1,\tau_2) + (\epsilon + R^{-1})B^1_{p,\geq R/2}[\psi](\tau_1,\tau_2)\\
&+ R^{p+2}\Big(E^1_{R/2\leq r\leq R}[\psi](\tau_1) + \mathrm{Mor}^1_{R/2\leq r\leq R}[\psi](\tau_1,\tau_2)\Big).
\end{aligned} \tag{10.3.26}$$

Step 4. Next, we derive and estimate for $r\nabla_4\psi$. To this end, we commute the wave equation (10.0.1) with $r\nabla_4$ and obtain, using Lemma 4.7.11,

$$\dot{\square}_2(r\nabla_4\psi) - Vr\nabla_4\psi = -\frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_T(r\nabla_4\psi) + N_{r\nabla_4},$$

where

$$\begin{aligned}
N_{r\nabla_4} = & \frac{1}{r}\nabla_4(r\nabla_4\psi) + O(1)\mathfrak{d}^{\leq 1}N + O(1)\cdot\dot{\square}_2\psi + O(r^{-2})\not{\mathfrak{d}}^2\psi + O(r^{-2})\mathfrak{d}^{\leq 1}\psi\\
&+O(r^{-3})\mathfrak{d}^{\leq 2}\psi + \mathfrak{d}^{\leq 2}(\Gamma_g\cdot\psi).
\end{aligned}$$

In view of (10.0.1), we may rewrite $N_{r\nabla_4}$ in the following form

$$\begin{aligned}
N_{r\nabla_4} &= \frac{1}{r}\nabla_4(r\nabla_4\psi) + \widetilde{N}_{r\nabla_4},\\
\widetilde{N}_{r\nabla_4} &= +O(1)\mathfrak{d}^{\leq 1}N + O(r^{-2})\not{\mathfrak{d}}^2\psi + O(r^{-2})\mathfrak{d}^{\leq 1}\psi + O(r^{-3})\mathfrak{d}^{\leq 2}\psi + \mathfrak{d}^{\leq 2}(\Gamma_g\cdot\psi).
\end{aligned}$$

Next, we apply (10.3.24) to the above wave equation for $r\nabla_4\psi$, and notice that the $\frac{1}{r}\nabla_4(r\nabla_4\psi)$ has a favorable sign in the estimate so that we may drop it and obtain

$$\begin{aligned}
BEF_{p,\geq R}[r\nabla_4\psi](\tau_1,\tau_2) \lesssim & E^1_{p,\geq R}[\psi](\tau_1) + \mathcal{N}_{p,\geq R/2}[r\nabla_4\psi, \widetilde{N}_{r\nabla_4}](\tau_1,\tau_2)\\
&+ R^{p+2}\Big(E^1_{R/2\leq r\leq R}[\psi](\tau_1) + \mathrm{Mor}^1_{R/2\leq r\leq R}[\psi](\tau_1,\tau_2)\Big).
\end{aligned}$$

Next, we have

$$
\begin{aligned}
& \mathcal{N}_{p,\geq R/2}[r\nabla_4, \widetilde{N}_{r\nabla_4}](\tau_1,\tau_2)\\
\lesssim\ & \sqrt{B_{p,\geq R/2}[r\nabla_4\psi](\tau_1,\tau_2)}\left(\sqrt{B_{p,\geq R/2}[\not{\mathfrak{d}}\psi](\tau_1,\tau_2)}+\sqrt{B_{p,\geq R/2}[\psi](\tau_1,\tau_2)}\right)\\
& +(\epsilon+R^{-1})B^1_{p,\geq R/2}[\psi](\tau_1,\tau_2)+\mathcal{N}^1_{p,\geq R/2}[\psi,N](\tau_1,\tau_2)
\end{aligned}
$$

where we have applied Lemma 10.3.7 in the particular case where $F=\not{\mathfrak{d}}\psi$. We deduce

$$
\begin{aligned}
& BEF_{p,\geq R}[r\nabla_4\psi](\tau_1,\tau_2)\\
\lesssim & E^1_{p,\geq R}[\psi](\tau_1)+\mathcal{N}^1_{p,\geq R/2}[\psi,N](\tau_1,\tau_2)+(\epsilon+R^{-1})B^1_{p,\geq R/2}[\psi](\tau_1,\tau_2)\\
& + R^{p+2}\Big(E^1_{R/2\leq r\leq R}[\psi](\tau_1)+\mathrm{Mor}^1_{R/2\leq r\leq R}[\psi](\tau_1,\tau_2)\Big)\\
& + \sqrt{B_{p,\geq R/2}[r\nabla_4\psi](\tau_1,\tau_2)}\left(\sqrt{B_{p,\geq R/2}[\not{\mathfrak{d}}\psi](\tau_1,\tau_2)}+\sqrt{B_{p,\geq R/2}[\psi](\tau_1,\tau_2)}\right).
\end{aligned}
$$

Together with (10.3.24) and (10.3.26), we infer

$$
\begin{aligned}
& BEF_{p,\geq R}[(\nabla_{\mathbf{T}}, \not{\mathfrak{d}}, r\nabla_4)\psi](\tau_1,\tau_2)+BEF_{p,\geq R}[\psi](\tau_1,\tau_2)\\
\lesssim & E^1_{p,\geq R}[\psi](\tau_1)+\mathcal{N}^1_{p,\geq R/2}[\psi,N](\tau_1,\tau_2)+(\epsilon+R^{-1})B^1_{p,\geq R/2}[\psi](\tau_1,\tau_2)\\
& + R^{p+2}\Big(E^1_{R/2\leq r\leq R}[\psi](\tau_1)+\mathrm{Mor}^1_{R/2\leq r\leq R}[\psi](\tau_1,\tau_2)\Big).
\end{aligned}
$$

Since $(\nabla_{\mathbf{T}}, \not{\mathfrak{d}}, r\nabla_4)$ spans $\mathfrak{d}$ away from the horizon, we infer

$$
\begin{aligned}
BEF^1_{p,\geq R}[\psi](\tau_1,\tau_2) \lesssim & E^1_{p,\geq R}[\psi](\tau_1)+\mathcal{N}^1_{p,\geq R/2}[\psi,N](\tau_1,\tau_2)+(\epsilon+R^{-1})B^1_{p,\geq R/2}[\psi](\tau_1,\tau_2)\\
& + R^{p+2}\Big(E^1_{R/2\leq r\leq R}[\psi](\tau_1)+\mathrm{Mor}^1_{R/2\leq r\leq R}[\psi](\tau_1,\tau_2)\Big).
\end{aligned}
$$

For ϵ and R^{-1} small enough, we may absorb the third term on the RHS from the LHS and we obtain

$$
\begin{aligned}
BEF^1_{p,\geq R}[\psi](\tau_1,\tau_2) \lesssim & E^1_{p,\geq R}[\psi](\tau_1)+\mathcal{N}^1_{p,\geq R/2}[\psi,N](\tau_1,\tau_2)\\
& + R^{p+2}\Big(E^1_{R/2\leq r\leq R}[\psi](\tau_1)+\mathrm{Mor}^1_{R/2\leq r\leq R}[\psi](\tau_1,\tau_2)\Big).
\end{aligned}
$$

which establishes the desired estimate (10.1.4) for $s=1$.

We have shown above how to go from $s=0$ to $s=1$. The procedure to recover the estimate for $s+1$ from the one for s is completely analogous. This concludes the proof of Proposition 10.1.2.

10.4 Proof of Theorem 6.2.2

For the convenience of the reader we restate Theorem 6.2.2.

Theorem 10.4.1 (Improved r^p-weighted estimates). *The following estimates hold true for the quantity $\check{\psi} = f_2\left(e_4\psi + \frac{r}{|q|^2}\psi\right)$ corresponding to solutions $\psi \in \mathfrak{s}_2$ of* (6.1.1) *on $\mathcal{M}$, for all $-1+\delta < q \leq 1-\delta$, $s \leq k_L - 1$,*

$$BEF_q^s[\check{\psi}](\tau_1,\tau_2) \lesssim \widetilde{E}_q^s[\check{\psi}](\tau_1) + \widetilde{\mathcal{N}}_q^s[\check{\psi}, N](\tau_1,\tau_2) + \mathcal{N}^{s+1}_{\max\{q,\delta\}}[\psi, N](\tau_1,\tau_2). \tag{10.4.1}$$

where the norms on the right are given by

$$\widetilde{E}_q^s[\check{\psi}](\tau) = E_q^s[\check{\psi}](\tau) + E^{s+1}_{\max\{q,\delta\}}[\psi](\tau) \tag{10.4.2}$$

and

$$\widetilde{\mathcal{N}}_q^s[\check{\psi}, N](\tau_1,\tau_2) = \left|\int_{\mathcal{M}_{\geq R}(\tau_1,\tau_2)} r^{q+2}\mathfrak{d}^{\leq s}\check{\psi}\cdot\left(\nabla_4\mathfrak{d}^{\leq s}N + \frac{3}{r}\mathfrak{d}^{\leq s}N\right)\right|. \tag{10.4.3}$$

Proof. We sketch the proof below in the case $s = 0$. Like the proof of the corresponding result in [50] (see Theorem 5.18 in [50]) the proof of the result rests on a commutation formula according to which, if ψ verifies equation (10.0.1), then $\check{\psi}$ verifies an equation of the form

$$\dot{\square}_2\check{\psi} - V\check{\psi} = -\frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_T\check{\psi} + \check{N} + f_2\left(\nabla_4 + \frac{3}{r}N\right), \tag{10.4.4}$$

where

$$\check{N} = \begin{cases} \frac{2}{r}\left(1-\frac{3m}{r}+O(r^{-2})\right)\nabla_4\check{\psi} + O(r^{-2})\mathfrak{d}^{\leq 1}\psi \\ +r\Gamma_b\cdot\nabla_4\mathfrak{d}\psi + \mathfrak{d}^{\leq 1}(\Gamma_b)\cdot\mathfrak{d}^{\leq 1}\psi + r\mathfrak{d}^{\leq 1}(\Gamma_g)\cdot\nabla_3\psi + \mathfrak{d}^{\leq 1}(\Gamma_g)\cdot\mathfrak{d}^2\psi, \quad r \geq R/2, \\ \\ O(1)\mathfrak{d}^{\leq 2}\psi, \qquad\qquad r \leq R. \end{cases} \tag{10.4.5}$$

Formula (10.4.5), which is the precise analogue of formula (10.3.2) of [50], can be verified by a straightforward calculation. We refer the reader to section 10.3.2 and appendix D.4 of [50] for the details.

We apply the results of Theorem 6.2.1 to equation (10.4.4) and derive[9] more details

$$BEF_q[\check{\psi}] \quad \lesssim \quad E_q[\check{\psi}](\tau_1) + \mathcal{N}_q[\check{\psi}, \check{N}](\tau_1,\tau_2).$$

[9]We refer the reader to section 10.3.2 of [50] for the same type of calculation in the proof of the analogous result, i.e. Theorem 5.18 in [50].

Note that the main term[10] in $\mathcal{N}_q[\check{\psi}, \check{N}]$ is given by

$$-\int_{\mathcal{M}_{\geq R}(\tau_1,\tau_2)} \frac{2}{r}\left(1-\frac{3m}{r}+O(r^{-2})\right) r^q |\check{\nabla}_4\check{\psi}|^2$$

Denoting $\mathcal{E}_q = \mathcal{E}[\check{\psi}][f_q, 2r^{-1}f_q, 2r^{-1}f_q' e_4]$ with $\mathcal{E}[\check{\psi}]$ as in (10.2.10) (for ψ replaced by $\check{\psi}$), we derive the following analogue of Proposition 10.48 in [50].

Proposition 10.4.2. *The following estimate holds true,*

$$\begin{aligned}\int_{\mathcal{M}_{\geq R}(\tau_1,\tau_2)} (\mathcal{E}_q + r^q \check{e}_4(\check{\psi})\check{N}) \geq \frac{1}{8}\int_{\mathcal{M}_{\geq R}(\tau_1,\tau_2)} & r^{q-1}\left((2+q)|\check{e}_4\check{\psi}|^2 + (2-q)|\nabla\!\!\!/\check{\psi}|^2 + 2r^{-2}|\check{\psi}|^2\right) \\ & - O(\epsilon)\sup_{\tau_1\leq\tau\leq\tau_2} \dot{E}_{q,R}[\check{\psi}](\tau) \\ & - O(1)\left(E^1_{\max(q,\delta)}[\check{\psi}](\tau_1) + \mathcal{N}^1_{\max(q,\delta)}[\psi, N]\right).\end{aligned} \tag{10.4.6}$$

The remaining part of the proof, based on choosing R large and making use of the result of Theorem 6.2.1, is exactly as in section 10.3.2 of [50]. $\square$

10.5 Conditional weighted estimate for scalar wave

Proposition 10.5.1. *Let ψ be a solution to the following scalar wave equation*

$$\square_{\mathbf{g}}\psi + V\psi = N, \tag{10.5.1}$$

where V is real and satisfies $V = O(r^{-3})$ for r large, in a spacetime $\mathcal{M} = \mathcal{M}(1,\tau_*)$ verifying the assumptions (6.1.6). *Then:*

1. *The following conditional Morawetz estimates hold true in $\mathcal{M} = \mathcal{M}(1,\tau_*)$*

$$\begin{aligned}B^k_\delta[\psi](1,\tau_*) \lesssim\ & EF^k_\delta[\psi](1,\tau_*) + B^{k-1}_\delta[\psi](1,\tau_*) + \int_{\mathcal{M}_{trap}(1,\tau_*)} |\mathfrak{d}^{\leq k}\psi|^2 \\ & + {}^{(mor)}\mathcal{N}^k[\psi,N](1,\tau_*) + {}^{(red)}\mathcal{N}^k[\psi,N](1,\tau_*) \\ & + \int_{^{(ext)}\mathcal{M}(1,\tau_*)} r^\delta\Big(\big|\nabla_4\mathfrak{d}^{\leq k}\psi\big| + r^{-1}\big|\mathfrak{d}^{\leq k}\psi\big|\Big)\big|\mathfrak{d}^{\leq k}N\big|.\end{aligned} \tag{10.5.2}$$

[10]The sign can be easily tracked down from the divergence formula, see for example (10.3.15).

2. The following conditional Energy-Morawetz estimates hold true

$$\begin{aligned} BEF_\delta[\psi](1,\tau_*) \lesssim\; & E^k_\delta[\psi](0) + BEF^{k-1}_\delta[\psi](1,\tau_*) \\ & + \int_{\mathcal{M}_{trap}(1,\tau_*)} |\mathfrak{d}^k\psi|^2 + \mathcal{N}^k_\delta[\psi,N](1,\tau_*). \end{aligned} \tag{10.5.3}$$

Remark 10.5.2. *Note that both estimates are conditional on the control of the terms* $BEF^{k-1}_\delta[\psi](1,\tau_*)$ *and* $\int_{\mathcal{M}_{trap}} |\mathfrak{d}^k\psi|^2$. *This result will be used to control* $\check{P}$ *in Chapter 14.*

The proof of the Proposition 10.5.1 relies on the conditional energy-Morawetz estimates of Proposition 9.6.1 and an analog of Proposition 10.1.2 for solutions to (10.5.1). Given that the estimate is only conditional, and in view of the the strong decay in r for the potential V, the proof for solutions to (10.5.1) is similar and in fact simpler that the one of Proposition 10.1.2.

Chapter 11

Estimates for the full Regge Wheeler equation

The goal of this chapter is to provide a complete proof for Theorem M1. To this end we proceed as follows:

1. We use the gRW equation for $\psi = \Re(\mathfrak{q})$ coupled with the transport equation provided by the definition of $\mathfrak{q}$ in terms of A to derive combined r-weighted estimates for (ψ, A), see Theorem 11.2.3.

2. We extend the results of Theorem 6.2.2 to the full gRW equations to derive improved r-weighted estimates based on the quantity $\check{\psi} = r^2(\nabla_4\psi + \frac{r}{|q|^2}\psi)$. The result, stated in Theorem 11.6.2, is the precise analogue of of Theorem 5.15 in [50].

3. We use the r weighted estimates of Theorems 11.2.3 and 11.6.2 to prove Theorem M1 by relying on the decay of flux arguments, based on mean value arguments, following the procedure detailed in section 5.4 of [50].

11.1 Preliminaries

The spacetime $\mathcal{M}$ we are dealing with here is precisely that described in section 6.1. We do however make stronger assumptions on (Γ_g, Γ_b). We assume in fact that for all $k \leq k_L$, with[1] $k_L = k_{small} + 120$

$$\begin{aligned}\left(r^2\tau^{\frac{1}{2}+\delta_{dec}} + r\tau^{1+\delta_{dec}}\right)|\mathfrak{d}^{\leq k}\Gamma_g| &\leq \epsilon,\\ r\tau^{1+\delta_{dec}}|\mathfrak{d}^{\leq k}\Gamma_b| &\leq \epsilon.\end{aligned} \tag{11.1.1}$$

We also assume that the curvature components A, B verify, for $k \leq k_L$,

$$r^{7/2+\delta_{dec}}|\mathfrak{d}^{\leq k}(A, B)| \quad \leq \epsilon. \tag{11.1.2}$$

It is important in this chapter that the frame is such that

$$\check{H} \in \Gamma_g \tag{11.1.3}$$

and that Ξ verifies

$$r^3|\mathfrak{d}^{\leq k}\Xi| \leq \epsilon, \qquad \nabla_3\Xi \in r^{-1}\mathfrak{d}^{\leq 1}\Gamma_g. \tag{11.1.4}$$

Remark 11.1.1. *The additional conditions* (11.1.3)- (11.1.4) *are verified by the global frame constructed in section 3.6.3 of [53]. These are crucial in deriving the correct structure of the nonlinear terms of the gRW equation for* $\mathfrak{q}$.

Remark 11.1.2. *We note that in reality the estimates* (11.1.1)-(11.1.2) *for* $k_L - 120 \leq s \leq k_L$ *should be relaxed by replacing* δ_{dec} *with* $\frac{3}{4}\delta_{dec}$*. This loss is due to the interpolation between the bootstrap estimates for* k_{large} *and those for* k_{small}*, see for example Lemma 5.1 in [50]. The loss is more than compensated by the fact that the resulting gain* $\frac{3}{4}\delta_{dec}$ *is doubled in nonlinear estimates. The remark also applies to Chapter 12.*

11.1.1 Definition of $\mathfrak{q}$

Recall, see Definition 5.2.2,

$$\begin{aligned}\mathfrak{q} &= q\overline{q}^3\left({}^{(c)}\nabla_3\,{}^{(c)}\nabla_3 A + C_1\,{}^{(c)}\nabla_3 A + C_2 A\right),\\ C_1 &= 2\mathrm{tr}\,\underline{\chi} - 2\frac{{}^{(a)}\mathrm{tr}\underline{\chi}^2}{\mathrm{tr}\,\underline{\chi}} - 4i\,{}^{(a)}\mathrm{tr}\underline{\chi},\\ C_2 &= \frac{1}{2}\mathrm{tr}\,\underline{\chi}^2 - 4\,{}^{(a)}\mathrm{tr}\underline{\chi}^2 + \frac{3}{2}\frac{{}^{(a)}\mathrm{tr}\underline{\chi}^4}{\mathrm{tr}\,\underline{\chi}^2} + i\left(-2\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\underline{\chi} + 4\frac{{}^{(a)}\mathrm{tr}\underline{\chi}^3}{\mathrm{tr}\,\underline{\chi}}\right).\end{aligned} \tag{11.1.5}$$

[1]This is consistent with the value of k_L used in the bootstrap assumption needed in the proof of Theorem M1 (see section 1.5.3).

11.1.2 Full Regge Wheeler equation for $\mathfrak{q}$

The real part of $\mathfrak{q}$, denoted $\psi = \Re(\mathfrak{q})$, verifies according to Proposition 5.2.14 the equation

$$\dot{\square}_2\psi - V_0\psi = -\frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_T\psi + N, \qquad V_0 = \frac{4\Delta}{(r^2+a^2)|q|^2}, \tag{11.1.6}$$

with the right hand side N being given by

$$N = N_0 + N_L + N_{\text{Err}} \tag{11.1.7}$$

where

- N_0 denotes the zero-th order term in ψ, i.e.

$$N_0 := \big(V - V_0\big)\psi = O\left(\frac{a}{r^4}\right)\psi. \tag{11.1.8}$$

- N_L denotes the linear term in A given by

$$\begin{aligned} N_L =\Re\Bigg(q\overline{q}^3\bigg[&-\frac{8a^2\Delta}{r^2|q|^4}\nabla_{\mathbf{T}}\nabla_3 A - \frac{8a\Delta}{r^2|q|^4}\nabla_{\mathbf{Z}}\nabla_3 A \\ &+ W_4\nabla_4 A + W_3\nabla_3 A + W\cdot\nabla A + W_0 A\bigg]\Bigg) \end{aligned} \tag{11.1.9}$$

where W_4, W_3, W_0 are complex functions of (r,θ), and W is the product of a complex function of (r,θ) with ${}^*\Re(\mathfrak{J})$, having the following fall-off in r

$$q\overline{q}^3 W_4 = q\overline{q}^3 W_3 = q\overline{q}^3 W = O(a), \qquad q\overline{q}^3 W_0 = O\left(\frac{a}{r}\right).$$

Away from the trapping, the following schematic structure will suffice

$$N_L = O(a)\mathfrak{d}^{\leq 1}\nabla_3 A + O(ar^{-1})\mathfrak{d}^{\leq 1}A. \tag{11.1.10}$$

- N_{Err} denotes the quadratic error terms, given schematically by the expression

$$\begin{aligned} N_{\text{Err}} &= N_g + \nabla_3(rN_g) + N_m[\mathfrak{q}], \\ N_g &= r^2\mathfrak{d}^{\leq 2}\big(\Gamma_g\cdot(A,B)\big), \quad N_m[\mathfrak{q}] = \mathfrak{d}^{\leq 1}\big(\Gamma_g\cdot\mathfrak{q}\big). \end{aligned} \tag{11.1.11}$$

We refer to solutions to (11.1.6), with N as above, as solutions to the real gRW.

11.1.3 Factorization of $\mathfrak{q}$

Lemma 11.1.3. *We have*

$$\begin{aligned}
&{}^{(c)}\nabla_3\left({}^{(c)}\nabla_3\left(\frac{(\overline{tr\underline{X}})^2}{(\Re(tr\underline{X}))^2(tr\underline{X})^2}A\right)-\frac{r^2}{2}F_2A\right)\\
=\;& \Big(O(r^{-2})+r^{-1}\Gamma_g\Big)\mathfrak{q}+r^2\mathfrak{d}^{\leq 1}\Gamma_b\,{}^{(c)}\nabla_3A+r\mathfrak{d}^{\leq 1}\Gamma_bA,
\end{aligned}\tag{11.1.12}$$

where F_2 *is given by*

$$\begin{aligned}
F_1 &:= {}^{(c)}\mathcal{D}\cdot\overline{\underline{\Xi}}+\underline{\Xi}\cdot\overline{\underline{H}}+\overline{\underline{\Xi}}\cdot H-\frac{1}{2}\widehat{\underline{X}}\cdot\overline{\widehat{\underline{X}}},\\
F_2 &:= -\frac{F_1}{tr\underline{X}}+\frac{\overline{F_1}}{\overline{tr\underline{X}}}-\frac{\Re(F1)}{\Re(tr\underline{X})}.
\end{aligned}\tag{11.1.13}$$

Proof. We compute

$$\begin{aligned}
&{}^{(c)}\nabla_3\,{}^{(c)}\nabla_3\left(\frac{(\overline{\mathrm{tr}\underline{X}})^2}{(\Re(\mathrm{tr}\underline{X}))^2(\mathrm{tr}\underline{X})^2}A\right)\\
=\;& {}^{(c)}\nabla_3\left(\frac{(\overline{\mathrm{tr}\underline{X}})^2}{(\Re(\mathrm{tr}\underline{X}))^2(\mathrm{tr}\underline{X})^2}\,{}^{(c)}\nabla_3A+{}^{(c)}\nabla_3\left(\frac{(\overline{\mathrm{tr}\underline{X}})^2}{(\Re(\mathrm{tr}\underline{X}))^2(\mathrm{tr}\underline{X})^2}\right)A\right)\\
=\;& \frac{(\overline{\mathrm{tr}\underline{X}})^2}{(\Re(\mathrm{tr}\underline{X}))^2(\mathrm{tr}\underline{X})^2}\Big({}^{(c)}\nabla_3\,{}^{(c)}\nabla_3A+\widetilde{C}_1\,{}^{(c)}\nabla_3A+\widetilde{C}_2A\Big)
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{C}_1 &:= 2\frac{{}^{(c)}\nabla_3\left(\frac{(\overline{\mathrm{tr}\underline{X}})^2}{(\Re(\mathrm{tr}\underline{X}))^2(\mathrm{tr}\underline{X})^2}\right)}{\frac{(\overline{\mathrm{tr}\underline{X}})^2}{(\Re(\mathrm{tr}\underline{X}))^2(\mathrm{tr}\underline{X})^2}},\\
\widetilde{C}_2 &:= \frac{{}^{(c)}\nabla_3\,{}^{(c)}\nabla_3\left(\frac{(\overline{\mathrm{tr}\underline{X}})^2}{(\Re(\mathrm{tr}\underline{X}))^2(\mathrm{tr}\underline{X})^2}\right)}{\frac{(\overline{\mathrm{tr}\underline{X}})^2}{(\Re(\mathrm{tr}\underline{X}))^2(\mathrm{tr}\underline{X})^2}}.
\end{aligned}$$

Together with the definition of $\mathfrak{q}$, we infer

$$\begin{aligned}
&{}^{(c)}\nabla_3\,{}^{(c)}\nabla_3\left(\frac{(\overline{\mathrm{tr}\underline{X}})^2}{(\Re(\mathrm{tr}\underline{X}))^2(\mathrm{tr}\underline{X})^2}A\right)\\
=\;& \frac{(\overline{\mathrm{tr}\underline{X}})^2}{(\Re(\mathrm{tr}\underline{X}))^2(\mathrm{tr}\underline{X})^2}\left(\frac{1}{q\overline{q}^3}\mathfrak{q}+(\widetilde{C}_1-C_1)\,{}^{(c)}\nabla_3A+(\widetilde{C}_2-C_2)A\right).
\end{aligned}$$

Next, recall the following null structure equation

$$
\begin{aligned}
{}^{(c)}\nabla_3 \mathrm{tr}\underline{X} + \frac{1}{2}(\mathrm{tr}\underline{X})^2 &= {}^{(c)}\mathcal{D}\cdot\overline{\underline{\Xi}} + \underline{\Xi}\cdot\overline{\underline{H}} + \overline{\underline{\Xi}}\cdot H - \frac{1}{2}\widehat{\underline{X}}\cdot\overline{\widehat{\underline{X}}}\\
&=: F_1.
\end{aligned}
$$

We have, in view of the definition of $\widetilde{C}_1$,

$$
\begin{aligned}
\frac{1}{4}\widetilde{C}_1 &= -\frac{{}^{(c)}\nabla_3\mathrm{tr}\underline{X}}{\mathrm{tr}\underline{X}} + \frac{{}^{(c)}\nabla_3\overline{\mathrm{tr}\underline{X}}}{\overline{\mathrm{tr}\underline{X}}} - \frac{{}^{(c)}\nabla_3\Re(\mathrm{tr}\underline{X})}{\Re(\mathrm{tr}\underline{X})}\\
&= \frac{1}{2}\mathrm{tr}\underline{X} - \frac{1}{2}\overline{\mathrm{tr}\underline{X}} + \frac{1}{4\Re(\mathrm{tr}\underline{X})}((\mathrm{tr}\underline{X})^2 + (\overline{\mathrm{tr}\underline{X}})^2) + F_2
\end{aligned}
$$

where

$$
F_2 := -\frac{F_1}{\mathrm{tr}\underline{X}} + \frac{\overline{F_1}}{\overline{\mathrm{tr}\underline{X}}} - \frac{\Re(F1)}{\Re(\mathrm{tr}\underline{X})}.
$$

Also, we have in view of the definition of $\widetilde{C}_2$

$$
\begin{aligned}
\widetilde{C}_2 &= \frac{1}{2}\frac{{}^{(c)}\nabla_3\left(\frac{(\overline{\mathrm{tr}\underline{X}})^2}{(\Re(\mathrm{tr}\underline{X}))^2(\mathrm{tr}\underline{X})^2}\widetilde{C}_1\right)}{\frac{(\overline{\mathrm{tr}\underline{X}})^2}{(\Re(\mathrm{tr}\underline{X}))^2(\mathrm{tr}\underline{X})^2}}\\
&= \frac{1}{2}{}^{(c)}\nabla_3\widetilde{C}_1 + \frac{1}{4}(\widetilde{C}_1)^2\\
&= \frac{1}{2}{}^{(c)}\nabla_3\left(2\mathrm{tr}\underline{X} - 2\overline{\mathrm{tr}\underline{X}} + \frac{1}{\Re(\mathrm{tr}\underline{X})}((\mathrm{tr}\underline{X})^2 + (\overline{\mathrm{tr}\underline{X}})^2) + 4F_2\right)\\
&\quad + \frac{1}{4}\left(2\mathrm{tr}\underline{X} - 2\overline{\mathrm{tr}\underline{X}} + \frac{1}{\Re(\mathrm{tr}\underline{X})}((\mathrm{tr}\underline{X})^2 + (\overline{\mathrm{tr}\underline{X}})^2) + 4F_2\right)^2.
\end{aligned}
$$

Note that $F_1 \in r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b$ and $F_2 \in \mathfrak{d}^{\leq 1}\Gamma_b$. In view of the above, and using the definition of C_1 and C_2 in (11.1.5), we infer

$$
\begin{aligned}
\widetilde{C}_1 &= 2\mathrm{tr}\underline{X} - 2\overline{\mathrm{tr}\underline{X}} + \frac{1}{\Re(\mathrm{tr}\underline{X})}((\mathrm{tr}\underline{X})^2 + (\overline{\mathrm{tr}\underline{X}})^2) + \mathfrak{d}^{\leq 1}\Gamma_b\\
&= -4i\,{}^{(a)}\mathrm{tr}\underline{\chi} + \frac{2}{\mathrm{tr}\,\underline{\chi}}((\mathrm{tr}\,\underline{\chi})^2 - ({}^{(a)}\mathrm{tr}\underline{\chi})^2) + \mathfrak{d}^{\leq 1}\Gamma_b\\
&= -4i\,{}^{(a)}\mathrm{tr}\underline{\chi} + 2\mathrm{tr}\,\underline{\chi} - \frac{2({}^{(a)}\mathrm{tr}\underline{\chi})^2}{\mathrm{tr}\,\underline{\chi}} + \mathfrak{d}^{\leq 1}\Gamma_b\\
&= C_1 + \mathfrak{d}^{\leq 1}\Gamma_b
\end{aligned}
$$

and

$$\begin{aligned}
\widetilde{C}_2 &= 2\,{}^{(c)}\nabla_3(F_2) + \frac{1}{2}\Bigg(-(\mathrm{tr}\underline{X})^2 + (\overline{\mathrm{tr}\underline{X}})^2 - \frac{1}{\Re(\mathrm{tr}\underline{X})}((\mathrm{tr}\underline{X})^3 + (\overline{\mathrm{tr}\underline{X}})^3) \\
&\quad + \frac{\Re((\mathrm{tr}\underline{X})^2)}{2(\Re(\mathrm{tr}\underline{X}))^2}((\mathrm{tr}\underline{X})^2 + (\overline{\mathrm{tr}\underline{X}})^2)\Bigg) \\
&\quad + \frac{1}{4}\left(2\mathrm{tr}\underline{X} - 2\overline{\mathrm{tr}\underline{X}} + \frac{1}{\Re(\mathrm{tr}\underline{X})}((\mathrm{tr}\underline{X})^2 + (\overline{\mathrm{tr}\underline{X}})^2)\right)^2 + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b \\
&= 2\,{}^{(c)}\nabla_3(F_2) + \frac{1}{2}\Bigg(4i\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\underline{\chi} - \frac{1}{\mathrm{tr}\,\underline{\chi}}(2\mathrm{tr}\,\underline{\chi}^3 - 6\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\underline{\chi}^2) \\
&\quad + \frac{\mathrm{tr}\,\underline{\chi}^2 - {}^{(a)}\mathrm{tr}\underline{\chi}^2}{\mathrm{tr}\,\underline{\chi}^2}(\mathrm{tr}\,\underline{\chi}^2 - {}^{(a)}\mathrm{tr}\underline{\chi}^2)\Bigg) + \frac{1}{4}\left(-4i\,{}^{(a)}\mathrm{tr}\underline{\chi} + \frac{2}{\mathrm{tr}\,\underline{\chi}}(\mathrm{tr}\,\underline{\chi}^2 - {}^{(a)}\mathrm{tr}\underline{\chi}^2)\right)^2 \\
&\quad + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b \\
&= C_2 + 2\,{}^{(c)}\nabla_3(F_2) + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b.
\end{aligned}$$

We deduce

$$\begin{aligned}
&{}^{(c)}\nabla_3\,{}^{(c)}\nabla_3\left(\frac{(\overline{\mathrm{tr}\underline{X}})^2}{(\Re(\mathrm{tr}\underline{X}))^2(\mathrm{tr}\underline{X})^2}A\right) \\
&= \frac{(\overline{\mathrm{tr}\underline{X}})^2}{(\Re(\mathrm{tr}\underline{X}))^2(\mathrm{tr}\underline{X})^2}\left(\frac{1}{q\overline{q}^3}\mathfrak{q} + \mathfrak{d}^{\leq 1}\Gamma_b\,{}^{(c)}\nabla_3 A + \left(2\,{}^{(c)}\nabla_3(F_2) + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b\right)A\right) \\
&= \frac{(\overline{\mathrm{tr}\underline{X}})^2}{(\Re(\mathrm{tr}\underline{X}))^2(\mathrm{tr}\underline{X})^2}\frac{1}{q\overline{q}^3}\mathfrak{q} + \frac{2(\overline{\mathrm{tr}\underline{X}})^2}{(\Re(\mathrm{tr}\underline{X}))^2(\mathrm{tr}\underline{X})^2}\,{}^{(c)}\nabla_3(F_2)A + r^2\mathfrak{d}^{\leq 1}\Gamma_b\,{}^{(c)}\nabla_3 A + r\mathfrak{d}^{\leq 1}\Gamma_b A \\
&= \left(O(r^{-2}) + r^{-1}\Gamma_g\right)\mathfrak{q} + \frac{r^2}{2}\,{}^{(c)}\nabla_3(F_2)A + r^2\mathfrak{d}^{\leq 1}\Gamma_b\,{}^{(c)}\nabla_3 A + r\mathfrak{d}^{\leq 1}\Gamma_b A \\
&= \left(O(r^{-2}) + r^{-1}\Gamma_g\right)\mathfrak{q} + {}^{(c)}\nabla_3\left(\frac{r^2}{2}F_2 A\right) + r^2\mathfrak{d}^{\leq 1}\Gamma_b\,{}^{(c)}\nabla_3 A + r\mathfrak{d}^{\leq 1}\Gamma_b A
\end{aligned}$$

and hence

$$\begin{aligned}
&{}^{(c)}\nabla_3\left({}^{(c)}\nabla_3\left(\frac{(\overline{\mathrm{tr}\underline{X}})^2}{(\Re(\mathrm{tr}\underline{X}))^2(\mathrm{tr}\underline{X})^2}A\right) - \frac{r^2}{2}F_2 A\right) \\
&= \left(O(r^{-2}) + r^{-1}\Gamma_g\right)\mathfrak{q} + r^2\mathfrak{d}^{\leq 1}\Gamma_b\,{}^{(c)}\nabla_3 A + r\mathfrak{d}^{\leq 1}\Gamma_b A
\end{aligned}$$

as stated. This concludes the proof of Lemma 11.1.3. □

Next, we introduce the tensor Ψ.

Definition 11.1.4. *Let $\Psi \in \mathfrak{s}_2(\mathbb{C})$ given by*

$$\Psi := {}^{(c)}\nabla_3\left(\frac{(\overline{tr\underline{X}})^2}{(\Re(tr\underline{X}))^2(tr\underline{X})^2}A\right) - \frac{r^2}{2}F_2 A,$$

where F_2 is given by (11.1.13)*.*

We have the following corollary of Lemma 11.1.3.

Corollary 11.1.5. *Let Ψ as in Definition 11.1.4. Then, (Ψ, A) satisfies the following system of transport equations*

$$\begin{aligned} {}^{(c)}\nabla_3\Psi &= \Big(O(r^{-2}) + r^{-1}\Gamma_g\Big)\mathfrak{q} + r^2\mathfrak{d}^{\leq 1}\Gamma_b\,{}^{(c)}\nabla_3 A + r\mathfrak{d}^{\leq 1}\Gamma_b A, \\ {}^{(c)}\nabla_3\left(\frac{(\overline{tr\underline{X}})^2}{(\Re(tr\underline{X}))^2(tr\underline{X})^2}A\right) &= \Psi + r^2\mathfrak{d}^{\leq 1}(\Gamma_b)\cdot A. \end{aligned}$$

Proof. This is an immediate consequence of Lemma 11.1.3, Definition 11.1.4, and the fact that $F_2 \in \mathfrak{d}^{\leq 1}\Gamma_b$. □

11.1.4 Norms for ψ

We recall that the norms $B^s_p[\psi]$, $E^s_p[\psi]$ and $F^s_p[\psi]$, respectively for the bulk, energy and flux of ψ were defined in section 6.1.5. To simplify notations, we make use of the combined norms

$$BEF^s_p(\tau_1,\tau_2) := B^s_p[\psi](\tau_1,\tau_2) + \sup_{\tau\in[\tau_1,\tau_2]} E^s_p[\psi](\tau) + F^s_p[\psi](\tau_1,\tau_2). \tag{11.1.14}$$

We also recall

$$\begin{aligned} \mathcal{N}_p[\psi, N](\tau_1,\tau_2) &= \int_{\mathcal{M}(\tau_1,\tau_2)}\big(|\nabla_{\widehat{R}}\psi| + r^{-1}|\psi|\big)|N| + \left|\int_{\mathcal{M}_{r\geq 4m}(\tau_1,\tau_2)} r^{p-1}\,\nabla_4(r\psi)\cdot N\right| \\ &+ \left|\int_{\mathcal{M}(\tau_1,\tau_2)}\nabla_{\widehat{T}_\delta}\psi\cdot N\right|, \end{aligned}$$

and the corresponding higher derivatives $\mathcal{N}^s_p[\psi, N]$ norms.

11.2 Control of the full gRW equation for $\mathfrak{q}$

11.2.1 Combined norms for ψ, A

Definition 11.2.1. *We define the following norms modified bulk and energy-flux norms for A and combined norms for (A, ψ)*

$$\begin{aligned}
B_p[A](\tau_1,\tau_2) &= \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+1}\Big(r^4|\nabla_3\,{}^{(c)}\nabla_3 A|^2 + r^4|\nabla_4\nabla_3 A|^2 + r^4|\nabla\nabla_3 A|^2 + r^2|\nabla_3 A|^2 \\
&\quad + r^2|\nabla_4 A|^2 + r^2|\nabla A|^2 + |A|^2\Big),\\
E_p[A](\tau) &= \int_{\Sigma(\tau)} r^{p+2}\Big(r^4\chi_{nt}^2|\nabla_4\nabla_3 A|^2 + r^2|\nabla_{\widehat{R}}\nabla_3 A|^2 + r^2|\nabla_3 A|^2 + r^2|\nabla_4 A|^2 \\
&\quad + |A|^2\Big),\\
F_p[A](\tau_1,\tau_2) &= \int_{\mathcal{A}\cup\Sigma_*(\tau_1,\tau_2)} r^{p+2}\Big(r^4\chi_{nt}^2|\nabla_4\nabla_3 A|^2 + r^2|\nabla_{\widehat{R}}\nabla_3 A|^2 + r^2|\nabla_3 A|^2 + r^2|\nabla_4 A|^2 \\
&\quad + |A|^2\Big),
\end{aligned}$$

where $\chi_{nt} = \chi_{nt}(r)$ denotes a smooth cut-off function equal to 0 on $\mathcal{M}_{trap}$ and equal to 1 on $r \geq 4m$.

The higher derivative norms are defined by the usual procedure

$$B_p^s[A] = B_p[\mathfrak{d}^{\leq s}A], \qquad E_p^s[A] = E_p^s[\mathfrak{d}^{\leq s}A], \qquad F_p^s[A] = F_p^s[\mathfrak{d}^{\leq s}A].$$

We also define the combined norms:

$$\begin{aligned}
E_p^s[\psi, A](\tau) &= E_p^s[\psi](\tau) + E_p^s[A](\tau),\\
B_p^s[\psi, A](\tau_1,\tau_2) &= B_p^s[\psi](\tau_1,\tau_2) + B_p^s[A](\tau_1,\tau_2),\\
F_p^s[\psi, A](\tau_1,\tau_2) &= F_p^s[\psi](\tau_1,\tau_2) + F_p^s[A](\tau_1,\tau_2).
\end{aligned}$$

We use the short hand notation

$$\begin{aligned}
BEF_p^s[A](\tau_1,\tau_2) &= B_p^s[A](\tau_1,\tau_2) + \sup_{\tau\in[\tau_1,\tau_2]} E_p^s[A](\tau) + F_p^s[A](\tau_1,\tau_2),\\
BEF_p^s[\psi, A] &= BEF_p^s[\psi] + BEF_p^s[A].
\end{aligned}$$

Remark 11.2.2. *Note that the derivatives $\nabla_4^2 A$, $\nabla\nabla_4 A$ and $\nabla^2 A$ are missing in the combined norm $BEF_p[A](\tau_1,\tau_2)$, as they cannot be derived by the transport equation methods* *used here. Fortunately, in view of the structure of the N_L term, they are not needed to close the estimates for ψ and thus for $\mathfrak{q}$. Additional derivatives in $E_p[A](\tau)$ and $F_p[A](\tau_1, \tau_2)$ are missing as well and are also not needed to close the estimates for $\mathfrak{q}$, with the exception of the ones recovered in* (11.2.7). *We also remark that*

$$\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+1}\Big(r^2|\mathfrak{d}^{\leq s+1}\nabla_3 A|^2 + |\mathfrak{d}^{\leq s+1}A|^2\Big) \quad \lesssim \quad B^s_p[A](\tau_1, \tau_2),$$

but the norm $B^s_p[A]$ is in fact stronger in powers of r for the ∇_3 derivative.

11.2.2 Weighted estimates for the full gRW system 11.1.6

The main technical results of Chapter 11, which extend Theorems 6.2.1 and 6.2.2 to the full real gRW system 11.1.6, are as follows.

Theorem 11.2.3. *The following holds true, for all $\delta \leq p \leq 2-\delta$ and $2 \leq s \leq k_L$,*

$$BEF^s_p[\psi, A](\tau_1, \tau_2) \lesssim E^s_p[\psi, A](\tau_1) + \mathcal{N}^s_p[\psi, N_{Err}](\tau_1, \tau_2) + \epsilon_0^2 \tau_1^{-2-3\delta_{dec}}, \qquad (11.2.1)$$

where N_{Err} is given by (11.1.11).

Theorem 11.2.4. *Under the same assumptions as before we have for $2 \leq s \leq k_L - 1$, for all $-1+\delta \leq q \leq 1-\delta$,*

$$\begin{aligned} BEF^s_q[\check{\psi}](\tau_1, \tau_2) &\lesssim E^s_q[\check{\psi}](\tau_1) + E^{s+1}_{\max(q,\delta)}[\psi, A](\tau_1) + \check{\mathcal{N}}^s_q[\check{\psi}, N_{Err}](\tau_1, \tau_2) \\ &\quad + \mathcal{N}^{s+1}_{\max(q,\delta)}[\psi, N_{Err}](\tau_1, \tau_2) + \epsilon_0^2 \tau_1^{-2-3\delta_{dec}}, \end{aligned} \qquad (11.2.2)$$

where $\check{\psi} = r^2(\nabla_4\psi + \frac{r}{|q|^2}\psi)$ and

$$\check{\mathcal{N}}^s_q[\check{\psi}, N_{Err}](\tau_1, \tau_2) \quad = \quad \int_{{}^{(ext)}\mathcal{M}} r^{q+2}\check{\nabla}_4 \mathfrak{d}^{\leq s}\check{\psi} \cdot \left(\nabla_4 \mathfrak{d}^{\leq s} N_{Err} + \frac{3}{r}\mathfrak{d}^{\leq s} N_{Err}\right).$$

We postpone the proof of Theorem 11.2.4 to section 11.5 and concentrate our attention to the proof of Theorem 11.2.3.

11.2.3 Proof of Theorem 11.2.3

We restate for convenience the result of Theorem 6.2.1, whose proof was completed in chapter 10.

Theorem 11.2.5 (Basic r^p-weighted estimates)**.** *The following estimates hold true for solutions $\psi \in \mathfrak{s}_2$ of the model gRW equation, see* (6.1.1)*, on $\mathcal{M}$, for all $\delta \leq p \leq 2-\delta$ and $2 \leq s \leq k_L$,*

$$BEF^s_p[\psi](\tau_1,\tau_2) \lesssim E^s_p[\psi](\tau_1) + \mathcal{N}^s_p[\psi, N](\tau_1,\tau_2). \tag{11.2.3}$$

The proof of Theorem 11.2.3 is done in steps as follows.

Step 1. Recall that $N = N_0 + N_L + N_{\text{Err}}$, see (11.1.7). We first eliminate $N_0 + N_L$ from the right hand side of (11.2.3).

Proposition 11.2.6. *The following estimate for solutions ψ of the full gRW equation hold true, for all $s \leq k_L$ and all $\delta \leq p \leq 2-\delta$,*

$$BEF^s_p[\psi](\tau_1,\tau_2) \lesssim E^s_p[\psi](\tau_1) + |a| BEF^s_\delta[\psi, A](\tau_1,\tau_2) + \mathcal{N}^s_p[\psi, N_{Err}](\tau_1,\tau_2). \tag{11.2.4}$$

The proof of Proposition 11.2.6 is an immediate consequence of (11.2.3) and the following lemma.

Lemma 11.2.7. *For $\delta \leq p \leq 2-\delta$, N given by* (11.1.7) *satisfies*

$$\mathcal{N}^s_p[\psi, N](\tau_1,\tau_2) \lesssim |a| BEF^s_\delta[\psi, A](\tau_1,\tau_2) + \mathcal{N}^s_p[\psi, N_{Err}](\tau_1,\tau_2) + \epsilon_0^2 \tau_1^{-2-3\delta_{dec}}. \tag{11.2.5}$$

The proof of Lemma 11.2.7 is given in section 11.3.

Step 2. We eliminate the terms $BEF^s_p[A]$ from the right hand side of (11.2.4) with the help of the proposition below.

Proposition 11.2.8. *The following estimates hold true, for all $s \leq k_L$ and for all $\delta \leq p \leq 2-\delta$,*

$$BEF^s_p[A](\tau_1,\tau_2) \quad \lesssim \quad B^s_p[\psi](\tau_1,\tau_2) + E^s_p[A](\tau_1) + \epsilon_0^2 \tau_1^{-2-3\delta_{dec}}. \tag{11.2.6}$$

Also, we have the following additional control on $\Sigma(\tau)$ with $\tau \in [\tau_1,\tau_2]$, for $s \leq k_L$, for all $\delta \leq p \leq 2-\delta$,

$$\begin{aligned} & \int_{\Sigma(\tau)} r^{p+2}\Big(r^{\min(4,5-\delta-p)} |\nabla_3 {}^{(c)}\nabla_3 \mathfrak{d}^{\leq s} A|^2 + r^4 |\nabla_4 \nabla_3 \mathfrak{d}^{\leq s} A|^2 \\ & + r^4 |\nabla\nabla_3 \mathfrak{d}^{\leq s} \underline{A}|^2 + r^2 |\nabla \mathfrak{d}^{\leq s} \underline{A}|^2 \Big) \\ \lesssim & EB^s_p[\psi](\tau_1,\tau_2) + E^s_p[A](\tau_1) + \epsilon_0^2 \tau_1^{-2-3\delta_{dec}}. \end{aligned} \tag{11.2.7}$$

The proof of Proposition 11.2.8 is given in section 11.4.

Step 3. As a consequence of Proposition 11.2.8 and Proposition 11.2.6, as well as the smallness of $|a|/m$, we deduce, for all $s \leq k_L$ and for all $\delta \leq p \leq 2-\delta$,

$$BEF^s_p[\psi, A](\tau_1, \tau_2) \lesssim E^s_p[\psi, A](\tau_1) + \mathcal{N}^s_p[\psi, N_{\text{Err}}](\tau_1, \tau_2) + \epsilon_0^2 \tau_1^{-2-3\delta_{dec}}$$

as stated. This ends the proof of Theorem 11.2.3.

11.3 Proof of Lemma 11.2.7

Recall that $N = N_0 + N_L + N_{\text{Err}}$, see (11.1.7). Therefore, we have

$$\mathcal{N}^s_p[\psi, N](\tau_1, \tau_2) \lesssim \mathcal{N}^s_p[\psi, N_0](\tau_1, \tau_2) + \mathcal{N}^s_p[\psi, N_L](\tau_1, \tau_2) + \mathcal{N}^s_p[\psi, N_{\text{Err}}](\tau_1, \tau_2).$$

In order to prove Lemma 11.2.7, we need to eliminate the terms $\mathcal{N}^s_p[\psi, N_0]$ and $\mathcal{N}^s_p[\psi, N_L]$.

Recall that by definition of $\mathcal{N}^s_p[\psi, N](\tau_1, \tau_2)$ we have:

$$\begin{aligned}
\mathcal{N}^s_p[\psi, N](\tau_1, \tau_2) &= {}^{(Mor)}\mathcal{N}^s[\psi, N](\tau_1, \tau_2) + {}^{(ext)}\mathcal{N}^s_p[\psi, N](\tau_1, \tau_2) + {}^{(En)}\mathcal{N}^s[\psi, N](\tau_1, \tau_2),\\
{}^{(Mor)}\mathcal{N}^s[\psi, N](\tau_1, \tau_2) &= \sum_{k\leq s} \int_{\mathcal{M}(\tau_1,\tau_2)} \Big(|\nabla_{\widehat{R}}(\mathfrak{d}^k\psi)| + r^{-1}|\mathfrak{d}^k\psi|\Big)|\mathfrak{d}^k N|,\\
{}^{(ext)}\mathcal{N}^s_p[\psi, N](\tau_1, \tau_2) &= \sum_{k\leq s} \left| \int_{{}^{(ext)}\mathcal{M}(\tau_1,\tau_2)} r^{p-1}\, \nabla_4(r\mathfrak{d}^k\psi) \cdot \mathfrak{d}^k N \right|,\\
{}^{(En)}\mathcal{N}^s[\psi, N](\tau_1, \tau_2) &= \sum_{k\leq s} \left| \int_{\mathcal{M}(\tau_1,\tau_2)} \nabla_{\widehat{T}_\delta}(\mathfrak{d}^k\psi) \cdot \mathfrak{d}^k N \right|.
\end{aligned}$$

We then estimate separately the above terms as follows.

Step 1. First, we estimate ${}^{(Mor)}\mathcal{N}^s[\psi, N_0 + N_L] + {}^{(ext)}\mathcal{N}^s_p[\psi, N_0 + N_L]$. We obtain, for $\delta \leq p \leq 2-\delta$,

$${}^{(Mor)}\mathcal{N}^s[\psi, N_0 + N_L](\tau_1, \tau_2) + {}^{(ext)}\mathcal{N}^s_p[\psi, N_0 + N_L](\tau_1, \tau_2) \lesssim |a| B^s_\delta[\psi, A](\tau_1, \tau_2). \quad (11.3.1)$$

This is done in section 11.3.1.

Step 2. Then, we estimate ${}^{(En)}\mathcal{N}^s[\psi, N_0]$ and obtain

$${}^{(En)}\mathcal{N}^s[\psi, N_0](\tau_1, \tau_2) \lesssim |a| \left(\sup_{\tau\in[\tau_1,\tau_2]} E^s[\psi](\tau) + B^s_\delta[\psi](\tau_1, \tau_2) \right). \quad (11.3.2)$$

This is done in section 11.3.2.

Step 3. Next, we estimate ${}^{(En)}\mathcal{N}^s[\psi, N_L]$ and obtain

$$
\begin{aligned}
{}^{(En)}\mathcal{N}^s[\psi, N_L](\tau_1,\tau_2) \;\lesssim\;& |a|\left(\sup_{\tau\in(\tau_1,\tau_2)} E^s_\delta[\psi, A](\tau) + B^s_\delta[\psi, A](\tau_1,\tau_2)\right)\\
&+\epsilon_0^2\tau_1^{-2-3\delta_{dec}}.
\end{aligned}
\tag{11.3.3}
$$

This is done in section 11.3.3.

Step 4. Combining the results of Steps 1-3, we obtain the following, for $\delta \le p \le 2-\delta$,

$$
\begin{aligned}
\mathcal{N}^s_p[\psi, N](\tau_1,\tau_2) \;\lesssim\;& {}^{(Mor)}\mathcal{N}^s[\psi, N_0+N_L](\tau_1,\tau_2) + {}^{(ext)}\mathcal{N}^s_p[\psi, N_0+N_L](\tau_1,\tau_2)\\
&+{}^{(En)}\mathcal{N}^s[\psi, N_0](\tau_1,\tau_2) + {}^{(En)}\mathcal{N}^s[\psi, N_L](\tau_1,\tau_2) + \mathcal{N}^s_p[\psi, N_{\text{Err}}](\tau_1,\tau_2)\\
\lesssim\;& |a|BEF^s_\delta[\psi, A](\tau_1,\tau_2) + \mathcal{N}^s_p[\psi, N_{\text{Err}}](\tau_1,\tau_2) + \epsilon_0^2\tau_1^{-2-3\delta_{dec}}
\end{aligned}
$$

as stated. This concludes the proof of Lemma 11.2.7.

It thus remains to prove estimates (11.3.1), (11.3.2) and (11.3.3). This is done in sections 11.3.1, 11.3.2 and 11.3.3 respectively.

11.3.1 Proof of the estimate (11.3.1)

In this section, we estimate ${}^{(Mor)}\mathcal{N}^s_p[\psi, N_0+N_L](\tau_1,\tau_2) + {}^{(ext)}\mathcal{N}^s_p[\psi, N_0+N_L](\tau_1,\tau_2)$.

We have

$$
\begin{aligned}
&{}^{(Mor)}\mathcal{N}^s[\psi, N_0+N_L](\tau_1,\tau_2)\\
=\;& \int_{\mathcal{M}(\tau_1,\tau_2)}\Big(|\nabla_{\widehat{R}}\mathfrak{d}^{\le s}\psi| + r^{-1}|\mathfrak{d}^{\le s}\psi|\Big)|\mathfrak{d}^{\le s}N|\\
\lesssim\;& \left(\int_{\mathcal{M}(\tau_1,\tau_2)} r^{-1-\delta}\Big(|\nabla_{\widehat{R}}\mathfrak{d}^{\le s}\psi|^2 + r^{-2}|\psi|^2\Big)\right)^{1/2}\left(\int_{\mathcal{M}(\tau_1,\tau_2)} r^{1+\delta}|\mathfrak{d}^{\le s}N|^2\right)^{1/2}\\
\lesssim\;& \left(\int_{\mathcal{M}(\tau_1,\tau_2)} r^{-1-\delta}\Big(|\nabla_{\widehat{R}}\mathfrak{d}^{\le s}\psi|^2\mathbb{1}_{r\le 4m} + \big(|\nabla_3\psi|^2 + r^{-2}|\mathfrak{d}\psi|^2\big)\mathbb{1}_{r\ge 4m} + r^{-2}|\psi|^2\Big)\right)^{1/2}\\
&\times\left(\int_{\mathcal{M}(\tau_1,\tau_2)} r^{1+\delta}|\mathfrak{d}^{\le s}N|^2\right)^{1/2}\\
\lesssim\;& \Big(B^s_\delta[\psi](\tau_1,\tau_2)\Big)^{1/2}\left(\int_{\mathcal{M}(\tau_1,\tau_2)} r^{1+\delta}|\mathfrak{d}^{\le s}N|^2\right)^{1/2}.
\end{aligned}
$$

Since $N_0 = O(ar^{-4})\psi$, we have

$$\int_{\mathcal{M}(\tau_1,\tau_2)} r^{1+\delta}|\mathfrak{d}^{\leq s}N_0|^2 \lesssim |a|\text{Mor}^s[\psi](\tau_1,\tau_2).$$

To estimate the term in N_L we make use of the schematic structure given in (11.1.10)

$$N_L = O(a)\mathfrak{d}^{\leq 1}\nabla_3 A + O(ar^{-1})\mathfrak{d}^{\leq 1}A.$$

Thus,

$$\int_{\mathcal{M}(\tau_1,\tau_2)} r^{1+\delta}|\mathfrak{d}^{\leq s}N_L|^2 \lesssim |a|\int_{\mathcal{M}(\tau_1,\tau_2)} r^{1+\delta}\Big(|\nabla_3\mathfrak{d}^{\leq s+1}A|^2 + r^{-2}|\mathfrak{d}^{\leq s+1}A|^2\Big).$$

Therefore,

$$\begin{aligned}
{}^{(Mor)}\mathcal{N}^s[\psi, N_0+N_L] &\lesssim |a|B^s_\delta[\psi](\tau_1,\tau_2) + |a|\Big(B^s_\delta[\psi](\tau_1,\tau_2)\Big)^{1/2}\\
&\quad\times\left(\int_{\mathcal{M}(\tau_1,\tau_2)} r^{1+\delta}\Big(|\nabla_3\mathfrak{d}^{\leq s+1}A|^2 + r^{-2}|\mathfrak{d}^{\leq s+1}A|^2\Big)\right)^{1/2}\\
&\lesssim |a|\left(B^s_\delta[\psi](\tau_1,\tau_2) + \int_{\mathcal{M}(\tau_1,\tau_2)} r^{1+\delta}\Big(|\nabla_3\mathfrak{d}^{\leq s+1}A|^2 + r^{-2}|\mathfrak{d}^{\leq s+1}A|^2\Big)\right)\\
&\lesssim |a|\Big(B^s_\delta[\psi](\tau_1,\tau_2) + B^s_\delta[A](\tau_1,\tau_2)\Big)\\
&\lesssim |a|B^s_\delta[\psi, A](\tau_1,\tau_2).
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
{}^{(ext)}\mathcal{N}^s_p[\psi, N_0+N_L] &= \left|\int_{{}^{(ext)}\mathcal{M}} r^{p-1}(r\nabla_4)\mathfrak{d}^{\leq s}\psi\cdot\mathfrak{d}^{\leq s}N\right|\\
&\lesssim \left(\int_{{}^{(ext)}\mathcal{M}} r^{p-3}|\mathfrak{d}^{\leq s+1}\psi|^2\right)^{1/2}\left(\int_{{}^{(ext)}\mathcal{M}} r^{p+1}|\mathfrak{d}^{\leq s}N|^2\right)^{1/2}\\
&\lesssim \left({}^{(ext)}B^s_p[\psi]\right)^{1/2}\left(\int_{{}^{(ext)}\mathcal{M}} r^{p+1}|\mathfrak{d}^{\leq s}N|^2\right)^{1/2}.
\end{aligned}$$

Now[2], for $\delta \leq p \leq 2-\delta$,

$$\begin{aligned}
\int_{{}^{(ext)}\mathcal{M}} r^{p+1}|\mathfrak{d}^{\leq s}N_0|^2 &\lesssim |a|\int_{{}^{(ext)}\mathcal{M}} r^{p-7}|\mathfrak{d}^{\leq s}\psi|^2 \lesssim |a|B^s_\delta[\psi],\\
\int_{{}^{(ext)}\mathcal{M}} r^{p+1}|\mathfrak{d}^{\leq s}N_L|^2 &\lesssim |a|\int_{{}^{(ext)}\mathcal{M}} r^{p+1}\big(|\mathfrak{d}^{\leq s+1}\nabla_3 A|^2 + r^{-2}|\mathfrak{d}^{\leq s+1}A|^2\big) \lesssim |a|B^s_\delta[A],
\end{aligned}$$

[2]Recall the definition of the $B_p[A]$ norms and Remark 12.2.2.

from which we deduce, for $\delta \leq p \leq 2-\delta$,

$$
{}^{(ext)}\mathcal{N}^s_p[\psi, N_0+N_L] \lesssim |a|\Big(B^s_\delta[\psi]+B^s_\delta[A]\Big) \lesssim |a| B^s_\delta[\psi, A](\tau_1,\tau_2).
$$

We conclude, for $\delta \leq p \leq 2-\delta$,

$$
{}^{(Mor)}\mathcal{N}^s[\psi, N_0+N_L] + {}^{(ext)}\mathcal{N}^s_p[\psi, N_0+N_L] \lesssim |a| B^s_\delta[\psi, A],
$$

which proves (11.3.1).

11.3.2 Proof of the estimate (11.3.2)

Here we estimate ${}^{(En)}\mathcal{N}^s[\psi, N_0]$. We first observe that

$$
\begin{aligned}
{}^{(En)}\mathcal{N}^s[\psi, N_0](\tau_1,\tau_2) &= \left|\int_{\mathcal{M}(\tau_1,\tau_2)} \nabla_{\widehat{T}_\delta}(\mathfrak{d}^{\leq s}\psi)\cdot\mathfrak{d}^{\leq s}N\right| \\
&\lesssim \left|\int_{\mathcal{M}(\tau_1,\tau_2)} \chi\nabla_{\widehat{T}_\delta}(\mathfrak{d}^{\leq s}\psi)\cdot\mathfrak{d}^{\leq s}N_0\right| \\
&\quad + \int_{\mathcal{M}(\tau_1,\tau_2)} (1-\chi)\big|\nabla_{\widehat{T}_\delta}(\mathfrak{d}^{\leq s}\psi)\big|\big|\mathfrak{d}^{\leq s}N_0\big|,
\end{aligned}
$$

where $\chi=\chi(r)$ is a smooth cut-off function which is 1 on $\mathcal{M}_{trap}$ and vanishes for $r\geq 4m$ and $r\leq r_+(1+\delta_{red})$.

Observe that the second integral above can be bounded just like ${}^{(Mor)}\mathcal{N}^s[\psi, N_0](\tau_1,\tau_2)$, and gives

$$
\int_{\mathcal{M}(\tau_1,\tau_2)} (1-\chi)\big|\nabla_{\widehat{T}_\delta}(\mathfrak{d}^{\leq s}\psi)\big|\big|\mathfrak{d}^{\leq s}N_0\big| \lesssim O(a) B^s_\delta[\psi](\tau_1,\tau_2).
$$

We are therefore left to estimate $|\int_{\mathcal{M}(\tau_1,\tau_2)}\chi\nabla_{\widehat{T}_\delta}(\mathfrak{d}^{\leq s}\psi)\cdot\mathfrak{d}^{\leq s}N_0|$. Using that $N_0=O(ar^{-4})\psi$, we have

$$
\begin{aligned}
\chi\nabla_{\widehat{T}_\delta}(\mathfrak{d}^{\leq s}\psi)\cdot\mathfrak{d}^{\leq s}N_0 &= O(ar^{-4})\chi\nabla_{\widehat{T}_\delta}(\mathfrak{d}^{\leq s}\psi)\cdot\mathfrak{d}^{\leq s}\psi \\
&= \mathbf{D}_\alpha\Big(O(ar^{-4})\chi|\mathfrak{d}^{\leq s}\psi|^2\widehat{T}^\alpha_\delta\Big) - O(ar^{-4})\chi|\mathfrak{d}^{\leq s}\psi|^2\mathbf{D}_\alpha\widehat{T}^\alpha_\delta \\
&\quad -O(ar^{-4})\big(\widehat{T}_\delta(r),\widehat{T}_\delta(\cos\theta)\big)\big(|\chi'|+r^{-1}|\chi|\big)|\mathfrak{d}^{\leq s}\psi|^2.
\end{aligned}
$$

Integrating by parts the first term, we therefore conclude[3]

$$\begin{aligned} {}^{(En)}\mathcal{N}^s[\psi, N_0](\tau_1,\tau_2) &\lesssim \left|\int_{\mathcal{M}(\tau_1,\tau_2)}\chi\nabla_{\widehat{T}_\delta}(\mathfrak{d}^{\leq s}\psi)\cdot\mathfrak{d}^{\leq s}N_0\right| + |a|B^s_\delta[\psi](\tau_1,\tau_2)\\ &\lesssim |a|\left(\sup_{\tau\in[\tau_1,\tau_2]}E^s[\psi](\tau_1,\tau_2) + B^s_\delta[\psi](\tau_1,\tau_2)\right), \end{aligned}$$

which proves (11.3.2).

11.3.3 Proof of the estimate (11.3.3)

Here we estimate ${}^{(En)}\mathcal{N}^s[\psi, N_L]$. As above, we write

$$\begin{aligned} {}^{(En)}\mathcal{N}^s[\psi, N_L](\tau_1,\tau_2) &\lesssim \left|\int_{\mathcal{M}(\tau_1,\tau_2)}\chi\nabla_{\widehat{T}_\delta}(\mathfrak{d}^{\leq s}\psi)\cdot\mathfrak{d}^{\leq s}N_L\right|\\ &\quad + \int_{\mathcal{M}(\tau_1,\tau_2)}(1-\chi)\big|\nabla_{\widehat{T}_\delta}(\mathfrak{d}^{\leq s}\psi)\big|\big|\mathfrak{d}^{\leq s}N_L\big|\\ &\lesssim \left|\int_{\mathcal{M}(\tau_1,\tau_2)}\chi\nabla_{\widehat{T}_\delta}(\mathfrak{d}^{\leq s}\psi)\cdot\mathfrak{d}^{\leq s}N_L\right| + |a|B^s_\delta[\psi, A](\tau_1,\tau_2), \end{aligned}$$

where $\chi = \chi(r)$ is a smooth cut-off function which is 1 on $\mathcal{M}_{trap}$ and vanishes for $r \geq 4m$ and $r \leq r_+(1+\delta_{red})$. Therefore, in what follows, we consider the first integral, and notice that r is bounded on the support of χ so that we can neglect the powers of r.

Also, recall that we have

$$\widehat{T}_\delta = \mathbf{T} + \frac{a}{r^2+a^2}\chi_0\left(\delta^{-1}\frac{\mathcal{T}}{r^3}\right)\mathbf{Z}$$

where $\chi_0(\delta^{-1}\frac{\mathcal{T}}{r^3}) = 0$ on $\mathcal{M}_{trap}$. In particular, we can estimate the term involving $\chi_0(\delta^{-1}\frac{\mathcal{T}}{r^3})$ as above, and we obtain

$${}^{(En)}\mathcal{N}^s[\psi, N_L](\tau_1,\tau_2) \lesssim \left|\int_{\mathcal{M}(\tau_1,\tau_2)}\chi\nabla_{\mathbf{T}}(\mathfrak{d}^{\leq s}\psi)\cdot\mathfrak{d}^{\leq s}N_L\right| + |a|B^s_\delta[\psi, A](\tau_1,\tau_2).$$

Furthermore, since $\psi = \Re(\mathfrak{q})$, and in view of the definition of $\mathfrak{q}$ in (11.1.5), we have

$$\begin{aligned} \psi &= \Re\left(q\overline{q}^3\left({}^{(c)}\nabla_3\,{}^{(c)}\nabla_3 A + C_1\,{}^{(c)}\nabla_3 A + C_2 A\right)\right)\\ &= \Re\left(q\overline{q}^3\nabla_3\nabla_3 A\right) + O(a)\mathfrak{d}^{\leq 1}A. \end{aligned}$$

[3]Notice that weights in r are irrelevant in this estimate since r is bounded on the support of χ.

We introduce the notation

$$\psi_0 \;:=\; \Re\big(q\bar{q}^3\nabla_3\nabla_3 A\big)$$

and obtain

$$\psi \;=\; \psi_0+O(a)\mathfrak{d}^{\leq 1}A.$$

Proceeding as above, we infer

$$^{(En)}\mathcal{N}^s[\psi, N_L](\tau_1,\tau_2) \;\lesssim\; \left|\int_{\mathcal{M}(\tau_1,\tau_2)}\chi\nabla_{\mathbf{T}}(\mathfrak{d}^{\leq s}\psi_0)\cdot\mathfrak{d}^{\leq s}N_L\right|+|a|B^s_\delta[\psi, A](\tau_1,\tau_2).$$

Next, we focus on the term $\chi\nabla_{\mathbf{T}}(\mathfrak{d}^{\leq s}\psi_0)\cdot\mathfrak{d}^{\leq s}N_L$. We have

$$\begin{aligned}
\chi\nabla_{\mathbf{T}}(\mathfrak{d}^{\leq s}\psi_0)\cdot\mathfrak{d}^{\leq s}N_L \;&=\; -\chi(\mathfrak{d}^{\leq s}\psi_0)\cdot\nabla_{\mathbf{T}}\mathfrak{d}^{\leq s}N_L+\mathbf{D}_\alpha\big(\chi(\mathfrak{d}^{\leq s}\psi_0)\cdot(\mathfrak{d}^{\leq s}N_L)\mathbf{T}^\alpha\big)\\
&\quad -\chi(\mathfrak{d}^{\leq s}\psi_0)\cdot(\mathfrak{d}^{\leq s}N_L)\mathbf{D}_\alpha\mathbf{T}^\alpha-\chi'(r)\mathbf{T}(r)(\mathfrak{d}^{\leq s}\psi_0)\cdot(\mathfrak{d}^{\leq s}N_L)\\
&=\; -\chi(\mathfrak{d}^{\leq s}\psi_0)\cdot\mathfrak{d}^{\leq s}\nabla_{\mathbf{T}}N_L+\chi(\mathfrak{d}^{\leq s}\psi_0)\cdot\mathfrak{d}^{\leq s}N_L\\
&\quad +\mathbf{D}_\alpha\big(\chi(\mathfrak{d}^{\leq s}\psi_0)\cdot(\mathfrak{d}^{\leq s}N_L)\mathbf{T}^\alpha\big)\\
&\quad -\chi(\mathfrak{d}^{\leq s}\psi_0)\cdot(\mathfrak{d}^{\leq s}N_L)\mathbf{D}_\alpha\mathbf{T}^\alpha-\chi'(r)\mathbf{T}(r)(\mathfrak{d}^{\leq s}\psi_0)\cdot(\mathfrak{d}^{\leq s}N_L)
\end{aligned}$$

where we used the fact that $[\nabla_{\mathbf{T}},\mathfrak{d}^k]=O(1)\mathfrak{d}^k$ in view of the control of Γ_b and Γ_g on the support of χ.

Observe that, since $N_L=O(a)\,\not{\mathfrak{d}}^{\leq 1}\nabla_3^{\leq 1}A$, we have, using integration by parts[4], and the fact that $\psi_0=\psi+O(a)\mathfrak{d}^{\leq 1}A$,

$$\begin{aligned}
&\Bigg|\int_{\mathcal{M}(\tau_1,\tau_2)}\Big[\chi(\mathfrak{d}^{\leq s}\psi_0)\cdot\mathfrak{d}^{\leq s}N_L+\mathbf{D}_\alpha\big(\chi(\mathfrak{d}^{\leq s}\psi_0)\cdot(\mathfrak{d}^{\leq s}N_L)\mathbf{T}^\alpha\big)\\
&-\chi(\mathfrak{d}^{\leq s}\psi_0)\cdot(\mathfrak{d}^{\leq s}N_L)\mathbf{D}_\alpha\mathbf{T}^\alpha-\chi'(r)\mathbf{T}(r)(\mathfrak{d}^{\leq s}\psi_0)\cdot(\mathfrak{d}^{\leq s}N_L)\Big]\Bigg|\\
\lesssim\;& |a|\left(\sup_{\tau\in[\tau_1,\tau_2]}E_\delta[\psi, A]+B^s_\delta[\psi, A](\tau_1,\tau_2)\right)+\epsilon_0^2\tau_1^{-2-3\delta_{dec}}.
\end{aligned}$$

We deduce from the above that

$$\begin{aligned}
^{(En)}\mathcal{N}^s[\psi_0, N_L](\tau_1,\tau_2) \;\lesssim&\; \left|\int_{\mathcal{M}(\tau_1,\tau_2)}\chi(\mathfrak{d}^{\leq s}\psi_0)\cdot\mathfrak{d}^{\leq s}\nabla_{\mathbf{T}}N_L\right|\\
&+|a|\left(\sup_{\tau\in[\tau_1,\tau_2]}E_\delta[\psi, A]+B^s_\delta[\psi, A](\tau_1,\tau_2)\right)+\epsilon_0^2\tau_1^{-2-3\delta_{dec}}.
\end{aligned}$$

[4]Notice that for the boundary terms, we use the control of A provided by $E_\delta[A]$ as well as the one provided by (11.2.7).

We now consider the term $\chi(\mathfrak{d}^{\leq s}\psi_0)\cdot\mathfrak{d}^{\leq s}\nabla_{\mathbf{T}}N_L$. Recall that, see (11.1.9),

$$N_L = -\Re\left(\frac{8q\overline{q}^3r^2}{|q|^6}(a^2\nabla_{\mathbf{T}}+a\nabla_{\mathbf{Z}})\nabla_3A\right)+O(a)\mathfrak{d}^{\leq 1}\alpha$$

and hence

$$\nabla_{\mathbf{T}}N_L = -\Re\left(\frac{8q\overline{q}^3r^2}{|q|^6}(a^2\nabla_{\mathbf{T}}^2+a\nabla_{\mathbf{T}}\nabla_{\mathbf{Z}})\nabla_3A\right)+O(a)\mathfrak{d}^{\leq 2}\alpha.$$

Since we have

$$\begin{aligned}\mathbf{T} &= \widehat{T}-\frac{a}{r^2+a^2}\mathbf{Z}\\ &= \widehat{R}+\frac{\Delta}{r^2+a^2}e_3-\frac{a}{r^2+a^2}\mathbf{Z},\end{aligned}$$

we infer

$$\begin{aligned}\nabla_{\mathbf{T}}N_L &= -\Re\left(\frac{8q\overline{q}^3r^2}{|q|^6}\left(a^2\nabla_{\mathbf{T}}^2+a\nabla_{\widehat{R}}\nabla_{\mathbf{Z}}+\frac{a\Delta}{r^2+a^2}\nabla_3\nabla_{\mathbf{Z}}-\frac{a^2}{r^2+a^2}\nabla_{\mathbf{Z}}^2\right)\nabla_3A\right)\\ &\quad +O(a)\mathfrak{d}^{\leq 2}\alpha\\ &= -\Re\left(\frac{8q\overline{q}^3r^2}{|q|^6}\left(a^2\nabla_{\mathbf{T}}^2+\frac{a\Delta}{r^2+a^2}\nabla_{\mathbf{Z}}\nabla_3-\frac{a^2}{r^2+a^2}\nabla_{\mathbf{Z}}^2\right)\nabla_3A\right)\\ &\quad +O(a)\nabla_{\widehat{R}}\mathfrak{d}^{\leq 2}\alpha+O(a)\mathfrak{d}^{\leq 2}\alpha\end{aligned}$$

and hence

$$\begin{aligned}&\chi(\mathfrak{d}^{\leq s}\psi_0)\cdot\mathfrak{d}^{\leq s}\nabla_{\mathbf{T}}N_L\\ =\; & -\chi(\mathfrak{d}^{\leq s}\psi_0)\cdot\mathfrak{d}^{\leq s}\Re\left(\frac{8q\overline{q}^3r^2}{|q|^6}\left(a^2\nabla_{\mathbf{T}}^2+\frac{a\Delta}{r^2+a^2}\nabla_{\mathbf{Z}}\nabla_3-\frac{a^2}{r^2+a^2}\nabla_{\mathbf{Z}}^2\right)\nabla_3A\right)\\ & +O(a)\chi\mathfrak{d}^{\leq s}\psi_0\cdot\nabla_{\widehat{R}}\mathfrak{d}^{\leq s+2}\alpha+O(a)\chi(\mathfrak{d}^{\leq s+2}\alpha)\cdot(\mathfrak{d}^{\leq s+2}\alpha)\\ =\; & \chi(\nabla_{\mathbf{T}}\mathfrak{d}^{\leq s}\psi_0)\cdot\mathfrak{d}^{\leq s}\Re\left(\frac{8q\overline{q}^3r^2}{|q|^6}a^2\nabla_{\mathbf{T}}\nabla_3A\right)\\ & -\chi(\mathfrak{d}^{\leq s}\psi_0)\cdot\mathfrak{d}^{\leq s}\Re\left(\frac{8q\overline{q}^3r^2}{|q|^6}\frac{a\Delta}{r^2+a^2}\nabla_{\mathbf{Z}}\nabla_3^2A\right)\\ & -\chi(\nabla_{\mathbf{Z}}\mathfrak{d}^{\leq s}\psi_0)\cdot\mathfrak{d}^{\leq s}\Re\left(\frac{8q\overline{q}^3r^2}{|q|^6}\frac{a^2}{r^2+a^2}\nabla_{\mathbf{Z}}\nabla_3A\right)\\ & +\mathbf{D}_\mu\Big(O(a)\chi\mathfrak{d}^{\leq s}\psi_0\cdot\mathfrak{d}^{\leq s+2}\alpha\big(\widehat{R}^\mu,\mathbf{T}^\mu,\mathbf{Z}^\mu\big)\Big)+O(a)\chi\nabla_{\widehat{R}}\mathfrak{d}^{\leq s}\psi_0\mathfrak{d}^{\leq s+2}\alpha\\ & +O(a)(|\chi'|+|\chi|)(\mathfrak{d}^{\leq s+2}\alpha)\cdot(\mathfrak{d}^{\leq s+2}\alpha)\end{aligned}$$

Recalling

$$\psi_0 = \Re\left(q\overline{q}^3\nabla_3\nabla_3 A\right), \qquad \psi = \psi_0 + O(a)\mathfrak{d}^{\leq 1}A,$$

we deduce

$$\begin{aligned}
& \chi(\mathfrak{d}^{\leq s}\psi_0)\cdot\mathfrak{d}^{\leq s}\nabla_{\mathbf{T}}N_L \\
= \;& \chi(\nabla_{\mathbf{T}}\mathfrak{d}^{\leq s}\Re\left(q\overline{q}^3\nabla_3\nabla_3 A\right))\cdot\mathfrak{d}^{\leq s}\Re\left(\frac{8q\overline{q}^3r^2}{|q|^6}a^2\nabla_{\mathbf{T}}\nabla_3 A\right) \\
& -\chi(\mathfrak{d}^{\leq s}\Re\left(q\overline{q}^3\nabla_3\nabla_3 A\right))\cdot\mathfrak{d}^{\leq s}\Re\left(\frac{8q\overline{q}^3r^2}{|q|^6}\frac{a\Delta}{r^2+a^2}\nabla_{\mathbf{Z}}\nabla_3^2 A\right) \\
& -\chi(\nabla_{\mathbf{Z}}\mathfrak{d}^{\leq s}\Re\left(q\overline{q}^3\nabla_3\nabla_3 A\right))\cdot\mathfrak{d}^{\leq s}\Re\left(\frac{8q\overline{q}^3r^2}{|q|^6}\frac{a^2}{r^2+a^2}\nabla_{\mathbf{Z}}\nabla_3 A\right) \\
& +\mathbf{D}_\mu\Big(O(a)\chi\mathfrak{d}^{\leq s}\psi_0\cdot\mathfrak{d}^{\leq s+2}\alpha\big(\widehat{R}^\mu,\mathbf{T}^\mu,\mathbf{Z}^\mu\big)\Big)+O(a)\chi\nabla_{\widehat{R}}\mathfrak{d}^{\leq s}\psi\mathfrak{d}^{\leq s+2}\alpha \\
& +O(a)(|\chi'|+|\chi|)(\mathfrak{d}^{\leq s+2}\alpha)\cdot(\mathfrak{d}^{\leq s+2}\alpha),
\end{aligned}$$

or

$$\begin{aligned}
& \chi(\mathfrak{d}^{\leq s}\psi_0)\cdot\mathfrak{d}^{\leq s}\nabla_{\mathbf{T}}N_L \\
= \;& \frac{8a^2r^2}{|q|^6}\chi\nabla_3\left(\mathfrak{d}^{\leq s}\Re\left(q\overline{q}^3\nabla_{\mathbf{T}}\nabla_3 A\right)\right)\cdot\mathfrak{d}^{\leq s}\Re\left(q\overline{q}^3\nabla_{\mathbf{T}}\nabla_3 A\right) \\
& -\frac{a\Delta}{r^2+a^2}\frac{8r^2}{|q|^6}\chi(\mathfrak{d}^{\leq s}\Re\left(q\overline{q}^3\nabla_3^2 A\right))\cdot\nabla_{\mathbf{Z}}\left(\mathfrak{d}^{\leq s}\Re\left(q\overline{q}^3\nabla_3^2 A\right)\right) \\
& -\frac{8r^2}{|q|^6}\frac{a^2}{r^2+a^2}\chi\nabla_3\left(\mathfrak{d}^{\leq s}\Re\left(q\overline{q}^3\nabla_{\mathbf{Z}}\nabla_3 A\right)\right)\cdot\mathfrak{d}^{\leq s}\Re\left(q\overline{q}^3\nabla_{\mathbf{Z}}\nabla_3 A\right) \\
& +\mathbf{D}_\mu\Big(O(a)\chi\mathfrak{d}^{\leq s}\psi_0\cdot\mathfrak{d}^{\leq s+2}\alpha\big(\widehat{R}^\mu,\mathbf{T}^\mu,\mathbf{Z}^\mu\big)\Big)+O(a)\chi\nabla_{\widehat{R}}\mathfrak{d}^{\leq s}\psi\mathfrak{d}^{\leq s+2}\alpha \\
& +O(a)(|\chi'|+|\chi|)(\mathfrak{d}^{\leq s+2}\alpha)\cdot(\mathfrak{d}^{\leq s+2}\alpha),
\end{aligned}$$

and hence

$$\begin{aligned}
& \chi(\mathfrak{d}^{\leq s}\psi_0)\cdot\mathfrak{d}^{\leq s}\nabla_{\mathbf{T}}N_L \\
= \;& \mathbf{D}_\mu\Big(O(a)\chi\mathfrak{d}^{\leq s+2}\alpha\cdot\mathfrak{d}^{\leq s+2}\alpha\big(e_3^\mu,\mathbf{Z}^\mu\big)\Big)+\mathbf{D}_\mu\Big(O(a)\chi\mathfrak{d}^{\leq s}\psi_0\cdot\mathfrak{d}^{\leq s+2}\alpha\big(\widehat{R}^\mu,\mathbf{T}^\mu,\mathbf{Z}^\mu\big)\Big) \\
& +O(a)\chi\nabla_{\widehat{R}}\mathfrak{d}^{\leq s}\psi\mathfrak{d}^{\leq s+2}\alpha+O(a)(|\chi'|+|\chi|)(\mathfrak{d}^{\leq s+2}\alpha)\cdot(\mathfrak{d}^{\leq s+2}\alpha).
\end{aligned}$$

We deduce from the above, using integration by parts[5],

$$\begin{aligned}
{}^{(En)}\mathcal{N}^s[\psi_0, N_L](\tau_1,\tau_2) &\lesssim \left|\int_{\mathcal{M}(\tau_1,\tau_2)} \chi(\mathfrak{d}^{\leq s}\psi_0)\cdot\mathfrak{d}^{\leq s}\nabla_{\mathbf{T}} N_L\right| \\
&\quad +|a|\left(\sup_{\tau\in[\tau_1,\tau_2]} E_\delta[\psi, A] + B^s_\delta[\psi, A](\tau_1,\tau_2)\right) + \epsilon_0^2\tau_1^{-2-3\delta_{dec}} \\
&\lesssim |a|\left(\sup_{\tau\in[\tau_1,\tau_2]} E_\delta[\psi, A] + B^s_\delta[\psi, A](\tau_1,\tau_2)\right) + \epsilon_0^2\tau_1^{-2-3\delta_{dec}}
\end{aligned}$$

which concludes the proof of (11.3.3).

11.4 Transport estimates for A

The goal of this section is to prove Proposition 11.2.8, i.e. show, for $s \leq k_L$ and for all $\delta \leq p \leq 2-\delta$,

$$BEF^s_p[A](\tau_1,\tau_2) \lesssim BEF^s_p[\psi](\tau_1,\tau_2) + E^s_p[A](\tau_1) + \epsilon_0^2\tau_1^{-2-3\delta_{dec}}. \tag{11.4.1}$$

To this end, we proceed as follows:

1. First, we state a general lemma for transport equation in ∇_3 in section 11.4.1, see Lemma 11.4.1.
2. Lemma 11.4.1 is then proved in section 11.4.2.
3. Next, we derive estimates for A, $\nabla_3 A$ and $\nabla_4 A$ in section 11.4.3.
4. Then, we control angular derivatives of A in section 11.4.4.
5. Finally, we conclude the proof of Proposition 11.2.8 in section 11.4.5.

11.4.1 General transport estimates

The main result of this section is the following general transport estimates.

[5]For the boundary terms, we use again the control of A provided by $E_\delta[A]$ as well as the one provided by (11.2.7).

Lemma 11.4.1. *Suppose $\Phi_1, \Phi_2 \in \mathfrak{s}_2(\mathbb{C})$ satisfy the differential relation*

$$
{}^{(c)}\nabla_3 \Phi_1 = \Phi_2. \tag{11.4.2}
$$

Also, let $\chi_{nt} = \chi_{nt}(r)$ a smooth cut-off function equal to 0 on $\mathcal{M}_{trap}$ and equal to 1 on $r \geq 4m$. Then, for every $p \geq \delta$, we have

$$
\begin{aligned}
&\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}\big(r^2|\nabla_3\Phi_1|^2 + r^2|\nabla_4\Phi_1|^2 + |\Phi_1|^2\big)\\
&+ \int_{\partial\mathcal{M}^+(\tau_1,\tau_2)} r^{p-2}\big(r^2\chi_{nt}^2|\nabla_4\Phi_1|^2 + |\nabla_{\widehat{R}}\Phi_1|^2 + |\Phi_1|^2\big)\\
\lesssim &\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}\Big(r^2\chi_{nt}^2|\nabla_4\Phi_2|^2 + |\nabla_{\widehat{R}}\Phi_2|^2 + |\Phi_2|^2\Big)\\
&+ \int_{\Sigma(\tau_1)} r^{p-2}\big(r^2\chi_{nt}^2|\nabla_4\Phi_1|^2 + |\nabla_{\widehat{R}}\Phi_1|^2 + |\Phi_1|^2\big)\\
&+ (a^2+\epsilon^2)\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}|\nabla\Phi_1|^2,
\end{aligned} \tag{11.4.3}
$$

where $\partial^+\mathcal{M}(\tau_1,\tau_2)$ denotes the future boundary of $\mathcal{M}(\tau_1,\tau_2)$, i.e.

$$
\partial^+\mathcal{M}(\tau_1,\tau_2) = \mathcal{A}(\tau_1,\tau_2) \cup \Sigma(\tau_2) \cup \Sigma_*(\tau_1,\tau_2).
$$

Remark 11.4.2. *Observe that estimate* (11.4.3) *is conditional with respect to the $\nabla\Phi_1$ appearing on the right hand side.*

In the proof we make us of the divergence theorem in the form, see Lemma 15.3.11 for a general vectorfield X,

$$
-\int_{\partial^+\mathcal{M}(\tau_1,\tau_2)} \mathbf{g}(X,N) + \int_{\partial^-\mathcal{M}(\tau_1,\tau_2)} \mathbf{g}(X,N) = \int_{\mathcal{M}(\tau_1,\tau_2)} \mathrm{Div}(X), \tag{11.4.4}
$$

where N is the normal to the boundary such that $\mathbf{g}(N, e_3) = -1$.

Remark 11.4.3. *We also make use of the following properties of the boundary, see Definition 6.1.5,*

- *On $\mathcal{A}$ we have*

$$
\mathbf{g}(N_{\mathcal{A}}, e_3) = -1, \qquad \mathbf{g}(N_{\mathcal{A}}, e_4) \leq -\frac{1}{10}\delta_{\mathcal{H}}, \qquad \mathbf{g}(N_{\mathcal{A}}, e_a) = O(\delta_{\mathcal{H}}),
$$

- *On the boundary Σ_* we have, with $N_* = N_{\Sigma_*}$,*

$$
\mathbf{g}(N_*, e_3) = -1, \qquad \mathbf{g}(N_*, e_4) \leq -1, \qquad \mathbf{g}(N_*, e_a) = O(r^{-1}).
$$

- *On the spacelike boundary* $\Sigma = \Sigma(\tau)$, $\mathbf{g}(N_\Sigma, N_\Sigma) \le -\frac{1}{100}\frac{m^2}{r^2}$,

$$\mathbf{g}(N_\Sigma, e_4) = -e_4(\tau) \le -\frac{1}{100}\frac{m^2}{r^2}, \quad \mathbf{g}(N_\Sigma, e_3) = -e_3(\tau) = -1.$$

The following basic lemma will be used in the proof of Lemma 11.4.1.

Lemma 11.4.4. *For any function* f, *we have*

$$Div\big(fe_3\big) = e_3(f) + \left(-\frac{2r}{|q|^2} + \Gamma_b\right) f.$$

Proof. We have

$$\begin{aligned}
\mathrm{Div}(e_3) &= \mathbf{g}^{43}\mathbf{g}(\mathbf{D}_4 e_3, e_3) + \mathbf{g}^{43}\mathbf{g}(\mathbf{D}_3 e_3, e_4) + \mathbf{g}^{bc}\mathbf{g}(\mathbf{D}_b e_3, e_c)\\
&= -\frac{1}{2}4\underline{\omega} + \mathrm{tr}\,\underline{\chi} = -\frac{2r}{|q|^2} + \widecheck{\mathrm{tr}\underline{\chi}} + \Gamma_b\\
&= -\frac{2r}{|q|^2} + \Gamma_b
\end{aligned}$$

and hence

$$\mathrm{Div}(fe_3) = f\mathrm{Div}(e_3) + e_3(f) = e_3(f) + \left(-\frac{2r}{|q|^2} + \Gamma_b\right) f$$

as stated. This concludes the proof of Lemma 11.4.4. □

11.4.2 Proof of Lemma 11.4.1

We start with the following lemma.

Lemma 11.4.5. *Suppose* $\Phi_1, \Phi_2 \in \mathfrak{s}_2(\mathbb{C})$ *satisfy the differential relation*

$$^{(c)}\nabla_3 \Phi_1 = \Phi_2. \tag{11.4.5}$$

Then for every $p \ge \delta$ *we have*

$$r|q|^{p-4}|\Phi_1|^2 \lesssim \frac{4}{p^2} r^{-1}|q|^p|\Phi_2|^2 - \frac{2}{p} Div(|q|^{p-2}|\Phi_1|^2 e_3) \tag{11.4.6}$$

and its integral form

$$\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}|\Phi_1|^2 + \int_{\partial\mathcal{M}^+(\tau_1,\tau_2)} r^{p-2}|\Phi_1|^2 \lesssim \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}|\Phi_2|^2 + \int_{\Sigma(\tau_1)} r^{p-2}|\Phi_1|^2.$$

Proof. Multiplying the relation ${}^{(c)}\nabla_3\Phi_1 = \Phi_2$ by $\overline{\Phi_1}$, we deduce

$$\begin{aligned} e_3(|\Phi_1|^2) &= 2\Re\big((\Phi_2 + 2s\underline{\omega}\Phi_1)\cdot\overline{\Phi_1}\big) \\ &= 2\Re\big(\Phi_2\cdot\overline{\Phi_1}\big) + \Gamma_b|\Phi_1|^2. \end{aligned}$$

Multiplying by $|q|^{p-2}$, and using

$$e_3(|q|) = \frac{e_3(|q|^2)}{2|q|} = -\frac{r}{|q|} + O(1)\widecheck{e_3(r)} + O(r^{-1})e_3(\cos\theta) = -\frac{r}{|q|} + r\Gamma_b,$$

we deduce

$$\begin{aligned} 2|q|^{p-2}\Re\big(\Phi_2\cdot\overline{\Phi_1}\big) &= e_3(|q|^{p-2}|\Phi_1|^2) - e_3(|q|^{p-2})|\Phi_1|^2 + |q|^{p-2}\Gamma_b|\Phi_1|^2 \\ &= e_3(|q|^{p-2}|\Phi_1|^2) + (p-2)r|q|^{p-4}|\Phi_1|^2 + r^{p-2}\Gamma_b|\Phi_1|^2. \end{aligned}$$

In view of Lemma 11.4.4, we write

$$\begin{aligned} \mathrm{Div}(|q|^{p-2}|\Phi_1|^2 e_3) &= e_3(|q|^{p-2}|\Phi_1|^2) + \left(-\frac{2r}{|q|^2} + \Gamma_b\right)|q|^{p-2}|\Phi_1|^2 \\ &= 2|q|^{p-2}\Re\big(\Phi_2\cdot\overline{\Phi_1}\big) - (p-2)r|q|^{p-4}|\Phi_1|^2 \\ &\quad + \left(-\frac{2r}{|q|^2} + \Gamma_b\right)|q|^{p-2}|\Phi_1|^2 + r^{p-2}\Gamma_b|\Phi_1|^2 \\ &= 2|q|^{p-2}\Re\big(\Phi_2\cdot\overline{\Phi_1}\big) - pr|q|^{p-4}|\Phi_1|^2 + r^{p-2}\Gamma_b|\Phi_1|^2. \end{aligned}$$

From the above identity we deduce

$$\begin{aligned} & pr|q|^{p-4}|\Phi_1|^2 \\ =\;& 2|q|^{p-2}\Re\big(\Phi_2\cdot\overline{\Phi_1}\big) - \mathrm{Div}(|q|^{p-2}|\Phi_1|^2 e_3) + r^{p-2}\Gamma_b|\Phi_1|^2 \\ =\;& 2\Re\big((\lambda r)^{-1/2}|q|^{p/2}\Phi_2\cdot(\lambda r)^{1/2}|q|^{p/2-2}\overline{\Phi_1}\big) - \mathrm{Div}(|q|^{p-2}|\Phi_1|^2 e_3) + r^{p-2}\Gamma_b|\Phi_1|^2 \\ \leq\;& \lambda r|q|^{p-4}|\Phi_1|^2 + \lambda^{-1}r^{-1}|q|^p|\Phi_2|^2 - \mathrm{Div}(|q|^{p-2}|\Phi_1|^2 e_3) + r^{p-2}\Gamma_b|\Phi_1|^2. \end{aligned}$$

We obtain, in view of the control of Γ_b,

$$pr|q|^{p-4}|\Phi_1|^2 \leq \lambda r|q|^{p-4}|\Phi_1|^2 + \lambda^{-1}r^{-1}|q|^p|\Phi_2|^2 + O(\epsilon)r|q|^{p-4}|\Phi_1|^2 - \mathrm{Div}(|q|^{p-2}|\Phi_1|^2 e_3).$$

Therefore, for $p \geq \delta$, choosing $\lambda = \frac{p}{2}$, we infer, for ϵ sufficiently small,

$$r|q|^{p-4}|\Phi_1|^2 \lesssim \frac{4}{p^2}r^{-1}|q|^p|\Phi_2|^2 - \frac{2}{p}\mathrm{Div}(|q|^{p-2}|\Phi_1|^2 e_3)$$

which is precisely (11.4.6).

The integral form (11.4.7) of the inequality then follows by the divergence theorem, see (11.4.4), and Remark 11.4.3. This concludes the proof of Lemma 11.4.5. □

Next, we need to control $\nabla_{\widehat{R}}\Phi_1$ for solutions Φ_1 of the transport equation ${}^{(c)}\nabla_3\Phi_1 = \Phi_2$. To this end, we start with the following commutation lemma.

Lemma 11.4.6. *Let $U \in \mathfrak{s}_k$. Then, we have*

$$\left[\nabla_3, \frac{r^2+a^2}{|q|^2}\nabla_{\widehat{R}}\right] U = O((a,\epsilon)r^{-1})\nabla U + O(r^{-1}\epsilon)\nabla_4 U + O(r^{-2})\nabla_3 U + O(r^{-3})U$$

Proof. Recall that $\widehat{R}$ is given by

$$\widehat{R} = \frac{1}{2}\left(\frac{|q|^2}{r^2+a^2}e_4 - \frac{\Delta}{r^2+a^2}e_3\right)$$

so that

$$\frac{r^2+a^2}{|q|^2}\widehat{R} = \frac{1}{2}\left(e_4 - \frac{\Delta}{|q|^2}e_3\right).$$

We infer

$$\begin{aligned}\left[\nabla_3, \frac{r^2+a^2}{|q|^2}\nabla_{\widehat{R}}\right] &= \frac{1}{2}[\nabla_3,\nabla_4] - \frac{1}{2}e_3\left(\frac{\Delta}{|q|^2}\right)\nabla_3\\ &= \frac{1}{2}[\nabla_3,\nabla_4] - \frac{1}{2}\left(\partial_r\left(\frac{\Delta}{|q|^2}\right)e_3(r) + O(r^{-2})e_3(\cos\theta)\right)\nabla_3\\ &= \frac{1}{2}[\nabla_3,\nabla_4] + \frac{1}{2}\left(\partial_r\left(\frac{\Delta}{|q|^2}\right) + r^{-1}\Gamma_b\right)\nabla_3.\end{aligned}$$

Also, note that the commutation formula for $[\nabla_4,\nabla_3]$ of Corollary A.1.1 implies

$$\begin{aligned}[\nabla_4,\nabla_3]U &= \big(O(ar^{-2})+\Gamma_b\big)\nabla U + 2\omega\nabla_3 U + \Gamma_b\nabla_4 U + O(r^{-3})U\\ &= \big(O(ar^{-2})+\Gamma_b\big)\nabla U + 2\left(-\frac{1}{2}\partial_r\left(\frac{\Delta}{|q|^2}\right) + \breve{\omega}\right)\nabla_3 U + \Gamma_b\nabla_4 U + O(r^{-3})U\\ &= \big(O(ar^{-2})+\Gamma_b\big)\nabla U + \left(-\partial_r\left(\frac{\Delta}{|q|^2}\right) + \Gamma_g\right)\nabla_3 U + \Gamma_b\nabla_4 U + O(r^{-3})U,\end{aligned}$$

where we used the definition of $\breve{\omega}$ and the fact that $\breve{\omega} \in \Gamma_g$. We deduce

$$\left[\nabla_3, \frac{r^2+a^2}{|q|^2}\nabla_{\widehat{R}}\right] U = \big(O(ar^{-2})+\Gamma_b\big)\nabla U + (O(r^{-2})+\Gamma_g)\nabla_3 U + \Gamma_b\nabla_4 U + O(r^{-3})U.$$

Together with the control of Γ_g and Γ_b, we deduce

$$\left[\nabla_3, \frac{r^2+a^2}{|q|^2}\nabla_{\widehat{R}}\right] U = O((a,\epsilon)r^{-1})\nabla U + O(r^{-1}\epsilon)\nabla_4 U + O(r^{-2})\nabla_3 U + O(r^{-3})U$$

as stated. This concludes the proof of Lemma 11.4.6. □

We next, we consider the following transport lemma.

Lemma 11.4.7. *Suppose $\Phi_1, \Phi_2 \in \mathfrak{s}_2(\mathbb{C})$ satisfy the relation ${}^{(c)}\nabla_3\Phi_1 = \Phi_2$. Then, for every $p \geq \delta$, we have*

$$\begin{aligned}
&\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}\big(|\nabla_{\widehat{R}}\Phi_1|^2 + |\Phi_1|^2\big) + \int_{\partial\mathcal{M}^+(\tau_1,\tau_2)} r^{p-2}\big(|\nabla_{\widehat{R}}\Phi_1|^2 + |\Phi_1|^2\big)\\
&\lesssim \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}\Big(|\nabla_{\widehat{R}}\Phi_2|^2 + |\Phi_2|^2\Big) + \int_{\Sigma(\tau_1)} r^{p-2}\big(|\nabla_{\widehat{R}}\Phi_1|^2 + |\Phi_1|^2\big)\\
&\quad + \epsilon^2 \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}|\nabla_4\Phi_1|^2 + (a^2+\epsilon^2)\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}|\nabla\Phi_1|^2.
\end{aligned} \tag{11.4.7}$$

Proof. We commute the transport equation for Φ_1 with $\frac{r^2+a^2}{|q|^2}\nabla_{\widehat{R}}$. Together with Lemma 11.4.6, we infer

$$\begin{aligned}
{}^{(c)}\nabla_3\left(\frac{r^2+a^2}{|q|^2}\nabla_{\widehat{R}}\Phi_1\right) &= \left[\nabla_3 - 2s\underline{\omega}, \frac{r^2+a^2}{|q|^2}\nabla_{\widehat{R}}\right]\Phi_1 + \frac{r^2+a^2}{|q|^2}\nabla_{\widehat{R}}\Phi_2\\
&= O((a,\epsilon)r^{-1})\nabla\Phi_1 + O(r^{-1}\epsilon)\nabla_4\Phi_1 + O(r^{-2})\nabla_3\Phi_1 + O(r^{-3})\Phi_1\\
&\quad +2s\frac{r^2+a^2}{|q|^2}(\nabla_{\widehat{R}}\underline{\omega})\Phi_1 + O(1)\nabla_{\widehat{R}}\Phi_2
\end{aligned}$$

and hence, since $\nabla_{\widehat{R}}\underline{\omega} = \mathfrak{d}\Gamma_b = O(r^{-1})$, we obtain, using also ${}^{(c)}\nabla_3\Phi_1 = \Phi_2$,

$$\begin{aligned}
{}^{(c)}\nabla_3\left(\frac{r^2+a^2}{|q|^2}\nabla_{\widehat{R}}\Phi_1\right) &= O((a,\epsilon)r^{-1})\nabla\Phi_1 + O(r^{-1}\epsilon)\nabla_4\Phi_1 + O(r^{-1})\Phi_1\\
&\quad +O(r^{-2})\Phi_2 + O(1)\nabla_{\widehat{R}}\Phi_2.
\end{aligned}$$

Applying (11.4.7) to this transport equation, we infer

$$\begin{aligned}
&\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}|\nabla_{\widehat{R}}\Phi_1|^2 + \int_{\partial\mathcal{M}^+(\tau_1,\tau_2)} r^{p-2}|\nabla_{\widehat{R}}\Phi_1|^2\\
&\lesssim \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}\Big((a^2+\epsilon^2)r^{-2}|\nabla\Phi_1|^2 + r^{-2}\epsilon^2|\nabla_4\Phi_1|^2 + r^{-2}|\Phi_1|^2 + r^{-4}|\Phi_2|^2 + |\nabla_{\widehat{R}}\Phi_2|^2\Big)\\
&\quad + \int_{\Sigma(\tau_1)} r^{p-2}|\nabla_{\widehat{R}}\Phi_1|^2.
\end{aligned}$$

Together with (11.4.7), this yields

$$\begin{aligned}
&\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}\big(|\nabla_{\widehat{R}}\Phi_1|^2 + |\Phi_1|^2\big) + \int_{\partial\mathcal{M}^+(\tau_1,\tau_2)} r^{p-2}\big(|\nabla_{\widehat{R}}\Phi_1|^2 + |\Phi_1|^2\big)\\
&\lesssim \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}\Big(|\nabla_{\widehat{R}}\Phi_2|^2 + |\Phi_2|^2\Big) + \int_{\Sigma(\tau_1)} r^{p-2}\big(|\nabla_{\widehat{R}}\Phi_1|^2 + |\Phi_1|^2\big)\\
&\quad + \epsilon^2 \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}|\nabla_4\Phi_1|^2 + (a^2+\epsilon^2)\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}|\nabla\Phi_1|^2
\end{aligned}$$

as stated. This concludes the proof of Lemma 11.4.7. □

We are now ready to prove Lemma 11.4.1.

Proof of Lemma 11.4.1. We need to derive an estimate for $\chi_{nt}\nabla_4\Phi_1$, where $\chi_{nt}=\chi_{nt}(r)$ denotes a smooth cut-off function equal to 0 on $\mathcal{M}_{trap}$ and equal to 1 on $r\geq 4m$. Note that, in view of Lemma 4.2.2,

$$[\,{}^{(c)}\nabla_3,\,{}^{(c)}\nabla_4]U \;=\; O\big((a+\epsilon)r^{-1}\big)\,{}^{(c)}\nabla U+O(r^{-3})U$$

so that

$$\begin{aligned}{}^{(c)}\nabla_3(\chi_{nt}\,{}^{(c)}\nabla_4\Phi_1) \;&=\; \chi_{nt}\,{}^{(c)}\nabla_4\Phi_2+\partial_r\chi_{nt}e_3(r)\,{}^{(c)}\nabla_4\Phi_1+O\big((a+\epsilon)r^{-1}\big)\,{}^{(c)}\nabla\Phi_1\\ &\quad +O(r^{-3})\Phi_1.\end{aligned}$$

Since ${}^{(c)}\nabla_3\Phi_1=\Phi_2$, and since $\nabla_4=O(1)\nabla_{\widehat{R}}+O(1)\nabla_3$, we infer

$$\begin{aligned}{}^{(c)}\nabla_3(\chi_{nt}\,{}^{(c)}\nabla_4\Phi_1) \;&=\; \chi_{nt}\,{}^{(c)}\nabla_4\Phi_2+O(1)\partial_r\chi_{nt}\nabla_{\widehat{R}}\Phi_1+O(1)\partial_r\chi_{nt}\Phi_2\\ &\quad +O\big((a+\epsilon)r^{-1}\big)\,{}^{(c)}\nabla\Phi_1+O(r^{-3})\Phi_1.\end{aligned}$$

Since $p+1>p\geq\delta$, applying (11.4.7) to this transport equation with p replaced by $p+1$, and using the control of Φ_1 provided by Lemma 11.4.7, we infer

$$\begin{aligned}&\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}\big(r^2\chi_{nt}^2|\nabla_4\Phi_1|^2+|\nabla_{\widehat{R}}\Phi_1|^2+|\Phi_1|^2\big)\\ &+\int_{\partial\mathcal{M}^+(\tau_1,\tau_2)} r^{p-2}\big(r^2\chi_{nt}^2|\nabla_4\Phi_1|^2+|\nabla_{\widehat{R}}\Phi_1|^2+|\Phi_1|^2\big)\\ \lesssim&\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}\Big(r^2\chi_{nt}^2|\nabla_4\Phi_2|^2+|\nabla_{\widehat{R}}\Phi_2|^2+|\Phi_2|^2\Big)\\ &+\int_{\Sigma(\tau_1)} r^{p-2}\big(r^2\chi_{nt}^2|\nabla_4\Phi_1|^2+|\nabla_{\widehat{R}}\Phi_1|^2+|\Phi_1|^2\big)\\ &+\epsilon^2\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}|\nabla_4\Phi_1|^2+(a^2+\epsilon^2)\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}|\nabla\Phi_1|^2.\end{aligned}$$

Together with the fact that ${}^{(c)}\nabla_3\Phi_1 = \Phi_2$, we deduce

$$\begin{aligned}
&\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}\big(r^2|\nabla_3\Phi_1|^2 + r^2\chi_{nt}^2|\nabla_4\Phi_1|^2 + |\nabla_{\widehat{R}}\Phi_1|^2 + |\Phi_1|^2\big)\\
&+\int_{\partial\mathcal{M}^+(\tau_1,\tau_2)} r^{p-2}\big(r^2\chi_{nt}^2|\nabla_4\Phi_1|^2 + |\nabla_{\widehat{R}}\Phi_1|^2 + |\Phi_1|^2\big)\\
\lesssim&\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}\Big(r^2\chi_{nt}^2|\nabla_4\Phi_2|^2 + |\nabla_{\widehat{R}}\Phi_2|^2 + |\Phi_2|^2\Big)\\
&+\int_{\Sigma(\tau_1)} r^{p-2}\big(r^2\chi_{nt}^2|\nabla_4\Phi_1|^2 + |\nabla_{\widehat{R}}\Phi_1|^2 + |\Phi_1|^2\big)\\
&+\epsilon^2\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}|\nabla_4\Phi_1|^2 + (a^2+\epsilon^2)\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}|\nabla\Phi_1|^2.
\end{aligned}$$

Since ∇_4 is spanned by $\nabla_{\widehat{R}}$ and ∇_3, and since $\chi_{nt} = 1$ in the region $r \geq 4m$, we infer

$$\begin{aligned}
&\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}\big(r^2|\nabla_3\Phi_1|^2 + r^2|\nabla_4\Phi_1|^2 + |\Phi_1|^2\big)\\
&+\int_{\partial\mathcal{M}^+(\tau_1,\tau_2)} r^{p-2}\big(r^2\chi_{nt}^2|\nabla_4\Phi_1|^2 + |\nabla_{\widehat{R}}\Phi_1|^2 + |\Phi_1|^2\big)\\
\lesssim&\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}\Big(r^2\chi_{nt}^2|\nabla_4\Phi_2|^2 + |\nabla_{\widehat{R}}\Phi_2|^2 + |\Phi_2|^2\Big)\\
&+\int_{\Sigma(\tau_1)} r^{p-2}\big(r^2\chi_{nt}^2|\nabla_4\Phi_1|^2 + |\nabla_{\widehat{R}}\Phi_1|^2 + |\Phi_1|^2\big)\\
&+\epsilon^2\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}|\nabla_4\Phi_1|^2 + (a^2+\epsilon^2)\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}|\nabla\Phi_1|^2.
\end{aligned}$$

For $\epsilon > 0$ small enough, this yields

$$\begin{aligned}
&\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}\big(r^2|\nabla_3\Phi_1|^2 + r^2|\nabla_4\Phi_1|^2 + |\Phi_1|^2\big)\\
&+\int_{\partial\mathcal{M}^+(\tau_1,\tau_2)} r^{p-2}\big(r^2\chi_{nt}^2|\nabla_4\Phi_1|^2 + |\nabla_{\widehat{R}}\Phi_1|^2 + |\Phi_1|^2\big)\\
\lesssim&\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}\Big(r^2\chi_{nt}^2|\nabla_4\Phi_2|^2 + |\nabla_{\widehat{R}}\Phi_2|^2 + |\Phi_2|^2\Big)\\
&+\int_{\Sigma(\tau_1)} r^{p-2}\big(r^2\chi_{nt}^2|\nabla_4\Phi_1|^2 + |\nabla_{\widehat{R}}\Phi_1|^2 + |\Phi_1|^2\big)\\
&+(a^2+\epsilon^2)\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}|\nabla\Phi_1|^2
\end{aligned}$$

as stated in (11.4.3). This concludes the proof of Lemma 11.4.1. □

11.4.3 Estimates for A, $\nabla_3 A$ and $\nabla_4 A$

Recall from Corollary 11.1.5 that (Ψ, A) satisfies the following system of transport equations

$$\begin{aligned}
{}^{(c)}\nabla_3\Psi &= \Big(O(r^{-2})+r^{-1}\Gamma_g\Big)\mathfrak{q}+r^2\mathfrak{d}^{\leq 1}\Gamma_b\,{}^{(c)}\nabla_3 A+r\mathfrak{d}^{\leq 1}\Gamma_b A,\\
{}^{(c)}\nabla_3\left(\frac{(\overline{\mathrm{tr}\underline{X}})^2}{(\Re(\mathrm{tr}\underline{X}))^2(\mathrm{tr}\underline{X})^2}A\right) &= \Psi+r^2\mathfrak{d}^{\leq 1}(\Gamma_b)\cdot A,
\end{aligned} \tag{11.4.8}$$

where Ψ is given in view of Definition 11.1.4 by

$$\Psi = {}^{(c)}\nabla_3\left(\frac{(\overline{\mathrm{tr}\underline{X}})^2}{(\Re(\mathrm{tr}\underline{X}))^2(\mathrm{tr}\underline{X})^2}A\right)-\frac{r^2}{2}F_2A,$$

with F_2 given by (11.1.13).

To state the next proposition, we introduce the following partial norms for A which do not provide control for angular derivatives.

Definition 11.4.8. *We define*

$$\begin{aligned}
\dot{B}_p[A](\tau_1,\tau_2) &= \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+1}\Big(r^4|\nabla_3\,{}^{(c)}\nabla_3 A|^2+r^4|\nabla_4\nabla_3 A|^2+r^2|\nabla_3 A|^2+r^2|\nabla_4 A|^2+|A|^2\Big),\\
\dot{E}_p[A](\tau) &= \int_{\Sigma(\tau)} r^{p+2}\Big(r^4\chi_{nt}^2|\nabla_4\nabla_3 A|^2+r^2|\nabla_{\widehat{R}}\nabla_3 A|^2+r^2|\nabla_3 A|^2+r^2|\nabla_4 A|^2\\
&\quad +|A|^2\Big),\\
\dot{F}_p[A](\tau_1,\tau_2) &= \int_{\mathcal{A}\cup\Sigma_*(\tau_1,\tau_2)} r^{p+2}\Big(r^4\chi_{nt}^2|\nabla_4\nabla_3 A|^2+r^2|\nabla_{\widehat{R}}\nabla_3 A|^2+r^2|\nabla_3 A|^2+r^2|\nabla_4 A|^2\\
&\quad +|A|^2\Big),
\end{aligned}$$

where $\chi_{nt}=\chi_{nt}(r)$ denotes a smooth cut-off function equal to 0 on $\mathcal{M}_{trap}$ and equal to 1 on $r\geq 4m$.

We also define the combined norms

$$\dot{BEF}_p[A](\tau_1,\tau_2) = \dot{B}_p[A](\tau_1,\tau_2)+\sup_{\tau\in[\tau_1,\tau_2]}\dot{E}_p[A]+\dot{F}_p[A](\tau_1,\tau_2). \tag{11.4.9}$$

Remark 11.4.9. *Note that the norms above do not contain angular derivatives which will have to be recovered later.*

We prove the following proposition.

Proposition 11.4.10. *The following estimates hold true, for all $\delta \leq p \leq 2-\delta$,*

$$\begin{aligned} \dot{BEF}_p[A](\tau_1,\tau_2) \lesssim & B_p[\mathfrak{q}](\tau_1,\tau_2) + \dot{E}_p[A](\tau_1) \\ & + (a^2+\epsilon^2)\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+3}\big(r^2|\nabla\nabla_3 A|^2 + |\nabla A|^2\big). \end{aligned} \tag{11.4.10}$$

Proof. Recall from (11.4.8) that we have

$$ {}^{(c)}\nabla_3\Psi = \Big(O(r^{-2}) + r^{-1}\Gamma_g\Big)\mathfrak{q} + r^2\mathfrak{d}^{\leq 1}\Gamma_b\, {}^{(c)}\nabla_3 A + r\mathfrak{d}^{\leq 1}\Gamma_b A. $$

By applying Lemma 11.4.1 with $\Phi_1 = \Psi$ and

$$\begin{aligned} \Phi_2 &= \Big(O(r^{-2}) + r^{-1}\Gamma_g\Big)\mathfrak{q} + r^2\mathfrak{d}^{\leq 1}\Gamma_b\, {}^{(c)}\nabla_3 A + r\mathfrak{d}^{\leq 1}\Gamma_b A \\ &= O(r^{-2})\mathfrak{q} + O(r\epsilon)\nabla_3 A + O(\epsilon)A, \end{aligned}$$

where we used the control of Γ_g and Γ_b, we obtain for $p' \geq \delta$,

$$\begin{aligned} & \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p'-3}\big(r^2|\nabla_3\Psi|^2 + r^2|\nabla_4\Psi|^2 + |\Psi|^2\big) \\ & + \int_{\partial\mathcal{M}^+(\tau_1,\tau_2)} r^{p'-2}\big(r^2\chi_{nt}^2|\nabla_4\Psi|^2 + |\nabla_{\widehat{R}}\Psi|^2 + |\Psi|^2\big) \\ \lesssim & \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p'-5}\Big(r^2\chi_{nt}^2|\nabla_4\mathfrak{q}|^2 + |\nabla_{\widehat{R}}\mathfrak{q}|^2 + |\mathfrak{q}|^2\Big) \\ & + \int_{\Sigma(\tau_1)} r^{p'-2}\big(r^2\chi_{nt}^2|\nabla_4\Psi|^2 + |\nabla_{\widehat{R}}\Psi|^2 + |\Psi|^2\big) + (a^2+\epsilon^2)\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p'-1}|\nabla\Psi|^2 \\ & + \epsilon^2\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p'+1}\Big(r^2\chi_{nt}^2|\nabla_4\nabla_3 A|^2 + |\nabla_{\widehat{R}}\nabla_3 A|^2 + |\nabla_3 A|^2\Big) \\ & + \epsilon^2\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p'-1}\Big(r^2\chi_{nt}^2|\nabla_4 A|^2 + |\nabla_{\widehat{R}} A|^2 + |A|^2\Big) \end{aligned}$$

We choose $p' = p+2$ and obtain for $\delta \leq p \leq 2-\delta$, noticing that $p' \geq 2+\delta > \delta$ in that case,

$$\begin{aligned}
&\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}\big(r^2|\nabla_3\Psi|^2+r^2|\nabla_4\Psi|^2+|\Psi|^2\big)\\
&+\int_{\partial\mathcal{M}^+(\tau_1,\tau_2)} r^p\big(r^2\chi_{nt}^2|\nabla_4\Psi|^2+|\nabla_{\widehat{R}}\Psi|^2+|\Psi|^2\big)\\
\lesssim&\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}\Big(r^2\chi_{nt}^2|\nabla_4\mathfrak{q}|^2+|\nabla_{\widehat{R}}\mathfrak{q}|^2+|\mathfrak{q}|^2\Big)\\
&+\int_{\Sigma(\tau_1)} r^p\big(r^2\chi_{nt}^2|\nabla_4\Psi|^2+|\nabla_{\widehat{R}}\Psi|^2+|\Psi|^2\big)+(a^2+\epsilon^2)\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+1}|\nabla\Psi|^2\\
&+\epsilon^2\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+3}\Big(r^2\chi_{nt}^2|\nabla_4\nabla_3 A|^2+|\nabla_{\widehat{R}}\nabla_3 A|^2+|\nabla_3 A|^2\Big)\\
&+\epsilon^2\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+1}\Big(r^2\chi_{nt}^2|\nabla_4 A|^2+|\nabla_{\widehat{R}} A|^2+|A|^2\Big)
\end{aligned}$$

In view of the definition of the norm $\dot{B}_p[A]$ and $B_p[\mathfrak{q}]$, we infer, for $\delta\leq p\leq 2-\delta$,

$$\begin{aligned}
&\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}\big(r^2|\nabla_3\Psi|^2+r^2|\nabla_4\Psi|^2+|\Psi|^2\big)\\
&+\int_{\partial\mathcal{M}^+(\tau_1,\tau_2)} r^p\big(r^2\chi_{nt}^2|\nabla_4\Psi|^2+|\nabla_{\widehat{R}}\Psi|^2+|\Psi|^2\big)\\
\lesssim& B_p[\mathfrak{q}](\tau_1,\tau_2)+\int_{\Sigma(\tau_1)} r^p\big(r^2\chi_{nt}^2|\nabla_4\Psi|^2+|\nabla_{\widehat{R}}\Psi|^2+|\Psi|^2\big)\\
&+(a^2+\epsilon^2)\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+1}|\nabla\Psi|^2+\epsilon^2\dot{B}_p[A](\tau_1,\tau_2).
\end{aligned}\tag{11.4.11}$$

Next, recall from (11.4.8) that we have

$$^{(c)}\nabla_3\left(\frac{(\overline{\mathrm{tr}\underline{X}})^2}{(\Re(\mathrm{tr}\underline{X}))^2(\mathrm{tr}\underline{X})^2}A\right) = \Psi+r^2\mathfrak{d}^{\leq 1}(\Gamma_b)\cdot A.$$

By applying Lemma 11.4.1 with $\Phi_1=\frac{(\overline{\mathrm{tr}\underline{X}})^2}{(\Re(\mathrm{tr}\underline{X}))^2(\mathrm{tr}\underline{X})^2}A$ and

$$\Phi_2 \;=\; \Psi+r^2\mathfrak{d}^{\leq 1}(\Gamma_b)\cdot A=\Psi+O(r\epsilon)A,$$

where we used the control of Γ_b, we obtain, for $p \geq \delta$,

$$\begin{aligned}
&\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+1}\big(r^2|\nabla_3 A|^2 + r^2|\nabla_4 A|^2 + |A|^2\big)\\
&+\int_{\partial\mathcal{M}^+(\tau_1,\tau_2)} r^{p+2}\big(r^2\chi_{nt}^2|\nabla_4 A|^2 + |\nabla_{\widehat{R}} A|^2 + |A|^2\big)\\
\lesssim & \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}\Big(r^2\chi_{nt}^2|\nabla_4\Psi|^2 + |\nabla_{\widehat{R}}\Psi|^2 + |\Psi|^2\Big)\\
&+\int_{\Sigma(\tau_1)} r^{p+2}\big(r^2\chi_{nt}^2|\nabla_4 A|^2 + |\nabla_{\widehat{R}} A|^2 + |A|^2\big) + (a^2+\epsilon^2)\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+3}|\nabla A|^2\\
&+\epsilon^2\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+1}\Big(r^2\chi_{nt}^2|\nabla_4 A|^2 + |\nabla_{\widehat{R}} A|^2 + |A|^2\Big)
\end{aligned}$$

and hence, for ϵ small enough,

$$\begin{aligned}
&\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+1}\big(r^2|\nabla_3 A|^2 + r^2|\nabla_4 A|^2 + |A|^2\big)\\
&+\int_{\partial\mathcal{M}^+(\tau_1,\tau_2)} r^{p+2}\big(r^2\chi_{nt}^2|\nabla_4 A|^2 + |\nabla_{\widehat{R}} A|^2 + |A|^2\big)\\
\lesssim & \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}\Big(r^2\chi_{nt}^2|\nabla_4\Psi|^2 + |\nabla_{\widehat{R}}\Psi|^2 + |\Psi|^2\Big)\\
&+\int_{\Sigma(\tau_1)} r^{p+2}\big(r^2\chi_{nt}^2|\nabla_4 A|^2 + |\nabla_{\widehat{R}} A|^2 + |A|^2\big) + (a^2+\epsilon^2)\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+3}|\nabla A|^2.
\end{aligned}$$

Together with (11.4.11), we infer, for $\delta \leq p \leq 2-\delta$,

$$\begin{aligned}
&\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}\big(r^2|\nabla_3\Psi|^2 + r^2|\nabla_4\Psi|^2 + |\Psi|^2\big) + \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+1}\big(r^2|\nabla_3 A|^2 + r^2|\nabla_4 A|^2 + |A|^2\big)\\
&+\int_{\partial\mathcal{M}^+(\tau_1,\tau_2)} r^{p}\big(r^2\chi_{nt}^2|\nabla_4\Psi|^2 + |\nabla_{\widehat{R}}\Psi|^2 + |\Psi|^2\big)\\
&+\int_{\partial\mathcal{M}^+(\tau_1,\tau_2)} r^{p+2}\big(r^2\chi_{nt}^2|\nabla_4 A|^2 + |\nabla_{\widehat{R}} A|^2 + |A|^2\big)\\
\lesssim & B_p[\mathfrak{q}](\tau_1,\tau_2) + \int_{\Sigma(\tau_1)} r^{p}\big(r^2\chi_{nt}^2|\nabla_4\Psi|^2 + |\nabla_{\widehat{R}}\Psi|^2 + |\Psi|^2\big)\\
&+\int_{\Sigma(\tau_1)} r^{p+2}\big(r^2\chi_{nt}^2|\nabla_4 A|^2 + |\nabla_{\widehat{R}} A|^2 + |A|^2\big)\\
&+(a^2+\epsilon^2)\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+1}|\nabla\Psi|^2 + (a^2+\epsilon^2)\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+3}|\nabla A|^2 + \epsilon^2\dot{B}_p[A](\tau_1,\tau_2).
\end{aligned}$$

In view of the definition of Ψ, we have

$$\begin{aligned}\Psi &= {}^{(c)}\nabla_3\left(\frac{(\overline{\mathrm{tr}\underline{X}})^2}{(\Re(\mathrm{tr}\underline{X}))^2(\mathrm{tr}\underline{X})^2}A\right)-\frac{r^2}{2}F_2A\\ &= {}^{(c)}\nabla_3\left(\left(\frac{\overline{q}^4}{r^2}+r\Gamma_g\right)A\right)+r^2\Gamma_b\cdot A\\ &= \left(\frac{\overline{q}^4}{r^2}+r\Gamma_g\right)\cdot\nabla_3A+(O(r)+r^2\mathfrak{d}^{\leq 1}\Gamma_b)\cdot A\end{aligned}$$

where we used the fact that F_2 given by (11.1.13) satisfies $F_2\in\Gamma_b$. Using the control of Γ_g and Γ_b, this yields

$$\begin{aligned}|\nabla_3A| &\lesssim r^{-2}|\Psi|+r^{-1}|A|,\\ |\nabla_4\nabla_3A| &\lesssim r^{-2}|\nabla_4\Psi|+r^{-3}|\Psi|+r^{-1}|\nabla_4A|+r^{-2}|A|,\\ |\nabla_{\widehat{R}}\nabla_3A| &\lesssim r^{-2}|\nabla_{\widehat{R}}\Psi|+r^{-2}|\Psi|+r^{-1}|\nabla_{\widehat{R}}A|+r^{-1}|A|.\end{aligned}$$

Also, using the definition of $\mathfrak{q}$, see (11.1.5), we have

$${}^{(c)}\nabla_3^2A = \frac{1}{q\overline{q}^3}\mathfrak{q}+O(r^{-1})\,{}^{(c)}\nabla_3A+O(r^{-2})A+\Gamma_g\,{}^{(c)}\nabla_3A+r^{-1}\Gamma_gA,$$

and hence, using also

$$\nabla_3\,{}^{(c)}\nabla_3A = {}^{(c)}\nabla_3^2A+\Gamma_b\,{}^{(c)}\nabla_3A={}^{(c)}\nabla_3^2A+\Gamma_b\nabla_3A+\Gamma_b\Gamma_bA$$

and the control of Γ_b, we infer

$$|\nabla_3\,{}^{(c)}\nabla_3A| \lesssim r^{-4}|\mathfrak{q}|+r^{-1}|\nabla_3A|+r^{-2}|A|.$$

Plugging in the above estimate, we infer, for $\delta\leq p\leq 2-\delta$,

$$\begin{aligned}&\int_{\mathcal{M}(\tau_1,\tau_2)}r^{p+1}\big(r^4|\nabla_3\,{}^{(c)}\nabla_3A|^2+r^4|\nabla_4\nabla_3A|^2+r^2|\nabla_3A|^2+r^2|\nabla_4A|^2+|A|^2\big)\\ &+\int_{\partial\mathcal{M}^+(\tau_1,\tau_2)}r^{p+2}\big(r^4\chi_{nt}^2|\nabla_4\nabla_3A|^2+r^2|\nabla_{\widehat{R}}\nabla_3A|^2+r^2|\nabla_3A|^2+r^2|\nabla_4A|^2+|A|^2\big)\\ \lesssim&B_p[\mathfrak{q}](\tau_1,\tau_2)+\int_{\Sigma(\tau_1)}r^{p+2}\big(r^4\chi_{nt}^2|\nabla_4\nabla_3A|^2+r^2|\nabla_{\widehat{R}}\nabla_3A|^2+r^2|\nabla_3A|^2+r^2|\nabla_4A|^2+|A|^2\big)\\ &+(a^2+\epsilon^2)\int_{\mathcal{M}(\tau_1,\tau_2)}r^{p+1}|\nabla\Psi|^2+(a^2+\epsilon^2)\int_{\mathcal{M}(\tau_1,\tau_2)}r^{p+3}|\nabla A|^2+\epsilon^2\dot{B}_p[A](\tau_1,\tau_2).\end{aligned}$$

In view of the definition of $\dot{BEF}_p[A]$, we obtain, for $\delta\leq p\leq 2-\delta$,

$$\begin{aligned}\dot{BEF}_p[A](\tau_1,\tau_2)\lesssim&B_p[\mathfrak{q}](\tau_1,\tau_2)+\dot{E}_p[A](\tau_1)+(a^2+\epsilon^2)\int_{\mathcal{M}(\tau_1,\tau_2)}r^{p+3}\big(r^2|\nabla\nabla_3A|^2+|\nabla A|^2\big)\\ &+\epsilon^2\dot{B}_p[A](\tau_1,\tau_2).\end{aligned}$$

For ϵ small enough, we deduce, for $\delta \leq p \leq 2-\delta$,

$$\dot{BEF}_p[A](\tau_1,\tau_2) \lesssim B_p[\mathfrak{q}](\tau_1,\tau_2) + \dot{E}_p[A](\tau_1) + (a^2+\epsilon^2)\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+3}\big(r^2|\nabla\nabla_3 A|^2 + |\nabla A|^2\big)$$

as stated. This concludes the proof of Proposition 11.4.10. □

11.4.4 Estimates for the angular derivatives of A

In order to control angular derivatives of A, we start with the following estimates for ${}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A)$ and ${}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot \nabla_3 A)$.

Lemma 11.4.11. *The quantity* ${}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A)$ *satisfies, for* $\delta \leq p \leq 2-\delta$,

$$\begin{aligned}\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+5}|{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A)|^2 &\lesssim a^2\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+1}|\nabla A|^2 + \dot{B}_p[A](\tau_1,\tau_2)\\ &\quad+\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+5}|Err_{\nabla A}|^2\end{aligned}$$

and

$$\begin{aligned}\int_{\Sigma(\tau)} r^{p+6}|{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A)|^2 &\lesssim a^2\int_{\Sigma(\tau)} r^{p+2}|\nabla A|^2 + \dot{E}_p[A](\tau)\\ &\quad+\int_{\Sigma(\tau)} r^{p+6}|Err_{\nabla A}|^2,\end{aligned}$$

where

$$Err_{\nabla A} := \Gamma_g\cdot\nabla_3 A + r^{-1}\mathfrak{d}^{\leq 1}\big(\Gamma_g\cdot(A,B)\big) + \Gamma_b\cdot\Gamma_g\cdot A.$$

Also, the quantity ${}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot \nabla_3 A)$ *satisfies, for* $\delta \leq p \leq 2-\delta$,

$$\begin{aligned}\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+7}|{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot \nabla_3 A)|^2 &\lesssim a^2\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+3}\big(|\nabla\nabla_3 A|^2 + |\nabla A|^2\big) + B_p[\mathfrak{q}](\tau_1,\tau_2)\\ &\quad+\dot{B}_p[A](\tau_1,\tau_2) + \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+7}|Err_{\nabla\nabla_3 A}|^2\end{aligned}$$

and

$$\begin{aligned}\int_{\Sigma(\tau)} r^{p+8}|{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot \nabla_3 A)|^2 &\lesssim a^2\int_{\Sigma(\tau)} r^{p+4}\big(|\nabla\nabla_3 A|^2 + |\nabla A|^2\big) + E_p[\mathfrak{q}](\tau)\\ &\quad+\dot{E}_p[A](\tau) + \int_{\Sigma(\tau)} r^{p+8}|Err_{\nabla\nabla_3 A}|^2,\end{aligned}$$

where

$$Err_{\nabla\nabla_3 A} := r^{-4}\Gamma_g\cdot\mathfrak{q} + r^{-1}\mathfrak{d}^{\leq 1}\big(\Gamma_g\cdot(\nabla_3 A, \nabla_3 B)\big) + r^{-2}\mathfrak{d}^{\leq 2}\big(\Gamma_b\cdot(A,B)\big) + \mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g\cdot A).$$

Proof. In view of Proposition 5.1.1, we have

$$\begin{aligned}\frac{1}{4}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A) &= {}^{(c)}\nabla_4\,{}^{(c)}\nabla_3 A - \left(-\frac{1}{2}\mathrm{tr}X - 2\overline{\mathrm{tr}X}\right){}^{(c)}\nabla_3 A + \frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(c)}\nabla_4 A\\ &\quad - \big(4H + \underline{H} + \overline{\underline{H}}\big)\cdot{}^{(c)}\nabla A - \big(-\overline{\mathrm{tr}X}\mathrm{tr}\underline{X} + 2\overline{P}\big)A - H\widehat{\otimes}(\overline{\underline{H}}\cdot A)\\ &\quad + r^{-1}\mathfrak{d}^{\leq 1}\big(\Gamma_g\cdot B\big) + \Gamma_b\cdot\Gamma_g\cdot A,\end{aligned}$$

where we used the fact that the frame used in this chapter satisfies $\check{H}\in\Gamma_g$ and $\nabla_3\Xi\in r^{-1}\mathfrak{d}^{\leq 1}\Gamma_g$. We infer

$$\begin{aligned}\frac{1}{4}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A) &= {}^{(c)}\nabla_4\,{}^{(c)}\nabla_3 A + O(r^{-1})\,{}^{(c)}\nabla_3 A + O(r^{-1})\,{}^{(c)}\nabla_4 A\\ &\quad + O(ar^{-2})\,{}^{(c)}\nabla A + O(r^{-2})A + \Gamma_g\cdot\nabla_3 A + r^{-1}\mathfrak{d}^{\leq 1}\big(\Gamma_g\cdot(A,B)\big)\\ &\quad + \Gamma_b\cdot\Gamma_g\cdot A,\end{aligned}$$

where we used again that fact $\check{H}\in\Gamma_g$. Thus, we obtain

$$\begin{aligned}\frac{1}{4}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A) &= {}^{(c)}\nabla_4\,{}^{(c)}\nabla_3 A + O(r^{-1})\,{}^{(c)}\nabla_3 A + O(r^{-1})\,{}^{(c)}\nabla_4 A\\ &\quad + O(ar^{-2})\,{}^{(c)}\nabla A + O(r^{-2})A + \mathrm{Err}_{\nabla A},\\ \mathrm{Err}_{\nabla A} &= \Gamma_g\cdot\nabla_3 A + r^{-1}\mathfrak{d}^{\leq 1}\big(\Gamma_g\cdot(A,B)\big) + \Gamma_b\cdot\Gamma_g\cdot A.\end{aligned}$$

This yields, for $\delta\leq p\leq 2-\delta$, in view of the definition of the norms $\dot{BEF}_p[A]$,

$$\begin{aligned}\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+5}|\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A)|^2 &\lesssim a^2\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+1}|\nabla A|^2 + \dot{B}_p[A](\tau_1,\tau_2)\\ &\quad + \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+5}|\mathrm{Err}_{\nabla A}|^2\end{aligned}$$

and

$$\begin{aligned}\int_{\Sigma(\tau)} r^{p+6}|\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A)|^2 &\lesssim a^2\int_{\Sigma(\tau)} r^{p+2}|\nabla A|^2 + \dot{E}_p[A](\tau)\\ &\quad + \int_{\Sigma(\tau)} r^{p+6}|\mathrm{Err}_{\nabla A}|^2\end{aligned}$$

which are the stated estimates for ${}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A)$.

Next, we consider the control of ${}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot\nabla_3 A)$. Recall from above the following identity

$$
\begin{aligned}
\frac{1}{4}{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A) &= {}^{(c)}\nabla_4{}^{(c)}\nabla_3 A-\left(-\frac{1}{2}\mathrm{tr}X-2\overline{\mathrm{tr}X}\right){}^{(c)}\nabla_3 A+\frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(c)}\nabla_4 A\\
&-\left(4H+\underline{H}+\overline{\underline{H}}\right)\cdot{}^{(c)}\nabla A-\left(-\overline{\mathrm{tr}X}\mathrm{tr}\underline{X}+2\overline{P}\right)A-H\widehat{\otimes}(\overline{\underline{H}}\cdot A)\\
&+r^{-1}\mathfrak{d}^{\leq 1}\left(\Gamma_g\cdot B\right)+\Gamma_b\cdot\Gamma_g\cdot A,
\end{aligned}
$$

which we differentiate w.r.t. ${}^{(c)}\nabla_3$. Using the following consequences of the null structure equations, taking into account that $\check{H}\in\Gamma_g$ in the frame used in this chapter,

$$
\begin{aligned}
{}^{(c)}\nabla_3\mathrm{tr}X &= O(r^{-2})+r^{-1}\mathfrak{d}^{\leq 1}\Gamma_g,\\
{}^{(c)}\nabla_3\mathrm{tr}\underline{X} &= O(r^{-2})+r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b,
\end{aligned}
$$

as well as

$$
\begin{aligned}
&\mathrm{tr}X=\frac{2}{r}+O(ar^{-2})+\Gamma_g,\qquad \mathrm{tr}\underline{X}=O(r^{-1})+\Gamma_g,\qquad P=O(r^{-3})+r^{-1}\Gamma_g,\\
&H=O(ar^{-2})+\Gamma_g,\qquad \underline{H}=O(ar^{-2})+\Gamma_g,
\end{aligned}
$$

and the fact that $\nabla_3\Gamma_g=r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b$, we have

$$
\begin{aligned}
&{}^{(c)}\nabla_3\Bigg\{-\left(-\frac{1}{2}\mathrm{tr}X-2\overline{\mathrm{tr}X}\right){}^{(c)}\nabla_3 A+\frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(c)}\nabla_4 A\\
&-\left(4H+\underline{H}+\overline{\underline{H}}\right)\cdot{}^{(c)}\nabla A-\left(-\overline{\mathrm{tr}X}\mathrm{tr}\underline{X}+2\overline{P}\right)A-H\widehat{\otimes}(\overline{\underline{H}}\cdot A)\\
&+r^{-1}\mathfrak{d}^{\leq 1}\left(\Gamma_g\cdot B\right)+\Gamma_b\cdot\Gamma_g\cdot A\Bigg\}\\
=\;&\left(\frac{5}{r}+O(ar^{-2})\right){}^{(c)}\nabla_3{}^{(c)}\nabla_3 A+O(r^{-2}){}^{(c)}\nabla_3 A+O(r^{-1}){}^{(c)}\nabla_4{}^{(c)}\nabla_3 A\\
&+O(r^{-1})[{}^{(c)}\nabla_3,{}^{(c)}\nabla_4]A+O(r^{-2}){}^{(c)}\nabla_4 A+O(ar^{-2}){}^{(c)}\nabla{}^{(c)}\nabla_3 A\\
&+O(ar^{-2})[{}^{(c)}\nabla_3,{}^{(c)}\nabla]A+O(ar^{-2})\nabla A+O(r^{-2}){}^{(c)}\nabla_3 A+O(r^{-3})A+\Gamma_g\cdot{}^{(c)}\nabla_3^2 A\\
&+r^{-1}\mathfrak{d}^{\leq 1}\big(\Gamma_g\cdot(\nabla_3 A,\nabla_3 B)\big)+r^{-2}\mathfrak{d}^{\leq 2}\big(\Gamma_b\cdot(A,B)\big)+\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g\cdot A).
\end{aligned}
$$

Using the following consequences of the commutation formulas of Lemma 4.2.2, taking into account that $\check{H}\in\Gamma_g$ in the frame used in this chapter,

$$
\begin{aligned}
[{}^{(c)}\nabla_3,{}^{(c)}\nabla_4]U &= O(ar^{-2})\nabla U+O(r^{-3})U+r^{-1}\Gamma_g\mathfrak{d}^{\leq 1}U,\\
[{}^{(c)}\nabla_3,{}^{(c)}\nabla]U &= O(ar^{-2})\nabla_3 U+O(r^{-1})\nabla U+O(ar^{-3})U+\Gamma_g\cdot\mathfrak{d}^{\leq 1}U,
\end{aligned}
$$

we infer

$$
\begin{aligned}
& {}^{(c)}\nabla_3\Bigg\{-\left(-\frac{1}{2}\mathrm{tr}X-2\overline{\mathrm{tr}X}\right){}^{(c)}\nabla_3A+\frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(c)}\nabla_4A\\
& -\big(4H+\underline{H}+\overline{\underline{H}}\big)\cdot{}^{(c)}\nabla A-\big(-\overline{\mathrm{tr}X}\mathrm{tr}\underline{X}+2\overline{P}\big)A-H\widehat{\otimes}(\overline{\underline{H}}\cdot A)\\
& +r^{-1}\mathfrak{d}^{\leq 1}\big(\Gamma_g\cdot B\big)+\Gamma_b\cdot\Gamma_g\cdot A\Bigg\}\\
= & \left(\frac{5}{r}+O(ar^{-2})\right){}^{(c)}\nabla_3\,{}^{(c)}\nabla_3A+O(r^{-1})\,{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3A+O(ar^{-2})\,{}^{(c)}\nabla\,{}^{(c)}\nabla_3A\\
& +O(r^{-2})\,{}^{(c)}\nabla_3A+O(r^{-2})\,{}^{(c)}\nabla_4A+O(ar^{-2})\nabla A+O(r^{-3})A\\
& +\Gamma_g\cdot{}^{(c)}\nabla_3^2A+r^{-1}\mathfrak{d}^{\leq 1}\big(\Gamma_g\cdot(\nabla_3A,\nabla_3B)\big)+r^{-2}\mathfrak{d}^{\leq 2}\big(\Gamma_b\cdot(A,B)\big)+\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g\cdot A).
\end{aligned}
$$

Next, using the definition of $\mathfrak{q}$, see (11.1.5), we have

$$
{}^{(c)}\nabla_3^2A = \left(\frac{1}{r^4}+O(ar^{-5})\right)\mathfrak{q}+O(r^{-1})\,{}^{(c)}\nabla_3A+O(r^{-2})A+\Gamma_g\,{}^{(c)}\nabla_3A+r^{-1}\Gamma_gA,
$$

and hence

$$
\begin{aligned}
& {}^{(c)}\nabla_3\Bigg\{-\left(-\frac{1}{2}\mathrm{tr}X-2\overline{\mathrm{tr}X}\right){}^{(c)}\nabla_3A+\frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(c)}\nabla_4A\\
& -\big(4H+\underline{H}+\overline{\underline{H}}\big)\cdot{}^{(c)}\nabla A-\big(-\overline{\mathrm{tr}X}\mathrm{tr}\underline{X}+2\overline{P}\big)A-H\widehat{\otimes}(\overline{\underline{H}}\cdot A)\\
& +r^{-1}\mathfrak{d}^{\leq 1}\big(\Gamma_g\cdot B\big)+\Gamma_b\cdot\Gamma_g\cdot A\Bigg\}\\
= & \left(\frac{5}{r}+O(ar^{-6})\right)\mathfrak{q}+O(r^{-1})\,{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3A+O(ar^{-2})\,{}^{(c)}\nabla\,{}^{(c)}\nabla_3A\\
& +O(r^{-2})\,{}^{(c)}\nabla_3A+O(r^{-2})\,{}^{(c)}\nabla_4A+O(ar^{-2})\nabla A+O(r^{-3})A\\
& +r^{-4}\Gamma_g\cdot\mathfrak{q}+r^{-1}\mathfrak{d}^{\leq 1}\big(\Gamma_g\cdot(\nabla_3A,\nabla_3B)\big)+r^{-2}\mathfrak{d}^{\leq 2}\big(\Gamma_b\cdot(A,B)\big)+\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g\cdot A).
\end{aligned}
$$

Coming back to the identity

$$
\begin{aligned}
\frac{1}{4}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A) = & \;{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3A-\left(-\frac{1}{2}\mathrm{tr}X-2\overline{\mathrm{tr}X}\right){}^{(c)}\nabla_3A+\frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(c)}\nabla_4A\\
& -\big(4H+\underline{H}+\overline{\underline{H}}\big)\cdot{}^{(c)}\nabla A-\big(-\overline{\mathrm{tr}X}\mathrm{tr}\underline{X}+2\overline{P}\big)A-H\widehat{\otimes}(\overline{\underline{H}}\cdot A)\\
& +r^{-1}\mathfrak{d}^{\leq 1}\big(\Gamma_g\cdot B\big)+\Gamma_b\cdot\Gamma_g\cdot A,
\end{aligned}
$$

and differentiating it w.r.t. ${}^{(c)}\nabla_3$, we infer

$$
\begin{aligned}
&\frac{1}{4}\,{}^{(c)}\nabla_3\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A)\\
=\;&{}^{(c)}\nabla_3\,{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3 A+\left(\frac{5}{r}+O(ar^{-2})\right)\mathfrak{q}+O(r^{-1})\,{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3 A+O(ar^{-2})\,{}^{(c)}\nabla\,{}^{(c)}\nabla_3 A\\
&+O(r^{-2})\,{}^{(c)}\nabla_3 A+O(r^{-2})\,{}^{(c)}\nabla_4 A+O(ar^{-2})\nabla A+O(r^{-3})A\\
&+r^{-4}\Gamma_g\cdot\mathfrak{q}+r^{-1}\mathfrak{d}^{\leq 1}\big(\Gamma_g\cdot(\nabla_3 A,\nabla_3 B)\big)+r^{-2}\mathfrak{d}^{\leq 2}\big(\Gamma_b\cdot(A,B)\big)+\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g\cdot A).
\end{aligned}
$$

Next, using again the commutation formula

$$
[\,{}^{(c)}\nabla_3,\,{}^{(c)}\nabla_4]U \;=\; O(ar^{-2})\nabla U+O(r^{-3})U+r^{-1}\Gamma_g\mathfrak{d}^{\leq 1}U,
$$

we have

$$
[\,{}^{(c)}\nabla_3,\,{}^{(c)}\nabla_4]\,{}^{(c)}\nabla_3 A \;=\; O(ar^{-2})\nabla\,{}^{(c)}\nabla_3 A+O(r^{-3})\,{}^{(c)}\nabla_3 A+r^{-1}\Gamma_g\mathfrak{d}^{\leq 1}\,{}^{(c)}\nabla_3 A
$$

and hence

$$
\begin{aligned}
&\frac{1}{4}\,{}^{(c)}\nabla_3\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A)\\
=\;&{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3^2 A+\left(\frac{5}{r}+O(ar^{-2})\right)\mathfrak{q}+O(r^{-1})\,{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3 A+O(ar^{-2})\,{}^{(c)}\nabla\,{}^{(c)}\nabla_3 A\\
&+O(r^{-2})\,{}^{(c)}\nabla_3 A+O(r^{-2})\,{}^{(c)}\nabla_4 A+O(ar^{-2})\nabla A+O(r^{-3})A\\
&+r^{-4}\Gamma_g\cdot\mathfrak{q}+r^{-1}\mathfrak{d}^{\leq 1}\big(\Gamma_g\cdot(\nabla_3 A,\nabla_3 B)\big)+r^{-2}\mathfrak{d}^{\leq 2}\big(\Gamma_b\cdot(A,B)\big)+\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g\cdot A).
\end{aligned}
$$

Using again the fact that

$$
{}^{(c)}\nabla_3^2 A \;=\; \left(\frac{1}{r^4}+O(ar^{-5})\right)\mathfrak{q}+O(r^{-1})\,{}^{(c)}\nabla_3 A+O(r^{-2})A+\Gamma_g\,{}^{(c)}\nabla_3 A+r^{-1}\Gamma_g A,
$$

we infer

$$
\begin{aligned}
&\frac{1}{4}\,{}^{(c)}\nabla_3\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A)\\
=\;&\left(\frac{1}{r^4}+O(ar^{-5})\right){}^{(c)}\nabla_4\mathfrak{q}+\left(\frac{1}{r^5}+O(ar^{-6})\right)\mathfrak{q}+O(r^{-1})\,{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3 A\\
&+O(ar^{-2})\,{}^{(c)}\nabla\,{}^{(c)}\nabla_3 A+O(r^{-2})\,{}^{(c)}\nabla_3 A+O(r^{-2})\,{}^{(c)}\nabla_4 A+O(ar^{-2})\nabla A+O(r^{-3})A\\
&+r^{-4}\Gamma_g\cdot\mathfrak{q}+r^{-1}\mathfrak{d}^{\leq 1}\big(\Gamma_g\cdot(\nabla_3 A,\nabla_3 B)\big)+r^{-2}\mathfrak{d}^{\leq 2}\big(\Gamma_b\cdot(A,B)\big)+\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g\cdot A)
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\frac{1}{4}\,{}^{(c)}\nabla_3\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A)\\
=\;&O(r^{-5})e_4(r\mathfrak{q})+O(r^{-6})\mathfrak{q}+O(r^{-1})\,{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3 A+O(ar^{-2})\,{}^{(c)}\nabla\,{}^{(c)}\nabla_3 A\\
&+O(r^{-2})\,{}^{(c)}\nabla_3 A+O(r^{-2})\,{}^{(c)}\nabla_4 A+O(ar^{-2})\nabla A+O(r^{-3})A\\
&+r^{-4}\Gamma_g\cdot\mathfrak{q}+r^{-1}\mathfrak{d}^{\leq 1}\big(\Gamma_g\cdot(\nabla_3 A,\nabla_3 B)\big)+r^{-2}\mathfrak{d}^{\leq 2}\big(\Gamma_b\cdot(A,B)\big)+\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g\cdot A).
\end{aligned}
$$

Next, using the following consequences of the commutation formulas of Lemma 4.2.2, taking into account that $\widecheck{H} \in \Gamma_g$ in the frame used in this chapter,

$$\begin{aligned}
[{}^{(c)}\nabla_3, \mathcal{D}\widehat{\otimes}]U &= -\frac{1}{2}\mathrm{tr}X\mathcal{D}\widehat{\otimes}U + O(ar^{-2})\nabla_3 U + O(ar^{-3})U + \Gamma_g\nabla_3 U + r^{-1}\Gamma_g \cdot \mathfrak{d}^{\leq 1}U,\\
[{}^{(c)}\nabla_3, \overline{\mathcal{D}}\cdot]U &= -\frac{1}{2}\overline{\mathrm{tr}X}\,\overline{\mathcal{D}}\cdot U + O(ar^{-2})\nabla_3 U + O(ar^{-3})U + \Gamma_g\nabla_3 U + r^{-1}\Gamma_g \cdot \mathfrak{d}^{\leq 1}U,
\end{aligned}$$

we have

$$\begin{aligned}
&{}^{(c)}\nabla_3\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A) - {}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot {}^{(c)}\nabla_3 A)\\
=\ & [{}^{(c)}\nabla_3, {}^{(c)}\mathcal{D}\widehat{\otimes}](\overline{{}^{(c)}\mathcal{D}}\cdot A) + {}^{(c)}\mathcal{D}\widehat{\otimes}[{}^{(c)}\nabla_3, \overline{{}^{(c)}\mathcal{D}}\cdot]A\\
=\ & \left(-\frac{1}{2}\mathrm{tr}X\mathcal{D}\widehat{\otimes} + O(ar^{-2})\nabla_3 + O(ar^{-3}) + \Gamma_g\nabla_3 + r^{-1}\Gamma_g\cdot\mathfrak{d}^{\leq 1}\right)(\overline{{}^{(c)}\mathcal{D}}\cdot A)\\
&+{}^{(c)}\mathcal{D}\widehat{\otimes}\left(-\frac{1}{2}\overline{\mathrm{tr}X}\,\overline{\mathcal{D}}\cdot A + O(ar^{-2})\nabla_3 A + O(ar^{-3})A + \Gamma_g\nabla_3 A + r^{-1}\Gamma_g\cdot\mathfrak{d}^{\leq 1}A\right)\\
=\ & -\mathrm{tr}\,\chi\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A) + O(ar^{-2})\nabla\nabla_3 A + O(ar^{-2})[\nabla_3, \nabla]A + O(ar^{-3})\nabla_3 A\\
&+O(ar^{-3})\nabla A + O(ar^{-4})A + r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_g\nabla_3 A) + r^{-2}\mathfrak{d}^{\leq 2}(\Gamma_g\cdot A)\\
=\ & O(r^{-1})\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A) + O(ar^{-2})\nabla\nabla_3 A + O(ar^{-3})\nabla_3 A + O(ar^{-3})\nabla A\\
&+O(ar^{-4})A + r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_g\nabla_3 A) + r^{-2}\mathfrak{d}^{\leq 2}(\Gamma_g\cdot A).
\end{aligned}$$

Together with the above, we infer

$$\begin{aligned}
&{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot {}^{(c)}\nabla_3 A)\\
=\ & O(r^{-5})e_4(r\mathfrak{q}) + O(r^{-6})\mathfrak{q} + O(r^{-1})\nabla_4\nabla_3 A + O(ar^{-2})\nabla\nabla_3 A\\
&+O(r^{-1})\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A) + O(r^{-2})\nabla_3 A + O(r^{-2})\nabla_4 A + O(ar^{-2})\nabla A + O(r^{-3})A\\
&+r^{-4}\Gamma_g\cdot\mathfrak{q} + r^{-1}\mathfrak{d}^{\leq 1}\big(\Gamma_g\cdot(\nabla_3 A, \nabla_3 B)\big) + r^{-2}\mathfrak{d}^{\leq 2}\big(\Gamma_b\cdot(A, B)\big) + \mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g\cdot A).
\end{aligned}$$

Recalling

$$\begin{aligned}
\frac{1}{4}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A) &= {}^{(c)}\nabla_4\,{}^{(c)}\nabla_3 A + O(r^{-1})\,{}^{(c)}\nabla_3 A + O(r^{-1})\,{}^{(c)}\nabla_4 A\\
&\quad +O(ar^{-2})\,{}^{(c)}\nabla A + O(r^{-2})A + \mathrm{Err}_{\nabla A},
\end{aligned}$$

we deduce

$$\begin{aligned}
&{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot {}^{(c)}\nabla_3 A)\\
=\ & O(r^{-5})e_4(r\mathfrak{q}) + O(r^{-6})\mathfrak{q} + O(r^{-1})\nabla_4\nabla_3 A + O(ar^{-2})\nabla\nabla_3 A\\
&+O(r^{-2})\nabla_3 A + O(r^{-2})\nabla_4 A + O(ar^{-2})\nabla A + O(r^{-3})A\\
&+r^{-4}\Gamma_g\cdot\mathfrak{q} + r^{-1}\mathfrak{d}^{\leq 1}\big(\Gamma_g\cdot(\nabla_3 A, \nabla_3 B)\big) + r^{-2}\mathfrak{d}^{\leq 2}\big(\Gamma_b\cdot(A, B)\big) + \mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g\cdot A)\\
&+O(r^{-1})\mathrm{Err}_{\nabla A}.
\end{aligned}$$

In view of the form of $\text{Err}_{\nabla A}$, i.e.

$$\text{Err}_{\nabla A} = \Gamma_g \cdot \nabla_3 A + r^{-1}\mathfrak{d}^{\leq 1}\big(\Gamma_g \cdot (A, B)\big) + \Gamma_b \cdot \Gamma_g \cdot A,$$

we obtain

$$\begin{aligned} & {}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}} \cdot {}^{(c)}\nabla_3 A) \\ = \; & O(r^{-5})e_4(r\mathfrak{q}) + O(r^{-6})\mathfrak{q} + O(r^{-1})\nabla_4\nabla_3 A + O(ar^{-2})\nabla\nabla_3 A \\ & + O(r^{-2})\nabla_3 A + O(r^{-2})\nabla_4 A + O(ar^{-2})\nabla A + O(r^{-3})A + \text{Err}_{\nabla\nabla_3 A} \end{aligned}$$

where

$$\text{Err}_{\nabla\nabla_3 A} = r^{-4}\Gamma_g \cdot \mathfrak{q} + r^{-1}\mathfrak{d}^{\leq 1}\big(\Gamma_g \cdot (\nabla_3 A, \nabla_3 B)\big) + r^{-2}\mathfrak{d}^{\leq 2}\big(\Gamma_b \cdot (A, B)\big) + \mathfrak{d}^{\leq 1}(\Gamma_b \cdot \Gamma_g \cdot A).$$

This yields, for $\delta \leq p \leq 2-\delta$, in view of the definition of the norms $BEF_p[\mathfrak{q}]$ and $\dot{BEF}_p[A]$,

$$\begin{aligned} \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+7}|{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}} \cdot \nabla_3 A)|^2 \lesssim \; & a^2 \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+3}\big(|\nabla\nabla_3 A|^2 + |\nabla A|^2\big) + B_p[\mathfrak{q}](\tau_1,\tau_2) \\ & + \dot{B}_p[A](\tau_1,\tau_2) + \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+7}|\text{Err}_{\nabla\nabla_3 A}|^2 \end{aligned}$$

and

$$\begin{aligned} \int_{\Sigma(\tau)} r^{p+8}|{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}} \cdot \nabla_3 A)|^2 \lesssim \; & a^2 \int_{\Sigma(\tau)} r^{p+4}\big(|\nabla\nabla_3 A|^2 + |\nabla A|^2\big) + E_p[\mathfrak{q}](\tau) \\ & + \dot{E}_p[A](\tau) + \int_{\Sigma(\tau)} r^{p+8}|\text{Err}_{\nabla\nabla_3 A}|^2 \end{aligned}$$

which are the stated estimates for ${}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}} \cdot \nabla_3 A)$. This concludes the proof of Lemma 11.4.11. □

To control angular derivatives of A, we will also need the following lemma.

Lemma 11.4.12. *The following estimate holds true, for any $U \in \mathfrak{s}_2(\mathbb{C})$ and $S \subset \mathcal{M}$,*

$$\int_S \Big(|\nabla U|^2 + r^{-2}|U|^2\Big) \lesssim \left|\int_S \overline{U} \cdot \mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}} \cdot U)\right| + O(a^2)\int_S r^{-2}|(\nabla_3, \nabla_4)U|^2.$$

Proof. In view of Lemma 2.4.7 we write

$$\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}} \cdot U) = 2\triangle_2 U - 4\,{}^{(h)}KU - i\big({}^{(a)}\text{tr}\chi\nabla_3 + {}^{(a)}\text{tr}\underline{\chi}\nabla_4\big)U$$

where ${}^{(h)}K = -\frac{1}{4}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} - \frac{1}{4}{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} + \frac{1}{2}\widehat{\chi}\cdot\widehat{\underline{\chi}} - \frac{1}{4}\rho$. Therefore

$$\begin{aligned}\overline{U}\cdot\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot U) &= 2\overline{U}\cdot\triangle_2 U - 4\,{}^{(h)}K|U|^2 - i\overline{U}\cdot\big({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4\big)U\\ &= -|\nabla U|^2 + \nabla^a(\overline{U}\cdot\nabla_a U) - 4\,{}^{(h)}K|U|^2 - i\overline{U}\cdot\big({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4\big)U.\end{aligned}$$

Proceeding as in the proof of Proposition 9.3.2, with the help of Lemma 9.3.1,

$$\nabla^a(\overline{U}\cdot\nabla_a U) = \mathrm{d}iv^{\mathbf{S}}\big(U\cdot\nabla U\big) + O(ar^{-1})\overline{U}\cdot(\nabla_3,\nabla_4,\nabla)U.$$

Thus, by integration on S, we deduce

$$\int_S\big(|\nabla U|^2 + 4\,{}^{(h)}K|U|^2\big) \lesssim \left|\int_S\overline{U}\cdot\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot U)\right| + a\int_S r^{-1}\big|U\big|\big|(\nabla_3,\nabla_4,\nabla)U\big|.$$

Since ${}^{(h)}K = r^{-2} + O(a^2r^{-4})$, we deduce

$$\int_S\big(|\nabla U|^2 + r^{-2}|U|^2\big) \lesssim \left|\int_S\overline{U}\cdot\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot U)\right| + a^2\int_S r^{-2}\big|(\nabla_3,\nabla_4)U\big|$$

as stated. □

We now obtain the desired control of ∇A and $\nabla\nabla_3 A$ as a corollary of Lemma 11.4.11 and Lemma 11.4.12.

Corollary 11.4.13. *We have, for all $\delta\le p\le 2-\delta$,*

$$\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+3}|\nabla A|^2 \lesssim \dot{B}_p[A](\tau_1,\tau_2) + \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+5}|Err_{\nabla A}|^2,$$

$$\int_{\Sigma(\tau)} r^{p+4}|\nabla A|^2 \lesssim \dot{E}_p[A](\tau) + \int_{\Sigma(\tau)} r^{p+6}|Err_{\nabla A}|^2,$$

$$\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+5}|\nabla\nabla_3 A|^2 \lesssim B_p[\mathfrak{q}](\tau_1,\tau_2) + \dot{B}_p[A](\tau_1,\tau_2) + \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+7}|Err_{\nabla\nabla_3 A}|^2,$$

and

$$\int_{\Sigma(\tau)} r^{p+6}|\nabla\nabla_3 A|^2 \lesssim E_p[\mathfrak{q}](\tau) + \dot{E}_p[A](\tau) + \int_{\Sigma(\tau)} r^{p+8}|Err_{\nabla\nabla_3 A}|^2,$$

where the error terms $Err_{\nabla A}$ and $Err_{\nabla\nabla_3 A}$ are defined in Lemma 11.4.11.

Proof. According to the elliptic type estimates of Lemma 11.4.12 we have for any $S \subset \mathcal{M}$.

$$\int_S \Big(|\nabla A|^2 + r^{-2}|A|^2\Big) \lesssim \left|\int_S A\cdot \mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot A)\right| + (a^2+\epsilon^2)\int_S r^{-2}|({}^{(c)}\nabla_3, {}^{(c)}\nabla_4)A|^2.$$

We deduce, for all $\delta \le p \le 2-\delta$,

$$\begin{aligned}\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+3}|\nabla A|^2 \lesssim& \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+1}|A|^2 + \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+5}\big|\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot A)\big|^2\\ &+ (a^2+\epsilon^2)\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+1}|(\nabla_3,\nabla_4)A|^2\\ \lesssim& \dot{B}_p[A](\tau_1,\tau_2) + \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+5}\big|\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot A)\big|^2,\end{aligned}$$

and

$$\begin{aligned}\int_{\Sigma(\tau)} r^{p+4}|\nabla A|^2 \lesssim& \int_{\Sigma(\tau)} r^{p+2}|A|^2 + \int_{\Sigma(\tau)} r^{p+6}\big|\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot A)\big|^2\\ &+ (a^2+\epsilon^2)\int_{\Sigma(\tau)} r^{p+2}|(\nabla_3,\nabla_4)A|^2\\ \lesssim& \dot{E}_p[A](\tau) + \int_{\Sigma(\tau)} r^{p+6}\big|\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot A)\big|^2\end{aligned}$$

Next, recall from Lemma 11.4.11 that we have, for $\delta \le p \le 2-\delta$,

$$\begin{aligned}\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+5}|{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A)|^2 \lesssim& a^2\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+1}|\nabla A|^2 + \dot{B}_p[A](\tau_1,\tau_2)\\ &+ \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+5}|\mathrm{Err}_{\nabla A}|^2\end{aligned}$$

and

$$\begin{aligned}\int_{\Sigma(\tau)} r^{p+6}|{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A)|^2 \lesssim& a^2\int_{\Sigma(\tau)} r^{p+2}|\nabla A|^2 + \dot{E}_p[A](\tau)\\ &+ \int_{\Sigma(\tau)} r^{p+6}|\mathrm{Err}_{\nabla A}|^2,\end{aligned}$$

see Lemma 11.4.11 for the definition of $\mathrm{Err}_{\nabla A}$. We infer, for all $\delta \le p \le 2-\delta$,

$$\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+3}|\nabla A|^2 \lesssim a^2\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+1}|\nabla A|^2 + \dot{B}_p[A](\tau_1,\tau_2) + \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+5}|\mathrm{Err}_{\nabla A}|^2,$$

and

$$\int_{\Sigma(\tau)} r^{p+4}|\nabla A|^2 \lesssim a^2\int_{\Sigma(\tau)} r^{p+2}|\nabla A|^2 + \dot{E}_p[A](\tau) + \int_{\Sigma(\tau)} r^{p+6}|\mathrm{Err}_{\nabla A}|^2.$$

For a small enough, we infer, for all $\delta \leq p \leq 2-\delta$,

$$\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+3}|\nabla A|^2 \lesssim \dot{B}_p[A](\tau_1,\tau_2) + \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+5}|\mathrm{Err}_{\nabla A}|^2,$$

and

$$\int_{\Sigma(\tau)} r^{p+4}|\nabla A|^2 \lesssim \dot{E}_p[A](\tau) + \int_{\Sigma(\tau)} r^{p+6}|\mathrm{Err}_{\nabla A}|^2,$$

as stated.

Next, proceeding as above, with $\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot A)$ replaced by $\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot\nabla_3 A)$, we have, for all $\delta \leq p \leq 2-\delta$,

$$\begin{aligned}\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+5}|\nabla\nabla_3 A|^2 \lesssim& \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+3}|\nabla_3 A|^2 + \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+7}\big|\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot\nabla_3 A)\big|^2\\ &+ (a^2+\epsilon^2)\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+3}|(\nabla_3^2, \nabla_4\nabla_3)A|^2\\ \lesssim& \dot{B}_p[A](\tau_1,\tau_2) + \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+7}\big|\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot\nabla_3 A)\big|^2,\end{aligned}$$

and

$$\begin{aligned}\int_{\Sigma(\tau)} r^{p+6}|\nabla\nabla_3 A|^2 \lesssim& \int_{\Sigma(\tau)} r^{p+4}|\nabla_3 A|^2 + \int_{\Sigma(\tau)} r^{p+8}\big|\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot\nabla_3 A)\big|^2\\ &+ (a^2+\epsilon^2)\int_{\Sigma(\tau)} r^{p+4}|(\nabla_3^2, \nabla_4\nabla_3)A|^2\\ \lesssim& \dot{E}_p[A](\tau) + \int_{\Sigma(\tau)} r^{p+8}\big|\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot\nabla_3 A)\big|^2\end{aligned}$$

Next, recall from Lemma 11.4.11 that we have, for $\delta \leq p \leq 2-\delta$,

$$\begin{aligned}\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+7}|{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot\nabla_3 A)|^2 \;\lesssim\;& a^2\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+3}\big(|\nabla\nabla_3 A|^2+|\nabla A|^2\big) + B_p[\mathfrak{q}](\tau_1,\tau_2)\\ &+\dot{B}_p[A](\tau_1,\tau_2) + \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+7}|\mathrm{Err}_{\nabla\nabla_3 A}|^2\end{aligned}$$

and

$$\begin{aligned}\int_{\Sigma(\tau)} r^{p+8}|{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot\nabla_3 A)|^2 \;\lesssim\;& a^2\int_{\Sigma(\tau)} r^{p+4}\big(|\nabla\nabla_3 A|^2+|\nabla A|^2\big) + E_p[\mathfrak{q}](\tau)\\ &+\dot{E}_p[A](\tau) + \int_{\Sigma(\tau)} r^{p+8}|\mathrm{Err}_{\nabla\nabla_3 A}|^2,\end{aligned}$$

see Lemma 11.4.11 for the definition of $\mathrm{Err}_{\nabla\nabla_3 A}$. We infer, for all $\delta \le p \le 2-\delta$,

$$\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+5}|\nabla\nabla_3 A|^2 \lesssim a^2 \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+3}\big(|\nabla\nabla_3 A|^2 + |\nabla A|^2\big) + B_p[\mathfrak{q}](\tau_1,\tau_2) + \dot{B}_p[A](\tau_1,\tau_2) + \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+7}|\mathrm{Err}_{\nabla\nabla_3 A}|^2$$

and

$$\int_{\Sigma(\tau)} r^{p+6}|\nabla\nabla_3 A|^2 \lesssim a^2 \int_{\Sigma(\tau)} r^{p+4}\big(|\nabla\nabla_3 A|^2 + |\nabla A|^2\big) + E_p[\mathfrak{q}](\tau) + \dot{E}_p[A](\tau) + \int_{\Sigma(\tau)} r^{p+8}|\mathrm{Err}_{\nabla\nabla_3 A}|^2.$$

For a small enough, we infer, for all $\delta \le p \le 2-\delta$,

$$\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+5}|\nabla\nabla_3 A|^2 \lesssim B_p[\mathfrak{q}](\tau_1,\tau_2) + \dot{B}_p[A](\tau_1,\tau_2) + \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+7}|\mathrm{Err}_{\nabla\nabla_3 A}|^2$$

and

$$\int_{\Sigma(\tau)} r^{p+6}|\nabla\nabla_3 A|^2 \lesssim E_p[\mathfrak{q}](\tau) + \dot{E}_p[A](\tau) + \int_{\Sigma(\tau)} r^{p+8}|\mathrm{Err}_{\nabla\nabla_3 A}|^2$$

as stated. This concludes the proof of Corollary 11.4.13. □

To conclude this section, we provide the control of the error terms $\mathrm{Err}_{\nabla A}$ and $\mathrm{Err}_{\nabla\nabla_3 A}$ defined in Lemma 11.4.11.

Lemma 11.4.14. *The error terms $Err_{\nabla A}$ and $Err_{\nabla\nabla_3 A}$ defined in Lemma 11.4.11 satisfy the following estimates, for all $s \le k_L$, and for all $\delta \le p \le 2-\delta$,*

$$\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+5}|\mathfrak{d}^s Err_{\nabla A}|^2 + \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+7}|\mathfrak{d}^s Err_{\nabla\nabla_3 A}|^2 \lesssim \epsilon_0^2 \tau_1^{-2-3\delta_{dec}},$$

and

$$\int_{\Sigma(\tau)} r^{p+6}|\mathfrak{d}^s Err_{\nabla A}|^2 + \int_{\Sigma(\tau)} r^{p+8}|\mathfrak{d}^s Err_{\nabla\nabla_3 A}|^2 \lesssim \epsilon_0^2 \tau^{-2-3\delta_{dec}}.$$

Proof. Recall that the error terms $\mathrm{Err}_{\nabla A}$ and $\mathrm{Err}_{\nabla\nabla_3 A}$ are given respectively by

$$\mathrm{Err}_{\nabla A} = \Gamma_g \cdot \nabla_3 A + r^{-1}\mathfrak{d}^{\le 1}\big(\Gamma_g \cdot (A,B)\big) + \Gamma_b \cdot \Gamma_g \cdot A$$

and

$$\mathrm{Err}_{\nabla\nabla_3 A} = r^{-4}\Gamma_g \cdot \mathfrak{q} + r^{-1}\mathfrak{d}^{\leq 1}\big(\Gamma_g \cdot (\nabla_3 A, \nabla_3 B)\big) + r^{-2}\mathfrak{d}^{\leq 2}\big(\Gamma_b \cdot (A, B)\big) + \mathfrak{d}^{\leq 1}(\Gamma_b \cdot \Gamma_g \cdot A).$$

In view of the Bianchi identities for $\nabla_3 A$ and $\nabla_3 B$, we have

$$\nabla_3 A = O(r^{-1})\mathfrak{d}^{\leq 1}B + O(r^{-1})A + r^{-3}\Gamma_g, \qquad \nabla_3 B = r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g.$$

Together with the fact that $\mathfrak{q} \in r\mathfrak{d}^{\leq 2}\Gamma_g$, we infer

$$\mathrm{Err}_{\nabla A} = r^{-1}\mathfrak{d}^{\leq 1}\big(\Gamma_g \cdot (A, B)\big) + r^{-3}\Gamma_g \cdot \Gamma_g + \Gamma_b \cdot \Gamma_g \cdot A$$

and

$$\mathrm{Err}_{\nabla\nabla_3 A} = r^{-3}\mathfrak{d}^{\leq 2}\big(\Gamma_g \cdot \Gamma_g\big) + r^{-2}\mathfrak{d}^{\leq 2}\big(\Gamma_b \cdot (A, B)\big) + \mathfrak{d}^{\leq 1}(\Gamma_b \cdot \Gamma_g \cdot A).$$

Using the bootstrap assumptions for Γ_g, Γ_b, A and B, we infer, for all $s \leq k_L$,

$$\begin{aligned}
&\int_{\mathcal{M}(\tau_1,\tau_2)} r^{7-\delta}|\mathfrak{d}^s \mathrm{Err}_{\nabla A}|^2 + \int_{\mathcal{M}(\tau_1,\tau_2)} r^{9-\delta}|\mathfrak{d}^s \mathrm{Err}_{\nabla\nabla_3 A}|^2 \\
\lesssim\;& \epsilon^4 \left(\int_{\tau_1}^{+\infty} \frac{d\tau}{\tau^{3+3\delta_{dec}}}\right)\left(\int_{r \geq r_+(1-\delta_{\mathcal{H}})} \frac{dr}{r^{1+\delta}}\right) \\
\lesssim\;& \epsilon_0^2 \tau_1^{-2-3\delta_{dec}}
\end{aligned}$$

and

$$\begin{aligned}
\int_{\Sigma(\tau)} r^{8-\delta}|\mathfrak{d}^s \mathrm{Err}_{\nabla A}|^2 + \int_{\Sigma(\tau)} r^{10-\delta}|\mathfrak{d}^s \mathrm{Err}_{\nabla\nabla_3 A}|^2 \;&\lesssim\; \epsilon^4 \tau^{-2-3\delta_{dec}} \left(\int_{r \geq r_+(1-\delta_{\mathcal{H}})} \frac{dr}{r^{1+\delta}}\right) \\
&\lesssim\; \epsilon_0^2 \tau^{-2-3\delta_{dec}}
\end{aligned}$$

as stated. □

11.4.5 End of the proof of Proposition 11.2.8

In view of Corollary 11.4.13 and Lemma 11.4.14, we have, for all $\delta \leq p \leq 2-\delta$,

$$\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+3}\big(r^2|\nabla\nabla_3 A|^2 + |\nabla A|^2\big) \lesssim B_p[\mathfrak{q}](\tau_1,\tau_2) + \dot{B}_p[A](\tau_1,\tau_2) + \epsilon_0^2 \tau_1^{-2-3\delta_{dec}}.$$

Also, recall from Proposition 11.4.10 that we have, for $\delta \leq p \leq 2-\delta$,

$$\begin{aligned}
\dot{BEF}_p[A](\tau_1,\tau_2) \lesssim\;& B_p[\mathfrak{q}](\tau_1,\tau_2) + \dot{E}_p[A](\tau_1) \\
&+ (a^2+\epsilon^2)\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+3}\big(r^2|\nabla\nabla_3 A|^2 + |\nabla A|^2\big).
\end{aligned}$$

Putting the two estimates together, and using the smallness of ϵ and a, we infer

$$\begin{aligned} & B\dot{E}F_p[A](\tau_1,\tau_2) + \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+3}\big(r^2|\nabla\nabla_3 A|^2 + |\nabla A|^2\big) \\ \lesssim & B_p[\mathfrak{q}](\tau_1,\tau_2) + \dot{E}_p[A](\tau_1) + \epsilon_0^2\tau_1^{-2-3\delta_{dec}}. \end{aligned}$$

Noticing that $F_p[A](\tau_1,\tau_2) = \dot{F}_p[A](\tau_1,\tau_2)$, $E_p[A](\tau) = \dot{E}_p[A](\tau)$ and

$$B_p[A](\tau_1,\tau_2) = \dot{B}_p[A](\tau_1,\tau_2) + \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+3}\big(r^2|\nabla\nabla_3 A|^2 + |\nabla A|^2\big),$$

we infer, since $\mathfrak{q} = \psi + i\,{}^*\psi$,

$$BEF_p[A](\tau_1,\tau_2) \lesssim B_p[\psi](\tau_1,\tau_2) + \dot{E}_p[A](\tau_1) + \epsilon_0^2\tau_1^{-2-3\delta_{dec}}$$

which is (11.2.6) in the case $s = 0$.

Next, in view of Corollary 11.4.13 and Lemma 11.4.14, we have, for all $\delta \le p \le 2-\delta$,

$$\int_{\Sigma(\tau)} r^{p+2}\big(r^4|\nabla\nabla_3 A|^2 + r^2|\nabla A|^2\big) \lesssim E_p[\mathfrak{q}](\tau) + \dot{E}_p[A](\tau) + \epsilon_0^2\tau^{-2-3\delta_{dec}}.$$

Moreover, we have in view of the above, for all $\delta \le p \le 2-\delta$,

$$\sup_{\tau\in[\tau_1,\tau_2]} E_p[A](\tau) \lesssim BEF_p[A](\tau_1,\tau_2) \lesssim B_p[\psi](\tau_1,\tau_2) + \dot{E}_p[A](\tau_1) + \epsilon_0^2\tau_1^{-2-3\delta_{dec}}$$

and hence, we have in particular, for any $\tau \in [\tau_1,\tau_2]$, , for all $\delta \le p \le 2-\delta$,

$$\begin{aligned} & \int_{\Sigma(\tau)} r^{p+2}\Big(r^4\chi_{nt}^2|\nabla_4\nabla_3 A|^2 + r^2|\nabla_{\widehat{R}}\nabla_3 A|^2 + r^2|\nabla_3 A|^2 + |A|^2\Big) \\ \lesssim\ & B_p[\psi](\tau_1,\tau_2) + \dot{E}_p[A](\tau_1) + \epsilon_0^2\tau_1^{-2-3\delta_{dec}}. \end{aligned}$$

Also, recall that we have

$${}^{(c)}\nabla_3^2 A = O(r^{-4})\mathfrak{q} + O(r^{-1})\,{}^{(c)}\nabla_3 A + O(r^{-2})A + \Gamma_g\,{}^{(c)}\nabla_3 A + r^{-1}\Gamma_g A,$$

and hence

$$\begin{aligned} \nabla_3\,{}^{(c)}\nabla_3 A &= {}^{(c)}\nabla_3^2 A + \Gamma_b\,{}^{(c)}\nabla_3 A \\ &= {}^{(c)}\nabla_3^2 A + \Gamma_b\nabla_3 A + \Gamma_b\Gamma_b A \\ &= O(r^{-4})\mathfrak{q} + O(r^{-1})\,{}^{(c)}\nabla_3 A + O(r^{-2})A + \Gamma_g\,{}^{(c)}\nabla_3 A + r^{-1}\Gamma_g A \\ &\quad +\Gamma_b\nabla_3 A + \Gamma_b\Gamma_b A \\ &= O(r^{-4})\mathfrak{q} + O(r^{-1})\nabla_3 A + O(r^{-2})A \end{aligned}$$

where we used the control of Γ_b and Γ_g. This yields, for any $\tau \in [\tau_1, \tau_2]$, for all $\delta \leq p \leq 1-\delta$,

$$\int_{\Sigma(\tau)} r^{p+6}|\nabla_3\, {}^{(c)}\nabla_3 A|^2 \lesssim E_p[\mathfrak{q}](\tau_1, \tau_2) + \int_{\Sigma(\tau)} r^{p+2}\Big(r^2|\nabla_3 A|^2 + |A|^2\Big).$$

Grouping the above estimates, we infer, for any $\tau \in [\tau_1, \tau_2]$, for all $\delta \leq p \leq 2-\delta$,

$$\begin{aligned}&\int_{\Sigma(\tau)} r^{p+2}\big(r^{\min(4,5-\delta-p)}|\nabla_3\, {}^{(c)}\nabla_3 A|^2 + r^4\chi_{nt}^2|\nabla_4\nabla_3 A|^2 + r^2|\nabla_{\widehat{R}}\nabla_3 A|^2 \\ &+ r^2|\nabla_3 A|^2 + |A|^2 + r^4|\nabla\nabla_3 A|^2 + r^2|\nabla A|^2\big) \\ &\lesssim EB_p[\psi](\tau_1, \tau_2) + E_p[A](\tau_1) + \epsilon_0^2\tau_1^{-2-3\delta_{dec}}.\end{aligned}$$

In particular, since $\chi_{nt} = 1$ on $r \geq 4m$, and since ∇_4 is spanned by ∇_3 and $\nabla_{\widehat{R}}$ on $r \leq 4m$, we infer, for any $\tau \in [\tau_1, \tau_2]$, for all $\delta \leq p \leq 2-\delta$,

$$\begin{aligned}&\int_{\Sigma(\tau)} r^{p+2}\big(r^{\min(4,5-\delta-p)}|\nabla_3\, {}^{(c)}\nabla_3 A|^2 + r^4|\nabla_4\nabla_3 A|^2 + r^4|\nabla\nabla_3 A|^2 + r^2|\nabla A|^2\big) \\ &\lesssim EB_p[\psi](\tau_1, \tau_2) + E_p[A](\tau_1) + \epsilon_0^2\tau_1^{-2-3\delta_{dec}}.\end{aligned}$$

which is (11.2.7) in the case $s = 0$.

It remains to recover (11.2.6) and (11.2.7) for $1 \leq s \leq k_L$. To this end, we proceed as follows:

1. We argue by iteration assuming that (11.2.6) and (11.2.7) hold for some $0 \leq s \leq k_L - 1$. It it true for $s = 0$ by the above, and our goal is to prove that (11.2.6) and (11.2.7) hold with s replaced by $s+1$.

2. We commute the system of transport equations (11.4.8), i.e.

$$\begin{aligned}{}^{(c)}\nabla_3\Psi &= \Big(O(r^{-2}) + r^{-1}\Gamma_g\Big)\mathfrak{q} + r^2\mathfrak{d}^{\leq 1}\Gamma_b\, {}^{(c)}\nabla_3 A + r\mathfrak{d}^{\leq 1}\Gamma_b A, \\ {}^{(c)}\nabla_3\left(\frac{(\overline{\mathrm{tr}\underline{X}})^2}{(\Re(\mathrm{tr}\underline{X}))^2(\mathrm{tr}\underline{X})^2}A\right) &= \Psi + r^2\mathfrak{d}^{\leq 1}(\Gamma_b)\cdot A,\end{aligned}$$

   with $\not{\mathcal{L}}_{\mathbf{T}}$, $\overline{q}\,\overline{{}^{(c)}\mathcal{D}}\cdot$ and ${}^{(c)}\nabla_4$. In view of the commutation formulas of Lemma 9.2.1 and Lemma 4.2.2, we have, for $U \in \mathfrak{s}_2$,

$$\begin{aligned}[{}^{(c)}\nabla_3, \not{\mathcal{L}}_{\mathbf{T}}]U &= [\nabla_3, \not{\mathcal{L}}_{\mathbf{T}}]U + 4[\underline{\omega}, \not{\mathcal{L}}_{\mathbf{T}}]U = [\not{\mathcal{L}}_{\mathbf{T}}, \mathfrak{d}]U + \mathfrak{d}^{\leq 1}(\Gamma_b)U \\ &= \mathfrak{d}^{\leq 1}(\Gamma_b U),\end{aligned}$$

$$\begin{aligned}
[\,{}^{(c)}\nabla_3, q\,\overline{{}^{(c)}\mathcal{D}}\cdot]U &= q[\,{}^{(c)}\nabla_3, \overline{q}\,\overline{{}^{(c)}\mathcal{D}}\cdot]U + e_3(q)\,\overline{{}^{(c)}\mathcal{D}}\cdot U\\
&= -\frac{1}{2}q\left(\overline{\mathrm{tr}\underline{X}} - \frac{2}{q}e_3(q)\right)\overline{{}^{(c)}\mathcal{D}}\cdot U + 2q\overline{\mathrm{tr}\underline{X}}\,\overline{H}\cdot U + q\overline{H}\cdot\,{}^{(c)}\nabla_3 U\\
&\quad +\Gamma_b\cdot\mathfrak{d}^{\leq 1}U\\
&= O(ar^{-1})\,{}^{(c)}\nabla_3 U + O(ar^{-2})U + r\Gamma_g\,{}^{(c)}\nabla_3 U + \Gamma_b\cdot\mathfrak{d}^{\leq 1}U,
\end{aligned}$$

and

$$[\,{}^{(c)}\nabla_3,\,{}^{(c)}\nabla_4]U = O(ar^{-2})\nabla U + O(r^{-3})U + r^{-1}\Gamma_g\mathfrak{d}^{\leq 1}U.$$

This yields the commuted systems

$$\begin{aligned}
{}^{(c)}\nabla_3\not{\mathcal{L}}_{\mathbf{T}}\Psi =& \Big(O(r^{-2}) + r^{-1}\Gamma_g\Big)\not{\mathcal{L}}_{\mathbf{T}}\mathfrak{q} + \Big(O(r^{-3}) + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_b\Big)\mathfrak{q}\\
&+ r^2\mathfrak{d}^{\leq 1}\Gamma_b\,{}^{(c)}\nabla_3\not{\mathcal{L}}_{\mathbf{T}}A + r\mathfrak{d}^{\leq 1}\Gamma_b\not{\mathcal{L}}_{\mathbf{T}}A\\
&+ r^2\mathfrak{d}^{\leq 2}\Gamma_b\,{}^{(c)}\nabla_3 A + r\mathfrak{d}^{\leq 2}\Gamma_b A,\\
{}^{(c)}\nabla_3\not{\mathcal{L}}_{\mathbf{T}}\left(\frac{(\overline{\mathrm{tr}\underline{X}})^2}{(\Re(\mathrm{tr}\underline{X}))^2(\mathrm{tr}\underline{X})^2}A\right) =& \not{\mathcal{L}}_{\mathbf{T}}\Psi + r^2\mathfrak{d}^{\leq 1}(\Gamma_b)\cdot\not{\mathcal{L}}_{\mathbf{T}}A + r^2\mathfrak{d}^{\leq 2}(\Gamma_b)\cdot A,
\end{aligned}$$

$$\begin{aligned}
{}^{(c)}\nabla_3 q\,\overline{{}^{(c)}\mathcal{D}}\cdot\Psi =& \Big(O(r^{-1}) + \Gamma_g\Big)\nabla\mathfrak{q} + \Big(O(r^{-2}) + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_g\Big)\mathfrak{q}\\
&+ r\mathfrak{d}^{\leq 1}\Gamma_b\cdot\nabla\,{}^{(c)}\nabla_3 A + r^2\mathfrak{d}^{\leq 2}\Gamma_b\,{}^{(c)}\nabla_3 A\\
&+ \mathfrak{d}^{\leq 1}\Gamma_b\nabla A + r\mathfrak{d}^{\leq 2}\Gamma_b A,\\
{}^{(c)}\nabla_3 q\,\overline{{}^{(c)}\mathcal{D}}\cdot\left(\frac{(\overline{\mathrm{tr}\underline{X}})^2}{(\Re(\mathrm{tr}\underline{X}))^2(\mathrm{tr}\underline{X})^2}A\right) =& q\,\overline{{}^{(c)}\mathcal{D}}\cdot\Psi + O(ar)\,{}^{(c)}\nabla_3 A + O(a)A\\
&+ r^2\Gamma_g\,{}^{(c)}\nabla_3 A + r^2\Gamma_b\cdot\mathfrak{d}^{\leq 1}A + r^2\mathfrak{d}^{\leq 2}(\Gamma_b)\cdot A,
\end{aligned}$$

and

$$\begin{aligned}
{}^{(c)}\nabla_3\,{}^{(c)}\nabla_4\Psi =& \Big(O(r^{-2}) + r^{-1}\Gamma_g\Big)\,{}^{(c)}\nabla_4\mathfrak{q} + \Big(O(r^{-3}) + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g\Big)\mathfrak{q}\\
&+ r^2\mathfrak{d}^{\leq 1}\Gamma_b\,{}^{(c)}\nabla_3 A + r\mathfrak{d}^{\leq 1}\Gamma_b A\\
&+ O(ar^{-2})\nabla\Psi + O(r^{-3})\Psi + r^{-1}\Gamma_g\mathfrak{d}^{\leq 1}\Psi,\\
{}^{(c)}\nabla_3\,{}^{(c)}\nabla_4\left(\frac{(\overline{\mathrm{tr}\underline{X}})^2}{(\Re(\mathrm{tr}\underline{X}))^2(\mathrm{tr}\underline{X})^2}A\right) =& {}^{(c)}\nabla_4\Psi + O(a)\nabla A + O(r^{-1})A + r^2\mathfrak{d}^{\leq 1}(\Gamma_b)\cdot A.
\end{aligned}$$

3. Using the iteration assumption for these commuted systems, and using the original system to recover the ∇_3 derivative in the redshift region, we infer that (11.2.6) and (11.2.7) hold for s derivatives with A replaced with $(\not{\mathcal{L}}_{\mathbf{T}}, q\,\overline{{}^{(c)}\mathcal{D}}\cdot, {}^{(c)}\nabla_4, \chi_{red}\,{}^{(c)}\nabla_3)\underline{A}$. Together with:

(a) the link between $\not{\mathcal{L}}_{\mathbf{T}}$ and $\nabla_{\mathbf{T}}$ of Lemma 9.2.1,

(b) the Hodge elliptic estimates of Proposition 9.3.2,

(c) the fact that $(\nabla_{\mathbf{T}}, r\nabla_4, \not{\mathfrak{d}}, \chi_{red}\nabla_3)$ span $\mathfrak{d}$,

and using the iteration assumption to absorb lower order terms in differentiability, we infer that (11.2.6) and (11.2.7) hold for s derivatives with A replaced with $\mathfrak{d}^{\leq 1}A$. In particular, (11.2.6) and (11.2.7) hold with s replaced by $s+1$. Thus, by iteration, (11.2.6) and (11.2.7) hold for all s such that $0 \leq s \leq k_L$. This end the proof of Proposition 11.2.8.

11.5 Proof of Theorem 11.2.4

In this section we prove Theorem 11.2.4, i.e. we establish the estimate, for $2 \leq s \leq k_L - 1$, for all $-1+\delta \leq q \leq 1-\delta$,

$$\begin{aligned} BEF^s_q[\check{\psi}](\tau_1,\tau_2) \lesssim\; & E^s_q[\check{\psi}](\tau_1) + E^{s+1}_{\max(q,\delta)}[\psi, A](\tau_1) + \check{\mathcal{N}}^s_q[\check{\psi}, N_{\text{Err}}](\tau_1,\tau_2) \\ & + \mathcal{N}^{s+1}_{\max(q,\delta)}[\psi, N_{\text{Err}}](\tau_1,\tau_2) + \epsilon_0^2 \tau_1^{-2-3\delta_{dec}}, \end{aligned}$$

where

$$\check{\mathcal{N}}^s_q[\check{\psi}, N_{\text{Err}}](\tau_1,\tau_2) \;=\; \int_{{}^{(ext)}\mathcal{M}} r^{q+2}\check{\nabla}_4 \mathfrak{d}^{\leq s}\check{\psi}\cdot\left(\nabla_4\mathfrak{d}^{\leq s}N_{\text{Err}} + \frac{3}{r}\mathfrak{d}^{\leq s}N_{\text{Err}}\right).$$

Proof of Theorem 11.2.4. According to Theorem 6.2.2 we have, for solutions $\psi \in \mathfrak{s}_2$ of (6.1.1) on $\mathcal{M}$, for all $-1+\delta < q \leq 1-\delta$, $s \leq k_L - 1$,

$$BEF^s_q[\check{\psi}](\tau_1,\tau_2) \lesssim \widetilde{E}^s_q[\check{\psi}](\tau_1) + \widetilde{\mathcal{N}}^s_q[\check{\psi}, N](\tau_1,\tau_2) + \mathcal{N}^{s+1}_{\max\{q,\delta\}}[\psi, N](\tau_1,\tau_2), \tag{11.5.1}$$

where $\check{\psi} = r^2(e_4\psi + \frac{r}{|q|^2}\psi)$, $\widetilde{E}^s_q[\check{\psi}](\tau) = E^s_q[\check{\psi}](\tau) + E^{s+1}_{\max\{q,\delta\}}[\psi](\tau)$ and

$$\widetilde{\mathcal{N}}^s_q[\check{\psi}, N](\tau_1,\tau_2) = \int_{\mathcal{M}_{\geq R}(\tau_1,\tau_2)} r^{q+2}\mathfrak{d}^{\leq s}\check{\psi}\cdot\left(\nabla_4\mathfrak{d}^{\leq s}N + \frac{3}{r}\mathfrak{d}^{\leq s}N\right),$$

where, see section 11.1.2, $N = N_0 + N_L + N_{\text{Err}}$. In view of (11.5.1), it remains to estimate the terms $\mathcal{N}^{s+1}_{\max(q,\delta)}[\psi, N_0+N_L](\tau_1,\tau_2)$ and $\check{\mathcal{N}}^s_q[\check{\psi}, N_0+N_L](\tau_1,\tau_2)$. This is done in the following steps.

Step 1. We first estimate the term $\mathcal{N}^{s+1}_{\max(q,\delta)}[\psi, N_0+N_L](\tau_1,\tau_2)$. Note that

$$\begin{aligned}
&\mathcal{N}^{s+1}_{\max(q,\delta)}[\psi, N_0+N_L](\tau_1,\tau_2)\\
\lesssim\;& {}^{(Mor)}\mathcal{N}^{s+1}[\psi, N_0+N_L](\tau_1,\tau_2) + {}^{(ext)}\mathcal{N}^{s+1}_{\max(q,\delta)}[\psi, N_0+N_L](\tau_1,\tau_2)\\
&+{}^{(En)}\mathcal{N}^{s+1}[\psi, N_0](\tau_1,\tau_2) + {}^{(En)}\mathcal{N}^{s+1}[\psi, N_L](\tau_1,\tau_2),
\end{aligned}$$

which together with (11.3.1)-(11.3.3) implies

$$\mathcal{N}^{s+1}_{\max(q,\delta)}[\psi, N_0+N_L](\tau_1,\tau_2) \lesssim BEF^{s+1}_{\max(q,\delta)}[\psi, A](\tau_1,\tau_2) + \epsilon_0^2\tau_1^{-2-3\delta_{dec}}. \quad (11.5.2)$$

Step 2. Next, we estimate the term $\check{\mathcal{N}}^s_q[\check{\psi}, N_0+N_L](\tau_1,\tau_2)$. Notice that

$$\begin{aligned}
\check{\mathcal{N}}^s_q[\check{\psi}, N](\tau_1,\tau_2) &= \left|\int_{{}^{(ext)}\mathcal{M}(\tau_1,\tau_2)} r^{q+2}\check{\nabla}_4\mathfrak{d}^{\leq s}\check{\psi}\cdot\left(\nabla_4\mathfrak{d}^{\leq s}N + \frac{3}{r}\mathfrak{d}^{\leq s}N\right)\right|\\
&\lesssim \left|\int_{{}^{(ext)}\mathcal{M}(\tau_1,\tau_2)} r^{q+1}\check{\nabla}_4\mathfrak{d}^{\leq s}\check{\psi}\cdot\left(\mathfrak{d}^{\leq s+1}N\right)\right|.
\end{aligned}$$

We thus have

$$\begin{aligned}
\check{\mathcal{N}}^s_q[\check{\psi}, N_0+N_L](\tau_1,\tau_2) &\lesssim \check{\mathcal{N}}^s_q[\check{\psi}, N_0](\tau_1,\tau_2) + \check{\mathcal{N}}^s_q[\check{\psi}, N_L](\tau_1,\tau_2),\\
\check{\mathcal{N}}^s_q[\check{\psi}, N_0](\tau_1,\tau_2) &\lesssim \left|\int_{{}^{(ext)}\mathcal{M}(\tau_1,\tau_2)} r^{q+1}\check{\nabla}_4\mathfrak{d}^{\leq s}\check{\psi}\cdot\left(\mathfrak{d}^{\leq s+1}N_0\right)\right|,\\
\check{\mathcal{N}}^s_q[\check{\psi}, N_L](\tau_1,\tau_2) &\lesssim \left|\int_{{}^{(ext)}\mathcal{M}(\tau_1,\tau_2)} r^{q+1}\check{\nabla}_4\mathfrak{d}^{\leq s}\check{\psi}\cdot\left(\mathfrak{d}^{\leq s+1}N_L\right)\right|,
\end{aligned}$$

and we estimate below the terms $\check{\mathcal{N}}^s_q[\check{\psi}, N_0](\tau_1,\tau_2)$ and $\check{\mathcal{N}}^s_q[\check{\psi}, N_L](\tau_1,\tau_2)$ separately.

Step 2a. We have

$$\begin{aligned}
\check{\mathcal{N}}^s_q[\check{\psi}, N_0](\tau_1,\tau_2) &\lesssim \left(\int_{{}^{(ext)}\mathcal{M}(\tau_1,\tau_2)} r^{q-3}|\mathfrak{d}^{\leq s+1}\check{\psi}|^2\right)^{1/2}\left(\int_{{}^{(ext)}\mathcal{M}(\tau_1,\tau_2)} r^{q+5}|\mathfrak{d}^{\leq s+1}(N_0)|^2\right)^{1/2}\\
&\lesssim \left({}^{(ext)}B^s_q[\check{\psi}](\tau_1,\tau_2)\right)^{1/2}\left(\int_{{}^{(ext)}\mathcal{M}(\tau_1,\tau_2)} r^{q+5}|\mathfrak{d}^{\leq s+1}(N_0)|^2\right)^{1/2},
\end{aligned}$$

and, since $N_0 = O(a^2r^{-4})\psi$,

$$\begin{aligned}
\int_{{}^{(ext)}\mathcal{M}(\tau_1,\tau_2)} r^{q+5}|\mathfrak{d}^{\leq s+1}N_0|^2 &\lesssim O(a^4)\int_{{}^{(ext)}\mathcal{M}(\tau_1,\tau_2)} r^{q-3}|\mathfrak{d}^{\leq s+1}\psi|^2\\
&\lesssim O(a^4)B^{s+1}_{\max(q,\delta)}[\psi](\tau_1,\tau_2)
\end{aligned}$$

which yields

$$\check{\mathcal{N}}^s_q[\check{\psi}, N_0](\tau_1,\tau_2) \lesssim \left({}^{(ext)}B^s_q[\check{\psi}](\tau_1,\tau_2)\right)^{1/2}\left(B^{s+1}_{\max(q,\delta)}[\psi](\tau_1,\tau_2)\right)^{1/2}. \quad (11.5.3)$$

Step 2b. Since, according to (11.1.10), we have

$$N_L = O(a)\mathfrak{d}^{\leq 1}\nabla_3 A + O(ar^{-1})\mathfrak{d}^{\leq 1}A,$$

we infer

$$\begin{aligned}
\check{\mathcal{N}}^s_q[\check{\psi}, N_L] &\lesssim \left|\int_{{}^{(ext)}\mathcal{M}} r^{q+1}\check{\nabla}_4\mathfrak{d}^{\leq s}\check{\psi}\cdot\left(\mathfrak{d}^{\leq s+1}(O(a)\mathfrak{d}^{\leq 1}\nabla_3 A)\right)\right| \\
&+\left|\int_{{}^{(ext)}\mathcal{M}} r^{q}\check{\nabla}_4\mathfrak{d}^{\leq s}\check{\psi}\cdot\left(\mathfrak{d}^{\leq s+1}(O(a)\mathfrak{d}^{\leq 1} A)\right)\right| \\
&\lesssim |a|\left(\int_{{}^{(ext)}\mathcal{M}} r^{q-1}|\check{\nabla}_4\mathfrak{d}^{\leq s}\check{\psi}|^2\right)^{\frac{1}{2}}\left(\int_{{}^{(ext)}\mathcal{M}} r^{q+1}\left(r^2|\mathfrak{d}^{\leq s+2}\nabla_3 A|^2+|\mathfrak{d}^{\leq s+2}A|^2\right)\right)^{\frac{1}{2}}
\end{aligned}$$

which yields, recalling the definition of the norms $B_q[A]$ in Definition 11.2.1,

$$\check{\mathcal{N}}^s_q[\check{\psi}, N_L](\tau_1,\tau_2) \lesssim \left({}^{(ext)}B^s_q[\check{\psi}](\tau_1,\tau_2)\right)^{1/2}\left(B^{s+1}_{\max(q,\delta)}[A](\tau_1,\tau_2)\right)^{1/2}. \quad (11.5.4)$$

Step 3. Combining the estimates of Step 1 and Step 2, we have

$$\begin{aligned}
&\mathcal{N}^{s+1}_{\max(q,\delta)}[\psi, N_0+N_L](\tau_1,\tau_2)+\check{\mathcal{N}}^s_q[\check{\psi}, N_0+N_L] \\
\lesssim &\left({}^{(ext)}B^s_q[\check{\psi}](\tau_1,\tau_2)\right)^{1/2}\left(B^{s+1}_{\max(q,\delta)}[\psi, A](\tau_1,\tau_2)\right)^{1/2} \\
&+BEF^{s+1}_{\max(q,\delta)}[\psi, A](\tau_1,\tau_2)+\epsilon_0^2\tau_1^{-2-3\delta_{dec}}.
\end{aligned}$$

Since $N = N_0 + N_L + N_{\text{Err}}$, this yields, together with (11.5.1),

$$\begin{aligned}
BEF^s_q[\check{\psi}](\tau_1,\tau_2) &\lesssim \widetilde{E}^s_q[\check{\psi}](\tau_1)+\widetilde{\mathcal{N}}^s_q[\check{\psi}, N_{\text{Err}}](\tau_1,\tau_2)+\mathcal{N}^{s+1}_{\max\{q,\delta\}}[\psi, N_{\text{Err}}](\tau_1,\tau_2) \\
&+\left({}^{(ext)}B^s_q[\check{\psi}](\tau_1,\tau_2)\right)^{1/2}\left(B^{s+1}_{\max(q,\delta)}[\psi, A](\tau_1,\tau_2)\right)^{1/2} \\
&+BEF^{s+1}_{\max(q,\delta)}[\psi, A](\tau_1,\tau_2)+\epsilon_0^2\tau_1^{-2-3\delta_{dec}}.
\end{aligned}$$

In view of the control of $BEF^{s+1}_{\max(q,\delta)}[\psi, A](\tau_1,\tau_2)$ provided by Theorem 11.2.1, we infer

$$\begin{aligned}
BEF^s_q[\check{\psi}](\tau_1,\tau_2) \lesssim\, & E^s_q[\check{\psi}](\tau_1)+E^{s+1}_{\max(q,\delta)}[\psi, A](\tau_1)+\check{\mathcal{N}}^s_q[\check{\psi}, N_{\text{Err}}](\tau_1,\tau_2) \\
&+\mathcal{N}^{s+1}_{\max(q,\delta)}[\psi, N_{\text{Err}}](\tau_1,\tau_2)+\epsilon_0^2\tau_1^{-2-3\delta_{dec}},
\end{aligned}$$

as stated. This concludes the proof of Theorem 11.2.4 . □

11.6 Eliminating $N_{\rm Err}$

The goal of this section is to eliminate the error term $N_{\rm Err}$ appearing in the RHS of the estimates of Theorems 11.2.3 and 11.2.4 and derive the following results.

Theorem 11.6.1. *Under the assumptions made in section 11.1, the following estimates hold true, for all $\delta \le p \le 2-\delta$ and $2 \le s \le k_L$,*

$$BEF^s_p[\psi, A](\tau_1,\tau_2) \lesssim E^s_p[\psi, A](\tau_1) + \epsilon_0^2 \tau_1^{p-2-3\delta_{dec}}. \tag{11.6.1}$$

Theorem 11.6.2. *Under the assumptions made in section 11.1, the following estimates hold true, for $2 \le s \le k_L - 1$ and $-1+\delta \le q \le 3\delta_{dec}$,*

$$BEF^s_q[\check{\psi}](\tau_1,\tau_2) \lesssim E^s_q[\check{\psi}](\tau_1) + E^{s+1}_{\max(q,\delta)}[\psi, A](\tau_1) + \epsilon_0^2 \tau_1^{q-3\delta_{dec}}. \tag{11.6.2}$$

Proof of Theorem 11.6.1. Recall, see (11.1.11),

$$N_{\rm Err} = N_g + \nabla_3(rN_g) + N_m[\mathfrak{q}], \quad N_g = r^2 \mathfrak{d}^{\le 2}\big(\Gamma_g\cdot(A,B)\big), \quad N_m[\mathfrak{q}] = \mathfrak{d}^{\le 1}\big(\Gamma_g \cdot \mathfrak{q}\big).$$

The proof of the theorem follows step by step the proof of Theorem 5.14 in [50]. As in that paper, the terms $N_m[\mathfrak{q}]$ and N_g are estimated directly using the bounds for Γ_g, A, B, making sure to absorb to the left the corresponding bulk contribution in $\mathfrak{q}$. The more difficult term $\nabla_3(rN_g)$ requires an integration by parts which is explained in detail in the last part of section 5.3.2 of [50]. □

Proof of Theorem 11.6.2. The proof is similar to the proof of Theorem 5.15 in [50], and we recall below the main steps of the proof.

Step 1. First observe that the control of $N_{\rm Err}$ in the proof of Theorem 11.6.1 yields in particular

$$\mathcal{N}^{s+1}_{\max(q,\delta)}[\psi, N_{\rm Err}](\tau_1,\tau_2) \lesssim \epsilon_0^2 \tau_1^{\max(q,\delta)-2-3\delta_{dec}}.$$

We therefore only need to estimate

$$\begin{aligned}\check{\mathcal{N}}^s_q[\check{\psi}, N_{\rm Err}](\tau_1,\tau_2) &= \int_{^{(ext)}\mathcal{M}} r^{q+2}\check{\nabla}_4 \mathfrak{d}^{\le s}\check{\psi}\cdot\left(\nabla_4\mathfrak{d}^{\le s}N_{\rm Err} + \frac{3}{r}\mathfrak{d}^{\le s}N_{\rm Err}\right)\\ &\lesssim \left|\int_{^{(ext)}\mathcal{M}} r^q (r\check{\nabla}_4\mathfrak{d}^{\le s}\check{\psi})\big(\mathfrak{d}^{\le s+1}N_{\rm Err}\big)\right|.\end{aligned}$$

Step 2. We use fact that

$$\mathfrak{d}^{\leq s+1}N_{\text{Err}} = \mathfrak{d}^{\leq s+1}N_g + e_3(\mathfrak{d}^{\leq s+1}rN_g) + \mathfrak{d}^{\leq s+1}N_m[\mathfrak{q}],$$

and estimate each contribution separately.

Step 2a. We have

$$\begin{aligned}
\check{\mathcal{N}}^s_q[\check{\psi}, N_g](\tau_1, \tau_2) &\lesssim \int_{{}^{(ext)}\mathcal{M}} r^q|r\check{\nabla}_4\mathfrak{d}^{\leq s}\check{\psi}||\mathfrak{d}^{\leq s+1}N_g| \\
&\lesssim \left(\int_{{}^{(ext)}\mathcal{M}} r^{q-3}|r\check{\nabla}_4\mathfrak{d}^{\leq s}\check{\psi}|^2\right)^{1/2}\left(\int_{{}^{(ext)}\mathcal{M}} r^{q+3}|\mathfrak{d}^{\leq s+1}N_g|^2\right)^{1/2} \\
&\lesssim \delta_1 B^s_q[\check{\psi}](\tau_1, \tau_2) + \delta_1^{-1}\mathcal{N}^{s+1}_{q+2}[\psi, N_g](\tau_1, \tau_2),
\end{aligned}$$

where $\delta_1 > 0$ is chosen sufficiently small so that we can later absorb the term $\delta_1 B^s_q[\check{\psi}](\tau_1, \tau_2)$ on the left hand side of the main estimate.

Step 2b. As in the proof of Theorem 11.6.1, the integral due to $\mathfrak{d}^{\leq s+1}N_m[\mathfrak{q}]$ can be bounded using the control of Γ_g. Since $\delta \leq q+1 \leq 2-\delta$, we finally infer

$$\check{\mathcal{N}}^s_q[\check{\psi}, N_m[\mathfrak{q}]](\tau_1, \tau_2) \lesssim \delta_1 B^s_q[\check{\psi}](\tau_1, \tau_2) + \delta_1^{-1}\mathcal{N}^{s+1}_{q+2}[\psi, N_g](\tau_1, \tau_2).$$

Step 2c. The integral due to $e_3(\mathfrak{d}^{\leq s+1}rN_g)$ is estimated through the integration by parts in e_3 and the use of the wave equation for $\check{\psi}$, as in the proof of Theorem 11.6.1. As in the proof of Theorem 5.15 in [50], we obtain

$$\begin{aligned}
\check{\mathcal{N}}^s_q[\check{\psi}, e_3(rN_g)]](\tau_1, \tau_2) &\lesssim \sum_{k\leq s}\left|\int_{{}^{(ext)}\mathcal{M}(\tau_1,\tau_2)} r^q e_3 e_4(r\mathfrak{d}^k\check{\psi})\cdot\mathfrak{d}^{k+1}(rN_g)\right| \\
&\quad +\delta_1 B^s_q[\check{\psi}](\tau_1, \tau_2) + \delta_1^{-1}\mathcal{N}^{s+1}_{q+2}[\psi, N_g](\tau_1, \tau_2), \\
&\lesssim \delta_1 B^s_q[\check{\psi}](\tau_1, \tau_2) + \delta_1^{-1}\mathcal{N}^{s+2}_{q+2}[\psi, N_g](\tau_1, \tau_2).
\end{aligned}$$

Step 2d. The estimates in Steps 2a, 2b and 2c imply, for any $\delta_1 > 0$,

$$\check{\mathcal{N}}^s_q[\check{\psi}, N_{\text{Err}}](\tau_1, \tau_2) \lesssim \delta_1 B^s_q[\check{\psi}](\tau_1, \tau_2) + \delta_1^{-1}\mathcal{N}^{s+2}_{q+2}[\psi, N_g](\tau_1, \tau_2).$$

Step 3. Together with Theorem 11.2.4, steps 1 and 2 imply, for all $-1+\delta \leq q \leq 1-\delta$ and any $\delta_1 > 0$,

$$\begin{aligned}
BEF^s_q[\check{\psi}](\tau_1, \tau_2) \lesssim{}& E^s_q[\check{\psi}](\tau_1) + E^{s+1}_{\max(q,\delta)}[\psi, A](\tau_1) + \epsilon_0^2\tau_1^{\max(q,\delta)-2-3\delta_{dec}} \\
&+ \delta_1 B^s_q[\check{\psi}](\tau_1, \tau_2) + \delta_1^{-1}\mathcal{N}^{s+2}_{q+2}[\psi, N_g](\tau_1, \tau_2).
\end{aligned}$$

Choosing $\delta_1 > 0$ small enough to absorb the fourth term on the RHS from the LHS, we infer, for all $-1+\delta \le q \le 1-\delta$,

$$\begin{aligned} BEF^s_q[\check{\psi}](\tau_1,\tau_2) \lesssim E^s_q[\check{\psi}](\tau_1) + E^{s+1}_{\max(q,\delta)}[\psi, A](\tau_1) + \epsilon_0^2 \tau_1^{\max(q,\delta)-2-3\delta_{dec}} \\ + \mathcal{N}^{s+2}_{q+2}[\psi, N_g](\tau_1,\tau_2). \end{aligned}$$

Now, arguing as in Proposition 5.10 of [50], we have, for all $-1+\delta \le q \le 3\delta_{dec}$,

$$\mathcal{N}^{s+2}_{q+2}[\psi, N_g](\tau_1,\tau_2) \quad \lesssim \quad \epsilon_0^2 \tau_1^{q-3\delta_{dec}}.$$

We infer, for all $-1+\delta \le q \le 3\delta_{dec}$,

$$BEF^s_q[\check{\psi}](\tau_1,\tau_2) \lesssim E^s_q[\check{\psi}](\tau_1) + E^{s+1}_{\max(q,\delta)}[\psi, A](\tau_1) + \epsilon_0^2 \tau_1^{q-3\delta_{dec}}$$

as stated. This concludes the proof of Theorem 11.6.2. □

11.7 Proof of Theorem M1

In this section we make use of the results of Theorems 11.6.1 and 11.6.2 to complete the proof of Theorem M1 of [53] which we restate below.

Theorem 11.7.1 (Theorem M1 in [53]). *Assume that the spacetime $\mathcal{M}$ verifies the assumptions* (11.1.1)–(11.1.4) *as well as the assumption* (1.5.7) *on the initial data. Then, if $\epsilon_0 > 0$ is sufficiently small, there exists $\delta_{extra} > \delta_{dec}$ such that we have the following estimates in $\mathcal{M}$, for all $s \le k_L - 10$,*

$$\sup_{\mathcal{M}} \left(\frac{r^2(2r+\tau)^{1+\delta_{extra}}}{\log(1+\tau)} + r^3(2r+\tau)^{\frac{1}{2}+\delta_{extra}} \right) \left(|\mathfrak{d}^{\le s} A| + r|\mathfrak{d}^{\le s-1}\nabla_3 A| \right) \lesssim \epsilon_0. \quad (11.7.1)$$

Also, for all $s \le k_L - 10$,

$$\int_{\Sigma_*(\ge\tau)} |\nabla_3 \mathfrak{d}^{s-1}\psi|^2 \quad \lesssim \quad \epsilon_0^2 \tau^{-2-2\delta_{extra}}. \quad (11.7.2)$$

Proof. We proceed according to the following steps.

Step 1. Starting with the result of Theorem 11.6.1, we run the basic mean value argument for $\delta \le p \le 2-\delta$ as in Theorem 5.21 of [50]. We obtain[6] for $\tau_1 \le \tau \le \tau_*$ and $s \le k_L - 2$,

$$BEF_p[\psi, A](\tau, \tau_*) \quad \lesssim \quad \epsilon_0^2 \tau^{-(2-p-\delta)}. \quad (11.7.3)$$

[6]With $\mathfrak{q}$ replaced here by the pair (ψ, A). Note that there is a loss one derivative for each application of the mean value theorem. See section 5.4.1 in [50], and in particular Theorem 5.21 in that paper.

Step 2. According to Theorem 11.6.2 we have for $2 \le s \le k_L - 1$, for all $-1+\delta \le q \le 3\delta_{dec}$,

$$BEF^s_q[\check{\psi}](\tau_1, \tau_2) \lesssim E^s_q[\check{\psi}](\tau_1) + E^{s+1}_{\max(q,\delta)}[\psi, A](\tau_1) + \epsilon_0^2 \tau_1^{q-3\delta_{dec}}.$$

In view of Step 1, we have

$$E^{s+1}_{\max(q,\delta)}[\psi, A](\tau_1) \lesssim \epsilon_0^2 \tau_1^{-(2-\max(q,\delta)-\delta)}$$

and hence, for $s \le k_L - 3$ and $-1 + \delta \le q \le 3\delta_{dec}$,

$$BEF^s_q[\check{\psi}](\tau_1, \tau_2) \quad \lesssim E^s_q[\check{\psi}](\tau_1) + \epsilon_0^2 \tau_1^{q-3\delta_{dec}}.$$

Applying the basic mean value theorem argument in the range $-1 + 3\delta_{dec} \le q \le 3\delta_{dec}$ as in the proof of Theorem 5.22 in [50] we deduce, for all $-1 + 3\delta_{dec} \le q \le 3\delta_{dec}$,

$$BEF^s_q[\check{\psi}](\tau_1, \tau_2) \quad \lesssim \quad \epsilon_0^2 \tau_1^{-(3\delta_{dec}-q)}.$$

Step 3. In view of Step 2, we have in particular, for $s \le k_L - 3$ and $q = -\delta$,

$$\sup_{\tau \in [\tau_1, \tau_2]} E^s_{-\delta}[\check{\psi}](\tau_1, \tau_2) \quad \lesssim \quad \epsilon_0^2 \tau_1^{-3\delta_{dec}-\delta}.$$

From the relation between ψ and $\check{\psi}$, we deduce, as in (5.4.20) of [50], for $s \le k_L - 3$,

$$\sup_{\tau \in [\tau_1, \tau_2]} E^s_{2-\delta}[\psi](\tau) \quad \lesssim \quad \epsilon_0^2 \tau_1^{-3\delta_{dec}-\delta}.$$

Step 4. Recall from Proposition 11.2.6 that the following estimate for solutions ψ of the full gRW equation hold true[7], for all $s \le k_L$ and all $\delta \le p \le 2 - \delta$,

$$BEF^s_p[\psi](\tau_1, \tau_2) \lesssim E^s_p[\psi](\tau_1) + BEF^s_\delta[\psi, A](\tau_1, \tau_2) + \mathcal{N}^s_p[\psi, N_{\text{Err}}](\tau_1, \tau_2).$$

Eliminating N_{Err} as before in Theorems 11.6.1 and Theorem 11.6.2, see also Proposition 5.10 and the proof of Theorem 5.14 in [50], we deduce, for $s \le k_L - 1$ and all $\delta \le p \le 2 - \delta$,

$$BEF^s_p[\psi](\tau_1, \tau_2) \lesssim E^s_p[\psi](\tau_1) + BEF^s_\delta[\psi, A](\tau_1, \tau_2) + \epsilon_0^2 \tau_1^{p-2-3\delta_{dec}}.$$

In view of Step 1 we have $BEF_\delta[\psi, A](\tau, \tau_*) \lesssim \epsilon_0^2 \tau^{-(2-2\delta)}$. Hence, for $s \le k_L - 1$ and all $\delta \le p \le 2 - \delta$,

$$BEF^s_p[\psi](\tau_1, \tau_2) \lesssim E^s_p[\psi](\tau_1) + \epsilon_0^2 \tau_1^{-(2-2\delta)} + \epsilon_0^2 \tau_1^{-(2+3\delta_{dec}-p)}.$$

[7]Here, we crucially exploit the fact that, in the estimate of the Proposition 11.2.6, the term in A on the right hand side appears in the norm $BEF^s_\delta[\psi, A]$ (rather than $BEF^s_p[\psi, A]$).

In particular, we infer for $s \leq k_L - 3$,

$$BEF_p^s[\psi](\tau_1, \tau_2) \lesssim E_p^s[\psi](\tau_1) + \epsilon_0^2 \tau_1^{-(2+3\delta_{dec}-p)}, \qquad 3\delta_{dec} + 2\delta \leq p \leq 2 - \delta.$$

Together with Step 3 and the usual mean value argument, we deduce for $s \leq k_L - 4$,

$$BEF_p^s[\psi](\tau_1, \tau_2) \lesssim \epsilon_0^2 \tau_1^{-(2+3\delta_{dec}-p)}, \qquad 1 - \delta \leq p \leq 2 - \delta. \tag{11.7.4}$$

In particular, for $s \leq k_L - 4$, we have

$$BEF_{1-\delta}^s[\psi](\tau_1, \tau_2) \lesssim \epsilon_0^2 \tau_1^{-(3\delta_{dec}+1+\delta)}. \tag{11.7.5}$$

Step 5. As in Proposition 5.12 in [50], we interpolate the control on $E_p[\psi]$ provided (11.7.4) between $p = 1 + \delta$ and $p = 1 - \delta$ and obtain

$$\tau^{1+3\delta_{dec}} \int_{S_r} |\mathfrak{d}^{\leq s}\psi|^2 \lesssim \epsilon_0^2.$$

Using Sobolev, we infer the following pointwise decay estimate for ψ, for all $s \leq k_L - 6$,

$$|\mathfrak{d}^{\leq s}\psi| \lesssim \epsilon_0 r^{-1} \tau^{-\frac{1+3\delta_{dec}}{2}}. \tag{11.7.6}$$

Step 6. We can now make use of the system of transport equations, see Corollary 11.1.5,

$$\begin{aligned} {}^{(c)}\nabla_3 \Psi &= \Big(O(r^{-2}) + r^{-1}\Gamma_g\Big)\mathfrak{q} + r^2 \mathfrak{d}^{\leq 1}\Gamma_b \, {}^{(c)}\nabla_3 A + r\mathfrak{d}^{\leq 1}\Gamma_b A, \\ {}^{(c)}\nabla_3 \left(\frac{(\overline{\mathrm{tr}\underline{X}})^2}{(\Re(\mathrm{tr}\underline{X}))^2 (\mathrm{tr}\underline{X})^2} A\right) &= \Psi + r^2 \mathfrak{d}^{\leq 1}(\Gamma_b) \cdot A. \end{aligned} \tag{11.7.7}$$

with Ψ introduced in Definition 11.1.4. Integrating the system of transport equations[8] and using the pointwise decay for $\psi = \Re(\mathfrak{q})$ in (11.7.6), we infer the following pointwise decay estimate for A, for all $s \leq k_L - 6$,

$$|\mathfrak{d}^{\leq s} A| \lesssim \epsilon_0 r^{-3} \tau^{-\frac{1+3\delta_{dec}}{2}}. \tag{11.7.8}$$

Integrating this estimate on $\Sigma(\tau)$ and recalling the definition of the $E_p^s[A]$ norms in Definition 11.2.1 we deduce, for all $s \leq k_L - 6$,

$$E_{1-\delta}^s[A](\tau) \lesssim \epsilon_0^2 \tau^{-(1+3\delta_{dec})}. \tag{11.7.9}$$

Step 7. The estimates (11.7.5) and (11.7.9) imply, for all $s \leq k_L - 6$,

$$E_{1-\delta}^s[A, \psi](\tau) \lesssim \epsilon_0^2 \tau^{-(1+3\delta_{dec})}.$$

[8]Proceeding as in sections 6.1.3 and 6.1.4 of [50].

We can thus run again the standard mean value argument and deduce from that estimate and Theorem 11.6.1, for $s \leq k_L - 7$,

$$BEF^s_\delta[A, \psi](\tau, \tau_*) \lesssim \epsilon_0^2 \tau^{-(2+3\delta_{dec}-2\delta)}.$$

In particular, for $s \leq k_L - 7$,

$$BEF^s_\delta[\psi](\tau, \tau_*) \lesssim \epsilon_0^2 \tau^{-(2+3\delta_{dec}-2\delta)} \tag{11.7.10}$$

and, in view of the definition of the flux norms $F^s_\delta[\psi]$ in section 6.1.5, we deduce

$$\int_{\Sigma_*(\geq\tau)} |\nabla_3 \mathfrak{d}^{s-1}\psi|^2 \lesssim \epsilon_0^2 \tau^{-(2+3\delta_{dec}-3\delta)}.$$

Hence, choosing $\delta_{extra} = \frac{3\delta_{dec}-2\delta}{2} > \delta_{dec}$, for $s \leq k_L - 7$,

$$\int_{\Sigma_*(\geq\tau)} |\nabla_3 \mathfrak{d}^{s-1}\psi|^2 \lesssim \epsilon_0^2 \tau^{-2-2\delta_{extra}}$$

which establishes the desired estimate (11.7.2).

Step 8. Making use of the estimate (11.7.10) and proceeding as in the derivation of the estimate (5.2.7) in Proposition 5.12 of [50], we derive the estimate, for any $S_r \subset \Sigma(\tau)$, $s \leq k_L - 8$,

$$r^{-1} \int_{S_r} |\mathfrak{d}^{\leq s}\psi|^2 \lesssim \epsilon_0^2 \tau^{-(2+3\delta_{dec}-2\delta)}. \tag{11.7.11}$$

Similarly, proceeding as in the estimate (5.2.9) in Proposition 5.13 in [50] we derive for any $S_r \subset \Sigma(\tau)$, $s \leq k_L - 8$,

$$\int_{S_r} |\mathfrak{d}^{\leq s-1}\nabla_3\psi|^2 \lesssim \epsilon_0^2 \tau^{-(2+3\delta_{dec}-2\delta)}. \tag{11.7.12}$$

Step 9. In view of (11.7.6), (11.7.11) and (11.7.12), and since $\psi = \Re(\mathfrak{q})$, we deduce the following estimate[9] for $\mathfrak{q}$, for all $s \leq k_L - 10$,

$$\sup_{\mathcal{M}} \left(r\tau^{\frac{1}{2}+\delta_{extra}} + \tau^{1+\delta_{extra}} \right) |\mathfrak{d}^{\leq s}\mathfrak{q}| + \sup_{\mathcal{M}} r\tau^{1+\delta_{extra}} |\mathfrak{d}^{\leq s-1}\nabla_3\mathfrak{q}| \lesssim \epsilon_0 \tag{11.7.13}$$

for $\delta_{extra} = \frac{3\delta_{dec}-2\delta}{2} > \delta_{dec}$.

[9]Note that this corresponds to the estimate for $\mathfrak{q}$ stated in section 3.6.1 of [50].

Step 10. Using (11.7.13), together with the system of transport equations in (11.7.7), and proceeding as in sections 6.1.3 and 6.1.4 of [50], we derive the following pointwise estimate for A, for all $s \leq k_L - 10$,

$$\sup_{\mathcal{M}} \left(\frac{r^2 (2r+\tau)^{1+\delta_{extra}}}{\log(1+\tau)} + r^3 (2r+\tau)^{\frac{1}{2}+\delta_{extra}} \right) \left(|\mathfrak{d}^{\leq s} A| + r|\mathfrak{d}^{\leq s-1} \nabla_3 A| \right) \lesssim \epsilon_0$$

as stated in (11.7.1). Together with the proof of (11.7.2) in Step 7, this concludes the proof of Theorem M1. $\square$

Chapter 12

Decay Estimates for $\underline{A}$

The goal of this chapter is to provide a complete proof for Theorem $M2$. To this end we proceed as follows:

1. We derive combined r^p weighted estimates for the pair $(\underline{\mathfrak{q}}, \underline{A})$ stated in Theorem 12.2.4. This, by far the most demanding result of the chapter, is proved in sections 12.2 and 12.3.

2. We then prove Theorem M2 in section 12.4, relying in particular on these combined r^p weighted estimates.

12.1 Preliminaries

The spacetime $\mathcal{M}$ we are dealing with here is precisely that described in section 6.1. As in Chapter 11, we make stronger assumptions on (Γ_g, Γ_b). We assume in fact for all for all $k \leq k_L$, with[1] $k_L = k_{small} + 120$,

$$\begin{aligned} \left(r^2 \tau^{\frac{1}{2}+\delta_{dec}} + r\tau^{1+\delta_{dec}}\right)|\mathfrak{d}^{\leq k}\Gamma_g| &\leq \epsilon, \\ r\tau^{1+\delta_{dec}}|\mathfrak{d}^{\leq k}\Gamma_b| &\leq \epsilon. \end{aligned} \tag{12.1.1}$$

We also assume that the curvature components A, B verify, for $k \leq k_L$,

$$r^{7/2+\delta_{dec}}|\mathfrak{d}^{\leq k}(A, B)| \quad \leq \epsilon. \tag{12.1.2}$$

[1]This is consistent with the value of k_L used in the bootstrap assumption needed in the proof of Theorem M2 (see section 1.5.3).

We make the additional gauge condition

$$\Xi \in r^{-2}\Gamma_g, \qquad \underline{\widecheck{H}} \in r^{-1}\Gamma_g, \tag{12.1.3}$$

condition which played an essential role in deriving the gRW equation for $\underline{\mathfrak{q}}$ in section 5.3. Recall that this choice of frame was necessary to derive the correct structure of the nonlinear terms $\text{Err}[\square_2\underline{\mathfrak{q}}]$ in Theorem 5.3.6.

Remark 12.1.1. *The additional conditions* (12.1.3) *are verified by the global frame constructed in section 3.6 of [53]. These are crucial in deriving the correct structure of the nonlinear terms N_{Err} of the gRW equation for $\underline{\mathfrak{q}}$.*

12.1.1 Full Regge Wheeler equation for $\underline{\mathfrak{q}}$

Recall the definition of $\underline{\mathfrak{q}}$, see Definition 5.3.3,

$$\underline{\mathfrak{q}} \;=\; \overline{q}q^3\Big({}^{(c)}\nabla_4\, {}^{(c)}\nabla_4\underline{A} + \underline{C}_1\, {}^{(c)}\nabla_4\underline{A} + \underline{C}_2\underline{A}\Big),$$

with complex scalars

$$\begin{aligned}
\underline{C}_1 &= 2\mathrm{tr}\,\chi - 2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi} - 4i\,{}^{(a)}\mathrm{tr}\chi,\\
\underline{C}_2 &= \frac{1}{2}\mathrm{tr}\,\chi^2 - 4\,{}^{(a)}\mathrm{tr}\chi^2 + \frac{3}{2}\frac{{}^{(a)}\mathrm{tr}\chi^4}{\mathrm{tr}\,\chi^2} + i\left(-2\mathrm{tr}\,\chi\,{}^{(a)}\mathrm{tr}\chi + 4\frac{{}^{(a)}\mathrm{tr}\chi^3}{\mathrm{tr}\,\chi}\right).
\end{aligned} \tag{12.1.4}$$

The real part of $\underline{\mathfrak{q}}$, denoted $\underline{\psi} = \Re(\underline{\mathfrak{q}})$, verifies the following real equation, see Theorem 5.3.6,

$$\dot{\square}_2\underline{\psi} - V_0\underline{\psi} = \frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_T\underline{\psi} + N, \qquad V_0 = \frac{4\Delta}{(r^2+a^2)|q|^2}, \tag{12.1.5}$$

with the right hand side N given by

$$N = N_0 + N_L + N_{\mathrm{Err}} \tag{12.1.6}$$

where

- N_0 denotes the zero-th order term in ψ, i.e.

$$N_0 := \big(V - V_0\big)\underline{\psi} = O\left(\frac{a}{r^4}\right)\underline{\psi}. \tag{12.1.7}$$

- N_L denotes[2]

$$N_L = \Re\left(q\overline{q}^3\Big(\frac{8a^2\Delta}{r^2|q|^4}\nabla_{\mathbf{T}}\underline{A}_4 + \frac{8a\Delta}{r^2|q|^4}\nabla_{\mathbf{Z}}\underline{A}_4 + \underline{W}_4\underline{A}_4 + \underline{W}_3\nabla_3\underline{A} + \underline{W}_a\nabla_a\underline{A} + \underline{W}_0\underline{A}\right)$$

where $\underline{W}_4, \underline{W}_3, \underline{W}_0$ are complex functions of (r,θ) and $\underline{W}$ is the product of a complex function of (r,θ) with ${}^*\Re(\mathfrak{J})$, with the following fall-off

$$q\overline{q}^3\underline{W}_4, q\overline{q}^3\underline{W} = O\left(\frac{a^2}{r}\right), \qquad q\overline{q}^3\underline{W}_3, q\overline{q}^3\underline{W}_0 = O\left(\frac{a^2}{r^2}\right).$$

Away from the trapping, the following schematic structure will suffice

$$N_L = O(ar^{-1})\mathfrak{d}^{\leq 1}\nabla_4(r\underline{A}) + O(ar^{-2})\mathfrak{d}^{\leq 1}\underline{A}. \tag{12.1.8}$$

- $N_{\text{Err}} = \text{Err}[\square_2\underline{\mathfrak{q}}]$ is the nonlinear quadratic error term, given schematically by the expression (5.3.10) which we recall below

$$N_{\text{Err}} = \widetilde{N}_{\text{Err}} + \mathfrak{d}^{\leq 3}(\Gamma_g\cdot\Gamma_b), \qquad \widetilde{N}_{\text{Err}} = r^2\mathfrak{d}^{\leq 2}\big(\Gamma_b\cdot(A,B)\big), \tag{12.1.9}$$

with $\widetilde{N}_{\text{Err}}$ the principal term in N_{Err} with respect to decay in r.

12.1.2 Factorizations of $\underline{\mathfrak{q}}$

Lemma 12.1.2. *Assume that $\Xi \in r^{-1}\Gamma_g$. Then, we have*

$$\underline{\mathfrak{q}} \;=\; \overline{q}q^3\left({}^{(c)}\nabla_4 + 2trX - \frac{|trX|^2}{2tr\,\chi}\right)\left({}^{(c)}\nabla_4 + 2trX - \frac{3|trX|^2}{2tr\,\chi}\right)\underline{A} + r^2\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b).$$

Proof. We have

$$\begin{aligned}
&\left({}^{(c)}\nabla_4 + 2\text{tr}X - \frac{|\text{tr}X|^2}{2\text{tr}\,\chi}\right)\left({}^{(c)}\nabla_4 + 2\text{tr}X - \frac{3|\text{tr}X|^2}{2\text{tr}\,\chi}\right)\underline{A}\\
=\;& \nabla_4^2\underline{A} + \left(4\text{tr}X - \frac{2|\text{tr}X|^2}{\text{tr}\,\chi}\right){}^{(c)}\nabla_4\underline{A} + {}^{(c)}\nabla_4\left(2\text{tr}X - \frac{3|\text{tr}X|^2}{2\text{tr}\,\chi}\right)\underline{A}\\
&+\left(2\text{tr}X - \frac{|\text{tr}X|^2}{2\text{tr}\,\chi}\right)\left(2\text{tr}X - \frac{3|\text{tr}X|^2}{2\text{tr}\,\chi}\right)\underline{A}
\end{aligned}$$

[2]Here $\underline{W}_4, \underline{W}_3, \underline{W}_a, \underline{W}_0$ are complex functions of (r,θ), all of which vanish for zero angular momentum, having the following fall-off in r.

Since

$$\begin{aligned}
4\mathrm{tr}X - \frac{2|\mathrm{tr}X|^2}{tr\,\chi} &= 4\mathrm{tr}\,\chi - 4i\,{}^{(a)}\mathrm{tr}\chi - \frac{2(\mathrm{tr}\,\chi^2 + {}^{(a)}\mathrm{tr}\chi^2)}{\mathrm{tr}\,\chi} \\
&= 2\mathrm{tr}\,\chi - 2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi} - 4i\,{}^{(a)}\mathrm{tr}\chi \\
&= \underline{C}_1,
\end{aligned}$$

where $\underline{C}_1$ is defined by (12.1.4), we deduce

$$\begin{aligned}
&\left({}^{(c)}\nabla_4 + 2\mathrm{tr}X - \frac{|\mathrm{tr}X|^2}{2\mathrm{tr}\,\chi}\right)\left({}^{(c)}\nabla_4 + 2\mathrm{tr}X - \frac{3|\mathrm{tr}X|^2}{2\mathrm{tr}\,\chi}\right)\underline{A} \\
&= \nabla_4^2\underline{A} + \underline{C}_1\,{}^{(c)}\nabla_4\underline{A} + {}^{(c)}\nabla_4\left(2\mathrm{tr}X - \frac{3|\mathrm{tr}X|^2}{2\mathrm{tr}\,\chi}\right)\underline{A} \\
&\quad + \left(2\mathrm{tr}X - \frac{|\mathrm{tr}X|^2}{2\mathrm{tr}\,\chi}\right)\left(2\mathrm{tr}X - \frac{3|\mathrm{tr}X|^2}{2\mathrm{tr}\,\chi}\right)\underline{A}.
\end{aligned}$$

Next, in view of the following consequence of the null structure equations and of the fact that $\Xi \in r^{-1}\Gamma_g$,

$$ {}^{(c)}\nabla_4\mathrm{tr}X + \frac{1}{2}(\mathrm{tr}X)^2 = r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g, $$

we have

$$\begin{aligned}
{}^{(c)}\nabla_4\left(2\mathrm{tr}X - \frac{3}{2}\frac{|\mathrm{tr}X|^2}{tr\,\chi}\right) &= -(\mathrm{tr}X)^2 - \frac{3}{2}\frac{-\frac{1}{2}(\mathrm{tr}X)^2\overline{\mathrm{tr}X} + \mathrm{tr}X\overline{-\frac{1}{2}(\mathrm{tr}X)^2}}{\mathrm{tr}\,\chi} \\
&\quad + \frac{3}{2}\frac{|\mathrm{tr}X|^2\Re(-\frac{1}{2}(\mathrm{tr}X)^2)}{tr\,\chi^2} + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g \\
&= -(\mathrm{tr}X)^2 + \frac{3}{2}|\mathrm{tr}X|^2 - \frac{3}{4}\frac{|\mathrm{tr}X|^2(tr\,\chi^2 - {}^{(a)}\mathrm{tr}\chi^2)}{tr\,\chi^2} + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g \\
&= -\frac{1}{4}tr\,\chi^2 + \frac{5}{2}\,{}^{(a)}\mathrm{tr}\chi^2 + \frac{3}{4}\frac{{}^{(a)}\mathrm{tr}\chi^4}{tr\,\chi^2} + 2itr\,\chi\,{}^{(a)}\mathrm{tr}\chi + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g
\end{aligned}$$

and hence

$$\begin{aligned}
& {}^{(c)}\nabla_4\left(2\mathrm{tr}X - \frac{3}{2}\frac{|\mathrm{tr}X|^2}{\mathrm{tr}\,\chi}\right) + \left(2\mathrm{tr}X - \frac{1}{2}\frac{|\mathrm{tr}X|^2}{\mathrm{tr}\,\chi}\right)\left(2\mathrm{tr}X - \frac{3}{2}\frac{|\mathrm{tr}X|^2}{\mathrm{tr}\,\chi}\right)\\
= & -\frac{1}{4}\mathrm{tr}\,\chi^2 + \frac{5}{2}{}^{(a)}\mathrm{tr}\chi^2 + \frac{3}{4}\frac{{}^{(a)}\mathrm{tr}\chi^4}{\mathrm{tr}\,\chi^2} + 2i\mathrm{tr}\,\chi\,{}^{(a)}\mathrm{tr}\chi + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g\\
& + \left(\frac{3}{2}\mathrm{tr}\,\chi - \frac{1}{2}\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi} - 2i\,{}^{(a)}\mathrm{tr}\chi\right)\left(\frac{1}{2}\mathrm{tr}\,\chi - \frac{3}{2}\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi} - 2i\,{}^{(a)}\mathrm{tr}\chi\right)\\
= & \frac{1}{2}\mathrm{tr}\,\chi^2 - 4\,{}^{(a)}\mathrm{tr}\chi^2 + \frac{3}{2}\frac{{}^{(a)}\mathrm{tr}\chi^4}{\mathrm{tr}\,\chi^2} + i\left(-2\mathrm{tr}\,\chi\,{}^{(a)}\mathrm{tr}\chi + 4\frac{{}^{(a)}\mathrm{tr}\chi^3}{\mathrm{tr}\,\chi}\right) + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g\\
= & \underline{C}_2 + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g
\end{aligned}$$

where $\underline{C}_2$ is defined by (12.1.4). We infer

$$\begin{aligned}
& \left({}^{(c)}\nabla_4 + 2\mathrm{tr}X - \frac{|\mathrm{tr}X|^2}{2\mathrm{tr}\,\chi}\right)\left({}^{(c)}\nabla_4 + 2\mathrm{tr}X - \frac{3|\mathrm{tr}X|^2}{2\mathrm{tr}\,\chi}\right)\underline{A}\\
= & \nabla_4^2\underline{A} + \underline{C}_1\,{}^{(c)}\nabla_4\underline{A} + \underline{C}_2\underline{A} + r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b).
\end{aligned}$$

In view of the definition of $\underline{\mathfrak{q}}$, we deduce

$$\underline{\mathfrak{q}} = \overline{q}q^3\left({}^{(c)}\nabla_4 + 2\mathrm{tr}X - \frac{|\mathrm{tr}X|^2}{2\mathrm{tr}\,\chi}\right)\left({}^{(c)}\nabla_4 + 2\mathrm{tr}X - \frac{3|\mathrm{tr}X|^2}{2\mathrm{tr}\,\chi}\right)\underline{A} + r^2\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b)$$

as stated. This concludes the proof of Lemma 12.1.2. □

Next, we introduce the tensor $\underline{\Psi}$.

Definition 12.1.3. *Let* $\underline{\Psi}\in\mathfrak{s}_2(\mathbb{C})$ *given by*

$$\underline{\Psi} := \frac{q^4}{r^2}\left({}^{(c)}\nabla_4 + 2trX - \frac{3|trX|^2}{2tr\,\chi}\right)\underline{A}.$$

We have the following corollary of Lemma 12.1.2.

Corollary 12.1.4. *Let* $\underline{\Psi}$ *as in Definition 12.1.3. Then,* $\underline{\Psi}\in\mathfrak{d}^{\leq 1}\Gamma_b$*, and* $(\underline{\Psi},\underline{A})$ *satisfies the following system of transport equations*

$${}^{(c)}\nabla_4(r\underline{\Psi}) = \frac{q}{r\overline{q}}\underline{\mathfrak{q}} + r\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b), \qquad {}^{(c)}\nabla_4\left(\frac{q^4}{r^3}\underline{A}\right) = \frac{1}{r}\underline{\Psi} + r\Gamma_g\cdot\Gamma_b.$$

Proof. We have

$$\begin{aligned}
\underline{\Psi} &= \frac{q^4}{r^2}\left({}^{(c)}\nabla_4 + 2\mathrm{tr}X - \frac{3|\mathrm{tr}X|^2}{2\mathrm{tr}\,\chi}\right)\underline{A}\\
&= \frac{q^4}{r^2}\left({}^{(c)}\nabla_4 + \frac{1}{2}\mathrm{tr}X + \frac{3(\mathrm{tr}\,\chi\mathrm{tr}X - |\mathrm{tr}X|^2)}{2\mathrm{tr}\,\chi}\right)\underline{A}\\
&= O(r^2)\left({}^{(c)}\nabla_4 + \frac{1}{2}\mathrm{tr}X\right)\underline{A} + O(r^2)\,{}^{(a)}\mathrm{tr}\chi\underline{A}\\
&= O(r^2)\left({}^{(c)}\nabla_4 + \frac{1}{2}\mathrm{tr}X\right)\underline{A} + \Gamma_b.
\end{aligned}$$

Together with the Bianchi identity for $({}^{(c)}\nabla_4\underline{A}$, we infer

$$\begin{aligned}
\underline{\Psi} &= O(r^2)\left(-\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\underline{B} - 2\,\underline{H}\widehat{\otimes}\underline{B} - 3P\underline{\widehat{X}}\right)\underline{A} + \Gamma_b\\
&= \mathfrak{d}^{\leq 1}\Gamma_b
\end{aligned}$$

as stated.

Next, we compute

$$\begin{aligned}
{}^{(c)}\nabla_4(r\underline{\Psi}) &= {}^{(c)}\nabla_4\left(\frac{q^4}{r}\left({}^{(c)}\nabla_4 + 2\mathrm{tr}X - \frac{3|\mathrm{tr}X|^2}{2\mathrm{tr}\,\chi}\right)\underline{A}\right)\\
&= \frac{q^4}{r}\left({}^{(c)}\nabla_4 + \frac{4e_4(q)}{q} - \frac{e_4(r)}{r}\right)\left({}^{(c)}\nabla_4 + 2\mathrm{tr}X - \frac{3|\mathrm{tr}X|^2}{2\mathrm{tr}\,\chi}\right)\underline{A}.
\end{aligned}$$

Since

$$\frac{e_4(q)}{q} = \frac{1}{2}\mathrm{tr}X + \Gamma_g, \qquad \frac{e_4(r)}{r} = \frac{1}{2}\frac{|\mathrm{tr}X|^2}{\mathrm{tr}\,\chi} + \Gamma_g,$$

we have

$$\begin{aligned}
{}^{(c)}\nabla_4(r\underline{\Psi}) &= \frac{q^4}{r}\left({}^{(c)}\nabla_4 + 2\mathrm{tr}X - \frac{|\mathrm{tr}X|^2}{2\mathrm{tr}\,\chi} + \Gamma_g\right)\left({}^{(c)}\nabla_4 + 2\mathrm{tr}X - \frac{3|\mathrm{tr}X|^2}{2\mathrm{tr}\,\chi}\right)\underline{A}\\
&= \frac{q^4}{r}\left({}^{(c)}\nabla_4 + 2\mathrm{tr}X - \frac{|\mathrm{tr}X|^2}{2\mathrm{tr}\,\chi}\right)\left({}^{(c)}\nabla_4 + 2\mathrm{tr}X - \frac{3|\mathrm{tr}X|^2}{2\mathrm{tr}\,\chi}\right)\underline{A} + r\Gamma_g\underline{\Psi}\\
&= \frac{q^4}{r}\left({}^{(c)}\nabla_4 + 2\mathrm{tr}X - \frac{|\mathrm{tr}X|^2}{2\mathrm{tr}\,\chi}\right)\left({}^{(c)}\nabla_4 + 2\mathrm{tr}X - \frac{3|\mathrm{tr}X|^2}{2\mathrm{tr}\,\chi}\right)\underline{A}\\
&\quad + r\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b)
\end{aligned}$$

where we used the fact that $\underline{\Psi} \in \mathfrak{d}^{\leq 1}\Gamma_b$. Now, recall from Lemma 12.1.2 that we have

$$\underline{\mathfrak{q}} = \overline{q}q^3\left({}^{(c)}\nabla_4 + 2\mathrm{tr}X - \frac{|\mathrm{tr}X|^2}{2\mathrm{tr}\,\chi}\right)\left({}^{(c)}\nabla_4 + 2\mathrm{tr}X - \frac{3|\mathrm{tr}X|^2}{2\mathrm{tr}\,\chi}\right)\underline{A} + r^2\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b).$$

We deduce

$$^{(c)}\nabla_4(r\underline{\Psi}) = \frac{q}{r\overline{q}}\underline{\mathfrak{q}} + r\mathfrak{d}^{\leq 1}(\Gamma_g \cdot \Gamma_b)$$

as stated.

Finally, in view of the definition of $\underline{\Psi}$, we have

$$\begin{aligned}
^{(c)}\nabla_4\left(\frac{q^4}{r^3}\underline{A}\right) &= \frac{q^4}{r^3}\left({}^{(c)}\nabla_4\underline{A} + \left(\frac{4e_4(q)}{q} - \frac{3e_3(r)}{r}\right)\underline{A}\right)\\
&= \frac{q^4}{r^3}\left({}^{(c)}\nabla_4 + 2\mathrm{tr}X - \frac{3|\mathrm{tr}X|^2}{2\mathrm{tr}\,\chi} + \Gamma_g\right)\underline{A}\\
&= \frac{1}{r}\underline{\Psi} + r\Gamma_g \cdot \Gamma_b
\end{aligned}$$

as stated. This concludes the proof of Corollary 12.1.4. □

12.2 Control of the full gRW equation for $\underline{\mathfrak{q}}$

12.2.1 Norms for $\underline{A}$

Definition 12.2.1. *We introduce the following norms for $\underline{A}$ in $\mathcal{M} = \mathcal{M}(\tau_1, \tau_2)$*

$$\begin{aligned}
B_p[\underline{A}](\tau_1, \tau_2) &= \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}\Big(r^4|\nabla_4\nabla_4(r\underline{A})|^2 + r^4|\nabla\nabla_4(r\underline{A})|^2 + r^2|\nabla_3\nabla_4(r\underline{A})|^2\\
&\quad + r^2|\nabla_4(r\underline{A})|^2 + r^2|\nabla\underline{A}|^2 + |\nabla_3\underline{A}|^2 + |\underline{A}|^2\Big)\\
&= \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}\Big(r^2|\mathfrak{d}^{\leq 1}\nabla_4(r\underline{A})|^2 + |\mathfrak{d}^{\leq 1}\underline{A}|^2\Big),\\
E_p[\underline{A}](\tau) &= \int_{\Sigma(\tau)} r^{p-4}\Big(r^2|\nabla_{\widehat{R}}\nabla_4(r\underline{A})|^2 + r^2|\nabla_4(r\underline{A})|^2 + \chi^2_{red}|\nabla_3\nabla_4(r\underline{A})|^2\Big)\\
&\quad + \int_{\Sigma(\tau)} r^{p-4}\Big(|\nabla_3\underline{A}|^2 + |\underline{A}|^2\Big),\\
F_p[\underline{A}](\tau_1, \tau_2) &= \int_{\mathcal{A}\cup\Sigma_*(\tau_1,\tau_2)} r^{p-2}\Big(r^2|\nabla_{\widehat{R}}\nabla_4(r\underline{A})|^2 + r^2|\nabla_4(r\underline{A})|^2 + |\nabla_3\underline{A}|^2 + |\underline{A}|^2\Big)\\
&\quad + \int_{\mathcal{A}(\tau_1,\tau_2)} |\nabla_3\nabla_4(r\underline{A})|^2,
\end{aligned}$$

where $\chi_{red} = \chi_{red}(r)$ *is a smooth function such that* $\chi_{red} = 1$ *for* $r \le r_+(1+2\delta_{red})$ *and* $\chi_{red} = 0$ *for* $r \ge r_+(1+2\delta_{red})$*, with the constant* $\delta_{red} > 0$ *small enough such that there holds* $\mathcal{M}_{trap} \subset \{r \ge r_+(1+2\delta_{red}\}$.

Remark 12.2.2. *Note that the derivatives* $\nabla_3^2\underline{A}$*,* $\nabla\nabla_3\underline{A}$ *and* $\nabla^2\underline{A}$ *are missing in* $BEF_p[\underline{A}](\tau_1,\tau_2)$ *but are fortunately not needed to close the estimates for* $\underline{\mathfrak{q}}$*. Additional derivatives in* $E_p[\underline{A}](\tau)$ *and* $F_p[\underline{A}](\tau_1,\tau_2)$ *are missing as well and are also not needed to close the estimates for* $\underline{\mathfrak{q}}$*, with the exception of the ones recovered in* (12.2.6).

The higher derivative norms are defined by the usual procedure

$$B_p^s[\underline{A}] = B_p[\mathfrak{d}^{\le s}\underline{A}], \qquad E_p^s[\underline{A}] = E_p^s[\mathfrak{d}^{\le s}\underline{A}], \qquad F_p^s[\underline{A}] = F_p^s[\mathfrak{d}^{\le s}\underline{A}].$$

For $\underline{\psi}$*, we use the norms for solutions of RW type equations introduced in section 6.1.5. We also define the combined* $(\underline{A},\underline{\psi})$ *norms as follows*

$$\begin{aligned} E_p^s[\underline{\psi},\underline{A}](\tau) &= E_p^s[\underline{\psi}](\tau) + E_p^s[\underline{A}](\tau), \\ B_p^s[\underline{\psi},\underline{A}](\tau_1,\tau_2) &= B_p^s[\underline{\psi}](\tau_1,\tau_2) + B_p^s[\underline{A}](\tau_1,\tau_2), \\ F_p^s[\underline{\psi},\underline{A}](\tau_1,\tau_2) &= F_p^s[\underline{\psi}](\tau_1,\tau_2) + F_p^s[\underline{A}](\tau_1,\tau_2). \end{aligned}$$

We use the short hand notation

$$\begin{aligned} BEF_p^s[\underline{A}](\tau_1,\tau_2) &= B_p^s[\underline{A}](\tau_1,\tau_2) + \sup_{\tau\in[\tau_1,\tau_2]} E_p^s[\underline{A}](\tau) + F_p^s[\underline{A}](\tau_1,\tau_2), \\ BEF_p^s[\underline{\psi}](\tau_1,\tau_2) &= B_p^s[\underline{\psi}](\tau_1,\tau_2) + \sup_{\tau\in[\tau_1,\tau_2]} E_p^s[\underline{\psi}](\tau) + F_p^s[\underline{\psi}](\tau_1,\tau_2), \\ BEF_p^s[\underline{\psi},\underline{A}] &= BEF_p^s[\underline{\psi}] + BEF_p^s[\underline{A}]. \end{aligned}$$

12.2.2 Statement of the main result of Chapter 12

First, note that Theorem 6.2.1 applies also to the gRW model problem (12.1.5) for $\underline{\psi}$.

Theorem 12.2.3 (Basic r^p-weighted estimates for $\underline{\psi}$)**.** *The following estimates hold true for solutions* $\underline{\psi} \in \mathfrak{s}_2$ *of the model gRW equation* (12.1.5)*, for all* $s \le k_L$*,* $\delta \le p \le 2-\delta$*.*

$$BEF_p^s[\underline{\psi}](\tau_1,\tau_2) \lesssim E_p^s[\underline{\psi}](\tau_1) + \mathcal{N}_p^s[\underline{\psi},N](\tau_1,\tau_2). \tag{12.2.1}$$

Proof. The proof of the estimate (12.2.1) is identical to the analogous for ψ in Theorem 6.2.1. Indeed, the only difference between the two RW model equations is the change of sign in front of the term $\frac{4a\cos\theta}{|q|^2}\ {}^*\nabla_T$ which is never used in the proof of Theorem 6.2.1. □

The main result of this chapter is to extend (12.2.1) to the full gRW system as follows.

Theorem 12.2.4. *The following holds true for $s \leq k_L$, for all $\delta \leq p \leq 2-\delta$,*

$$BEF^s_p[\underline{\psi}, \underline{A}](\tau_1, \tau_2) \lesssim E^s_p[\underline{\psi}, \underline{A}](\tau_1) + \mathcal{N}^s_p[\underline{\psi}, \widetilde{N}_{Err}](\tau_1, \tau_2) + \epsilon_0^2 \tau_1^{-2-2\delta_{dec}}, \tag{12.2.2}$$

where $\widetilde{N}_{Err}$ is defined in (12.1.9).

12.2.3 Proof of Theorem 12.2.4

As in the case of the analogous result for ψ in Chapter 11, the proof is done in steps as follows.

Step 1. Recall that $N = N_0 + N_L + N_{\text{Err}}$, see (12.1.6). Also, recall that $N_{\text{Err}} = \widetilde{N}_{\text{Err}} + \mathfrak{d}^{\leq 3}(\Gamma_b \cdot \Gamma_g)$, see (12.1.9). We first eliminate $N - \widetilde{N}_{\text{Err}}$ from the right hand side of (12.2.1).

Proposition 12.2.5. *The following estimate for solutions $\underline{\psi}$ of the full gRW equation hold true for all $s \leq k_L$ and all $\delta \leq p \leq 2-\delta$.*

$$\begin{aligned} BEF^s_p[\underline{\psi}](\tau_1, \tau_2) \quad &\lesssim \quad E^s_p[\underline{\psi}](\tau_1) + O(a) BEF^s_p[\underline{\psi}, \underline{A}](\tau_1, \tau_2) \\ &\quad + \mathcal{N}^s_p[\underline{\psi}, \widetilde{N}_{Err}](\tau_1, \tau_2) + \epsilon_0^2 \tau_1^{-2-2\delta_{dec}}. \end{aligned} \tag{12.2.3}$$

The proof of Proposition 12.2.5 is an immediate consequence of (12.2.1) and the following lemma.

Lemma 12.2.6. *For $\delta \leq p \leq 2-\delta$, N given by* (12.1.6) *satisfies*

$$\begin{aligned} \mathcal{N}^s_p[\underline{\psi}, N](\tau_1, \tau_2) \quad &\lesssim \quad |a| BEF^s_p[\underline{\psi}, \underline{A}](\tau_1, \tau_2) + \mathcal{N}^s_p[\underline{\psi}, \widetilde{N}_{Err}](\tau_1, \tau_2) \\ &\quad + \epsilon_0 \tau_1^{-1-\delta_{dec}} \Big(BEF_p[\underline{\psi}](\tau_1, \tau_2) \Big)^{\frac{1}{2}} + \epsilon_0^2 \tau_1^{-2-2\delta_{dec}}, \end{aligned} \tag{12.2.4}$$

where $\widetilde{N}_{Err}$ is defined in (12.1.9).

The proof of Lemma 12.2.6 is given in section 12.2.4.

Step 2. We control the term $BEF^s_p[\underline{A}]$ with the help of the proposition below.

Proposition 12.2.7. *The following estimates hold true, for $s \leq k_L$, for all $\delta \leq p \leq 2-\delta$,*

$$BEF^s_p[\underline{A}](\tau_1,\tau_2) \lesssim B^s_\delta[\underline{\psi}](\tau_1,\tau_2) + E^s_p[\underline{A}](\tau_1) + \epsilon_0^2\tau_1^{-2-2\delta_{dec}}. \tag{12.2.5}$$

Also, we have the following additional control on $\Sigma(\tau)$ with $\tau\in[\tau_1,\tau_2]$, for $s\leq k_L$, for all $\delta\leq p\leq 1-\delta$,

$$\begin{aligned}&\int_{\Sigma(\tau)} r^{p-2}\Big(r^2|\nabla_4\nabla_4(r\mathfrak{d}^{\leq s}\underline{A})|^2 + r^2|\nabla\nabla_4(r\mathfrak{d}^{\leq s}\underline{A})|^2 + |\nabla_3\nabla_4(r\mathfrak{d}^{\leq s}\underline{A})|^2 + |\nabla\mathfrak{d}^{\leq s}\underline{A}|^2\Big)\\ \lesssim\ & EB^s_p[\underline{\psi}](\tau_1,\tau_2) + E^s_p[\underline{A}](\tau_1) + \epsilon_0^2\tau_1^{-2-2\delta_{dec}}.\end{aligned} \tag{12.2.6}$$

The proof of Proposition 12.2.7 is given in section 12.3.

Step 3. As a consequence of Proposition 12.2.7 and Proposition 12.2.5, as well as the smallness of $|a|/m$, we deduce, for all $s\leq k_L$ and and all $\delta\leq p\leq 2-\delta$,

$$BEF^s_p[\underline{\psi},\underline{A}](\tau_1,\tau_2) \lesssim E^s_p[\underline{\psi},\underline{A}](\tau_1) + \mathcal{N}^s_p[\underline{\psi},\widetilde{N}_{\text{Err}}](\tau_1,\tau_2) + \epsilon_0^2\tau_1^{-2-2\delta_{dec}}$$

as stated. This ends the proof of Theorem 12.2.4.

12.2.4 Proof of Lemma 12.2.6

Since $N = N_0 + N_L + N_{\text{Err}}$, it suffices to prove, for $\delta\leq p\leq 2-\delta$, the following estimates

$$\begin{aligned}\mathcal{N}^s_p[\underline{\psi}, N_0+N_L](\tau_1,\tau_2) &\lesssim |a| BEF^s_p[\underline{\psi},\underline{A}](\tau_1,\tau_2) + \epsilon_0\tau_1^{-2-2\delta_{dec}},\\ \mathcal{N}^s_p[\underline{\psi}, N_{\text{Err}}-\widetilde{N}_{\text{Err}}](\tau_1,\tau_2) &\lesssim \epsilon_0\tau_1^{-1-\delta_{dec}}\Big(BEF^s_p[\underline{\psi}](\tau_1,\tau_2)\Big)^{\frac{1}{2}}.\end{aligned} \tag{12.2.7}$$

We start with the control of $N_{\text{Err}}-\widetilde{N}_{\text{Err}}$. Recalling the definition of the norms $\mathcal{N}_p[\underline{\psi},N]$, see section 6.1.5 we have, for $\delta \leq p \leq 2-\delta$,

$$
\begin{aligned}
& \mathcal{N}_p[\underline{\psi}, N](\tau_1,\tau_2)\\
= & \int_{\mathcal{M}(\tau_1,\tau_2)}\big(|\nabla_{\widehat{R}}\underline{\psi}|+r^{-1}|\underline{\psi}|\big)|N| + \left|\int_{{}^{(ext)}\mathcal{M}(\tau_1,\tau_2)} r^{p-1}\nabla_4(r\underline{\psi})\cdot N\right| + \left|\int_{\mathcal{M}(\tau_1,\tau_2)}\nabla_{\widehat{T}_\delta}\underline{\psi}\cdot N\right|\\
\lesssim & \int_{\mathcal{M}_{trap}}(|\mathfrak{d}\underline{\psi}|+r^{-1}|\underline{\psi}|)|N| + \int_{\mathcal{M}_{\not{trap}}}\Big((|\nabla_3\underline{\psi}|+r^{-1}|\mathfrak{d}^{\leq 1}\underline{\psi}|)|N| + r^{p-1}|\nabla_4(r\underline{\psi})||N|\Big)\\
\lesssim & \left(\sup_{[\tau_1,\tau_2]}E[\underline{\psi}](\tau)\right)^{\frac{1}{2}}\int_{\tau_1}^{\tau_2}\|N\|_{L^2(\Sigma_{trap}(\tau))}\\
+ & \left(\int_{\mathcal{M}_{\not{trap}}(\tau_1,\tau_2)}\Big(r^{-1-\delta}|\nabla_3\underline{\psi}|^2 + r^{p-3}|\mathfrak{d}^{\leq 1}\underline{\psi}|^2\Big)\right)^{\frac{1}{2}}\left(\int_{\mathcal{M}_{\not{trap}}(\tau_1,\tau_2)} r^{p+1}|N|^2\right)^{\frac{1}{2}}\\
\lesssim & \Big(BEF_p[\underline{\psi}](\tau_1,\tau_2)\Big)^{\frac{1}{2}}\left(\int_{\tau_1}^{\tau_2}\|N\|_{L^2(\Sigma_{trap}(\tau))} + \left(\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+1}|N|^2\right)^{\frac{1}{2}}\right).
\end{aligned}
$$

Since $N_{\text{Err}} - \widetilde{N}_{\text{Err}} = \mathfrak{d}^{\leq 3}(\Gamma_b\cdot\Gamma_g)$ we have, according to our bootstrap assumptions, both

$$
N_{\text{Err}} - \widetilde{N}_{\text{Err}} = \epsilon^2 r^{-3}\tau^{-3/2-2\delta_{dec}}, \qquad N_{\text{Err}} - \widetilde{N}_{\text{Err}} = \epsilon^2 r^{-2}\tau^{-2-2\delta_{dec}}.
$$

Thus, for $\delta \leq p \leq 2-\delta$,

$$
\begin{aligned}
& \int_{\tau_1}^{\tau_2}\left\|N_{\text{Err}} - \widetilde{N}_{\text{Err}}\right\|_{L^2(\Sigma_{trap}(\tau))} + \left(\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+1}\left|N_{\text{Err}} - \widetilde{N}_{\text{Err}}\right|^2\right)^{\frac{1}{2}}\\
\lesssim & \ \epsilon^2\int_{\tau_1}^{+\infty}\frac{d\tau}{\tau^{2+2\delta_{dec}}} + \epsilon^2\left(\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+1-6}\tau^{-3-4\delta_{dec}}\right)^{\frac{1}{2}}\\
\lesssim & \ \epsilon_0\tau_1^{-1-2\delta_{dec}}.
\end{aligned}
$$

We deduce, for $\delta \leq p \leq 2-\delta$,

$$
\mathcal{N}_p[\underline{\psi}, N_{\text{Err}} - \widetilde{N}_{\text{Err}}](\tau_1,\tau_2) \ \lesssim \ \epsilon_0\tau_1^{-1-2\delta_{dec}}\Big(BEF_p[\underline{\psi}](\tau_1,\tau_2)\Big)^{\frac{1}{2}}
$$

which proves the estimate for $N_{\text{Err}} - \widetilde{N}_{\text{Err}}$ in (12.2.7) in the case $s = 0$.

For higher derivatives, $s \leq k_L$, we write schematically

$$
\mathfrak{d}^{\leq s}(N_{\text{Err}} - \widetilde{N}_{\text{Err}}) \ = \ \mathfrak{d}^{\leq 3+s}(\Gamma_g\cdot\Gamma_b) = \mathfrak{d}^{\leq 3+s}\Gamma_g\cdot\mathfrak{d}^{\leq\frac{k_L}{2}}\Gamma_b + \mathfrak{d}^{\leq\frac{k_L}{2}}\Gamma_g\cdot\mathfrak{d}^{\leq 3+s}\Gamma_b
$$

and we use an additional $\tau_1^{\delta_0}$ by a standard interpolation argument, see Lemma 5.1 in [50]. Hence, for $\delta\leq p\leq 2-\delta$,

$$\mathcal{N}^s_p[\underline{\psi}, N_{\text{Err}}-\widetilde{N}_{\text{Err}}](\tau_1,\tau_2) \lesssim \epsilon_0\tau_1^{-1-\delta_{dec}}\Big(BEF^s_p[\underline{\psi}](\tau_1,\tau_2)\Big)^{\frac{1}{2}}$$

which concludes the proof of the estimate for $N_{\text{Err}}-\widetilde{N}_{\text{Err}}$ in (12.2.7).

It remains to prove the estimate in (12.2.7) for the linear terms N_0+N_L. We decompose in two parts

$$\mathcal{N}^s_p[\psi, N_0+N_L](\tau_1,\tau_2) = \mathcal{N}^s_{p,r\leq 4m}[\psi, N_0+N_L](\tau_1,\tau_2)+\mathcal{N}^s_{p,r\geq 4m}[\psi, N_0+N_L](\tau_1,\tau_2).$$

The control of the term $\mathcal{N}^s_{p,r\leq 4m}[\psi, N_0+N_L](\tau_1,\tau_2)$ can be done exactly as in section 11.3. In particular, for the treatment of ${}^{(En)}\mathcal{N}^s_{r\leq 4m}[\psi, N_L](\tau_1,\tau_2)$, we proceed exactly as in section 11.3.3, with ${}^{(c)}\nabla_3$ replaced by ${}^{(c)}\nabla_4$, where the boundary terms in the integrations by parts require to control all first order derivatives of $\underline{A}$ and $\nabla_4(r\underline{A})$ on $\Sigma(\tau)\cap\{r\leq 4m\}$ which follows immediately from the control of $E_p[\underline{A}](\tau)$ and the additional control on $\Sigma(\tau)$ provided by (12.2.6). We infer

$$\mathcal{N}^s_{p,r\leq 4m}[\psi, N_0+N_L](\tau_1,\tau_2) \lesssim |a|BEF^s_\delta[\underline{\psi}](\tau_1,\tau_2)+\epsilon_0\tau_1^{-2-2\delta_{dec}}$$

and hence

$$\mathcal{N}^s_p[\psi, N_0+N_L](\tau_1,\tau_2)\lesssim \mathcal{N}^s_{p,r\geq 4m}[\psi, N_0+N_L](\tau_1,\tau_2)+|a|BEF^s_\delta[\underline{\psi}](\tau_1,\tau_2)+\epsilon_0\tau_1^{-2-2\delta_{dec}}.$$

It thus remains to control $\mathcal{N}^s_{p,r\geq 4m}[\psi, N_0+N_L](\tau_1,\tau_2)$. We have, for $\delta\leq p\leq 2-\delta$,

$$\begin{aligned}
&\mathcal{N}^s_{p,r\geq 4m}[\psi, N_0+N_L](\tau_1,\tau_2)\\
\lesssim\ & \int_{\mathcal{M}_{r\geq 4m}(\tau_1,\tau_2)}\Big(|\nabla_{\widehat{R}}\mathfrak{d}^{\leq s}\underline{\psi}|+r^{-1}|\mathfrak{d}^{\leq s}\underline{\psi}|+|\nabla_{\widehat{T}_\delta}\mathfrak{d}^{\leq s}\underline{\psi}|+r^{p-1}|\nabla_4(r\mathfrak{d}^{\leq s}\underline{\psi})|\Big)|\mathfrak{d}^{\leq s}(N_0+N_L)|\\
\lesssim\ & \int_{\mathcal{M}_{r\geq 4m}(\tau_1,\tau_2)}\Big(|\nabla_3\mathfrak{d}^{\leq s}\underline{\psi}|+r^{p-1}|\mathfrak{d}^{\leq s+1}\underline{\psi}|\Big)|\mathfrak{d}^{\leq s}(N_0+N_L)|\\
\lesssim\ & \left(\int_{\mathcal{M}_{r\geq 4m}(\tau_1,\tau_2)}\Big(r^{-1-\delta}|\nabla_3\mathfrak{d}^{\leq s}\underline{\psi}|^2+r^{p-3}|\mathfrak{d}^{\leq s+1}\underline{\psi}|^2\Big)\right)^{\frac{1}{2}}\\
&\times\left(\int_{\mathcal{M}_{r\geq 4m}(\tau_1,\tau_2)}\Big(r^{1+\delta}+r^{1+p}\Big)\Big(|\mathfrak{d}^{\leq s}N_0|^2+|\mathfrak{d}^{\leq s}N_L|^2\Big)\right)^{\frac{1}{2}}\\
\lesssim\ & \Big(BEF^s_p[\underline{\psi}](\tau_1,\tau_2)\Big)^{\frac{1}{2}}\left(\int_{\mathcal{M}_{r\geq 4m}(\tau_1,\tau_2)}r^{1+p}\Big(|\mathfrak{d}^{\leq s}N_0|^2+|\mathfrak{d}^{\leq s}N_L|^2\Big)\right)^{\frac{1}{2}}
\end{aligned}$$

and hence

$$\begin{aligned}
&\mathcal{N}_p^s[\psi, N_0+N_L](\tau_1,\tau_2)\\
\lesssim\;& \Big(BEF_p^s[\underline{\psi}](\tau_1,\tau_2)\Big)^{\frac12}\left(\int_{\mathcal{M}_{r\geq 4m}(\tau_1,\tau_2)} r^{1+p}\Big(|\mathfrak{d}^{\leq s}N_0|^2+|\mathfrak{d}^{\leq s}N_L|^2\Big)\right)^{\frac12}\\
&+|a|BEF_\delta^s[\underline{\psi}](\tau_1,\tau_2)+\epsilon_0\tau_1^{-2-2\delta_{dec}}.
\end{aligned}$$

Since $N_0 = O(ar^{-4})\underline{\psi}$, we have, for $\delta\leq p\leq 2-\delta$,

$$\int_{\mathcal{M}_{r\geq 4m}(\tau_1,\tau_2)} r^{1+p}|\mathfrak{d}^{\leq s}N_0|^2 \;\lesssim\; a^2 B_p[\underline{\psi}](\tau_1,\tau_2).$$

For the control of N_L, we write, see (12.1.8),

$$N_L = O(ar^{-1})\mathfrak{d}^{\leq 1}\nabla_4(r\underline{A}) + O(ar^{-2})\mathfrak{d}^{\leq 1}\underline{A},$$

and the definition of the $B_p[\underline{A}]$ norms (see Definition 12.2.1),

$$\begin{aligned}
\int_{\mathcal{M}_{r\geq 4m}(\tau_1,\tau_2)} r^{p+1}|\mathfrak{d}^{\leq s}N_L|^2 \;&\lesssim\; a^2\int_{\mathcal{M}_{r\geq 4m}(\tau_1,\tau_2)} r^{p-1}\big|\mathfrak{d}^{\leq 1}\nabla_4(r\underline{A})\big|^2 + a^4\int_{\mathcal{M}_{r\geq 4m}(\tau_1,\tau_2)} r^{p-3}|\underline{A}|^2\\
&\lesssim\; a^2 B_p^s[\underline{A}](\tau_1,\tau_2).
\end{aligned}$$

We deduce, for all $\delta\leq p\leq 2-\delta$,

$$\mathcal{N}_p^s[\underline{\psi}, N_0+N_L](\tau_1,\tau_2) \;\lesssim |a|BEF_p^s[\underline{\psi},\underline{A}](\tau_1,\tau_2)+\epsilon_0\tau_1^{-2-2\delta_{dec}}$$

as stated in (12.2.7). This concludes the proof of Lemma 12.2.6.

12.3 Transport Estimates for $\underline{A}$

In this section we prove Proposition 12.2.7. i.e. we prove, for $\delta\leq p\leq 2-\delta$, $s\leq k_L$, the following estimate

$$BEF_p^s[\underline{A}](\tau_1,\tau_2) \;\lesssim\; BEF_\delta^s[\underline{\psi}](\tau_1,\tau_2)+E_p^s[\underline{A}](\tau_1)+\epsilon_0^2(1+\tau)^{-2-2\delta_{dec}}.$$

To this end, we proceed as follows:

1. First, we derive a basic lemma for transport equation in ∇_4 in section 12.3.1.

2. Next, we derive estimates for $\underline{A}$, $\nabla_4\underline{A}$ and $\nabla_3\underline{A}$ in section 12.3.2.

3. It remains to control angular derivatives. To this end, we fist derive some algebraic identities involving angular derivatives of $\underline{A}$ in section 12.3.3.

4. Next, we derive estimates for $\nabla\underline{A}$ in section 12.3.4 and for $\nabla\nabla_4(r\underline{A})$ in section 12.3.5.

5. Finally, we conclude the proof of Proposition 12.2.7 in section 12.3.6.

12.3.1 Basic transport lemma in ∇_4

We prove below the ${}^{(c)}\nabla_4$-transport lemma counterpart of Lemma 11.4.5 and 11.4.7.

Lemma 12.3.1. *Suppose $\Phi_1, \Phi_2 \in \mathfrak{s}_2(\mathbb{C})$ with signature $s \leq -1$ satisfy the differential relation*

$$ {}^{(c)}\nabla_4\Phi_1 = \Phi_2. \tag{12.3.1}$$

Also, let $\chi_{red} = \chi_{red}(r)$ is a smooth function such that $\chi_{red} = 1$ for $r \leq r_+(1+2\delta_{red})$ and $\chi_{red} = 0$ for $r \geq r_+(1+2\delta_{red})$, with the constant $\delta_{red} > 0$ small enough such that there holds $\mathcal{M}_{trap} \subset \{r \geq r_+(1+2\delta_{red}\}$. Then, for every $p \leq -\delta$:

1. *The pointwise inequality holds true*

$$ r|q|^{p-4}|\Phi_1|^2 \quad \lesssim \quad \frac{4}{p^2}r^{-1}|q|^p|\Phi_2|^2 - \frac{2}{p}Div(|q|^{p-2}|\Phi_1|^2 e_4) \tag{12.3.2}$$

and its integral form

$$\begin{aligned}\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}|\Phi_1|^2 + \int_{\Sigma(\tau_2)} r^{p-4}|\Phi_1|^2 + \int_{\mathcal{A}\cup\Sigma_*(\tau_1,\tau_2)} r^{p-2}|\Phi_1|^2 \\ \lesssim \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}|\Phi_2|^2 + \int_{\Sigma(\tau_1)} r^{p-4}|\Phi_1|^2.\end{aligned} \tag{12.3.3}$$

2. We also have

$$
\begin{aligned}
&\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}\big(r^2|\nabla_4\Phi_1|^2+|\nabla_3\Phi_1|^2+|\Phi_1|^2\big)\\
&+\int_{\Sigma(\tau_2)} r^{p-4}\big(|\nabla_{\widehat{R}}\Phi_1|^2+|\Phi_1|^2+\chi_{red}^2|\nabla_3\Phi_1|^2\big)\\
&+\int_{\mathcal{A}\cup\Sigma_*(\tau_1,\tau_2)} r^{p-2}\big(|\nabla_{\widehat{R}}\Phi_1|^2+|\Phi_1|^2\big)+\int_{\mathcal{A}(\tau_1,\tau_2)}|\nabla_3\Phi_1|^2 \qquad (12.3.4)\\
&\lesssim \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}\big(|\nabla_{\widehat{R}}\Phi_2|^2+|\Phi_2|^2\big)+\int_{\mathcal{M}_{r\leq r_+(1+2\delta_{red})}(\tau_1,\tau_2)}|\nabla_3\Phi_2|^2\\
&+\int_{\Sigma(\tau_1)} r^{p-4}\big(|\nabla_{\widehat{R}}\Phi_1|^2+|\Phi_1|^2+\chi_{red}^2|\nabla_3\Phi_1|^2\big)+(a^2+\epsilon^2)\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}|\nabla\Phi_1|^2.
\end{aligned}
$$

Proof. We prove first the following analogue of Lemma 11.4.4.

Lemma 12.3.2. *For any smooth scalar function f on $\mathcal{M}$, we have*

$$
Div\big(fe_4\big) \;=\; e_4(f)+\left(\frac{2r\Delta}{|q|^4}-2\omega+\Gamma_g\right)f.
$$

Proof. We have

$$
\begin{aligned}
\mathrm{Div}(e_4) &= \mathbf{g}^{43}\mathbf{g}(\mathbf{D}_4e_4,e_3)+\mathbf{g}^{43}\mathbf{g}(\mathbf{D}_3e_4,e_4)+\mathbf{g}^{bc}\mathbf{g}(\mathbf{D}_be_4,e_c)\\
&= -\frac{1}{2}4\omega+\mathrm{tr}\,\chi=-2\omega+\frac{2\Delta r}{|q|^4}+\widecheck{\mathrm{tr}\chi}\\
&= -2\omega+\frac{2\Delta r}{|q|^4}+\Gamma_g
\end{aligned}
$$

and hence

$$
\mathrm{Div}(fe_4)=f\mathrm{Div}(e_4)+e_4(f)=e_4(f)+\left(\frac{2r\Delta}{|q|^4}-2\omega+\Gamma_g\right)f
$$

as stated. This concludes the proof of Lemma 12.3.2. □

We continue the proof of (12.3.2) as follows. Multiplying the relation ${}^{(c)}\nabla_4\Phi_1=\Phi_2$ by $\overline{\Phi_1}$, we deduce, since Φ_1 has signature s,

$$
e_4(|\Phi_1|^2) \;=\; 2\Re\big((\Phi_2-2s\omega\Phi_1)\cdot\overline{\Phi_1}\big).
$$

Multiplying by $|q|^{p-2}$, and using

$$e_4(|q|) = \frac{e_4(|q|^2)}{2|q|} = \frac{r\Delta}{|q|^3} + O(1)\widetilde{e_4(r)} + O(r^{-1})e_4(\cos\theta) = \frac{r\Delta}{|q|^3} + \Gamma_g,$$

we deduce

$$\begin{aligned} 2|q|^{p-2}\Re\big(\Phi_2\cdot\overline{\Phi_1}\big) &= e_4(|q|^{p-2}|\Phi_1|^2) - e_4(|q|^{p-2})|\Phi_1|^2 + 4s\omega|q|^{p-2}|\Phi_1|^2 \\ &= e_4(|q|^{p-2}|\Phi_1|^2) - (p-2)r\Delta|q|^{p-6}|\Phi_1|^2 + 4s\omega|q|^{p-2}|\Phi_1|^2 \\ &\quad +|q|^{p-3}\Gamma_g|\Phi_1|^2. \end{aligned}$$

In view of Lemma 12.3.2, we write

$$\begin{aligned} \mathrm{Div}(|q|^{p-2}|\Phi_1|^2 e_4) &= e_4(|q|^{p-2}|\Phi_1|^2) + \left(\frac{2r\Delta}{|q|^4} - 2\omega + \Gamma_g\right)|q|^{p-2}|\Phi_1|^2 \\ &= 2|q|^{p-2}\Re\big(\Phi_2\cdot\overline{\Phi_1}\big) + (p-2)r\Delta|q|^{p-6}|\Phi_1|^2 - 4s\omega|q|^{p-2}|\Phi_1|^2 \\ &\quad + \left(\frac{2r\Delta}{|q|^4} - 2\omega\right)|q|^{p-2}|\Phi_1|^2 + |q|^{p-2}\Gamma_g|\Phi_1|^2 \\ &= 2|q|^{p-2}\Re\big(\Phi_2\cdot\overline{\Phi_1}\big) + pr\Delta|q|^{p-6}|\Phi_1|^2 - 2(2s+1)\omega|q|^{p-2}|\Phi_1|^2 \\ &\quad +|q|^{p-2}\Gamma_g|\Phi_1|^2. \end{aligned}$$

From the above identity we deduce

$$\begin{aligned} & -pr\Delta|q|^{p-6}|\Phi_1|^2 + 2(2s+1)\omega|q|^{p-2}|\Phi_1|^2 \\ =\;& 2|q|^{p-2}\Re\big(\Phi_2\cdot\overline{\Phi_1}\big) - \mathrm{Div}(|q|^{p-2}|\Phi_1|^2e_4) + |q|^{p-2}\Gamma_g|\Phi_1|^2 \\ =\;& 2\Re\Big((\lambda r)^{-1/2}|q|^{p/2}\Phi_2\cdot(\lambda r)^{1/2}|q|^{p/2-2}\overline{\Phi_1}\Big) - \mathrm{Div}(|q|^{p-2}|\Phi_1|^2e_4) + |q|^{p-2}\Gamma_g|\Phi_1|^2 \\ \leq\;& \lambda r|q|^{p-4}|\Phi_1|^2 + \lambda^{-1}r^{-1}|q|^p|\Phi_2|^2 - \mathrm{Div}(|q|^{p-2}|\Phi_1|^2e_4) + |q|^{p-2}\Gamma_g|\Phi_1|^2. \end{aligned}$$

Since $-\omega \gtrsim \frac{m}{r^2} + \Gamma_g$ and $s\leq -1$, we deduce, using also the control of Γ_g,

$$pr|q|^{p-4}|\Phi_1|^2 \leq \lambda r|q|^{p-4}|\Phi_1|^2 + \lambda^{-1}r^{-1}|q|^p|\Phi_2|^2 + O(\epsilon)|q|^{p-4}|\Phi_1|^2 - \mathrm{Div}(|q|^{p-2}|\Phi_1|^2e_4).$$

Therefore, for $p\leq -\delta$, choosing $\lambda = \frac{p}{2}$, we deduce, for ϵ sufficiently small,

$$r|q|^{p-4}|\Phi_1|^2 \lesssim \frac{4}{p^2}r^{-1}|q|^p|\Phi_2|^2 - \frac{2}{p}\mathrm{Div}(|q|^{p-2}|\Phi_1|^2e_4)$$

which is precisely (12.3.2).

The integral form (12.3.3) of the inequality then follows by the divergence theorem, see (11.4.4), and Remark[3] 11.4.3.

Next, we focus on deriving (12.3.4). To this end, we start with the following commutation lemma.

[3]Note in particular that $\mathbf{g}(N_\Sigma, e_4) = -\frac{m^2}{r^2}$, which is responsible for the energy integral on $\Sigma(\tau)$.

Lemma 12.3.3. *Let $U \in \mathfrak{s}_k$. Then, we have*

$$\left[\nabla_4, \frac{r^2+a^2}{|q|^2}\nabla_{\widehat{R}}\right]U = O((a,\epsilon)r^{-1})\nabla U + O(r^{-1}\epsilon)\nabla_4 U + O(r^{-2}\epsilon)\nabla_3 U + O(r^{-3})U.$$

Proof. Recall that $\widehat{R}$ is given by

$$\widehat{R} = \frac{1}{2}\left(\frac{|q|^2}{r^2+a^2}e_4 - \frac{\Delta}{r^2+a^2}e_3\right)$$

so that

$$\frac{r^2+a^2}{|q|^2}\widehat{R} = \frac{1}{2}\left(e_4 - \frac{\Delta}{|q|^2}e_3\right).$$

We infer

$$\begin{aligned}\left[\nabla_4, \frac{r^2+a^2}{|q|^2}\nabla_{\widehat{R}}\right] &= -\frac{1}{2}e_4\left(\frac{\Delta}{|q|^2}\right)\nabla_3 - \frac{1}{2}\frac{\Delta}{|q|^2}[\nabla_4, \nabla_3]\\ &= -\frac{1}{2}\left(\partial_r\left(\frac{\Delta}{|q|^2}\right)e_4(r) + O(r^{-2})e_4(\cos\theta)\right)\nabla_3 - \frac{1}{2}\frac{\Delta}{|q|^2}[\nabla_4, \nabla_3]\\ &= -\frac{1}{2}\left(\partial_r\left(\frac{\Delta}{|q|^2}\right)\frac{\Delta}{|q|^2} + r^{-2}\Gamma_g\right)\nabla_3 - \frac{1}{2}\frac{\Delta}{|q|^2}[\nabla_4, \nabla_3].\end{aligned}$$

Also, note that the commutation formula for $[\nabla_4, \nabla_3]$ of Corollary A.1.1 implies

$$\begin{aligned}[\nabla_4, \nabla_3]U &= \big(O(ar^{-2}) + \Gamma_b\big)\nabla U + 2\omega\nabla_3 U + \Gamma_b\nabla_4 U + O(r^{-3})U\\ &= \big(O(ar^{-2}) + \Gamma_b\big)\nabla U + 2\left(-\frac{1}{2}\partial_r\left(\frac{\Delta}{|q|^2}\right) + \check{\omega}\right)\nabla_3 U + \Gamma_b\nabla_4 U + O(r^{-3})U\\ &= \big(O(ar^{-2}) + \Gamma_b\big)\nabla U + \left(-\partial_r\left(\frac{\Delta}{|q|^2}\right) + \Gamma_g\right)\nabla_3 U + \Gamma_b\nabla_4 U + O(r^{-3})U,\end{aligned}$$

where we used the definition of $\check{\omega}$ and the fact that $\check{\omega} \in \Gamma_g$. We deduce

$$\left[\nabla_4, \frac{r^2+a^2}{|q|^2}\nabla_{\widehat{R}}\right]U = \big(O(ar^{-2}) + \Gamma_b\big)\nabla U + \Gamma_g\nabla_3 U + \Gamma_b\nabla_4 U + O(r^{-3})U.$$

Together with the control of Γ_g and Γ_b, we deduce

$$\left[\nabla_4, \frac{r^2+a^2}{|q|^2}\nabla_{\widehat{R}}\right]U = O((a,\epsilon)r^{-1})\nabla U + O(r^{-1}\epsilon)\nabla_4 U + O(r^{-2}\epsilon)\nabla_3 U + O(r^{-3})U$$

as stated. This concludes the proof of Lemma 12.3.3. □

We commute the transport equation for Φ_1 with $\frac{r^2+a^2}{|q|^2}\nabla_{\widehat{R}}$. Together with Lemma 12.3.3, we infer

$$\begin{aligned}
{}^{(c)}\nabla_4\left(\frac{r^2+a^2}{|q|^2}\nabla_{\widehat{R}}\Phi_1\right) &= \left[\nabla_4+2s\omega, \frac{r^2+a^2}{|q|^2}\nabla_{\widehat{R}}\right]\Phi_1+\frac{r^2+a^2}{|q|^2}\nabla_{\widehat{R}}\Phi_2\\
&= O((a,\epsilon)r^{-1})\nabla\Phi_1+O(r^{-1}\epsilon)\nabla_4\Phi_1+O(r^{-2}\epsilon)\nabla_3\Phi_1+O(r^{-3})\Phi_1\\
&\quad -2s\frac{r^2+a^2}{|q|^2}(\nabla_{\widehat{R}}\omega)\Phi_1+O(1)\nabla_{\widehat{R}}\Phi_2
\end{aligned}$$

and hence, since $\nabla_{\widehat{R}}\omega = O(r^{-3})+\mathfrak{d}\Gamma_g = O(r^{-2})$, we obtain, using also ${}^{(c)}\nabla_4\Phi_1=\Phi_2$,

$$\begin{aligned}
{}^{(c)}\nabla_4\left(\frac{r^2+a^2}{|q|^2}\nabla_{\widehat{R}}\Phi_1\right) &= O((a,\epsilon)r^{-1})\nabla\Phi_1+O(r^{-2}\epsilon)\nabla_3\Phi_1+O(r^{-2})\Phi_1\\
&\quad +O(\epsilon r^{-1})\Phi_2+O(1)\nabla_{\widehat{R}}\Phi_2.
\end{aligned}$$

Applying (12.3.3) to this transport equation, we infer

$$\begin{aligned}
&\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}|\nabla_{\widehat{R}}\Phi_1|^2+\int_{\Sigma(\tau_2)} r^{p-4}|\nabla_{\widehat{R}}\Phi_1|^2+\int_{\mathcal{A}\cup\Sigma_*(\tau_1,\tau_2)} r^{p-2}|\nabla_{\widehat{R}}\Phi_1|^2\\
&\lesssim \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}\Big((a^2+\epsilon^2)r^{-2}|\nabla\Phi_1|^2+r^{-4}\epsilon^2|\nabla_3\Phi_1|^2+r^{-4}|\Phi_1|^2+r^{-2}|\Phi_2|^2+|\nabla_{\widehat{R}}\Phi_2|^2\Big)\\
&+\int_{\Sigma(\tau_1)} r^{p-4}|\nabla_{\widehat{R}}\Phi_1|^2.
\end{aligned}$$

Together with (12.3.3) and the fact that ${}^{(c)}\nabla_4\Phi_1=\Phi_2$, this yields

$$\begin{aligned}
&\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}\big(r^2|\nabla_4\Phi_1|^2+|\nabla_{\widehat{R}}\Phi_1|^2+|\Phi_1|^2\big)+\int_{\Sigma(\tau_2)} r^{p-4}\big(|\nabla_{\widehat{R}}\Phi_1|^2+|\Phi_1|^2\big)\\
&+\int_{\mathcal{A}\cup\Sigma_*(\tau_1,\tau_2)} r^{p-2}\big(|\nabla_{\widehat{R}}\Phi_1|^2+|\Phi_1|^2\big)\\
&\lesssim \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}\big(|\nabla_{\widehat{R}}\Phi_2|^2+|\Phi_2|^2\big)+\int_{\Sigma(\tau_1)} r^{p-4}\big(|\nabla_{\widehat{R}}\Phi_1|^2+|\Phi_1|^2\big)\\
&+\epsilon^2\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-5}|\nabla_3\Phi_1|^2+(a^2+\epsilon^2)\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}|\nabla\Phi_1|^2.
\end{aligned}$$

It remains to recover $\nabla_3\underline{A}$ in the redshift region. To this end, we commute the transport equation ${}^{(c)}\nabla_4\Phi_1=\Phi_2$ with $\chi_{red}\,{}^{(c)}\nabla_3$ where χ_{red} is a smooth cut-off function equal to 1 in $r\le r_+(1+\delta_{red})$ and 0 for $r\ge r_+(1+2\delta_{red})$. Note that, in view of Lemma 4.2.2,

$$[\,{}^{(c)}\nabla_3,\,{}^{(c)}\nabla_4]U = O\big((a+\epsilon)r^{-1}\big)\,{}^{(c)}\nabla U+O(r^{-3})U$$

so that

$$
\begin{aligned}
{}^{(c)}\nabla_4(\chi_{red}\,{}^{(c)}\nabla_3\Phi_1) &= \chi_{red}\nabla_3\Phi_2 + \partial_r\chi_{red}e_4(r)\,{}^{(c)}\nabla_3\Phi_1 + O\big((a+\epsilon)r^{-1}\big)\,{}^{(c)}\nabla\Phi_1 \\
&\quad + O(r^{-3})\Phi_1.
\end{aligned}
$$

Since $s-1<s\leq -1$, applying (12.3.3) to this transport equation, and using the above control of Φ_1, we infer

$$
\begin{aligned}
&\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}\big(r^2|\nabla_4\Phi_1|^2 + |\nabla_{\widehat{R}}\Phi_1|^2 + |\Phi_1|^2\big) + \int_{\mathcal{M}(r\leq r_+(1+\delta_{red}))} |\nabla_3\Phi_1|^2 \\
&+ \int_{\Sigma(\tau_2)} r^{p-4}\big(|\nabla_{\widehat{R}}\Phi_1|^2 + |\Phi_1|^2 + \chi_{red}^2|\nabla_3\Phi_1|^2\big) \\
&+ \int_{\mathcal{A}\cup\Sigma_*(\tau_1,\tau_2)} r^{p-2}\big(|\nabla_{\widehat{R}}\Phi_1|^2 + |\Phi_1|^2\big) + \int_{\mathcal{A}(\tau_1,\tau_2)} |\nabla_3\Phi_1|^2 \\
&\lesssim \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}\big(|\nabla_{\widehat{R}}\Phi_2|^2 + |\Phi_2|^2\big) + \int_{\mathcal{M}_{r\leq r_+(1+2\delta_{red})}(\tau_1,\tau_2)} |\nabla_3\Phi_2|^2 \\
&+ \int_{\Sigma(\tau_1)} r^{p-4}\big(|\nabla_{\widehat{R}}\Phi_1|^2 + |\Phi_1|^2 + \chi_{red}^2|\nabla_3\Phi_1|^2\big) \\
&+ \epsilon^2\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-5}|\nabla_3\Phi_1|^2 + (a^2+\epsilon^2)\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}|\nabla\Phi_1|^2
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}\big(r^2|\nabla_4\Phi_1|^2 + |\nabla_3\Phi_1|^2 + |\Phi_1|^2\big) \\
&+ \int_{\Sigma(\tau_2)} r^{p-4}\big(|\nabla_{\widehat{R}}\Phi_1|^2 + |\Phi_1|^2 + \chi_{red}^2|\nabla_3\Phi_1|^2\big) \\
&+ \int_{\mathcal{A}\cup\Sigma_*(\tau_1,\tau_2)} r^{p-2}\big(|\nabla_{\widehat{R}}\Phi_1|^2 + |\Phi_1|^2\big) + \int_{\mathcal{A}(\tau_1,\tau_2)} |\nabla_3\Phi_1|^2 \\
&\lesssim \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}\big(|\nabla_{\widehat{R}}\Phi_2|^2 + |\Phi_2|^2\big) + \int_{\mathcal{M}_{r\leq r_+(1+2\delta_{red})}(\tau_1,\tau_2)} |\nabla_3\Phi_2|^2 \\
&+ \int_{\Sigma(\tau_1)} r^{p-4}\big(|\nabla_{\widehat{R}}\Phi_1|^2 + |\Phi_1|^2 + \chi_{red}^2|\nabla_3\Phi_1|^2\big) \\
&+ \epsilon^2\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-5}|\nabla_3\Phi_1|^2 + (a^2+\epsilon^2)\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}|\nabla\Phi_1|^2.
\end{aligned}
$$

For $\epsilon > 0$ small enough, we deduce

$$
\begin{aligned}
&\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}\big(r^2|\nabla_4\Phi_1|^2 + |\nabla_3\Phi_1|^2 + |\Phi_1|^2\big)\\
&+\int_{\Sigma(\tau_2)} r^{p-4}\big(|\nabla_{\widehat{R}}\Phi_1|^2 + |\Phi_1|^2 + \chi_{red}^2|\nabla_3\Phi_1|^2\big)\\
&+\int_{\mathcal{A}\cup\Sigma_*(\tau_1,\tau_2)} r^{p-2}\big(|\nabla_{\widehat{R}}\Phi_1|^2 + |\Phi_1|^2\big) + \int_{\mathcal{A}(\tau_1,\tau_2)} |\nabla_3\Phi_1|^2\\
&\lesssim \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}\big(|\nabla_{\widehat{R}}\Phi_2|^2 + |\Phi_2|^2\big) + \int_{\mathcal{M}_{r\leq r_+(1+2\delta_{red})}(\tau_1,\tau_2)} |\nabla_3\Phi_2|^2\\
&+\int_{\Sigma(\tau_1)} r^{p-4}\big(|\nabla_{\widehat{R}}\Phi_1|^2 + |\Phi_1|^2 + \chi_{red}^2|\nabla_3\Phi_1|^2\big) + (a^2+\epsilon^2)\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}|\nabla\Phi_1|^2
\end{aligned}
$$

as stated in (12.3.4). This concludes the proof of Lemma 12.3.1. □

12.3.2 Estimates for $\underline{A}$, $\nabla_4\underline{A}$ and $\nabla_3\underline{A}$

Recall from Corollary 12.1.4 that $(\underline{\Psi}, \underline{A})$ satisfies the following system of transport equations

$$
{}^{(c)}\nabla_4(r\underline{\Psi}) = \frac{q}{r\overline{q}}\underline{\mathfrak{q}} + r\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b), \qquad {}^{(c)}\nabla_4\left(\frac{q^4}{r^3}\underline{A}\right) = \frac{1}{r}\underline{\Psi} + r\Gamma_g\cdot\Gamma_b, \tag{12.3.5}
$$

where $\underline{\Psi}$ is given in view of Definition 12.1.3 by

$$
\underline{\Psi} \;=\; \frac{q^4}{r^2}\left({}^{(c)}\nabla_4 + 2\mathrm{tr}X - \frac{3|\mathrm{tr}X|^2}{2\mathrm{tr}\,\chi}\right)\underline{A}.
$$

To state the next proposition, we introduce the following partial norms for $\underline{A}$ which do not provide control for angular derivatives.

Definition 12.3.4. *We define, for all p,*

1. *In $\mathcal{M}(\tau_1,\tau_2)$*

$$
\begin{aligned}
\dot{B}_p[\underline{A}](\tau_1,\tau_2) \;=\;& \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}\Big(r^2|\nabla_4\nabla_4(r\underline{A})|^2 + |\nabla_3\nabla_4(r\underline{A})|^2 + |\nabla_4(r\underline{A})|^2\Big)\\
& \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}\Big(r^2|\nabla_4\underline{A}|^2 + |\nabla_3\underline{A}|^2 + |\underline{A}|^2\Big).
\end{aligned}
$$

2. *On* $\Sigma(\tau)$, $\dot{E}_p[\underline{A}](\tau) = E_p[\underline{A}](\tau)$,

3. *On* $\mathcal{A}\cup\Sigma_*(\tau_1,\tau_2)$, $\dot{F}_p[\underline{A}](\tau_1,\tau_2) = F_p[\underline{A}](\tau_1,\tau_2)$.

Proposition 12.3.5. *The following estimates hold true for* $p \leq 2-\delta$

$$
\begin{aligned}
B\dot{E}F_p[\underline{A}](\tau_1,\tau_2) \;\lesssim\;& B_\delta[\underline{\psi}](\tau_1,\tau_2) + \dot{E}_p[\underline{A}](\tau_1)\\
&+(a^2+\epsilon^2)\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}\Big(r^2|\nabla\nabla_4(r\underline{A})|^2 + |\nabla\underline{A}|^2\Big) + \epsilon_0^2\tau^{-2-2\delta_{dec}}.
\end{aligned}
$$

Proof. For $p\leq -\delta$ we apply (12.3.4) of Lemma 12.3.1 with $\Phi_1 = r\underline{\Psi}$ and $\Phi_2 = r^{-1}\frac{q}{\overline{q}}\underline{\mathfrak{q}} + r\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b)$ and we infer

$$
\begin{aligned}
&\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}\big(r^4|\nabla_4\underline{\Psi}|^2 + r^2|\nabla_3\underline{\Psi}|^2 + r^2|\underline{\Psi}|^2\big)\\
&+\int_{\Sigma(\tau_2)} r^{p-4}\big(r^2|\nabla_{\widehat{R}}\underline{\Psi}|^2 + r^2|\underline{\Psi}|^2 + \chi_{red}^2|\nabla_3\underline{\Psi}|^2\big)\\
&+\int_{\mathcal{A}\cup\Sigma_*(\tau_1,\tau_2)} r^{p-2}\big(r^2|\nabla_{\widehat{R}}\underline{\Psi}|^2 + r^2|\underline{\Psi}|^2\big) + \int_{\mathcal{A}(\tau_1,\tau_2)}|\nabla_3\underline{\Psi}|^2\\
&\lesssim \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}\big(r^{-2}|\nabla_{\widehat{R}}\underline{\mathfrak{q}}|^2 + r^{-2}|\underline{\mathfrak{q}}|^2 + r^2|\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\Gamma_b)|^2\big)\\
&+\int_{\mathcal{M}_{r\leq r_+(1+2\delta_{red})}(\tau_1,\tau_2)}\Big(|\nabla_3\underline{\mathfrak{q}}|^2 + |\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\Gamma_b)|^2\Big)\\
&+\int_{\Sigma(\tau_1)} r^{p-4}\big(r^2|\nabla_{\widehat{R}}\underline{\Psi}|^2 + r^2|\underline{\Psi}|^2 + \chi_{red}^2|\nabla_3\underline{\Psi}|^2\big) + (a^2+\epsilon^2)\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}r^2|\nabla\underline{\Psi}|^2.
\end{aligned}
$$

i.e., since $\underline{\psi} = \Re(\underline{\mathfrak{q}})$, and in view of the control of Γ_g and Γ_b, using also $p\leq -\delta$,

$$
\begin{aligned}
&\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}\big(r^2|\nabla_4\underline{\Psi}|^2 + |\nabla_3\underline{\Psi}|^2 + |\underline{\Psi}|^2\big)\\
&+\int_{\Sigma(\tau_2)} r^{p-2}\big(|\nabla_{\widehat{R}}\underline{\Psi}|^2 + |\underline{\Psi}|^2 + \chi_{red}^2|\nabla_3\underline{\Psi}|^2\big)\\
&+\int_{\mathcal{A}\cup\Sigma_*(\tau_1,\tau_2)} r^{p}\big(|\nabla_{\widehat{R}}\underline{\Psi}|^2 + |\underline{\Psi}|^2\big) + \int_{\mathcal{A}(\tau_1,\tau_2)}|\nabla_3\underline{\Psi}|^2\\
&\lesssim B_\delta[\underline{\psi}](\tau_1,\tau_2) + \int_{\Sigma(\tau_1)} r^{p-2}\big(|\nabla_{\widehat{R}}\underline{\Psi}|^2 + |\underline{\Psi}|^2 + \chi_{red}^2|\nabla_3\underline{\Psi}|^2\big)\\
&+(a^2+\epsilon^2)\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}|\nabla\underline{\Psi}|^2 + \epsilon_0^2\tau^{-2-2\delta_{dec}}.
\end{aligned}
\tag{12.3.6}
$$

For $p \leq -\delta$, we apply estimate (12.3.4) with $\Phi_1 = \frac{q^4}{r^3}\underline{A}$ and $\Phi_2 = r^{-1}\underline{\Psi} + r\Gamma_g \cdot \Gamma_b$ and deduce

$$
\begin{aligned}
&\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}\big(r^4|\nabla_4\underline{A}|^2 + r^2|\nabla_3\underline{A}|^2 + r^2|\underline{A}|^2\big)\\
&+\int_{\Sigma(\tau_2)} r^{p-4}\big(r^2|\nabla_{\widehat{R}}\underline{A}|^2 + r^2|\underline{A}|^2 + \chi_{red}^2|\nabla_3\underline{A}|^2\big)\\
&+\int_{\mathcal{A}\cup\Sigma_*(\tau_1,\tau_2)} r^{p-2}\big(r^2|\nabla_{\widehat{R}}\underline{A}|^2 + r^2|\underline{A}|^2\big) + \int_{\mathcal{A}(\tau_1,\tau_2)} |\nabla_3\underline{A}|^2\\
\lesssim\ &\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}\big(r^{-2}|\nabla_{\widehat{R}}\underline{\Psi}|^2 + r^{-2}|\underline{\Psi}|^2 + r^2|\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\Gamma_b)|^2\big)\\
&+\int_{\mathcal{M}_{r\leq r_+(1+2\delta_{red})}(\tau_1,\tau_2)} \Big(|\nabla_3\underline{\Psi}|^2 + |\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\Gamma_b)|^2\Big)\\
&+\int_{\Sigma(\tau_1)} r^{p-4}\big(r^2|\nabla_{\widehat{R}}\underline{A}|^2 + r^2|\underline{A}|^2 + \chi_{red}^2|\nabla_3\underline{A}|^2\big) + (a^2+\epsilon^2)\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}r^2|\nabla\underline{A}|^2
\end{aligned}
$$

i.e., in view of the control of Γ_g and Γ_b, using also $p \leq -\delta$,

$$
\begin{aligned}
&\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}\big(r^2|\nabla_4\underline{A}|^2 + |\nabla_3\underline{A}|^2 + |\underline{A}|^2\big)\\
&+\int_{\Sigma(\tau_2)} r^{p-2}\big(|\nabla_{\widehat{R}}\underline{A}|^2 + |\underline{A}|^2 + \chi_{red}^2|\nabla_3\underline{A}|^2\big)\\
&+\int_{\mathcal{A}\cup\Sigma_*(\tau_1,\tau_2)} r^{p}\big(|\nabla_{\widehat{R}}\underline{A}|^2 + |\underline{A}|^2\big) + \int_{\mathcal{A}(\tau_1,\tau_2)} |\nabla_3\underline{A}|^2\\
\lesssim\ &\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}\big(|\nabla_{\widehat{R}}\underline{\Psi}|^2 + |\underline{\Psi}|^2\big) + \int_{\mathcal{M}_{r\leq r_+(1+2\delta_{red})}(\tau_1,\tau_2)} |\nabla_3\underline{\Psi}|^2\\
&+\int_{\Sigma(\tau_1)} r^{p-2}\big(|\nabla_{\widehat{R}}\underline{A}|^2 + |\underline{A}|^2 + \chi_{red}^2|\nabla_3\underline{A}|^2\big) + (a^2+\epsilon^2)\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}|\nabla\underline{A}|^2 + \epsilon_0^2\tau^{-2-2\delta_{dec}}.
\end{aligned}
$$

Combining with (12.3.6) we deduce, for all $p \leq -\delta$,

$$\begin{aligned}
&\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}\Big(r^2|\nabla_4\underline{\Psi}|^2 + |\nabla_3\underline{\Psi}|^2 + |\underline{\Psi}|^2 + r^2|\nabla_4\underline{A}|^2 + |\nabla_3\underline{A}|^2 + |\underline{A}|^2\Big)\\
&+\int_{\Sigma(\tau_2)} r^{p-2}\Big(|\nabla_{\widehat{R}}\underline{\Psi}|^2 + |\underline{\Psi}|^2 + \chi_{red}^2|\nabla_3\underline{\Psi}|^2 + |\nabla_{\widehat{R}}\underline{A}|^2 + |\underline{A}|^2 + \chi_{red}^2|\nabla_3\underline{A}|^2\Big)\\
&+\int_{\mathcal{A}\cup\Sigma_*(\tau_1,\tau_2)} r^p\Big(|\nabla_{\widehat{R}}\underline{\Psi}|^2 + |\underline{\Psi}|^2 + |\nabla_{\widehat{R}}\underline{A}|^2 + |\underline{A}|^2\Big) + \int_{\mathcal{A}(\tau_1,\tau_2)}\Big(|\nabla_3\underline{\Psi}|^2 + |\nabla_3\underline{A}|^2\Big)\\
\lesssim\;& B_\delta[\underline{\psi}] + \int_{\Sigma(\tau_1)} r^{p-2}\Big(|\nabla_{\widehat{R}}\underline{\Psi}|^2 + |\underline{\Psi}|^2 + \chi_{red}^2|\nabla_3\underline{\Psi}|^2 + |\nabla_{\widehat{R}}\underline{A}|^2 + |\underline{A}|^2 + \chi_{red}^2|\nabla_3\underline{A}|^2\Big)\\
&+(a^2+\epsilon^2)\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}\Big(|\nabla\underline{\Psi}|^2 + |\nabla\underline{A}|^2\Big) + \epsilon_0^2\tau^{-2-2\delta_{dec}}.
\end{aligned}$$

Replacing p with $p-2$ we rewrite, for all $p \leq 2-\delta$,

$$\begin{aligned}
&\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}\Big(r^2|\nabla_4\underline{\Psi}|^2 + |\nabla_3\underline{\Psi}|^2 + |\underline{\Psi}|^2 + r^2|\nabla_4\underline{A}|^2 + |\nabla_3\underline{A}|^2 + |\underline{A}|^2\Big)\\
&+\int_{\Sigma(\tau_2)} r^{p-4}\Big(|\nabla_{\widehat{R}}\underline{\Psi}|^2 + |\underline{\Psi}|^2 + \chi_{red}^2|\nabla_3\underline{\Psi}|^2 + |\nabla_{\widehat{R}}\underline{A}|^2 + |\underline{A}|^2 + \chi_{red}^2|\nabla_3\underline{A}|^2\Big)\\
&+\int_{\mathcal{A}\cup\Sigma_*(\tau_1,\tau_2)} r^{p-2}\Big(|\nabla_{\widehat{R}}\underline{\Psi}|^2 + |\underline{\Psi}|^2 + |\nabla_{\widehat{R}}\underline{A}|^2 + |\underline{A}|^2\Big) + \int_{\mathcal{A}(\tau_1,\tau_2)}\Big(|\nabla_3\underline{\Psi}|^2 + |\nabla_3\underline{A}|^2\Big)\\
\lesssim\;& B_\delta[\underline{\psi}] + \int_{\Sigma(\tau_1)} r^{p-4}\Big(|\nabla_{\widehat{R}}\underline{\Psi}|^2 + |\underline{\Psi}|^2 + \chi_{red}^2|\nabla_3\underline{\Psi}|^2 + |\nabla_{\widehat{R}}\underline{A}|^2 + |\underline{A}|^2 + \chi_{red}^2|\nabla_3\underline{A}|^2\Big)\\
&+(a^2+\epsilon^2)\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}\Big(|\nabla\underline{\Psi}|^2 + |\nabla\underline{A}|^2\Big) + \epsilon_0^2\tau^{-2-2\delta_{dec}}.
\end{aligned}$$

We now note that

$$\begin{aligned}
\underline{\Psi} &= \frac{q^4}{r^2}\left({}^{(c)}\nabla_4 + 2\mathrm{tr}X - \frac{3|\mathrm{tr}X|^2}{2\mathrm{tr}\,\chi}\right)\underline{A}\\
&= r^2(1+O(r^{-1}))\left(\nabla_4 - 4\omega + \frac{4}{q} - \frac{3}{r} + \Gamma_g\right)\underline{A}\\
&= r^2(1+O(r^{-1}))\left(\nabla_4 + \frac{1}{r} + O(r^{-2}) + \Gamma_g\right)\underline{A}\\
&= r(1+O(r^{-1}))\left(\nabla_4(r\underline{A}) + \left(-\widetilde{e_4(r)} + O(r^{-1}) + r\Gamma_g\right)\underline{A}\right)\\
&= r(1+O(r^{-1}))\big(\nabla_4(r\underline{A}) + \big(O(r^{-1}) + r\Gamma_g\big)\underline{A}\big)\\
&= (1+O(r^{-1}))\big(r\nabla_4(r\underline{A}) + O(1)\underline{A}\big),
\end{aligned}$$

where we have used the control of Γ_g. We can thus replace $\underline{\Psi}$ with $r\nabla_4(r\underline{A})$ in formula above. Together with the fact that

$$r^2|\nabla_4\underline{A}|^2 + |\nabla_3\underline{A}|^2 + |\underline{A}|^2 \lesssim |\nabla_4(r\underline{A})|^2 + |\nabla_{\widehat{R}}\underline{A}|^2 + |\underline{A}|^2 + \chi_{red}^2|\nabla_3\underline{A}|^2,$$

we deduce, for $p \leq 2-\delta$,

$$\begin{aligned} \dot{BEF}_p[\underline{A}](\tau_1,\tau_2) &\lesssim B_\delta[\underline{\psi}](\tau_1,\tau_2) + \dot{E}_p[\underline{A}](\tau_1) \\ &\quad +(a^2+\epsilon^2)\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}\Big(r^2|\nabla\nabla_4(r\underline{A})|^2 + |\nabla\underline{A}|^2\Big) + \epsilon_0^2\tau^{-2-2\delta_{dec}} \end{aligned}$$

which ends the proof of Proposition 12.3.5. □

12.3.3 Identities for angular derivatives of $\underline{A}$

In addition to the estimates of Proposition 12.3.5, we need to control angular derivatives of $\underline{A}$. To this end, we derive in this section several identities. We start with the following identity for ${}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A})$.

Lemma 12.3.6. *${}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A})$ satisfies the following identity*

$$\begin{aligned} \frac{1}{4}{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}) &= {}^{(c)}\nabla_3\left({}^{(c)}\nabla_4\underline{A} + \frac{1}{2}trX\underline{A}\right) + \left(\frac{1}{2}tr\underline{X} + 2\overline{tr\underline{X}}\right)\left({}^{(c)}\nabla_4\underline{A} + \frac{1}{2}trX\underline{A}\right) \\ &\quad + (H+\overline{H})\cdot\nabla\underline{A} - \frac{1}{2}({}^{(c)}\mathcal{D}\cdot\overline{H})\underline{A} + O(ar^{-2})\nabla\underline{A} + O(r^{-3})\underline{A} \\ &\quad + r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g). \end{aligned} \tag{12.3.7}$$

Proof. Recall that $\underline{A}$ verifies the Teukolsky equation

$$\begin{aligned} &-{}^{(c)}\nabla_3{}^{(c)}\nabla_4\underline{A} + \frac{1}{4}{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}) - \frac{1}{2}\mathrm{tr}X\,{}^{(c)}\nabla_3\underline{A} - \frac{1}{2}\Big(\mathrm{tr}\underline{X} + 4\overline{\mathrm{tr}\underline{X}}\Big){}^{(c)}\nabla_4\underline{A} \\ &+\Big(4\,\underline{H} + H + \overline{H}\Big)\cdot\nabla\underline{A} + \Big(-\mathrm{tr}X\overline{\mathrm{tr}\underline{X}} + 2P\Big)\underline{A} + 2\,\underline{H}\widehat{\otimes}\Big(\overline{H}\cdot\underline{A}\Big) = r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_b) \end{aligned}$$

and hence

$$\begin{aligned} \frac{1}{4}{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}) &= {}^{(c)}\nabla_3{}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\,{}^{(c)}\nabla_3\underline{A} + \frac{1}{2}\Big(\mathrm{tr}\underline{X} + 4\overline{\mathrm{tr}\underline{X}}\Big){}^{(c)}\nabla_4\underline{A} \\ &\quad - \Big(4\,\underline{H} + H + \overline{H}\Big)\cdot\nabla\underline{A} - \Big(-\mathrm{tr}X\overline{\mathrm{tr}\underline{X}} + 2P\Big)\underline{A} - 2\,\underline{H}\widehat{\otimes}\Big(\overline{H}\cdot\underline{A}\Big) \\ &\quad + r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_b) \\ &= {}^{(c)}\nabla_3{}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\,{}^{(c)}\nabla_3\underline{A} + \frac{1}{2}\Big(\mathrm{tr}\underline{X} + 4\overline{\mathrm{tr}\underline{X}}\Big){}^{(c)}\nabla_4\underline{A} + \overline{\mathrm{tr}\underline{X}}\mathrm{tr}X\underline{A} \\ &\quad + (H+\overline{H})\cdot\nabla\underline{A} + O(ar^{-2})\nabla\underline{A} + O(r^{-3})\underline{A} + r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g). \end{aligned}$$

Next, using the null structure equation for ${}^{(c)}\nabla_3\mathrm{tr}X$, i.e.

$$\begin{aligned}
{}^{(c)}\nabla_3\mathrm{tr}X + \frac{1}{2}\mathrm{tr}\underline{X}\mathrm{tr}X &= {}^{(c)}\mathcal{D}\cdot\overline{H} + H\cdot\overline{H} + 2P + \underline{\Xi}\cdot\overline{\Xi} - \frac{1}{2}\widehat{\underline{X}}\cdot\overline{\widehat{X}}\\
&= {}^{(c)}\mathcal{D}\cdot\overline{H} + O(r^{-3}) + r^{-1}\Gamma_g + \Gamma_g\cdot\Gamma_b,
\end{aligned}$$

we compute

$$\begin{aligned}
&{}^{(c)}\nabla_3\left({}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A}\right) + \left(\frac{1}{2}\mathrm{tr}\underline{X} + 2\overline{\mathrm{tr}\underline{X}}\right)\left({}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A}\right)\\
= \;& {}^{(c)}\nabla_3\,{}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\,{}^{(c)}\nabla_3\underline{A} + \frac{1}{2}\,{}^{(c)}\nabla_3(\mathrm{tr}X)\underline{A} + \left(\frac{1}{2}\mathrm{tr}\underline{X} + 2\overline{\mathrm{tr}\underline{X}}\right){}^{(c)}\nabla_4\underline{A}\\
&+ \left(\frac{1}{2}\mathrm{tr}\underline{X} + 2\overline{\mathrm{tr}\underline{X}}\right)\frac{1}{2}\mathrm{tr}X\underline{A}\\
= \;& {}^{(c)}\nabla_3\,{}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\,{}^{(c)}\nabla_3\underline{A} - \frac{1}{4}\mathrm{tr}\underline{X}\mathrm{tr}X\underline{A} + \left(\frac{1}{2}\mathrm{tr}\underline{X} + 2\overline{\mathrm{tr}\underline{X}}\right){}^{(c)}\nabla_4\underline{A}\\
&+ \left(\frac{1}{2}\mathrm{tr}\underline{X} + 2\overline{\mathrm{tr}\underline{X}}\right)\frac{1}{2}\mathrm{tr}X\underline{A} + \frac{1}{2}({}^{(c)}\mathcal{D}\cdot\overline{H})\underline{A} + O(r^{-3})\underline{A} + r^{-1}\Gamma_b\cdot\Gamma_g\\
= \;& {}^{(c)}\nabla_3\,{}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\,{}^{(c)}\nabla_3\underline{A} + \left(\frac{1}{2}\mathrm{tr}\underline{X} + 2\overline{\mathrm{tr}\underline{X}}\right){}^{(c)}\nabla_4\underline{A} + \overline{\mathrm{tr}\underline{X}}\mathrm{tr}X\underline{A}\\
&+ \frac{1}{2}({}^{(c)}\mathcal{D}\cdot\overline{H})\underline{A} + O(r^{-3})\underline{A} + r^{-1}\Gamma_b\cdot\Gamma_g.
\end{aligned}$$

We deduce

$$\begin{aligned}
\frac{1}{4}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}) = \;& {}^{(c)}\nabla_3\left({}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A}\right) + \left(\frac{1}{2}\mathrm{tr}\underline{X} + 2\overline{\mathrm{tr}\underline{X}}\right)\left({}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A}\right)\\
&+ (H+\overline{H})\cdot\nabla\underline{A} - \frac{1}{2}({}^{(c)}\mathcal{D}\cdot\overline{H})\underline{A} + O(ar^{-2})\nabla\underline{A} + O(r^{-3})\underline{A}\\
&+ r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g)
\end{aligned}$$

and hence

$$\begin{aligned}
\frac{1}{4}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}) = \;& {}^{(c)}\nabla_3\left({}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A}\right) + \left(\frac{1}{2}\mathrm{tr}\underline{X} + 2\overline{\mathrm{tr}\underline{X}}\right)\left({}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A}\right)\\
&+ (H+\overline{H})\cdot\nabla\underline{A} - \frac{1}{2}({}^{(c)}\mathcal{D}\cdot\overline{H})\underline{A} + O(ar^{-2})\nabla\underline{A} + O(r^{-3})\underline{A}\\
&+ r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g)
\end{aligned}$$

as stated. This concludes the proof of Lemma 12.3.6. □

We will need to differentiate the identity of Lemma 12.3.6. To this end, we first derive the following identity.

Lemma 12.3.7. *We have*

$$\begin{aligned}
& \overline{q}q^3\left({}^{(c)}\nabla_4+\frac{3}{2}trX-2\frac{{}^{(a)}tr\chi^2}{tr\,\chi}-2i\,{}^{(a)}tr\chi\right)\left({}^{(c)}\nabla_4\underline{A}+\frac{1}{2}trX\underline{A}\right)\\
= & \ \underline{\mathfrak{q}}+O(a^2)\underline{A}+r^2\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b).
\end{aligned}$$

Proof. Let h a scalar function to be chosen below. We have

$$\begin{aligned}
& \left({}^{(c)}\nabla_4+\frac{3}{2}\mathrm{tr}X+h\right)\left({}^{(c)}\nabla_4\underline{A}+\frac{1}{2}\mathrm{tr}X\underline{A}\right)\\
= & \ {}^{(c)}\nabla_4^2\underline{A}+(2\mathrm{tr}X+h)\,{}^{(c)}\nabla_4\underline{A}+\left(\frac{3}{4}(\mathrm{tr}X)^2+\frac{1}{2}\mathrm{tr}Xh+\frac{1}{2}\,{}^{(c)}\nabla_4(\mathrm{tr}X)\right)\underline{A}\\
= & \ {}^{(c)}\nabla_4^2\underline{A}+(2\mathrm{tr}X+h)\,{}^{(c)}\nabla_4\underline{A}+\left(\frac{3}{4}(\mathrm{tr}X)^2+\frac{1}{2}\mathrm{tr}Xh+\frac{1}{2}\left(-\frac{1}{2}(\mathrm{tr}X)^2+r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g\right)\right)\underline{A}\\
= & \ {}^{(c)}\nabla_4^2\underline{A}+(2\mathrm{tr}X+h)\,{}^{(c)}\nabla_4\underline{A}+\left(\frac{1}{2}(\mathrm{tr}X)^2+\frac{1}{2}\mathrm{tr}Xh\right)\underline{A}+r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b)
\end{aligned}$$

where we used the null structure equation for ${}^{(c)}\nabla_4\mathrm{tr}X$ and the fact that $\Xi\in r^{-1}\Gamma_g$ in this chapter. Recall from (12.1.4) that $\underline{C}_1$ is given by

$$\underline{C}_1 \ = \ 2\mathrm{tr}\,\chi-2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}-4i\,{}^{(a)}\mathrm{tr}\chi.$$

We choose

$$h:=\underline{C}_1-2\mathrm{tr}X$$

so that

$$\begin{aligned}
h \ = & \ 2\mathrm{tr}\,\chi-2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}-4i\,{}^{(a)}\mathrm{tr}\chi-2\mathrm{tr}\,\chi+2i\,{}^{(a)}\mathrm{tr}\chi\\
= & \ -2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}-2i\,{}^{(a)}\mathrm{tr}\chi.
\end{aligned}$$

With this choice of h, we deduce

$$\begin{aligned}
& \left({}^{(c)}\nabla_4+\frac{3}{2}\mathrm{tr}X+h\right)\left({}^{(c)}\nabla_4\underline{A}+\frac{1}{2}\mathrm{tr}X\underline{A}\right)\\
= & \ {}^{(c)}\nabla_4^2\underline{A}+\underline{C}_1\,{}^{(c)}\nabla_4\underline{A}+\left(\frac{1}{2}(\mathrm{tr}X)^2+\frac{1}{2}\mathrm{tr}X\left(-2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}-2i\,{}^{(a)}\mathrm{tr}\chi\right)\right)\underline{A}+r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b).
\end{aligned}$$

Next, recall from (12.1.4) that $\underline{C}_2$ is given by

$$\underline{C}_2 = \frac{1}{2}\mathrm{tr}\,\chi^2 - 4\,{}^{(a)}\mathrm{tr}\chi^2 + \frac{3}{2}\frac{{}^{(a)}\mathrm{tr}\chi^4}{\mathrm{tr}\,\chi^2} + i\left(-2\mathrm{tr}\,\chi\,{}^{(a)}\mathrm{tr}\chi + 4\frac{{}^{(a)}\mathrm{tr}\chi^3}{\mathrm{tr}\,\chi}\right).$$

We compute

$$\begin{aligned}
&\underline{C}_2 - \left(\frac{1}{2}(\mathrm{tr}X)^2 + \frac{1}{2}\mathrm{tr}X\left(-2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi} - 2i\,{}^{(a)}\mathrm{tr}\chi\right)\right)\\
=\;& \frac{1}{2}\mathrm{tr}\,\chi^2 - 4\,{}^{(a)}\mathrm{tr}\chi^2 + \frac{3}{2}\frac{{}^{(a)}\mathrm{tr}\chi^4}{\mathrm{tr}\,\chi^2} + i\left(-2\mathrm{tr}\,\chi\,{}^{(a)}\mathrm{tr}\chi + 4\frac{{}^{(a)}\mathrm{tr}\chi^3}{\mathrm{tr}\,\chi}\right)\\
&- \left(\frac{1}{2}(\mathrm{tr}X)^2 + \frac{1}{2}\mathrm{tr}X\left(-2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi} - 2i\,{}^{(a)}\mathrm{tr}\chi\right)\right)\\
=\;& -\frac{3}{2}\,{}^{(a)}\mathrm{tr}\chi^2 + \frac{3}{2}\frac{{}^{(a)}\mathrm{tr}\chi^4}{\mathrm{tr}\,\chi^2} + \frac{3i\,{}^{(a)}\mathrm{tr}\chi^3}{\mathrm{tr}\,\chi}
\end{aligned}$$

where both the $\mathrm{tr}\,\chi^2$ in the real part and the $\mathrm{tr}\,\chi\,{}^{(a)}\mathrm{tr}\chi$ terms in the imaginary part cancel. We infer

$$\begin{aligned}
&\left({}^{(c)}\nabla_4 + \frac{3}{2}\mathrm{tr}X + h\right)\left({}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A}\right)\\
=\;& {}^{(c)}\nabla_4^2\underline{A} + \underline{C}_1\,{}^{(c)}\nabla_4\underline{A} + \left(\underline{C}_2 + \frac{3}{2}\,{}^{(a)}\mathrm{tr}\chi^2 - \frac{3}{2}\frac{{}^{(a)}\mathrm{tr}\chi^4}{\mathrm{tr}\,\chi^2} - \frac{3i\,{}^{(a)}\mathrm{tr}\chi^3}{\mathrm{tr}\,\chi}\right)\underline{A} + r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b).
\end{aligned}$$

Since

$$\frac{3}{2}\,{}^{(a)}\mathrm{tr}\chi^2 - \frac{3}{2}\frac{{}^{(a)}\mathrm{tr}\chi^4}{\mathrm{tr}\,\chi^2} - \frac{3i\,{}^{(a)}\mathrm{tr}\chi^3}{\mathrm{tr}\,\chi} = O(a^2r^{-4}) + r^{-2}\Gamma_g,$$

we deduce

$$\begin{aligned}
&\left({}^{(c)}\nabla_4 + \frac{3}{2}\mathrm{tr}X + h\right)\left({}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A}\right)\\
=\;& {}^{(c)}\nabla_4^2\underline{A} + \underline{C}_1\,{}^{(c)}\nabla_4\underline{A} + \underline{C}_2\underline{A} + O(a^2r^{-4})\underline{A} + r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b)
\end{aligned}$$

and hence, using the definition of $\underline{\mathfrak{q}}$,

$$\overline{q}q^3\left({}^{(c)}\nabla_4 + \frac{3}{2}\mathrm{tr}X + h\right)\left({}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A}\right) = \underline{\mathfrak{q}} + O(a^2)\underline{A} + r^2\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b).$$

In view of the definition of h, this concludes the proof of Lemma 12.3.7. □

Lemma 12.3.8. *We have*

$$
\begin{aligned}
&\frac{1}{4}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\left(\overline{{}^{(c)}\mathcal{D}}\cdot\left({}^{(c)}\nabla_4\underline{A}+\left(\frac{1}{2}trX-2\frac{{}^{(a)}tr\chi^2}{tr\,\chi}-2i\,{}^{(a)}tr\chi\right)\underline{A}\right)\right)\\
=&O(ar^{-3})\nabla\nabla_4(r\underline{A})+O(ar^{-3})\nabla\underline{A}+O(r^{-4})\nabla_3\underline{\mathfrak{q}}+O(r^{-5})\underline{\mathfrak{q}}+O(r^{-3})\nabla_4(r\underline{A})\\
&+O(r^{-4})\nabla_3\underline{A}+O(r^{-4})\underline{A}+r^{-2}\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\Gamma_g).
\end{aligned}
$$

Proof. Recall from Lemma 12.3.6 that we have

$$
\begin{aligned}
\frac{1}{4}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}) =&\, {}^{(c)}\nabla_3\left({}^{(c)}\nabla_4\underline{A}+\frac{1}{2}\mathrm{tr}X\underline{A}\right)+\left(\frac{1}{2}\mathrm{tr}\underline{X}+2\overline{\mathrm{tr}\underline{X}}\right)\left({}^{(c)}\nabla_4\underline{A}+\frac{1}{2}\mathrm{tr}X\underline{A}\right)\\
&+(H+\overline{H})\cdot\nabla\underline{A}-\frac{1}{2}({}^{(c)}\mathcal{D}\cdot\overline{H})\underline{A}+O(ar^{-2})\nabla\underline{A}+O(r^{-3})\underline{A}\\
&+r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g).
\end{aligned}
$$

We will differentiate this identity by ${}^{(c)}\nabla_4+\frac{3}{2}\mathrm{tr}X-2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}-2i\,{}^{(a)}\mathrm{tr}\chi$. To this end, first, note that

$$
\begin{aligned}
&\left({}^{(c)}\nabla_4+\frac{3}{2}\mathrm{tr}X-2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}-2i\,{}^{(a)}\mathrm{tr}\chi\right)\Bigg((H+\overline{H})\cdot\nabla\underline{A}-\frac{1}{2}({}^{(c)}\mathcal{D}\cdot\overline{H})\underline{A}\\
&+O(ar^{-2})\nabla\underline{A}+O(r^{-3})\underline{A}+r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g)\Bigg)\\
=&\left({}^{(c)}\nabla_4+\frac{3}{2}\mathrm{tr}X\right)\left((H+\overline{H})\cdot\nabla\underline{A}-\frac{1}{2}({}^{(c)}\mathcal{D}\cdot\overline{H})\underline{A}\right)\\
&+O(ar^{-2})\nabla\nabla_4\underline{A}+O(ar^{-3})\nabla\underline{A}+O(r^{-3})\nabla_4\underline{A}+O(r^{-4})\underline{A}+r^{-2}\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\Gamma_g)\\
=&\left(\left({}^{(c)}\nabla_4+\frac{1}{2}\mathrm{tr}X\right)(H+\overline{H})\right)\cdot\nabla\underline{A}+(H+\overline{H})\cdot\nabla\left(\left({}^{(c)}\nabla_4+\frac{1}{2}\mathrm{tr}X\right)\underline{A}\right)\\
&-\frac{1}{2}\left({}^{(c)}\mathcal{D}\cdot\left(\left({}^{(c)}\nabla_4+\frac{1}{2}\mathrm{tr}X\right)\overline{H}\right)\right)\underline{A}-\frac{1}{2}({}^{(c)}\mathcal{D}\cdot\overline{H})\left({}^{(c)}\nabla_4+\frac{1}{2}\mathrm{tr}X\right)\underline{A}\\
&+O(ar^{-2})\nabla\nabla_4\underline{A}+O(ar^{-3})\nabla\underline{A}+O(r^{-3})\nabla_4\underline{A}+O(r^{-4})\underline{A}+r^{-2}\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\Gamma_g)
\end{aligned}
$$

where we used the fact that, in view of Lemma 4.2.1 and the fact that $\Xi\in r^{-1}\Gamma_g$ in this chapter, we have

$$
[\nabla_4,\nabla]F \;=\; -\frac{1}{2}\mathrm{tr}X\nabla F+O(ar^{-2})\nabla_4 F+O(ar^{-3})F+r^{-1}\Gamma_g\cdot\mathfrak{d}^{\leq 1}F.
$$

In view of the following consequence of the null structure equation

$$
\begin{aligned}
{}^{(c)}\nabla_4 H + \frac{1}{2}\overline{\mathrm{tr}X} H &= {}^{(c)}\nabla_3 \Xi + \frac{1}{2}\overline{\mathrm{tr}X}\,\underline{H} - \frac{1}{2}\widehat{X}\cdot(\overline{H} - \overline{\underline{H}}) - B \\
&= O(ar^{-3}) + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_g, \\
{}^{(c)}\nabla_4 H + \frac{1}{2}\mathrm{tr}X H &= {}^{(c)}\nabla_4 H + \frac{1}{2}\overline{\mathrm{tr}X} H + O(a^2r^{-4}) + r^{-2}\Gamma_b \\
&= O(ar^{-3}) + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_g,
\end{aligned}
$$

where we used again the fact that $\Xi \in r^{-1}\Gamma_g$ in this chapter, we infer

$$
\begin{aligned}
& \left({}^{(c)}\nabla_4 + \frac{3}{2}\mathrm{tr}X - 2\frac{{}^{(a)}\mathrm{tr}\chi^2}{tr\,\chi} - 2i\,{}^{(a)}\mathrm{tr}\chi\right)\Bigg((H+\overline{H})\cdot\nabla\underline{A} - \frac{1}{2}({}^{(c)}\mathcal{D}\cdot\overline{H})\underline{A} \\
& +O(ar^{-2})\nabla\underline{A} + O(r^{-3})\underline{A} + r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g)\Bigg) \\
= \;& (H+\overline{H})\cdot\nabla\left(\left({}^{(c)}\nabla_4 + \frac{1}{2}\mathrm{tr}X\right)\underline{A}\right) - \frac{1}{2}({}^{(c)}\mathcal{D}\cdot\overline{H})\left({}^{(c)}\nabla_4 + \frac{1}{2}\mathrm{tr}X\right)\underline{A} \\
& +O(ar^{-2})\nabla\nabla_4\underline{A} + O(ar^{-3})\nabla\underline{A} + O(r^{-3})\nabla_4\underline{A} + O(r^{-4})\underline{A} + r^{-2}\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\Gamma_g).
\end{aligned}
$$

Also, since $({}^{(c)}\nabla_4 + \frac{1}{2}\mathrm{tr}X)\underline{A} \in r^{-2}\mathfrak{d}^{\leq 1}\Gamma_b$ in view of Bianchi, we obtain

$$
\begin{aligned}
& \left({}^{(c)}\nabla_4 + \frac{3}{2}\mathrm{tr}X - 2\frac{{}^{(a)}\mathrm{tr}\chi^2}{tr\,\chi} - 2i\,{}^{(a)}\mathrm{tr}\chi\right)\Bigg((H+\overline{H})\cdot\nabla\underline{A} - \frac{1}{2}({}^{(c)}\mathcal{D}\cdot\overline{H})\underline{A} \\
& +O(ar^{-2})\nabla\underline{A} + O(r^{-3})\underline{A} + r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g)\Bigg) \\
= \;& O(ar^{-2})\nabla\nabla_4\underline{A} + O(ar^{-3})\nabla\underline{A} + O(r^{-3})\nabla_4\underline{A} + O(r^{-4})\underline{A} + r^{-2}\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\Gamma_g).
\end{aligned}
$$

Recalling

$$
\begin{aligned}
\frac{1}{4}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}) = {}^{(c)}\nabla_3\left({}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A}\right) + \left(\frac{1}{2}\mathrm{tr}\underline{X} + 2\overline{\mathrm{tr}\underline{X}}\right)\left({}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A}\right) \\
+ (H+\overline{H})\cdot\nabla\underline{A} - \frac{1}{2}({}^{(c)}\mathcal{D}\cdot\overline{H})\underline{A} + O(ar^{-2})\nabla\underline{A} + O(r^{-3})\underline{A} \\
+ r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g),
\end{aligned}
$$

and differentiating this identity by ${}^{(c)}\nabla_4 + \frac{3}{2}\mathrm{tr}X - 2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi} - 2i\,{}^{(a)}\mathrm{tr}\chi$, we deduce

$$\begin{aligned}
&\frac{1}{4}\left({}^{(c)}\nabla_4 + \frac{3}{2}\mathrm{tr}X - 2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi} - 2i\,{}^{(a)}\mathrm{tr}\chi\right){}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A})\\
&= {}^{(c)}\nabla_3\left(\left({}^{(c)}\nabla_4 + \frac{3}{2}\mathrm{tr}X - 2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi} - 2i\,{}^{(a)}\mathrm{tr}\chi\right)\left({}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A}\right)\right)\\
&\quad + \left[{}^{(c)}\nabla_4 + \frac{3}{2}\mathrm{tr}X - 2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi} - 2i\,{}^{(a)}\mathrm{tr}\chi,\ {}^{(c)}\nabla_3\right]\left({}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A}\right)\\
&\quad + \left(\frac{1}{2}\mathrm{tr}\underline{X} + 2\overline{\mathrm{tr}\underline{X}}\right)\left({}^{(c)}\nabla_4 + \frac{3}{2}\mathrm{tr}X - 2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi} - 2i\,{}^{(a)}\mathrm{tr}\chi\right)\left({}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A}\right)\\
&\quad + {}^{(c)}\nabla_4\left(\frac{1}{2}\mathrm{tr}\underline{X} + 2\overline{\mathrm{tr}\underline{X}}\right)\left({}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A}\right)\\
&\quad + O(ar^{-2})\nabla\nabla_4\underline{A} + O(ar^{-3})\nabla\underline{A} + O(r^{-3})\nabla_4\underline{A} + O(r^{-4})\underline{A} + r^{-2}\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\Gamma_g).
\end{aligned}$$

Also, recalling from Lemma 12.3.7 that we have

$$\begin{aligned}
&\overline{q}q^3\left({}^{(c)}\nabla_4 + \frac{3}{2}\mathrm{tr}X - 2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi} - 2i\,{}^{(a)}\mathrm{tr}\chi\right)\left({}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A}\right)\\
=\ &\underline{\mathfrak{q}} + O(a^2)\underline{A} + r^2\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b),
\end{aligned}$$

we infer

$$\begin{aligned}
&\frac{1}{4}\left({}^{(c)}\nabla_4 + \frac{3}{2}\mathrm{tr}X - 2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi} - 2i\,{}^{(a)}\mathrm{tr}\chi\right){}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A})\\
&= {}^{(c)}\nabla_3\left(\frac{1}{\overline{q}q^3}\underline{\mathfrak{q}}\right) + \left[{}^{(c)}\nabla_4 + \frac{3}{2}\mathrm{tr}X - 2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi} - 2i\,{}^{(a)}\mathrm{tr}\chi,\ {}^{(c)}\nabla_3\right]\left({}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A}\right)\\
&\quad + \frac{1}{\overline{q}q^3}\left(\frac{1}{2}\mathrm{tr}\underline{X} + 2\overline{\mathrm{tr}\underline{X}}\right)\underline{\mathfrak{q}} + {}^{(c)}\nabla_4\left(\frac{1}{2}\mathrm{tr}\underline{X} + 2\overline{\mathrm{tr}\underline{X}}\right)\left({}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A}\right)\\
&\quad + O(ar^{-2})\nabla\nabla_4\underline{A} + O(ar^{-3})\nabla\underline{A} + O(r^{-3})\nabla_4\underline{A} + O(r^{-4})\nabla_3\underline{A} + O(r^{-4})\underline{A}\\
&\quad + r^{-2}\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\Gamma_g)
\end{aligned}$$

and hence

$$\begin{aligned}
&\frac{1}{4}\left({}^{(c)}\nabla_4 + \frac{3}{2}\mathrm{tr}X - 2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi} - 2i\,{}^{(a)}\mathrm{tr}\chi\right){}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A})\\
&= \left[{}^{(c)}\nabla_4 + \frac{3}{2}\mathrm{tr}X - 2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi} - 2i\,{}^{(a)}\mathrm{tr}\chi,\ {}^{(c)}\nabla_3\right]\left({}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A}\right)\\
&\quad + {}^{(c)}\nabla_4\left(\frac{1}{2}\mathrm{tr}\underline{X} + 2\overline{\mathrm{tr}\underline{X}}\right)\left({}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A}\right)\\
&\quad + O(ar^{-2})\nabla\nabla_4\underline{A} + O(ar^{-3})\nabla\underline{A} + O(r^{-4})\nabla_3\underline{\mathfrak{q}} + O(r^{-5})\underline{\mathfrak{q}}\\
&\quad + O(r^{-3})\nabla_4\underline{A} + O(r^{-4})\nabla_3\underline{A} + O(r^{-4})\underline{A} + r^{-2}\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\Gamma_g).
\end{aligned}$$

Since we have, in view of the null structure equations,

$$ {}^{(c)}\nabla_4 \text{tr}\underline{X} = O(r^{-2}) + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_g, $$

and since

$$ {}^{(c)}\nabla_4 \underline{A} + \frac{1}{2}\text{tr}X\underline{A} = r^{-1}\nabla_4(r\underline{A}) + O(r^{-2})\underline{A} + \Gamma_g \cdot \Gamma_b, $$

we obtain

$$ \begin{aligned} & \frac{1}{4}\left({}^{(c)}\nabla_4 + \frac{3}{2}\text{tr}X - 2\frac{{}^{(a)}\text{tr}\chi^2}{tr\,\chi} - 2i\,{}^{(a)}\text{tr}\chi \right) {}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}} \cdot \underline{A}) \\ = & \left[{}^{(c)}\nabla_4 + \frac{3}{2}\text{tr}X - 2\frac{{}^{(a)}\text{tr}\chi^2}{tr\,\chi} - 2i\,{}^{(a)}\text{tr}\chi,\ {}^{(c)}\nabla_3 \right] \left({}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\text{tr}X\underline{A} \right) \\ & + O(ar^{-2})\nabla\nabla_4\underline{A} + O(ar^{-3})\nabla\underline{A} + O(r^{-4})\nabla_3\underline{\mathfrak{q}} + O(r^{-5})\underline{\mathfrak{q}} + O(r^{-3})\nabla_4(r\underline{A}) \\ & + O(r^{-4})\nabla_3\underline{A} + O(r^{-4})\underline{A} + r^{-2}\mathfrak{d}^{\leq 2}(\Gamma_b \cdot \Gamma_g). \end{aligned} $$

Next, using the null structure equation for ${}^{(c)}\nabla_3\text{tr}X$, i.e.

$$ \begin{aligned} {}^{(c)}\nabla_3\text{tr}X + \frac{1}{2}\text{tr}\underline{X}\text{tr}X &= {}^{(c)}\mathcal{D} \cdot \overline{H} + H \cdot \overline{H} + 2P + \underline{\Xi} \cdot \overline{\Xi} - \frac{1}{2}\widehat{\underline{X}} \cdot \overline{\widehat{X}} \\ &= O(r^{-3}) + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b + \Gamma_g \cdot \Gamma_b, \end{aligned} $$

as well as the commutator formula

$$ [\,{}^{(c)}\nabla_3,\ {}^{(c)}\nabla_4]U = O(ar^{-2})\nabla U + O(r^{-3})U + \Gamma_b \cdot \nabla U + r^{-1}\Gamma_g U, $$

we infer, using again $({}^{(c)}\nabla_4 + \frac{1}{2}\text{tr}X)\underline{A} \in r^{-2}\mathfrak{d}^{\leq 1}\Gamma_b$ in view of Bianchi,

$$ \begin{aligned} & \left[{}^{(c)}\nabla_4 + \frac{3}{2}\text{tr}X - 2\frac{{}^{(a)}\text{tr}\chi^2}{tr\,\chi} - 2i\,{}^{(a)}\text{tr}\chi,\ {}^{(c)}\nabla_3 \right] \left({}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\text{tr}X\underline{A} \right) \\ = &\ O(ar^{-2})\nabla\left({}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\text{tr}X\underline{A} \right) + O(r^{-2})\left({}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\text{tr}X\underline{A} \right) + r^{-3}\mathfrak{d}^{\leq 2}(\Gamma_b \cdot \Gamma_b). \end{aligned} $$

Using again

$$ {}^{(c)}\nabla_4 \underline{A} + \frac{1}{2}\text{tr}X\underline{A} = r^{-1}\nabla_4(r\underline{A}) + O(r^{-2})\underline{A} + \Gamma_g \cdot \Gamma_b, $$

this yields

$$ \begin{aligned} & \left[{}^{(c)}\nabla_4 + \frac{3}{2}\text{tr}X - 2\frac{{}^{(a)}\text{tr}\chi^2}{tr\,\chi} - 2i\,{}^{(a)}\text{tr}\chi,\ {}^{(c)}\nabla_3 \right] \left({}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\text{tr}X\underline{A} \right) \\ = &\ O(ar^{-3})\nabla(\nabla_4(r\underline{A})) + O(ar^{-4})\nabla\underline{A} + O(r^{-3})\nabla_4(r\underline{A}) + O(r^{-4})\underline{A} + r^{-2}\mathfrak{d}^{\leq 2}(\Gamma_g \cdot \Gamma_b) \end{aligned} $$

and hence

$$\begin{aligned}
&\frac{1}{4}\left({}^{(c)}\nabla_4+\frac{3}{2}\mathrm{tr}X-2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}-2i\,{}^{(a)}\mathrm{tr}\chi\right){}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A})\\
=&O(ar^{-3})\nabla\nabla_4(r\underline{A})+O(ar^{-3})\nabla\underline{A}+O(r^{-4})\nabla_3\underline{\mathfrak{q}}+O(r^{-5})\underline{\mathfrak{q}}+O(r^{-3})\nabla_4(r\underline{A})\\
&+O(r^{-4})\nabla_3\underline{A}+O(r^{-4})\underline{A}+r^{-2}\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\Gamma_g).
\end{aligned}$$

Next, we use again the following commutation formula

$$[\nabla_4,\nabla]F \;=\; -\frac{1}{2}\mathrm{tr}X\nabla F+O(ar^{-2})\nabla_4F+O(ar^{-3})F+r^{-1}\Gamma_g\cdot\mathfrak{d}^{\leq 1}F$$

twice to obtain

$$\begin{aligned}
{}^{(c)}\nabla_4\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}) \;=&\; {}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot{}^{(c)}\nabla_4\underline{A})-\frac{1}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A})\\
&-\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\mathrm{tr}X\,\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A})+O(ar^{-3})\nabla\nabla_4(r\underline{A})+O(ar^{-3})\nabla\underline{A}\\
&+O(ar^{-3})\nabla_4(r\underline{A})+O(ar^{-4})\underline{A}+r^{-2}\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\Gamma_g)\\
=&\; {}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot{}^{(c)}\nabla_4\underline{A})-\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A})+O(ar^{-3})\nabla\nabla_4(r\underline{A})\\
&+O(ar^{-3})\nabla\underline{A}+O(ar^{-3})\nabla_4(r\underline{A})+O(ar^{-4})\underline{A}+r^{-2}\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\Gamma_g).
\end{aligned}$$

We infer

$$\begin{aligned}
&\left({}^{(c)}\nabla_4+\frac{3}{2}\mathrm{tr}X-2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}-2i\,{}^{(a)}\mathrm{tr}\chi\right){}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A})\\
=&\;{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot{}^{(c)}\nabla_4\underline{A})+\left(\frac{1}{2}\mathrm{tr}X-2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}-2i\,{}^{(a)}\mathrm{tr}\chi\right){}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A})\\
&+O(ar^{-3})\nabla\nabla_4(r\underline{A})+O(ar^{-3})\nabla\underline{A}+O(ar^{-3})\nabla_4(r\underline{A})+O(ar^{-4})\underline{A}+r^{-2}\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\Gamma_g)\\
=&\;{}^{(c)}\mathcal{D}\widehat{\otimes}\left(\overline{{}^{(c)}\mathcal{D}}\cdot\left({}^{(c)}\nabla_4\underline{A}+\left(\frac{1}{2}\mathrm{tr}X-2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}-2i\,{}^{(a)}\mathrm{tr}\chi\right)\underline{A}\right)\right)\\
&+O(ar^{-3})\nabla\nabla_4(r\underline{A})+O(ar^{-3})\nabla\underline{A}+O(ar^{-3})\nabla_4(r\underline{A})+O(ar^{-4})\underline{A}+r^{-2}\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\Gamma_g).
\end{aligned}$$

This implies

$$\begin{aligned}
&\frac{1}{4}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\left(\overline{{}^{(c)}\mathcal{D}}\cdot\left({}^{(c)}\nabla_4\underline{A}+\left(\frac{1}{2}\mathrm{tr}X-2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}-2i\,{}^{(a)}\mathrm{tr}\chi\right)\underline{A}\right)\right)\\
=&O(ar^{-3})\nabla\nabla_4(r\underline{A})+O(ar^{-3})\nabla\underline{A}+O(r^{-4})\nabla_3\underline{\mathfrak{q}}+O(r^{-5})\underline{\mathfrak{q}}+O(r^{-3})\nabla_4(r\underline{A})\\
&+O(r^{-4})\nabla_3\underline{A}+O(r^{-4})\underline{A}+r^{-2}\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\Gamma_g).
\end{aligned}$$

as stated. This concludes the proof of Lemma 12.3.8. □

12.3.4 Estimates for $\nabla\underline{A}$

The following lemma provides the control of $\nabla\underline{A}$.

Lemma 12.3.9. *The following estimates hold true, for all $p \leq 2-\delta$,*

$$\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}|\nabla\underline{A}|^2 \lesssim \dot{B}_p[\underline{A}](\tau_1,\tau_2) + \epsilon_0^2\tau_1^{-2-2\delta_{dec}}. \tag{12.3.8}$$

Also, we have, for all $p \leq 2-\delta$,

$$\int_{\Sigma(\tau)} r^{p-2}|\nabla\underline{A}|^2 \lesssim \dot{E}_p[\underline{A}](\tau) + \int_{\Sigma(\tau)} r^{p-2}|\nabla_3\nabla_4(r\underline{A})|^2 + \epsilon_0^2\tau^{-2-2\delta_{dec}}. \tag{12.3.9}$$

Proof. According to the elliptic type estimates of Lemma 11.4.12 we have for any $S \subset \mathcal{M}$.

$$\int_S \Big(|\nabla\underline{A}|^2 + r^{-2}|\underline{A}|^2\Big) \lesssim \left|\int_S \underline{A}\cdot\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot\underline{A})\right| + (a^2+\epsilon^2)\int_S r^{-2}|({}^{(c)}\nabla_3, {}^{(c)}\nabla_4)\underline{A}|^2.$$

We deduce, on $\mathcal{M}(\tau_1,\tau_2)$,

$$\begin{aligned}\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}\big(|\nabla\underline{A}|^2 + r^{-2}|\underline{A}|^2\big) \lesssim{}& \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}|\underline{A}|^2 + \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+1}\big|\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot\underline{A})\big|^2\\ &+ (a^2+\epsilon^2)\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}|({}^{(c)}\nabla_3, {}^{(c)}\nabla_4)\underline{A}|^2.\end{aligned} \tag{12.3.10}$$

Next, recall (12.3.7)

$$\begin{aligned}\frac{1}{4}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}) ={}& {}^{(c)}\nabla_3\left({}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A}\right) + \left(\frac{1}{2}\mathrm{tr}\underline{X} + 2\overline{\mathrm{tr}\underline{X}}\right)\left({}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A}\right)\\ &+ (H+\overline{H})\cdot\nabla\underline{A} - \frac{1}{2}({}^{(c)}\mathcal{D}\cdot\overline{H})\underline{A} + O(ar^{-2})\nabla\underline{A} + O(r^{-3})\underline{A}\\ &+ r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g),\end{aligned}$$

which together with

$${}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A} = r^{-1}\nabla_4(r\underline{A}) + O(r^{-2})\underline{A} + \Gamma_g\cdot\Gamma_b,$$

yields

$$\begin{aligned}|{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A})| \lesssim{}& ar^{-2}|\nabla\underline{A}| + r^{-1}|\nabla_3(\nabla_4(r\underline{A}))| + r^{-2}|\nabla_4(r\underline{A})| + r^{-2}|\nabla_3\underline{A}| + r^{-3}|\underline{A}|\\ &+ r^{-1}|\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_b)|.\end{aligned}$$

In view of the definition of the norms $\dot{B}_p[\underline{A}]$, see Definition 12.3.4, we infer

$$\begin{aligned}
& \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+1}\big|\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot A)\big|^2 \\
\lesssim & \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}\Big(|\nabla_3\big(\nabla_4(r\underline{A})\big)|^2+|\nabla_4(r\underline{A})|^2\Big)+a^2\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}|\nabla\underline{A}|^2 \\
+ & \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}\Big(|\nabla_3\underline{A}|^2+|\underline{A}|^2\Big)+\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}|\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_b)|^2 \\
\lesssim & \dot{B}_p[\underline{A}]+a^2\int_{\mathcal{M}} r^{p-1}|\nabla\underline{A}|^2+\int_{\mathcal{M}} r^{p-1}|\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g)|^2
\end{aligned}$$

where we recall

$$\begin{aligned}
\dot{B}_p[\underline{A}](\tau_1,\tau_2) \;=\; & \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}\Big(r^2|\nabla_4\nabla_4(r\underline{A})|^2+|\nabla_3\nabla_4(r\underline{A})|^2+|\nabla_4(r\underline{A})|^2\Big) \\
& +\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}\Big(r^2|\nabla_4\underline{A}|^2+|\nabla_3\underline{A}|^2+|\underline{A}|^2\Big).
\end{aligned}$$

Also

$$\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}|\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_b)|^2 \lesssim \epsilon^4\tau_1^{-2-2\delta_{dec}} \tag{12.3.11}$$

which holds true for $p\leq 2-\delta$. Hence, for all $p\leq 2-\delta$,

$$\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+1}\big|\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot A)\big|^2 \;\lesssim\; \dot{B}_p[\underline{A}]+a^2\int_{\mathcal{M}} r^{p-1}|\nabla\underline{A}|^2+\epsilon_0^2\tau_1^{-2-2\delta_{dec}}.$$

Back to (12.3.10), we deduce, after absorbing the term $a^2\int_{\mathcal{M}} r^{p-1}|\nabla\underline{A}|^2$ on the left for a small enough, for all $p\leq 2-\delta$,

$$\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}\big(|\nabla\underline{A}|^2+r^{-2}|\underline{A}|^2\big) \;\lesssim\; \dot{B}_p[\underline{A}]+\epsilon_0^2\tau_1^{-2-2\delta_{dec}}$$

which yields (12.3.8). In the same vein, we derive

$$\int_{\Sigma(\tau)} r^{p-2}\big(|\nabla\underline{A}|^2+r^{-2}|\underline{A}|^2\big)\lesssim \dot{E}_p[\underline{A}](\tau)+\int_{\Sigma(\tau)} r^{p-2}|\nabla_3\nabla_4(r\underline{A})|^2+\epsilon_0^2\tau^{-2-2\delta_{dec}},$$

which yields (12.3.9). This concludes the proof of Lemma 12.3.9. □

12.3.5 Estimates for $\nabla\nabla_4(r\underline{A})$

It remains to estimate the terms involving $\nabla\nabla_4(r\underline{A})$. This is achieved in the lemma below.

Lemma 12.3.10. *The following estimates hold true in $\mathcal{M} = \mathcal{M}(\tau_1, \tau_2)$ for all $p \leq 2 - \delta$*

$$\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+1}|\nabla\nabla_4(r\underline{A})|^2 \lesssim B_\delta[\underline{\mathfrak{q}}](\tau_1,\tau_2) + \dot{B}_p[\underline{A}](\tau_1,\tau_2) + \epsilon_0^2\tau_1^{-2-2\delta_{dec}},$$

$$\int_{\Sigma(\tau)} r^p|\nabla\nabla_4(r\underline{A})|^2 \lesssim E_\delta[\underline{\mathfrak{q}}](\tau) + \dot{E}_p[\underline{A}](\tau) + \int_{\Sigma(\tau)} r^{p-2}|\nabla_3\nabla_4(r\underline{A})|^2 + \epsilon_0^2\tau^{-2-2\delta_{dec}}.$$

Proof. Let us introduce the notation

$$Y[\underline{A}] \;:=\; {}^{(c)}\nabla_4\underline{A} + \left(\frac{1}{2}\mathrm{tr}X - 2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi} - 2i\,{}^{(a)}\mathrm{tr}\chi\right)\underline{A}.$$

According to the elliptic type estimates of Lemma 11.4.12 we have for any $S \subset \mathcal{M}$.

$$\begin{aligned}\int_S \Big(|\nabla Y[\underline{A}]|^2 + r^{-2}|Y[\underline{A}]|^2\Big) \;&\lesssim\; \left|\int_S \underline{A}\cdot\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot Y[\underline{A}])\right| \\ &\quad +(a^2+\epsilon^2)\int_S r^{-2}|({}^{(c)}\nabla_3,\, {}^{(c)}\nabla_4)Y[\underline{A}]|^2.\end{aligned}$$

We deduce, on $\mathcal{M}(\tau_1, \tau_2)$,

$$\begin{aligned}&\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+3}\big(|\nabla Y[\underline{A}]|^2 + r^{-2}|Y[\underline{A}]|^2\big) \\ &\lesssim \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+1}|Y[\underline{A}]|^2 + \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+5}\big|\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot Y[\underline{A}])\big|^2 \\ &+ (a^2+\epsilon^2)\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+1}|({}^{(c)}\nabla_3,\, {}^{(c)}\nabla_4)Y[\underline{A}]|^2.\end{aligned} \tag{12.3.12}$$

Next, recall from Lemma 12.3.8 that we have, using also the above definition of $Y[\underline{A}]$,

$$\begin{aligned}\frac{1}{4}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\left(\overline{{}^{(c)}\mathcal{D}}\cdot(Y[\underline{A}])\right) =& O(ar^{-3})\nabla\nabla_4(r\underline{A}) + O(ar^{-3})\nabla\underline{A} + O(r^{-4})\nabla_3\underline{\mathfrak{q}} + O(r^{-5})\underline{\mathfrak{q}} \\ &+ O(r^{-3})\nabla_4(r\underline{A}) + O(r^{-4})\nabla_3\underline{A} + O(r^{-4})\underline{A} + r^{-2}\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\Gamma_g).\end{aligned}$$

which yields

$$\begin{aligned}&\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+5}\big|\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot Y[\underline{A}])\big|^2 \\ \lesssim\; &\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}\Big(|\nabla_3\underline{\mathfrak{q}}|^2 + r^{-2}|\underline{\mathfrak{q}}|^2 + |\nabla_3\underline{A}|^2 + |\underline{A}|^2\Big) + \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}|\nabla_4(r\underline{A})|^2 \\ &+a^2\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}\Big(|\nabla\nabla_4(r\underline{A})|^2 + |\nabla\underline{A}|^2\Big) + \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+1}|\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\Gamma_g)|^2.\end{aligned}$$

In view of the definition of $\dot{B}_p[\underline{A}](\tau_1,\tau_2)$ and $B_p[\underline{\mathfrak{q}}](\tau_1,\tau_2)$, we infer, for $p\leq 2-\delta$,

$$\begin{aligned}
&\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+5}\big|\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot Y[\underline{A}])\big|^2\\
\lesssim\;& B_\delta[\underline{\mathfrak{q}}](\tau_1,\tau_2)+\dot{B}_p[\underline{A}](\tau_1,\tau_2)\\
&+a^2\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}\Big(|\nabla\nabla_4(r\underline{A})|^2+|\nabla\underline{A}|^2\Big)+\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+1}|\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\Gamma_g)|^2.
\end{aligned}$$

Together with the control of Γ_g and Γ_b, we infer, for $p\leq 2-\delta$,

$$\begin{aligned}
&\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+5}\big|\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot Y[\underline{A}])\big|^2\\
\lesssim\;& B_\delta[\underline{\mathfrak{q}}](\tau_1,\tau_2)+\dot{B}_p[\underline{A}](\tau_1,\tau_2)+a^2\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}\Big(|\nabla\nabla_4(r\underline{A})|^2+|\nabla\underline{A}|^2\Big)+\epsilon^4\tau_1^{-2-2\delta_{dec}}.
\end{aligned}$$

Pluggin in (12.3.12), we deduce, for $p\leq 2-\delta$,

$$\begin{aligned}
&\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+3}\big(|\nabla Y[\underline{A}]|^2+r^{-2}|Y[\underline{A}]|^2\big)\\
&\lesssim \int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+1}|Y[\underline{A}]|^2+B_\delta[\underline{\mathfrak{q}}](\tau_1,\tau_2)+\dot{B}_p[\underline{A}](\tau_1,\tau_2)\\
&+a^2\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}\Big(|\nabla\nabla_4(r\underline{A})|^2+|\nabla\underline{A}|^2\Big)\\
&+(a^2+\epsilon^2)\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+1}|({}^{(c)}\nabla_3,\,{}^{(c)}\nabla_4)Y[\underline{A}]|^2+\epsilon_0^2\tau_1^{-2-2\delta_{dec}}.
\end{aligned}$$

Noticing that

$$\begin{aligned}
Y[\underline{A}] &= {}^{(c)}\nabla_4\underline{A}+\left(\frac{1}{2}\mathrm{tr}X-2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}-2i\,{}^{(a)}\mathrm{tr}\chi\right)\underline{A}\\
&= r^{-1}\nabla_4(r\underline{A})+O(ar^{-2})\underline{A}+\Gamma_g\cdot\Gamma_b,
\end{aligned}$$

we infer, together with the control of Γ_g and Γ_b and the definition of $\dot{B}_p[\underline{A}](\tau_1,\tau_2)$, for $p\leq 2-\delta$,

$$\begin{aligned}
\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+1}|\nabla\nabla_4(r\underline{A})|^2 &\lesssim B_\delta[\underline{\mathfrak{q}}](\tau_1,\tau_2)+\dot{B}_p[\underline{A}](\tau_1,\tau_2)\\
&+a^2\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}\Big(|\nabla\nabla_4(r\underline{A})|^2+|\nabla\underline{A}|^2\Big)+\epsilon_0^2\tau_1^{-2-2\delta_{dec}}.
\end{aligned}$$

Absorbing the term $\nabla\nabla_4(r\underline{A})$ on the RHS from a small enough, and using the control provided by (12.3.8) for the term $\nabla\underline{A}$, we infer

$$\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+1}|\nabla\nabla_4(r\underline{A})|^2 \lesssim B_\delta[\underline{\mathfrak{q}}](\tau_1,\tau_2) + \dot{B}_p[\underline{A}](\tau_1,\tau_2) + \epsilon_0^2\tau_1^{-2-2\delta_{dec}}$$

as stated.

In the same vein, we derive

$$\int_{\Sigma(\tau)} r^p\big(|\nabla\nabla_4(r\underline{A})|^2 + r^{-2}|\nabla_4(r\underline{A})|^2\big) \lesssim E_\delta[\underline{\mathfrak{q}}](\tau) + \dot{E}_p[\underline{A}](\tau) + \int_{\Sigma(\tau)} r^{p-2}|\nabla_3\nabla_4(r\underline{A})|^2 + \epsilon_0^2\tau^{-2-2\delta_{dec}},$$

as stated. This ends the proof of Lemma 12.3.10. □

12.3.6 End of the proof of Proposition 12.2.7

We combine (12.3.8) and the first estimate of Lemma 12.3.10, i.e., for any $p \leq 2-\delta$,

$$\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-1}|\nabla\underline{A}|^2 \lesssim \dot{B}_p[\underline{A}](\tau_1,\tau_2) + \epsilon_0^2\tau_1^{-2-2\delta_{dec}},$$
$$\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+1}|\nabla\nabla_4(r\underline{A})|^2 \lesssim B_\delta[\underline{\mathfrak{q}}](\tau_1,\tau_2) + \dot{B}_p[\underline{A}](\tau_1,\tau_2) + \epsilon_0^2\tau_1^{-2-2\delta_{dec}},$$

with those of Proposition 12.3.5, i.e.

$$\begin{aligned} \dot{BEF}_p[\underline{A}](\tau_1,\tau_2) \lesssim\;& B_\delta[\underline{\psi}](\tau_1,\tau_2) + \dot{E}_p[\underline{A}](\tau_1) \\ & +(a^2+\epsilon^2)\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p-3}\Big(r^2|\nabla\nabla_4(r\underline{A})|^2 + |\nabla\underline{A}|^2\Big) + \epsilon_0^2\tau^{-2-2\delta_{dec}}. \end{aligned}$$

This yields, recalling the definition of the norms $BEF_p[\underline{A}]$ in Definition 12.2.1 and after absorbing the terms proportional to a on the left, for all $p \leq 2-\delta$,

$$BEF_p[\underline{A}](\tau_1,\tau_2) \lesssim B_\delta[\underline{\psi}](\tau_1,\tau_2) + E_p[\underline{A}](\tau_1) + \epsilon_0^2\tau_1^{-2-2\delta_{dec}}$$

which is (12.2.5) in the case $s = 0$.

Next, notice that

$$\begin{aligned}
&\int_{\Sigma(\tau)} r^{p-2}\Big(r^2|\nabla_4\nabla_4(r\underline{A})|^2+|\nabla_3\nabla_4(r\underline{A})|^2\Big)\\
\lesssim\ &\int_{\Sigma(\tau)} r^{p-2}\Big(r^2|\nabla_4\nabla_4(r\underline{A})|^2+|\nabla_{\widehat{R}}\nabla_4(r\underline{A})|^2+\chi_{red}^2|\nabla_3\nabla_4(r\underline{A})|^2\Big)\\
\lesssim\ &\int_{\Sigma(\tau)} r^{p-2}r^2|\nabla_4\nabla_4(r\underline{A})|^2+E_p[\underline{A}](\tau).
\end{aligned}$$

Since we have, in view of the definition of $\underline{\Psi}$ and $\underline{\mathfrak{q}}$,

$$\begin{aligned}
\nabla_4\nabla_4(r\underline{A}) &= \nabla_4(r^{-1}\underline{\Psi}+O(ar^{-1})\underline{A}+r\Gamma_b\cdot\Gamma_g)\\
&= r^{-2}\nabla_4(r\underline{\Psi})-\frac{1}{r^2}\underline{\Psi}+O(ar^{-2})\nabla_4(r\underline{A})+O(ar^{-2})\underline{A}+\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g)\\
&= O(r^{-1})\underline{\mathfrak{q}}+O(r^{-1})\nabla_4(r\underline{A})+O(ar^{-2})\underline{A}+\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g),
\end{aligned}$$

we infer, for $p\leq 1-\delta$,

$$\int_{\Sigma(\tau)} r^p|\nabla_4\nabla_4(r\underline{A})|^2 \lesssim E_p[\underline{\mathfrak{q}}](\tau)+\epsilon_0^2\tau^{-2-2\delta_{dec}}.$$

Together with (12.3.9) and the second estimate of Lemma 12.3.10, i.e., for any $p\leq 2-\delta$,

$$\begin{aligned}
\int_{\Sigma(\tau)} r^{p-2}|\nabla\underline{A}|^2 &\lesssim \dot{E}_p[\underline{A}](\tau)+\int_{\Sigma(\tau)} r^{p-2}|\nabla_3\nabla_4(r\underline{A})|^2+\epsilon_0^2\tau^{-2-2\delta_{dec}},\\
\int_{\Sigma(\tau)} r^p|\nabla\nabla_4(r\underline{A})|^2 &\lesssim E_\delta[\underline{\mathfrak{q}}](\tau)+\dot{E}_p[\underline{A}](\tau)+\int_{\Sigma(\tau)} r^{p-2}|\nabla_3\nabla_4(r\underline{A})|^2+\epsilon_0^2\tau^{-2-2\delta_{dec}},
\end{aligned}$$

and together with (12.2.5) derived above, this yields, for any $p\leq 1-\delta$,

$$\begin{aligned}
&\sup_{\tau\in[\tau_1,\tau_2]}\int_{\Sigma(\tau)} r^{p-2}\Big(r^2|\nabla_4\nabla_4(r\underline{A})|^2+r^2|\nabla\nabla_4(r\underline{A})|^2+|\nabla_3\nabla_4(r\underline{A})|^2+|\nabla\underline{A}|^2\Big)\\
\lesssim\ & EB_p[\underline{\psi}](\tau_1,\tau_2)+E_p[\underline{A}](\tau_1)+\epsilon_0^2\tau_1^{-2-2\delta_{dec}}.
\end{aligned}$$

which is (12.2.6) in the case $s=0$.

It remains to recover (12.2.5) and (12.2.6) for $1\leq s\leq k_L$. To this end, we proceed as follows:

1. We argue by iteration assuming that (12.2.5) and (12.2.6) hold for some $0\leq s\leq k_L-1$. It it true for $s=0$ by the above, and our goal is to prove that (12.2.5) and (12.2.6) hold with s replaced by $s+1$.

2. We commute the system of transport equations (12.3.5), i.e.

$$^{(c)}\nabla_4(r\underline{\Psi}) = \frac{q}{r\overline{q}}\underline{\mathfrak{q}} + r\mathfrak{d}^{\leq 1}(\Gamma_g \cdot \Gamma_b), \qquad ^{(c)}\nabla_4\left(\frac{q^4}{r^3}\underline{A}\right) = \frac{1}{r}\underline{\Psi} + r\Gamma_g \cdot \Gamma_b,$$

with $\not{\mathcal{L}}_{\mathbf{T}}$, $\overline{q}\,\overline{^{(c)}\mathcal{D}}\cdot$ and $\chi_{red}\,^{(c)}\nabla_3$. In view of the commutation formulas of Lemma 9.2.1 and Lemma 4.2.2, we have, for $U \in \mathfrak{s}_2$,

$$\begin{aligned}
[\,^{(c)}\nabla_4, \not{\mathcal{L}}_{\mathbf{T}}]U &= [\nabla_4, \not{\mathcal{L}}_{\mathbf{T}}]U - 4[\omega, \not{\mathcal{L}}_{\mathbf{T}}]U = r^{-1}[\not{\mathcal{L}}_{\mathbf{T}}, \mathfrak{d}]U + r^{-1}\Gamma_b U \\
&= r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b U),
\end{aligned}$$

and

$$\begin{aligned}
[\,^{(c)}\nabla_4, \overline{q}\,\overline{^{(c)}\mathcal{D}}\cdot]U &= \overline{q}\,[\,^{(c)}\nabla_4, \overline{q}\,\overline{^{(c)}\mathcal{D}}\cdot]U + e_4(\overline{q})\,\overline{^{(c)}\mathcal{D}}\cdot U \\
&= -\frac{1}{2}\overline{q}\left(\overline{\mathrm{tr}X} - \frac{2}{\overline{q}}e_4(\overline{q})\right)\overline{^{(c)}\mathcal{D}}\cdot U + \overline{q}\,\overline{\underline{H}}\cdot\,^{(c)}\nabla_4 U + \Gamma_g \cdot \mathfrak{d}^{\leq 1}U \\
&= O(ar^{-1})\,^{(c)}\nabla_4 U + \Gamma_g \cdot \mathfrak{d}^{\leq 1}U.
\end{aligned}$$

This yields the commuted systems

$$\begin{aligned}
^{(c)}\nabla_4\left(\not{\mathcal{L}}_{\mathbf{T}}(r\underline{\Psi})\right) &= \frac{q}{r\overline{q}}\not{\mathcal{L}}_{\mathbf{T}}\underline{\mathfrak{q}} + r\mathfrak{d}^{\leq 2}(\Gamma_g \cdot \Gamma_b), \\
^{(c)}\nabla_4\left(\not{\mathcal{L}}_{\mathbf{T}}\left(\frac{q^4}{r^3}\underline{A}\right)\right) &= \frac{1}{r}\not{\mathcal{L}}_{\mathbf{T}}\underline{\Psi} + r\mathfrak{d}^{\leq 1}(\Gamma_g \cdot \Gamma_b),
\end{aligned}$$

$$\begin{aligned}
^{(c)}\nabla_4\left(\overline{q}\,\overline{^{(c)}\mathcal{D}}\cdot(r\underline{\Psi})\right) &= \frac{q}{r\overline{q}}\,\not{\mathfrak{d}}^{\leq 1}\underline{\mathfrak{q}} + r\mathfrak{d}^{\leq 2}(\Gamma_g \cdot \Gamma_b), \\
^{(c)}\nabla_4\left(\overline{q}\,\overline{^{(c)}\mathcal{D}}\cdot\left(\frac{q^4}{r^3}\underline{A}\right)\right) &= \frac{1}{r}\,\not{\mathfrak{d}}^{\leq 1}\underline{\Psi} + r\mathfrak{d}^{\leq 1}(\Gamma_g \cdot \Gamma_b),
\end{aligned}$$

and

$$\begin{aligned}
^{(c)}\nabla_4\left(\chi_{red}\,^{(c)}\nabla_3(r\underline{\Psi})\right) &= \chi_{red}\,^{(c)}\nabla_3\left(\frac{q}{r\overline{q}}\underline{\mathfrak{q}}\right) + \partial_r\chi_{red}e_4(r)\,^{(c)}\nabla_3(r\underline{\Psi}) \\
&\quad + O\big((a+\epsilon)r^{-1}\big)\chi_{red}\,^{(c)}\nabla(r\underline{\Psi}) + O(r^{-2})\chi_{red}\underline{\Psi}, \\
^{(c)}\nabla_4\left(\chi_{red}\,^{(c)}\nabla_3\left(\frac{q^4}{r^3}\underline{A}\right)\right) &= \chi_{red}\,^{(c)}\nabla_3\left(\frac{1}{r}\underline{\Psi}\right) + \partial_r\chi_{red}e_4(r)\,^{(c)}\nabla_3\left(\frac{q^4}{r^3}\underline{A}\right) \\
&\quad + O\big((a+\epsilon)r^{-1}\big)\chi_{red}\,^{(c)}\nabla\left(\frac{q^4}{r^3}\underline{A}\right) + O(r^{-2})\chi_{red}\underline{A}.
\end{aligned}$$

3. Using the iteration assumption for these commuted systems, and using the original system to recover the ∇_4 derivative, we infer that (12.2.5) and (12.2.6) hold for s derivatives with $\underline{A}$ replaced with $(\not{\mathcal{L}}_{\mathbf{T}}, \overline{q}\,\overline{^{(c)}\mathcal{D}}\cdot, \nabla_4, \chi_{red}\,^{(c)}\nabla_3)\underline{A}$. Together with:

(a) the link between $\not{\mathcal{L}}_{\mathbf{T}}$ and $\nabla_{\mathbf{T}}$ of Lemma 9.2.1,

(b) the Hodge elliptic estimates of Proposition 9.3.2,

(c) the fact that $(\nabla_{\mathbf{T}}, r\nabla_4, \not{\mathfrak{d}}, \chi_{red}\,{}^{(c)}\nabla_3)$ span $\mathfrak{d}$,

and using the iteration assumption to absorb lower order terms in differentiability, we infer that (12.2.5) and (12.2.6) hold for s derivatives with $\underline{A}$ replaced with $\mathfrak{d}^{\leq 1}\underline{A}$. In particular, (12.2.5) and (12.2.6) hold with s replaced by $s+1$. Thus, by iteration, (12.2.5) and (12.2.6) hold for all s such that $0 \leq s \leq k_L$. This end the proof of Proposition 12.2.7.

12.4 Proof of Theorem M2

In this section, we prove Theorem M2 of [53] by relying on Theorem 12.2.4.

12.4.1 Statement of Theorem M2

In this section 12.4.1, we restate Theorem M2 on the decay of the flux of $\underline{\alpha}$ on Σ_*. Note that the global frame used for the proof of Theorem M2 is constructed in section 3.6 of [53] and satisfies in particular the assumptions of section 12.1.

Definition of the r-foliation of Σ_*

We consider the foliation on Σ_* induced by the scalar function r.

Definition 12.4.1 (r-foliation of Σ_*)**.** *The foliation on Σ_* induced by the scalar function r is such that:*

1. *The function r foliates Σ_* by spheres $S_{\Sigma_*}(r)$.*

2. *We have $\tau = \tau(r)$ on Σ_*, i.e. the restriction to Σ_* of τ is a function of r.*

3. *We consider a null pair $(e_3^{\Sigma_*}, e_4^{\Sigma_*})$ and and orthonormal basis $e_a^{\Sigma_*}$, $a = 1, 2$, of the tangent space to $S_{\Sigma_*}(r)$ such that $(e_3^{\Sigma_*}, e_4^{\Sigma_*}, e_1^{\Sigma_*}, e_2^{\Sigma_*})$ forms a null frame on Σ_*.*

4. *We denote by ν the unique vectorfield tangent to Σ_* and normal to $S_{\Sigma_*}(r)$ such that ν is given on Σ_* by*

$$\nu \;=\; e_3^{\Sigma_*} + b^{\Sigma_*} e_4^{\Sigma_*} \tag{12.4.1}$$

for some scalar function b^{Σ_*}.

5. *The Ricci coefficients associated with the null frame $(e_3^{\Sigma_*}, e_4^{\Sigma_*}, e_1^{\Sigma_*}, e_2^{\Sigma_*})$ satisfy the following transversality conditions*[4]

$$\xi^{\Sigma_*} = 0, \qquad \omega^{\Sigma_*} = 0, \qquad \underline{\eta}^{\Sigma_*} = -\zeta^{\Sigma_*}, \qquad on \quad \Sigma_*. \tag{12.4.2}$$

6. *We introduce the following linearized Ricci and curvature components associated to the null frame $(e_3^{\Sigma_*}, e_4^{\Sigma_*}, e_1^{\Sigma_*}, e_2^{\Sigma_*})$:*

$$\widetilde{tr\,\chi^{\Sigma_*}} := tr\,\chi^{\Sigma_*} - \frac{2}{r}, \qquad \widetilde{tr\underline{\chi}^{\Sigma_*}} := tr\underline{\chi}^{\Sigma_*} + \frac{2\left(1-\frac{2m}{r}\right)}{r},$$
$$\widecheck{\underline{\omega}^{\Sigma_*}} := \underline{\omega}^{\Sigma_*} - \frac{m}{r^2}, \qquad \widecheck{\rho}^{\Sigma_*} := \rho^{\Sigma_*} + \frac{2m}{r^3}.$$

We also linearize the scalar function b^{Σ_*} appearing in (12.4.1) and $\nu(r)$ as follows

$$\widecheck{b^{\Sigma_*}} := b^{\Sigma_*} + 1 + \frac{2m}{r}, \qquad \widecheck{\nu(r)} := \nu(r) + 2.$$

7. *We group the above linearized quantities as follows*

$$\Gamma_g^{\Sigma_*} = \left\{\widetilde{tr\,\chi^{\Sigma_*}},\, \widehat{\chi}^{\Sigma_*},\, \eta^{\Sigma_*},\, \underline{\eta}^{\Sigma_*},\, \zeta^{\Sigma_*},\, \widetilde{tr\underline{\chi}^{\Sigma_*}},\, \widehat{\underline{\chi}}^{\Sigma_*},\, \widecheck{\underline{\omega}^{\Sigma_*}},\, \underline{\xi}^{\Sigma_*},\, r\alpha^{\Sigma_*},\, r\beta^{\Sigma_*},\, r\widecheck{\rho}^{\Sigma_*},\, r\,{}^*\rho^{\Sigma_*}\right\},$$
$$\Gamma_b^{\Sigma_*} = \left\{\widehat{\underline{\chi}}^{\Sigma_*},\, \widecheck{\underline{\omega}^{\Sigma_*}},\, \underline{\xi}^{\Sigma_*},\, \underline{\alpha}^{\Sigma_*},\, r\underline{\beta}^{\Sigma_*},\, r^{-1}\widecheck{b^{\Sigma_*}},\, r^{-1}\widecheck{\nu(r)}\right\}.$$

Remark 12.4.2. *Definition 12.4.1 is compatible with the definition of the r-foliation on the last slice Σ_* in [53], see for example section 5.1 in [53].*

Assumptions on the r-foliation of Σ_*

We will need the following assumptions on the r-foliation of Σ_*:

- The function r satisfies along Σ_* the following lower bound

$$\min_{\Sigma_*} r \;\gtrsim\; \epsilon_0^{-1}\tau_*^{1+\delta_{dec}}. \tag{12.4.3}$$

[4]Note that, in view of these transversality conditions, all Ricci coefficients associated to the null frame $(e_3^{\Sigma_*}, e_4^{\Sigma_*}, e_1^{\Sigma_*}, e_2^{\Sigma_*})$ are defined on Σ_*.

- $\Gamma_g^{\Sigma_*}$ and $\Gamma_b^{\Sigma_*}$ verify the following estimates on Σ_*, for $0 \leq k \leq k_L$,

$$\left(r^2 u^{\frac{1}{2}+\delta_{dec}} + ru^{1+\delta_{dec}}\right)|\mathfrak{d}^k \Gamma_g^{\Sigma_*}| + ru^{1+\delta_{dec}}|\mathfrak{d}^k \Gamma_b^{\Sigma_*}| \lesssim \epsilon. \tag{12.4.4}$$

- The functions τ and r satisfy the following estimate on Σ_*

$$|\nu(\tau) - 2| \lesssim \frac{1}{r}, \qquad \nabla^{\Sigma_*}(\tau) = 0, \qquad \nabla^{\Sigma_*}(r) = 0, \tag{12.4.5}$$

where that last two identities follow from the fact that ∇^{Σ_*} is tangent to $S_{\Sigma_*}(r)$ and the fact that τ is a function of r on Σ_*.

Remark 12.4.3. *The assumptions above on the r-foliation are compatible with the ones on the last slice Σ_* in [53], see (3.4.5) in [53] for* (12.4.3) *and section 5.1.4 in [53] for* (12.4.4) *and* (12.4.5).

Comparison of the global frame of $\mathcal{M}$ with the r-foliation of Σ_*

We assume that the global frame of $\mathcal{M}$ and the null frame $(e_3^{\Sigma_*}, e_4^{\Sigma_*}, e_1^{\Sigma_*}, e_2^{\Sigma_*})$ adapted to the r-foliation of Σ_* are related on Σ_* by the following formulas

$$\begin{aligned}
e_4 &= \lambda\left(e_4^{\Sigma_*} + f^b e_b^{\Sigma_*} + \frac{1}{4}|f|^2 e_3^{\Sigma_*}\right),\\
e_a &= \left(\delta_a^b + \frac{1}{2}\underline{f}_a f^b\right) e_b^{\Sigma_*} + \frac{1}{2}\underline{f}_a e_4^{\Sigma_*} + \left(\frac{1}{2}f_a + \frac{1}{8}|f|^2\underline{f}_a\right) e_3^{\Sigma_*}, \quad a = 1, 2,\\
e_3 &= \lambda^{-1}\left(\left(1 + \frac{1}{2}f\cdot\underline{f} + \frac{1}{16}|f|^2|\underline{f}|^2\right) e_3^{\Sigma_*} + \left(\underline{f}^b + \frac{1}{4}|\underline{f}|^2 f^b\right) e_b^{\Sigma_*} + \frac{1}{4}|\underline{f}|^2 e_4^{\Sigma_*}\right),
\end{aligned} \tag{12.4.6}$$

for some scalar λ and some 1-forms $(f, \underline{f})$, see (3.2.4) in [53].

To compare the global from of $\mathcal{M}$ with the null frame $(e_3^{\Sigma_*}, e_4^{\Sigma_*}, e_1^{\Sigma_*}, e_2^{\Sigma_*})$ adapted to the r-foliation of Σ_*, we assume that $(f, \underline{f}, \lambda)$ appearing in (12.4.6) satisfies on Σ_*

$$|\mathfrak{d}^k f| \lesssim \frac{1}{r}, \qquad |\mathfrak{d}^k \underline{f}| \lesssim \frac{1}{r}, \qquad |\mathfrak{d}^k(\lambda - 1)| \lesssim \frac{1}{r}, \quad k \leq k_L, \tag{12.4.7}$$

see (3.2.5) and (3.2.6) in [53].

Statement of Theorem M2

We are now ready to restate Theorem M2 of [53], see also Theorem 1.5.2 in the Introduction.

Theorem 12.4.4 (Theorem M2 in [53]). *Assume that the global frame of $\mathcal{M}$ satisfies the assumptions of section 12.1. Assume in addition that the assumptions* (12.4.3)–(12.4.5) *and* (12.4.7) *hold on Σ_*. Finally, assume that the control of the flux of $\mathfrak{q}$ provided by Theorem M1 holds, i.e.*

$$\int_{\Sigma_*(\geq\tau)} |\nabla_3 \mathfrak{d}^k \mathfrak{q}|^2 \lesssim \epsilon_0^2 \tau^{-2-2\delta_{dec}}. \tag{12.4.8}$$

Then, $\underline{\alpha}$ satisfies the following estimate on Σ_*, for all $1 \leq \tau \leq \tau_*$ and $k \leq k_L - 7$,

$$\int_{\Sigma_*(\geq\tau)} |\mathfrak{d}^k \underline{\alpha}|^2 \lesssim \epsilon_0^2 \tau^{-2-2\delta_{dec}}.$$

To prove Theorem M2, restated here as Theorem 12.4.4, one starts with Theorem 12.2.4 from which, using the structure of the error term $\widetilde{N}_{\text{Err}}$ in (12.1.9), one can only derive estimates of the form (see (12.4.10) below) for $\delta \leq p \leq 1 - \delta$,

$$BEF_p^s[\underline{\psi}, \underline{A}](\tau_1, \tau_2) \lesssim E_p^s[\underline{\psi}, \underline{A}](\tau_1) + \epsilon_0^2 \tau_1^{-2-3\delta_{dec}+p+\delta}.$$

These estimates can be improved by replacing $(\underline{\psi}, \underline{A})$ with their $\not{\mathcal{L}}_{\mathbf{T}}$ derivatives. This fact is crucial in the proof of Theorem M2. We proceed as follows:

1. In section 12.4.2, we obtain decay estimates for $\not{\mathcal{L}}_{\mathbf{T}}^2 \underline{\alpha}$ on $\mathcal{M}$ by relying on Theorem 12.2.4.

2. In section 12.4.3, we deduce from the decay estimates of $\not{\mathcal{L}}_{\mathbf{T}}^2 \underline{\alpha}$ on $\mathcal{M}$ of section 12.4.2 a decay estimates for the flux of $\not{\mathcal{L}}_{\mathbf{T}}^2 \underline{\alpha}$ on Σ_*.

3. In section 12.4.4, we derive an identity on Σ_* involving $\mathfrak{q}$ and $\underline{\alpha}$.

4. Finally, in section 12.4.5, we rely on the control of $\not{\mathcal{L}}_{\mathbf{T}}^2 \underline{\alpha}$ on Σ_* and the control of $\mathfrak{q}$ provided by Theorem M1, which together with the above mentioned identity involving $\underline{\alpha}$ and $\mathfrak{q}$ yields an elliptic equation for $\underline{\alpha}$ along Σ_*. We then rely on this elliptic equation for $\underline{\alpha}$ along Σ_* to prove Theorem M2.

12.4.2 Decay estimate for $\not{\mathcal{L}}_{\mathbf{T}}^2 \underline{A}$

The goal of this section is to prove the following improved decay estimate for $\not{\mathcal{L}}_{\mathbf{T}}^2 \underline{A}$.

Proposition 12.4.5. *The following decay estimate holds for $\mathcal{L}\mkern-10mu/_{\mathbf{T}}^2\underline{A}$, for $s \leq k_L - 9$,*

$$B^s_{2-\delta}[\mathcal{L}\mkern-10mu/_{\mathbf{T}}^2\underline{A}](\tau_1, \tau_*) \lesssim \epsilon_0^2 \tau_1^{-2-2\delta_{dec}}.$$

Furthermore, there exists a sequence of times $\tau^{(j)}$ such that, for $s \leq k_L - 9$,

$$E^s_2[\mathcal{L}\mkern-10mu/_{\mathbf{T}}^2\underline{A}](\tau^{(j)}) \lesssim \epsilon_0^2 (\tau^{(j)})^{-2-2\delta_{dec}}, \qquad \tau^{(j)} \sim 2^j.$$

We will rely on the following lemma.

Lemma 12.4.6. *We have for all $\delta \leq p \leq 1 - \delta$ and $s \leq k_L$,*

$$\mathcal{N}^s_p[\underline{\psi}, \widetilde{N}_{Err}](\tau_1, \tau_2) \lesssim \epsilon_0 \tau_1^{-1-\frac{3}{2}\delta_{dec}+\frac{p+\delta}{2}} \Big(BEF^s_p[\underline{\psi}](\tau_1, \tau_2) \Big)^{\frac{1}{2}}. \tag{12.4.9}$$

Proof. Recall that $\widetilde{N}_{\text{Err}} = r^2 \mathfrak{d}^{\leq 2}\big(\Gamma_b \cdot (A, B)\big)$, see (12.1.9). In the particular case $s = 0$, we use as in the proof of Lemma 12.2.6, see section 12.2.4, for $\delta \leq p \leq 2 - \delta$,

$$\mathcal{N}_p[\underline{\psi}, N](\tau_1, \tau_2) \lesssim \Big(BEF_p[\underline{\psi}](\tau_1, \tau_2) \Big)^{\frac{1}{2}} \left(\int_{\tau_1}^{\tau_2} \|N\|_{L^2(\Sigma_{trap}(\tau))} + \left(\int_{\mathcal{M}(\tau_1, \tau_2)} r^{p+1} |N|^2 \right)^{\frac{1}{2}} \right).$$

In view of $\widetilde{N}_{\text{Err}} = r^2 \mathfrak{d}^{\leq 2}\big(\Gamma_b \cdot (A, B)\big)$, we immediately have, using (12.1.1),

$$\int_{\tau_1}^{\tau_2} \|N\|_{L^2(\Sigma_{trap}(\tau))} \lesssim \epsilon^4 \tau_1^{-2-3\delta_{dec}} \lesssim \epsilon_0^2 \tau_1^{-2-3\delta_{dec}}.$$

Also, we have, for $\delta \leq p \leq 1 - \delta$,

$$\begin{aligned} \int_{\mathcal{M}(\tau_1, \tau_2)} r^{p+1} |N|^2 &\lesssim \epsilon^2 \int_{\mathcal{M}(\tau_1, \tau_2)} \frac{r^{p+3}}{\tau^{2+2\delta_{dec}}} |\mathfrak{d}^{\leq 2}(A, B)|^2 \\ &\lesssim \frac{\epsilon^2}{\tau_1^{1+2\delta_{dec}}} \int_{\mathcal{M}(\tau_1, \tau_2)} \frac{r^{p+3}}{\tau} |\mathfrak{d}^{\leq 2}(A, B)|^2. \end{aligned}$$

Next, interpolating between (12.1.1) and (12.1.2) to control (A, B), we infer, for $\delta \leq p \leq 1 - \delta$,

$$\begin{aligned} \int_{\mathcal{M}(\tau_1, \tau_2)} r^{p+1} |N|^2 &\lesssim \frac{\epsilon^4}{\tau_1^{1+2\delta_{dec}}} \int_{\mathcal{M}(\tau_1, \tau_2)} \frac{r^{p+3}}{\tau} \left(\frac{1}{r^6 \tau^{1+2\delta_{dec}}} \right)^{1-\frac{p+\delta}{1+2\delta_{dec}}} \left(\frac{1}{r^{7+2\delta_{dec}}} \right)^{\frac{p+\delta}{1+2\delta_{dec}}} \\ &\lesssim \frac{\epsilon^4}{\tau_1^{1+2\delta_{dec}}} \left(\int \frac{dr}{r^{1+\delta}} \right) \left(\int_{\tau \geq \tau_1} \frac{d\tau}{\tau^{2+2\delta_{dec}-(p+\delta)}} \right) \end{aligned}$$

and hence, for $\delta \le p \le 1-\delta$,

$$\int_{\mathcal{M}(\tau_1,\tau_2)} r^{p+1}|N|^2 \quad \lesssim \quad \epsilon_0^2\tau_1^{-2-3\delta_{dec}-(p+\delta)}.$$

In view of the above, we deduce, for $\delta \le p \le 1-\delta$,

$$\mathcal{N}_p[\underline{\psi}, \widetilde{N}_{\text{Err}}](\tau_1,\tau_2) \quad \lesssim \quad \epsilon_0\tau_1^{-1-\frac{3}{2}\delta_{dec}+\frac{p+\delta}{2}}\Big(BEF_p[\underline{\psi}](\tau_1,\tau_2)\Big)^{\frac{1}{2}},$$

which proves (12.4.9) in the particular case $s=0$. The general case can be shown in the same manner which concludes the proof of Lemma 12.4.6. □

Proof of Proposition 12.4.5. We proceed in steps as follows.

Step 1. We derive the estimate for $s \le k_L$,

$$BEF_p^s[\underline{\psi},\underline{A}](\tau_1,\tau_2) \lesssim E_p^s[\underline{\psi},\underline{A}](\tau_1) + \epsilon_0^2\tau_1^{-2-3\delta_{dec}+p+\delta}, \qquad \delta \le p \le 1-\delta. \tag{12.4.10}$$

This is an immediate consequence of Theorem 12.2.4 and Lemma 12.4.6.

Step 2. Next, applying the standard mean value procedure, see for instance the statement and proof of Theorem 5.21 in [50], we infer from (12.4.10), for $s \le k_L - 1$,

$$BEF_\delta^s[\underline{\psi},\underline{A}](\tau_1,\tau_*) \lesssim \epsilon_0^2\tau_1^{-1+2\delta}. \tag{12.4.11}$$

Step 3. Next, note that we have[5]

$$\begin{aligned}\int_{\mathcal{M}(\tau_1,\tau_*)} r^{-1-\delta}|\not{\mathcal{L}}_{\mathbf{T}}\mathfrak{d}^{\le s}\underline{\mathfrak{q}}|^2 \quad &\lesssim \quad \int_{\mathcal{M}(\tau_1,\tau_*)}\Big(r^{-1-\delta}|\nabla_3\mathfrak{d}^{\le s}\underline{\mathfrak{q}}|^2 + r^{-3-\delta}|\mathfrak{d}^{\le s+1}\underline{\mathfrak{q}}|^2\Big)\\ &\lesssim \quad B_\delta^{s+1}[\underline{\psi}](\tau_1,\tau_*)\end{aligned}$$

which together with (12.4.11) implies, for $s \le k_L - 2$,

$$\int_{\mathcal{M}(\tau_1,\tau_*)} r^{-1-\delta}|\not{\mathcal{L}}_{\mathbf{T}}\mathfrak{d}^{\le s}\underline{\mathfrak{q}}|^2 \quad \lesssim \quad \epsilon_0^2\tau_1^{-1+2\delta}.$$

In view of the definition of $B_p^s[\not{\mathcal{L}}_{\mathbf{T}}\underline{\psi}]$, see section 6.1.5, we infer for $s \le k_L - 3$ and for all $\delta \le p \le 2-\delta$,

$$B_p^s[\not{\mathcal{L}}_{\mathbf{T}}\underline{\psi}](\tau_1,\tau_*) \quad \lesssim \quad \epsilon_0^2\tau_1^{-1+2\delta}. \tag{12.4.12}$$

[5] Note that the B_δ norms are stronger than the Morr norms, see definition of both in section 6.1.5.

Step 4. Next, we commute the wave equation (12.1.5) for $\underline{\psi}$ and the system of transport equations (12.3.5) with $\not{\mathcal{L}}_{\mathbf{T}}$ and obtain

$$\dot{\square}_2\not{\mathcal{L}}_{\mathbf{T}}\underline{\psi} - V_0\not{\mathcal{L}}_{\mathbf{T}}\underline{\psi} = \frac{4a\cos\theta}{|q|^2}\,{}^*\nabla_T\not{\mathcal{L}}_{\mathbf{T}}\underline{\psi} + N_{\not{\mathcal{L}}_{\mathbf{T}}}$$

and

$$\nabla_4(r\not{\mathcal{L}}_{\mathbf{T}}\underline{\Psi}) = \frac{q}{r\overline{q}}\not{\mathcal{L}}_{\mathbf{T}}\underline{\mathfrak{q}} + F_{1,\not{\mathcal{L}}_{\mathbf{T}}}, \qquad \nabla_4\left(\frac{q^4}{r^3}\not{\mathcal{L}}_{\mathbf{T}}\underline{A}\right) = r^{-1}\not{\mathcal{L}}_{\mathbf{T}}\underline{\Psi} + F_{2,\not{\mathcal{L}}_{\mathbf{T}}},$$

where

$$\begin{aligned}
N_{\not{\mathcal{L}}_{\mathbf{T}}} &= \not{\mathcal{L}}_{\mathbf{T}}N + [\dot{\square}_2, \not{\mathcal{L}}_{\mathbf{T}}]\underline{\psi},\\
F_{1,\not{\mathcal{L}}_{\mathbf{T}}} &= [\nabla_4(r\cdot), \not{\mathcal{L}}_{\mathbf{T}}]\underline{\Psi} + r\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\Gamma_b),\\
F_{2,\not{\mathcal{L}}_{\mathbf{T}}} &= \left[\nabla_4\left(\frac{q^4}{r^3}\cdot\right), \not{\mathcal{L}}_{\mathbf{T}}\right]\underline{A} + O(r^{-2})\mathbf{T}(r)\underline{\Psi} + r\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\Gamma_b).
\end{aligned}$$

Now, in view of Corollary 4.3.4, Lemma 9.2.1 and the above definition of $N_{\not{\mathcal{L}}_{\mathbf{T}}}$ and $F_{\not{\mathcal{L}}_{\mathbf{T}}}$, we have

$$\begin{aligned}
N_{\not{\mathcal{L}}_{\mathbf{T}}} &= \not{\mathcal{L}}_{\mathbf{T}}N + \mathfrak{d}(\Gamma_g\cdot\mathfrak{d}\underline{\psi}) + \Gamma_b\cdot\dot{\square}_2\underline{\psi},\\
&= \not{\mathcal{L}}_{\mathbf{T}}N + \mathfrak{d}^{\leq 1}(\Gamma_g\cdot\underline{\psi}) + \Gamma_b\cdot N,\\
F_{1,\not{\mathcal{L}}_{\mathbf{T}}} &= r\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\Gamma_b),\\
F_{2,\not{\mathcal{L}}_{\mathbf{T}}} &= r\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\Gamma_b).
\end{aligned}$$

The above system for $(\not{\mathcal{L}}_{\mathbf{T}}\underline{\psi}, \not{\mathcal{L}}_{\mathbf{T}}\underline{A})$ has error terms of the same type as before, so that the estimates established for $(\underline{\psi}, \underline{A})$ hold for $(\not{\mathcal{L}}_{\mathbf{T}}\underline{\psi}, \not{\mathcal{L}}_{\mathbf{T}}\underline{A})$ as well. In particular, we have the following analog of (12.2.2) for $s \leq k_L - 1$ and for all $\delta \leq p \leq 2-\delta$,

$$\begin{aligned}
BEF^s_p[\not{\mathcal{L}}_{\mathbf{T}}\underline{\psi}, \not{\mathcal{L}}_{\mathbf{T}}\underline{A}](\tau_1,\tau_2) \lesssim\; & E^s_p[\not{\mathcal{L}}_{\mathbf{T}}\underline{\psi}, \not{\mathcal{L}}_{\mathbf{T}}\underline{A}](\tau_1) + \mathcal{N}^s_p[\not{\mathcal{L}}_{\mathbf{T}}\underline{\psi}, \not{\mathcal{L}}_{\mathbf{T}}\widetilde{N}_{\text{Err}}](\tau_1,\tau_2)\\
&+\epsilon_0^2\tau_1^{-2-2\delta_{dec}},
\end{aligned} \tag{12.4.13}$$

and the following analog of (12.2.5) for $s \leq k_L - 1$ and for all $\delta \leq p \leq 2-\delta$,

$$B^s_p[\not{\mathcal{L}}_{\mathbf{T}}\underline{A}](\tau_1,\tau_2) \lesssim B^s_\delta[\not{\mathcal{L}}_{\mathbf{T}}\underline{\psi}](\tau_1,\tau_2) + E^s_p[\not{\mathcal{L}}_{\mathbf{T}}\underline{A}](\tau_1) + \epsilon_0^2\tau_1^{-2-2\delta_{dec}}. \tag{12.4.14}$$

Step 5. Next, we deduce from (12.4.12) and (12.4.14), for $s \leq k_L - 3$ and for all $\delta \leq p \leq 2-\delta$,

$$B^s_p[\not{\mathcal{L}}_{\mathbf{T}}\underline{A}](\tau_1,\tau_2) \lesssim E^s_p[\not{\mathcal{L}}_{\mathbf{T}}\underline{A}](\tau_1) + \epsilon_0^2\tau_1^{-1+2\delta}.$$

Now, in view of the Definition 12.2.1 of the norms $B^s_p[\underline{A}]$ and $E^s_p[\not{\mathcal{L}}_{\mathbf{T}}\underline{A}]$, we have for all $\delta \le p \le 2-\delta$

$$\int_{\tau_1}^{\tau_2} E^s_p[\not{\mathcal{L}}_{\mathbf{T}}\underline{A}](\tau) \;\; \lesssim \;\; B^s_{p-1}[\not{\mathcal{L}}_{\mathbf{T}}\underline{A}](\tau_1,\tau_2)$$

and hence, for $s \le k_L - 3$,

$$\int_{\tau_1}^{\tau_2} E^s_{1+\delta}[\not{\mathcal{L}}_{\mathbf{T}}\underline{A}](\tau) \;\; \lesssim \;\; B^s_{\delta}[\not{\mathcal{L}}_{\mathbf{T}}\underline{A}](\tau_1,\tau_2) \lesssim E^s_\delta[\not{\mathcal{L}}_{\mathbf{T}}\underline{A}](\tau_1) + \epsilon_0^2\tau_1^{-1+2\delta}.$$

Together with (12.4.11), we infer, for $s \le k_L - 3$ and for all $\delta \le p \le 1+\delta$,

$$\int_{\tau_1}^{\tau_2} E^s_p[\not{\mathcal{L}}_{\mathbf{T}}\underline{A}](\tau) \;\; \lesssim \;\; \epsilon_0^2\tau_1^{-1+2\delta}.$$

Thus, since

$$\int_{\tau_1}^{\tau_2} E^s_p[\not{\mathcal{L}}_{\mathbf{T}}\underline{\psi}](\tau_1) \;\; \lesssim \;\; B^{s+1}_{p+1}[\not{\mathcal{L}}_{\mathbf{T}}\underline{\psi}](\tau_1,\tau_2),$$

we deduce, in view of (12.4.12), for $s \le k_L - 4$ and for all $\delta \le p \le 1-\delta$,

$$\int_{\tau_1}^{\tau_2} E^s_p[\not{\mathcal{L}}_{\mathbf{T}}\underline{\psi}, \not{\mathcal{L}}_{\mathbf{T}}\underline{A}](\tau) \;\; \lesssim \;\; \epsilon_0^2\tau_1^{-1+2\delta}.$$

We infer the existence of a sequence of times $\tau^{(j)}$ such that, for $s \le k_L - 4$,

$$E^s_{1-\delta}[\not{\mathcal{L}}_{\mathbf{T}}\underline{\psi}, \not{\mathcal{L}}_{\mathbf{T}}\underline{A}](\tau^{(j)}) \;\; \lesssim \;\; \epsilon_0^2(\tau^{(j)})^{-1+2\delta}, \qquad \tau^{(j)} \sim 2^j. \tag{12.4.15}$$

Step 6. Next, arguing as in Step 1, using the structure of $\not{\mathcal{L}}_{\mathbf{T}}\widetilde{N}_{\text{Err}}$ and Lemma 12.4.6 to deduce, for $\delta \le p \le 1-\delta$ and $s \le k_L - 1$,

$$\mathcal{N}^s_p[\not{\mathcal{L}}_{\mathbf{T}}\underline{\psi}, \not{\mathcal{L}}_{\mathbf{T}}\widetilde{N}_{\text{Err}}](\tau_1,\tau_2) \;\; \lesssim \;\; \epsilon_0\tau_1^{-1-\frac{3}{2}\delta_{dec}+\frac{p+\delta}{2}}\Big(BEF^s_p[\underline{\psi}](\tau_1,\tau_2)\Big)^{\frac{1}{2}}.$$

Together with (12.4.13) and the estimate (12.4.10) for $BEF^s_p[\underline{\psi}](\tau_1,\tau_2)$, we deduce, for $s \le k_L - 1$ and for all $\delta \le p \le 1-\delta$,

$$BEF^s_p[\not{\mathcal{L}}_{\mathbf{T}}\underline{\psi}, \not{\mathcal{L}}_{\mathbf{T}}\underline{A}](\tau_1,\tau_2) \lesssim E^s_p[\not{\mathcal{L}}_{\mathbf{T}}\underline{\psi}, \not{\mathcal{L}}_{\mathbf{T}}\underline{A}](\tau_1) + \epsilon_0^2\tau_1^{-2-3\delta_{dec}+p+\delta}. \tag{12.4.16}$$

We then apply again the standard mean value procedure starting with (12.4.16) and making use of (12.4.15). We thus infer, for $s \le k_L - 5$,

$$BEF^s_\delta[\not{\mathcal{L}}_{\mathbf{T}}\underline{\psi}, \not{\mathcal{L}}_{\mathbf{T}}\underline{A}](\tau_1,\tau_*) \lesssim \epsilon_0^2\tau_1^{-2+4\delta}. \tag{12.4.17}$$

Step 7. We now run the procedure of Step 3 to Step 6 again, with $(\not{\mathcal{L}}_{\mathbf{T}}\underline{\psi}, \not{\mathcal{L}}_{\mathbf{T}}\underline{A})$ replaced by $(\not{\mathcal{L}}^2_{\mathbf{T}}\underline{\psi}, \not{\mathcal{L}}^2_{\mathbf{T}}\underline{A})$. More precisely:

1. Starting from (12.4.17), as analogous to (12.4.11), and proceeding as in Step 3, we obtain, for $s \leq k_L - 7$ and for all $\delta \leq p \leq 2-\delta$,
$$B^s_p[\not{\mathcal{L}}^2_{\mathbf{T}}\underline{\psi}](\tau_1, \tau_*) \lesssim \epsilon_0^2\tau_1^{-2+4\delta}.$$

2. Next, commuting the system for $(\not{\mathcal{L}}_{\mathbf{T}}\underline{\psi}, \not{\mathcal{L}}_{\mathbf{T}}\underline{\alpha})$ with another $\not{\mathcal{L}}_{\mathbf{T}}$, and proceeding as in Step 4, we obtain the analog of (12.4.13) and (12.4.14) for $(\not{\mathcal{L}}^2_{\mathbf{T}}\underline{\psi}, \not{\mathcal{L}}^2_{\mathbf{T}}\underline{\alpha})$, i.e. for $s \leq k_L - 2$ and for all $\delta \leq p \leq 2-\delta$, we have
$$\begin{aligned} BEF^s_p[\not{\mathcal{L}}^2_{\mathbf{T}}\underline{\psi}, \not{\mathcal{L}}^2_{\mathbf{T}}\underline{A}](\tau_1,\tau_2) &\lesssim E^s_p[\not{\mathcal{L}}^2_{\mathbf{T}}\underline{\psi}, \not{\mathcal{L}}^2_{\mathbf{T}}\underline{A}](\tau_1) + \mathcal{N}^s_p[\not{\mathcal{L}}^2_{\mathbf{T}}\underline{\psi}, \not{\mathcal{L}}^2_{\mathbf{T}}\widetilde{N}_{\mathrm{Err}}](\tau_1,\tau_2) \\ &\quad + \epsilon_0^2\tau_1^{-2-2\delta_{dec}}, \end{aligned}$$
and
$$B^s_p[\not{\mathcal{L}}^2_{\mathbf{T}}\underline{A}](\tau_1,\tau_2) \lesssim B^s_\delta[\not{\mathcal{L}}^2_{\mathbf{T}}\underline{\psi}](\tau_1,\tau_2) + E^s_p[\not{\mathcal{L}}^2_{\mathbf{T}}\underline{A}](\tau_1) + \epsilon_0^2\tau_1^{-2-2\delta_{dec}}.$$

3. Next, proceeding as in Step 5, we obtain the analog of (12.4.15), i.e. we infer the existence of a sequence of times $\tau_1^{(j)}$ such that, for $s \leq k_L - 8$,
$$E^s_{1-\delta}[\not{\mathcal{L}}^2_{\mathbf{T}}\underline{\psi}, \not{\mathcal{L}}^2_{\mathbf{T}}\underline{A}](\tau_1^{(j)}) \lesssim \epsilon_0^2(\tau_1^{(j)})^{-2+4\delta}, \qquad \tau_1^{(j)} \sim 2^j.$$

4. Next, proceeding as in Step 6, we obtain the analog of (12.4.17), for $s \leq k_L - 9$,
$$BEF^s_\delta[\not{\mathcal{L}}^2_{\mathbf{T}}\underline{\psi}, \not{\mathcal{L}}^2_{\mathbf{T}}\underline{A}](\tau_1,\tau_*) \lesssim \epsilon_0^2\tau_1^{-2-2\delta_{dec}} + \epsilon_0^2\tau_1^{-2-3\delta_{dec}+2\delta} + \epsilon_0^2\tau_1^{-3+6\delta},$$
where:
   - the first term in the RHS comes from the contribution of all nonlinear terms except $\not{\mathcal{L}}^2_{\mathbf{T}}\widetilde{N}_{\mathrm{Err}}$, i.e. the terms of type $\mathfrak{d}^{\leq 5}(\Gamma_b \cdot \Gamma_g)$,
   - the second term in the RHS comes from the contribution of $\not{\mathcal{L}}^2_{\mathbf{T}}\widetilde{N}_{\mathrm{Err}}$,
   - the last term comes from the mean value argument.

   Choosing $\delta > 0$ such that $2\delta \leq \delta_{dec}$, we infer, for $s \leq k_L - 9$ and δ_{dec} small enough,
$$BEF^s_\delta[\not{\mathcal{L}}^2_{\mathbf{T}}\underline{\psi}, \not{\mathcal{L}}^2_{\mathbf{T}}\underline{A}](\tau_1,\tau_*) \lesssim \epsilon_0^2\tau_1^{-2-2\delta_{dec}}.$$

Step 8. Recall that we have obtained in Step 7, for all $\delta \leq p \leq 2-\delta$,
$$B^s_p[\not{\mathcal{L}}^2_{\mathbf{T}}\underline{A}](\tau_1,\tau_2) \lesssim B^s_\delta[\not{\mathcal{L}}^2_{\mathbf{T}}\underline{\psi}](\tau_1,\tau_2) + E^s_p[\not{\mathcal{L}}^2_{\mathbf{T}}\underline{A}](\tau_1) + \epsilon_0^2\tau_1^{-2-2\delta_{dec}}.$$
Together with the final estimate of Step 7 for $\underline{\psi}$, this yields, for all $\delta \leq p \leq 2-\delta$ and $s \leq k_L - 9$,
$$B^s_p[\not{\mathcal{L}}^2_{\mathbf{T}}\underline{A}](\tau_1,\tau_*) \lesssim E^s_p[\not{\mathcal{L}}^2_{\mathbf{T}}\underline{A}](\tau_1) + \epsilon_0^2\tau_1^{-2-2\delta_{dec}}. \tag{12.4.18}$$

As in Step 5, we use the fact that we have for all $\delta \leq p \leq 2-\delta$

$$\int_{\tau_1}^{\tau_2} E_p^s[\not{\mathcal{L}}_{\mathbf{T}}^2 \underline{A}](\tau) \lesssim B_{p-1}^s[\not{\mathcal{L}}_{\mathbf{T}}^2 \underline{A}](\tau_1, \tau_2).$$

Together with the final estimate of Step 7 for $BEF_\delta^s[\not{\mathcal{L}}_{\mathbf{T}}^2 \underline{A}](\tau_1, \tau_*)$, we infer the existence of a sequence of times $\tau_1^{(j)}$ such that, for $s \leq k_L - 9$,

$$E_{1+\delta}^s[\not{\mathcal{L}}_{\mathbf{T}}^2 \underline{A}](\tau_1^{(j)}) \lesssim \epsilon_0^2 (\tau_1^{(j)})^{-3-2\delta_{dec}}, \qquad \tau_1^{(j)} \sim 2^j.$$

Plugging in (12.4.18), we infer, for all $s \leq k_L - 9$,

$$B_{1+\delta}^s[\not{\mathcal{L}}_{\mathbf{T}}^2 \underline{A}](\tau_1, \tau_*) \lesssim \epsilon_0^2 \tau_1^{-2-2\delta_{dec}}. \tag{12.4.19}$$

Running again the same argument, we deduce the existence of a sequence of times $\tau_2^{(j)}$ such that, for $s \leq k_L - 9$,

$$E_2^s[\not{\mathcal{L}}_{\mathbf{T}}^2 \underline{A}](\tau_2^{(j)}) \lesssim \epsilon_0^2 (\tau_2^{(j)})^{-3-2\delta_{dec}}, \qquad \tau_2^{(j)} \sim 2^j.$$

Plugging in (12.4.18), we finally obtain, for all $s \leq k_L - 9$,

$$B_{2-\delta}^s[\not{\mathcal{L}}_{\mathbf{T}}^2 \underline{A}](\tau_1, \tau_*) \lesssim \epsilon_0^2 \tau_1^{-2-2\delta_{dec}}.$$

as stated. This concludes the proof of Proposition 12.4.5. $\square$

12.4.3 Decay of the flux on Σ_* of $\not{\mathcal{L}}_{\mathbf{T}}^2 \underline{A}$

The goal of this section is to prove the following decay estimate for the flux on Σ_* of $\not{\mathcal{L}}_{\mathbf{T}}^2 \underline{A}$.

Proposition 12.4.7. *The following decay estimate holds for $\not{\mathcal{L}}_{\mathbf{T}}^2 \underline{A}$, for $s \leq k_L - 10$,*

$$F_{\Sigma_*}^s[\not{\mathcal{L}}_{\mathbf{T}}^2 \underline{A}](\tau, \tau_*) \lesssim \epsilon_0^2 \tau^{-2-2\delta_{dec}}, \tag{12.4.20}$$

where $F_{\Sigma_*}^s$ denotes the flux on Σ_*.

To prove Proposition 12.4.7, we first derive the following extension of Lemma 12.3.1.

Lemma 12.4.8. *Suppose $\Phi_1, \Phi_2 \in \mathfrak{s}_2(\mathbb{C})$ with signature $s \leq -1$ satisfy the differential relation*

$${}^{(c)}\nabla_4 \Phi_1 = \Phi_2,$$

Then, we have

$$
\begin{aligned}
&\int_{\Sigma(\tau_2)} r^{-4}|\Phi_1|^2 + \int_{\mathcal{A}\cup\Sigma_*(\tau_1,\tau_2)} r^{-2}|\Phi_1|^2 \\
&\lesssim \int_{\mathcal{M}(\tau_1,\tau_2)} |\Phi_2|^2 + \int_{\Sigma(\tau_1)} r^{-4}|\Phi_1|^2 + \int_{\mathcal{M}(\tau_1,\tau_2)} r^{-4}|\Phi_1|^2 \qquad (12.4.21)\\
&+ \int_{\mathcal{M}(\tau_1,\tau_2)} r^{-2}|\Gamma_g||\Phi_1|^2.
\end{aligned}
$$

Proof. Recall from the proof of Lemma 12.3.1 that we have

$$
\begin{aligned}
\mathrm{Div}(|q|^{p-2}|\Phi_1|^2 e_4) &= 2|q|^{p-2}\Re\big(\Phi_2\cdot\overline{\Phi_1}\big) + pr\Delta|q|^{p-6}|\Phi_1|^2 - 2(2s+1)\omega|q|^{p-2}|\Phi_1|^2\\
&\quad +|q|^{p-2}\Gamma_g|\Phi_1|^2.
\end{aligned}
$$

We choose $p=0$ which yields

$$
\mathrm{Div}(|q|^{-2}|\Phi_1|^2 e_4) = 2|q|^{-2}\Re\big(\Phi_2\cdot\overline{\Phi_1}\big) - 2(2s+1)\omega|q|^{-2}|\Phi_1|^2 + |q|^{-2}\Gamma_g|\Phi_1|^2.
$$

Since $-\omega \gtrsim \frac{m}{r^2} + \Gamma_g$ and $s\le -1$, we deduce

$$
\begin{aligned}
\mathrm{Div}(|q|^{-2}|\Phi_1|^2 e_4) &\le 2|q|^{-2}\Re\big(\Phi_2\cdot\overline{\Phi_1}\big) + |q|^{-2}\Gamma_g|\Phi_1|^2\\
&\le |\Phi_2|^2 + |q|^{-4}|\Phi_1|^2 + |q|^{-2}\Gamma_g|\Phi_1|^2.
\end{aligned}
$$

We now apply the divergence theorem as in the proof of Lemma 12.3.1 to obtain (12.4.21). This concludes the proof of Lemma 12.4.8. □

We have the following higher derivatives version of Lemma 12.4.8.

Corollary 12.4.9. *Suppose $\Phi_1, \Phi_2 \in \mathfrak{s}_2(\mathbb{C})$ with signature $s\le -1$ satisfy the differential relation*

$$
{}^{(c)}\nabla_4\Phi_1 = \Phi_2, \qquad (12.4.22)
$$

Then, for any $0\le s\le k_L$, we have

$$
\begin{aligned}
&\int_{\Sigma(\tau_2)} r^{-4}|\mathfrak{d}^s\Phi_1|^2 + \int_{\mathcal{A}\cup\Sigma_*(\tau_1,\tau_2)} r^{-2}|\mathfrak{d}^s\Phi_1|^2 \\
&\lesssim \int_{\mathcal{M}(\tau_1,\tau_2)} |\mathfrak{d}^s\Phi_2|^2 + \int_{\Sigma(\tau_1)} r^{-4}|\mathfrak{d}^s\Phi_1|^2 \qquad (12.4.23)\\
&+ \int_{\mathcal{M}(\tau_1,\tau_2)} r^{-4}|\mathfrak{d}^s\Phi_1|^2 + \int_{\mathcal{M}(\tau_1,\tau_2)} r^{-2}|\mathfrak{d}^{\le s}(\Gamma_g\cdot\Phi_1)||\mathfrak{d}^s\Phi_1|.
\end{aligned}
$$

Proof. The case $s=0$ holds true by Lemma 12.4.8. We then argue by iteration on s, and pass from s to $s+1$ by following the procedure outlined at the end of section 12.3.6. $\Box$

We are now ready to prove Proposition 12.4.7.

Proof of Proposition 12.4.7. Recall the second equation of (12.3.5), i.e.

$$
{}^{(c)}\nabla_4\left(\frac{q^4}{r^3}\underline{A}\right) = \frac{1}{r}\underline{\Psi} + r\Gamma_g\cdot\Gamma_b,
$$

where $\underline{\Psi}$ is given by

$$
\underline{\Psi} \;=\; \frac{q^4}{r^2}\left({}^{(c)}\nabla_4 + 2\mathrm{tr}X - \frac{3|\mathrm{tr}X|^2}{2\mathrm{tr}\,\chi}\right)\underline{A}.
$$

We commute this equation with $\not{\mathcal{L}}_{\mathbf{T}}^2\underline{A}$ and obtain, as in the end of section 12.3.6,

$$
{}^{(c)}\nabla_4\left(\frac{q^4}{r^3}\not{\mathcal{L}}_{\mathbf{T}}^2\underline{A}\right) = \frac{1}{r}\not{\mathcal{L}}_{\mathbf{T}}^2\underline{\Psi} + r\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\Gamma_b).
$$

We apply Corollary 12.4.9 to this transport equation, i.e. we choose

$$
\Phi_1 = \frac{q^4}{r^3}\not{\mathcal{L}}_{\mathbf{T}}^2\underline{A}, \qquad \Phi_2 = \frac{1}{r}\not{\mathcal{L}}_{\mathbf{T}}^2\underline{\Psi} + r\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\Gamma_b).
$$

We infer, for $0\leq s\leq k_L-2$

$$
\begin{aligned}
&\int_{\Sigma(\tau_2)} r^{-2}|\mathfrak{d}^s\not{\mathcal{L}}_{\mathbf{T}}^2\underline{A}|^2 + \int_{\mathcal{A}\cup\Sigma_*(\tau_1,\tau_2)}|\mathfrak{d}^s\not{\mathcal{L}}_{\mathbf{T}}^2\underline{A}|^2\\
&\lesssim \int_{\mathcal{M}(\tau_1,\tau_2)} r^{-2}|\mathfrak{d}^s\not{\mathcal{L}}_{\mathbf{T}}^2\underline{\Psi}|^2 + \int_{\Sigma(\tau_1)} r^{-2}|\mathfrak{d}^s\not{\mathcal{L}}_{\mathbf{T}}^2\underline{A}|^2\\
&+\int_{\mathcal{M}(\tau_1,\tau_2)} r^{-2}|\mathfrak{d}^s\not{\mathcal{L}}_{\mathbf{T}}^2\underline{A}|^2 + \int_{\mathcal{M}(\tau_1,\tau_2)}|\mathfrak{d}^{\leq s}(\Gamma_g\cdot\Gamma_b)||\mathfrak{d}^s\Gamma_b|.
\end{aligned}
$$

Using the definition of $\underline{\Psi}$, we infer

$$
\begin{aligned}
&\int_{\Sigma(\tau_2)} r^{-2}|\mathfrak{d}^s\not{\mathcal{L}}_{\mathbf{T}}^2\underline{A}|^2 + \int_{\mathcal{A}\cup\Sigma_*(\tau_1,\tau_2)}|\mathfrak{d}^s\not{\mathcal{L}}_{\mathbf{T}}^2\underline{A}|^2\\
&\lesssim \int_{\mathcal{M}(\tau_1,\tau_2)}|\mathfrak{d}^s\not{\mathcal{L}}_{\mathbf{T}}^2\nabla_4(r\underline{A})|^2 + \int_{\Sigma(\tau_1)} r^{-2}|\mathfrak{d}^s\not{\mathcal{L}}_{\mathbf{T}}^2\underline{A}|^2\\
&+\int_{\mathcal{M}(\tau_1,\tau_2)} r^{-2}|\mathfrak{d}^s\not{\mathcal{L}}_{\mathbf{T}}^2\underline{A}|^2 + \int_{\mathcal{M}(\tau_1,\tau_2)}|\mathfrak{d}^{\leq s}(\Gamma_g\cdot\Gamma_b)||\mathfrak{d}^s\Gamma_b|.
\end{aligned}
$$

In view of the control of Γ_g and Γ_b, as well as the definition of $B_1^s[\not{\mathcal{L}}_{\mathbf{T}}^2\underline{A}](\tau_1,\tau_2)$ and $E_2^s[\not{\mathcal{L}}_{\mathbf{T}}^2\underline{A}](\tau)$, we deduce

$$\begin{aligned}&\int_{\Sigma(\tau_2)} r^{-2}|\mathfrak{d}^s\not{\mathcal{L}}_{\mathbf{T}}^2\underline{A}|^2+\int_{\mathcal{A}\cup\Sigma_*(\tau_1,\tau_2)}|\mathfrak{d}^s\not{\mathcal{L}}_{\mathbf{T}}^2\underline{A}|^2\\ &\lesssim B_1^s[\not{\mathcal{L}}_{\mathbf{T}}^2\underline{A}](\tau_1,\tau_2)+E_2^s[\not{\mathcal{L}}_{\mathbf{T}}^2\underline{A}](\tau_1)+\epsilon_0^2\tau_1^{-2-2\delta_{dec}}.\end{aligned}$$

Finally, recall from Proposition 12.4.5 that we have, for $s\leq k_L-9$,

$$B_{2-\delta}^s[\not{\mathcal{L}}_{\mathbf{T}}^2\underline{A}](\tau_1,\tau_*)\ \lesssim\ \epsilon_0^2\tau_1^{-2-2\delta_{dec}}.$$

and that there exists a sequence of times $\tau^{(j)}$ such that, for $s\leq k_L-9$,

$$E_2^s[\not{\mathcal{L}}_{\mathbf{T}}^2\underline{A}](\tau^{(j)})\ \lesssim\ \epsilon_0^2(\tau^{(j)})^{-2-2\delta_{dec}},\qquad \tau^{(j)}\sim 2^j.$$

Plugging in the above, we deduce, for $s\leq k_L-9$,

$$\int_{\mathcal{A}\cup\Sigma_*(\tau_1,\tau_*)}|\mathfrak{d}^s\not{\mathcal{L}}_{\mathbf{T}}^2\underline{A}|^2\lesssim\epsilon_0^2\tau_1^{-2-2\delta_{dec}}.$$

In particular, we infer The following decay estimate holds for $\not{\mathcal{L}}_{\mathbf{T}}^2\underline{A}$, for $s\leq k_L-10$,

$$F_{\Sigma_*}^s[\not{\mathcal{L}}_{\mathbf{T}}^2\underline{A}](\tau,\tau_*)\ \lesssim\ \epsilon_0^2\tau^{-2-2\delta_{dec}},$$

as stated. This concludes the proof of Proposition 12.4.7. □

12.4.4 An identity on Σ_* involving $\mathfrak{q}$ and $\underline{\alpha}$

Recall from Definition 5.2.2 that the quantity $\mathfrak{q}$ is given by

$$\begin{aligned}\mathfrak{q}&=q\overline{q}^3\left({}^{(c)}\nabla_3\,{}^{(c)}\nabla_3A+C_1\,{}^{(c)}\nabla_3A+C_2A\right),\\ C_1&=2\mathrm{tr}\,\underline{\chi}-2\frac{{}^{(a)}\mathrm{tr}\underline{\chi}^2}{\mathrm{tr}\,\underline{\chi}}-4i\,{}^{(a)}\mathrm{tr}\underline{\chi},\\ C_2&=\frac{1}{2}\mathrm{tr}\,\underline{\chi}^2-4\,{}^{(a)}\mathrm{tr}\underline{\chi}^2+\frac{3}{2}\frac{{}^{(a)}\mathrm{tr}\underline{\chi}^4}{\mathrm{tr}\,\underline{\chi}^2}+i\left(-2\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\underline{\chi}+4\frac{{}^{(a)}\mathrm{tr}\underline{\chi}^3}{\mathrm{tr}\,\underline{\chi}}\right).\end{aligned}$$

We first derive the following identities involving $\mathfrak{q}$ which are the analog of the ones of section 2.3.4 in [50].

Proposition 12.4.10. *We have*

$$\mathfrak{q} = \frac{1}{2}r^4\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\overline{\underline{P}} - 6m\widehat{\underline{X}} + \mathfrak{d}^{\leq 2}\Gamma_g + r^2\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g), \tag{12.4.24}$$

$${}^{(c)}\nabla_3(r\mathfrak{q}) = -\frac{1}{4}r^5\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\,{}^{(c)}\mathcal{D}\cdot\overline{\underline{B}} + 6mr\underline{A} + \mathfrak{d}^{\leq 3}\Gamma_b + r^3\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\Gamma_g), \tag{12.4.25}$$

and

$$\begin{aligned}{}^{(c)}\nabla_3(r^2\,{}^{(c)}\nabla_3(r\mathfrak{q})) &= \frac{1}{8}r^7\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\,{}^{(c)}\mathcal{D}\cdot\,{}^{(c)}\mathcal{D}\cdot\overline{\underline{A}} + 6mr^3\nabla_3\underline{A} + r^2\mathfrak{d}^{\leq 4}\Gamma_b\\ &\quad + r^5\mathfrak{d}^{\leq 3}(\Gamma_b\cdot\Gamma_g).\end{aligned} \tag{12.4.26}$$

Proof. We start with the proof of (12.4.24). Note that C_1 and C_2 satisfy

$$C_1 = -\frac{4}{r} + O(r^{-2}) + \Gamma_g,$$
$$C_2 = \frac{2}{r^2} + O(r^{-3}) + r^{-1}\Gamma_g,$$

so that

$$\begin{aligned}\mathfrak{q} &= q\overline{q}^3\left({}^{(c)}\nabla_3\,{}^{(c)}\nabla_3 A + C_1\,{}^{(c)}\nabla_3 A + C_2 A\right)\\ &= r^4\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_3 A - 4r^3\,{}^{(c)}\nabla_3 A + 2r^2 A + O(r^3)\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_3 A\\ &\quad + \Big(O(r^2) + r^4\Gamma_g\Big)\,{}^{(c)}\nabla_3 A + \Big(O(r) + r^3\Gamma_g\Big)A.\end{aligned}$$

Since $A \in r^{-1}\Gamma_g$, and since ${}^{(c)}\nabla_3 A \in r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g$ and ${}^{(c)}\nabla_3^2 A \in r^{-3}\mathfrak{d}^{\leq 2}\Gamma_g$ by Bianchi, we infer

$$\mathfrak{q} = r^4\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_3 A - 4r^3\,{}^{(c)}\nabla_3 A + 2r^2 A + \mathfrak{d}^{\leq 2}\Gamma_g.$$

Next, we have in view of the Bianchi identity for ${}^{(c)}\nabla_3 A$ of Proposition 2.4.14

$$\begin{aligned}{}^{(c)}\nabla_3 A &= \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}B - \frac{1}{2}\mathrm{tr}\underline{X}A + 2H\widehat{\otimes}B - 3\overline{P}\widehat{X}\\ &= \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}B + \frac{1}{r}A + O(r^{-2})(A,B) + \frac{6m}{r^3}\widehat{X} + r^{-4}\Gamma_g\end{aligned}$$

and hence

$$\begin{aligned}
\mathfrak{q} &= r^4\,{}^{(c)}\nabla_3\left(\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}B+\frac{1}{r}A+O(r^{-2})(A,B)+\frac{6m}{r^3}\widehat{X}+r^{-4}\Gamma_g\right)\\
&\quad -4r^3\left(\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}B+\frac{1}{r}A+O(r^{-2})(A,B)+\frac{6m}{r^3}\widehat{X}+r^{-4}\Gamma_g\right)+2r^2A+\mathfrak{d}^{\leq 2}\Gamma_g\\
&= \frac{1}{2}r^4\,{}^{(c)}\nabla_3\,{}^{(c)}\mathcal{D}\widehat{\otimes}B+r^3\,{}^{(c)}\nabla_3A+r^2A+6mr\,{}^{(c)}\nabla_3\widehat{X}-2r^3\,{}^{(c)}\mathcal{D}\widehat{\otimes}B-4r^2A\\
&\quad +2r^2A+\mathfrak{d}^{\leq 2}\Gamma_g+r^2\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g)\\
&= \frac{1}{2}r^4\,{}^{(c)}\nabla_3\,{}^{(c)}\mathcal{D}\widehat{\otimes}B+r^3\,{}^{(c)}\nabla_3A+6mr\,{}^{(c)}\nabla_3\widehat{X}-2r^3\,{}^{(c)}\mathcal{D}\widehat{\otimes}B-r^2A+\mathfrak{d}^{\leq 2}\Gamma_g\\
&\quad +r^2\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g).
\end{aligned}$$

Using again the Bianchi identity for ${}^{(c)}\nabla_3A$, as well as the following consequence of the null structure equation for ${}^{(c)}\nabla_3\widehat{X}$ of Proposition 2.4.13,

$$ {}^{(c)}\nabla_3\widehat{X} = -\frac{1}{r}\widehat{\underline{X}}+r^{-1}\mathfrak{d}^{\leq 1}\Gamma_g, $$

we infer

$$\begin{aligned}
\mathfrak{q} &= \frac{1}{2}r^4\,{}^{(c)}\nabla_3\,{}^{(c)}\mathcal{D}\widehat{\otimes}B+r^3\left(\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}B+\frac{1}{r}A+O(r^{-2})(A,B)+\frac{6m}{r^3}\widehat{X}+r^{-4}\Gamma_g\right)\\
&\quad +6mr\left(-\frac{1}{r}\widehat{\underline{X}}+r^{-1}\mathfrak{d}^{\leq 1}\Gamma_g\right)-2r^3\,{}^{(c)}\mathcal{D}\widehat{\otimes}B-r^2A+\mathfrak{d}^{\leq 2}\Gamma_g+r^2\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g)\\
&= \frac{1}{2}r^4\,{}^{(c)}\nabla_3\,{}^{(c)}\mathcal{D}\widehat{\otimes}B-\frac{3}{2}r^3\,{}^{(c)}\mathcal{D}\widehat{\otimes}B-6m\widehat{\underline{X}}+\mathfrak{d}^{\leq 2}\Gamma_g+r^2\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g).
\end{aligned}$$

Together with the following commutator identity, see Lemma 4.2.2,

$$\begin{aligned}
[\,{}^{(c)}\nabla_3,\,{}^{(c)}\mathcal{D}\widehat{\otimes}]B &= -\frac{1}{2}\mathrm{tr}\underline{X}\left({}^{(c)}\mathcal{D}\widehat{\otimes}B+2H\widehat{\otimes}B\right)+H\widehat{\otimes}\,{}^{(c)}\nabla_3B+r^{-1}\Gamma_b\cdot\mathfrak{d}^{\leq 1}B\\
&= \frac{1}{r}\,{}^{(c)}\mathcal{D}\widehat{\otimes}B+r^{-4}\mathfrak{d}^{\leq 1}\Gamma_g+r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g)
\end{aligned}$$

where we used in particular the fact that $\check{H}\in\Gamma_g$ and ${}^{(c)}\nabla_3B\in r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g$. We deduce

$$\begin{aligned}
\mathfrak{q} &= \frac{1}{2}r^4\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\nabla_3B+\frac{1}{2}r^4[\,{}^{(c)}\nabla_3,\,{}^{(c)}\mathcal{D}\widehat{\otimes}]B-\frac{3}{2}r^3\,{}^{(c)}\mathcal{D}\widehat{\otimes}B-6m\widehat{\underline{X}}+\mathfrak{d}^{\leq 2}\Gamma_g\\
&\quad +r^2\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g)\\
&= \frac{1}{2}r^4\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\nabla_3B-r^3\,{}^{(c)}\mathcal{D}\widehat{\otimes}B-6m\widehat{\underline{X}}+\mathfrak{d}^{\leq 2}\Gamma_g+r^2\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g).
\end{aligned}$$

In view of the following consequence of the Bianchi identity for $\nabla_3 B$

$$
\begin{aligned}
{}^{(c)}\nabla_3 B &= {}^{(c)}\mathcal{D}\overline{P} + 3\overline{P}H + \frac{2}{r}B + r^{-3}\mathfrak{d}^{\le 1}\Gamma_g + r^{-1}\Gamma_b\cdot\Gamma_g\\
&= {}^{(c)}\mathcal{D}\overline{\check{P}} + \frac{2}{r}B + r^{-3}\mathfrak{d}^{\le 1}\Gamma_g + r^{-1}\Gamma_b\cdot\Gamma_g,
\end{aligned}
$$

we infer

$$
\begin{aligned}
\mathfrak{q} &= \frac{1}{2}r^4\,{}^{(c)}\mathcal{D}\widehat{\otimes}\left({}^{(c)}\mathcal{D}\overline{\check{P}} + \frac{2}{r}B\right) - r^3\,{}^{(c)}\mathcal{D}\widehat{\otimes}B - 6m\widehat{\underline{X}} + \mathfrak{d}^{\le 2}\Gamma_g + r^2\mathfrak{d}^{\le 1}(\Gamma_b\cdot\Gamma_g)\\
&= \frac{1}{2}r^4\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\overline{\check{P}} - 6m\widehat{\underline{X}} + \mathfrak{d}^{\le 2}\Gamma_g + r^2\mathfrak{d}^{\le 1}(\Gamma_b\cdot\Gamma_g)
\end{aligned}
$$

which concludes the proof of (12.4.24).

Next, we prove (12.4.25). First, we multiply by r and differentiate (12.4.24) w.r.t. ${}^{(c)}\nabla_3$ and obtain, using $\nabla_3\Gamma_g = r^{-1}\mathfrak{d}^{\le 1}\Gamma_b$,

$$
\begin{aligned}
{}^{(c)}\nabla_3(r\mathfrak{q}) &= \frac{1}{2}r^5\,{}^{(c)}\nabla_3\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\overline{\check{P}} + \frac{5}{2}r^4 e_3(r)\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\overline{\check{P}} - 6mr\,{}^{(c)}\nabla_3\widehat{\underline{X}} + \mathfrak{d}^{\le 3}\Gamma_b\\
&\quad + r^3\mathfrak{d}^{\le 2}(\Gamma_b\cdot\Gamma_g)\\
&= \frac{1}{2}r^5\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\overline{{}^{(c)}\nabla_3\check{P}} + \frac{1}{2}r^5[\,{}^{(c)}\nabla_3,\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}]\overline{\check{P}} - \frac{5}{2}r^4\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\overline{\check{P}}\\
&\quad - 6mr\,{}^{(c)}\nabla_3\widehat{\underline{X}} + \mathfrak{d}^{\le 3}\Gamma_b + r^3\mathfrak{d}^{\le 2}(\Gamma_b\cdot\Gamma_g).
\end{aligned}
$$

Now, we have, in view of the commutator identities for $[\,{}^{(c)}\nabla_3,\,{}^{(c)}\mathcal{D}\widehat{\otimes}]$ and $[\,{}^{(c)}\nabla_3,\,{}^{(c)}\mathcal{D}]$, see Lemma 4.2.2,

$$
\begin{aligned}
&[\,{}^{(c)}\nabla_3,\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}]\overline{\check{P}}\\
=\;& [\,{}^{(c)}\nabla_3,\,{}^{(c)}\mathcal{D}\widehat{\otimes}]\,{}^{(c)}\mathcal{D}\overline{\check{P}} + {}^{(c)}\mathcal{D}\widehat{\otimes}[\,{}^{(c)}\nabla_3,\,{}^{(c)}\mathcal{D}]\overline{\check{P}}\\
=\;& -\frac{1}{2}\mathrm{tr}\underline{X}\left({}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\overline{\check{P}} + H\widehat{\otimes}\,{}^{(c)}\mathcal{D}\overline{\check{P}}\right) + H\widehat{\otimes}\,{}^{(c)}\nabla_3\,{}^{(c)}\mathcal{D}\overline{\check{P}} + r^{-1}\Gamma_b\cdot\mathfrak{d}^{\le 1}\,{}^{(c)}\mathcal{D}\overline{\check{P}}\\
&+ {}^{(c)}\mathcal{D}\widehat{\otimes}\left(-\frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(c)}\mathcal{D}\overline{\check{P}} + H\widehat{\otimes}\,{}^{(c)}\nabla_3\overline{\check{P}} + r^{-1}\Gamma_b\cdot\mathfrak{d}^{\le 1}\overline{\check{P}}\right)\\
=\;& \frac{2}{r}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\overline{\check{P}} + r^{-5}\mathfrak{d}^{\le 2}\Gamma_b + r^{-3}\mathfrak{d}^{\le 2}(\Gamma_b\cdot\Gamma_g)
\end{aligned}
$$

where we used in particular the fact that ${}^{(c)}\nabla_3\check{P}\in r^{-2}\mathfrak{d}^{\le 1}\Gamma_b$ in view of Bianchi. We deduce

$$
\begin{aligned}
{}^{(c)}\nabla_3(r\mathfrak{q}) &= \frac{1}{2}r^5\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\overline{{}^{(c)}\nabla_3\check{P}} - \frac{3}{2}r^4\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\overline{\check{P}} - 6mr\,{}^{(c)}\nabla_3\widehat{\underline{X}} + \mathfrak{d}^{\le 3}\Gamma_b\\
&\quad + r^3\mathfrak{d}^{\le 2}(\Gamma_b\cdot\Gamma_g).
\end{aligned}
$$

Next, using the following consequence of Bianchi and the null structure equations

$$\begin{aligned}
{}^{(c)}\nabla_3\check{P} &= -\frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot\underline{B}+\frac{3}{r}\check{P}+r^{-3}\Gamma_b+\Gamma_b\cdot\Gamma_g\\
{}^{(c)}\nabla_3\underline{\widehat{X}} &= -\underline{A}+r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b+\Gamma_b\cdot\Gamma_g,
\end{aligned}$$

we infer

$$\begin{aligned}
{}^{(c)}\nabla_3(r\mathfrak{q}) &= \frac{1}{2}r^5\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\overline{\Big(-\frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot\underline{B}+\frac{3}{r}\check{P}+r^{-3}\Gamma_b+\Gamma_b\cdot\Gamma_g\Big)}-\frac{3}{2}r^4\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\overline{\check{P}}\\
&\quad -6mr\Big(-\underline{A}+r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b+\Gamma_b\cdot\Gamma_g\Big)+\mathfrak{d}^{\leq 3}\Gamma_b+r^3\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\Gamma_g)\\
&= -\frac{1}{4}r^5\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\,{}^{(c)}\mathcal{D}\cdot\overline{\underline{B}}+6mr\underline{A}+\mathfrak{d}^{\leq 3}\Gamma_b+r^3\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\Gamma_g)
\end{aligned}$$

which concludes the proof of (12.4.25).

Finally, we prove (12.4.26). First, we multiply by r^2 and differentiate (12.4.25) w.r.t. ${}^{(c)}\nabla_3$ and obtain

$$\begin{aligned}
{}^{(c)}\nabla_3(r^2\,{}^{(c)}\nabla_3(r\mathfrak{q})) &= -\frac{1}{4}r^7\,{}^{(c)}\nabla_3\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\,{}^{(c)}\mathcal{D}\cdot\overline{\underline{B}}+6mr^3\nabla_3\underline{A}+r^2\mathfrak{d}^{\leq 4}\Gamma_b\\
&\quad +r^5\mathfrak{d}^{\leq 3}(\Gamma_b\cdot\Gamma_g)\\
&= -\frac{1}{4}r^7\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\,{}^{(c)}\mathcal{D}\cdot\overline{{}^{(c)}\nabla_3\underline{B}}-\frac{1}{4}r^7[\,{}^{(c)}\nabla_3,\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\,{}^{(c)}\mathcal{D}\cdot]\overline{\underline{B}}\\
&\quad +6mr^3\nabla_3\underline{A}+r^2\mathfrak{d}^{\leq 4}\Gamma_b+r^5\mathfrak{d}^{\leq 3}(\Gamma_b\cdot\Gamma_g).
\end{aligned}$$

Now, we have, in view of the commutator identities for $[\,{}^{(c)}\nabla_3,\,{}^{(c)}\mathcal{D}\widehat{\otimes}]$, $[\,{}^{(c)}\nabla_3,\,{}^{(c)}\mathcal{D}]$ and $[\,{}^{(c)}\nabla_3,\,{}^{(c)}\mathcal{D}\cdot]$, see Lemma 4.2.2,

$$\begin{aligned}
&[\,{}^{(c)}\nabla_3,\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\,{}^{(c)}\mathcal{D}\cdot]\overline{\underline{B}}\\
=\;& [\,{}^{(c)}\nabla_3,\,{}^{(c)}\mathcal{D}\widehat{\otimes}]\,{}^{(c)}\mathcal{D}\,{}^{(c)}\mathcal{D}\cdot\overline{\underline{B}}+\,{}^{(c)}\mathcal{D}\widehat{\otimes}[\,{}^{(c)}\nabla_3,\,{}^{(c)}\mathcal{D}]\,{}^{(c)}\mathcal{D}\cdot\overline{\underline{B}}+\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}[\,{}^{(c)}\nabla_3,\,{}^{(c)}\mathcal{D}\cdot]\overline{\underline{B}}\\
=\;& r^{-5}\mathfrak{d}^{\leq 3}\Gamma_b+r^{-3}\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\Gamma_g).
\end{aligned}$$

We deduce

$$\begin{aligned}
{}^{(c)}\nabla_3(r^2\,{}^{(c)}\nabla_3(r\mathfrak{q})) &= -\frac{1}{4}r^7\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\,{}^{(c)}\mathcal{D}\cdot\overline{{}^{(c)}\nabla_3\underline{B}}+6mr^3\nabla_3\underline{A}\\
&\quad +r^2\mathfrak{d}^{\leq 4}\Gamma_b+r^5\mathfrak{d}^{\leq 3}(\Gamma_b\cdot\Gamma_g).
\end{aligned}$$

Next, using the following consequence of Bianchi

$$
{}^{(c)}\nabla_3\underline{B} = -\frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+r^{-2}\Gamma_b+\Gamma_b\cdot\Gamma_g,
$$

we infer

$$
\begin{aligned}
{}^{(c)}\nabla_3(r^2\,{}^{(c)}\nabla_3(r\mathfrak{q})) &= \frac{1}{8}r^7\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\,{}^{(c)}\mathcal{D}\cdot\,{}^{(c)}\mathcal{D}\cdot\overline{\underline{A}}+6mr^3\nabla_3\underline{A}\\
&\quad +r^2\mathfrak{d}^{\leq 4}\Gamma_b+r^5\mathfrak{d}^{\leq 3}(\Gamma_b\cdot\Gamma_g)
\end{aligned}
$$

which concludes the proof of (12.4.26), and hence of Proposition 12.4.10. □

Proposition 12.4.10 implies the following corollary.

Corollary 12.4.11. *We have*

$$
\begin{aligned}
3m\nabla_3\underline{\alpha}+r^4\,{\not{\mathcal{D}}}_2^*{\not{\mathcal{D}}}_1^*{\not{\mathcal{D}}}_1\,{\not{\mathcal{D}}}_2\underline{\alpha} &= \frac{1}{2}r^{-3}\Re\big({}^{(c)}\nabla_3(r^2\,{}^{(c)}\nabla_3(r\mathfrak{q}))\big)+r^{-1}\mathfrak{d}^{\leq 4}\Gamma_b\\
&\quad +r^2\mathfrak{d}^{\leq 3}(\Gamma_b\cdot\Gamma_g).
\end{aligned}
\tag{12.4.27}
$$

Proof. Recall (12.4.26)

$$
\begin{aligned}
{}^{(c)}\nabla_3(r^2\,{}^{(c)}\nabla_3(r\mathfrak{q})) &= \frac{1}{8}r^7\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\,{}^{(c)}\mathcal{D}\cdot\,{}^{(c)}\mathcal{D}\cdot\overline{\underline{A}}+6mr^3\nabla_3\underline{A}\\
&\quad +r^2\mathfrak{d}^{\leq 4}\Gamma_b+r^5\mathfrak{d}^{\leq 3}(\Gamma_b\cdot\Gamma_g).
\end{aligned}
$$

Taking the real part, we infer

$$
\begin{aligned}
\Re\big({}^{(c)}\nabla_3(r^2\,{}^{(c)}\nabla_3(r\mathfrak{q}))\big) &= \frac{1}{8}r^7\Re\Big({}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\,{}^{(c)}\mathcal{D}\cdot\,{}^{(c)}\mathcal{D}\cdot\overline{\underline{A}}\Big)+6mr^3\nabla_3\underline{\alpha}\\
&\quad +r^2\mathfrak{d}^{\leq 4}\Gamma_b+r^5\mathfrak{d}^{\leq 3}(\Gamma_b\cdot\Gamma_g).
\end{aligned}
$$

Now, we have

$$
\begin{aligned}
{}^{(c)}\mathcal{D}\cdot\,{}^{(c)}\mathcal{D}\cdot\overline{\underline{A}} &= {}^{(c)}\mathcal{D}\cdot\big(2\mathrm{d}iv\,\underline{\alpha}-i2\,{}^*(\mathrm{d}iv\,\underline{\alpha})\big)\\
&= 4\mathrm{d}iv\,\mathrm{d}iv\,\underline{\alpha}-4i\,curl\,\mathrm{d}iv\,\underline{\alpha},
\end{aligned}
$$

and, for two scalar functions $(f,\ {}^*f)$,

$$
\begin{aligned}
\Re\Big({}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}(f-i\,{}^*f)\Big) &= 2\nabla\widehat{\otimes}\Re({}^{(c)}\mathcal{D}(f-i\,{}^*f))\\
&= 2\nabla\widehat{\otimes}\nabla f+2\nabla\widehat{\otimes}\,{}^*\nabla({}^*f)
\end{aligned}
$$

and hence, using the definition of the Hodge operators ${\not{\mathcal{D}}}_1$, ${\not{\mathcal{D}}}_2$, ${\not{\mathcal{D}}}_1^*$ and ${\not{\mathcal{D}}}_2^*$, see section 2.1.3, we infer

$$
\begin{aligned}
\Re\Big({}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\,{}^{(c)}\mathcal{D}\cdot\,{}^{(c)}\mathcal{D}\cdot\overline{\underline{A}}\Big) &= 8\nabla\widehat{\otimes}\nabla\mathrm{d}iv\,\mathrm{d}iv\,\underline{\alpha}-8\nabla\widehat{\otimes}\,{}^*\nabla curl\,\mathrm{d}iv\,\underline{\alpha}\\
&= 16\,{\not{\mathcal{D}}}_2^*\Big(-\nabla\mathrm{d}iv\,\mathrm{d}iv\,\underline{\alpha}+{}^*\nabla curl\,\mathrm{d}iv\,\underline{\alpha}\Big)\\
&= 16\,{\not{\mathcal{D}}}_2^*{\not{\mathcal{D}}}_1^*\Big(\mathrm{d}iv\,\mathrm{d}iv\,\underline{\alpha},curl\,\mathrm{d}iv\,\underline{\alpha}\Big)\\
&= 16\,{\not{\mathcal{D}}}_2^*{\not{\mathcal{D}}}_1^*{\not{\mathcal{D}}}_1\mathrm{d}iv\,\underline{\alpha}\\
&= 16\,{\not{\mathcal{D}}}_2^*{\not{\mathcal{D}}}_1^*{\not{\mathcal{D}}}_1\,{\not{\mathcal{D}}}_2\underline{\alpha}.
\end{aligned}
$$

This yields

$$\Re\big({}^{(c)}\nabla_3(r^{2\,(c)}\nabla_3(r\mathfrak{q}))\big) = 2r^7 \,{\not{\mathcal{D}}}_2^*{\not{\mathcal{D}}}_1^*{\not{\mathcal{D}}}_1\,{\not{\mathcal{D}}}_2\underline{\alpha} + 6mr^3\nabla_3\underline{\alpha} + r^2\mathfrak{d}^{\leq 4}\Gamma_b + r^5\mathfrak{d}^{\leq 3}(\Gamma_b\cdot\Gamma_g)$$

and finally

$$3m\nabla_3\underline{\alpha} + r^4\,{\not{\mathcal{D}}}_2^*{\not{\mathcal{D}}}_1^*{\not{\mathcal{D}}}_1\,{\not{\mathcal{D}}}_2\underline{\alpha} = \frac{1}{2}r^{-3}\Re\big({}^{(c)}\nabla_3(r^{2\,(c)}\nabla_3(r\mathfrak{q}))\big) + r^{-1}\mathfrak{d}^{\leq 4}\Gamma_b + r^2\mathfrak{d}^{\leq 3}(\Gamma_b\cdot\Gamma_g)$$

as stated. This concludes the proof of Corollary 12.4.11. □

The following corollary of Corollary 12.4.11 is the goal of this section.

Corollary 12.4.12. *We have on* Σ_*

$$\begin{aligned} r^4\,{\not{\mathcal{D}}}_2^{*\Sigma_*}\,{\not{\mathcal{D}}}_1^{*\Sigma_*}\,{\not{\mathcal{D}}}_1^{\Sigma_*}\,{\not{\mathcal{D}}}_2^{\Sigma_*}\,{\not{\mathcal{L}}}_{\mathbf{T}}^j\underline{\alpha} &= O(1)\mathfrak{d}^{\leq 3}\nabla_3{\not{\mathcal{L}}}_{\mathbf{T}}^j\underline{\alpha} + O(1)\mathfrak{d}^{\leq 1}\nabla_3{\not{\mathcal{L}}}_{\mathbf{T}}^j\mathfrak{q} + r^{-1}\mathfrak{d}^{\leq 4+j}\Gamma_b \\ &\quad + r^2\mathfrak{d}^{\leq 3+j}(\Gamma_b\cdot\Gamma_g). \end{aligned} \tag{12.4.28}$$

Proof. In view of Corollary 12.4.11, and since $\mathfrak{q}\in r\mathfrak{d}^{\leq 2}\Gamma_g$ and $\nabla_3\mathfrak{q}\in\mathfrak{d}^{\leq 3}\Gamma_b$, we have

$$r^4\,{\not{\mathcal{D}}}_2^*{\not{\mathcal{D}}}_1^*{\not{\mathcal{D}}}_1\,{\not{\mathcal{D}}}_2\underline{\alpha} = -3m\nabla_3\underline{\alpha} + O(1)\mathfrak{d}^{\leq 1}\nabla_3\mathfrak{q} + r^{-1}\mathfrak{d}^{\leq 4}\Gamma_b + r^2\mathfrak{d}^{\leq 3}(\Gamma_b\cdot\Gamma_g).$$

Differentiating this identity with respect to ${\not{\mathcal{L}}}_{\mathbf{T}}$, using the fact that $\mathbf{T}(r)\in r\Gamma_b$, and using the commutation formula $[{\not{\mathcal{L}}}_{\mathbf{T}},\mathfrak{d}]U = \mathfrak{d}(\Gamma_b U)$, see Lemma 9.2.1, we infer

$$r^4\,{\not{\mathcal{D}}}_2^*{\not{\mathcal{D}}}_1^*{\not{\mathcal{D}}}_1\,{\not{\mathcal{D}}}_2{\not{\mathcal{L}}}_{\mathbf{T}}^j\underline{\alpha} = -3m\nabla_3{\not{\mathcal{L}}}_{\mathbf{T}}^j\underline{\alpha} + O(1)\mathfrak{d}^{\leq 1}\nabla_3{\not{\mathcal{L}}}_{\mathbf{T}}^j\mathfrak{q} + r^{-1}\mathfrak{d}^{\leq 4+j}\Gamma_b + r^2\mathfrak{d}^{\leq 3+j}(\Gamma_b\cdot\Gamma_g).$$

Next, in view of the comparison provided by (12.4.6) and (12.4.7) between the global frame of $\mathcal{M}$ and the null frame $(e_3^{\Sigma_*}, e_4^{\Sigma_*}, e_1^{\Sigma_*}, e_2^{\Sigma_*})$ adapted to the r-foliation of Σ_*, we have

$$\begin{aligned} r^4\,{\not{\mathcal{D}}}_2^*{\not{\mathcal{D}}}_1^*{\not{\mathcal{D}}}_1\,{\not{\mathcal{D}}}_2{\not{\mathcal{L}}}_{\mathbf{T}}^j\underline{\alpha} &= r^4\,{\not{\mathcal{D}}}_2^{*\Sigma_*}\,{\not{\mathcal{D}}}_1^{*\Sigma_*}\,{\not{\mathcal{D}}}_1^{\Sigma_*}\,{\not{\mathcal{D}}}_2^{\Sigma_*}\,{\not{\mathcal{L}}}_{\mathbf{T}}^j\underline{\alpha} + O(1)\mathfrak{d}^{\leq 3}\nabla_3{\not{\mathcal{L}}}_{\mathbf{T}}^j\underline{\alpha} + O(r^{-1})\mathfrak{d}^{\leq 4}{\not{\mathcal{L}}}_{\mathbf{T}}^j\underline{\alpha} \\ &= r^4\,{\not{\mathcal{D}}}_2^{*\Sigma_*}\,{\not{\mathcal{D}}}_1^{*\Sigma_*}\,{\not{\mathcal{D}}}_1^{\Sigma_*}\,{\not{\mathcal{D}}}_2^{\Sigma_*}\,{\not{\mathcal{L}}}_{\mathbf{T}}^j\underline{\alpha} + O(1)\mathfrak{d}^{\leq 3}\nabla_3{\not{\mathcal{L}}}_{\mathbf{T}}^j\underline{\alpha} + r^{-1}\mathfrak{d}^{\leq 4+j}\Gamma_b, \end{aligned}$$

where ${\not{\mathcal{D}}}_2^{*\Sigma_*}$, ${\not{\mathcal{D}}}_1^{*\Sigma_*}$, ${\not{\mathcal{D}}}_1^{\Sigma_*}$ and ${\not{\mathcal{D}}}_2^{\Sigma_*}$ denote the Hodge operators tangent to the spheres $S^{\Sigma_*}(r)$ of the r-foliation of Σ_*. In view of the above, we infer

$$\begin{aligned} r^4\,{\not{\mathcal{D}}}_2^{*\Sigma_*}\,{\not{\mathcal{D}}}_1^{*\Sigma_*}\,{\not{\mathcal{D}}}_1^{\Sigma_*}\,{\not{\mathcal{D}}}_2^{\Sigma_*}\,{\not{\mathcal{L}}}_{\mathbf{T}}^j\underline{\alpha} &= O(1)\mathfrak{d}^{\leq 3}\nabla_3{\not{\mathcal{L}}}_{\mathbf{T}}^j\underline{\alpha} + O(1)\mathfrak{d}^{\leq 1}\nabla_3{\not{\mathcal{L}}}_{\mathbf{T}}^j\mathfrak{q} + r^{-1}\mathfrak{d}^{\leq 4+j}\Gamma_b \\ &\quad + r^2\mathfrak{d}^{\leq 3+j}(\Gamma_b\cdot\Gamma_g) \end{aligned}$$

as stated. This concludes the proof of Corollary 12.4.12. □

12.4.5 End of the proof of Theorem M2

We are now ready to prove Theorem M2. We proceed in several steps.

Step 1. Recall from Corollary 12.4.12 that we have

$$
\begin{aligned}
r^4 \,{}^{(c)}\!\mathcal{D}_2^{*\Sigma_*}\, {}^{(c)}\!\mathcal{D}_1^{*\Sigma_*}\, {}^{(c)}\!\mathcal{D}_1^{\Sigma_*}\, {}^{(c)}\!\mathcal{D}_2^{\Sigma_*} \mathcal{L}_{\mathbf{T}}^j \underline{\alpha} &= O(1)\mathfrak{d}^{\leq 3}\nabla_3 \mathcal{L}_{\mathbf{T}}^j \underline{\alpha} + O(1)\mathfrak{d}^{\leq 1}\nabla_3 \mathcal{L}_{\mathbf{T}}^j \mathfrak{q} + r^{-1}\mathfrak{d}^{\leq 4+j}\Gamma_b \\
&\quad + r^2 \mathfrak{d}^{\leq 3+j}(\Gamma_b \cdot \Gamma_g).
\end{aligned}
$$

We take the $L^2(S^{\Sigma_*}(r))$ norm of this identity which yields

$$
\begin{aligned}
& \|r^4 \,{}^{(c)}\!\mathcal{D}_2^{*\Sigma_*}\, {}^{(c)}\!\mathcal{D}_1^{*\Sigma_*}\, {}^{(c)}\!\mathcal{D}_1^{\Sigma_*}\, {}^{(c)}\!\mathcal{D}_2^{\Sigma_*} \mathcal{L}_{\mathbf{T}}^j \underline{\alpha}\|_{L^2(S^{\Sigma_*}(r))} \\
\lesssim\; & \|\mathfrak{d}^{\leq 3}\nabla_3 \mathcal{L}_{\mathbf{T}}^j \underline{\alpha}\|_{L^2(S^{\Sigma_*}(r))} + \|\mathfrak{d}^{\leq 1}\nabla_3 \mathcal{L}_{\mathbf{T}}^j \mathfrak{q}\|_{L^2(S^{\Sigma_*}(r))} + r^{-1}\|\mathfrak{d}^{\leq 4+j}\Gamma_b\|_{L^2(S^{\Sigma_*}(r))} \\
& + r^2\|\mathfrak{d}^{\leq 3+j}(\Gamma_b \cdot \Gamma_g)\|_{L^2(S^{\Sigma_*}(r))}.
\end{aligned}
$$

Next, note the identity

$$
\begin{aligned}
{}^{(c)}\!\mathcal{D}_1^{*\Sigma_*}\, {}^{(c)}\!\mathcal{D}_1^{\Sigma_*} &= {}^{(c)}\!\mathcal{D}_2^{\Sigma_*}\, {}^{(c)}\!\mathcal{D}_2^{*\Sigma_*} + 2K(S^{\Sigma_*}) \\
&= {}^{(c)}\!\mathcal{D}_2^{\Sigma_*}\, {}^{(c)}\!\mathcal{D}_2^{*\Sigma_*} + \frac{2}{r^2} + r^{-1}\Gamma_g
\end{aligned}
$$

which yields

$$
r^4 \,{}^{(c)}\!\mathcal{D}_2^{*\Sigma_*}\, {}^{(c)}\!\mathcal{D}_1^{*\Sigma_*}\, {}^{(c)}\!\mathcal{D}_1^{\Sigma_*}\, {}^{(c)}\!\mathcal{D}_2^{\Sigma_*} U = r^2 \,{}^{(c)}\!\mathcal{D}_2^{*\Sigma_*}\, {}^{(c)}\!\mathcal{D}_2^{\Sigma_*}\Big(r^2 \,{}^{(c)}\!\mathcal{D}_2^{*\Sigma_*}\, {}^{(c)}\!\mathcal{D}_2^{\Sigma_*} + 2\Big)U + r\mathfrak{d}^{\leq 2}(\Gamma_g U)
$$

and hence

$$
\begin{aligned}
& \Big\| r^2 \,{}^{(c)}\!\mathcal{D}_2^{*\Sigma_*}\, {}^{(c)}\!\mathcal{D}_2^{\Sigma_*}\Big(r^2 \,{}^{(c)}\!\mathcal{D}_2^{*\Sigma_*}\, {}^{(c)}\!\mathcal{D}_2^{\Sigma_*} + 2\Big)\mathcal{L}_{\mathbf{T}}^j \underline{\alpha}\Big\|_{L^2(S^{\Sigma_*}(r))} \\
\lesssim\; & \|\mathfrak{d}^{\leq 3}\nabla_3 \mathcal{L}_{\mathbf{T}}^j \underline{\alpha}\|_{L^2(S^{\Sigma_*}(r))} + \|\mathfrak{d}^{\leq 1}\nabla_3 \mathcal{L}_{\mathbf{T}}^j \mathfrak{q}\|_{L^2(S^{\Sigma_*}(r))} + r^{-1}\|\mathfrak{d}^{\leq 4+j}\Gamma_b\|_{L^2(S^{\Sigma_*}(r))} \\
& + r^2\|\mathfrak{d}^{\leq 3+j}(\Gamma_b \cdot \Gamma_g)\|_{L^2(S^{\Sigma_*}(r))}.
\end{aligned}
$$

Thanks to Hodge estimates relying on Proposition 2.1.35, $r^2 \,{}^{(c)}\!\mathcal{D}_2^{*\Sigma_*}\, {}^{(c)}\!\mathcal{D}_2^{\Sigma_*}$ is an elliptic coercive operator and we infer, for $j \leq 2$,

$$
\begin{aligned}
& \big\|(r\nabla^{\Sigma_*})^{\leq 4}\mathcal{L}_{\mathbf{T}}^j \underline{\alpha}\big\|_{L^2(S^{\Sigma_*}(r))} \\
\lesssim\; & \|\mathfrak{d}^{\leq 3}\nabla_3 \mathcal{L}_{\mathbf{T}}^j \underline{\alpha}\|_{L^2(S^{\Sigma_*}(r))} + \|\mathfrak{d}^{\leq 1}\nabla_3 \mathcal{L}_{\mathbf{T}}^j \mathfrak{q}\|_{L^2(S^{\Sigma_*}(r))} + r^{-1}\|\mathfrak{d}^{\leq 4+j}\Gamma_b\|_{L^2(S^{\Sigma_*}(r))} \\
& + r^2\|\mathfrak{d}^{\leq 3+j}(\Gamma_b \cdot \Gamma_g)\|_{L^2(S^{\Sigma_*}(r))}.
\end{aligned}
$$

Now, in view of the comparison provided by (12.4.6) and (12.4.7) between the global frame of $\mathcal{M}$ and the null frame $(e_3^{\Sigma_*}, e_4^{\Sigma_*}, e_1^{\Sigma_*}, e_2^{\Sigma_*})$ adapted to the r-foliation of Σ_*, we have, for $j \leq 2$,

$$
\begin{aligned}
\big\|\, \not{\mathfrak{d}}^{\leq 4}\mathcal{L}_{\mathbf{T}}^j \underline{\alpha}\big\|_{L^2(S^{\Sigma_*}(r))} &\lesssim \big\|(r\nabla^{\Sigma_*})^{\leq 4}\mathcal{L}_{\mathbf{T}}^j \underline{\alpha}\big\|_{L^2(S^{\Sigma_*}(r))} + \|\mathfrak{d}^{\leq 3}\nabla_3 \mathcal{L}_{\mathbf{T}}^j \underline{\alpha}\|_{L^2(S^{\Sigma_*}(r))} \\
&\quad + r^{-1}\|\mathfrak{d}^{\leq 4+j}\Gamma_b\|_{L^2(S^{\Sigma_*}(r))}.
\end{aligned}
$$

We deduce, for $j\leq 2$,

$$
\begin{aligned}
& \|\,\not{\mathfrak{d}}^{\leq 4}\not{\mathcal{L}}_{\mathbf{T}}^{j}\underline{\alpha}\|_{L^2(S^{\Sigma_*}(r))}\\
\lesssim\ & \|\mathfrak{d}^{\leq 3}\nabla_3\not{\mathcal{L}}_{\mathbf{T}}^{j}\underline{\alpha}\|_{L^2(S^{\Sigma_*}(r))}+\|\mathfrak{d}^{\leq 1}\nabla_3\not{\mathcal{L}}_{\mathbf{T}}^{j}\mathfrak{q}\|_{L^2(S^{\Sigma_*}(r))}+r^{-1}\|\mathfrak{d}^{\leq 4+j}\Gamma_b\|_{L^2(S^{\Sigma_*}(r))}\\
& +r^2\|\mathfrak{d}^{\leq 3+j}(\Gamma_b\cdot\Gamma_g)\|_{L^2(S^{\Sigma_*}(r))}.
\end{aligned}
$$

Step 2. Next, we square the final identity of Step 1 and integrate it in τ which implies, for $j\leq 2$,

$$
\begin{aligned}
\int_{\Sigma_*(\geq\tau)}|\,\not{\mathfrak{d}}^{\leq 4}\not{\mathcal{L}}_{\mathbf{T}}^{j}\underline{\alpha}|^2 \ \lesssim\ & \int_{\Sigma_*(\geq\tau)}|\mathfrak{d}^{\leq 3}\nabla_3\not{\mathcal{L}}_{\mathbf{T}}^{j}\underline{\alpha}|^2+\int_{\Sigma_*(\geq\tau)}|\mathfrak{d}^{\leq 1}\nabla_3\not{\mathcal{L}}_{\mathbf{T}}^{j}\mathfrak{q}|^2\\
& +\int_{\Sigma_*(\geq\tau)}r^{-2}|\mathfrak{d}^{\leq 4+j}\Gamma_b|^2+\int_{\Sigma_*(\geq\tau)}r^4|\mathfrak{d}^{\leq 3+j}(\Gamma_b\cdot\Gamma_g)|^2.
\end{aligned}
$$

Together with the bootstrap assumptions on Γ_g and Γ_b, the dominant condition (12.4.3) for r on Σ_*, and the control (12.4.8) for $\mathfrak{q}$ provided by Theorem M1, we infer, for $j\leq 2$,

$$
\begin{aligned}
\int_{\Sigma_*(\geq\tau)}|\,\not{\mathfrak{d}}^{\leq 4}\not{\mathcal{L}}_{\mathbf{T}}^{j}\underline{\alpha}|^2 \ \lesssim\ & \int_{\Sigma_*(\geq\tau)}|\mathfrak{d}^{\leq 3}\nabla_3\not{\mathcal{L}}_{\mathbf{T}}^{j}\underline{\alpha}|^2+\epsilon_0^2\tau^{-2-2\delta_{dec}}\\
& +\epsilon^2\left(\max_{\Sigma_*}r^{-2}\right)\left(\int_{\geq\tau}\frac{d\tau'}{\tau'^{2+2\delta_{dec}}}\right)+\epsilon^4\left(\int_{\geq\tau}\frac{d\tau'}{\tau'^{3+2\delta_{dec}}}\right)\\
\lesssim\ & \int_{\Sigma_*(\geq\tau)}|\mathfrak{d}^{\leq 3}\nabla_3\not{\mathcal{L}}_{\mathbf{T}}^{j}\underline{\alpha}|^2+\epsilon_0^2\tau^{-2-2\delta_{dec}}+\epsilon^2\tau^{-1-2\delta_{dec}}\epsilon_0^2\tau_*^{-2-2\delta_{dec}}
\end{aligned}
$$

and hence, for $j\leq 2$,

$$
\int_{\Sigma_*(\geq\tau)}|\,\not{\mathfrak{d}}^{\leq 4}\not{\mathcal{L}}_{\mathbf{T}}^{j}\underline{\alpha}|^2 \ \lesssim\ \int_{\Sigma_*(\geq\tau)}|\mathfrak{d}^{\leq 3}\nabla_3\not{\mathcal{L}}_{\mathbf{T}}^{j}\underline{\alpha}|^2+\epsilon_0^2\tau^{-2-2\delta_{dec}}.
$$

Step 3. We consider the final identity of Step 2 with $j=2$, i.e.

$$
\begin{aligned}
\int_{\Sigma_*(\geq\tau)}|\,\not{\mathfrak{d}}^{\leq 4}\not{\mathcal{L}}_{\mathbf{T}}^{2}\underline{\alpha}|^2 \ \lesssim\ & \int_{\Sigma_*(\geq\tau)}|\mathfrak{d}^{\leq 3}\nabla_3\not{\mathcal{L}}_{\mathbf{T}}^{2}\underline{\alpha}|^2+\epsilon_0^2\tau^{-2-2\delta_{dec}}\\
\lesssim\ & F^3_{\Sigma_*}[\not{\mathcal{L}}_{\mathbf{T}}^{2}\underline{A}](\tau,\tau_*)+\epsilon_0^2\tau^{-2-2\delta_{dec}}.
\end{aligned}
$$

Together with the control of $F^3_{\Sigma_*}[\not{\mathcal{L}}_{\mathbf{T}}^{2}\underline{A}](\tau,\tau_*)$ provided by Proposition 12.4.7, we infer

$$
\int_{\Sigma_*(\geq\tau)}|\,\not{\mathfrak{d}}^{\leq 4}\not{\mathcal{L}}_{\mathbf{T}}^{2}\underline{\alpha}|^2+\int_{\Sigma_*(\geq\tau)}|\mathfrak{d}^{\leq 3}\nabla_3\not{\mathcal{L}}_{\mathbf{T}}^{2}\underline{\alpha}|^2 \ \lesssim\ \epsilon_0^2\tau^{-2-2\delta_{dec}}.
$$

Also, as a consequence of Bianchi, we have

$$\nabla_4 \underline{A} = -\frac{1}{r}\underline{A} + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_b$$

and hence, we infer, for $k \leq k_L - 1$,

$$\begin{aligned}\int_{\Sigma_*(\geq\tau)} r^2|\mathfrak{d}^k\nabla_4\underline{A}|^2 &\lesssim \int_{\Sigma_*(\geq\tau)}|\mathfrak{d}^k\underline{A}|^2 + \int_{\Sigma_*(\geq\tau)} r^{-2}|\mathfrak{d}^{k+1}\Gamma_b|^2\\ &\lesssim \int_{\Sigma_*(\geq\tau)}|\mathfrak{d}^k\underline{A}|^2 + \epsilon^2\left(\max_{\Sigma_*} r^{-2}\right)\left(\int_{\geq\tau}\frac{d\tau'}{\tau'^{2+2\delta_{dec}}}\right),\end{aligned}$$

which together with the dominant condition (12.4.3) for r on Σ_* implies

$$\int_{\Sigma_*(\geq\tau)} r^2|\mathfrak{d}^k\nabla_4\underline{A}|^2 \lesssim \int_{\Sigma_*(\geq\tau)}|\mathfrak{d}^k\underline{A}|^2 + \epsilon_0^2\tau^{-2-2\delta_{dec}}. \tag{12.4.29}$$

In view of the above estimates, we easily infer

$$\int_{\Sigma_*(\geq\tau)}|\mathfrak{d}^{\leq 4}\not{\mathcal{L}}_{\mathbf{T}}^2\underline{\alpha}|^2 \lesssim \epsilon_0^2\tau^{-2-2\delta_{dec}}. \tag{12.4.30}$$

Next, we consider the following iteration assumption, for $4 \leq l \leq k_L - 10$, that we have

$$\int_{\Sigma_*(\geq\tau)}|\mathfrak{d}^{\leq l}\not{\mathcal{L}}_{\mathbf{T}}^2\underline{\alpha}|^2 \lesssim \epsilon_0^2\tau^{-2-2\delta_{dec}}. \tag{12.4.31}$$

In view of (12.4.30), (12.4.31) holds for $l = 4$. We assume from now on that (12.4.31) holds for $4 \leq l \leq k_L - 10$.

Next, note that (12.4.29) and the iteration assumption (12.4.31) imply

$$\int_{\Sigma_*(\geq\tau)} r^2|\nabla_4\mathfrak{d}^{\leq l}\not{\mathcal{L}}_{\mathbf{T}}^2\underline{\alpha}|^2 \lesssim \epsilon_0^2\tau^{-2-2\delta_{dec}}.$$

Also, Proposition 12.4.7 implies

$$\int_{\Sigma_*(\geq\tau)}|\nabla_3\mathfrak{d}^{\leq l}\not{\mathcal{L}}_{\mathbf{T}}^2\underline{\alpha}|^2 \lesssim \epsilon_0^2\tau^{-2-2\delta_{dec}}.$$

We deduce

$$\int_{\Sigma_*(\geq\tau)}|\mathfrak{d}^{\leq l+1}\not{\mathcal{L}}_{\mathbf{T}}^2\underline{\alpha}|^2 \lesssim \int_{\Sigma_*(\geq\tau)}|\not{\mathfrak{d}}^{l+1}\not{\mathcal{L}}_{\mathbf{T}}^2\underline{\alpha}|^2 + \epsilon_0^2\tau^{-2-2\delta_{dec}}.$$

In view of the elliptic-Hodge estimates of Proposition 9.3.2, and using again the above control of $\nabla_3\mathfrak{d}^{\leq l}\mathcal{L}_{\mathbf{T}}^2$ and $\nabla_4\mathfrak{d}^{\leq l}\mathcal{L}_{\mathbf{T}}^2\underline{\alpha}$, as well as the and the iteration assumption (12.4.31), we have

$$\begin{aligned}\int_{\Sigma_*(\geq\tau)}|\not{\mathfrak{d}}^{l+1}\mathcal{L}_{\mathbf{T}}^2\underline{\alpha}|^2 &\lesssim \int_{\Sigma_*(\geq\tau)}|r^4\,\not{\mathcal{D}}_2^*\not{\mathcal{D}}_1^*\not{\mathcal{D}}_1\,\not{\mathcal{D}}_2\,\not{\mathfrak{d}}^{l-3}\mathcal{L}_{\mathbf{T}}^2\underline{\alpha}|^2+\int_{\Sigma_*(\geq\tau)}|\nabla_4\mathfrak{d}^{\leq l}\mathcal{L}_{\mathbf{T}}^2\underline{\alpha}|\\ &\quad+\int_{\Sigma_*(\geq\tau)}|\nabla_3\mathfrak{d}^{\leq l}\mathcal{L}_{\mathbf{T}}^2\underline{\alpha}|^2+\int_{\Sigma_*(\geq\tau)}|\mathfrak{d}^{\leq l}\mathcal{L}_{\mathbf{T}}^2\underline{\alpha}|^2\\ &\lesssim \int_{\Sigma_*(\geq\tau)}|r^4\,\not{\mathcal{D}}_2^*\not{\mathcal{D}}_1^*\not{\mathcal{D}}_1\,\not{\mathcal{D}}_2\,\not{\mathfrak{d}}^{l-3}\mathcal{L}_{\mathbf{T}}^2\underline{\alpha}|^2+\epsilon_0^2\tau^{-2-2\delta_{dec}}\\ &\lesssim \int_{\Sigma_*(\geq\tau)}|\not{\mathfrak{d}}^{l-3}r^4\,\not{\mathcal{D}}_2^*\not{\mathcal{D}}_1^*\not{\mathcal{D}}_1\,\not{\mathcal{D}}_2\mathcal{L}_{\mathbf{T}}^2\underline{\alpha}|^2+\int_{\Sigma_*(\geq\tau)}|\mathfrak{d}^{\leq l}\mathcal{L}_{\mathbf{T}}^2\underline{\alpha}|^2\\ &\quad+\epsilon_0^2\tau^{-2-2\delta_{dec}}\end{aligned}$$

and hence

$$\int_{\Sigma_*(\geq\tau)}|\mathfrak{d}^{\leq l+1}\mathcal{L}_{\mathbf{T}}^2\underline{\alpha}|^2 \lesssim \int_{\Sigma_*(\geq\tau)}|\not{\mathfrak{d}}^{l-3}r^4\,\not{\mathcal{D}}_2^*\not{\mathcal{D}}_1^*\not{\mathcal{D}}_1\,\not{\mathcal{D}}_2\mathcal{L}_{\mathbf{T}}^2\underline{\alpha}|^2+\epsilon_0^2\tau^{-2-2\delta_{dec}}.$$

On the other hand, we have, in view Corollary 12.4.11 and the commutator formula for $[\mathcal{L}_{\mathbf{T}},\mathfrak{d}]$ of Lemma 9.2.1,

$$\begin{aligned}r^4\,\not{\mathcal{D}}_2^*\not{\mathcal{D}}_1^*\not{\mathcal{D}}_1\,\not{\mathcal{D}}_2\mathcal{L}_{\mathbf{T}}^2\underline{\alpha} &= -3m\nabla_3\mathcal{L}_{\mathbf{T}}^2\underline{\alpha}+\frac{1}{2}r^{-3}\Re\big(\mathcal{L}_{\mathbf{T}}^2\,{}^{(c)}\nabla_3(r^2\,{}^{(c)}\nabla_3(r\mathfrak{q}))\big)+r^{-1}\mathfrak{d}^{\leq 6}\Gamma_b\\ &\quad+r^2\mathfrak{d}^{\leq 5}(\Gamma_b\cdot\Gamma_g)\end{aligned}$$

which yields, in view of the above,

$$\begin{aligned}\int_{\Sigma_*(\geq\tau)}|\mathfrak{d}^{\leq l+1}\mathcal{L}_{\mathbf{T}}^2\underline{\alpha}|^2 &\lesssim \int_{\Sigma_*(\geq\tau)}|\mathfrak{d}^{\leq l}\mathcal{L}_{\mathbf{T}}^2\underline{\alpha}|^2+\int_{\Sigma_*(\geq\tau)}|\mathfrak{d}^{\leq l}\nabla_3\mathfrak{q}|^2\\ &\quad+\int_{\Sigma_*(\geq\tau)}r^{-2}|\mathfrak{d}^{\leq l+3}\Gamma_b|^2+\int_{\Sigma_*(\geq\tau)}r^4|\mathfrak{d}^{\leq l+2}(\Gamma_b\cdot\Gamma_g)|^2\\ &\quad+\epsilon_0^2\tau^{-2-2\delta_{dec}}.\end{aligned}$$

Together with the iteration assumption (12.4.31), the bootstrap assumptions on Γ_g and Γ_b, the dominant condition (12.4.3) for r on Σ_*, and the control (12.4.8) for $\mathfrak{q}$ provided by Theorem M1, we infer

$$\int_{\Sigma_*(\geq\tau)}|\mathfrak{d}^{\leq l+1}\mathcal{L}_{\mathbf{T}}^2\underline{\alpha}|^2 \lesssim \epsilon_0^2\tau^{-2-2\delta_{dec}}$$

which is (12.4.31) with l replaced by $l+1$. This implies by iteration that (12.4.31) holds for any $4\leq l\leq k_L-9$, i.e.

$$\int_{\Sigma_*(\geq\tau)}|\mathfrak{d}^{\leq k_L-9}\mathcal{L}_{\mathbf{T}}^2\underline{\alpha}|^2 \lesssim \epsilon_0^2\tau^{-2-2\delta_{dec}}.$$

Step 4. According to the final identity of Step 2 with $j = 1$, we have

$$\int_{\Sigma_*(\geq\tau)} |\not{\mathfrak{d}}^{\leq 4}\not{\mathcal{L}}_{\mathbf{T}}\underline{\alpha}|^2 \lesssim \int_{\Sigma_*(\geq\tau)} |\mathfrak{d}^{\leq 3}\nabla_3\not{\mathcal{L}}_{\mathbf{T}}\underline{\alpha}|^2 + \epsilon_0^2\tau^{-2-2\delta_{dec}}.$$

In view of the definition of $\mathbf{T}$, as well as the comparison of $\not{\mathcal{L}}_{\mathbf{T}}$ and $\nabla_{\mathbf{T}}$ provided by Lemma 9.2.1, we have on Σ_*

$$\nabla_3\not{\mathcal{L}}_{\mathbf{T}}\underline{\alpha} = \nabla_{\mathbf{T}}\not{\mathcal{L}}_{\mathbf{T}}\underline{\alpha} + r^{-1}\mathfrak{d}\Gamma_b = \not{\mathcal{L}}_{\mathbf{T}}^2\underline{\alpha} + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b$$

which yields

$$\int_{\Sigma_*(\geq\tau)} |\mathfrak{d}^{\leq k_L-9}\nabla_3\not{\mathcal{L}}_{\mathbf{T}}\underline{\alpha}|^2 \lesssim \int_{\Sigma_*(\geq\tau)} |\mathfrak{d}^{\leq k_L-9}\not{\mathcal{L}}_{\mathbf{T}}^2\underline{\alpha}|^2 + \int_{\Sigma_*(\geq\tau)} r^{-2}|\mathfrak{d}^{\leq k_L-10}\Gamma_b|^2$$

and hence, in view of the final estimate of Step 3, the bootstrap assumptions on Γ_b and the dominant condition (12.4.3) for r on Σ_*, we obtain

$$\int_{\Sigma_*(\geq\tau)} |\mathfrak{d}^{\leq k_L-9}\nabla_3\not{\mathcal{L}}_{\mathbf{T}}\underline{\alpha}|^2 \lesssim \epsilon_0^2\tau^{-2-2\delta_{dec}}.$$

In particular, in view of the above, we infer

$$\int_{\Sigma_*(\geq\tau)} |\not{\mathfrak{d}}^{\leq 4}\not{\mathcal{L}}_{\mathbf{T}}\underline{\alpha}|^2 + \int_{\Sigma_*(\geq\tau)} |\mathfrak{d}^{\leq k_L-9}\nabla_3\not{\mathcal{L}}_{\mathbf{T}}\underline{\alpha}|^2 \lesssim \epsilon_0^2\tau^{-2-2\delta_{dec}}.$$

Together with (12.4.29), we easily infer

$$\int_{\Sigma_*(\geq\tau)} |\mathfrak{d}^{\leq 4}\not{\mathcal{L}}_{\mathbf{T}}\underline{\alpha}|^2 \lesssim \epsilon_0^2\tau^{-2-2\delta_{dec}}. \tag{12.4.32}$$

which is the analog of (12.4.30) for $\not{\mathcal{L}}_{\mathbf{T}}\underline{\alpha}$.

We then proceed by iteration, exactly as in Step 3, with $\not{\mathcal{L}}_{\mathbf{T}}^2\underline{\alpha}$ replaced by $\not{\mathcal{L}}_{\mathbf{T}}\underline{\alpha}$, and we deduce the following analog of the final estimate of Step 3

$$\int_{\Sigma_*(\geq\tau)} |\mathfrak{d}^{\leq k_L-8}\not{\mathcal{L}}_{\mathbf{T}}\underline{\alpha}|^2 \lesssim \epsilon_0^2\tau^{-2-2\delta_{dec}}.$$

Step 5. According to the final identity of Step 2 with $j = 0$, we have

$$\int_{\Sigma_*(\geq\tau)} |\not{\mathfrak{d}}^{\leq 4}\underline{\alpha}|^2 \lesssim \int_{\Sigma_*(\geq\tau)} |\mathfrak{d}^{\leq 3}\nabla_3\underline{\alpha}|^2 + \epsilon_0^2\tau^{-2-2\delta_{dec}}.$$

Proceeding as in Step 4, with $\not{\mathcal{L}}_{\mathbf{T}}\underline{\alpha}$ replaced by $\underline{\alpha}$, and relying on the final estimate of Step 4 for $\not{\mathcal{L}}_{\mathbf{T}}\underline{\alpha}$, we obtain

$$\int_{\Sigma_*(\geq\tau)} |\mathfrak{d}^{\leq 4}\underline{\alpha}|^2 \quad \lesssim \quad \epsilon_0^2\tau^{-2-2\delta_{dec}}.$$

which is the analog of (12.4.32) for $\underline{\alpha}$.

We then proceed by iteration, exactly as in Step 3, with $\not{\mathcal{L}}_{\mathbf{T}}^2\underline{\alpha}$ replaced by $\underline{\alpha}$, and we deduce the following analog of the final estimate of Step 3

$$\int_{\Sigma_*(\geq\tau)} |\mathfrak{d}^{\leq k_L-7}\underline{\alpha}|^2 \quad \lesssim \quad \epsilon_0^2\tau^{-2-2\delta_{dec}}$$

as stated. This concludes the proof of Theorem M2.

Part III

Top curvature estimates for Theorem M8

Chapter 13

Main results of Part III

13.1 Properties of $\mathcal{M}$

Recall, see section 6.1, that the spacetime $\mathcal{M}$ consists of two regions

$$\mathcal{M} = {}^{(int)}\mathcal{M} \cup {}^{(ext)}\mathcal{M}, \qquad {}^{(int)}\mathcal{M} = \mathcal{M} \cap \{r \leq r_0\}, \qquad {}^{(ext)}\mathcal{M} = \mathcal{M} \cap \{r \geq r_0\},$$

and comes together with:

1. a pair of constants (a, m),
2. scalar functions (r, τ, θ), where τ is a time function[1], and with the complex scalar q defined by $q = r + ia\cos\theta$,
3. a global regular null pair (e_4, e_3) and its associated horizontal structure,
4. a complex horizontal 1-form $\mathfrak{J}$.

13.1.1 Boundaries of $\mathcal{M}$

The boundaries of $\mathcal{M}$ are given by

$$\partial\mathcal{M} \;=\; \mathcal{A} \cup \Sigma(\tau_*) \cup \Sigma_* \cup \Sigma(1)$$

where:

[1]The hypersurfaces $\Sigma(\tau)$ are spacelike but not strictly spacelike. It is tied to the horizontal structure by properties which will be discussed below in section 13.1.5.

1. the hypersurface $\mathcal{A}$ is spacelike and given by $\mathcal{A} = \{r = r_+(1-\delta_{\mathcal{H}})\}$ for some small constant $\delta_{\mathcal{H}} > 0$,
2. the hypersurfaces $\Sigma(1)$ and $\Sigma(\tau_*)$ are spacelike level hypersurfaces of τ, with $1 \leq \tau \leq \tau_*$ on $\mathcal{M}$,
3. Σ_* is a uniformly spacelike hypersurface connecting $\Sigma(\tau_*)$ to $\Sigma(1)$.

With respect to the horizontal structure of $\mathcal{M}$, we have for the normal of $\mathcal{A}$

$$\mathbf{g}(N_{\mathcal{A}}, e_3) = -1, \qquad \mathbf{g}(N_{\mathcal{A}}, e_4) \leq -\frac{1}{10}\delta_{\mathcal{H}}, \qquad \mathbf{g}(N_{\mathcal{A}}, e_a) = O(\delta_{\mathcal{H}}),$$

and for the normal of Σ_*

$$\mathbf{g}(N_{\Sigma_*}, e_3) = -1, \qquad \mathbf{g}(N_{\Sigma_*}, e_4) \leq -1, \qquad \mathbf{g}(N_{\Sigma_*}, e_a) = O(r^{-1}).$$

13.1.2 Linearized quantities

We make use of the definitions and conventions of section 4.1.1 in Chapter 4. In particular, recall that the following quantities in vanish in Kerr and are thus linear quantities on $\mathcal{M}$.

1. The quantities

$$\widehat{X}, \quad \underline{\widehat{X}}, \quad \Xi, \quad \underline{\Xi}, \quad A, \quad B, \quad \underline{B}, \quad \underline{A}, \quad \nabla(r), \quad e_4(\theta), \quad e_3(\theta), \qquad \mathcal{D}\widehat{\otimes}\mathfrak{J}.$$

2. The renormalization of 0-conformally invariant quantities

$$\begin{aligned}
&\check{H} = H - \frac{aq}{|q|^2}\mathfrak{J}, \qquad \underline{\check{H}} = \underline{H} + \frac{a\overline{q}}{|q|^2}\mathfrak{J}, \qquad \check{P} = P + \frac{2m}{q^3},\\
&\widecheck{\mathcal{D}q} = \mathcal{D}q + a\mathfrak{J}, \qquad \widecheck{\mathcal{D}\overline{q}} = \mathcal{D}\overline{q} - a\mathfrak{J}, \qquad \widecheck{\mathcal{D}(\cos\theta)} = \mathcal{D}(\cos\theta) - i\mathfrak{J},\\
&\widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}} = \overline{\mathcal{D}}\cdot\mathfrak{J} - \frac{4i(r^2+a^2)\cos\theta}{|q|^4}.
\end{aligned}$$

3. The remaining renormalized quantities[2]

$$\begin{aligned}
&\widecheck{\mathrm{tr}X} = \mathrm{tr}X - \frac{2\overline{q}\Delta}{|q|^4}, \quad \widecheck{\mathrm{tr}\underline{X}} = \mathrm{tr}\underline{X} + \frac{2}{\overline{q}}, \quad \check{Z} = Z - \frac{aq}{|q|^2}\mathfrak{J}, \quad \check{\omega} = \omega + \frac{1}{2}\partial_r\left(\frac{\Delta}{|q|^2}\right),\\
&\widecheck{e_3(r)} = e_3(r) + 1, \qquad \widecheck{e_4(r)} = e_4(r) - \frac{\Delta}{|q|^2},\\
&\widecheck{\nabla_3\mathfrak{J}} = \nabla_3\mathfrak{J} - \frac{1}{\overline{q}}\mathfrak{J}, \qquad \widecheck{\nabla_4\mathfrak{J}} = \nabla_4\mathfrak{J} + \frac{\Delta\overline{q}}{|q|^4}\mathfrak{J}.
\end{aligned}$$

[2]With respect to the ingoing renormalization, see Definition 4.1.3.

Finally we provide a definition[3] for Γ_b, Γ_g, $\check{R}_b$ and $\check{R}_g$.

Definition 13.1.1. *We define the quantities* Γ_g, Γ_b

$$\begin{aligned}\Gamma_g &= \Big\{\Xi,\, \check{\omega},\, \widecheck{trX},\, \widehat{X},\, \check{Z},\, \underline{\check{H}},\, \widecheck{tr\underline{X}},\, \widecheck{re_4(r)},\, r^{-1}\nabla(r),\, re_4(\cos\theta),\, r^2\widecheck{\nabla_4\mathfrak{J}}\Big\},\\ \Gamma_b &= \Big\{\check{H},\, \underline{\widehat{X}},\, \underline{\check{\omega}},\, \underline{\Xi},\, r^{-1}\widecheck{e_3(r)},\, \widecheck{\mathcal{D}(\cos\theta)},\, e_3(\cos\theta),\, r\,\widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}},\, r\,\mathcal{D}\widehat{\otimes}\mathfrak{J},\, r\,\widecheck{\nabla_3\mathfrak{J}}\Big\}.\end{aligned}$$

We also define the following sets of curvature quantities.

$$\check{R}_g = \Big\{\check{P},\, B,\, A\Big\}, \qquad \check{R}_b = \Big\{r\underline{B}, \underline{A}\Big\}.$$

For any of the sets $\check{\Gamma} = \Gamma_g, \Gamma_b, \check{R}_g, \check{R}_b$ *we denote by* $\mathfrak{d}^{\leq k}\check{\Gamma}$ *all derivatives up to order k with respect to* $\mathfrak{d} = \{\nabla_3, r\nabla_4, \not{\mathfrak{d}} = r\nabla\}$.

Remark 13.1.2. *In the norms* $\mathfrak{G}_k$ *for Ricci and metric coefficients introduced in section 13.5,* Γ_g *behaves precisely as* $r^{-1}\Gamma_b$*. Also, in the norms* $\mathfrak{R}_k$ *for curvature components introduced in section 13.5,* $\check{R}_g$ *behaves precisely as* $r^{-2}\check{R}_b$*. Since in Part 3 we are only interested in deriving estimates for the top derivatives of curvature using the norms* $\mathfrak{G}_k$ *and* $\mathfrak{R}_k$*, we will often identify*[4] Γ_g *with* $r^{-1}\Gamma_b$ *and* $\check{R}_g$ *with* $r^{-2}\check{R}_b$.

13.1.3 The vectorfields T, Z and $\widehat{R}$

We recall below the definition of $\mathbf{T}, \mathbf{Z}$ in the global frame of $\mathcal{M}$.

$$\begin{aligned}\mathbf{T} &= \frac{1}{2}\left(e_4 + \frac{\Delta}{|q|^2}e_3 - 2a\Re(\mathfrak{J})^b e_b\right),\\ \mathbf{Z} &= \frac{1}{2}\left(2(r^2+a^2)\Re(\mathfrak{J})^b e_b - a(\sin\theta)^2 e_4 - \frac{a(\sin\theta)^2\Delta}{|q|^2}e_3\right)\\ &= (r^2+a^2)\Re(\mathfrak{J})^b e_b - \frac{a\sin\theta}{2}\left(e_4 + \frac{\Delta}{|q|^2}e_3\right).\end{aligned}$$

Remark 13.1.3. *Recall, see* (4.1.14)*, the formula*

$$^{(a)}tr\chi e_3 + {}^{(a)}tr\underline{\chi} e_4 + 2(\eta+\underline{\eta})\cdot {}^*\nabla = \frac{4a\cos\theta}{|q|^2}\mathbf{T} + \Gamma_g\cdot\mathfrak{d}. \tag{13.1.1}$$

[3] The definition here differs from that in section 4.1.2 in that we separate the linearized curvature quantities, denoted $\check{R}_g, \check{R}_b$, from the Ricci and metric coefficients, denoted by Γ_g, Γ_b.

[4] The sole properties for which one might have to distinguish are the low derivatives decay rates in τ that are only needed when estimating quadratic error terms. But, even in that case, we in fact only need $\tau^{-1-\delta_{dec}}$ decay in $\mathcal{M}_{trap}$ which is again consistent with these identifications.

In Part III, we will use the following definition of $\widehat{R}$.

Definition 13.1.4. *We define $\widehat{R}$ in the global frame of $\mathcal{M}$ to be*

$$\widehat{R} \ := \ \frac{1}{2}\left(e_4 - \frac{\Delta}{|q|^2}e_3\right).$$

Remark 13.1.5. *Note that the above definition of $\widehat{R}$ differs by a factor of $\frac{|q|^2}{r^2+a^2}$ from the one used in Part II.*

13.1.4 Conformally invariant operators

Recall, see Lemma 2.2.18, that the conformal operators ${}^{(c)}\nabla_3$, ${}^{(c)}\nabla_4$, ${}^{(c)}\nabla$ for an s-conformally invariant tensor f are defined by

- ${}^{(c)}\nabla_3 f := \nabla_3 f - 2s\underline{\omega} f$ has signature $(s-1)$.
- ${}^{(c)}\nabla_4 f := \nabla_4 f + 2s\omega f$ has signature $(s+1)$.
- ${}^{(c)}\nabla f := \nabla_A f + s\zeta f$ has signature s.

We also introduce the following operator

$$ {}^{(c)}\nabla_{\widehat{R}} := -\frac{|q|^2}{4r}\big(\mathrm{tr}\,\chi\,{}^{(c)}\nabla_3 + \mathrm{tr}\,\underline{\chi}\,{}^{(c)}\nabla_4\big) \tag{13.1.2}$$

which preserves the signature.

Remark 13.1.6. *In view of the definition of $\widehat{R}$, $\widecheck{tr\chi}$ and $\widecheck{tr\underline{\chi}}$, and since $\widecheck{tr\chi}, \widecheck{tr\underline{\chi}} \in \Gamma_g$, we have, for an s-conformally invariant tensor f,*

$$ {}^{(c)}\nabla_{\widehat{R}} f = \nabla_{\widehat{R}} f + s\omega f + r\Gamma_g\cdot(\nabla_3, \nabla_4)f + \Gamma_b f. \tag{13.1.3}$$

We have the following commutator lemma for conformal derivatives.

Lemma 13.1.7. *For any tensor horizontal tensorfield U we have*

$$\begin{aligned}
[{}^{(c)}\nabla_3, {}^{(c)}\nabla_4]U &= O(ar^{-2})\,{}^{(c)}\nabla^{\le 1}U + O(r^{-3})U + r^{-1}\Gamma_b\cdot\mathfrak{d}^{\le 1}U,\\
[{}^{(c)}\nabla_4, {}^{(c)}\mathcal{D}]U &= -\frac{1}{2}trX\,{}^{(c)}\mathcal{D}U + O(ar^{-3})\mathfrak{d}^{\le 1}U + \Xi\,{}^{(c)}\nabla_3 U + r^{-2}\Gamma_b\cdot\mathfrak{d}^{\le 1}U,\\
[{}^{(c)}\nabla_3, {}^{(c)}\mathcal{D}]U &= -\frac{1}{2}tr\underline{X}\,{}^{(c)}\mathcal{D}U + O(ar^{-2})\mathfrak{d}^{\le 1}U + \Gamma_b\cdot\mathfrak{d}^{\le 1}U.
\end{aligned} \tag{13.1.4}$$

Proof. This is an immediate consequence of Lemma 4.2.2. □

As a consequence of Lemma 13.1.7, the definition of $\widehat{R}$ and $\omega = -\frac{1}{2}\partial_r\left(\frac{\Delta}{|q|^2}\right) + \Gamma_g$, we derive the following lemma.

Lemma 13.1.8. *The following commutation relations hold*

$$\begin{aligned}
[\,{}^{(c)}\nabla_3, {}^{(c)}\nabla_{\widehat{R}}]U &= -\omega\,{}^{(c)}\nabla_3 U + O(ar^{-3})\,\mathfrak{\not{d}}^{\le 1}U + O(r^{-3})U + r^{-1}\Gamma_b\cdot\mathfrak{d}^{\le 1}U,\\
[\,{}^{(c)}\nabla_4, {}^{(c)}\nabla_{\widehat{R}}]U &= \frac{\Delta}{|q|^2}\omega\,{}^{(c)}\nabla_3 U + O(ar^{-3})\,\mathfrak{\not{d}}^{\le 1}U + O(r^{-3})U + r^{-1}\Gamma_b\cdot\mathfrak{d}^{\le 1}U,\\
[\,{}^{(c)}\mathcal{D}, {}^{(c)}\nabla_{\widehat{R}}]U &= -\frac{\Delta}{2|q|^2}tr\underline{X}\,{}^{(c)}\mathcal{D}U + O(ar^{-2})\mathfrak{d}^{\le 1}U + \Gamma_b\cdot\mathfrak{d}^{\le 1}U.
\end{aligned} \tag{13.1.5}$$

13.1.5 Properties of the function τ

We recall below Definition 6.1.5.

Definition 13.1.9 (Choice of τ)**.** *Let $\delta_{\mathcal{H}} > 0$ small enough. We choose the smooth scalar function τ on $r \ge r_+(1-\delta_{\mathcal{H}})$ such that we have on $r \ge r_+(1-\delta_{\mathcal{H}})$*

1. *We have for all $r \ge r_+(1-\delta_{\mathcal{H}})$*

$$e_4(\tau) > 0, \qquad e_3(\tau) > 0, \qquad |\nabla\tau|^2 \le \frac{8}{9}e_4(\tau)e_3(\tau).$$

In addition, we have the following asymptotic behavior for r large

$$\frac{m^2}{r^2} \lesssim e_4(\tau) \lesssim \frac{m^2}{r^2}, \qquad 1 \lesssim e_3(\tau) \lesssim 1.$$

2. *The unit normal N_Σ to $\Sigma = \Sigma(\tau)$, normalized by the condition $\mathbf{g}(N_\Sigma, e_3) = -1$, verifies*

$$\mathbf{g}(N_\Sigma, N_\Sigma) \le -\frac{m^2}{8r^2}.$$

3. *Finally, we assume on $\mathcal{M}$*

$$\mathbf{T}(\tau) = 1 + r\Gamma_b, \qquad \nabla(\tau) = a\Re(\mathfrak{J}) + \Gamma_b.$$

13.2 Some structure equations

We recall the following null structure equations with complex notations and conformally invariant operators that will be useful later, see Proposition 2.4.13.

$$
\begin{aligned}
{}^{(c)}\nabla_3\underline{\widehat{X}} + \Re(\mathrm{tr}\underline{X})\underline{\widehat{X}} &= \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\underline{\Xi} + \frac{1}{2}\underline{\Xi}\widehat{\otimes}(H+\underline{H}) - \underline{A},\\
{}^{(c)}\nabla_4\widehat{X} + \Re(\mathrm{tr}X)\widehat{X} &= \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\Xi + \frac{1}{2}\Xi\widehat{\otimes}(\underline{H}+H) - A,
\end{aligned}
$$

$$
\begin{aligned}
{}^{(c)}\nabla_3\underline{H} - {}^{(c)}\nabla_4\underline{\Xi} &= -\frac{1}{2}\overline{\mathrm{tr}\underline{X}}(\underline{H}-H) - \frac{1}{2}\underline{\widehat{X}}\cdot(\overline{\underline{H}}-\overline{H}) + \underline{B},\\
{}^{(c)}\nabla_4 H - {}^{(c)}\nabla_3\Xi &= -\frac{1}{2}\overline{\mathrm{tr}X}(H-\underline{H}) - \frac{1}{2}\widehat{X}\cdot(\overline{H}-\overline{\underline{H}}) - B.
\end{aligned}
$$

Linearizing the equations for H, $\underline{H}$ and writing schematically, we obtain

$$
\begin{aligned}
{}^{(c)}\nabla_3\underline{\widehat{X}} + \Re(\mathrm{tr}\underline{X})\underline{\widehat{X}} &= \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\underline{\Xi} + O(ar^{-2})\underline{\Xi} - \underline{A} + \Gamma_b\cdot\Gamma_b,\\
{}^{(c)}\nabla_4\widehat{X} + \Re(\mathrm{tr}X)\widehat{X} &= \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\Xi + O(ar^{-2})\Xi - A + \Gamma_b\cdot\Gamma_g,\\
{}^{(c)}\nabla_3\underline{\check{H}} - {}^{(c)}\nabla_4\underline{\Xi} &= \underline{B} + O(r^{-1})\Gamma_b + \Gamma_b\cdot\Gamma_b,\\
{}^{(c)}\nabla_4\check{H} - {}^{(c)}\nabla_3\Xi &= -B + O(r^{-1})\Gamma_b + \Gamma_b\cdot\Gamma_g.
\end{aligned}
\tag{13.2.1}
$$

For the convenience of the the reader we recall below the Bianchi identities in complex notation, see Proposition 2.4.11.

Remark 13.2.1. *For simplicity we make the simplifying convention below $\Gamma_g = r^{-1}\Gamma_b$ and $\check{R}_g = r^{-2}\check{R}_b$, see Remark 13.1.2. This is convenient for the interior estimates in chapter 4. We note however that in Chapter 16 we will need to rely on the more precise structure of the equations as stated in Proposition 2.4.11.*

$$
\begin{aligned}
{}^{(c)}\nabla_3 A + \frac{1}{2}\mathrm{tr}\underline{X} A &= \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes} B + O(ar^{-2})B + O(r^{-3})\widehat{X} + r^{-2}\Gamma_b\cdot\check{R}_b,\\
{}^{(c)}\nabla_4 B + 2\overline{\mathrm{tr}X} B &= \frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot A + O(ar^{-2})A + O(r^{-3})\Xi + r^{-2}\Gamma_b\cdot\check{R}_b,\\
{}^{(c)}\nabla_3 B + \mathrm{tr}\underline{X} B &= {}^{(c)}\mathcal{D}\overline{P} + 3\overline{P}H + r^{-2}\Gamma_b\cdot\check{R}_b,\\
{}^{(c)}\nabla_4 P + \frac{3}{2}\mathrm{tr}X P &= \frac{1}{2}\,{}^{(c)}\mathcal{D}\cdot\overline{B} + O(ar^{-2})B + r^{-2}\Gamma_b\cdot\check{R}_b,\\
{}^{(c)}\nabla_3 P + \frac{3}{2}\overline{\mathrm{tr}\underline{X}} P &= -\frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot\underline{B} + O(ar^{-2})\underline{B} + r^{-1}\Gamma_b\cdot\check{R}_b,\\
{}^{(c)}\nabla_4 \underline{B} + \mathrm{tr}X\underline{B} &= -\,{}^{(c)}\mathcal{D}P - 3P\,\underline{H} + r^{-1}\Gamma_b\cdot\check{R}_b,\\
{}^{(c)}\nabla_3 \underline{B} + 2\overline{\mathrm{tr}\underline{X}}\,\underline{B} &= -\frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A} + O(ar^{-2})\underline{A} - O(r^{-3})\,\underline{\Xi} + \Gamma_b\cdot\check{R}_b,\\
{}^{(c)}\nabla_4 \underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A} &= -\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\underline{B} + O(ar^{-2})\underline{B} + O(r^{-3})\widehat{\underline{X}} + \Gamma_b\cdot\check{R}_b.
\end{aligned}
$$

13.3 Useful commutation formulas

We state some of the commutation formulas which will be used later.

Lemma 13.3.1. *Let $U_A = U_{a_1\dots a_k}$ be a general k-horizontal tensorfield.*

1. *We have*

$$
\begin{aligned}
[\nabla_3, r\nabla_b]U_A &= \big(O(a^2r^{-2}) + r\Gamma_b\big)\nabla_b U_A - \frac{1}{2}r\,{}^{(a)}tr\underline{\chi}\,{}^*\nabla_b U_A + r(\eta_b-\zeta_b)\nabla_3 U_A\\
&\quad + r\underline{\xi}_b\nabla_4 U_A + r\widehat{\underline{\chi}}_{bc}\nabla_c U_A + r\sum_{i=1}^k\Big(-\in_{a_ic}\;{}^*\underline{\beta}_b + \frac{1}{2}\mathbf{B}_{a_ic3b}\Big)U_{a_1\dots}{}^{c}{}_{\dots a_k}.
\end{aligned}
$$

2. *Also, we have*

$$
\begin{aligned}
[\nabla_4, r\nabla_b]U_A &= \big(O(a^2r^{-2}) + r\Gamma_g\big)\nabla_b U_A - \frac{1}{2}r\,{}^{(a)}tr\chi\,{}^*\nabla_b U_A + r(\underline{\eta}_b+\zeta_b)\nabla_4 U_A\\
&\quad + r\xi_b\nabla_3 U_A + r\widehat{\chi}_{bc}\nabla_c U_A + r\sum_{i=1}^k\Big(\in_{a_ic}\;{}^*\beta_b + \frac{1}{2}\mathbf{B}_{a_ic4b}\Big)U_{a_1\dots}{}^{c}{}_{\dots a_k}.
\end{aligned}
$$

Proof. According to Lemma 2.2.7 we have

$$\begin{aligned}
[\nabla_3, r\nabla_b]U_A &= r[\nabla_3, \nabla_b]U_A + e_3(r)\nabla_b U_a \\
&= -\frac{1}{2} r \mathrm{tr}\,\underline{\chi}\nabla_b U_A + e_3(r)\nabla_b U_A - \frac{1}{2} r\, {}^{(a)}\mathrm{tr}\underline{\chi}\, {}^*\nabla_b U_A + r(\eta_b - \zeta_b)\nabla_3 U_A + \\
& r\underline{\xi}_b \nabla_4 U_A + r\widehat{\underline{\chi}}_{bc}\nabla_c U_A + r\sum_{i=1}^k \Big(- \in_{a_i c} \ {}^*\underline{\beta}_b + \frac{1}{2}\mathbf{B}_{a_i c 3 b}\Big) U_{a_1 \dots}{}^c{}_{\dots a_k}.
\end{aligned}$$

Since $e_3(r) = -1 + r\Gamma_b$ and $\mathrm{tr}\,\underline{\chi} = -\frac{2}{r} + O(a^2 r^{-3}) + \Gamma_g$, we deduce

$$-\frac{1}{2} r \mathrm{tr}\,\underline{\chi}\nabla_b U_A + e_3(r)\nabla_b U_A = \big(O(ar^{-2}) + r\Gamma_b\big)\nabla_b U_A$$

which implies the first commutator identity of the lemma.

Similarly, since $e_4(r) = \frac{\Delta}{|q|^2} + \Gamma_g$ and $\mathrm{tr}\,\chi = \frac{2\Delta}{r|q|^2} + O(a^2 r^{-3}) + \Gamma_g$, we have

$$-\frac{1}{2} r \mathrm{tr}\,\chi\nabla_b U_A + e_4(r)\nabla_b U_A = \big(O(a^2 r^{-2}) + r\Gamma_g\big)\nabla_b U_A$$

which together with the formula for $[\nabla_4, \nabla_b]U_A$ in Lemma 2.2.7 implies the second commutator identity of the lemma. □

We infer the following corollary.

Corollary 13.3.2. *Given U a general k horizontal tensor, we have*

$$\begin{aligned}
[\nabla_3, r\nabla_b]U_A &= \big(O(ar^{-1}) + r\Gamma_b\big)\nabla_3 U + O(ar^{-2})\,\not{\mathfrak{d}}^{\le 1} U + \Gamma_b \cdot \mathfrak{d}^{\le 1} U + \check{R}_b \cdot U, \\
[\nabla_4, r\nabla_b]U &= O(ar^{-2})\mathfrak{d}^{\le 1} U + r\Gamma_g \nabla_3 U + \Gamma_g \cdot \mathfrak{d}^{\le 1} U + r^{-1}\check{R}_b \cdot U.
\end{aligned}$$

Proof. Recall, see Proposition 2.2.4, that we have

$$\begin{aligned}
\mathbf{B}_{abc3} &= -\mathrm{tr}\,\underline{\chi}\big(\delta_{ca}\eta_b - \delta_{cb}\eta_a\big) - {}^{(a)}\mathrm{tr}\underline{\chi}\big(\in_{ca} \eta_b - \in_{cb} \eta_a\big) + 2\big(- \widehat{\underline{\chi}}_{ca}\eta_b + \widehat{\underline{\chi}}_{cb}\eta_a - \chi_{ca}\underline{\xi}_b + \chi_{cb}\underline{\xi}_a\big), \\
&= O(r^{-1})(O(ar^{-2}) + \Gamma_b) + \Gamma_b \cdot \Gamma_b, \\
\mathbf{B}_{abc4} &= -\mathrm{tr}\,\chi\big(\delta_{ca}\underline{\eta}_b - \delta_{cb}\underline{\eta}_a\big) - {}^{(a)}\mathrm{tr}\chi\big(\in_{ca} \underline{\eta}_b - \in_{cb} \underline{\eta}_a\big) + 2\big(- \widehat{\chi}_{ca}\underline{\eta}_b + \widehat{\chi}_{cb}\underline{\eta}_a - \underline{\chi}_{ca}\xi_b + \underline{\chi}_{cb}\xi_a\big) \\
&= O(r^{-1})(O(ar^{-2}) + \Gamma_g) + \Gamma_b \cdot \Gamma_g.
\end{aligned}$$

Hence

$$\begin{aligned}
[\nabla_3, r\nabla_b]U_A &= \big(O(ar^{-2}) + r\Gamma_b\big)\nabla_b U_A - \frac{1}{2} r\, {}^{(a)}\mathrm{tr}\underline{\chi}\, {}^*\nabla_b U_A + r(\eta_b - \zeta_b)\nabla_3 U_A \\
& + r\underline{\xi}_b \nabla_4 U_A + r\widehat{\underline{\chi}}_{bc}\nabla_c U_A + r\sum_{i=1}^k \Big(- \in_{a_i c} \ {}^*\underline{\beta}_b + \frac{1}{2}\mathbf{B}_{a_i c 3 b}\Big) U_{a_1 \dots}{}^c{}_{\dots a_k} \\
&= \big(O(ar^{-3}) + \Gamma_b\big)\,\not{\mathfrak{d}} U + r\eta\nabla_3 U + \Gamma_b \mathfrak{d} U + \check{R}_b \cdot U + (O(ar^{-2}) + \Gamma_b) \cdot U \\
&= r\eta\nabla_3 U + O(ar^{-2})\,\not{\mathfrak{d}}^{\le 1} U + \Gamma_b \cdot \mathfrak{d}^{\le 1} U + \check{R}_b \cdot U
\end{aligned}$$

and hence

$$[\nabla_3, r\nabla_b]U_A = \big(O(ar^{-1}) + r\Gamma_b\big)\nabla_3 U + O(ar^{-2})\,\mathfrak{\not{d}}^{\leq 1}U + \Gamma_b \cdot \mathfrak{d}^{\leq 1}U + \check{R}_b \cdot U$$

as stated.

Similarly

$$\begin{aligned}
[\nabla_4, r\nabla_b]U_A &= \big(O(a^2r^{-2}) + r\Gamma_g\big)\nabla_b U_A - \frac{1}{2}r\,{}^{(a)}\mathrm{tr}\chi\,{}^*\nabla_b U_A + r(\underline{\eta}_b + \zeta_b)\nabla_4 U_A\\
&\quad + r\xi_b\nabla_3 U_A + r\widehat{\chi}_{bc}\nabla_c U_A + r\sum_{i=1}^{k}\Big(\in_{a_i c}\ {}^*\beta_b + \frac{1}{2}\mathbf{B}_{a_i c 4 b}\Big)U_{a_1\ldots}{}^{c}{}_{\ldots a_k}\\
&= \big(O(a^2r^{-3}) + \Gamma_g\big)\,\mathfrak{\not{d}}U + O(ar^{-2})\mathfrak{d}U + r^{-1}\check{R}_b \cdot U + (O(ar^{-2}) + \Gamma_g)\cdot U\\
&= O(ar^{-2})\mathfrak{d}^{\leq 1}U + r\Gamma_g\nabla_3 U + \Gamma_g \cdot \mathfrak{d}^{\leq 1}U + r^{-1}\check{R}_b \cdot U
\end{aligned}$$

as stated. □

13.3.1 Commutation formulas with $\not{\mathcal{L}}_{\mathbf{T}}$

We recall below some the following commutation Lemma with $\not{\mathcal{L}}_{\mathbf{T}}$. The definition of $\not{\mathcal{L}}_X$ for an arbitrary vectorfield X was given in section 2.2.8.

Lemma 13.3.3. *For a horizontal covariant k-tensor U we have*

$$\begin{aligned}
\nabla_b(\not{\mathcal{L}}_{\mathbf{T}}U_A) - \not{\mathcal{L}}_{\mathbf{T}}(\nabla_b U_A) &= r^{-1}\mathfrak{d}\Gamma_b \cdot U,\\
\nabla_4(\not{\mathcal{L}}_{\mathbf{T}}U_A) - \not{\mathcal{L}}_{\mathbf{T}}(\nabla_4 U_A) &= r^{-1}\mathfrak{d}(\Gamma_b \cdot U),\\
\nabla_3(\not{\mathcal{L}}_{\mathbf{T}}U_A) - \not{\mathcal{L}}_{\mathbf{T}}(\nabla_3 U_A) &= \mathfrak{d}(\Gamma_b \cdot U).
\end{aligned}$$

Proof. The proof, based on Lemma 2.2.13 and the fact that ${}^{(\mathbf{T})}\pi \in \Gamma_b$, see Lemma 4.3.2, appears in the proof of Lemma 9.2.1. □

13.3.2 Commutation formulas with $\overline{q}\nabla_4$ and $q\nabla_4$

Lemma 13.3.4. *The following commutation formulas hold true for an arbitrary horizontal tensor U*

$$\begin{aligned}
[\,^{(c)}\nabla_{\overline{q}e_4},\,^{(c)}\nabla_3]U &= -\frac{1}{2}tr\underline{X}\,^{(c)}\nabla_{\overline{q}e_4}U + O(ar^{-1})\nabla^{\leq 1}U + O(r^{-2})U + \Gamma_b\cdot\mathfrak{d}^{\leq 1}U,\\
[\,^{(c)}\nabla_{\overline{q}e_4},\,^{(c)}\nabla_4]U &= -\frac{1}{2}\overline{trX}\,^{(c)}\nabla_{\overline{q}e_4}U + r^{-1}\Gamma_b\cdot\mathfrak{d}^{\leq 1}U,\\
[\,^{(c)}\nabla_{qe_4},\,^{(c)}\nabla_3]U &= -\frac{1}{2}\overline{tr\underline{X}}\,^{(c)}\nabla_{qe_4}U + O(ar^{-1})\nabla^{\leq 1}U + O(r^{-2})U + \Gamma_b\cdot\mathfrak{d}^{\leq 1}U,\\
[\,^{(c)}\nabla_{qe_4},\,^{(c)}\nabla_4]U &= -\frac{1}{2}trX\,^{(c)}\nabla_{qe_4}\,^{(c)}\nabla_4U + r^{-1}\Gamma_b\cdot\mathfrak{d}^{\leq 1}U.
\end{aligned}\tag{13.3.1}$$

Also,

$$\begin{aligned}
[q\,^{(c)}\nabla_4,\,^{(c)}\mathcal{D}]U &= -\frac{1}{2}trXq\,^{(c)}\mathcal{D}U + O(ar^{-2})\mathfrak{d}^{\leq 1}U + r\Xi\,^{(c)}\nabla_3U + r^{-1}\Gamma_b\cdot\mathfrak{d}^{\leq 1}U,\\
[\overline{q}\,^{(c)}\nabla_4,\,^{(c)}\mathcal{D}]U &= -\frac{1}{2}trX\overline{q}\,^{(c)}\mathcal{D}U + O(ar^{-2})\mathfrak{d}^{\leq 1}U + r\Xi\,^{(c)}\nabla_3U + r^{-1}\Gamma_b\cdot\mathfrak{d}^{\leq 1}U,\\
[\overline{q}\,^{(c)}\nabla_4,\,\overline{^{(c)}\mathcal{D}}]U &= -\frac{1}{2}\overline{trX}\,\overline{q}\,\overline{^{(c)}\mathcal{D}}U + O(ar^{-2})\mathfrak{d}^{\leq 1}U + r\overline{\Xi}\,^{(c)}\nabla_3U + r^{-1}\Gamma_b\cdot\mathfrak{d}^{\leq 1}U,\\
[q\,^{(c)}\nabla_4,\,\overline{^{(c)}\mathcal{D}}]U &= -\frac{1}{2}\overline{trX}\,q\,\overline{^{(c)}\mathcal{D}}U + O(ar^{-2})\mathfrak{d}^{\leq 1}U + r\overline{\Xi}\,^{(c)}\nabla_3U + r^{-1}\Gamma_b\cdot\mathfrak{d}^{\leq 1}U.
\end{aligned}\tag{13.3.2}$$

Proof. We have

$$[\,^{(c)}\nabla_{\overline{q}e_4},\,^{(c)}\nabla_3]U = \overline{q}[\,^{(c)}\nabla_4,\,^{(c)}\nabla_3]U - e_3(\overline{q})\,^{(c)}\nabla_4U.$$

Making use of $\nabla_3\overline{q} = \frac{1}{2}\mathrm{tr}\underline{X}\overline{q} + r\Gamma_b$ and Lemma 13.1.7 we deduce

$$\begin{aligned}
[\,^{(c)}\nabla_{\overline{q}e_4},\,^{(c)}\nabla_3]U &= \left(-\frac{1}{2}\mathrm{tr}\underline{X}\overline{q} + r\Gamma_b\right)\cdot\,^{(c)}\nabla_4U + O(ar^{-1})\nabla^{\leq 1}U + O(r^{-2})U + \Gamma_b\cdot\mathfrak{d}^{\leq 1}U\\
&= -\frac{1}{2}\mathrm{tr}\underline{X}\,^{(c)}\nabla_{\overline{q}e_4}U + O(ar^{-1})\nabla^{\leq 1}U + O(r^{-2})U + \Gamma_b\cdot\mathfrak{d}^{\leq 1}U
\end{aligned}$$

as stated. Similarly, using $\nabla_4\overline{q} = \frac{1}{2}\overline{\mathrm{tr}X}\overline{q} + \Gamma_b$, we deduce

$$[\,^{(c)}\nabla_{\overline{q}e_4},\,^{(c)}\nabla_4]U = -\nabla_4(\overline{q})\,^{(c)}\nabla_4U = -\frac{1}{2}\overline{\mathrm{tr}X}\,^{(c)}\nabla_{\overline{q}e_4}U + r^{-1}\Gamma_b\cdot\mathfrak{d}^{\leq 1}U.$$

The other results in (13.3.1) follow by conjugation.

To check (13.3.2) we make use of the formula for $[\,^{(c)}\nabla_4, \,^{(c)}\nabla]U$ in Lemma 13.1.7 and the fact that $\mathcal{D}q = -a\mathfrak{J} + \Gamma_b + r\Gamma_g = O(ar^{-1}) + \Gamma_b$ we derive

$$\begin{aligned}
[q\,^{(c)}\nabla_4, \,^{(c)}\mathcal{D}]U &= q[\,^{(c)}\nabla_4, \,^{(c)}\mathcal{D}]U - \mathcal{D}(q)\cdot\,^{(c)}\nabla_4 U\\
&= -\frac{1}{2}\mathrm{tr}X q\,^{(c)}\mathcal{D}U + O(ar^{-2})\mathfrak{d}^{\leq 1}U + r\Xi\,^{(c)}\nabla_3 U + r^{-1}\Gamma_b\cdot\mathfrak{d}^{\leq 1}U.
\end{aligned}$$

The second identity in (13.3.2) is derived in the same way, and the last two identities in (13.3.2) follow by conjugation. This concludes the proof of Lemma 13.3.4. □

13.4 Non-integrable Hodge estimates

The following is an immediate corollary of Proposition 9.3.2.

Corollary 13.4.1. *Given $f \in \mathfrak{s}_p$, $p = 0, 1, 2$, and $\mathcal{D}$ any of the Hodge operators acting on f. Then, for any q,*

$$\int_S r^q|\nabla f|^2 \lesssim \int_S r^q|\mathcal{D}f|^2 + \int_S r^{q-2}|f|^2 + O(a^2,\epsilon^2)\int_S r^{q-2}|\mathfrak{d}f|^2. \tag{13.4.1}$$

Also, for higher derivatives,

$$\int_S r^q|\nabla\mathfrak{d}^{\leq k}f|^2 \lesssim \int_S r^q\Big(|\mathfrak{d}^{\leq k}\mathcal{D}f|^2 + r^{-2}|\mathfrak{d}^{\leq k}f|^2\Big) + O(a^2,\epsilon^2)\int_S r^{q-2}|\mathfrak{d}^{\leq k+1}f|^2. \tag{13.4.2}$$

13.5 Main norms

We recall here the main norms appearing in section 9.4.1 of [53] in connection with the higher order curvatures estimates of Theorem 9.4.10 in [53].

13.5.1 Norms of $^{(ext)}\mathcal{M}$

Definition 13.5.1. *We define the following norms for the Ricci coefficients in $^{(ext)}\mathcal{M}$ with respect to the global frame of $\mathcal{M}$*

$$\begin{aligned}
^{(ext)}\mathfrak{G}_k^2 &:= \sup_{\lambda\geq r_0}\int_{r=\lambda}\Big(r^2\big|\mathfrak{d}^{\leq k}(\Gamma_g\setminus\{\widetilde{tr\underline{X}}\})|^2 + \big|\mathfrak{d}^{\leq k}(\Gamma_b\setminus\{\Xi\})\big|^2\\
&\qquad + r^{2-\delta_B}|\mathfrak{d}^{\leq k}\widetilde{tr\underline{X}}|^2 + r^{-\delta_B}|\mathfrak{d}^{\leq k}\underline{\Xi}|^2\Big).
\end{aligned} \tag{13.5.1}$$

For convenience we introduce the notation

$$\Gamma'_b := \Gamma_b \setminus \{\underline{\Xi}\}, \qquad \Gamma'_g := \Gamma_g \setminus \{\widecheck{tr\underline{X}}\}. \tag{13.5.2}$$

Definition 13.5.2. *We define the following norms for the curvature coefficients in ${}^{(ext)}\mathcal{M}$, with respect to the global frame of $\mathcal{M}$.*

$$\begin{aligned} {}^{(ext)}\mathfrak{R}_k^2 := \int_{{}^{(ext)}\mathcal{M}} & r^{3+\delta_B}|\mathfrak{d}^{\leq k}(A,B)|^2 \\ & + r^{3-\delta_B}\big(|\mathfrak{d}^{\leq k}\check{P}|^2 + r^{-2}|\mathfrak{d}^{\leq k}\underline{B}|^2 + r^{-4}|\mathfrak{d}^{\leq k}\underline{A}|^2\big). \end{aligned} \tag{13.5.3}$$

13.5.2 Norms of ${}^{(int)}\mathcal{M}$

Definition 13.5.3. *We define the following norms for the Ricci coefficients in ${}^{(int)}\mathcal{M}$ with respect to the global frame of $\mathcal{M}$*

$${}^{(int)}\mathfrak{G}_k^2 := \int_{{}^{(int)}\mathcal{M}} \big|\mathfrak{d}^{\leq k}\check{\Gamma}\big|^2,$$

where $\check{\Gamma}$ denotes the set of all linearized Ricci and metric coefficients with respect to the global frame of $\mathcal{M}$, i.e.

$$\begin{aligned} \check{\Gamma} \;\; := \;\; & \Big\{\underline{\Xi},\, \underline{\omega},\, \widecheck{tr\underline{X}},\, \widehat{\underline{X}},\, \widecheck{e_3(r)},\, e_3(\cos\theta),\, \widecheck{\nabla_3\mathfrak{J}},\, \check{Z},\, \widecheck{\underline{H}},\, \check{H},\, \mathcal{D}r,\, \widecheck{\mathcal{D}\cos\theta},\, \widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}},\, \mathcal{D}\widehat{\otimes}\mathfrak{J}, \\ & \widecheck{trX},\, \widehat{X},\, \check{\omega},\, \widecheck{e_4(r)},\, e_4(\cos\theta),\, \widecheck{\nabla_4\mathfrak{J}},\, \Xi\Big\}. \end{aligned}$$

For the curvature norms in ${}^{(int)}\mathcal{M}$, we rely in particular on the scalar function τ of Definition 13.1.9. We also recall the definition, see Definition 9.1.2, of the non trapped region of ${}^{(int)}\mathcal{M}$ given by

$${}^{(int)}\mathcal{M}_{\not{trap}} \;\; := \;\; {}^{(int)}\mathcal{M} \cap \left\{\frac{|\mathcal{T}|}{r^3} \geq \delta_{trap}\right\}, \tag{13.5.4}$$

where $\mathcal{T}$ is the polynomial in r defined in (3.8.5), i.e.

$$\mathcal{T} = r^3 - 3mr^2 + a^2r + ma^2,$$

and where we choose $\delta_{trap} = \frac{1}{10}$ as in Lemma 6.1.12.

Definition 13.5.4. *We define the following norms for the null curvature components in* ${}^{(int)}\mathcal{M}$

$$
\begin{aligned}
{}^{(int)}\mathfrak{R}_k^2 &= \int_{{}^{(int)}\mathcal{M}} \left(\left|\nabla_{\widehat{R}} \mathfrak{d}^{\leq k-1} \check{R}\right|^2 + |\mathfrak{d}^{\leq k-1} \check{R}|^2 \right) + \int_{{}^{(int)}\mathcal{M}_{\not{trap}}} \left|\mathfrak{d}^{\leq k} \check{R}\right|^2 \\
&\quad + \sup_\tau \int_{{}^{(int)}\mathcal{M} \cap \Sigma(\tau)} |\mathfrak{d}^{\leq k} \check{R}|^2,
\end{aligned}
$$

where, see Definition 13.1.4, $\widehat{R} = \frac{1}{2}\left(e_4 - \frac{\Delta}{|q|^2} e_3\right)$, *and where* $\check{R}$ *denotes the set of all linearized curvature components with respect to the global frame of* $\mathcal{M}$, *i.e.*

$$
\check{R} := \left\{ \underline{A},\ \underline{B},\ \check{P},\ B,\ A \right\}.
$$

Global norms

We define the global norms of $\mathcal{M}$ as follows

$$
\mathfrak{G}_k = {}^{(ext)}\mathfrak{G}_k + {}^{(int)}\mathfrak{G}_k, \qquad \mathfrak{R}_k = {}^{(ext)}\mathfrak{R}_k + {}^{(int)}\mathfrak{R}_k. \tag{13.5.5}
$$

The following lemma will be used frequently in this chapter.

Lemma 13.5.5. *The following estimates hold true:*

1. *We have, in* ${}^{(ext)}\mathcal{M}$, *for* $p > 1 + \delta_B$.

$$
\int_{{}^{(ext)}\mathcal{M}} \left(r^{-p} |\mathfrak{d}^{\leq k} \Gamma_b|^2 + r^{-p+2} |\mathfrak{d}^{\leq k} \Gamma_g|^2 \right) \lesssim r_0^{1-p+\delta_B}\, {}^{(ext)}\mathfrak{G}_k^2. \tag{13.5.6}
$$

2. *We also have the stronger estimate for the notations* Γ'_b *and* Γ'_g *introduced in* (13.5.2), *for* $p > 1$,

$$
\int_{{}^{(ext)}\mathcal{M}} \left(r^{-p} |\mathfrak{d}^{\leq k} \Gamma'_b|^2 + r^{-p+2} |\mathfrak{d}^{\leq k} \Gamma'_g|^2 \right) \lesssim r_0^{1-p}\, {}^{(ext)}\mathfrak{G}_k^2. \tag{13.5.7}
$$

3. *We have*

$$
\begin{aligned}
&\sup_{\tau \in [1, \tau_*]} \int_{\Sigma(\tau)} r^{3-\delta_B} \left(|\mathfrak{d}^{\leq k} \check{R}_g|^2 + r^{-4} |\mathfrak{d}^{\leq k} \check{R}_b|^2 \right) \\
&\quad + \int_{\mathcal{A} \cup \Sigma_*} r^{3-\delta_B} \left(|\mathfrak{d}^{\leq k} \check{R}_g|^2 + r^{-4} |\mathfrak{d}^{\leq k} \check{R}_b|^2 \right) \lesssim \mathfrak{R}_{k+1} \mathfrak{R}_k,
\end{aligned} \tag{13.5.8}
$$

and, for $p > 1$,

$$\begin{aligned}
&\sup_{\tau\in[1,\tau_*]}\int_{\Sigma(\tau)}\Big(r^{-p}|\mathfrak{d}^{\leq k}\Gamma'_b|^2+r^{-p+2}|\mathfrak{d}^{\leq k}\Gamma'_g|^2\Big)\\
&+\int_{\mathcal{A}\cup\Sigma_*}\Big(r^{-p}|\mathfrak{d}^{\leq k}\Gamma'_b|^2+r^{-p+2}|\mathfrak{d}^{\leq k}\Gamma'_g|^2\Big)\\
&+\sup_{\tau\in[1,\tau_*]}\int_{\Sigma(\tau)}\Big(r^{-p-\delta_B}|\mathfrak{d}^{\leq k}\underline{\Xi}|^2+r^{-p-\delta_B+2}|\mathfrak{d}^{\leq k}\widetilde{tr\underline{X}}|^2\Big)\\
&+\int_{\mathcal{A}\cup\Sigma_*}\Big(r^{-p-\delta_B}|\mathfrak{d}^{\leq k}\underline{\Xi}|^2+r^{-p-\delta_B+2}|\mathfrak{d}^{\leq k}\widetilde{tr\underline{X}}|^2\Big)\ \lesssim\ \mathfrak{G}_{k+1}\mathfrak{G}_k. \qquad (13.5.9)
\end{aligned}$$

Proof. We write for $p > 1+\delta_B$, in view of the definition of ${}^{(ext)}\mathfrak{G}$ norms.

$$\int_{{}^{(ext)}\mathcal{M}} r^{-p}|\mathfrak{d}^{\leq k}\Gamma_b|^2 \ \lesssim\ \int_{r_0}^{\infty}\lambda^{-p+\delta_B}\left(\int_{r=\lambda} r^{-\delta_B}|\mathfrak{d}^{\leq k}\Gamma_b|^2\right)d\lambda \lesssim r_0^{1-p+\delta_B}\,{}^{(ext)}\mathfrak{G}_k^2,$$

and similarly for Γ_g. Also, we have, for $p > 1$,

$$\int_{{}^{(ext)}\mathcal{M}} r^{-p}|\mathfrak{d}^{\leq k}\Gamma'_b|^2 \ \lesssim\ \int_{r_0}^{\infty}\lambda^{-p}\Big(\int_{r=\lambda}|\mathfrak{d}^{\leq k}\Gamma'_b|^2\Big)d\lambda \lesssim r_0^{1-p}\,{}^{(ext)}\mathfrak{G}_k^2,$$

and similarly for Γ'_g. Finally, the last estimates of the lemma follow from the standard trace theorem, using the definition of $\mathfrak{R}_k$ for curvature and the two first estimates for Γ_b. □

Initial data norms

The initial data is set on the spacelike hypersurface Σ_1 as follows.

Definition 13.5.6. *We define the following initial data norms on* Σ_1

$$\begin{aligned}
\mathfrak{I}_k :=& \sup_{S\subset\Sigma_1} r^{\frac{5}{2}+\delta_B}\Big(\big\|\mathfrak{d}^k\,(A,B)\big\|_{L^2(S)}+\big\|\mathfrak{d}^k\,B\big\|_{L^2(S)}\Big)\\
&+\sup_{S\subset\Sigma_1}\Big(r^2\big\|\mathfrak{d}^k\,\check{P}\big\|_{L^2(S)}+r\big\|\mathfrak{d}^k\,\underline{B}\big\|_{L^2(S)}+\big\|\mathfrak{d}^k\,\underline{A}\big\|_{L^2(S)}\Big).
\end{aligned} \qquad (13.5.10)$$

13.6 Top curvature estimates for Theorem M8

13.6.1 Main assumptions

The goal of this part, i.e. Part III, is to provide the control of high order curvature estimates needed to complete the proof of Theorem M8 of [53]. In this section, we recall the assumptions on which this proof rests.

Control of the initial data

We have the following control of the initial data norm of Definition 13.5.6

$$\mathfrak{I}_{k_L} \lesssim \epsilon_0. \tag{13.6.1}$$

This results has been proved in Theorem 9.4.12 of [53].

Bootstrap assumptions

Relative to the global norms (13.5.5), we make the following bootstrap assumption

$$\mathfrak{G}_k + \mathfrak{R}_k \quad \leq \quad \epsilon, \quad k \leq k_L, \tag{13.6.2}$$

see the bootstrap assumption (9.4.32) in [53].

In addition, we make a bootstrap assumption on decay for low derivatives, weaker that the corresponding one in [53], to deal with trapping. Recall the scalar function τ_{trap} defined by

$$\tau_{trap} := \begin{cases} 1+\tau & \text{on} \quad \mathcal{M}_{trap}, \\ 1 & \text{on} \quad \mathcal{M}_{\not{trap}}. \end{cases}$$

Then, we assume that (Γ_g, Γ_b) and $(A, B, \check{P}, \check{R}_b)$ satisfy the following estimates on $\mathcal{M}$

$$r^{\frac{7}{2}+\delta_{dec}}|\mathfrak{d}^{\leq k}(A,B)| + r^3|\mathfrak{d}^{\leq k}\check{P}| + r^2|\mathfrak{d}^{\leq k}\Gamma_g| + r|\mathfrak{d}^{\leq k}(\check{R}_b, \Gamma_b)| \leq \frac{\epsilon}{\tau_{trap}^{1+\delta_{dec}}}, \quad k \leq \frac{k_L}{2}. \tag{13.6.3}$$

Iteration assumption

We make the iteration assumption for J in the range $\frac{k_L}{2} \leq J \leq k_L - 1$,

$$\mathfrak{G}_J + \mathfrak{R}_J \lesssim \epsilon_J, \tag{13.6.4}$$

see the iteration assumption (9.4.33) in [53].

Remark 13.6.1. *We refer to (9.4.34) in [53] for the specific choice of ϵ_J. We do not recall it here since it is irrelevant for the statements and proofs of Part III.*

Identities satisfied by the global frame of $\mathcal{M}$

We assume that the following identities hold for the global frame of $\mathcal{M}$

$$\Xi = 0, \qquad \widecheck{\underline{H}} = 0, \quad \text{for} \quad r \geq r_0 + 1. \tag{13.6.5}$$

Remark 13.6.2. *The global frame used in Part III is constructed in section 9.4 of [53] and indeed satisfies* (13.6.5)*. The identity $\widecheck{\underline{H}} = 0$ is only used for the control of $\underline{\mathfrak{q}}$ in Theorem 14.1.6, while the identity $\Xi = 0$ is used for the control of $\underline{\mathfrak{q}}$ in Theorem 14.1.6 and implies in particular $\Xi \in r^{-1}\Gamma_g$ which is used to control the error terms of some commutators appearing in Part III.*

13.6.2 Control of high order curvature estimates

The following is our main result of this Part on the control of high order curvature estimate for the proof of Theorem M8 of [53].

Theorem 13.6.3 (Control of Curvature)**.** *Let J such that $\frac{k_L}{2} \leq J \leq k_L - 1$. Assume*

- *the control of initial data in* (13.6.1)*,*
- *the bootstrap assumptions* (13.6.2) *and* (13.6.3)*,*
- *the iteration assumption* (13.6.4)*.*

Then the following estimates hold in $\mathcal{M}$

$$\begin{aligned} {}^{(int)}\mathfrak{R}^2_{J+1} &\lesssim r_0^{18}\Big(\epsilon_J(\mathfrak{G}_{J+1} + \mathfrak{R}_{J+1}) + \epsilon_J^2 + \epsilon_0^2\Big) + |a| r_0^3 \mathfrak{G}^2_{J+1} \\ &\quad + r_0^{\frac{27}{4}} \mathfrak{G}^{\frac{3}{2}}_{J+1}\Big(\epsilon_0 + \sqrt{\epsilon_J}\sqrt{\mathfrak{G}_{J+1} + \mathfrak{R}_{J+1}}\Big)^{\frac{1}{2}}, \qquad (13.6.6) \\ {}^{(ext)}\mathfrak{R}^2_{J+1} &\lesssim r_0^{3+\delta_B}\,{}^{(int)}\mathfrak{R}^2_{J+1} + r_0^{-\delta_B}\,{}^{(ext)}\mathfrak{G}^2_{J+1} + \epsilon_J^2 + \epsilon_0^2, \qquad (13.6.7) \end{aligned}$$

where the constant in $\lesssim$ is independent of r_0.

Remark 13.6.4. (13.6.6) *and* (13.6.7) *imply*

$$\begin{aligned}\mathfrak{R}_{J+1} \lesssim\ & r_0^{21+\delta_B}\epsilon_J + r_0^{\frac{21}{2}+\frac{\delta_B}{2}}\left(\sqrt{\mathfrak{G}_{J+1}}\sqrt{\epsilon_J}+\epsilon_0\right) + \sqrt{|a|}r_0^{3+\frac{\delta_B}{2}}\mathfrak{G}_{J+1}\\ &+r_0^{-\frac{\delta_B}{2}}\,{}^{(ext)}\mathfrak{G}_{J+1} + r_0^{\frac{39}{8}+\frac{\delta_B}{2}}\mathfrak{G}_{J+1}^{\frac{3}{4}}\left(\epsilon_0+\sqrt{\epsilon_J}\sqrt{\mathfrak{G}_{J+1}}\right)^{\frac{1}{4}}\\ &+r_0^{\frac{39}{7}+\frac{4\delta_B}{7}}\mathfrak{G}_{J+1}^{\frac{6}{7}}\epsilon_J^{\frac{1}{7}} \end{aligned} \tag{13.6.8}$$

where the constant in $\lesssim$ is independent of r_0. (13.6.8) *proves Theorem 9.4.15 of [53]. The control of $\mathfrak{R}_{J+1}$ provided by* (13.6.8)*, together with the control for $\mathfrak{G}_{J+1}$ provided by Proposition 9.4.17–9.4.20 in [53], allows to obtain the iteration assumption* (13.6.4) *with J replaced by $J+1$ for ϵ_J given by (9.4.34) in [53]. This iteration procedure then concludes the proof of Theorem M8 of [53], see section 9.4.7 in [53] for the iteration procedure, and section 9.4.3 in [53] for the proof of Theorem M8.*

Remark 13.6.5. *The estimate* (13.6.6) *results in fact from global energy-Morawetz estimates on $\mathcal{M}$ which are then restricted to ${}^{(int)}\mathcal{M}$.*

13.6.3 Structure of the proof of Theorem 13.6.3

Theorem 13.6.3 is proved as follows:

1. **Energy-Morawetz estimates.** Recall from Remark 13.6.5 that the interior estimates (13.6.6) follow in fact from global energy-Morawetz estimates on $\mathcal{M}$ which are then restricted to ${}^{(int)}\mathcal{M}$. These energy-Morawetz estimates are given in Chapters 14 and 15 as follows:

   (a) We start by deriving the following energy-Morawetz estimates for $\check{P}$ in $\mathcal{M}$ stated in Theorem 14.1.3

   $$BEF_\delta^J[r^2\check{P}] \lesssim r_0^{15}\Big(\mathfrak{G}_{J+1}\mathfrak{G}_J + \mathfrak{R}_{J+1}\mathfrak{R}_J + \epsilon_J^2+\epsilon_0^2\Big), \tag{13.6.9}$$

   where the BEF_δ^J norms are defined in section 14.1.1. The proof requires a linearized version of the wave equation verified by P, see section 5.5, the general Morawetz- Energy estimates for scalar wave equations derived in Part II and recalled here in Proposition 14.1.4, as well as the main energy-Morawetz estimates for $\mathfrak{q}$, derived also in Part II.

(b) Next, we derive the following energy-Morawetz estimates for B and $\underline{B}$ in $\mathcal{M}$ stated in Proposition 15.1.1

$$BEF^J_\delta[r^2 B] + BEF^J_\delta[\underline{B}] \lesssim BEF^J_\delta[r^2 \check{P}] + \epsilon_0^2 + \epsilon_J^2 + |a|\mathfrak{G}^2_{J+1}. \quad (13.6.10)$$

These estimates are based on integral estimates for Bianchi pairs stated in Proposition 15.3.12.

(c) Finally, we derive the following energy-Morawetz estimates for A and $\underline{A}$ in $\mathcal{M}$ stated in Proposition 15.1.2

$$\begin{aligned} BEF^J_\delta[A] &\lesssim BEF^J_\delta[r^2 B], +\epsilon_0^2 + \epsilon_J^2 + |a|\mathfrak{G}^2_{J+1}, \\ BEF^J_\delta[\underline{A}] &\lesssim BEF^J_\delta[\underline{B}] + \epsilon_0^2 + \epsilon_J^2 + |a|\mathfrak{G}^2_{J+1}. \end{aligned} \quad (13.6.11)$$

These estimates are again based on integral estimates for Bianchi pairs stated in Proposition 15.3.12.

The combination of the estimates (13.6.9), (13.6.10) and (13.6.11) then yields (13.6.6), see section 15.1.3.

2. **Exterior estimates.** The exterior estimates (13.6.7) are proved in Chapter 16 based on r^p weighted estimates for Bianchi pairs.

Chapter 14

Energy-Morawetz estimates for $\check{P}$

14.1 Control of $\check{P}$

14.1.1 Morawetz-Energy norms

For the convenience of the reader we recall the following norms, see section 6.1.5,

$$
\begin{aligned}
B^k_\delta[\psi](\tau_1,\tau_2) &= \int_{\mathcal{M}_{trap}(\tau_1,\tau_2)} \big(|\nabla_{\widehat{R}}\mathfrak{d}^{\le k}\psi|^2 + |\mathfrak{d}^{\le k}\psi|^2\big) + \int_{\mathcal{M}_{\not{trap}}(\tau_1,\tau_2)} r^{\delta-3}|\mathfrak{d}^{\le k+1}\psi|^2,\\
E^k_\delta[\psi](\tau) &= \int_{\Sigma(\tau)} \Big(r^\delta\big(|\nabla_4\mathfrak{d}^{\le k}\psi|^2 + r^{-2}|\mathfrak{d}^{\le k}\psi|^2\big) + |\nabla\mathfrak{d}^{\le k}\psi|^2 + r^{-2}|\nabla_3\mathfrak{d}^{\le k}\psi|^2\Big),\\
F^k_\delta[\psi](\tau_1,\tau_2) &= \int_{\Sigma_*(\tau_1,\tau_2)} \Big(r^\delta\big(|\nabla_4\mathfrak{d}^{\le k}\psi|^2 + |\nabla\mathfrak{d}^{\le k}\psi|^2 + r^{-2}|\mathfrak{d}^{\le k}\psi|^2\big) + |\nabla_3\mathfrak{d}^{\le k}\psi|^2\Big)\\
&\quad + \int_{\mathcal{A}(\tau_1,\tau_2)} |\mathfrak{d}^{\le k+1}\psi|^2.
\end{aligned}
$$

Throughout the chapter we set $\tau_1 = 1$ and $\tau_2 = \tau_*$ and we simplify our expressions by writing $\mathcal{M} = \mathcal{M}(1,\tau_*)$, $\mathcal{A} = \mathcal{A}(1,\tau_*)$, $\Sigma_* = \Sigma_*(1,\tau_*)$, $B^k_\delta = B^k_\delta(1,\tau_*)$, $F^k_\delta = F^k_\delta(1,\tau_*)$. We introduce the short hand notation

$$
\begin{aligned}
BEF^k_\delta[\psi] &:= B^k_\delta[\psi] + \sup_{\tau\in[1,\tau_*]} E^k_\delta[\psi](\tau) + F^k_\delta[\psi],\\
EF^k_\delta[\psi] &:= \sup_{\tau\in[1,\tau_*]} E^k_\delta[\psi](\tau) + F^k_\delta[\psi].
\end{aligned}
$$

We also recall the $\mathcal{N}$ norm

$$\mathcal{N}^k[\psi, N] \;\; := \;\; {}^{(mor)}\mathcal{N}^k[\psi, N] + {}^{(red)}\mathcal{N}^k[\psi, N] + {}^{(en)}\mathcal{N}^k[\psi, N]$$

where

$$\begin{aligned}
{}^{(mor)}\mathcal{N}^k[\psi, N] &:= \int_{\mathcal{M}(\tau_1,\tau_2)} \big(|\mathfrak{d}^{\leq k}\nabla_{\widehat{R}}\psi| + r^{-1}|\mathfrak{d}^{\leq k}\psi|\big)|\mathfrak{d}^{\leq k}N|,\\
{}^{(red)}\mathcal{N}^k[\psi, N] &:= \int_{{}^{(red)}\mathcal{M}(\tau_1,\tau_2)} \big|\mathfrak{d}^{\leq k+1}\psi\big|\big|\mathfrak{d}^{\leq k}N\big|,\\
{}^{(en)}\mathcal{N}^k[\psi, N] &:= \left|\int_{\mathcal{M}(\tau_1,\tau_2)} \nabla_{\widehat{T}_\delta}\mathfrak{d}^{\leq k}\psi\cdot\mathfrak{d}^{\leq k}N\right|,
\end{aligned}$$

Here, ${}^{(red)}\mathcal{M}$ is the region of $\mathcal{M}$ where $r \leq r_+(1+2\delta_{red})$ with δ_{red} a universal constant verifying $\delta_{\mathcal{H}} \ll \delta_{red} \ll m - |a|$, and $\widehat{T}_\delta$ is given in Definition 6.1.13. Also, the $\mathcal{N}_\delta$ norm is given by

$$\mathcal{N}^k_\delta[\psi, N] \;\; := \;\; \mathcal{N}^k[\psi, N] + \int_{{}^{(ext)}\mathcal{M}} r^\delta\Big(\big|\nabla_4\mathfrak{d}^{\leq k}\psi\big| + r^{-1}\big|\mathfrak{d}^{\leq k}\psi\big|\Big)\big|\mathfrak{d}^{\leq k}N\big|.$$

These norms will be used in this chapter to estimate $\check{P}$. With these goal in mind we need to compare the norms *BEF* with the standard curvature norms $\mathfrak{R}_k$, see section 13.5.

Lemma 14.1.1. *We have*

$$B^{k-1}_\delta[r^2\check{R}_g] + B^{k-1}_\delta[\check{R}_b] \lesssim \mathfrak{R}_k^2, \qquad EF^{k-1}_\delta[r^2\check{R}_g] + EF^{k-1}_\delta[\check{R}_b] \lesssim \mathfrak{R}_k\mathfrak{R}_{k+1}. \quad (14.1.1)$$

Proof. In view of the definition of *BEF* norms, we have

$$\begin{aligned}
B^k_\delta[r^2\check{R}_g] &= \int_{\mathcal{M}_{trap}} \big(|\nabla_{\widehat{R}}\mathfrak{d}^{\leq k}\check{R}_g|^2 + |\mathfrak{d}^{\leq k}\check{R}_g|^2\big) + \int_{{\mathcal{M}}_{\not{trap}}} \Big(r^{1+\delta}|\mathfrak{d}^{\leq k+1}\check{R}_g|^2 + r^{3-\delta}|\mathfrak{d}^{\leq k}\nabla_3\check{R}_g|^2\Big),\\
E^k_\delta[r^2\check{R}_g] &= \sup_\tau\left(\int_{\Sigma(\tau)} r^4\Big(r^\delta\big(|\nabla_4\mathfrak{d}^{\leq k}\check{R}_g|^2 + r^{-2}|\mathfrak{d}^{\leq k}\check{R}_g|^2\big) + |\nabla\mathfrak{d}^{\leq k}\check{R}_g|^2 + r^{-2}|\nabla_3\mathfrak{d}^{\leq k}\check{R}_g|^2\Big)\right),\\
F^k_\delta[r^2\check{R}_g] &= \int_{\Sigma_*} r^4\Big(r^\delta\big(|\nabla_4\mathfrak{d}^{\leq k}\check{R}_g|^2 + |\nabla\mathfrak{d}^{\leq k}\check{R}_g|^2 + r^{-2}|\mathfrak{d}^{\leq k}\check{R}_g|^2\big) + |\nabla_3\mathfrak{d}^{\leq k}\check{R}_g|^2\Big)\\
&\quad + \int_{\mathcal{A}} |\mathfrak{d}^{\leq k+1}\check{R}_g|^2.
\end{aligned}$$

Hence, in view of the definition of the $\mathfrak{R}_k$ norms in section 13.5 and (13.5.8), and the fact that $\check{R}_g = (A, B, \check{P})$, we have $B^{k-1}_\delta[r^2\check{R}_g] \lesssim \mathfrak{R}_k^2$ and $EF^{k-1}[r^2\check{R}_g] \lesssim \mathfrak{R}_k\mathfrak{R}_{k+1}$ as stated. The estimate for $\check{R}_b$ can be proved similarly. $\square$

We will also need to compare the norms *BEF* for Ricci coefficients with those given by $\mathfrak{G}_k$.

Lemma 14.1.2. *The following estimates hold true*

$$B_\delta^{k-1}[\Gamma_b'] + B_\delta^{k-1}[r^{-\delta_B}\Gamma_b] \lesssim \mathfrak{G}_k^2, \qquad EF_\delta^{k-1}[\Gamma_b'] + EF_\delta^{k-1}[r^{-\delta_B}\Gamma_b] \lesssim \mathfrak{G}_k \mathfrak{G}_{k+1}.$$

Proof. The proof follows easily from Lemma 13.5.5. □

14.1.2 Statement of main results on the control of $\check{P}$

We assume that all the bootstrap, initial data assumptions and the induction hypothesis made in section 13.6.1 hold true. The goal of this chapter is to derive the following Morawetz energy estimates for $\check{P}$ in $\mathcal{M} = \mathcal{M}(1, \tau_*)$.

Theorem 14.1.3 (Morawetz-Energy for $\check{P}$)**.** *The following estimates hold true in $\mathcal{M} = \mathcal{M}(1, \tau_*)$, for sufficiently small a,*

$$BEF_\delta^J[r^2\check{P}] \lesssim r_0^{15}\Big(\mathfrak{G}_{J+1}\mathfrak{G}_J + \mathfrak{R}_{J+1}\mathfrak{R}_J + \epsilon_J^2 + \epsilon_0^2\Big). \tag{14.1.2}$$

The proof is based on the conditional weighted estimates for scalar wave equations of Proposition 10.5.1 which we recall below.

Proposition 14.1.4. *Let ψ be a solution to the following scalar wave equation*

$$\square_{\mathbf{g}}\psi + V\psi = N, \tag{14.1.3}$$

where V is real and satisfies $V = O(r^{-3})$ for r large, in a spacetime $\mathcal{M} = \mathcal{M}(1, \tau_*)$ verifying the assumptions of section 13.6.1. Then:

1. *The following conditional Morawetz estimates hold true in $\mathcal{M}$*

$$\begin{aligned} B_\delta^k[\psi] \lesssim\; & EF_\delta^k[\psi] + B_\delta^{k-1}[\psi] + \int_{\mathcal{M}_{trap}} |\mathfrak{d}^{\le k}\psi|^2 + {}^{(mor)}\mathcal{N}^k[\psi, N] + {}^{(red)}\mathcal{N}^k[\psi, N] \\ & + \int_{{}^{(ext)}\mathcal{M}} r^\delta\Big(\big|\nabla_4\mathfrak{d}^{\le k}\psi\big| + r^{-1}\big|\mathfrak{d}^{\le k}\psi\big|\Big)\big|\mathfrak{d}^{\le k}N\big|. \end{aligned} \tag{14.1.4}$$

2. *The following conditional Energy-Morawetz estimates hold true*

$$\begin{aligned} B_\delta^k[\psi] + \sup_{\tau\in[1,\tau_*]} E_\delta^k[\psi](\tau) + F_\delta^k[\psi] \lesssim\; & E_\delta^k[\psi](0) + BEF_\delta^{k-1}[\psi] \\ & + \int_{\mathcal{M}_{trap}} |\mathfrak{d}^k\psi|^2 + \mathcal{N}_\delta^k[\psi, N]. \end{aligned} \tag{14.1.5}$$

Remark 14.1.5. *Note that both estimates are conditional on the control of the lower order term of $Mor^k[\psi]$ in $\mathcal{M}_{trap}$, i.e. the term $\int_{\mathcal{M}_{trap}} |\mathfrak{d}^k\psi|^2$.*

We will also make use of the following control of $\underline{\mathfrak{q}}$ which follows from a variant of Theorem 12.2.4.

Theorem 14.1.6. *We have, for $k \leq k_L - 3$,*

$$\sup_{\tau\in[1,\tau_*]} E_\delta^k[\underline{\mathfrak{q}}](\tau) + B_\delta^k[\underline{\mathfrak{q}}] + F_\delta^k[\underline{\mathfrak{q}}] \lesssim \epsilon_0^2. \tag{14.1.6}$$

Proof. Recall from (13.6.5) that the global frame of $\mathcal{M}$ satisfies in Part III the identities $\Xi = \underline{\widecheck{H}} = 0$ for $r \geq r_0 + 1$. In particular, $\underline{\psi} = \Re(\underline{\mathfrak{q}})$ satisfies the generalized RW equation (12.1.4). One can then easily adapt the proof of Theorem 12.2.4 to obtain that the following holds true, for $s \leq k_L - 3$,

$$BEF_\delta^s[\underline{\psi}, \underline{A}](1,\tau_*) \lesssim E_\delta^s[\underline{\psi}, \underline{A}](1) + \mathcal{N}_\delta^s[\underline{\psi}, N_{\text{Err}}](1,\tau_*),$$

where N_{Err} is defined in (12.1.9), i.e.

$$N_{\text{Err}} = r^2 \not{\mathfrak{d}}^{\leq 1}\underline{\alpha} \cdot \mathfrak{d}^{\leq 2}(\alpha,\beta) + \mathfrak{d}^{\leq 3}(\Gamma_b \cdot \Gamma_g).$$

Using the control of the initial data provided by (13.6.1) and the definition of $BEF_\delta^s[\underline{\psi}, \underline{A}](1,\tau_*)$ and of $\underline{\psi}$, we infer, for $s \leq k_L - 3$,

$$BEF_\delta^s[\underline{\mathfrak{q}}] \lesssim \epsilon_0^2 + \mathcal{N}_\delta^s[\underline{\psi}, N_{\text{Err}}](1,\tau_*).$$

Also, as established in the proof of Lemma 12.2.6, see section 12.2.4, we have

$$\begin{aligned}
&\mathcal{N}_\delta^s[\underline{\psi}, N](\tau_1,\tau_2)\\
\lesssim\ & \Big(BEF_\delta^s[\underline{\psi}](\tau_1,\tau_2)\Big)^{\frac{1}{2}} \left(\int_{\tau_1}^{\tau_2} \|\mathfrak{d}^{\leq s}N\|_{L^2(\Sigma_{trap}(\tau))} + \left(\int_{\mathcal{M}(\tau_1,\tau_2)} r^{\delta+1}|\mathfrak{d}^{\leq s}N|^2\right)^{\frac{1}{2}}\right).
\end{aligned}$$

Now, in view of the above form of N_{Err} and the control provided by the bootstrap assumptions (13.6.2) (13.6.3), we have, for $s \leq k_L - 3$,

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2} \|\mathfrak{d}^{\leq s}N_{\text{Err}}\|_{L^2(\Sigma_{trap}(\tau))} + \left(\int_{\mathcal{M}(\tau_1,\tau_2)} r^{\delta+1}|\mathfrak{d}^{\leq s}N_{\text{Err}}|^2\right)^{\frac{1}{2}}\\
\lesssim\ & (\mathfrak{G}_s + \mathfrak{R}_s)\int_{\tau_1}^{\tau_2} \|\mathfrak{d}^{\leq \frac{s}{2}}(\check{\Gamma}, \check{R})\|_{L^2(\Sigma_{trap}(\tau))} + \epsilon\int_{\mathcal{M}(\tau_1,\tau_2)} \Big(r^{3+\delta}|\mathfrak{d}^{\leq s+2}(\alpha,\beta)|^2\\
&+r^{-2-2\delta_{dec}+\delta}|\mathfrak{d}^{\leq s+1}\underline{\alpha}|^2 + r^{-1+\delta}|\mathfrak{d}^{\leq s+3}\Gamma_g|^2 + r^{-3+\delta}|\mathfrak{d}^{\leq s+3}\Gamma_b|^2\Big)\\
\lesssim\ & \epsilon^2\int_{\tau_1}^{\tau_2} \frac{d\tau}{\tau^{1+\delta_{dec}}} + \epsilon(\mathfrak{G}_{s+3} + \mathfrak{R}_{s+3}) \lesssim \epsilon^2 \lesssim \epsilon_0,
\end{aligned}$$

where we used the fact that $s+3 \leq k_L$ and $\delta < \delta_{dec} < \delta_B$. We infer

$$\mathcal{N}_\delta^s[\underline{\psi}, N](\tau_1, \tau_2) \lesssim \epsilon_0 \Big(BEF_\delta^s[\underline{\psi}](\tau_1, \tau_2) \Big)^{\frac{1}{2}}$$

and hence, for $s \leq k_L - 3$,

$$BEF_\delta^s[\underline{\mathfrak{q}}] \lesssim \epsilon_0^2$$

as stated. This concludes the proof of Theorem 14.1.6. □

These results will be used together with the following lemma relating ${}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\check{P}$ to $\underline{\mathfrak{q}}$, to prove Theorem 14.1.3.

Lemma 14.1.7. *The following relation between* $\underline{\mathfrak{q}}$ *and* $\check{P}$ *holds true.*

$$\underline{\mathfrak{q}} = \frac{1}{2}\overline{q}q^3\, {}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\check{P} + \mathfrak{d}^{\leq 1}\Gamma_b' + O(a)\mathfrak{d}^{\leq 1}\check{R}_b + O(ar)\mathfrak{d}^{\leq 1}\check{P} + r\mathfrak{d}^{\leq 1}(\Gamma_b \cdot \check{R}_b), \quad (14.1.7)$$

where we recall that $\Gamma_b' = \Gamma_b \setminus \{\Xi\}$.

Proof. This follows immediately from Proposition 5.6.1 and the fact that $r\underline{B} \in \check{R}_b$, $\Gamma_g' = \Gamma_g \setminus \{\widecheck{\mathrm{tr}\underline{X}}\}$ and the fact that we identity $\check{R}_g$ with $r^{-2}\check{R}_b$, Γ_g with $r^{-1}\Gamma_b$, and Γ_g' with $r^{-1}\Gamma_b'$ in Part III. □

The main part in the proof of Theorem 14.1.3 is to derive the following result for the pair of scalars $\psi = q^2(\mathbf{T}(P), \mathbf{Z}(P))$ stated below.

Proposition 14.1.8. *The pair of scalars*

$$\psi := q^2(\mathbf{T}(P), \mathbf{Z}(P))$$

verify the following estimates

$$BEF_\delta^{J-1}[\psi] \lesssim r_0^{10}\Big(\mathfrak{G}_{J+1}\mathfrak{G}_J + \mathfrak{R}_{J+1}\mathfrak{R}_J + \epsilon_J^2 + \epsilon_0^2 \Big). \quad (14.1.8)$$

14.2 Proof of the Morawetz-Energy estimates for $\check{P}$

The goal of this section is to prove Theorem 14.1.3 on Morawetz-Energy estimates for $\check{P}$. We start by proving Proposition 14.1.8 on Morawetz-Energy estimates for the pair of scalars $\psi = q^2(\mathbf{T}(P), \mathbf{Z}(P))$.

14.2.1 Proof of Proposition 14.1.8

The proof of Proposition 14.1.8 proceeds in several steps.

Step 0. We start with the equation of Lemma 5.5.1 for P

$$\begin{aligned}\square_{\mathbf{g}} P &= \operatorname{tr} X \nabla_3 P + \overline{\operatorname{tr} \underline{X}} \nabla_4 P - \overline{H} \cdot \mathcal{D} P - \underline{H} \cdot \overline{\mathcal{D}} P + VP \\ &\quad + r^{-3} \mathfrak{d}^{\leq 1}(\Gamma_b \cdot \check{R}_b) - \underline{A} \cdot \overline{A},\end{aligned} \tag{14.2.1}$$

see also Remark 5.5.2, and recall that we identify $\check{R}_g$ with $r^{-2}\check{R}_b$ and Γ_g with $r^{-1}\Gamma_b$.

Remark 14.2.1. *We will need to linearize* (14.2.1). *A possibility consists in deriving a wave equation for $\check{P}$, where we recall that $\check{P} = P + \frac{2m}{q^3}$. In view of* (14.2.1), *this leads to*

$$\begin{aligned}\square_{\mathbf{g}} \check{P} &= trX \nabla_3 \check{P} + \overline{tr\underline{X}} \nabla_4 \check{P} - \overline{H} \cdot \mathcal{D}\check{P} - \underline{H} \cdot \overline{\mathcal{D}} \check{P} + V\check{P} + r^{-4}\mathfrak{d}^{\leq 1}\Gamma_b \\ &+ r^{-3}\mathfrak{d}^{\leq 1}(\Gamma_b \cdot \check{R}_b) - \underline{A} \cdot \overline{A}.\end{aligned} \tag{14.2.2}$$

This linearization leads however to terms of the form $\mathfrak{d}^{J+1}\check{\Gamma}$ in estimates for $\mathfrak{d}^{J+1}\check{P}$, so that the iteration assumption (13.6.4) *on $\mathfrak{G}_J$ cannot be used.*

To avoid the linearization issue outlined in Remark 14.2.1, we proceed by the linearization procedure discussed in Step 1 below which consists in deriving Morawetz-Energy estimates for the linearized quantity

$$\psi := q^2(\mathbf{T}P, \mathbf{Z}P). \tag{14.2.3}$$

To apply the induction hypothesis we need to compare the Morawetz-Energy norms $BEF^k_\delta[\psi]$ with $\mathfrak{R}^2_{k+1}[\check{P}]$.

Lemma 14.2.2. *The induction hypothesis* (13.6.4) *implies*

$$B^{J-2}_\delta[\psi] \lesssim \epsilon^2_J, \qquad EF^{J-2}_\delta[\psi] \lesssim \epsilon^2_J + \mathfrak{R}_J \mathfrak{R}_{J+1}. \tag{14.2.4}$$

Proof. Note that

$$\begin{aligned}q^2 \mathbf{T} P &= q^2 \mathbf{T}\left(\check{P} - \frac{2m}{q^3}\right) = q^2 \mathbf{T}\check{P} + r^{-1}\Gamma_b, \\ q^2 \mathbf{Z} P &= q^2 \mathbf{Z}\left(\check{P} - \frac{2m}{q^3}\right) = q^2 \mathbf{Z}\check{P} + r^{-1}\Gamma_b.\end{aligned}$$

We deduce, in view of Lemmas 14.1.1 and 14.1.2 and the induction hypothesis.

$$\begin{aligned} B_\delta^{J-2}[\psi] &\lesssim B_\delta^{J-2}[r^2(\mathbf{T}\check{P}, \mathbf{Z}\check{P})] + BEF_\delta^{J-2}[r^{-1}\Gamma_b] \\ &\lesssim B_\delta^{J-1}[r^2\check{P}] + BEF_\delta^{J-2}[r^{-1}\Gamma_b] \\ &\lesssim \mathfrak{R}_J^2 + BEF_\delta^{J-2}[r^{-1}\Gamma_b] \lesssim \epsilon_J^2 + \mathfrak{G}_J\mathfrak{G}_{J-1} \\ &\lesssim \epsilon_J^2 \end{aligned}$$

and similarly

$$\begin{aligned} EF_\delta^{J-2}[\psi] &\lesssim EF_\delta^{J-2}[r^2(\mathbf{T}\check{P}, \mathbf{Z}\check{P})] + BEF_\delta^{J-2}[r^{-1}\Gamma_b] \\ &\lesssim EF_\delta^{J-1}[r^2\check{P}] + BEF_\delta^{J-2}[r^{-1}\Gamma_b] \\ &\lesssim \mathfrak{R}_J\mathfrak{R}_{J+1} + BEF_\delta^{J-2}[r^{-1}\Gamma_b] \lesssim \mathfrak{R}_J\mathfrak{R}_{J+1} + \mathfrak{G}_J\mathfrak{G}_{J-1} \\ &\lesssim \epsilon_J^2 + \mathfrak{R}_J\mathfrak{R}_{J+1} \end{aligned}$$

as stated. □

Step 1. Next, we derive a wave equation for the pair of scalars $\psi = q^2(\mathbf{T}P, \mathbf{Z}P)$.

Lemma 14.2.3. *The pair of scalars $\psi = q^2(\mathbf{T}P, \mathbf{Z}P)$ satisfies a wave equation of the schematic form*[1]

$$\square_{\mathbf{g}}\psi = W\psi + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_b + r^{-1}\mathfrak{d}^{\leq 2}(\Gamma_b \cdot \check{R}_b) - r^2\mathfrak{d}^{\leq 1}(\underline{A} \cdot \overline{A}) \qquad (14.2.5)$$

where W is a complex potential of the form

$$\Re(W) = O(mr^{-3}), \qquad \Im(W) = O(mar^{-4}). \qquad (14.2.6)$$

Proof. This is done by making use of the commutation formulas (4.3.1)

$$\begin{aligned} [\mathbf{T}, \square_{\mathbf{g}}]\psi &= \mathfrak{d}\big(\Gamma_g \cdot \mathfrak{d}\psi\big) + \Gamma_b \cdot \square_{\mathbf{g}}\psi, \\ [\mathbf{Z}, \square_{\mathbf{g}}]\psi &= \mathfrak{d}\big(\Gamma_g \cdot \mathfrak{d}\psi\big) + r\Gamma_b \cdot \square_{\mathbf{g}}\psi, \end{aligned}$$

as well as the renormalization Lemma 5.5.3. See section 14.3.1. □

Step 2. We apply the first conditional estimate of Proposition 14.1.4 to equation (14.2.5) to derive a conditional Morawetz estimate for ψ of the form.

$$B_\delta^{J-1}[\psi] \lesssim EF_\delta^{J-1}[\psi] + \int_{\mathcal{M}_{trap}} |\mathfrak{d}^{J-1}\psi|^2 + \epsilon_J^2 + \epsilon_0^2. \qquad (14.2.7)$$

[1] Recall that $\check{R}_b$ is a curvature term which behaves like Γ_b in terms of powers of r.

Remark 14.2.4. *Note that we cannot estimate the energy flux EF at this step because of the presence of the complex potential* W *in* (14.2.5).

The proof of (14.2.7) is a straightforward application of the estimate (14.1.4) of Proposition 14.1.4 which yields, for $k = J-1$,

$$\begin{aligned} B^{J-1}_\delta[\psi] &\lesssim EF^{J-1}_\delta[\psi] + B^{J-2}_\delta[\psi] + \int_{\mathcal{M}_{trap}} |\mathfrak{d}^{J-1}\psi|^2 \\ &\quad + {}^{(mor)}\mathcal{N}^{J-1}[\psi, N] + {}^{(red)}\mathcal{N}^{J-1}[\psi, N] \\ &\quad + \int_{{}^{(ext)}\mathcal{M}} r^\delta \Big(\big|\nabla_4 \mathfrak{d}^{\leq J-1}\psi\big| + r^{-1}\big|\mathfrak{d}^{\leq J-1}\psi\big| \Big) \big|\mathfrak{d}^{\leq J-1} N\big|, \end{aligned}$$

where we have, in view of (14.2.5), $V = -\Re(W)$ and

$$N := i\Im(W)\psi + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_b + r^{-1}\mathfrak{d}^{\leq 2}(\Gamma_b \cdot \check{R}_b) - r^2\mathfrak{d}^{\leq 1}(\underline{A} \cdot \overline{A}).$$

We have

$$\begin{aligned} &{}^{(mor)}\mathcal{N}^{J-1}[\psi, N] + {}^{(red)}\mathcal{N}^{J-1}[\psi, N] \\ &+ \int_{{}^{(ext)}\mathcal{M}} r^\delta \Big(\big|\nabla_4 \mathfrak{d}^{\leq J-1}\psi\big| + r^{-1}\big|\mathfrak{d}^{\leq J-1}\psi\big| \Big) \big|\mathfrak{d}^{\leq J-1} N\big| \\ \lesssim\ & \Big(B^{J-1}_\delta[\psi] \Big)^{\frac{1}{2}} \left(\int_{\mathcal{M}} r^{\delta+1} |\mathfrak{d}^{\leq J-1} N|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

and, using the definition of N, the fact that $\Im(W) = O(mar^{-4})$, the induction hypothesis for the linear term involving Γ_b and the bootstrap assumptions for the nonlinear one, and Lemma 13.5.5, we estimate

$$\begin{aligned} \int_{\mathcal{M}} r^{\delta+1}|\mathfrak{d}^{\leq J-1}N|^2 &\lesssim |a| B^{J-1}_\delta[\psi] + \int_{\mathcal{M}} r^{-3+\delta}|\mathfrak{d}^{\leq J}\Gamma_b|^2 + \epsilon^2 \int_{\mathcal{M}} r^{-3+\delta}|\mathfrak{d}^{\leq J+1}\check{R}_b|^2 \\ &\quad + \epsilon^2 \int_{\mathcal{M}} r^{-2+\delta}|\mathfrak{d}^{\leq J+1}\Gamma_b|^2 + \epsilon^2 \int_{\mathcal{M}} r^{3+\delta}|\mathfrak{d}^{\leq J+1}A|^2 \\ &\lesssim |a| B^{J-1}_\delta[\psi] + \mathfrak{G}_J^2 + \epsilon^2(\mathfrak{G}_{J+1} + \mathfrak{R}_{J+1})^2 \\ &\lesssim |a| B^{J-1}_\delta[\psi] + \epsilon_J^2 + \epsilon_0^2. \end{aligned}$$

We deduce

$$\begin{aligned} B^{J-1}_\delta[\psi] &\lesssim EF^{J-1}_\delta[\psi] + B^{J-2}_\delta[\psi] + \int_{\mathcal{M}_{trap}} |\mathfrak{d}^{J-1}\psi|^2 \\ &\quad + \Big(B^{J-1}_\delta[\psi] \Big)^{\frac{1}{2}} \left(|a| B^{J-1}_\delta[\psi] + \epsilon_J^2 + \epsilon_0^2 \right)^{\frac{1}{2}}, \end{aligned}$$

and hence, using also the fact that $B_\delta^{J-2}[\psi] \lesssim \epsilon_J^2$ in view of Lemma 14.2.2, we obtain

$$B_\delta^{J-1}[\psi] \lesssim |a| B_\delta^{J-1}[\psi] + EF_\delta^{J-1}[\psi] + \int_{\mathcal{M}_{trap}} |\mathfrak{d}^{J-1}\psi|^2 + \epsilon_J^2 + \epsilon_0^2.$$

For a small enough, this yields (14.2.7) as stated.

Step 3. To control the energy-flux term $EF_\delta^{J-1}[\psi]$ in (14.2.7), we take a circuitous route by commuting the wave equation for ψ with $\nabla_{\widehat{R}}^2$. We rely on on the following commutation lemma.

Lemma 14.2.5. *Assume $\square_{\mathbf{g}}\psi = N$. Then, we have*

$$\begin{aligned}\square_{\mathbf{g}}\nabla_{\widehat{R}}^2\psi &= O(r^{-1})\nabla_{\widehat{R}}\triangle\psi + O(ar^{-3})\nabla_{\widehat{R}}\mathfrak{d}^{\leq 2}\psi + O(r^{-2})\triangle\psi + O(r^{-2})\nabla_{\widehat{R}}\psi \\ &\quad + O(ar^{-3})\mathfrak{d}^{\leq 2}\psi + r^{-1}\mathfrak{d}^{\leq 3}\big(\Gamma_b\cdot\psi\big) + \nabla_{\widehat{R}}^2 N + r^{-1}\nabla_{\widehat{R}}N + r^{-2}N.\end{aligned} \tag{14.2.8}$$

Proof. See section 14.3.2. □

Remark 14.2.6. *The reason for commuting the wave equation for ψ with $\nabla_{\widehat{R}}^2$ is to ensure that all linear terms involving top order curvature components on the RHS of the wave equation of Lemma 14.2.5 contain at least on $\nabla_{\widehat{R}}$ derivative.*

We rewrite the wave equation for ψ in (14.2.5) as

$$\begin{aligned}\square_{\mathbf{g}}\psi &= N,\\ N &:= W\psi + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_b + r^{-1}\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\check{R}_b) - r^2\mathfrak{d}^{\leq 1}(\underline{A}\cdot\overline{A}),\end{aligned}$$

and compute, using $W = O(r^{-3})$,

$$\begin{aligned}\nabla_{\widehat{R}}^2 N + r^{-1}\nabla_{\widehat{R}}N + r^{-2}N &= W\nabla_{\widehat{R}}^2\psi + O(r^{-3})\mathfrak{d}^{\leq 1}\psi + r^{-2}\mathfrak{d}^{\leq 3}\Gamma_b \\ &\quad + r^{-1}\mathfrak{d}^{\leq 4}(\Gamma_b\cdot\check{R}_b) - r^2\mathfrak{d}^{\leq 3}(\underline{A}\cdot\overline{A}).\end{aligned}$$

Commuting $\square_{\mathbf{g}}\psi = N$ with $\nabla_{\widehat{R}}^2$, and relying on Lemma 14.2.5, we obtain the following wave equation for $\nabla_{\widehat{R}}^2\psi$

$$\begin{aligned}\square_{\mathbf{g}}(\nabla_{\widehat{R}}^2\psi) &= W\nabla_{\widehat{R}}^2\psi + O(r^{-2})\triangle\psi + O(r^{-1})\nabla_{\widehat{R}}\triangle\psi + O(ar^{-3})\Big(\nabla_{\widehat{R}}\mathfrak{d}^{\leq 2}\psi + \mathfrak{d}^{\leq 2}\psi\Big) \\ &\quad + O(r^{-2})\mathfrak{d}^{\leq 1}\psi + N^{\leq 3}, \\ N^{\leq 3} &= r^{-2}\mathfrak{d}^{\leq 3}\Gamma_b + r^{-1}\mathfrak{d}^{\leq 4}(\Gamma_b\cdot\check{R}_b) - r^2\mathfrak{d}^{\leq 3}(\underline{A}\cdot\overline{A}).\end{aligned} \tag{14.2.9}$$

Step 4. We apply the second conditional estimate of Proposition 14.1.4 to derive the following lemma.

Lemma 14.2.7. *For any $0 < \delta_1 \leq 1$, we have*

$$BEF_\delta^{J-3}[\nabla_{\widehat{R}}^2\psi] \lesssim (\delta_1 + |a|)B_\delta^{J-1}[\psi] + \delta_1^{-1}B_\delta^{J-3}[r\triangle\psi] + \delta_1^{-1}\Big(\epsilon_J^2 + \epsilon_0^2\Big) + \mathfrak{R}_J\mathfrak{R}_{J+1}(14.2.10)$$

Proof. See section 14.3.3. □

Remark 14.2.8. *In the proof of Lemma 14.2.7 in section 14.3.3, when applying the second conditional estimate of Proposition 14.1.4, we will have to take into account the additional term $\int_{\mathcal{M}_{trap}} |\mathfrak{d}^{J-3}\nabla_{\widehat{R}}^2\psi|^2$ on the right hand side. Fortunately, thanks to the $\nabla_{\widehat{R}}$ derivative, this term is bounded by $B_\delta^{J-1}[\psi]$ and thus can be controlled by the induction hypothesis.*

Step 5. Recall the following commutator formula, see Lemma 4.5.4,

$$[|q|^2\triangle, |q|^2\square_{\mathbf{g}}]\psi = |q|^2\Big[O(a^2r^{-3})\mathfrak{d}^{\leq 2}\psi + \mathfrak{d}^2\big(\Gamma_g \cdot \mathfrak{d}\psi\big) + \dot{\mathbf{D}}_3\mathfrak{d}\big(|q|^2\xi \cdot \dot{\mathbf{D}}_a\psi\big)\Big].$$

In view of (13.6.5), we have in particular $\Xi \in r^{-1}\Gamma_g$, and hence

$$[|q|^2\triangle, |q|^2\square_{\mathbf{g}}]\psi = |q|^2\Big[O(a^2r^{-3})\mathfrak{d}^{\leq 2}\psi + \mathfrak{d}^2\big(\Gamma_g \cdot \mathfrak{d}\psi\big)\Big].$$

We infer

$$\begin{aligned}\square_{\mathbf{g}}(|q|^2\triangle\psi) &= \frac{1}{|q|^2}\Big(|q|^2\triangle(|q|^2\square_{\mathbf{g}}\psi) - [|q|^2\triangle, |q|^2\square_{\mathbf{g}}]\psi\Big)\\ &= \triangle(|q|^2\square_{\mathbf{g}}\psi) + O(a^2r^{-3})\mathfrak{d}^{\leq 2}\psi + \mathfrak{d}^2\big(\Gamma_g \cdot \mathfrak{d}\psi\big).\end{aligned}$$

Plugging the wave equation for ψ, see (14.2.5), in the RHS, and using $W = O(r^{-3})$, we infer

$$\square_{\mathbf{g}}(|q|^2\triangle\psi) = W(|q|^2\triangle\psi) + O(ar^{-4})\mathfrak{d}^{\leq 2}\psi + O(r^{-3})\mathfrak{d}^{\leq 1}\psi + N^{\leq 3}, \quad (14.2.11)$$

where $N^{\leq 3}$ is as in (14.2.9). We can then proceed as in Step 2, using the first conditional estimate of Proposition 14.1.4, to derive the estimate

$$B_\delta^{J-3}[r^2\triangle\psi] \lesssim |a|B_\delta^{J-1}[\psi] + EF_\delta^{J-3}[r^2\triangle\psi] + \int_{\mathcal{M}_{trap}} |\mathfrak{d}^{J-3}\triangle\psi|^2 + \epsilon_J^2 + \epsilon_0^2. \quad (14.2.12)$$

Next, using the Hodge type estimate of Corollary 13.4.1, integrated on $\mathcal{M}$, we derive

$$B_\delta^{J-3}[r^2\nabla^2\psi] \lesssim B_\delta^{J-3}[r^2\triangle\psi] + O(a,\epsilon)B_\delta^{J-1}[\psi].$$

Combining with (14.2.12) we thus infer

$$\begin{aligned} B_\delta^{J-3}[r^2\nabla^2\psi] \lesssim\; & EF_\delta^{J-3}[r^2\triangle\psi] + O(a,\epsilon)B_\delta^{J-1}[\psi] + \int_{\mathcal{M}_{trap}} |\mathfrak{d}^{J-3}\triangle\psi|^2 \\ & + \epsilon_J^2 + \epsilon_0^2. \end{aligned} \tag{14.2.13}$$

Step 6. Combining (14.2.10) with (14.2.13) we derive, for any $0 < \delta_1 \le 1$,

$$\begin{aligned} BEF_\delta^{J-3}[\nabla_{\widehat{R}}^2\psi] &\lesssim (\delta_1 + |a|)B_\delta^{J-1}[\psi] + \delta_1^{-1}B_\delta^{J-3}[r\triangle\psi] + \delta_1^{-1}\Big(\epsilon_J^2 + \epsilon_0^2\Big) + \mathfrak{R}_J\mathfrak{R}_{J+1} \\ &\lesssim (\delta_1 + |a|)B_\delta^{J-1}[\psi] + \delta_1^{-1}EF_\delta^{J-3}[r^2\triangle\psi] + \delta_1^{-1}O(a,\epsilon)B_\delta^{J-1}[\psi] \\ & \quad +\delta_1^{-1}\int_{\mathcal{M}_{trap}} |\mathfrak{d}^{J-3}\triangle\psi|^2 + \delta_1^{-1}\big(\epsilon_J^2 + \epsilon_0^2\big) + \mathfrak{R}_J\mathfrak{R}_{J+1}. \end{aligned}$$

Hence, for any $0 < \delta_1 \le 1$,

$$\begin{aligned} BEF_\delta^{J-3}[(\nabla_{\widehat{R}}^2, r^2\nabla^2)\psi] \lesssim\; & \Big(\delta_1 + O(a,\epsilon)\delta_1^{-1}\Big)B_\delta^{J-1}[\psi] + \delta_1^{-1}EF_\delta^{J-3}[r^2\triangle\psi] \\ & + \delta_1^{-1}\int_{\mathcal{M}_{trap}} |\mathfrak{d}^{J-3}\triangle\psi|^2 + \delta_1^{-1}\big(\epsilon_J^2 + \epsilon_0^2\big) + \mathfrak{R}_J\mathfrak{R}_{J+1}. \end{aligned} \tag{14.2.14}$$

Step 7. We make use of the following consequence of Lemma 4.7.6

$$-\nabla_{\widehat{T}}^2\psi + \nabla_{\widehat{R}}^2\psi = O(1)\square\psi + O(1)\triangle\psi + O(r^{-1})\nabla_{\widehat{R}}\psi + O(ar^{-2})\nabla\psi + \Gamma_g\cdot\mathfrak{d}\psi$$

to derive the estimates

$$\begin{aligned} EF_\delta^{J-3}[\nabla_{\widehat{T}}^2\psi] &\lesssim EF_\delta^{J-3}[\nabla_{\widehat{R}}^2\psi] + EF_\delta^{J-3}[\triangle\psi] + \epsilon_J^2 + \epsilon_0^2, \\ B_\delta^{J-3}[\nabla_{\widehat{T}}^2\psi] &\lesssim B_\delta^{J-3}[\nabla_{\widehat{R}}^2\psi] + B_\delta^{J-3}[\triangle\psi] + \epsilon_J^2 + \epsilon_0^2. \end{aligned}$$

Combining this with (14.2.14), we infer that, for any $0 < \delta_1 \le 1$,

$$\begin{aligned} BEF_\delta^{J-3}[\big(\nabla_{\widehat{T}}^2, \nabla_{\widehat{R}}^2, r^2\nabla^2\big)\psi] \lesssim\; & \Big(\delta_1 + O(a,\epsilon)\delta_1^{-1}\Big)B_\delta^{J-1}[\psi] + \delta_1^{-1}EF_\delta^{J-3}[r^2\triangle\psi] \\ & + \delta_1^{-1}\int_{\mathcal{M}_{trap}} |\mathfrak{d}^{J-3}\triangle\psi|^2 + \delta_1^{-1}\Big(\epsilon_J^2 + \epsilon_0^2\Big) + \mathfrak{R}_J\mathfrak{R}_{J+1}. \end{aligned}$$

Moreover, since $J \ge \frac{k_L}{2}$ and k_L is large, we may assume that $J \ge 5$. In particular, we have

$$BEF_\delta^{J-5}[\big(\nabla_{\widehat{T}}, \nabla_{\widehat{R}}, r\nabla\big)^{\le 4}\psi] \lesssim BEF_\delta^{J-3}[\big(\nabla_{\widehat{T}}^2, \nabla_{\widehat{R}}^2, r^2\nabla^2\big)\psi] + BEF_\delta^{J-2}[\psi]$$

and hence, for any $0 < \delta_1 \le 1$,

$$\begin{aligned}BEF_\delta^{J-5}[\big(\nabla_{\widehat{T}}, \nabla_{\widehat{R}}, r\nabla\big)^{\le 4}\psi] \lesssim & \Big(\delta_1 + O(a,\epsilon)\delta_1^{-1}\Big)B_\delta^{J-1}[\psi] + \delta_1^{-1}EF_\delta^{J-3}[r^2\triangle\psi]\\ & + \delta_1^{-1}\int_{\mathcal{M}_{trap}} |\mathfrak{d}^{J-3}\triangle\psi|^2 + \delta_1^{-1}\Big(\epsilon_J^2 + \epsilon_0^2\Big) + \mathfrak{R}_J\mathfrak{R}_{J+1}\\ & + BEF_\delta^{J-2}[\psi].\end{aligned}$$

Together with (14.2.4), we infer, for any $0 < \delta_1 \le 1$,

$$\begin{aligned}BEF_\delta^{J-5}[\big(\nabla_{\widehat{T}}, \nabla_{\widehat{R}}, r\nabla\big)^{\le 4}\psi] \lesssim & \Big(\delta_1 + O(a,\epsilon)\delta_1^{-1}\Big)B_\delta^{J-1}[\psi] + \delta_1^{-1}EF_\delta^{J-3}[r^2\triangle\psi]\\ & + \delta_1^{-1}\int_{\mathcal{M}_{trap}} |\mathfrak{d}^{J-3}\triangle\psi|^2 + \delta_1^{-1}\Big(\epsilon_J^2 + \epsilon_0^2\Big) + \mathfrak{R}_J\mathfrak{R}_{J+1}.\end{aligned} \tag{14.2.15}$$

Step 8. Next, we derive an estimate for $BEF_{\delta;r\ge r_0}^{J-2}[re_4\psi]$. To this end, we commute the wave equation for ψ in (14.2.5) with $r\nabla_4$. We obtain

$$\begin{aligned}\square_{\mathbf{g}}\big(re_4\psi\big) &= re_4\big(\square_{\mathbf{g}}\psi\big) - [re_4, \square_{\mathbf{g}}]\psi\\ &= O(r^{-3})\mathfrak{d}^{\le 1}\psi + r^{-2}\mathfrak{d}^{\le 2}\Gamma_b + r^{-1}\mathfrak{d}^{\le 3}(\Gamma_b\cdot\check{R}_b) - r^2\mathfrak{d}^{\le 2}(\underline{A}\cdot\overline{A}) - [re_4, \square_{\mathbf{g}}]\psi,\end{aligned}$$

where we used the fact that $W = O(r^{-3})$. Together with the commutator formula of Lemma 4.7.11, we infer

$$\begin{aligned}\square_{\mathbf{g}}\big(re_4\psi\big) =& \frac{1}{r}\nabla_4(r\nabla_4\psi) + N_{re_4},\\ N_{re_4} :=& O(r^{-2})\,\not{\mathfrak{d}}^2\psi + O(r^{-2})\mathfrak{d}^{\le 1}\psi + O(r^{-3})\mathfrak{d}^{\le 2}\psi\\ & + r^{-2}\mathfrak{d}^{\le 2}\Gamma_b + r^{-1}\mathfrak{d}^{\le 3}(\Gamma_b\cdot\check{R}_b) - r^2\mathfrak{d}^{\le 2}(\underline{A}\cdot\overline{A}).\end{aligned}$$

Applying the r^p weighted estimates of Proposition 10.1.2, and noticing that the first term on the RHS of the above wave equation for $\square_{\mathbf{g}}\big(re_4\psi\big)$ has the right sign in the estimate, we infer

$$\begin{aligned}BEF_{\delta;r\ge r_0}^{J-2}[re_4\psi] \lesssim & \ \epsilon_0^2 + r_0 B_{\delta;r_0/2\le r\le r_0}^{J-2}[re_4\psi]\\ & + \int_{\mathcal{M}(r\ge r_0/2)} r^\delta\Big(|re_4\mathfrak{d}^{\le J-2}re_4\psi| + |\mathfrak{d}^{\le J-2}re_4\psi|\Big)|\mathfrak{d}^{\le J-2}N_{re_4}|.\end{aligned}$$

In view of the form of N_{re_4}, we infer

$$\begin{aligned}BEF_{\delta;r\ge r_0}^{J-2}[re_4\psi] \lesssim & \ \epsilon_0^2 + \epsilon_J^2 + r_0 B_{\delta;r_0/2\le r\le r_0}^{J-2}[re_4\psi] + (\epsilon_0 + \epsilon_J)\sqrt{BEF_{\delta;r\ge r_0}^{J-2}[re_4\psi]}\\ & + \sqrt{BEF_{\delta;r\ge r_0}^{J-3}[(r\nabla)^2\psi]}\sqrt{BEF_{\delta;r\ge r_0}^{J-2}[re_4\psi]} + r_0^{-1}BEF_{\delta;r\ge r_0}^{J-1}[\psi].\end{aligned}$$

For r_0 large enough, we may absorb the part of the last term on the RHS that has e_4 derivatives and obtain

$$\begin{aligned} BEF^{J-2}_{\delta;r\geq r_0}[re_4\psi] &\lesssim \epsilon_0^2+\epsilon_J^2+r_0B^{J-2}_{\delta;r_0/2\leq r\leq r_0}[re_4\psi] \\ &+\sqrt{BEF^{J-3}_{\delta;r\geq r_0}[(r\nabla)^2\psi]}\sqrt{BEF^{J-2}_{\delta;r\geq r_0}[re_4\psi]}+BEF^{J-5}_{\delta}[\big(\nabla_{\widehat{T}},r\nabla\big)^{\leq 4}\psi]. \end{aligned}$$

Also, integrating by parts, one easily obtains

$$\begin{aligned} BEF^{J-3}_{\delta;r\geq r_0}[(r\nabla)^2\psi] &\lesssim BEF^{J-5}_{\delta}[\big(\nabla_{\widehat{T}},r\nabla\big)^{\leq 4}\psi]+\epsilon_J\sqrt{EF^{J-2}_{\delta;r\geq r_0}[re_4\psi]} \\ &+\sqrt{BEF^{J-5}_{\delta}[\big(\nabla_{\widehat{T}},r\nabla\big)^{\leq 4}\psi]}\sqrt{BEF^{J-2}_{\delta;r\geq r_0}[re_4\psi]} \end{aligned}$$

and hence

$$BEF^{J-2}_{\delta;r\geq r_0}[re_4\psi] \lesssim \epsilon_0^2+\epsilon_J^2+r_0B^{J-2}_{\delta;r_0/2\leq r\leq r_0}[re_4\psi]+BEF^{J-5}_{\delta}[\big(\nabla_{\widehat{T}},r\nabla\big)^{\leq 4}\psi].$$

Also, since e_4 is spanned by $\widehat{T}$ and $\widehat{R}$,

$$\begin{aligned} BEF^{J-2}_{\delta;r\geq r_0}[re_4\psi] &\lesssim \epsilon_0^2+\epsilon_J^2+r_0B^{J-2}_{\delta;r_0/2\leq r\leq r_0}[re_4\psi]+BEF^{J-5}_{\delta}[\big(\nabla_{\widehat{T}},r\nabla\big)^{\leq 4}\psi] \\ &\lesssim \epsilon_0^2+\epsilon_J^2+r_0^5BEF^{J-5}_{\delta}[\big(\nabla_{\widehat{T}},\nabla_{\widehat{R}},r\nabla\big)^{\leq 4}\psi]. \end{aligned}$$

Since

$$BEF^{J-5}_{\delta}[\big(\nabla_{\widehat{T}},r\nabla_4,r\nabla\big)^{\leq 4}\psi] \lesssim r_0^5BEF^{J-5}_{\delta}[\big(\nabla_{\widehat{T}},\nabla_{\widehat{R}},r\nabla\big)^{\leq 4}\psi]+BEF^{J-2}_{\delta;r\geq r_0}[re_4\psi],$$

we infer

$$BEF^{J-5}_{\delta}[\big(\nabla_{\widehat{T}},r\nabla_4,r\nabla\big)^{\leq 4}\psi] \lesssim \epsilon_0^2+\epsilon_J^2+r_0^5BEF^{J-5}_{\delta}[\big(\nabla_{\widehat{T}},\nabla_{\widehat{R}},r\nabla\big)^{\leq 4}\psi].$$

Together with (14.2.15), we deduce, for any $0<\delta_1\leq 1$,

$$\begin{aligned} BEF^{J-5}_{\delta}[\big(\nabla_{\widehat{T}},r\nabla_4,r\nabla\big)^{\leq 4}\psi] \lesssim & r_0^5\Big(\delta_1+O(a,\epsilon)\delta_1^{-1}\Big)B^{J-1}_{\delta}[\psi]+r_0^5\delta_1^{-1}EF^{J-3}_{\delta}[r^2\triangle\psi] \\ &+r_0^5\delta_1^{-1}\int_{\mathcal{M}_{trap}}|\mathfrak{d}^{J-3}\triangle\psi|^2+r_0^5\delta_1^{-1}\Big(\epsilon_J^2+\epsilon_0^2\Big) \\ &+r_0^5\mathfrak{R}_J\mathfrak{R}_{J+1}. \end{aligned} \tag{14.2.16}$$

Step 9. Next, we derive an estimate for $BEF^{J-2}_{\delta;r\leq r_+(1+\delta_{red})}[e_3\psi]$. To this end, we commute the wave equation for ψ in (14.2.5) with ∇_3. In the region $r\leq 4m$, we obtain

$$\begin{aligned} \square_{\mathbf{g}}\big(e_3\psi\big) &= e_3\big(\square_{\mathbf{g}}\psi\big)-[e_3,\square_{\mathbf{g}}]\psi \\ &= O(1)\mathfrak{d}^{\leq 1}\psi+r^{-2}\mathfrak{d}^{\leq 2}\Gamma_b+r^{-1}\mathfrak{d}^{\leq 3}(\Gamma_b\cdot\check{R}_b)-r^2\mathfrak{d}^{\leq 2}(\underline{A}\cdot\overline{A})-[e_3,\square_{\mathbf{g}}]\psi. \end{aligned}$$

Together with the commutator formula of Lemma 9.4.5 which states that we have, for $r \leq 4m$,

$$\begin{aligned}
[\nabla_3, \square_{\mathbf{g}}] &= -\partial_r\left(\frac{\Delta}{|q|^2}\right)\nabla_3^2\psi + O(1)\nabla\nabla_3\psi + O(1)\nabla_4\nabla_3\psi \\
&\quad + O(1)\square_{\mathbf{g}}\psi + O(1)\mathfrak{d}^{\leq 1}\psi + \Gamma_b\mathfrak{d}^{\leq 2}\psi,
\end{aligned}$$

we infer

$$\begin{aligned}
\square_{\mathbf{g}}\big(e_3\psi\big) =& N_{e_3}, \\
N_{e_3} :=& \partial_r\left(\frac{\Delta}{|q|^2}\right)\nabla_3^2\psi + O(1)\nabla\nabla_3\psi + O(1)\nabla_4\nabla_3\psi + O(1)\mathfrak{d}^{\leq 1}\psi \\
& + r^{-2}\mathfrak{d}^{\leq 2}\Gamma_b + r^{-1}\mathfrak{d}^{\leq 3}(\Gamma_b\cdot\check{R}_b) - r^2\mathfrak{d}^{\leq 2}(\underline{A}\cdot\overline{A}) + \Gamma_b\mathfrak{d}^{\leq 2}\psi.
\end{aligned}$$

Making use of the favorable sign of $\partial_r\left(\frac{|\Delta|}{|q|^2}\right)$ in the red shift region $r \leq r_+(1+\delta_{red})$, and proceeding as in the redshift estimates of section 9.5, we easily infer

$$BEF^{J-2}_{\delta;r\leq r_+(1+\delta_{red})}[\nabla_3\psi] \lesssim \epsilon_0^2 + \epsilon_J^2 + \delta_{red}^{-1}BEF^{J-2}_{\delta;r_+(1+\delta_{red})\leq r\leq r_+(1+2\delta_{red})}[\nabla_3\psi].$$

This yields

$$BEF^{J-2}_{\delta;r\leq r_+(1+\delta_{red})}[\nabla_3\psi] \lesssim \epsilon_0^2 + \epsilon_J^2 + \delta_{red}^{-5}BEF^{J-5}_{\delta}[\big(\nabla_{\widehat{T}}, r\nabla_4, r\nabla\big)^{\leq 4}\psi].$$

Since

$$BEF^{J-1}_{\delta}[\psi] \lesssim \delta_{red}^{-5}BEF^{J-5}_{\delta}[\big(\nabla_{\widehat{T}}, r\nabla_4, r\nabla\big)^{\leq 4}\psi] + BEF^{J-2}_{\delta;r\leq r_+(1+\delta_{red})}[\nabla_3\psi]$$

we infer, fixing the value of $\delta_{red} > 0$ small enough for the redshift estimate used above to hold,

$$BEF^{J-1}_{\delta}[\psi] \lesssim \epsilon_0^2 + \epsilon_J^2 + BEF^{J-5}_{\delta}[\big(\nabla_{\widehat{T}}, r\nabla_4, r\nabla\big)^{\leq 4}\psi].$$

Together with (14.2.16), this yields, for any $0 < \delta_1 \leq 1$,

$$\begin{aligned}
BEF^{J-1}_{\delta}[\psi] \lesssim& r_0^5\Big(\delta_1 + O(a,\epsilon)\delta_1^{-1}\Big)B^{J-1}_{\delta}[\psi] + r_0^5\delta_1^{-1}EF^{J-3}_{\delta}[r^2\triangle\psi] \\
& + r_0^5\delta_1^{-1}\int_{\mathcal{M}_{trap}}|\mathfrak{d}^{J-3}\triangle\psi|^2 + r_0^5\delta_1^{-1}\Big(\epsilon_J^2 + \epsilon_0^2\Big) + r_0^5\mathfrak{R}_J\mathfrak{R}_{J+1}.
\end{aligned} \tag{14.2.17}$$

Step 10. In view of (14.2.17), we need to estimate $EF^{J-3}_{\delta}[r^2\Delta\psi] + \int_{\mathcal{M}_{trap}}|\mathfrak{d}^{J-3}\triangle\psi|^2$. To this end, we rely on the following identity, see Lemma 14.1.7,

$$\underline{\mathfrak{q}} = \frac{1}{2}\overline{q}q^3\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\check{P} + \mathfrak{d}^{\leq 1}\Gamma_b' + O(a)\mathfrak{d}^{\leq 1}\check{R}_b + O(ar)\mathfrak{d}^{\leq 1}\check{P} + r\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\check{R}_b).$$

Commuting with $(\not{\mathcal{L}}_{\bf T}, \not{\mathcal{L}}_{\bf Z})$, and using, as in the proof of Lemma 14.2.2,

$$\psi = q^2\big({\bf T}P, {\bf T}Z\big) = q^2({\bf T}\check{P}, {\bf Z}\check{P}) + r^{-1}\Gamma_b,$$

we easily derive the following identity for $\psi = q^2(\mathcal{L}_{\bf T}P, \mathcal{L}_{\bf Z}P)$

$$\begin{aligned} {}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\psi &= O(r^{-2})\not{\mathcal{L}}_{\bf T,Z}\underline{\mathfrak{q}} + O(r^{-2})\mathfrak{d}^{\leq 2}\Gamma_b' + O(ar^{-2})\not{\mathcal{L}}_{\bf T,Z}\mathfrak{d}^{\leq 1}\check{R}_b + O(ar^{-1})\mathfrak{d}^{\leq 1}\not{\mathcal{L}}_{\bf T,Z}\check{P} \\ &\quad + r^{-1}\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\check{R}_b). \end{aligned}$$

Making use of our bootstrap assumptions, we deduce

$$\begin{aligned} EF_\delta^{J-3}[r^{2\,(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\psi] &\lesssim EF_\delta^{J-3}[\not{\mathcal{L}}_{\bf T}\underline{\mathfrak{q}}, \not{\mathcal{L}}_{\bf Z}\underline{\mathfrak{q}}] + EF_\delta^{J-3}[\mathfrak{d}^{\leq 2}\Gamma_b'] + EF_\delta^{J-3}[r\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\check{R}_b)] \\ &\quad + O(a)EF_\delta^{J-3}[\not{\mathcal{L}}_{\bf T,Z}\mathfrak{d}^{\leq 1}\check{R}_b] + O(a)EF_\delta^{J-3}[r\mathfrak{d}^{\leq 1}\not{\mathcal{L}}_{\bf T,Z}\check{P}] \\ &\lesssim EF_\delta^{J-2}[\underline{\mathfrak{q}}] + EF_\delta^{J-1}[\Gamma_b'] + EF_\delta^{J-1}[\check{R}_b] + EF_\delta^{J-1}[r\check{P}] + \epsilon_0^2. \end{aligned}$$

Together with Lemmas 14.1.1 and 14.1.2, and the control of $\underline{\mathfrak{q}}$ in Theorem 14.1.6, we obtain

$$EF_\delta^{J-3}[r^{2\,(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\psi] \lesssim \mathfrak{G}_{J+1}\mathfrak{G}_J + \mathfrak{R}_{J+1}\mathfrak{R}_J + \epsilon_J^2 + \epsilon_0^2.$$

Using the Hodge type estimates of Corollary 13.4.1 over spheres of fixed r in $\Sigma(\tau)\cup\mathcal{A}\cup\Sigma_*$, integrating then over these regions, and making use of (14.2.4), we deduce

$$\begin{aligned} EF_\delta^{J-3}[r^2\nabla^2\psi] &\lesssim EF_\delta^{J-3}[r^{2\,(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\psi] + EF_\delta^{J-2}[\psi] + O(a+\epsilon)EF_\delta^{J-1}[\psi] \\ &\lesssim \mathfrak{G}_{J+1}\mathfrak{G}_J + \mathfrak{R}_{J+1}\mathfrak{R}_J + \epsilon_J^2 + \epsilon_0^2 + O(a+\epsilon)EF_\delta^{J-1}[\psi]. \end{aligned}$$

The term $\int_{\mathcal{M}_{trap}}|\mathfrak{d}^{J-3}\triangle\psi|^2$ is estimated by the same procedure, and is even easier to control. We obtain

$$\begin{aligned} \int_{\mathcal{M}_{trap}}|\mathfrak{d}^{J-3}\Delta\psi|^2 &\lesssim B_\delta^{J-3}[\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\psi] + O(a+\epsilon)B_\delta^{J-1}[\psi] + B_\delta^{J-2}[\psi] \\ &\lesssim \int_{\mathcal{M}_{trap}}|\nabla^2\mathfrak{d}^{J-3}\psi|^2 + O(a+\epsilon)B_\delta^{J-1}[\psi] + B_\delta^{J-2}[\psi] \\ &\lesssim B_\delta^{J-2}[\underline{\mathfrak{q}}] + B_\delta^{J-1}[\Gamma_b'] + B_\delta^{J-1}[\check{R}_b] + B_\delta^{J-1}[r\check{P}] + O(a+\epsilon)B_\delta^{J-1}[\psi] \\ &\quad + \epsilon_J^2 + \epsilon_0^2 \\ &\lesssim \mathfrak{G}_{J+1}\mathfrak{G}_J + \epsilon_J^2 + \epsilon_0^2 + O(a+\epsilon)B_\delta^{J-1}[\psi]. \end{aligned}$$

Therefore,

$$\begin{aligned} EF_\delta^{J-3}[r^2\triangle\psi] + \int_{\mathcal{M}_{trap}}|\mathfrak{d}^{J-3}\triangle\psi|^2 \lesssim& \mathfrak{G}_{J+1}\mathfrak{G}_J + \mathfrak{R}_{J+1}\mathfrak{R}_J + \epsilon_J^2 + \epsilon_0^2 \\ & + O(a+\epsilon)BEF_\delta^{J-1}[\psi]. \end{aligned} \tag{14.2.18}$$

Step 11. Combining (14.2.17) with (14.2.18), we infer, for any $0 < \delta_1 \leq 1$,

$$BEF_\delta^{J-1}[\psi] \lesssim r_0^5\Big(\delta_1 + O(a+\epsilon)\delta_1^{-1}\Big) BEF_\delta^{J-1}[\psi] + r_0^5\delta_1^{-1}\Big(\mathfrak{G}_{J+1}\mathfrak{G}_J + \mathfrak{R}_{J+1}\mathfrak{R}_J + \epsilon_J^2 + \epsilon_0^2\Big).$$

We may now fix δ_1 such that $r_0^5\delta_1$ is small, and then a and ϵ, such that both $r_0^5\delta_1$ and $r_0^5\delta_1^{-1}(a+\epsilon)$ are small enough to absorb the first term on the RHS from the LHS. We infer

$$BEF_\delta^{J-1}[\psi] \lesssim r_0^{10}\Big(\mathfrak{G}_{J+1}\mathfrak{G}_J + \mathfrak{R}_{J+1}\mathfrak{R}_J + \epsilon_J^2 + \epsilon_0^2\Big)$$

which ends the proof of Proposition 14.1.8.

14.2.2 Proof of Theorem 14.1.3

The proof of Theorem 14.1.3 proceeds along the following steps.

Step 1. Recalling the definition of ψ, we write as in the proof of Lemma 14.2.2,

$$\psi = q^2\big(\mathbf{T}P, \mathbf{T}Z\big) = q^2(\mathbf{T}\check{P}, \mathbf{Z}\check{P}) + r^{-1}\Gamma_b.$$

According to Proposition 14.1.8 and Lemma 14.1.2, we deduce

$$BEF_\delta^{J-1}[r^2(\mathbf{T}\check{P}, \mathbf{Z}\check{P})] \lesssim r_0^{10}\Big(\mathfrak{G}_{J+1}\mathfrak{G}_J + \mathfrak{R}_{J+1}\mathfrak{R}_J + \epsilon_J^2 + \epsilon_0^2\Big).$$

Recalling that $\widehat{T} = \mathbf{T} + \frac{a}{r^2+a^2}\mathbf{Z}$, we infer

$$BEF_\delta^{J-1}[r^2\widehat{T}(\check{P})] \lesssim r_0^{10}\Big(\mathfrak{G}_{J+1}\mathfrak{G}_J + \mathfrak{R}_{J+1}\mathfrak{R}_J + \epsilon_J^2 + \epsilon_0^2\Big). \tag{14.2.19}$$

Step 2. Next, renormalizing the wave equation (14.2.2) for $\check{P}$ using Lemma 5.5.3, we obtain

$$\square_{\mathbf{g}}(q^2\check{P}) = Wq^2\check{P} + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_b + \Gamma_b\cdot\mathfrak{d}^{\leq 1}\check{R}_b$$

which we rewrite as

$$\square_{\mathbf{g}}(\Psi) = W\Psi + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_b + \Gamma_b\cdot\mathfrak{d}^{\leq 1}\check{R}_b, \qquad \Psi := q^2\check{P}.$$

Therefore, using the formula of Lemma 4.7.6, we infer

$$\begin{aligned}\nabla_{\widehat{R}}^2\Psi &= \nabla_{\widehat{T}}^2\Psi + O(1)\square_{\mathbf{g}}\Psi + O(1)\triangle\Psi + O(r^{-1})\nabla_{\widehat{R}}\Psi + O(ar^{-2})\nabla\Psi + \Gamma_g\cdot\mathfrak{d}\Psi\\ &= \nabla_{\widehat{T}}^2\Psi + O(1)\triangle\Psi + O(r^{-1})\nabla_{\widehat{R}}\Psi + O(ar^{-2})\nabla\Psi + O(r^{-2})\Psi\\ &\quad + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_b + \Gamma_b\cdot\mathfrak{d}^{\leq 1}\check{R}_b.\end{aligned}$$

Arguing as in (14.2.4), we have

$$BEF_\delta^{J-1}[\Psi] \lesssim \epsilon_J^2 + \mathfrak{R}_J\mathfrak{R}_{J+1}.$$

Also, (14.2.19) yields for $\Psi = q^2\check{P}$

$$BEF_\delta^{J-1}[\widehat{T}(\Psi)] \lesssim r_0^{10}\Big(\mathfrak{G}_{J+1}\mathfrak{G}_J + \mathfrak{R}_{J+1}\mathfrak{R}_J + \epsilon_J^2 + \epsilon_0^2\Big).$$

Consequently

$$\begin{aligned} BEF_\delta^{J-2}[\nabla_{\widehat{R}}^2\Psi] &\lesssim BEF_\delta^{J-2}[\nabla_{\widehat{T}}^2\Psi] + BEF_\delta^{J-2}[\triangle\Psi] + BEF_\delta^{J-1}[\Psi] \\ &\quad + BEF_\delta^{J-2}[r^{-2}\mathfrak{d}^{\leq 1}\Gamma_b] + BEF_\delta^{J-2}[\Gamma_b\cdot\mathfrak{d}^{\leq 1}\check{R}_b] \\ &\lesssim BEF_\delta^{J-2}[\triangle\Psi] + r_0^{10}\Big(\mathfrak{G}_{J+1}\mathfrak{G}_J + \mathfrak{R}_{J+1}\mathfrak{R}_J + \epsilon_J^2 + \epsilon_0^2\Big). \end{aligned}$$

We thus deduce

$$BEF_\delta^{J-2}[\nabla_{\widehat{T}}^2\Psi, \nabla_{\widehat{R}}^2\Psi] \lesssim BEF_\delta^{J-2}[r^2\triangle\Psi] + r_0^{10}\Big(\mathfrak{G}_{J+1}\mathfrak{G}_J + \mathfrak{R}_{J+1}\mathfrak{R}_J + \epsilon_J^2 + \epsilon_0^2\Big).$$

Step 3. We can then proceed exactly as in Step 10 of section 14.2.1, with the help of the identity (14.1.7) and the Hodge estimate of Corollary 13.4.1 over spheres of fixed r either in $\mathcal{M}$ or $\Sigma(\tau)\cup\mathcal{A}\cup\Sigma_*$, to derive

$$BEF_\delta^{J-2}[r^2\nabla^2\Psi] \lesssim \mathfrak{G}_{J+1}\mathfrak{G}_J + \mathfrak{R}_{J+1}\mathfrak{R}_J + \epsilon_J^2 + \epsilon_0^2 + O(a+\epsilon)BEF_\delta^J[\Psi].$$

Together with Step 2, we infer

$$BEF_\delta^{J-2}[\nabla_{\widehat{T}}^2\Psi, \nabla_{\widehat{R}}^2\Psi, r^2\nabla^2\Psi] \lesssim r_0^{10}\Big(\mathfrak{G}_{J+1}\mathfrak{G}_J + \mathfrak{R}_{J+1}\mathfrak{R}_J + \epsilon_J^2 + \epsilon_0^2\Big) + O(a+\epsilon)BEF_\delta^J[\Psi].$$

Since $\Psi = q^2\check{P}$, we have, using Lemma 14.1.1,

$$\begin{aligned} BEF_\delta^{J-2}[r^2(\nabla_{\widehat{T}}^2, \nabla_{\widehat{R}}^2, r^2\nabla^2)\check{P}] &\lesssim BEF_\delta^{J-2}[\nabla_{\widehat{T}}^2\Psi, \nabla_{\widehat{R}}^2\Psi, r^2\nabla^2\Psi] + BEF^{J-1}[r^2\check{P}] \\ &\lesssim BEF_\delta^{J-2}[\nabla_{\widehat{T}}^2\Psi, \nabla_{\widehat{R}}^2\Psi, r^2\nabla^2\Psi] + \mathfrak{R}_J\mathfrak{R}_{J+1}, \end{aligned}$$

and hence

$$BEF_\delta^{J-2}[r^2(\nabla_{\widehat{T}}^2, \nabla_{\widehat{R}}^2, r^2\nabla^2)\check{P}] \lesssim r_0^{10}\Big(\mathfrak{G}_{J+1}\mathfrak{G}_J + \mathfrak{R}_{J+1}\mathfrak{R}_J + \epsilon_J^2 + \epsilon_0^2\Big) + O(a+\epsilon)BEF_\delta^J[r^2\check{P}].$$

Moreover, since $J \geq \frac{k_L}{2}$ and k_L is large, we may assume that $J \geq 4$. In particular, we have

$$BEF_\delta^{J-4}[r^2\big(\nabla_{\widehat{T}}, \nabla_{\widehat{R}}, r\nabla\big)^{\leq 4}\check{P}] \lesssim BEF_\delta^{J-2}[r^2\big(\nabla_{\widehat{T}}^2, \nabla_{\widehat{R}}^2, r^2\nabla^2\big)\check{P}] + BEF_\delta^{J-1}[\check{P}]$$

which yields

$$BEF_\delta^{J-4}[r^2\big(\nabla_{\widehat{T}}, \nabla_{\widehat{R}}, r\nabla\big)^{\leq 4}\check{P}] \lesssim r_0^{10}\Big(\mathfrak{G}_{J+1}\mathfrak{G}_J + \mathfrak{R}_{J+1}\mathfrak{R}_J + \epsilon_J^2 + \epsilon_0^2\Big) + O(a+\epsilon)BEF_\delta^J[r^2\check{P}]. \tag{14.2.20}$$

Step 4. (14.2.20) degenerates in the redshift region. To improve on it, we next derive estimates for $\nabla_3\check{P}$ in that region. These can be obtained, as in Step 9 in the proof of Proposition 14.1.8, according to the following steps[2].

1. First, commute the wave equation (14.2.1) for P with ∇_3 using Lemma 9.4.5 and linearize it to derive a wave equation for the linearized quantity $\widetilde{e_3(P)} = e_3(P) + \frac{6m}{q^4}$. This yields, for $r \leq 4m$,

$$\begin{aligned}\square_{\mathbf{g}}(\widetilde{e_3(P)}) &= \left(\partial_r\left(\frac{|\Delta|}{|q|^2}\right) + O\left(\frac{\Delta}{r^2}\right)\right)\nabla_3(\widetilde{e_3(P)}) + O(1)\nabla(\widetilde{e_3(P)}) \\ &\quad + O(1)e_4(\widetilde{e_3(P)}) + O(1)\mathfrak{d}^{\leq 1}\check{P} + \mathfrak{d}^{\leq 1}\Gamma_b + O(\epsilon)\mathfrak{d}^{\leq 2}\check{P}.\end{aligned}$$

2. Then, we commute further with $J-1$ non degenerate derivatives and apply the redshift estimate to the resulting commuted equation. Making use of the favorable sign of $\partial_r\left(\frac{|\Delta|}{|q|^2}\right)$ in the red shift region $r \leq r_+(1+\delta_{red})$, and proceeding as in the redshift estimates of section 9.5, we easily infer

$$BEF^{J-1}_{\delta; r\leq r_+(1+\delta_{red})}[\widetilde{e_3(P)}] \lesssim \epsilon_0^2 + \epsilon_J^2 + \delta_{red}^{-1}BEF^{J-1}_{\delta; r_+(1+\delta_{red})\leq r\leq r_+(1+2\delta_{red})}[\widetilde{e_3(P)}].$$

   This yields

$$BEF^{J-1}_{\delta; r\leq r_+(1+\delta_{red})}[\widetilde{e_3(P)}] \lesssim \epsilon_0^2 + \epsilon_J^2 + \delta_{red}^{-5}BEF_\delta^{J-4}[\big(\nabla_{\widehat{T}}, r\nabla_4, r\nabla\big)^{\leq 4}\check{P}].$$

3. Using the fact that

$$\begin{aligned}\widetilde{e_3(P)} &= e_3(P) + \frac{6m}{q^4} = e_3\left(\check{P} - \frac{2m}{q^3}\right) + \frac{6m}{q^4} \\ &= e_3(\check{P}) + r^{-3}\Gamma_b,\end{aligned}$$

   we infer

$$BEF^{J-1}_{\delta; r\leq r_+(1+\delta_{red})}[e_3(\check{P})] \lesssim \epsilon_0^2 + \epsilon_J^2 + \delta_{red}^{-5}BEF_\delta^{J-4}[\big(\nabla_{\widehat{T}}, r\nabla_4, r\nabla\big)^{\leq 4}\check{P}].$$

[2]Note that we avoid using the direct linearization (14.2.2) of the wave equation for P for the same reason as before, i.e. it leads to $\mathfrak{d}^{J+1}\Gamma_b$ on the right hand side. We thus first commute the wave equation for P with ∇_3 and then linearize.

Together with (14.2.20), we obtain, fixing the value of $\delta_{red} > 0$ small enough for the redshift estimate used above to hold,

$$\begin{aligned} BEF_\delta^{J-4}[r^2(\nabla_3, \nabla_4, r\nabla)^{\leq 4}\check{P}] \lesssim & r_0^{10}\Big(\mathfrak{G}_{J+1}\mathfrak{G}_J + \mathfrak{R}_{J+1}\mathfrak{R}_J + \epsilon_J^2 + \epsilon_0^2\Big) \\ & + O(a+\epsilon)BEF_\delta^J[r^2\check{P}]. \end{aligned} \tag{14.2.21}$$

Step 5. In view of (14.2.21), it remains to recover $r\nabla_4$ derivatives of $\check{P}$ in $r \geq r_0$. We only sketch this step:

1. As in Step 4, we do not use the direct linearization (14.2.2) of the wave equation for P. Instead, we commute the wave equation for P with re_4, and linearize it using $\widetilde{re_4(P)} = re_4(P) - \frac{6m}{q^4}\frac{r\Delta}{|q|^2}$. Finally, we then renormalize the wave equation for $\widetilde{re_4(P)}$ using Lemma 5.5.3, thus yielding a wave equation for $q^2\widetilde{re_4(P)}$.
2. We then use an r^p weighted estimate in the region $r \geq r_0$ with $p = \delta$ for the wave equation satisfied by $q^2\widetilde{re_4(P)}$ as in Step 8 of section 14.2.1.
3. As in Step 4, we relate $\widetilde{re_4(P)}$ and $re_4(\check{P})$ and deduce an r^p weighted estimate in the region $r \geq r_0$ with $p = \delta$ for $q^2 re_4(\check{P})$.

This r^p weighted procedure yields

$$BEF_{\delta; r\geq r_0}^{J-1}[r^3 e_4(\check{P})] \quad \lesssim \quad \epsilon_0^2 + \epsilon_J^2 + r_0^5 BEF_\delta^{J-4}[r^2(\nabla_3, \nabla_4, r\nabla)^{\leq 4} re_4(\check{P})]$$

which together with (14.2.21) yields

$$BEF_\delta^J[r^2\check{P}] \lesssim r_0^{15}\Big(\mathfrak{G}_{J+1}\mathfrak{G}_J + \mathfrak{R}_{J+1}\mathfrak{R}_J + \epsilon_J^2 + \epsilon_0^2\Big) + r_0^5 O(a+\epsilon)BEF_\delta^J[r^2\check{P}].$$

For a and ϵ small enough compared to r_0^{-5}, we infer

$$BEF_\delta^J[r^2\check{P}] \lesssim r_0^{15}\Big(\mathfrak{G}_{J+1}\mathfrak{G}_J + \mathfrak{R}_{J+1}\mathfrak{R}_J + \epsilon_J^2 + \epsilon_0^2\Big)$$

as stated. This ends the proof of Theorem 14.1.3

14.3 Proof of Lemmas 14.2.3, 14.2.5 and 14.2.7

In this section, we provide the proof of Lemmas 14.2.3, 14.2.5 and 14.2.7 which are used in the proof of Proposition 14.1.8 in section 14.2.1.

14.3.1 Proof of Lemma 14.2.3

The proof of Lemma 14.2.3 follows from checking that the linearized quantities $q^2\mathbf{T}P$, $q^2\mathbf{Z}P$ satisfy the following wave equations

$$\begin{aligned}\square_{\mathbf{g}}(q^2\mathbf{T}P) &= Wq^2\mathbf{T}P + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_b + r^{-1}\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\check{R}_b) - r^2\mathfrak{d}^{\leq 1}(\underline{A}\cdot\overline{A}),\\ \square_{\mathbf{g}}(q^2\mathbf{Z}P) &= Wq^2\mathbf{Z}P + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_b + r^{-1}\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\check{R}_b) - r^2\mathfrak{d}^{\leq 1}(\underline{A}\cdot\overline{A}),\end{aligned} \tag{14.3.1}$$

where the complex potential W verifies

$$\Re(W) = O(mr^{-3}), \qquad \Im(W) = O(amr^{-4}).$$

Recall the wave equation for P, see (14.2.1),

$$\square_{\mathbf{g}}P = \mathrm{tr}X\nabla_3P + \overline{\mathrm{tr}\underline{X}}\nabla_4P - \overline{H}\cdot\mathcal{D}P - \underline{H}\cdot\overline{\mathcal{D}}P + VP + r^{-3}\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\check{R}_b) - \underline{A}\cdot\overline{A}.$$

Using the commutator formula (4.3.1) for $[\mathbf{T},\square_{\mathbf{g}}]$, i.e.

$$[\mathbf{T},\square_{\mathbf{g}}]P = \mathfrak{d}\big(\Gamma_g\cdot\mathfrak{d}P\big) + \Gamma_b\cdot\square_{\mathbf{g}}P,$$

and, $[\mathbf{T},e_3] = [\mathbf{T},e_4] = [\mathbf{T},e_a] = \Gamma_b\cdot\mathfrak{d}$, we deduce

$$\begin{aligned}\square_{\mathbf{g}}(\mathbf{T}P) &= \mathbf{T}\square_{\mathbf{g}}P + \mathfrak{d}\big(\Gamma_g\cdot\mathfrak{d}P\big) + \Gamma_b\cdot\square_{\mathbf{g}}P\\ &= \mathbf{T}\Big(\mathrm{tr}X\nabla_3P + \overline{\mathrm{tr}\underline{X}}\nabla_4P - \overline{H}\cdot\mathcal{D}P - \underline{H}\cdot\overline{\mathcal{D}}P + VP + r^{-3}\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\check{R}_b) - \underline{A}\cdot\overline{A}\Big)\\ &\quad +\mathfrak{d}\big(\Gamma_g\cdot\mathfrak{d}P\big) + \Gamma_b\cdot\Big(r^{-1}\mathfrak{d}^{\leq 1}P + r^{-3}\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\check{R}_b) - \underline{A}\cdot\overline{A}\Big)\\ &= \mathrm{tr}X\nabla_3(\mathbf{T}P) + \overline{\mathrm{tr}\underline{X}}\nabla_4(\mathbf{T}P) - \overline{H}\cdot\mathcal{D}(\mathbf{T}P) - \underline{H}\cdot\overline{\mathcal{D}}(\mathbf{T}P) + V\mathbf{T}P\\ &\quad +r^{-3}\mathfrak{d}^{\leq 1}\Gamma_g + r^{-3}\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\check{R}_b) - \mathfrak{d}^{\leq 1}(\underline{A}\cdot\overline{A})\\ &\quad + \mathbf{T}(\mathrm{tr}X)\nabla_3P + \mathbf{T}(\overline{\mathrm{tr}\underline{X}})\nabla_4P - \mathbf{T}(\overline{H})\cdot\mathcal{D}P - \mathbf{T}(\underline{H})\cdot\overline{\mathcal{D}}P + \mathbf{T}(V)P\\ &\quad + \mathrm{tr}X[\mathbf{T},\nabla_3]P + \overline{\mathrm{tr}\underline{X}}[\mathbf{T},\nabla_4]P - \overline{H}\cdot[\mathbf{T},\mathcal{D}]P - \underline{H}\cdot[\mathbf{T},\overline{\mathcal{D}}]P\\ &= \mathrm{tr}X\nabla_3(\mathbf{T}P) + \overline{\mathrm{tr}\underline{X}}\nabla_4(\mathbf{T}P) - \overline{H}\cdot\mathcal{D}(\mathbf{T}P) - \underline{H}\cdot\overline{\mathcal{D}}(\mathbf{T}P) + V(\mathbf{T}P)\\ &\quad +r^{-3}\mathfrak{d}^{\leq 1}\Gamma_g + r^{-3}\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\check{R}_b) - \mathfrak{d}^{\leq 1}(\underline{A}\cdot\overline{A}).\end{aligned}$$

We are now in a position to apply Lemma 5.5.3 and deduce

$$\begin{aligned}\square_{\mathbf{g}}(q^2\mathbf{T}P) &= \Big[V + q^{-2}\square_{\mathbf{g}}(q^2)\Big]q^2\mathbf{T}P + r\Gamma_b\cdot\mathfrak{d}(\mathbf{T}P) + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_g\\ &\quad +r^{-1}\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\check{R}_b) - r^2\mathfrak{d}^{\leq 1}(\underline{A}\cdot\overline{A}).\end{aligned}$$

Thus, since $\mathbf{T}P = \mathbf{T}(\check{P} - \frac{2m}{q^3}) = \mathfrak{d}\check{P} + r^{-1}\mathfrak{d}\Gamma_g$, and $\Gamma_g = r^{-1}\Gamma_b$ in view of Remark 13.1.2,

$$\square_{\mathbf{g}}(q^2\mathbf{T}P) = W|q|^2\mathbf{T}P + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_b + r^{-1}\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\check{R}_b) - r^2\mathfrak{d}^{\leq 1}(\underline{A}\cdot\overline{A})$$

which is the first identity in (14.3.1). Also, we have $W = V + q^{-2}\square_{\mathbf{g}}(q^2)$ which satisfies indeed $\Re(W) = O(mr^{-3})$ and $\Im(W) = O(amr^{-4})$. Finally, the second identity in (14.3.1) can be derived in the same manner. This concludes the proof of (14.3.1), and hence of Lemma 14.2.3.

14.3.2 Proof of Lemma 14.2.5

Starting with the equation $\square_{\mathbf{g}}\psi = N$, and using the following commutation formula, see Lemma 4.7.12,

$$\begin{aligned}[\nabla_{\widehat{R}}, |q|^2\square_{\mathbf{g}}]\psi &= O(r)\square_{\mathbf{g}}\psi + O(r)\triangle\psi + O(ar^{-1})\mathfrak{d}^{\leq 2}\psi + O(1)\nabla_{\widehat{R}}\psi + O(ar^{-1})\nabla\psi \\ &\quad + r\mathfrak{d}\big(\Gamma_b\cdot\mathfrak{d}\psi\big),\end{aligned}$$

we obtain

$$\square_{\mathbf{g}}\psi(\nabla_{\widehat{R}}\psi) = \frac{1}{|q|^2}\nabla_{\widehat{R}}(|q|^2N) + \frac{1}{|q|^2}[\nabla_{\widehat{R}}, |q|^2\square_{\mathbf{g}}]\psi$$

and hence

$$\square_{\mathbf{g}}\nabla_{\widehat{R}}\psi = [N],$$

where

$$\begin{aligned}[N] &:= \frac{1}{|q|^2}\nabla_{\widehat{R}}(|q|^2N) + \frac{1}{|q|^2}[\nabla_{\widehat{R}}, |q|^2\square_{\mathbf{g}}]\psi \\ &= O(r^{-1})\square_{\mathbf{g}}\psi + O(r^{-1})\triangle\psi + O(ar^{-3})\mathfrak{d}^{\leq 2}\psi + O(r^{-2})\nabla_{\widehat{R}}\psi + O(ar^{-3})\nabla\psi \\ &\quad + r^{-1}\mathfrak{d}\big(\Gamma_b\cdot\mathfrak{d}\psi\big) + \nabla_{\widehat{R}}N + r^{-1}N \\ &= O(r^{-1})\triangle\psi + O(ar^{-3})\mathfrak{d}^{\leq 2}\psi + O(r^{-2})\nabla_{\widehat{R}}\psi + O(ar^{-4})\nabla\psi \\ &\quad + r^{-1}\mathfrak{d}\big(\Gamma_b\cdot\mathfrak{d}\psi\big) + \nabla_{\widehat{R}}N + r^{-1}N.\end{aligned}$$

Repeating the process, we find

$$\square_{\mathbf{g}}\nabla^2_{\widehat{R}}\psi = [[N]]$$

where

$$\begin{aligned}[[N]] &:= \frac{1}{|q|^2}\nabla_{\widehat{R}}(|q|^2[N]) + \frac{1}{|q|^2}[\nabla_{\widehat{R}}, |q|^2\square_{\mathbf{g}}]\nabla_{\widehat{R}}\psi \\ &= O(r^{-1})\triangle\widehat{R}\psi + O(ar^{-3})\mathfrak{d}^{\leq 2}\nabla_{\widehat{R}}\psi + O(r^{-2})\nabla^2_{\widehat{R}}\psi + O(ar^{-4})\nabla\nabla_{\widehat{R}}\psi \\ &\quad + r^{-1}\mathfrak{d}^{\leq 3}\big(\Gamma_b\cdot\psi\big) + \nabla_{\widehat{R}}[N] + r^{-1}[N] \\ &= O(r^{-1})\nabla_{\widehat{R}}\triangle\psi + O(r^{-2})\triangle\psi + O(ar^{-3})\nabla_{\widehat{R}}\mathfrak{d}^{\leq 2}\psi + O(r^{-2})\nabla_{\widehat{R}}\psi + O(ar^{-3})\mathfrak{d}^{\leq 2}\psi \\ &\quad + r^{-1}\mathfrak{d}^{\leq 3}\big(\Gamma_b\cdot\psi\big) + \nabla_{\widehat{R}}[N] + r^{-1}[N]\end{aligned}$$

where we used in particular the commutation formula for $[\nabla_{\widehat{R}}, \Delta]$ in Lemma 4.7.12, i.e.

$$[\nabla_{\widehat{R}}, \triangle]\psi = O(r^{-1})\triangle\psi + O(ar^{-5})\,\mathfrak{\not{\partial}}\psi + r^{-1}\mathfrak{d}\big(\Gamma_g \cdot \mathfrak{d}\psi\big).$$

Now

$$\begin{aligned}\nabla_{\widehat{R}}[N] &= \nabla_{\widehat{R}}\Big(O(r^{-1})\triangle\psi + O(ar^{-3})\mathfrak{d}^{\leq 2}\psi + O(r^{-2})\nabla_{\widehat{R}}\psi + O(ar^{-4})\nabla\psi\Big)\\ &\quad +r^{-1}\mathfrak{d}^{\leq 3}\big(\Gamma_b\cdot\psi\big) + \nabla^2_{\widehat{R}}N + r^{-1}\nabla_{\widehat{R}}N + r^{-2}N\\ &= O(r^{-1})\nabla_{\widehat{R}}\triangle\psi + O(ar^{-3})\nabla_{\widehat{R}}\mathfrak{d}^{\leq 2}\psi + O(ar^{-4})\nabla_{\widehat{R}}\nabla\psi\\ &\quad +O(r^{-2})\triangle\psi + O(r^{-2})\nabla_{\widehat{R}}\psi + O(ar^{-4})\mathfrak{d}^{\leq 2}\psi + r^{-1}\mathfrak{d}^{\leq 3}\big(\Gamma_b\cdot\psi\big)\\ &\quad +\nabla^2_{\widehat{R}}N + r^{-1}\nabla_{\widehat{R}}N + r^{-2}N.\end{aligned}$$

Therefore,

$$\begin{aligned}\square_{\mathbf{g}}\nabla^2_{\widehat{R}}\psi &= O(r^{-1})\nabla_{\widehat{R}}\triangle\psi + O(r^{-2})\triangle\psi + O(ar^{-3})\nabla_{\widehat{R}}\mathfrak{d}^{\leq 2}\psi + O(r^{-2})\nabla_{\widehat{R}}\psi + O(ar^{-3})\mathfrak{d}^{\leq 2}\psi\\ &\quad +O(r^{-1})\nabla_{\widehat{R}}\triangle\psi + O(ar^{-3})\nabla_{\widehat{R}}\mathfrak{d}^{\leq 2}\psi + O(ar^{-4})\nabla_{\widehat{R}}\nabla\psi\\ &\quad +O(r^{-2})\triangle\psi + O(r^{-2})\nabla_{\widehat{R}}\psi + O(ar^{-4})\mathfrak{d}^{\leq 2}\psi + r^{-1}\mathfrak{d}^{\leq 3}\big(\Gamma_b\cdot\psi\big)\\ &\quad +O(r^{-2})\triangle\psi + O(ar^{-4})\mathfrak{d}^{\leq 2}\psi + O(r^{-3})\nabla_{\widehat{R}}\psi + O(ar^{-5})\nabla\psi\\ &\quad +r^{-1}\mathfrak{d}^{\leq 3}\big(\Gamma_b\cdot\psi\big) + \nabla^2_{\widehat{R}}N + r^{-1}\nabla_{\widehat{R}}N + r^{-2}N\end{aligned}$$

and hence

$$\begin{aligned}\square_{\mathbf{g}}\nabla^2_{\widehat{R}}\psi &= O(r^{-1})\nabla_{\widehat{R}}\triangle\psi + O(ar^{-3})\nabla_{\widehat{R}}\mathfrak{d}^{\leq 2}\psi + O(r^{-2})\triangle\psi\\ &\quad +O(r^{-2})\nabla_{\widehat{R}}\psi + O(ar^{-3})\mathfrak{d}^{\leq 2}\psi + r^{-1}\mathfrak{d}^{\leq 3}\big(\Gamma_b\cdot\psi\big) + \nabla^2_{\widehat{R}}N + r^{-1}\nabla_{\widehat{R}}N + r^{-2}N\end{aligned}$$

as desired. This concludes the proof of Lemma 14.2.5.

14.3.3 Proof of Lemma 14.2.7

According to Lemma 14.2.2 we have

$$BEF^{J-2}_{\delta}[\psi] \lesssim \epsilon_J^2 + \mathfrak{R}_J\mathfrak{R}_{J+1}. \tag{14.3.2}$$

We apply the second estimate of Proposition 14.1.4 to equation (14.2.9) with $V = -\Re(W)$, $k = J-3$, and an inhomogeneous term N given by

$$\begin{aligned}N &= i\Im(W)\nabla^2_{\widehat{R}}\psi + O(r^{-2})\triangle\psi + O(r^{-1})\nabla_{\widehat{R}}\triangle\psi + O(ar^{-3})\Big(\nabla_{\widehat{R}}\mathfrak{d}^{\leq 2}\psi + \mathfrak{d}^{\leq 2}\psi\Big)\\ &\quad +O(r^{-2})\mathfrak{d}^{\leq 1}\psi + N^{\leq 3},\end{aligned}$$

where $N^{\leq 3}$ is defined in (14.2.9). We obtain

$$\begin{aligned}
BEF_\delta^{J-3}[\nabla^2_{\widehat{R}}\psi] &\lesssim \epsilon_0^2 + BEF_\delta^{J-4}[\nabla^2_{\widehat{R}}\psi] + \int_{\mathcal{M}_{trap}} |\mathfrak{d}^{J-3}\nabla^2_{\widehat{R}}\psi|^2 + \mathcal{N}_\delta^{J-3}[\nabla^2_{\widehat{R}}\psi, N]\\
&\lesssim \epsilon_0^2 + BEF_\delta^{J-2}[\psi] + \int_{\mathcal{M}_{trap}} \Big(|\nabla_{\widehat{R}}\mathfrak{d}^{\leq J-2}\psi|^2 + |\mathfrak{d}^{\leq J-2}\psi|^2\Big)\\
&+ \mathcal{N}_\delta^{J-3}[\nabla^2_{\widehat{R}}\psi, N]\\
&\lesssim \epsilon_0^2 + BEF_\delta^{J-2}[\psi] + \mathcal{N}_\delta^{J-3}[\nabla^2_{\widehat{R}}\psi, N]
\end{aligned}$$

which together with (14.3.2) implies

$$BEF_\delta^{J-3}[\nabla^2_{\widehat{R}}\psi] \lesssim \epsilon_0^2 + \epsilon_J^2 + \mathfrak{R}_J\mathfrak{R}_{J+1} + \mathcal{N}_\delta^{J-3}[\nabla^2_{\widehat{R}}\psi, N]. \tag{14.3.3}$$

It remains to estimate $\mathcal{N}_\delta^{J-3}[\nabla^2_{\widehat{R}}\psi, N]$. We have

$$\begin{aligned}
&\mathcal{N}_\delta^{J-3}[\nabla^2_{\widehat{R}}\psi, N]\\
=\;& {}^{(mor)}\mathcal{N}^{J-3}[\nabla^2_{\widehat{R}}\psi, N] + {}^{(en)}\mathcal{N}^{J-3}[\nabla^2_{\widehat{R}}\psi, N] + {}^{(red)}\mathcal{N}^{J-3}[\nabla^2_{\widehat{R}}\psi, N]\\
&+ \int_{{}^{(ext)}\mathcal{M}} r^\delta\Big(\big|\nabla_4\mathfrak{d}^{\leq J-3}\nabla^2_{\widehat{R}}\psi\big| + r^{-1}\big|\mathfrak{d}^{\leq J-3}\nabla^2_{\widehat{R}}\psi\big|\Big)\big|\mathfrak{d}^{\leq J-3}N\big|\\
\lesssim\;& \Big(B_\delta^{J-3}[\nabla^2_{\widehat{R}}\psi]\Big)^{\frac12}\left(\int_{\mathcal{M}} r^{\delta+1}|\mathfrak{d}^{\leq J-3}N|^2\right)^{\frac12} + {}^{(en)}\mathcal{N}^{J-3}[\nabla^2_{\widehat{R}}\psi, N]\\
\lesssim\;& \Big(B_\delta^{J-1}[\psi]\Big)^{\frac12}\left(\int_{\mathcal{M}} r^{\delta+1}|\mathfrak{d}^{\leq J-3}N|^2\right)^{\frac12} + \int_{\mathcal{M}_{trap}} \big|\nabla_{\widehat{T}_\delta}\nabla^2_{\widehat{R}}\mathfrak{d}^{\leq J-3}\psi\big|\big|\mathfrak{d}^{\leq J-3}N\big|.
\end{aligned}$$

Since $\nabla_{\widehat{T}_\delta}\nabla^2_{\widehat{R}}\mathfrak{d}^{\leq J-3}\psi = \nabla_{\widehat{R}}\mathfrak{d}^{\leq J-1}\psi + O(r^{-1})\mathfrak{d}^{\leq J-1}\psi$, we infer

$$\mathcal{N}_\delta^{J-3}[\nabla^2_{\widehat{R}}\psi, N] \lesssim \Big(B_\delta^{J-1}[\psi]\Big)^{\frac12}\left(\int_{\mathcal{M}} r^{\delta+1}|\mathfrak{d}^{\leq J-3}N|^2\right)^{\frac12}.$$

Now, using the definition of N, the fact that $\Im(W) = O(mar^{-4})$, and (14.3.2), we have

$$\begin{aligned}
\int_{\mathcal{M}} r^{\delta+1}|\mathfrak{d}^{\leq J-3}N|^2 &\lesssim |a|B_\delta^{J-1}[\psi] + B_\delta^{J-3}[r\Delta\psi] + B_\delta^{J-2}[\psi] + \int_{\mathcal{M}} r^{\delta+1}|\mathfrak{d}^{\leq J-3}N^{\leq 3}|^2\\
&\lesssim |a|B_\delta^{J-1}[\psi] + B_\delta^{J-3}[r\Delta\psi] + \epsilon_J^2 + \int_{\mathcal{M}} r^{\delta+1}|\mathfrak{d}^{\leq J-3}N^{\leq 3}|^2.
\end{aligned}$$

Together with the definition of $N^{\leq 3}$ in (14.2.9), i.e.

$$N^{\leq 3} = r^{-2}\mathfrak{d}^{\leq 3}\Gamma_b + r^{-1}\mathfrak{d}^{\leq 4}(\Gamma_b\cdot\check{R}_b) - r^2\mathfrak{d}^{\leq 3}(\underline{A}\cdot\overline{A}),$$

the induction hypothesis for the linear term involving Γ_b and the bootstrap assumptions for the nonlinear one, and Lemma 13.5.5, we infer

$$\begin{aligned}\int_{\mathcal{M}} r^{\delta+1}|\mathfrak{d}^{\leq J-3}N|^2 &\lesssim |a|B_\delta^{J-1}[\psi] + B_\delta^{J-3}[r\Delta\psi] + \epsilon_J^2 + \int_{\mathcal{M}} r^{-3+\delta}|\mathfrak{d}^{\leq J}\Gamma_b|^2 \\ &\quad +\epsilon^2\int_{\mathcal{M}} r^{-3+\delta}|\mathfrak{d}^{\leq J+1}\check{R}_b|^2 + \epsilon^2\int_{\mathcal{M}} r^{-2+\delta}|\mathfrak{d}^{\leq J+1}\Gamma_b|^2 \\ &\quad +\epsilon^2\int_{\mathcal{M}} r^{3+\delta}|\mathfrak{d}^{\leq J+1}A|^2 \\ &\lesssim |a|B_\delta^{J-1}[\psi] + B_\delta^{J-3}[r\Delta\psi] + \epsilon_J^2 + \mathfrak{G}_J^2 + \epsilon^2(\mathfrak{G}_{J+1} + \mathfrak{R}_{J+1})^2 \\ &\lesssim |a|B_\delta^{J-1}[\psi] + B_\delta^{J-3}[r\Delta\psi] + \epsilon_J^2 + \epsilon_0^2.\end{aligned}$$

This yields

$$\begin{aligned}\mathcal{N}_\delta^{J-3}[\nabla_{\widehat{R}}^2\psi, N] &\lesssim \Big(B_\delta^{J-1}[\psi]\Big)^{\frac{1}{2}}\left(\int_{\mathcal{M}} r^{\delta+1}|\mathfrak{d}^{\leq J-3}N|^2\right)^{\frac{1}{2}} \\ &\lesssim \Big(B_\delta^{J-1}[\psi]\Big)^{\frac{1}{2}}\Big(|a|B_\delta^{J-1}[\psi] + B_\delta^{J-3}[r\Delta\psi] + \epsilon_J^2 + \epsilon_0^2\Big)^{\frac{1}{2}}.\end{aligned}$$

Together with (14.3.3), we deduce

$$\begin{aligned}BEF_\delta^{J-3}[\nabla_{\widehat{R}}^2\psi] &\lesssim \epsilon_0^2 + \epsilon_J^2 + \mathfrak{R}_J\mathfrak{R}_{J+1} + \mathcal{N}_\delta^{J-3}[\nabla_{\widehat{R}}^2\psi, N] \\ &\lesssim \epsilon_0^2 + \epsilon_J^2 + \mathfrak{R}_J\mathfrak{R}_{J+1} + \Big(B_\delta^{J-1}[\psi]\Big)^{\frac{1}{2}}\Big(|a|B_\delta^{J-1}[\psi] + B_\delta^{J-3}[r\Delta\psi] + \epsilon_J^2 + \epsilon_0^2\Big)^{\frac{1}{2}}.\end{aligned}$$

Hence, we have for any $0 < \delta_1 \leq 1$

$$BEF_\delta^{J-3}[\nabla_{\widehat{R}}^2\psi] \lesssim (\delta_1 + |a|)B_\delta^{J-1}[\psi] + \delta_1^{-1}B_\delta^{J-3}[r\triangle\psi] + \delta_1^{-1}\Big(\epsilon_J^2 + \epsilon_0^2\Big) + \mathfrak{R}_J\mathfrak{R}_{J+1}$$

as stated in (14.2.10). This concludes the proof of Lemma 14.2.7.

Chapter 15

Energy-Morawetz for $A, B, \underline{B}, \underline{A}$

The goal of this chapter is to derive estimates Energy-Morawetz for the curvature components $A, B, \underline{B}, \underline{A}$ using the Bianchi identities as well as the estimates for $\check{P}$ derived in Chapter 14. This will complete the proof of the estimate (13.6.6) in Theorem 13.6.3, see section 15.1.3.

15.1 Statement of the main results of Chapter 15

In order to derive Energy-Morawetz for the curvature components $A, B, \underline{B}, \underline{A}$, we first control $(B, \underline{B})$, and then $(A, \underline{A})$.

15.1.1 Energy-Morawetz for $B, \underline{B}$

The following proposition provides energy-Morawetz estimates for $(B, \underline{B})$.

Proposition 15.1.1. *The following estimates hold true in $\mathcal{M} = \mathcal{M}(1, \tau_*)$*

$$\begin{aligned} BEF^J_\delta[r^2 B] + BEF^J_\delta[\underline{B}] \lesssim\ & \delta_{J+1}[\check{P}] + \epsilon_0^2 + \epsilon_J^2 + |a|^2\mathfrak{G}^2_{J+1} \\ & + \epsilon_J \mathfrak{R}_{J+1} + \left(\sqrt{\delta_{J+1}[\check{P}]} + \epsilon_0 + \epsilon_J\right)\mathfrak{G}_{J+1}, \end{aligned} \quad (15.1.1)$$

where

$$\delta_{J+1}[\check{P}] := BEF^J_\delta[r^2\check{P}].$$

Proposition 15.1.1 will be proved in section 15.5.

15.1.2 Energy-Morawetz for $A, \underline{A}$

The following proposition provides energy-Morawetz estimates for $(A, \underline{A})$.

Proposition 15.1.2. *The following estimates hold true in $\mathcal{M} = \mathcal{M}(1, \tau_*)$*

$$\begin{aligned} BEF^J_\delta[r^2 A] &\lesssim \delta_{J+1}[B] + \epsilon_0^2 + \epsilon_J^2 + \left(\sqrt{\delta_{J+1}[B]} + \epsilon_0 + \epsilon_J\right)\mathfrak{G}_{J+1} + |a|^2\mathfrak{G}^2_{J+1}, \\ BEF^J_\delta[\underline{A}] &\lesssim \delta_{J+1}[\underline{B}] + \epsilon_0^2 + \epsilon_J^2 + \left(\sqrt{\delta_{J+1}[\underline{B}]} + \epsilon_0 + \epsilon_J\right)\mathfrak{G}_{J+1} + |a|^2\mathfrak{G}^2_{J+1}, \end{aligned} \tag{15.1.2}$$

where

$$\delta_{J+1}[B] := BEF^J_\delta[r^2 B], \qquad \delta_{J+1}[\underline{B}] := BEF^J_\delta[\underline{B}].$$

Proposition 15.1.2 will be proved in section 15.6.

15.1.3 Proof of (13.6.6) in Theorem 13.6.3

We are now ready to prove the estimate (13.6.6) in Theorem 13.6.3 on the control of the curvature norm ${}^{(int)}\mathfrak{R}$.

Proof of (13.6.6) *of Theorem 13.6.3.* First, in view of Theorem 14.1.3 and the definition of $\delta_{J+1}[\check{P}]$, we have

$$\delta_{J+1}[\check{P}] = BEF^J_\delta[r^2\check{P}] \quad\lesssim\quad r_0^{15}\Big(\mathfrak{G}_{J+1}\mathfrak{G}_J + \mathfrak{R}_{J+1}\mathfrak{R}_J + \epsilon_J^2 + \epsilon_0^2\Big).$$

Also, in view of Proposition 15.1.1, we have

$$\begin{aligned} BEF^J_\delta[r^2 B] + BEF^J_\delta[\underline{B}] \quad&\lesssim\quad \delta_{J+1}[\check{P}] + \epsilon_0^2 + \epsilon_J^2 + |a|^2\mathfrak{G}^2_{J+1} \\ &\quad +\epsilon_J\mathfrak{R}_{J+1} + \left(\sqrt{\delta_{J+1}[\check{P}]} + \epsilon_0 + \epsilon_J\right)\mathfrak{G}_{J+1}. \end{aligned}$$

Together with the definition of $\delta_{J+1}[B]$ and $\delta_{J+1}[\underline{B}]$, and the above control of $\delta_{J+1}[\check{P}]$, we infer

$$\begin{aligned} \delta_{J+1}[\check{P}] + \delta_{J+1}[B] + \delta_{J+1}[\underline{B}] \quad&\lesssim\quad r_0^{15}\Big(\epsilon_J(\mathfrak{G}_{J+1} + \mathfrak{R}_{J+1}) + \epsilon_J^2 + \epsilon_0^2\Big) + |a|^2\mathfrak{G}^2_{J+1} \\ &\quad + r_0^{\frac{15}{2}}\mathfrak{G}_{J+1}\Big(\epsilon_0 + \sqrt{\epsilon_J}\sqrt{\mathfrak{G}_{J+1} + \mathfrak{R}_{J+1}}\Big). \end{aligned}$$

Next, in view of Proposition 15.1.2, we have

$$BEF^J_\delta[r^2A] \lesssim \delta_{J+1}[B] + \epsilon_0^2 + \epsilon_J^2 + \left(\sqrt{\delta_{J+1}[B]} + \epsilon_0 + \epsilon_J\right)\mathfrak{G}_{J+1} + |a|^2\mathfrak{G}^2_{J+1},$$
$$BEF^J_\delta[\underline{A}] \lesssim \delta_{J+1}[\underline{B}] + \epsilon_0^2 + \epsilon_J^2 + \left(\sqrt{\delta_{J+1}[\underline{B}]} + \epsilon_0 + \epsilon_J\right)\mathfrak{G}_{J+1} + |a|^2\mathfrak{G}^2_{J+1}.$$

We infer

$$\begin{aligned}
&\delta_{J+1}[\check{P}] + \delta_{J+1}[B] + \delta_{J+1}[\underline{B}] + BEF^J_\delta[r^2A] + BEF^J_\delta[\underline{A}]\\
\lesssim\ & r_0^{15}\Big(\epsilon_J(\mathfrak{G}_{J+1} + \mathfrak{R}_{J+1}) + \epsilon_J^2 + \epsilon_0^2\Big) + |a|\mathfrak{G}^2_{J+1}\\
&+ r_0^{\frac{15}{4}}\mathfrak{G}^{\frac{3}{2}}_{J+1}\Big(\epsilon_0 + \sqrt{\epsilon_J}\sqrt{\mathfrak{G}_{J+1} + \mathfrak{R}_{J+1}}\Big)^{\frac{1}{2}}.
\end{aligned}$$

Together with the definition of $\delta_{J+1}[\check{P}]$, $\delta_{J+1}[B]$ and $\delta_{J+1}[\underline{B}]$, this yields

$$\begin{aligned}
BEF^J_\delta[A, B, \check{P}, \underline{B}, \underline{A}] \ \lesssim\ & r_0^{15}\Big(\epsilon_J(\mathfrak{G}_{J+1} + \mathfrak{R}_{J+1}) + \epsilon_J^2 + \epsilon_0^2\Big) + |a|\mathfrak{G}^2_{J+1}\\
&+ r_0^{\frac{15}{4}}\mathfrak{G}^{\frac{3}{2}}_{J+1}\Big(\epsilon_0 + \sqrt{\epsilon_J}\sqrt{\mathfrak{G}_{J+1} + \mathfrak{R}_{J+1}}\Big)^{\frac{1}{2}}.
\end{aligned}$$

Also, in view of the definition of ${}^{(int)}\mathfrak{R}_k$ in section 13.5, and the one of B^k_δ in section 14.1.1, we have

$${}^{(int)}\mathfrak{R}^2_{J+1} \ \lesssim\ r_0^3 BEF^J_\delta[A, B, \check{P}, \underline{B}, \underline{A}].$$

We deduce

$$\begin{aligned}
{}^{(int)}\mathfrak{R}^2_{J+1} \ \lesssim\ & r_0^{18}\Big(\epsilon_J(\mathfrak{G}_{J+1} + \mathfrak{R}_{J+1}) + \epsilon_J^2 + \epsilon_0^2\Big) + |a| r_0^3\mathfrak{G}^2_{J+1}\\
&+ r_0^{\frac{27}{4}}\mathfrak{G}^{\frac{3}{2}}_{J+1}\Big(\epsilon_0 + \sqrt{\epsilon_J}\sqrt{\mathfrak{G}_{J+1} + \mathfrak{R}_{J+1}}\Big)^{\frac{1}{2}}
\end{aligned}$$

which concludes the proof of (13.6.6) of Theorem 13.6.3. □

The rest of the chapter is devoted to the proof of Proposition 15.1.1 and Proposition 15.1.2. We first exhibit useful properties of Bianchi pairs in sections 15.2, 15.3 and 15.4. Proposition 15.1.1 and Proposition 15.1.2 are then proved respectively in section 15.5 and 15.6.

15.2 Linearization of second and third Bianchi pairs

Observe that the first and fourth Bianchi pair, see section 13.2, involve A, B, $\underline{B}$, $\underline{A}$, $\widehat{X}$ and Ξ, as well as quadratic terms, and are therefore already in linearized form. On the other hand, in addition to B and $\underline{B}$, the second and third Bianchi pairs contain also P, tr$X$, tr$\underline{X}$, H and $\underline{H}$ which are not linearized quantities. In this section, we linearize the second and third Bianchi pairs based on commutation[1] with $\not{\mathcal{L}}_{\mathbf{T}}$. The result is stated in the following[2].

Lemma 15.2.1. *The quantities*

$$\dot{B} := \not{\mathcal{L}}_{\mathbf{T}}B, \qquad \dot{P} := \not{\mathcal{L}}_{\mathbf{T}}P = \mathbf{T}(P), \qquad \underline{\dot{B}} := \not{\mathcal{L}}_{\mathbf{T}}\underline{B},$$

verify the following equations:

1. *The linearization by $\not{\mathcal{L}}_{\mathbf{T}}$ of the second Bianchi pair can be written in the form*

$$\begin{aligned}
{}^{(c)}\nabla_3\dot{B} + tr\underline{X}\dot{B} =& \mathcal{D}\overline{\dot{P}} + O(ar^{-2})\overline{\dot{P}} + O(r^{-4})\Gamma_b + O(r^{-3})\not{\mathcal{L}}_{\mathbf{T}}\check{H} \\
&+ r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_b \cdot \check{R}_b), \\
{}^{(c)}\nabla_4\dot{P} + \frac{3}{2}trX\dot{P} =& \frac{1}{2}\mathcal{D}\cdot\overline{\dot{B}} + O(ar^{-2})\overline{\dot{B}} + O(r^{-3})\mathfrak{d}^{\leq 1}\Gamma_b + r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_b \cdot \check{R}_b).
\end{aligned} \tag{15.2.1}$$

2. *The linearization by $\not{\mathcal{L}}_{\mathbf{T}}$ of the third Bianchi pair can be written in the form*

$$\begin{aligned}
{}^{(c)}\nabla_3\dot{P} + \frac{3}{2}\overline{tr\underline{X}}\dot{P} =& -\frac{1}{2}\overline{\mathcal{D}}\cdot\underline{\dot{B}} + O(ar^{-2})\underline{\dot{B}} + O(r^{-3})\mathfrak{d}^{\leq 1}\Gamma_b + r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b \cdot \check{R}_b), \\
{}^{(c)}\nabla_4\underline{\dot{B}} + trX\underline{\dot{B}} =& -\mathcal{D}\dot{P} + O(ar^{-2})\dot{P} + O(r^{-4})\Gamma_b + O(r^{-3})\not{\mathcal{L}}_{\mathbf{T}}\check{\underline{H}} \\
&+ r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b \cdot \check{R}_b).
\end{aligned} \tag{15.2.2}$$

Remark 15.2.2. *Note that we can replace the operators $\mathcal{D}, \overline{\mathcal{D}}$ by the corresponding conformal ones ${}^{(c)}\mathcal{D}$, $\overline{{}^{(c)}\mathcal{D}}$ without changing the structure of the equations. Indeed, since $\dot{P}$ has signature 0, we have $\mathcal{D}\overline{\dot{P}} = {}^{(c)}\mathcal{D}\overline{\dot{P}}$ and $\mathcal{D}\dot{P} = {}^{(c)}\mathcal{D}\dot{P}$, while for $\dot{A}$, $\dot{B}$, $\underline{\dot{B}}$ and $\underline{\dot{A}}$, we use the fact that ${}^{(c)}\mathcal{D} = \mathcal{D} + sZ = \mathcal{D} + O(ar^{-2}) + \Gamma_g$ so that the extra terms do indeed not change the structure of the equations.*

Proof. We apply first $\not{\mathcal{L}}_{\mathbf{T}}$ to the Bianchi identity involving ${}^{(c)}\nabla_3 B$, i.e., see section 13.2,

$${}^{(c)}\nabla_3 B + \mathrm{tr}\underline{X}B = \mathcal{D}\overline{P} + 3\overline{P}H + r^{-2}\Gamma_b \cdot \check{R}_b,$$

where we have used the fact that ${}^{(c)}\mathcal{D}\overline{P} = \mathcal{D}\overline{P}$ by definition since P has signature 0. Using the commutator estimates, see Lemma 2.2.13, we have $[\not{\mathcal{L}}_{\mathbf{T}}, \mathcal{D}]P = 0$ since P is a scalar. Also, $[\not{\mathcal{L}}_{\mathbf{T}}, \nabla_3]B = r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_b \cdot \check{R}_b)$ in view of Lemma 13.3.3. Therefore

$$\begin{aligned}
{}^{(c)}\nabla_3\dot{B} + \mathrm{tr}\underline{X}\dot{B} &= \mathcal{D}\overline{\dot{P}} + 3\overline{\dot{P}}H + 3\overline{P}\not{\mathcal{L}}_{\mathbf{T}}H + r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_b \cdot \check{R}_b) \\
&= \mathcal{D}\overline{\dot{P}} + O(ar^{-2})\overline{\dot{P}} + O(r^{-3})\not{\mathcal{L}}_{\mathbf{T}}H + r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_b \cdot \check{R}_b).
\end{aligned}$$

[1]Recall the definition of $\not{\mathcal{L}}_X$ in section 2.2.8.

[2]See also Remark 13.2.1.

We write $\not{\mathcal{L}}_{\mathbf{T}} H = \not{\mathcal{L}}_{\mathbf{T}} \check{H} + \not{\mathcal{L}}_{\mathbf{T}}\left(\frac{aq}{|q|^2}\mathfrak{J}\right) = \not{\mathcal{L}}_{\mathbf{T}} \check{H} + O(r^{-1})\Gamma_b$ and hence

$$ {}^{(c)}\nabla_3 \dot{B} + \mathrm{tr}\underline{X}\dot{B} = \mathcal{D}\overline{\dot{P}} + O(ar^{-2})\overline{\dot{P}} + O(r^{-3})\not{\mathcal{L}}_{\mathbf{T}}\check{H} + O(r^{-4})\Gamma_b + r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_b \cdot \check{R}_b) $$

as stated.

Next, we apply $\not{\mathcal{L}}_{\mathbf{T}}$ to the Bianchi identity involving ${}^{(c)}\nabla_4 \underline{B}$, i.e., see section 13.2, and we deduce as above

$$ \begin{aligned} {}^{(c)}\nabla_4 \dot{\underline{B}} + \mathrm{tr}X\dot{\underline{B}} &= -\mathcal{D}\dot{P} - 3\dot{P}\,\underline{H} - 3P\not{\mathcal{L}}_{\mathbf{T}}\,\underline{H} + r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b \cdot \check{R}_b) \\ &= -\mathcal{D}\dot{P} + O(ar^{-2})\dot{P} + O(r^{-3})\not{\mathcal{L}}_{\mathbf{T}}\,\underline{H} + O(r^{-4})\Gamma_b + r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b \cdot \check{R}_b). \end{aligned} $$

The two remaining equations can be obtained in the same manner and are in fact easier. This concludes the proof of Lemma 15.2.1. □

15.3 Hyperbolic estimates for Bianchi pairs

15.3.1 Complex Hodge operators

Consider the following complex Hodge operators

$$ \not{\mathcal{D}}_2^* = -\frac{1}{2}\mathcal{D}\widehat{\otimes}, \qquad \not{\mathcal{D}}_2 = \frac{1}{2}\overline{\mathcal{D}}\cdot, \qquad \not{\mathcal{D}}_1^* = -\mathcal{D}, \qquad \not{\mathcal{D}}_1 = \frac{1}{2}\overline{\mathcal{D}}\cdot. \tag{15.3.1} $$

Remark 15.3.1. *Observe that these operators are the complexified version of the real Hodge operators* $\not{\mathcal{D}}_k$ *and* $\not{\mathcal{D}}_k^*$*, see section 2.1.3. More precisely, for* $\Psi = \psi + i\,{}^*\psi$ *a complex* $\mathfrak{s}_p$*-tensor with* $p = 1, 2$*, or for a complex scalar* $\Psi = (\psi + i\psi_*)$*, we have*

$$ \begin{aligned} \Re(\not{\mathcal{D}}_2^*\Psi) &= 2\not{\mathcal{D}}_2^*\Re(\Psi) = 2\not{\mathcal{D}}_2^*\psi, \\ \Re(\not{\mathcal{D}}_2\Psi) &= \not{\mathcal{D}}_2\Re(\Psi) = \not{\mathcal{D}}_2\psi, \\ \Re(\not{\mathcal{D}}_1^*\Psi) &= \not{\mathcal{D}}_1^*(\psi, \psi_*), \\ \not{\mathcal{D}}_1\Psi &= div\,\psi + i\,curl\,\psi. \end{aligned} $$

In particular, above and in the rest of Part III, the same notation $\not{\mathcal{D}}_k$ *and* $\not{\mathcal{D}}_k^*$ *is used for two different operators, i.e. if* Ψ *is complex, then* $\not{\mathcal{D}}_k\Psi$ *and* $\not{\mathcal{D}}_k^*\Psi$ *should be interpreted as in* (15.3.1)*, while if* ψ *is real,* $\not{\mathcal{D}}_k\psi$ *and* $\not{\mathcal{D}}_k^*\psi$ *should be interpreted as in section 2.1.3.*

The following lemma gives a sense to the fact that $\not{\mathcal{D}}_p^*$ is a formal adjoint to $\not{\mathcal{D}}_p$.

Lemma 15.3.2. *Given $\Psi_{(1)} \in \mathfrak{s}_p$, $\Psi_{(2)} \in \mathfrak{s}_{p-1}$, or $\Psi_{(2)} \in \mathfrak{s}_p$, $\Psi_{(1)} \in \mathfrak{s}_{p-1}$, we have, in both cases, for $p = 1, 2$,*

$$\Re\Big(\mathcal{D}\mkern-10mu /_p \Psi_{(1)} \cdot \overline{\Psi_{(2)}}\Big) - \frac{1}{2}\Re\Big(\Psi_{(1)} \cdot \overline{\mathcal{D}\mkern-10mu /_p^* \Psi_{(2)}}\Big) = \nabla \cdot \Re\Big(\Psi_{(1)} \cdot \overline{\Psi_{(2)}}\Big). \tag{15.3.2}$$

Proof. We give below the proof for both cases $p = 1, 2$.

Case $p = 2$: We have for $\Psi_{(1)} = \psi_{(1)} + i\,{}^*\psi_{(1)} \in \mathfrak{s}_2$ and $\Psi_{(2)} = \psi_{(2)} + i\,{}^*\psi_{(2)} \in \mathfrak{s}_1$,

$$\begin{aligned}
&\Re\Big(\mathcal{D}\mkern-10mu /_2 \Psi_{(1)} \cdot \overline{\Psi_{(2)}}\Big) - \frac{1}{2}\Re\Big(\Psi_{(1)} \cdot \overline{\mathcal{D}\mkern-10mu /_2^* \Psi_{(2)}}\Big)\\
&= \frac{1}{2}\Re\Big((\overline{\mathcal{D}} \cdot \Psi_{(1)}) \cdot \overline{\Psi_{(2)}}\Big) + \frac{1}{4}\Re\Big(\Psi_{(1)} \cdot \overline{\mathcal{D}\widehat{\otimes}\, \Psi_{(2)}}\Big)\\
&= \Re\Big((\mathrm{d}iv\,\psi_{(1)} + i\,{}^*(\mathrm{d}iv\,\psi_{(1)})) \cdot \overline{\Psi_{(2)}}\Big) + \frac{1}{2}\Re\Big(\Psi_{(1)} \cdot (\nabla\widehat{\otimes}\psi_{(2)} - i\,{}^*(\nabla\widehat{\otimes}\psi_{(2)}))\Big)\\
&= 2(\mathrm{d}iv\,\psi_{(1)}) \cdot \psi_{(2)} + \psi_{(1)} \cdot (\nabla\widehat{\otimes}\psi_{(2)}).
\end{aligned}$$

Using that we have, for $f \in \mathfrak{s}_1$ and $u \in \mathfrak{s}_2$,

$$(\nabla\widehat{\otimes} f) \cdot u = \Big(\nabla_a f_b + \nabla_b f_a - \delta_{ab}\mathrm{d}iv\, f\Big)u_{ab} = 2(\nabla_a f_b)u_{ab} = 2\nabla_a(u_{ab} f_b) - 2(\mathrm{d}iv\, u) \cdot f,$$

we obtain

$$\Re\Big(\mathcal{D}\mkern-10mu /_2 \Psi_{(1)} \cdot \overline{\Psi_{(2)}}\Big) - \frac{1}{2}\Re\Big(\Psi_{(1)} \cdot \overline{\mathcal{D}\mkern-10mu /_2^* \Psi_{(2)}}\Big) = 2\nabla \cdot (\psi_{(1)} \cdot \psi_{(2)}) = \nabla \cdot \Re(\Psi_{(1)} \cdot \overline{\Psi_{(2)}})$$

as stated.

Case $p = 1$: we have for $\Psi_{(1)} = \psi_{(1)} + i\,{}^*\psi_{(1)} \in \mathfrak{s}_1$ and $\Psi_{(2)} = a + ib \in \mathfrak{s}_0$,

$$\begin{aligned}
&\Re\Big(\mathcal{D}\mkern-10mu /_1 \Psi_{(1)} \cdot \overline{\Psi_{(2)}}\Big) - \frac{1}{2}\Re\Big(\Psi_{(1)} \cdot \overline{\mathcal{D}\mkern-10mu /_1^* \Psi_{(2)}}\Big)\\
&= \frac{1}{2}\Re\Big((\overline{\mathcal{D}} \cdot \Psi_{(1)}) \cdot \overline{\Psi_{(2)}}\Big) + \frac{1}{2}\Re\Big(\Psi_{(1)} \cdot \overline{\mathcal{D}\, \Psi_{(2)}}\Big)\\
&= \Re\Big((\mathrm{d}iv\,\psi_{(1)} + i curl\,\psi_{(1)}) \cdot \overline{\Psi_{(2)}}\Big) + \frac{1}{2}\Re\Big(\Psi_{(1)} \cdot (\nabla a - {}^*\nabla b - i({}^*\nabla a + \nabla b))\Big)\\
&= a\ \mathrm{d}iv\,\psi_{(1)} + b\ curl\,\psi_{(1)} + \psi_{(1)} \cdot (\nabla a - {}^*\nabla b).
\end{aligned}$$

Using that we have, for $f \in \mathfrak{s}_1$,

$$\begin{aligned}
f \cdot \nabla a &= f_c \nabla_c a = \nabla_c(a f_c) - a(\mathrm{d}iv\, f),\\
f \cdot {}^*\nabla b &= -{}^*f_c \nabla_c b = -\nabla_c(b\,{}^*f_c) + b(curl\, f),
\end{aligned}$$

we obtain

$$\Re\Big(\mathcal{D}\mkern-10mu /_1 \Psi_{(1)} \cdot \overline{\Psi_{(2)}}\Big) - \frac{1}{2}\Re\Big(\Psi_{(1)} \cdot \overline{\mathcal{D}\mkern-10mu /_1^* \Psi_{(2)}}\Big) = \nabla \cdot (a\psi_{(1)} + b\,{}^*\psi_{(1)}) = \nabla \cdot \Re(\Psi_{(1)} \cdot \overline{\Psi_{(2)}}),$$

as stated. This concludes the proof of Lemma 15.3.2. ∎

We also derive the following complex, non-integrable version of the identities (2.1.16).

Lemma 15.3.3. *The following formulas hold true*

$$\begin{aligned}
\not{\mathcal{D}}_1 \not{\mathcal{D}}_1^* \Psi &= -\triangle_0 \Psi - \frac{1}{2} i ({}^{(a)}tr\chi \nabla_3 + {}^{(a)}tr\underline{\chi} \nabla_4) \Psi, \\
\not{\mathcal{D}}_1^* \not{\mathcal{D}}_1 \Psi &= -\triangle_1 \Psi + \frac{1}{2} i ({}^{(a)}tr\chi \nabla_3 + {}^{(a)}tr\underline{\chi} \nabla_4) \Psi + {}^{(h)}K \Psi, \\
\not{\mathcal{D}}_2 \not{\mathcal{D}}_2^* \Psi &= -\triangle_1 \Psi - \frac{1}{2} i ({}^{(a)}tr\chi \nabla_3 + {}^{(a)}tr\underline{\chi} \nabla_4) \Psi - {}^{(h)}K \Psi, \\
\not{\mathcal{D}}_2^* \not{\mathcal{D}}_2 \Psi &= -\triangle_2 \psi + \frac{1}{2} i ({}^{(a)}tr\chi \nabla_3 + {}^{(a)}tr\underline{\chi} \nabla_4) \Psi + 2\, {}^{(h)}K \Psi,
\end{aligned} \tag{15.3.3}$$

with the scalar ${}^{(h)}K$ *given by the formula*[3] (2.1.30).

Proof. We start with the identities

$$\begin{aligned}
\not{\mathcal{D}}_1 \not{\mathcal{D}}_1^* \Psi &= -\triangle_0 \Psi - \frac{1}{2} i \in^{ab} [\nabla_a, \nabla_b] \Psi, \\
\not{\mathcal{D}}_1^* \not{\mathcal{D}}_1 \Psi &= -\triangle_1 \Psi + \frac{1}{2} i \in_{ab} [\nabla_a, \nabla_b] \Psi, \\
\not{\mathcal{D}}_2 \not{\mathcal{D}}_2^* \Psi &= -\triangle_1 \Psi - \frac{1}{2} i \in_{ab} [\nabla_a, \nabla_b] \Psi, \\
\not{\mathcal{D}}_2^* \not{\mathcal{D}}_2 \Psi &= -\triangle_2 \Psi + \frac{1}{2} i \in_{ab} [\nabla_a, \nabla_b] \Psi.
\end{aligned}$$

The last three identities are immediate consequences of Lemma 2.1.36 and Remark 15.3.1. It thus remains to check the first one. For $\Psi \in \mathfrak{s}_0$, we have

$$\begin{aligned}
\not{\mathcal{D}}_1 \not{\mathcal{D}}_1^* \Psi &= -\frac{1}{2} \overline{\mathcal{D}}^a \mathcal{D}_a \Psi = -\frac{1}{2} (\nabla^a - i\, {}^*\nabla^a)(\nabla_a + i\, {}^*\nabla_a) \Psi \\
&= -\frac{1}{2} \triangle \Psi - \frac{1}{2}\, {}^*\nabla^a ({}^*\nabla_a \Psi) + \frac{1}{2} i \in^{ab} (\nabla_b \nabla_a \Psi - \nabla_a \nabla_b \Psi) \\
&= -\triangle \Psi - \frac{1}{2} i \in_{ab} [\nabla_a, \nabla_b] \Psi,
\end{aligned}$$

as stated.

We rewrite the formulas above by making use of the Gauss type formula of Proposition 2.1.43 according to which we have, for $\Psi \in \mathfrak{s}_p(\mathbb{C})$ with $p = 0, 1, 2$,

$$\in^{ab} [\nabla_a, \nabla_b] \Psi = ({}^{(a)}\mathrm{tr}\chi \nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi} \nabla_4) \Psi - 2 i p\, {}^{(h)}K \Psi.$$

[3] Recall that ${}^{(h)}K$ coincides with the Gauss curvature K in the integrable case.

We deduce

$$\begin{aligned}
\mathcal{D}_1\,\mathcal{D}_1^*\Psi &= -\triangle_0\Psi - i\frac{1}{2}\in^{ab}[\nabla_a,\nabla_b]\Psi = -\triangle_0\Psi - \frac{1}{2}i({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4)\Psi,\\
\mathcal{D}_1^*\,\mathcal{D}_1\Psi &= -\triangle_1\Psi + \frac{1}{2}i\in^{ab}[\nabla_a,\nabla_b]\Psi = -\triangle_1\Psi + \frac{1}{2}i({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4)\Psi + {}^{(h)}K\Psi,\\
\mathcal{D}_2\,\mathcal{D}_2^*\Psi &= -\triangle_1\Psi - \frac{1}{2}i\in_{ab}[\nabla_a,\nabla_b]\Psi = -\triangle_1\Psi - \frac{1}{2}i({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4)\Psi - {}^{(h)}K\Psi,\\
\mathcal{D}_2^*\mathcal{D}_2\Psi &= -\triangle_2\Psi + \frac{1}{2}i\in_{ab}[\nabla_a,\nabla_b]\Psi = -\triangle_2\psi + \frac{1}{2}i({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4)\Psi + 2\,{}^{(h)}K\Psi,
\end{aligned}$$

as stated. This concludes the proof of Lemma 15.3.3. □

15.3.2 Bianchi pairs using the Hodge operators $\mathcal{D}_p$, $\mathcal{D}_p^*$

We rewrite below the Bianchi equations, see section 13.2, using the complex Hodge operators introduced in section 15.3.1. We split them in pairs as follows.

Definition 15.3.4. *We define the following pairs of Bianchi identities:*

1. *The first pair, involving A and B:*

$$\begin{aligned}
{}^{(c)}\nabla_3 A + \frac{1}{2}tr\underline{X}A &= -\mathcal{D}_2^* B + O(ar^{-2})B + O(r^{-3})\widehat{X} + r^{-2}\Gamma_b\cdot\check{R}_b,\\
{}^{(c)}\nabla_4 B + 2\overline{trX}B &= \mathcal{D}_2 A + O(ar^{-2})A + O(r^{-3})\Xi + r^{-2}\Gamma_b\cdot\check{R}_b.
\end{aligned} \tag{15.3.4}$$

2. *The second pair, involving B and $\overline{P}$:*

$$\begin{aligned}
{}^{(c)}\nabla_3 B + tr\underline{X}B &= -\mathcal{D}_1^*\overline{P} + 3\overline{P}H + r^{-2}\Gamma_b\cdot\check{R}_b,\\
{}^{(c)}\nabla_4\overline{P} + \frac{3}{2}\overline{trX}\,\overline{P} &= \mathcal{D}_1 B + O(ar^{-2})\overline{B} + r^{-2}\Gamma_b\cdot\check{R}_b.
\end{aligned} \tag{15.3.5}$$

3. *The third pair, involving P and $\underline{B}$:*

$$\begin{aligned}
{}^{(c)}\nabla_3 P + \frac{3}{2}\overline{tr\underline{X}}P &= -\mathcal{D}_1\underline{B} + O(ar^{-2})\underline{B} + r^{-1}\Gamma_b\cdot\check{R}_b,\\
{}^{(c)}\nabla_4\underline{B} + trX\underline{B} &= \mathcal{D}_1^* P - 3P\,\underline{H} + r^{-1}\Gamma_b\cdot\check{R}_b.
\end{aligned} \tag{15.3.6}$$

4. *The fourth pair, involving $\underline{B}$ and $\underline{A}$:*

$$\begin{aligned}
{}^{(c)}\nabla_3\underline{B} + 2\overline{tr\underline{X}}\,\underline{B} &= -\mathcal{D}_2\underline{A} + O(ar^{-2})\underline{A} - O(r^{-3})\,\underline{\Xi} + \Gamma_b\cdot\check{R}_b,\\
{}^{(c)}\nabla_4\underline{A} + \frac{1}{2}trX\underline{A} &= \mathcal{D}_2^*\underline{B} + O(ar^{-2})\underline{B} + O(r^{-3})\underline{\widehat{X}} + \Gamma_b\cdot\check{R}_b.
\end{aligned} \tag{15.3.7}$$

15.3.3 Main lemma for Bianchi pairs

We start with the following definition exhibiting the general form of Bianchi pairs.

Definition 15.3.5. *We consider the following general Bianchi pairs in $\mathcal{M}$, which generalize the Bianchi pairs written as in Definition 15.3.4:*

- *For $\Psi_{(1)} \in \mathfrak{s}_p$, $\Psi_{(2)} \in \mathfrak{s}_{p-1}$, and $F_{(1)} \in \mathfrak{s}_p$, $F_{(2)} \in \mathfrak{s}_{p-1}$,*

$$\begin{aligned} {}^{(c)}\nabla_3(\Psi_{(1)}) + c_{(1)} tr\underline{X}\Psi_{(1)} &= -\,{\not{\mathcal{D}}}_p^*\Psi_{(2)} + F_{(1)}, \\ {}^{(c)}\nabla_4(\Psi_{(2)}) + c_{(2)}\overline{trX}\Psi_{(2)} &= \,{\not{\mathcal{D}}}_p\,\Psi_{(1)} + F_{(2)}. \end{aligned} \tag{15.3.8}$$

- *For $\Psi_{(1)} \in \mathfrak{s}_{p-1}$, $\Psi_{(2)} \in \mathfrak{s}_p$, and $F_{(1)} \in \mathfrak{s}_{p-1}$, $F_{(2)} \in \mathfrak{s}_p$,*

$$\begin{aligned} {}^{(c)}\nabla_3(\Psi_{(1)}) + c_{(1)}\overline{tr\underline{X}}\Psi_{(1)} &= -\,{\not{\mathcal{D}}}_p\,\Psi_{(2)} + F_{(1)}, \\ {}^{(c)}\nabla_4(\Psi_{(2)}) + c_{(2)} trX\Psi_{(2)} &= \,{\not{\mathcal{D}}}_p^*\Psi_{(1)} + F_{(2)}. \end{aligned} \tag{15.3.9}$$

Remark 15.3.6. *Note that the third and fourth Bianchi pairs, according to Definition 15.3.4, are of the type* (15.3.9)*. Also, the first and second Bianch pairs are of the type* (15.3.8) *provided we write $\Psi_{(1)} = B$, $\Psi_{(2)} = \overline{P}$ for the second Bianchi pair.*

Remark 15.3.7. *We can also define the general Bianchi pairs in a conformally invariant way. Recall the conformal operators defined for tensors f of signature s by ${}^{(c)}\nabla_a f = \nabla_a f + s\zeta_a f$ and ${}^{(c)}\mathcal{D} = {}^{(c)}\nabla_a + i\,{}^*\,{}^{(c)}\nabla_a$. By introducing the conformal operators*

$$^{(c)}\,{\not{\mathcal{D}}}_2^* \;=\; -\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}, \qquad ^{(c)}\,{\not{\mathcal{D}}}_2 = \frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot, \qquad ^{(c)}\,{\not{\mathcal{D}}}_1^* = -\,{}^{(c)}\mathcal{D}, \qquad ^{(c)}\,{\not{\mathcal{D}}}_1 = \frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot,$$

we can define the above pairs with ${}^{(c)}\,{\not{\mathcal{D}}}_p^*$ and ${}^{(c)}\,{\not{\mathcal{D}}}_p$, i.e.

$$\begin{aligned} {}^{(c)}\nabla_3(\Psi_{(1)}) + c_{(1)} tr\underline{X}\Psi_{(1)} &= -\,{}^{(c)}\,{\not{\mathcal{D}}}_p^*\Psi_{(2)} + F_{(1)}, \\ {}^{(c)}\nabla_4(\Psi_{(2)}) + c_{(2)}\overline{trX}\Psi_{(2)} &= \,{}^{(c)}\,{\not{\mathcal{D}}}_p\,\Psi_{(1)} + F_{(2)}, \end{aligned} \tag{15.3.10}$$

and

$$\begin{aligned} {}^{(c)}\nabla_3(\Psi_{(1)}) + c_{(1)}\overline{tr\underline{X}}\Psi_{(1)} &= -\,{}^{(c)}\,{\not{\mathcal{D}}}_p\,\Psi_{(2)} + F_{(1)}, \\ {}^{(c)}\nabla_4(\Psi_{(2)}) + c_{(2)} trX\Psi_{(2)} &= \,{}^{(c)}\,{\not{\mathcal{D}}}_p^*\Psi_{(1)} + F_{(2)}. \end{aligned} \tag{15.3.11}$$

Note that the use of these conformally invariant horizontal Hodge operators does not modify the structure of the terms $F_{(1)}$ and $F_{(2)}$, see Remark 15.2.2.

Lemma 15.3.8. *Let $\Psi_{(1)}, \Psi_{(2)}$ verifying either one of the equations* (15.3.8), (15.3.9) *for positive real numbers $c_{(1)}$ and $c_{(2)}$, with $\Psi_{(1)}$ of signature k and $\Psi_{(2)}$ of signature $k-1$. Then denoting*

$$\Lambda_{(1)} := -2c_{(1)} + 1 + \frac{b}{2}, \qquad \Lambda_{(2)} := -2c_{(2)} + 1 + \frac{b}{2},$$

the following pointwise identity holds true for any real b:

1. *If $\Psi_{(1)}, \Psi_{(2)}$ verify equation* (15.3.8), *then*

$$\begin{aligned}
&Div\left(\frac{1}{2}|q|^b|\Psi_{(1)}|^2 e_3 + |q|^b|\Psi_{(2)}|^2 e_4 - 2|q|^b\Re(\Psi_{(1)}\cdot\overline{\Psi_{(2)}})\right)\\
=\ & \frac{1}{2}|q|^b\Lambda_{(1)}tr\underline{\chi}|\Psi_{(1)}|^2 + |q|^b\Lambda_{(2)}tr\,\chi|\Psi_{(2)}|^2 + |q|^b(2k-1)\big(\underline{\omega}|\Psi_{(1)}|^2 - 2\omega|\Psi_{(2)}|^2\big)\\
&+O(ar^{b-2})\cdot\Re(\Psi_{(1)}\cdot\overline{\Psi_{(2)}}) + |q|^b\Re\Big(F_{(1)}\cdot\overline{\Psi_{(1)}}\Big) + 2|q|^b\Re\Big(F_{(2)}\cdot\overline{\Psi_{(2)}}\Big)\\
&+r^b\Gamma_b\Big(|\Psi_{(1)}|^2 + |\Psi_{(1)}||\Psi_{(2)}| + r^{-1}|\Psi_{(2)}|^2\Big). \qquad (15.3.12)
\end{aligned}$$

2. *If $\Psi_{(1)}, \Psi_{(2)}$ verify equation* (15.3.9), *then*

$$\begin{aligned}
&Div\left(|q|^b|\Psi_{(1)}|^2 e_3 + \frac{1}{2}|q|^b|\Psi_{(2)}|^2 e_4 + 2|q|^b\Re(\Psi_{(1)}\cdot\overline{\Psi_{(2)}})\right)\\
=\ & |q|^b\Lambda_{(1)}tr\underline{\chi}|\Psi_{(1)}|^2 + \frac{1}{2}|q|^b\Lambda_{(2)}tr\,\chi|\Psi_{(2)}|^2 + |q|^b(2k-1)\big(2\underline{\omega}|\Psi_{(1)}|^2 - \omega|\Psi_{(2)}|^2\big)\\
&+O(ar^{b-2})\cdot\Re(\Psi_{(1)}\cdot\overline{\Psi_{(2)}}) + 2|q|^b\Re\Big(F_{(1)}\cdot\overline{\Psi_{(1)}}\Big) + |q|^b\Re\Big(F_{(2)}\cdot\overline{\Psi_{(2)}}\Big)\\
&+r^b\Gamma_b\Big(|\Psi_{(1)}|^2 + |\Psi_{(1)}||\Psi_{(2)}| + r^{-1}|\Psi_{(2)}|^2\Big). \qquad (15.3.13)
\end{aligned}$$

Remark 15.3.9. *Note that the two identities differ by a factor of $\frac{1}{2}$ when interchanging $\Psi_{(1)}$ and $\Psi_{(2)}$, and by the sign of the third term in the first line.*

Proof. We note first

$$\begin{aligned}
\mathbf{D}_\gamma e_4^\gamma &= -\frac{1}{2}\mathbf{g}(\mathbf{D}_4 e_4, e_3) - \frac{1}{2}\mathbf{g}(\mathbf{D}_3 e_4, e_4) + \gamma^{ab}\mathbf{g}(\mathbf{D}_a e_4, e_b) = \mathrm{tr}\,\chi - 2\omega,\\
\mathbf{D}_\gamma e_3^\gamma &= -\frac{1}{2}\mathbf{g}(\mathbf{D}_4 e_3, e_3) - \frac{1}{2}\mathbf{g}(\mathbf{D}_3 e_3, e_4) + \gamma^{ab}\mathbf{g}(\mathbf{D}_a e_3, e_b) = \mathrm{tr}\,\underline{\chi} - 2\underline{\omega}.
\end{aligned}$$

We now calculate,

$$
\begin{aligned}
& \mathrm{Div}\Big(|q|^b|\Psi_{(1)}|^2 e_3\Big)\\
= \; & 2|q|^b\Re\Big(\nabla_3\Psi_{(1)}\cdot\overline{\Psi_{(1)}}\Big)+\frac{b}{2}|q|^{b-2}\big(\overline{q}e_3(q)+qe_3(\overline{q})\big)|\Psi_{(1)}|^2+|q|^b|\Psi_{(1)}|^2\mathbf{D}_\gamma e_3^\gamma\\
= \; & 2|q|^b\Re\Big(\nabla_3\Psi_{(1)}\cdot\overline{\Psi_{(1)}}\Big)+\frac{b}{2}|q|^{b-2}\big(\overline{q}e_3(q)+qe_3(\overline{q})\big)|\Psi_{(1)}|^2+|q|^b(\mathrm{tr}\,\underline{\chi}-2\underline{\omega})|\Psi_{(1)}|^2\\
= \; & 2|q|^b\Re\Big(\nabla_3\Psi_{(1)}\cdot\overline{\Psi_{(1)}}\Big)+|q|^b\left(1+\frac{b}{2}\right)\mathrm{tr}\,\underline{\chi}|\Psi_{(1)}|^2-2\underline{\omega}|q|^b|\Psi_{(1)}|^2\\
& +\frac{b}{2}|q|^{b-2}\big(\overline{q}e_3(q)+qe_3(\overline{q})-\mathrm{tr}\,\underline{\chi}|q|^2\big)|\Psi_{(1)}|^2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\mathrm{Div}\Big(|q|^b|\Psi_{(2)}|^2 e_4\Big) &= 2|q|^b\Re\Big(\nabla_4\Psi_{(2)}\cdot\overline{\Psi_{(2)}}\Big)+|q|^b\left(1+\frac{b}{2}\right)tr\,\chi|\Psi_{(2)}|^2-2\omega|q|^b|\Psi_{(2)}|^2\\
& +\frac{b}{2}|q|^{b-2}\big(\overline{q}e_4(q)+qe_4(\overline{q})-tr\,\chi|q|^2\big)|\Psi_{(2)}|^2.
\end{aligned}
$$

Note that[4]

$$
\overline{q}e_3(q)+qe_3(\overline{q})-\mathrm{tr}\,\underline{\chi}|q|^2=r^2\Gamma_b, \qquad \overline{q}e_4(q)+qe_4(\overline{q})-tr\,\chi|q|^2=r^2\Gamma_g.
$$

Hence

$$
\begin{aligned}
\mathrm{Div}\Big(|q|^b|\Psi_{(1)}|^2 e_3\Big) &= 2|q|^b\Re\Big(\nabla_3\Psi_{(1)}\cdot\overline{\Psi_{(1)}}\Big)+|q|^b\left(1+\frac{b}{2}\right)\mathrm{tr}\,\underline{\chi}|\Psi_{(1)}|^2-2\underline{\omega}|q|^b|\Psi_{(1)}|^2\\
& +r^b\Gamma_b|\Psi_{(1)}|^2,\\
\mathrm{Div}\Big(|q|^b|\Psi_{(2)}|^2 e_4\Big) &= 2|q|^b\Re\Big(\nabla_4\Psi_{(2)}\cdot\overline{\Psi_{(2)}}\Big)+|q|^b\left(1+\frac{b}{2}\right)tr\,\chi|\Psi_{(2)}|^2-2\omega|q|^b|\Psi_{(2)}|^2\\
& +r^{b-1}\Gamma_b|\Psi_{(2)}|^2.
\end{aligned}
$$

We now consider the two cases:

Case 1. Consider the case when $\Psi_{(1)}, \Psi_{(2)}$ verify equations (15.3.8), with $\Psi_{(1)}$ of signature k and $\Psi_{(2)}$ of signature $k-1$. Then

$$
\begin{aligned}
{}^{(c)}\nabla_3\Psi_{(1)} &= \nabla_3\Psi_{(1)}-2k\underline{\omega}\Psi_{(1)},\\
{}^{(c)}\nabla_4\Psi_{(2)} &= \nabla_4\Psi_{(2)}+2(k-1)\omega\Psi_{(2)}.
\end{aligned}
$$

[4]In particular, we use the fact that $-(q+\overline{q})-\Re(-\frac{2}{q})|q|^2=-(q+\overline{q})+2\Re(q)=0$.

Hence, using the equation for $\Psi_{(1)}$,

$$\begin{aligned}
&\Re\Big(\nabla_3\Psi_{(1)}\cdot\overline{\Psi_{(1)}}\Big)\\
=&\ \Re\Big(\big(-c_{(1)}\mathrm{tr}\underline{X}\Psi_{(1)}+2k\underline{\omega}\Psi_{(1)}-\,{\not{\mathcal{D}}}_p^*\Psi_{(2)}+F_{(1)}\big)\cdot\overline{\Psi_{(1)}}\Big)\\
=&\ -c_{(1)}\mathrm{tr}\,\underline{\chi}|\Psi_{(1)}|^2+2k\underline{\omega}|\Psi_{(1)}|^2-\Re\big(\,{\not{\mathcal{D}}}_p^*\Psi_{(2)}\cdot\overline{\Psi_{(1)}}\big)+\Re\Big(F_{(1)}\cdot\overline{\Psi_{(1)}}\Big),
\end{aligned}$$

and using the equation for $\Psi_{(2)}$,

$$\begin{aligned}
&\Re\Big(\nabla_4\Psi_{(2)}\cdot\overline{\Psi_{(2)}}\Big)\\
=&\ \Re\Big(\big(-c_{(2)}\overline{\mathrm{tr}X}\Psi_{(2)}-2(k-1)\omega\Psi_{(2)}+\,{\not{\mathcal{D}}}_p\Psi_{(1)}\cdot\overline{\Psi_{(2)}}+F_{(2)}\big)\cdot\overline{\Psi_{(2)}}\Big)\\
=&\ -c_{(2)}tr\,\chi|\Psi_{(2)}|^2-2(k-1)\omega|\Psi_{(2)}|^2+\Re\big(\,{\not{\mathcal{D}}}_p\Psi_{(1)}\cdot\overline{\Psi_{(2)}}\big)+\Re\big(F_{(2)}\cdot\overline{\Psi_{(2)}}\big),
\end{aligned}$$

we deduce,

$$\begin{aligned}
&\mathrm{Div}\Big(|q|^b|\Psi_{(1)}|^2e_3\Big)\\
=&\ 2|q|^b\Big(-c_{(1)}\mathrm{tr}\,\underline{\chi}|\Psi_{(1)}|^2+2k\underline{\omega}|\Psi_{(1)}|^2-\Re\big(\,{\not{\mathcal{D}}}_p^*\Psi_{(2)}\cdot\overline{\Psi_{(1)}}\big)\Big)\\
&+|q|^b\left(1+\frac{b}{2}\right)\mathrm{tr}\,\underline{\chi}|\Psi_{(1)}|^2-2\underline{\omega}|q|^b|\Psi_{(1)}|^2+2|q|^b\Re\big(F_{(1)}\cdot\overline{\Psi_{(1)}}\big)+r^b\Gamma_b|\Psi_{(1)}|^2\\
=&\ |q|^b\mathrm{tr}\,\underline{\chi}|\Psi_{(1)}|^2\left(-2c_{(1)}+1+\frac{b}{2}\right)+2(2k-1)|q|^b\underline{\omega}|\Psi_{(1)}|^2-2|q|^b\Re\big(\,{\not{\mathcal{D}}}_p^*\Psi_{(2)}\cdot\overline{\Psi_{(1)}}\big)\\
&+2|q|^b\Re\big(F_{(1)}\cdot\overline{\Psi_{(1)}}\big)+r^b\Gamma_b|\Psi_{(2)}|^2
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Div}\Big(|q|^b|\Psi_{(2)}|^2e_4\Big)\\
=&\ 2|q|^b\Big(-c_{(2)}tr\,\chi|\Psi_{(2)}|^2-2(k-1)\omega|\Psi_{(2)}|^2+\Re\big(\,{\not{\mathcal{D}}}_p\Psi_{(1)}\cdot\overline{\Psi_{(2)}}\big)\Big)\\
&2|q|^b\Re\big(F_{(2)}\cdot\overline{\Psi_{(2)}}\big)+|q|^b\left(1+\frac{b}{2}\right)tr\,\chi|\Psi_{(2)}|^2-2\omega|q|^b|\Psi_{(2)}|^2+r^b|\Psi_{(2)}|^2\\
=&\ |q|^btr\,\chi|\Psi_{(2)}|^2\left(-2c_{(2)}+1+\frac{b}{2}\right)-2(2k-1)\omega|\Psi_{(2)}|^2+2|q|^b\Re\big(\,{\not{\mathcal{D}}}_p\Psi_{(1)}\cdot\overline{\Psi_{(2)}}\big)\\
&+2|q|^b\Re\big(F_{(2)}\cdot\overline{\Psi_{(2)}}\big)+r^{b-1}\Gamma_b|\Psi_{(2)}|^2.
\end{aligned}$$

We deduce, making use of the relation (15.3.2),

$$\begin{aligned}
&\frac{1}{2}\mathrm{Div}\Big(|q|^b|\Psi_{(1)}|^2e_3\Big)+\mathrm{Div}\Big(|q|^b|\Psi_{(2)}|^2e_4\Big)-2|q|^b\nabla\cdot\Re\Big(\Psi_{(1)}\cdot\overline{\Psi_{(2)}}\Big)\\
=&\ \frac{1}{2}|q|^b\Lambda_{(1)}\mathrm{tr}\,\underline{\chi}|\Psi_{(1)}|^2+|q|^b\Lambda_{(2)}tr\,\chi|\Psi_{(2)}|^2+(2k-1)|q|^b\Big(\underline{\omega}|\Psi_{(1)}|^2-2\omega|\Psi_{(2)}|^2\Big)\\
&+|q|^b\Re\Big(F_{(1)}\cdot\overline{\Psi_{(1)}}\Big)+2|q|^b\Re\Big(F_{(2)}\cdot\overline{\Psi_{(2)}}\Big)+r^b\Gamma_b\Big(|\Psi_{(1)}|^2+r^{-1}|\Psi_{(2)}|^2\Big).
\end{aligned}$$

Finally, using Lemma 2.1.40, i.e. $\mathbf{D}^\alpha f_\alpha = \nabla^a f_a + (\eta + \underline{\eta}) \cdot f$,

$$\begin{aligned}
|q|^b \nabla \cdot \Re\Big(\Psi_{(1)} \cdot \overline{\Psi_{(2)}}\Big) &= \nabla \cdot \Big(|q|^b \Re(\Psi_{(1)} \cdot \overline{\Psi_{(2)}})\Big) - \nabla(|q|^b) \cdot \Re(\Psi_{(1)} \cdot \overline{\Psi_{(2)}}) \\
&= \mathrm{Div}\Big(|q|^b \Re(\Psi_{(1)} \cdot \overline{\Psi_{(2)}})\Big) - \left(1 + \frac{b}{2}\right) |q|^b (\eta + \underline{\eta}) \cdot \Re(\Psi_{(1)} \cdot \overline{\Psi_{(2)}}) \\
&\quad + r^b \Gamma_b \cdot \Re(\Psi_{(1)} \cdot \overline{\Psi_{(2)}})
\end{aligned}$$

where we used that $\nabla(|q|^b) = \frac{b}{2}(\eta + \underline{\eta})|q|^b + r^b \Gamma_b$. Combining with the above, we obtain (15.3.12).

Case 2. Consider the case when $\Psi_{(1)}, \Psi_{(2)}$ verify equation (15.3.9), with $\Psi_{(1)}$ of signature k and $\Psi_{(2)}$ of signature $k - 1$. Then

$$\begin{aligned}
& \Re\Big(\nabla_3 \Psi_{(1)} \cdot \overline{\Psi_{(1)}}\Big) \\
= \;& \Re\Big(\big(- c_{(1)} \overline{\mathrm{tr}\underline{X}} \Psi_{(1)} + 2k\underline{\omega}\Psi_{(1)} - \,{\not{\mathcal{D}}}_p \Psi_{(2)} + F_{(1)}\big) \cdot \overline{\Psi_{(1)}}\Big) \\
= \;& -c_{(1)} \mathrm{tr}\,\underline{\chi} |\Psi_{(1)}|^2 + 2k\underline{\omega}|\Psi_{(1)}|^2 - \Re\Big({\not{\mathcal{D}}}_p \Psi_{(2)} \cdot \overline{\Psi_{(1)}}\Big) + \Re\Big(F_{(1)} \cdot \overline{\Psi_{(1)}}\Big),
\end{aligned}$$

$$\begin{aligned}
& \Re\Big(\nabla_4 \Psi_{(2)} \cdot \overline{\Psi_{(2)}}\Big) \\
= \;& \Re\Big(\big(- c_{(2)} \mathrm{tr} X \Psi_{(2)} - 2(k-1)\omega\Psi_{(2)} + \,{\not{\mathcal{D}}}_p^\star \Psi_{(1)}\big) \cdot \overline{\Psi_{(2)}} + F_{(2)} \cdot \overline{\Psi_{(2)}}\Big) \\
= \;& -c_{(2)} \mathrm{tr}\,\chi |\Psi_{(2)}|^2 - 2(k-1)\omega|\Psi_{(2)}|^2 + \Re\Big({\not{\mathcal{D}}}_p^\star \Psi_{(1)} \cdot \overline{\Psi_{(2)}}\Big) + \Re\Big(F_{(2)} \cdot \overline{\Psi_{(2)}}\Big).
\end{aligned}$$

Using again relation (15.3.2) and Lemma 2.1.40, we obtain (15.3.13), which concludes the proof of Lemma 15.3.8. □

Remark 15.3.10. *Recall that in Kerr, we have*

$$\underline{\omega} = 0, \quad \omega = -\frac{a^2 \cos^2\theta (r - m) + mr^2 - a^2 r}{|q|^4} < 0, \quad tr\underline{\chi} = -\frac{2r}{|q|^2}, \quad tr\,\chi = \frac{2r\Delta}{|q|^4}.$$

The dominant terms on the right hand side of (15.3.12) *and* (15.3.13) *are given by, modulo a factor of* 2*,*

$$|q|^{-b} J \;\; := \;\; \Lambda_{(1)} tr\underline{\chi} |\Psi_{(1)}|^2 + \frac{1}{2}\Lambda_{(2)} tr\,\chi |\Psi_{(2)}|^2 + (2k-1)\big(2\underline{\omega}|\Psi_{(1)}|^2 - \omega|\Psi_{(2)}|^2\big).$$

Then:

1. *The first two Bianchi pairs are applied in situations where $\Psi_{(2)}$ is already under control and $2k - 1$ is strictly positive, see part 1 in Proposition 15.3.12 below. To*

derive estimates for $\Psi_{(1)}$, we need the coefficient $\Lambda_{(1)} tr\underline{\chi} + 2(2k-1)\underline{\omega}$ to be strictly negative. Thus, since $\underline{\omega} = \Gamma_b$ and $tr\underline{\chi} < 0$, we need to choose b such that

$$\Lambda_{(1)} = -2c_{(1)} + 1 + \frac{b}{2} > 0.$$

2. *By contrast, the last two Bianchi pairs are applied in situations where $\Psi_{(1)}$ is under control and $2k-1$ is strictly negative, see part 2 in Proposition 15.3.12 below. To derive estimates for $\Psi_{(2)}$, we need the coefficient $\Lambda_{(2)} tr\,\chi - 2(2k-1)\omega$ to be strictly negative. Since $\omega < 0$ and is non degenerate near $r = r_+$, and since $tr\,\chi > 0$ for $r > r_+$, it thus suffices to choose b such that*

$$\Lambda_{(2)} = -2c_{(2)} + 1 + \frac{b}{2} < 0.$$

15.3.4 Main integrated estimates for the Bianchi pairs

Lemma 15.3.11 (Divergence lemma). *Consider a vectorfield X in $\mathcal{M}(\tau_1,\tau_2)$. We have*

$$-\int_{\mathcal{A}(\tau_1,\tau_2)} \mathbf{g}(X,N) - \int_{\Sigma(\tau_2)} \mathbf{g}(X,N) - \int_{\Sigma_*(\tau_1,\tau_2)} \mathbf{g}(X,N) + \int_{\Sigma(\tau_1)} \mathbf{g}(X,N) = \int_{\mathcal{M}(\tau_1,\tau_2)} Div(X),$$

where N is the normal to the boundary such that $\mathbf{g}(N, e_3) = -1$.

We rewrite it in the form

$$-\int_{\partial^+\mathcal{M}(\tau_1,\tau_2)} \mathbf{g}(X,N) + \int_{\partial^-\mathcal{M}(\tau_1,\tau_2)} \mathbf{g}(X,N) = \int_{\mathcal{M}(\tau_1,\tau_2)} Div(X),$$

where

$$\partial^+\mathcal{M}(\tau_1,\tau_2) = \mathcal{A}(\tau_1,\tau_2) \cup \Sigma(\tau_2) \cup \Sigma_*(\tau_1,\tau_2), \qquad \partial^-\mathcal{M}(\tau_1,\tau_2) = \Sigma(\tau_1).$$

Proof. Immediate consequence of the standard divergence lemma. □

We use the divergence lemma to obtain integrated estimates for the Bianchi pairs in the following proposition.

Proposition 15.3.12. *The following estimates[5] hold true in $\mathcal{M} = \mathcal{M}(\tau_1,\tau_2)$.*

[5]Note that the roles of $\Psi_{(1)}$ and $\Psi_{(2)}$ are inverted in the two estimates. This is due to the fact that, in applications, $\Psi_{(2)}$ is already under control for the first Bianchi pair, while $\Psi_{(1)}$ is already under control for the second Bianchi pair.

1. *Let* $\Psi_{(1)}, \Psi_{(2)}$ *verifying equations* (15.3.8) *for positive real numbers* $c_{(1)}$ *and* $c_{(2)}$ *with* $2k-1>0$. *Let* b *such that* $\Lambda_{(1)} = -2c_{(1)} + 1 + \frac{b}{2} > 0$. *Then, the following integrated estimate holds:*

$$\begin{aligned}
&\int_{\mathcal{M}} r^{b-1}|\Psi_{(1)}|^2 + \int_{\partial^+\mathcal{M}} r^b|\Psi_{(1)}|^2 \\
\lesssim &\int_{\mathcal{M}} r^{b-1}|\Psi_{(2)}|^2 + \left|\int_{\mathcal{M}} |q|^b \Re(F_{(1)} \cdot \overline{\Psi_{(1)}})\right| + \int_{\mathcal{M}} r^b|F_{(2)}||\Psi_{(2)}| \\
&+ \int_{\partial^-\mathcal{M}} \left(r^b|\Psi_{(1)}|^2 + r^{b-2}|\Psi_{(2)}|^2\right).
\end{aligned} \tag{15.3.14}$$

2. *Let* $\Psi_{(1)}, \Psi_{(2)}$ *verifying equations* (15.3.9) *for positive real numbers* $c_{(1)}$ *and* $c_{(2)}$, *with* $2k-1<0$. *Let* b *such that* $\Lambda_{(2)} = -2c_{(2)} + 1 + \frac{b}{2} < 0$. *Then the following integrated estimate holds:*

$$\begin{aligned}
&\int_{\mathcal{M}} r^{b-1}\left(1 + |2k-1|\frac{m}{r}\right)|\Psi_{(2)}|^2 + \int_{\partial^+\mathcal{M}} r^{b-2}|\Psi_{(2)}|^2 \\
\lesssim &\int_{\mathcal{M}} r^{b-1}|\Psi_{(1)}|^2 + \left|\int_{\mathcal{M}} |q|^b \Re(F_{(2)} \cdot \overline{\Psi_{(2)}})\right| + \int_{\mathcal{M}} r^b|F_{(1)}||\Psi_{(1)}| \\
&+ \int_{\partial^-\mathcal{M}} \left(r^b|\Psi_{(1)}|^2 + r^{b-2}|\Psi_{(2)}|^2\right).
\end{aligned} \tag{15.3.15}$$

Proof. We proceed as follows.

Case 1. We first consider $\Psi_{(1)}, \Psi_{(2)}$ verifying equation (15.3.8) with $2k-1>0$, and we apply Lemma 15.3.11 to the vectorfield

$$X := \frac{1}{2}|q|^b|\Psi_{(1)}|^2 e_3 + |q|^b|\Psi_{(2)}|^2 e_4 - 2|q|^b\Re\Big(\Psi_{(1)} \cdot \overline{\Psi_{(2)}}\Big).$$

Using (15.3.12), we obtain

$$\begin{aligned}
\text{Div}X =& \frac{1}{2}|q|^b\Lambda_{(1)}\text{tr}\,\underline{\chi}|\Psi_{(1)}|^2 + |q|^b\Lambda_{(2)}\text{tr}\,\chi|\Psi_{(2)}|^2 + |q|^b(2k-1)\big(\underline{\omega}|\Psi_{(1)}|^2 - 2\omega|\Psi_{(2)}|^2\big) \\
&+ O(ar^{b-2}) \cdot \Re(\Psi_{(1)} \cdot \overline{\Psi_{(2)}}) + |q|^b\Re\Big(F_{(1)} \cdot \overline{\Psi_{(1)}}\Big) + 2|q|^b\Re\Big(F_{(2)} \cdot \overline{\Psi_{(2)}}\Big) \\
&+ r^b\Gamma_b\Big(|\Psi_{(1)}|^2 + |\Psi_{(2)}|^2\Big).
\end{aligned} \tag{15.3.16}$$

Let b such that $\Lambda_{(1)} = -2c_{(1)} + 1 + \frac{b}{2} > 0$. Since $\text{tr}\,\underline{\chi} = -\frac{2r}{|q|^2} + \Gamma_g$ and $\underline{\omega} \in \Gamma_b$, we can bound the above by

$$\begin{aligned}
\text{Div}(X) \leq& -r|q|^{b-2}\Big(\Lambda_{(1)} - r\Gamma_b\Big)|\Psi_{(1)}|^2 + O(r^{b-1})|\Psi_{(2)}|^2 \\
&+ |q|^b\Re\Big(F_{(1)} \cdot \overline{\Psi_{(1)}}\Big) + 2|q|^b\Re\Big(F_{(2)} \cdot \overline{\Psi_{(2)}}\Big).
\end{aligned}$$

We deduce, since $r \le |q| \le \sqrt{2}r$,

$$\mathrm{Div}(X) \le -\frac{\Lambda_{(1)}}{2} r^{b-1}|\Psi_{(1)}|^2 + O(r^{b-1})|\Psi_{(2)}|^2 + |q|^b \Re\Big(F_{(1)} \cdot \overline{\Psi_{(1)}}\Big) + 2|q|^b \Re\Big(F_{(2)} \cdot \overline{\Psi_{(2)}}\Big).$$

In view of Lemma 15.3.11, integrated in the region $\mathcal{M}(\tau_1, \tau_2)$, we obtain

$$\begin{aligned} -\int_{\partial^+\mathcal{M}} \mathbf{g}(X,N) + \int_{\partial^-\mathcal{M}} \mathbf{g}(X,N) \;\; &\lesssim \;\; -\int_{\mathcal{M}} r^{b-1}|\Psi_{(1)}|^2 + \int_{\mathcal{M}} r^{b-1}|\Psi_{(2)}|^2 \\ &+ \left| \int_{\mathcal{M}} |q|^b \Re(F_{(1)} \cdot \overline{\Psi_{(1)}}) \right| + \int_{\mathcal{M}} r^b |F_{(2)}||\Psi_{(2)}|. \end{aligned}$$

Hence

$$\begin{aligned} \int_{\mathcal{M}} r^{b-1}|\Psi_{(1)}|^2 - \int_{\partial^+\mathcal{M}} \mathbf{g}(X,N) \;\; &\lesssim \;\; -\int_{\partial^-\mathcal{M}} \mathbf{g}(X,N) + \int_{\mathcal{M}} r^{b-1}|\Psi_{(2)}|^2 \\ &+ \left| \int_{\mathcal{M}} |q|^b \Re(F_{(1)} \cdot \overline{\Psi_{(1)}}) \right| + \int_{\mathcal{M}} r^b |F_{(2)}||\Psi_{(2)}|. \end{aligned}$$

For the boundary terms, we compute

$$\mathbf{g}(X,N) \;\; = \;\; |q|^b \left(\frac{1}{2}|\Psi_{(1)}|^2 \mathbf{g}(e_3, N) + |\Psi_{(2)}|^2 \mathbf{g}(e_4, N) - 2\mathbf{g}(N, e_a)\Re\big(\Psi_{(1)} \cdot \overline{\Psi_{(2)}}\big)_a \right),$$

and recall that $\partial^+\mathcal{M}(\tau_1, \tau_2) = \mathcal{A}(\tau_1, \tau_2) \cup \Sigma(\tau_2) \cup \Sigma_*(\tau_1, \tau_2)$ and $\partial^-\mathcal{M}(\tau_1, \tau_2) = \Sigma(\tau_1)$. Based on the assumptions made in sections 13.1.1 and 13.1.5, we have the following lemma.

Lemma 15.3.13. *The following inequalities hold on $\mathcal{A}$, Σ_* and $\Sigma(\tau)$:*

- *On $\mathcal{A}$ we have*

  $$\mathbf{g}(N_{\mathcal{A}}, e_3) = -1, \qquad \mathbf{g}(N_{\mathcal{A}}, e_4) \le -\frac{1}{10}\delta_{\mathcal{H}}, \qquad \mathbf{g}(N_{\mathcal{A}}, e_a) = O(\delta_{\mathcal{H}}).$$

  Therefore

  $$\begin{aligned} -\mathbf{g}(X,N) \;\; &= \;\; -|q|^b \left(\frac{1}{2}|\Psi_{(1)}|^2 \mathbf{g}(e_3, N) + |\Psi_{(2)}|^2 \mathbf{g}(e_4, N) - 2\mathbf{g}(N, e_a)\Re\big(\Psi_{(1)} \cdot \overline{\Psi_{(2)}}\big)_a \right) \\ &\gtrsim \;\; |\Psi_{(1)}|^2 - O(\delta_{\mathcal{H}})|\Psi_{(2)}|^2. \end{aligned}$$

- *On the boundary Σ_* we have, with $N_* = N_{\Sigma_*}$,*

  $$\mathbf{g}(N_*, e_3) = -1, \qquad \mathbf{g}(N_*, e_4) \le -1, \qquad \mathbf{g}(N_*, e_a) = O(r^{-1}).$$

  Therefore

  $$\begin{aligned} -\mathbf{g}(X,N_*) \;\; &= \;\; -|q|^b \left(\frac{1}{2}|\Psi_{(1)}|^2 \mathbf{g}(e_3, N_*) + |\Psi_{(2)}|^2 \mathbf{g}(e_4, N) - 2\mathbf{g}(N_*, e_a)\Re\big(\Psi_{(1)} \cdot \overline{\Psi_{(2)}}\big)_a \right) \\ &\gtrsim \;\; r^b\big(|\Psi_{(1)}|^2 + |\Psi_{(2)}|^2\big). \end{aligned}$$

- *On the boundary* $\Sigma = \Sigma(\tau)$

$$\mathbf{g}(N_\Sigma, N_\Sigma) \leq -\frac{1}{100}\frac{m^2}{r^2}, \quad e_4(\tau) \geq \frac{1}{100}\frac{m^2}{r^2}, \quad e_3(\tau) = 1, \quad |\nabla\tau|^2 \leq \frac{99}{100}e_4(\tau)e_3(\tau).$$

Therefore[6]

$$\begin{aligned} -\mathbf{g}(X, N) &= -|q|^b\left(\frac{1}{2}|\Psi_{(1)}|^2\mathbf{g}(e_3, N) + |\Psi_{(2)}|^2\mathbf{g}(e_4, N) - 2\mathbf{g}(N, e_a)\Re\big(\Psi_{(1)}\cdot\overline{\Psi_{(2)}}\big)_a\right) \\ &\gtrsim r^b|\Psi_{(1)}|^2 + r^{b-2}|\Psi_{(2)}|^2. \end{aligned}$$

Using Lemma 15.3.13 to control the boundary terms, we finally obtain

$$\begin{aligned} \int_{\mathcal{M}} r^{b-1}|\Psi_{(1)}|^2 + \int_{\partial^+\mathcal{M}} r^b|\Psi_{(1)}|^2 \lesssim \int_{\mathcal{M}} r^{b-1}|\Psi_{(2)}|^2 + \left|\int_{\mathcal{M}} |q|^b\Re(F_{(1)}\cdot\overline{\Psi_{(1)}})\right| + \int_{\mathcal{M}} r^b|F_{(2)}||\Psi_{(2)}| \\ + \int_{\partial^-\mathcal{M}} \big(r^b|\Psi_{(1)}|^2 + r^{b-2}|\Psi_{(2)}|^2\big). \end{aligned}$$

Case 2. To obtain the second part of the proposition, we consider $\Psi_{(1)}, \Psi_{(2)}$ verifying equation (15.3.9) with $2k-1<0$ and apply Lemma 15.3.11 to the vectorfield

$$X := |q|^b|\Psi_{(1)}|^2e_3 + \frac{1}{2}|q|^b|\Psi_{(2)}|^2e_4 + 2|q|^b\Re\Big(\Psi_{(2)}\cdot\overline{\Psi_{(1)}}\Big).$$

According to identity (15.3.13), we have

$$\begin{aligned} \mathrm{Div}X &= |q|^b\Lambda_{(1)}\mathrm{tr}\,\underline{\chi}|\Psi_{(1)}|^2 + \frac{1}{2}|q|^b\Lambda_{(2)}\mathrm{tr}\,\chi|\Psi_{(2)}|^2 + |q|^b(2k-1)\big(2\underline{\omega}|\Psi_{(1)}|^2 - \omega|\Psi_{(2)}|^2\big) \\ &+ O(ar^{b-2})\cdot\Re(\Psi_{(1)}\cdot\overline{\Psi_{(2)}}) + 2|q|^b\Re\Big(F_{(1)}\cdot\overline{\Psi_{(1)}}\Big) + |q|^b\Re\Big(F_{(2)}\cdot\overline{\Psi_{(2)}}\Big) \\ &+ r^b\Gamma_b\Big(|\Psi_{(1)}|^2 + |\Psi_{(2)}|^2\Big). \end{aligned}$$

Since $\Lambda_{(2)} < 0$, $\mathrm{tr}\,\chi = \frac{2r\Delta}{|q|^4} + \Gamma_g$, $2k-1<0$ and $\omega \gtrsim -\frac{m}{r^2} + \Gamma_g$, the coefficient C of $|q|^b|\Psi_{(2)}|^2$ is given by[7]

$$\begin{aligned} C &= \frac{1}{2}\Lambda_{(2)}\mathrm{tr}\,\chi - (2k-1)\omega = \frac{r\Delta}{|q|^4}\Lambda_{(2)} + |2k-1|\omega + \Gamma_g \\ &\gtrsim -\frac{|\Lambda_{(2)}|}{r} - \frac{m}{r^2}|2k-1|. \end{aligned}$$

---

[6]Note that

$$2|\mathbf{g}(N, e_a)\Re\big(\Psi_{(1)}\cdot\overline{\Psi_{(2)}}\big)_a| \leq \sqrt{2}|\nabla\tau||\Psi_{(1)}||\Psi_{(2)}| \leq \frac{99}{100}\left(\frac{1}{2}|\Psi_{(1)}|^2e_3(\tau) + |\Psi_{(2)}|^2e_4(\tau)\right).$$

[7]In particular, we use the fact that $\Delta \geq 0$ on $r \geq r_+$, that $\Delta = O(\delta_{\mathcal{H}})$ on $\mathcal{M}(r \leq r_+)$, and that $\omega \gtrsim -\frac{m}{r^2}$ on $\mathcal{M}$.

The desired estimate follows, as in the first case, by integration on $\mathcal{M}(\tau_1, \tau_2)$, with the boundary terms being treated in the same manner as before. This concludes the proof of Proposition 15.3.12. □

15.4 Bianchi pairs for higher derivatives

The goal of this section is to commute the equations (15.3.8), (15.3.9) with higher derivatives in ${}^{(c)}\nabla_3$, ${}^{(c)}\nabla_4$, ${}^{(c)}\nabla_{\widehat{R}}$, see section 13.1.4 for the definition of these conformally invariant operators, and prove the following two propositions.

Proposition 15.4.1. *The following higher order Bianchi identities hold true:*

- *If* $\Psi_{(1)}, \Psi_{(2)}$ *verify the equations* (15.3.8)*, then the quantities*

$$\widetilde{\Psi}_{(1,k)} = (\overline{q}\,{}^{(c)}\nabla_4)^k\,{}^{(c)}\nabla^2_{\widehat{R}}\Psi_{(1)}, \qquad \widetilde{\Psi}_{(2,k)} = (\overline{q}\,{}^{(c)}\nabla_4)^k\,{}^{(c)}\nabla^2_{\widehat{R}}\Psi_{(2)},$$

verify the equations

$$\begin{aligned}
{}^{(c)}\nabla_3\widetilde{\Psi}_{(1,k)} + \left(c_{(1)} - \frac{k}{2}\right) tr\underline{X}\,\widetilde{\Psi}_{(1,k)} &= -\,\mathcal{D}\!\!\!/_p^*\widetilde{\Psi}_{(2,k)} + \widetilde{F}_{(1,k)},\\
{}^{(c)}\nabla_4\widetilde{\Psi}_{(2,k)} + \left(c_{(2)} - \frac{k}{2}\right) \overline{trX}\,\widetilde{\Psi}_{(2,k)} &= \mathcal{D}\!\!\!/_p\widetilde{\Psi}_{(1,k)} + \widetilde{F}_{(2,k)},
\end{aligned} \tag{15.4.1}$$

where

$$\begin{aligned}
\widetilde{F}_{(1,k)} &= \left(\overline{q}\,{}^{(c)}\nabla_4\right)^k\,{}^{(c)}\nabla^2_{\widehat{R}}F_{(1)} - 2\omega\left(\overline{q}\,{}^{(c)}\nabla_4\right)^k\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_{\widehat{R}}\Psi_{(1)}\\
&\quad + O(r^{-1})\mathfrak{d}^{\leq k+1}\,{}^{(c)}\nabla_{\widehat{R}}\Psi_{(2)} + O(ar^{-2})\mathfrak{d}^{\leq k}\,\mathfrak{d}\!\!\!/\,{}^{(c)}\nabla_{\widehat{R}}\Psi_{(1)}\\
&\quad + O(r^{-1})\mathfrak{d}^{\leq k+1}\left(\Psi_{(1)}, \Psi_{(2)}\right) + \mathfrak{d}^{\leq k+2}\Big(\Gamma_b\cdot(\Psi_{(1)}, \Psi_{(2)})\Big),\\
\widetilde{F}_{(2,k)} &= \left(\overline{q}\,{}^{(c)}\nabla_4\right)^k\,{}^{(c)}\nabla^2_{\widehat{R}}F_{(2)} + 2\frac{\Delta}{|q|^2}\omega\left(\overline{q}\,{}^{(c)}\nabla_4\right)^k\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_{\widehat{R}}\Psi_{(2)}\\
&\quad + O(r^{-1})\mathfrak{d}^{\leq k+1}\,{}^{(c)}\nabla_{\widehat{R}}\Psi_{(1)} + O(ar^{-2})\mathfrak{d}^{\leq k}\,\mathfrak{d}\!\!\!/\,{}^{(c)}\nabla_{\widehat{R}}\Psi_{(2)}\\
&\quad + O(r^{-1})\mathfrak{d}^{\leq k+1}\left(\Psi_{(1)}, \Psi_{(2)}\right) + \mathfrak{d}^{\leq k+2}\Big(\Gamma_b\cdot(\Psi_{(1)}, \Psi_{(2)})\Big).
\end{aligned} \tag{15.4.2}$$

- *If* $\Psi_{(1)}, \Psi_{(2)}$ *verify the equations* (15.3.9)*, then the quantities*

$$\widetilde{\Psi}_{(1,k)} = (q\,{}^{(c)}\nabla_4)^k\,{}^{(c)}\nabla^2_{\widehat{R}}\Psi_{(1)}, \qquad \widetilde{\Psi}_{(2,k)} = (q\,{}^{(c)}\nabla_4)^k\,{}^{(c)}\nabla^2_{\widehat{R}}\Psi_{(2)},$$

verify the equations

$$
\begin{aligned}
{}^{(c)}\nabla_3\widetilde{\Psi}_{(1,k)} + \left(c_{(1)} - \frac{k}{2}\right)\overline{tr\underline{X}}\widetilde{\Psi}_{(1,k)} &= -\,\not{\mathcal{D}}_p\,\widetilde{\Psi}_{(2,k)} + \widetilde{F}_{(1,k)},\\
{}^{(c)}\nabla_4\widetilde{\Psi}_{(2,k)} + \left(c_{(2)} - \frac{k}{2}\right) trX\,\widetilde{\Psi}_{(2,k)} &= \not{\mathcal{D}}_p^*\widetilde{\Psi}_{(1,k)} + \widetilde{F}_{(2,k)},
\end{aligned}
\tag{15.4.3}
$$

with $\widetilde{F}_{(1,k)}, \widetilde{F}_{(2,k)}$ *as in* (15.4.2).

Proposition 15.4.2. *The following higher order Bianchi identities hold true:*

- *If* $\Psi_{(1)}, \Psi_{(2)}$ *verify the equations* (15.3.8), *then the quantities*

$$
\widetilde{\Psi}_{(1,k)} = ({}^{(c)}\nabla_3)^k\,{}^{(c)}\nabla^2_{\widehat{R}}\Psi_{(1)}, \qquad \widetilde{\Psi}_{(2,k)} = ({}^{(c)}\nabla_3)^k\,{}^{(c)}\nabla^2_{\widehat{R}}\Psi_{(2)},
$$

verify the equations

$$
\begin{aligned}
{}^{(c)}\nabla_3\widetilde{\Psi}_{(1,k)} + c_{(1)}tr\underline{X}\widetilde{\Psi}_{(1,k)} &= \not{\mathcal{D}}_p^*\widetilde{\Psi}_{(2,k)} + \widetilde{F}_{(1,k)},\\
{}^{(c)}\nabla_4\widetilde{\Psi}_{(2,k)} + c_{(2)}\overline{trX}\widetilde{\Psi}_{(2,k)} &= \not{\mathcal{D}}_p\widetilde{\Psi}_{(1,k)} + \widetilde{F}_{(2,k)},
\end{aligned}
\tag{15.4.4}
$$

where

$$
\begin{aligned}
\widetilde{F}_{(1,k)} &= \big({}^{(c)}\nabla_3\big)^k\,{}^{(c)}\nabla^2_{\widehat{R}}F_{(1)} - 2\omega\big({}^{(c)}\nabla_3\big)^k\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_{\widehat{R}}\Psi_{(1)}\\
&\quad + O(r^{-1})\mathfrak{d}^{\leq k+1}\,{}^{(c)}\nabla_{\widehat{R}}\Psi_{(2)} + O(ar^{-2})\mathfrak{d}^{\leq k}\,\not{\mathfrak{d}}\,{}^{(c)}\nabla_{\widehat{R}}\Psi_{(1)}\\
&\quad + O(r^{-1})\mathfrak{d}^{\leq k+1}\big(\Psi_{(1)}, \Psi_{(2)}\big) + \mathfrak{d}^{\leq k+2}\Big(\Gamma_b\cdot(\Psi_{(1)}, \Psi_{(2)})\Big),\\
\widetilde{F}_{(2,k)} &= \big({}^{(c)}\nabla_3\big)^k\,{}^{(c)}\nabla^2_{\widehat{R}}F_{(2)} + 2\frac{\Delta}{|q|^2}\omega\big({}^{(c)}\nabla_3\big)^k\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_{\widehat{R}}\Psi_{(2)}\\
&\quad + O(r^{-1})\mathfrak{d}^{\leq k+1}\,{}^{(c)}\nabla_{\widehat{R}}\Psi_{(1)} + O(ar^{-2})\mathfrak{d}^{\leq k}\,\not{\mathfrak{d}}\,{}^{(c)}\nabla_{\widehat{R}}\Psi_{(2)}\\
&\quad + O(r^{-1})\mathfrak{d}^{\leq k+1}\big(\Psi_{(1)}, \Psi_{(2)}\big) + \mathfrak{d}^{\leq k+2}\Big(\Gamma_b\cdot(\Psi_{(1)}, \Psi_{(2)})\Big).
\end{aligned}
\tag{15.4.5}
$$

- *If* $\Psi_{(1)}, \Psi_{(2)}$ *verify the equations* (15.3.9), *then the quantities*

$$
\widetilde{\Psi}_{(1,k)} = ({}^{(c)}\nabla_3)^k\,{}^{(c)}\nabla^2_{\widehat{R}}\Psi_{(1)}, \qquad \widetilde{\Psi}_{(2,k)} = ({}^{(c)}\nabla_3)^k\,{}^{(c)}\nabla^2_{\widehat{R}}\Psi_{(2)},
$$

verify the equations

$$
\begin{aligned}
{}^{(c)}\nabla_3\widetilde{\Psi}_{(1,k)} + c_{(1)}\overline{tr\underline{X}}\widetilde{\Psi}_{(1,k)} &= -\,\not{\mathcal{D}}_p\,\widetilde{\Psi}_{(2,k)} + \widetilde{F}_{(1,k)},\\
{}^{(c)}\nabla_4\widetilde{\Psi}_{(2,k)} + c_{(2)}trX\widetilde{\Psi}_{(2,k)} &= \not{\mathcal{D}}_p^*\widetilde{\Psi}_{(1,k)} + \widetilde{F}_{(2,k)},
\end{aligned}
\tag{15.4.6}
$$

with $\widetilde{F}_{(1,k)}, \widetilde{F}_{(2,k)}$ *as in* (15.4.5).

Propositions 15.4.1 and 15.4.2 will be proved in section 15.4.3 by relying on the commutation lemmas of sections 15.4.1 and 15.4.2.

15.4.1 Commutation of Bianchi pairs with ${}^{(c)}\nabla_{\widehat{R}}$

In the following lemma, we commute the Bianchi identities with ${}^{(c)}\nabla_{\widehat{R}}$, see (13.1.2) for the definition of ${}^{(c)}\nabla_{\widehat{R}}$.

Lemma 15.4.3. *The following identities hold true:*

- *If $\Psi_{(1)}, \Psi_{(2)}$ verify the equations* (15.3.8)*, then $\dot{\Psi}_{(1)} = {}^{(c)}\nabla_{\widehat{R}}\Psi_{(1)}, \dot{\Psi}_{(2)} = {}^{(c)}\nabla_{\widehat{R}}\Psi_{(2)}$ verify the following*

$$\begin{aligned} {}^{(c)}\nabla_3\dot{\Psi}_{(1)} + c_{(1)}tr\underline{X}\dot{\Psi}_{(1)} &= -\mathcal{D}_p^*\dot{\Psi}_{(2)} + \dot{F}_{(1)}, \\ {}^{(c)}\nabla_4\dot{\Psi}_{(2)} + c_{(2)}\overline{trX}\dot{\Psi}_{(2)} &= \mathcal{D}_p\dot{\Psi}_{(1)} + \dot{F}_{(2)}, \end{aligned} \tag{15.4.7}$$

where

$$\begin{aligned} \dot{F}_{(1)} =& {}^{(c)}\nabla_{\widehat{R}}F_{(1)} - \omega\, {}^{(c)}\nabla_3\Psi_{(1)} + O(r^{-2})\mathfrak{d}^{\le 1}\Psi_{(2)} + O(ar^{-3})\,\not{\mathfrak{d}}\Psi_{(1)} + O(r^{-2})\Psi_{(1)} \\ &+ \Gamma_b\cdot\mathfrak{d}^{\le 1}\Psi_{(2)} + r^{-1}\Gamma_b\cdot\mathfrak{d}^{\le 1}\Psi_{(1)}, \\ \dot{F}_{(2)} =& {}^{(c)}\nabla_{\widehat{R}}F_{(2)} + \frac{\Delta}{|q|^2}\omega\, {}^{(c)}\nabla_3\Psi_{(2)} + O(r^{-2})\mathfrak{d}^{\le 1}\Psi_{(1)} + O(ar^{-3})\,\not{\mathfrak{d}}^{\le 1}\Psi_{(2)} \\ &+ O(r^{-2})\Psi_{(2)} + \Gamma_b\cdot\mathfrak{d}^{\le 1}\Psi_{(1)} + r^{-1}\Gamma_b\cdot\Psi_{(2)}. \end{aligned} \tag{15.4.8}$$

- *If $\Psi_{(1)}, \Psi_{(2)}$ verify the equations* (15.3.9)*, then $\dot{\Psi}_{(1)} = {}^{(c)}\nabla_{\widehat{R}}\Psi_{(1)}, \dot{\Psi}_{(2)} = {}^{(c)}\nabla_{\widehat{R}}\Psi_{(2)}$ verify the following*

$$\begin{aligned} {}^{(c)}\nabla_3(\dot{\Psi}_{(1)}) + c_{(1)}\overline{tr\underline{X}}\dot{\Psi}_{(1)} &= -\mathcal{D}_p\,\dot{\Psi}_{(2)} + \dot{F}_{(1)}, \\ {}^{(c)}\nabla_4(\dot{\Psi}_{(2)}) + c_{(2)}trX\dot{\Psi}_{(2)} &= \mathcal{D}_p^*\dot{\Psi}_{(1)} + \dot{F}_{(2)}, \end{aligned} \tag{15.4.9}$$

with $\dot{F}_{(1)}, \dot{F}_{(2)}$ as in (15.4.8).

Proof. We start with the equation ${}^{(c)}\nabla_3\Psi_{(1)} + c_{(1)}\mathrm{tr}\underline{X}\Psi_{(1)} = -\mathcal{D}_p^*\Psi_{(2)} + F_{(1)}$ and write

$$\begin{aligned} {}^{(c)}\nabla_3\dot{\Psi}_{(1)} &= {}^{(c)}\nabla_3\,{}^{(c)}\nabla_{\widehat{R}}\Psi_{(1)} = {}^{(c)}\nabla_{\widehat{R}}\,{}^{(c)}\nabla_3\Psi_{(1)} + [\,{}^{(c)}\nabla_3, {}^{(c)}\nabla_{\widehat{R}}]\Psi_{(1)} \\ &= {}^{(c)}\nabla_{\widehat{R}}\Big(-c_{(1)}\mathrm{tr}\underline{X}\Psi_{(1)} - \mathcal{D}_p^*\Psi_{(2)} + F_{(1)}\Big) + [\,{}^{(c)}\nabla_3, {}^{(c)}\nabla_{\widehat{R}}]\Psi_{(1)} \\ &= -c_{(1)}\mathrm{tr}\underline{X}\dot{\Psi}_{(1)} + O(r^{-2})\Psi_{(1)} - \mathcal{D}_p^*\dot{\Psi}_{(2)} - [\nabla_{\widehat{R}}, \mathcal{D}_p^*]\Psi_{(2)} + [\,{}^{(c)}\nabla_3, {}^{(c)}\nabla_{\widehat{R}}]\Psi_{(1)} \\ &\quad + {}^{(c)}\nabla_{\widehat{R}}F_{(1)}. \end{aligned}$$

Henceforth

$${}^{(c)}\nabla_3\dot{\Psi}_{(1)} + c_{(1)}\mathrm{tr}\underline{X}\dot{\Psi}_{(1)} = -\mathcal{D}_p^*\dot{\Psi}_{(2)} + \dot{F}_{(1)}$$

where

$$\dot{F}_{(1)} = \nabla_{\widehat{R}} F_{(1)} - [\nabla_{\widehat{R}}, \,\mathcal{D}_p^*]\Psi_{(2)} + [\,{}^{(c)}\nabla_3, \,{}^{(c)}\nabla_{\widehat{R}}]\Psi_{(1)} + O(r^{-2})\Psi_{(1)}.$$

In view of the commutation Lemma 13.1.8

$$\begin{aligned}
[\,{}^{(c)}\nabla_3, \,{}^{(c)}\nabla_{\widehat{R}}]\Psi_{(1)} &= -\omega\,{}^{(c)}\nabla_3\Psi_{(1)} + O(ar^{-3})\,\not{\mathfrak{d}}^{\leq 1}\Psi_{(1)} + O(r^{-3})\Psi_{(1)} + r^{-1}\Gamma_b\cdot\mathfrak{d}^{\leq 1}\Psi_{(1)},\\
[\nabla_{\widehat{R}}, \,\mathcal{D}_p^*]\Psi_{(2)} &= \frac{\Delta}{2|q|^2}\mathrm{tr}\underline{X}\,\mathcal{D}_p^*\Psi_{(2)} + O(ar^{-2})\mathfrak{d}^{\leq 1}\Psi_{(2)} + \Gamma_b\cdot\mathfrak{d}^{\leq 1}\Psi_{(2)}.
\end{aligned}$$

Hence

$$\begin{aligned}
\dot{F}_{(1)} &= {}^{(c)}\nabla_{\widehat{R}} F_{(1)} - \omega\,{}^{(c)}\nabla_3\Psi_{(1)} + O(r^{-2})\mathfrak{d}^{\leq 1}\Psi_{(2)} + O(ar^{-3})\,\not{\mathfrak{d}}\Psi_{(1)} + O(r^{-2})\Psi_{(1)}\\
&\quad + \Gamma_b\cdot\mathfrak{d}^{\leq 1}\Psi_{(2)} + r^{-1}\Gamma_b\cdot\mathfrak{d}^{\leq 1}\Psi_{(1)}.
\end{aligned}$$

Similarly, starting with the second equation ${}^{(c)}\nabla_4\Psi_{(2)} + c_{(2)}\overline{\mathrm{tr}X}\Psi_{(2)} = \mathcal{D}_p\Psi_{(1)} + F_{(2)}$, we write

$$\begin{aligned}
{}^{(c)}\nabla_4\dot{\Psi}_{(2)} &= {}^{(c)}\nabla_4\,{}^{(c)}\nabla_{\widehat{R}}\Psi_{(2)} = {}^{(c)}\nabla_{\widehat{R}}\,{}^{(c)}\nabla_4\Psi_{(2)} + [\,{}^{(c)}\nabla_4, \,{}^{(c)}\nabla_{\widehat{R}}]\Psi_{(2)}\\
&= {}^{(c)}\nabla_{\widehat{R}}\Big(- c_{(2)}\overline{\mathrm{tr}X}\Psi_{(2)} + \mathcal{D}_p\,\Psi_{(1)} + F_{(2)}\Big) + [\,{}^{(c)}\nabla_4, \,{}^{(c)}\nabla_{\widehat{R}}]\Psi_{(2)}\\
&= -c_{(2)}\overline{\mathrm{tr}X}\dot{\Psi}_{(2)} + O(r^{-2})\Psi_{(2)} + \mathcal{D}_p\,\dot{\Psi}_{(1)} + [\nabla_{\widehat{R}}, \,\mathcal{D}_p\,]\Psi_{(1)} + [\,{}^{(c)}\nabla_4, \,{}^{(c)}\nabla_{\widehat{R}}]\Psi_{(2)}\\
&\quad + {}^{(c)}\nabla_{\widehat{R}} F_{(2)}.
\end{aligned}$$

Hence ${}^{(c)}\nabla_4\dot{\Psi}_{(2)} + c_{(2)}\overline{\mathrm{tr}X}\dot{\Psi}_{(2)} = \mathcal{D}_p\dot{\Psi}_{(1)} + \dot{F}_{(2)}$ with

$$\begin{aligned}
\dot{F}_{(2)} &= {}^{(c)}\nabla_{\widehat{R}} F_{(2)} + [\nabla_{\widehat{R}}, \,\mathcal{D}_p\,]\Psi_{(1)} + [\,{}^{(c)}\nabla_4, \,{}^{(c)}\nabla_{\widehat{R}}]\Psi_{(2)} + O(r^{-2})\Psi_{(2)}\\
&= {}^{(c)}\nabla_{\widehat{R}} F_{(2)} + \frac{\Delta}{2|q|^2}\mathrm{tr}\underline{X}\,\mathcal{D}_p\Psi_{(1)} + O(ar^{-2})\mathfrak{d}^{\leq 1}\Psi_{(1)} + \Gamma_b\cdot\mathfrak{d}^{\leq 1}\Psi_{(1)}\\
&\quad + \frac{\Delta}{|q|^2}\omega\,{}^{(c)}\nabla_3\Psi_{(2)} + O(ar^{-3})\,\not{\mathfrak{d}}^{\leq 1}\Psi_{(2)} + O(r^{-2})\Psi_{(2)} + r^{-1}\Gamma_b\cdot\mathfrak{d}^{\leq 1}\Psi_{(2)}\\
&= {}^{(c)}\nabla_{\widehat{R}} F_{(2)} + \frac{\Delta}{|q|^2}\omega\,{}^{(c)}\nabla_3\Psi_{(2)} + O(r^{-2})\mathfrak{d}^{\leq 1}\Psi_{(1)} + O(ar^{-3})\,\not{\mathfrak{d}}^{\leq 1}\Psi_{(2)} + O(r^{-2})\Psi_{(2)}\\
&\quad + \Gamma_b\cdot\mathfrak{d}^{\leq 1}\Psi_{(1)} + r^{-1}\Gamma_b\cdot\Psi_{(2)}.
\end{aligned}$$

This concludes the proof of Lemma 15.4.3. □

In the next lemma we commute the Bianchi pairs (15.3.8), (15.3.9) once more with ${}^{(c)}\nabla_{\widehat{R}}$.

Lemma 15.4.4. *The following identities hold true:*

- *If* $\Psi_{(1)}, \Psi_{(2)}$ *verify the equations* (15.3.8), *then* $\ddot{\Psi}_{(1)} = {}^{(c)}\nabla^2_{\widehat{R}}\Psi_{(1)}, \ddot{\Psi}_{(2)} = {}^{(c)}\nabla^2_{\widehat{R}}\Psi_{(2)}$ *verify the following*

$$\begin{aligned} {}^{(c)}\nabla_3\ddot{\Psi}_{(1)} + c_{(1)}tr\underline{X}\ddot{\Psi}_{(1)} &= -\not{\mathcal{D}}_p^*\ddot{\Psi}_{(2)} + \ddot{F}_{(1)}, \\ {}^{(c)}\nabla_4\ddot{\Psi}_{(2)} + c_{(2)}\overline{trX}\ddot{\Psi}_{(2)} &= \not{\mathcal{D}}_p\ddot{\Psi}_{(1)} + \ddot{F}_{(2)}, \end{aligned} \tag{15.4.10}$$

with

$$\begin{aligned} \ddot{F}_{(1)} &= {}^{(c)}\nabla^2_{\widehat{R}}F_{(1)} - 2\omega\,{}^{(c)}\nabla_3\dot{\Psi}_{(1)} + O(r^{-2})\mathfrak{d}^{\leq 1}\dot{\Psi}_{(2)} + O(ar^{-3})\,\not{\mathfrak{d}}\dot{\Psi}_{(1)} \\ &\quad + O(r^{-2})\mathfrak{d}^{\leq 1}\big(\Psi_{(1)}, \Psi_{(2)}\big) + \mathfrak{d}^{\leq 2}\big(\Gamma_b\cdot\Psi_{(2)}\big) + r^{-1}\mathfrak{d}^{\leq 2}\big(\Gamma_b\cdot\Psi_{(1)}\big), \\ \ddot{F}_{(2)} &= {}^{(c)}\nabla^2_{\widehat{R}}F_{(2)} + 2\frac{\Delta}{|q|^2}\omega\,{}^{(c)}\nabla_3\dot{\Psi}_{(2)} + O(r^{-2})\mathfrak{d}^{\leq 1}\dot{\Psi}_{(1)} + O(ar^{-3})\,\not{\mathfrak{d}}\dot{\Psi}_{(2)} \\ &\quad + O(r^{-2})\mathfrak{d}^{\leq 1}\big(\Psi_{(1)}, \Psi_{(2)}\big) + \mathfrak{d}^{\leq 2}\big(\Gamma_b\cdot\Psi_{(1)}\big) + r^{-1}\mathfrak{d}^{\leq 2}\big(\Gamma_b\cdot\Psi_{(2)}\big). \end{aligned} \tag{15.4.11}$$

- *If* $\Psi_{(1)}, \Psi_{(2)}$ *verify the equations* (15.3.9), *then* $\ddot{\Psi}_{(1)} = {}^{(c)}\nabla^2_{\widehat{R}}\Psi_{(1)}, \ddot{\Psi}_{(2)} = {}^{(c)}\nabla^2_{\widehat{R}}\Psi_{(2)}$ *verify the following*

$$\begin{aligned} {}^{(c)}\nabla_3(\ddot{\Psi}_{(1)}) + c_{(1)}\overline{tr\underline{X}}\ddot{\Psi}_{(1)} &= -\not{\mathcal{D}}_p\,\ddot{\Psi}_{(2)} + \ddot{F}_{(1)}, \\ {}^{(c)}\nabla_4(\ddot{\Psi}_{(2)}) + c_{(2)}trX\ddot{\Psi}_{(2)} &= \not{\mathcal{D}}_p^*\ddot{\Psi}_{(1)} + \ddot{F}_{(2)}, \end{aligned} \tag{15.4.12}$$

with $\ddot{F}_{(1)}, \ddot{F}_{(2)}$ *as in* (15.4.11).

Proof. Starting with the first Bianchi pair of Lemma 15.4.3, we commute once more with ${}^{(c)}\nabla_{\widehat{R}}$ and deduce

$$\begin{aligned} {}^{(c)}\nabla_3(\ddot{\Psi}_{(1)}) + c_{(1)}\mathrm{tr}\underline{X}\ddot{\Psi}_{(1)} &= -\not{\mathcal{D}}_p^*\,\ddot{\Psi}_{(2)} + \ddot{F}_{(1)}, \\ {}^{(c)}\nabla_4(\ddot{\Psi}_{(2)}) + c_{(2)}\overline{\mathrm{tr}X}\ddot{\Psi}_{(2)} &= \not{\mathcal{D}}_p\,\ddot{\Psi}_{(1)} + \ddot{F}_{(2)}, \end{aligned}$$

where

$$\begin{aligned} \ddot{F}_{(1)} &= {}^{(c)}\nabla_{\widehat{R}}\dot{F}_{(1)} - \omega\,{}^{(c)}\nabla_3\dot{\Psi}_{(1)} + O(r^{-2})\mathfrak{d}^{\leq 1}\dot{\Psi}_{(2)} + O(ar^{-3})\,\not{\mathfrak{d}}\dot{\Psi}_{(1)} + O(r^{-2})\dot{\Psi}_{(1)} \\ &\quad +\Gamma_b\cdot\mathfrak{d}^{\leq 1}\dot{\Psi}_{(2)} + r^{-1}\Gamma_b\cdot\mathfrak{d}^{\leq 1}\dot{\Psi}_{(1)} \\ &= {}^{(c)}\nabla_{\widehat{R}}\Big({}^{(c)}\nabla_{\widehat{R}}F_{(1)} - \omega\,{}^{(c)}\nabla_3\Psi_{(1)} + O(r^{-2})\mathfrak{d}^{\leq 1}\Psi_{(2)} + O(ar^{-3})\,\not{\mathfrak{d}}\Psi_{(1)} + O(r^{-2})\Psi_{(1)}\Big) \\ &\quad +\mathfrak{d}^{\leq 1}\big(\Gamma_b\cdot\mathfrak{d}^{\leq 1}\Psi_{(2)}\big) + r^{-1}\mathfrak{d}^{\leq 1}\big(\Gamma_b\cdot\mathfrak{d}^{\leq 1}\Psi_{(1)}\big) \\ &\quad -\omega\,{}^{(c)}\nabla_3\dot{\Psi}_{(1)} + O(r^{-2})\mathfrak{d}^{\leq 1}\dot{\Psi}_{(2)} + O(ar^{-3})\,\not{\mathfrak{d}}\dot{\Psi}_{(1)} + O(r^{-2})\dot{\Psi}_{(1)} \\ &\quad +\Gamma_b\cdot\mathfrak{d}^{\leq 1}\dot{\Psi}_{(2)} + r^{-1}\Gamma_b\cdot\mathfrak{d}^{\leq 1}\dot{\Psi}_{(1)} \\ &= {}^{(c)}\nabla^2_{\widehat{R}}F_{(1)} - 2\omega\,{}^{(c)}\nabla_3\dot{\Psi}_{(1)} + O(r^{-2})\mathfrak{d}^{\leq 1}\dot{\Psi}_{(2)} + O(ar^{-3})\,\not{\mathfrak{d}}\dot{\Psi}_{(1)} \\ &\quad +O(r^{-2})\mathfrak{d}^{\leq 1}\big(\Psi_{(1)}, \Psi_{(2)}\big) + \mathfrak{d}^{\leq 2}\big(\Gamma_b\cdot\Psi_{(2)}\big) + r^{-1}\mathfrak{d}^{\leq 2}\big(\Gamma_b\cdot\Psi_{(1)}\big). \end{aligned}$$

Similarly

$$
\begin{aligned}
\ddot{F}_{(2)} &= {}^{(c)}\nabla^2_{\widehat{R}} F_{(2)} + 2\frac{\Delta}{|q|^2}\omega\, {}^{(c)}\nabla_3 \dot{\Psi}_{(2)} + O(r^{-2})\mathfrak{d}^{\leq 1}\dot{\Psi}_{(1)} + O(ar^{-3})\, \not{\mathfrak{d}}\dot{\Psi}_{(2)} \\
&\quad + O(r^{-2})\mathfrak{d}^{\leq 1}\big(\Psi_{(1)}, \Psi_{(2)}\big) + \mathfrak{d}^{\leq 2}\big(\Gamma_b \cdot \Psi_{(1)}\big) + r^{-1}\mathfrak{d}^{\leq 2}\big(\Gamma_b \cdot \Psi_{(2)}\big).
\end{aligned}
$$

The proof of (15.4.12) is similar and left to the reader. □

15.4.2 Commutation of Bianchi pairs with ${}^{(c)}\nabla_3$ and ${}^{(c)}\nabla_4$

The following two lemmas concern commutation of the Bianchi pairs respectively with $\overline{q}\,{}^{(c)}\nabla_4$, $q\,{}^{(c)}\nabla_4$, and ${}^{(c)}\nabla_3$.

Lemma 15.4.5. *The following identities hold true:*

- *If* $\Psi_{(1)}, \Psi_{(2)}$ *verify the equations* (15.3.8), *then* $\widetilde{\Psi}_{(1)} = \overline{q}\,{}^{(c)}\nabla_4\Psi_{(1)}$, $\widetilde{\Psi}_{(2)} = \overline{q}\,{}^{(c)}\nabla_4\Psi_{(2)}$ *verify the equations*

$$
\begin{aligned}
{}^{(c)}\nabla_3\widetilde{\Psi}_{(1)} + \left(c_{(1)} - \frac{1}{2}\right) tr\underline{X}\,\widetilde{\Psi}_{(1)} &= -\,\not{\mathcal{D}}_p^{*}\widetilde{\Psi}_{(2)} + \widetilde{F}_{(1)}, \\
{}^{(c)}\nabla_4\widetilde{\Psi}_{(2)} + \left(c_{(2)} - \frac{1}{2}\right) \overline{trX}\,\widetilde{\Psi}_{(2)} &= \not{\mathcal{D}}_p\widetilde{\Psi}_{(1)} + \widetilde{F}_{(2)},
\end{aligned}
\tag{15.4.13}
$$

  where

$$
\begin{aligned}
\widetilde{F}_{(1)} &= \overline{q}\,{}^{(c)}\nabla_4 F_{(1)} + O(ar^{-2})\,\not{\mathfrak{d}}\Psi_{(1)} + O(r^{-1})\Psi_{(1)} + O(r^{-1})\mathfrak{d}^{\leq 1}\Psi_{(2)} \\
&\quad + \Gamma_b \cdot \mathfrak{d}^{\leq 1}(\Psi_{(1)}, \Psi_{(2)}), \\
\widetilde{F}_{(2)} &= \overline{q}\,{}^{(c)}\nabla_4 F_{(2)} + O(r^{-1})\Psi_{(2)} + O(r^{-1})\mathfrak{d}^{\leq 1}\Psi_{(1)} + r^{-1}\Gamma_b \cdot \mathfrak{d}^{\leq 1}\big(\Psi_{(1)}, \Psi_{(2)}\big).
\end{aligned}
\tag{15.4.14}
$$

- *If* $\Psi_{(1)}, \Psi_{(2)}$ *verify* (15.3.9), *then* $\widetilde{\Psi}_{(1)} = q\,{}^{(c)}\nabla_4\Psi_{(1)}$, $\widetilde{\Psi}_{(2)} = q\,{}^{(c)}\nabla_4\Psi_{(2)}$ *verify the equations*

$$
\begin{aligned}
{}^{(c)}\nabla_3\widetilde{\Psi}_{(1)} + \left(c_{(1)} - \frac{1}{2}\right) \overline{tr\underline{X}}\widetilde{\Psi}_{(1)} &= -\,\not{\mathcal{D}}_p\,\widetilde{\Psi}_{(2)} + \widetilde{F}_{(1)}, \\
{}^{(c)}\nabla_4\widetilde{\Psi}_{(2)} + \left(c_{(2)} - \frac{1}{2}\right) trX\widetilde{\Psi}_{(2)} &= \not{\mathcal{D}}_p^{*}\widetilde{\Psi}_{(1)} + \widetilde{F}_{(2)},
\end{aligned}
\tag{15.4.15}
$$

  with $\widetilde{F}_{(1)}, \widetilde{F}_{(2)}$ *defined from* $F_{(1)}, F_{(2)}$ *as in* (15.4.14).

Proof. We start with the first equation in (15.4.13). Using Lemma 13.3.4, we have

$$\begin{aligned}
{}^{(c)}\nabla_3\widetilde{\Psi}_{(1)} =& {}^{(c)}\nabla_{\overline{q}e_4}\,{}^{(c)}\nabla_3\Psi_{(1)} + [\,{}^{(c)}\nabla_3,\,{}^{(c)}\nabla_{\overline{q}e_4}]\Psi_{(1)}\\
=&\overline{q}\,{}^{(c)}\nabla_4\Big(-c_{(1)}\mathrm{tr}\underline{X}\Psi_{(1)} - \mathcal{D}_p^*\Psi_{(2)} + F_{(1)}\Big)\\
&+\frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(c)}\nabla_{\overline{q}e_4}\Psi_{(1)} + O(ar^{-1})\nabla^{\leq 1}\Psi_{(1)} + O(r^{-2})\Psi_{(1)} + \Gamma_b\cdot\mathfrak{d}^{\leq 1}\Psi_{(1)}\\
=&-\left(c_{(1)}-\frac{1}{2}\right)\mathrm{tr}\underline{X}\widetilde{\Psi}_{(1)} - \mathcal{D}_p^*\widetilde{\Psi}_{(2)} + O(ar^{-2})\,\nabla\!\!\!/\Psi_{(1)} + O(r^{-1})\Psi_{(1)} + O(r^{-1})\mathfrak{d}^{\leq 1}\Psi_{(2)}\\
&+\overline{q}\,{}^{(c)}\nabla_4F_{(1)} + \Gamma_b\cdot\mathfrak{d}^{\leq 1}(\Psi_{(1)},\Psi_{(2)}),
\end{aligned}$$

and hence

$$\widetilde{F}_{(1)} = \overline{q}\,{}^{(c)}\nabla_4F_{(1)} + O(ar^{-2})\,\nabla\!\!\!/\Psi_{(1)} + O(r^{-1})\Psi_{(1)} + O(r^{-1})\mathfrak{d}^{\leq 1}\Psi_{(2)} + \Gamma_b\cdot\mathfrak{d}^{\leq 1}(\Psi_{(1)},\Psi_{(2)})$$

as stated.

Also, using again Lemma 13.3.4, we have

$$\begin{aligned}
{}^{(c)}\nabla_4\widetilde{\Psi}_{(2)} =& {}^{(c)}\nabla_{\overline{q}e_4}\,{}^{(c)}\nabla_4\Psi_{(2)} + [\,{}^{(c)}\nabla_4,\,{}^{(c)}\nabla_{\overline{q}e_4}]\Psi_{(2)}\\
=&{}^{(c)}\nabla_{\overline{q}e_4}\Big(-c_{(2)}\overline{\mathrm{tr}X}\Psi_{(2)} + \mathcal{D}_p\,\Psi_{(1)} + F_{(2)}\Big) + \frac{1}{2}\overline{\mathrm{tr}X}\,{}^{(c)}\nabla_{\overline{q}e_4}\Psi_{(2)} + r^{-1}\Gamma_b\cdot\mathfrak{d}^{\leq 1}\Psi_{(2)}\\
=&-c_{(2)}\overline{\mathrm{tr}X}\widetilde{\Psi}_{(2)} + O(r^{-1})\Psi_{(2)} + \mathcal{D}_p\,\widetilde{\Psi}_{(1)} + [\nabla_{\overline{q}e_4},\,\mathcal{D}_p]\,\Psi_{(1)} + {}^{(c)}\nabla_{\overline{q}e_4}(F_{(2)})\\
&+\frac{1}{2}\overline{\mathrm{tr}X}\,{}^{(c)}\nabla_{\overline{q}e_4}\Psi_{(2)} + r^{-1}\Gamma_b\cdot\mathfrak{d}^{\leq 1}\Psi_{(2)}\\
=&-\left(c_{(2)}-\frac{1}{2}\right)\overline{\mathrm{tr}X}\widetilde{\Psi}_{(2)} + O(r^{-1})\Psi_{(2)} + \mathcal{D}_p\,\widetilde{\Psi}_{(1)}\\
&+O(r^{-1})\mathfrak{d}^{\leq 1}\Psi_{(1)} + {}^{(c)}\nabla_{\overline{q}e_4}(F_{(2)}) + r^{-1}\Gamma_b\cdot\mathfrak{d}^{\leq 1}\big(\Psi_{(1)},\Psi_{(2)}\big)
\end{aligned}$$

and hence

$$\widetilde{F}_{(2)} = \overline{q}\,{}^{(c)}\nabla_4F_{(2)} + O(r^{-1})\Psi_{(2)} + O(r^{-1})\mathfrak{d}^{\leq 1}\Psi_{(1)} + r^{-1}\Gamma_b\cdot\mathfrak{d}^{\leq 1}\big(\Psi_{(1)},\Psi_{(2)}\big)$$

as stated.

(15.4.15) is derived in the same manner. This concludes the proof of Lemma 15.4.5. □

Lemma 15.4.6. *The following identities hold true:*

- *If* $\Psi_{(1)},\Psi_{(2)}$ *verify the equations* (15.3.8) *then* $\widetilde{\Psi}_{(1)} = {}^{(c)}\nabla_3\Psi_{(1)}, \widetilde{\Psi}_{(2)} = {}^{(c)}\nabla_3\Psi_{(2)}$ *verify the equations*

$$\begin{aligned}
{}^{(c)}\nabla_3\widetilde{\Psi}_{(1)} + c_{(1)}tr\underline{X}\widetilde{\Psi}_{(1)} &= -\mathcal{D}_p^*\widetilde{\Psi}_{(2)} + \widetilde{F}_{(1)},\\
{}^{(c)}\nabla_4\widetilde{\Psi}_{(2)} + c_{(1)}\overline{trX}\widetilde{\Psi}_{(2)} &= \mathcal{D}_p\widetilde{\Psi}_{(1)} + \widetilde{F}_{(2)},
\end{aligned} \tag{15.4.16}$$

where

$$
\begin{aligned}
\widetilde{F}_{(1)} &= {}^{(c)}\nabla_3 F_{(1)} + O(r^{-1})\mathfrak{d}^{\leq 1}\Psi_{(2)} + O(r^{-2})\Psi_{(1)} + \Gamma_b\cdot\mathfrak{d}^{\leq 1}(\Psi_{(1)},\Psi_{(2)}),\\
\widetilde{F}_{(2)} &= {}^{(c)}\nabla_3 F_{(2)} + O(r^{-1})\mathfrak{d}^{\leq 1}\Psi_{(1)} + O(ar^{-2})\,\not{\mathfrak{d}}^{\leq 1}\Psi_{(2)} + O(r^{-2})\Psi_{(2)} \qquad (15.4.17)\\
&\quad + \Gamma_b\cdot\mathfrak{d}^{\leq 1}(\Psi_{(1)},\Psi_{(2)}).
\end{aligned}
$$

- *If* $\Psi_{(1)}, \Psi_{(2)}$ *verify* (15.3.9) *then* $\widetilde{\Psi}_{(1)} = {}^{(c)}\nabla_3\Psi_{(1)}, \widetilde{\Psi}_{(2)} = {}^{(c)}\nabla_3\Psi_{(2)}$ *verify the equations*

$$
\begin{aligned}
{}^{(c)}\nabla_3\widetilde{\Psi}_{(1)} + c_{(1)}\overline{tr\underline{X}}\widetilde{\Psi}_{(1)} &= -\,\not{\mathcal{D}}_p\,\widetilde{\Psi}_{(2)} + \widetilde{F}_{(1)},\\
{}^{(c)}\nabla_4\widetilde{\Psi}_{(2)} + c_{(2)}trX\widetilde{\Psi}_{(2)} &= \not{\mathcal{D}}_p^*\widetilde{\Psi}_{(1)} + \widetilde{F}_{(2)},
\end{aligned} \qquad (15.4.18)
$$

with $\widetilde{F}_{(1)}, \widetilde{F}_{(2)}$ *defined from* $F_{(1)}, F_{(2)}$ *as in* (15.4.17).

Proof. The proof relies on the commutators in Lemma 13.1.7. It is similar to that of Lemma 15.4.5 and is in fact simpler. □

15.4.3 Proof of Propositions 15.4.1 and 15.4.2

The proof of Propositions 15.4.1 and 15.4.2 are similar, so we focus on the one of Proposition 15.4.1 and consider first the case $k = 1$.

To check $k = 1$ in the case of the Bianchi pair (15.3.8), we apply the result of Lemma 15.4.5 to the Bianchi pair derived in Lemma 15.4.4. More precisely we start with the equation (15.4.10)

$$
\begin{aligned}
{}^{(c)}\nabla_3\ddot{\Psi}_{(1)} + c_{(1)}\mathrm{tr}\underline{X}\ddot{\Psi}_{(1)} &= -\,\not{\mathcal{D}}_p^*\ddot{\Psi}_{(2)} + \ddot{F}_{(1)},\\
{}^{(c)}\nabla_4\ddot{\Psi}_{(2)} + c_{(2)}\mathrm{tr}X\ddot{\Psi}_{(2)} &= \not{\mathcal{D}}_p\ddot{\Psi}_{(1)} + \ddot{F}_{(2)},
\end{aligned}
$$

with $\ddot{F}_{(1)}, \ddot{F}_{(2)}$ given by (15.4.11) and apply to it the result of the first part of Lemma 15.4.5 to deduce

$$
\begin{aligned}
{}^{(c)}\nabla_3\widetilde{\Psi}_{(1,1)} + \left(c_{(1)} - \frac{1}{2}\right)\mathrm{tr}\underline{X}\widetilde{\Psi}_{(1,1)} &= -\,\not{\mathcal{D}}_p^*\widetilde{\Psi}_{(2,1)} + \widetilde{F}_{(1,1)},\\
{}^{(c)}\nabla_4\widetilde{\Psi}_{(2,1)} + \left(c_{(2)} - \frac{1}{2}\right)\mathrm{tr}X\widetilde{\Psi}_{(2,1)} &= \not{\mathcal{D}}_p\widetilde{\Psi}_{(1,1)} + \widetilde{F}_{(2,1)},
\end{aligned} \qquad (15.4.19)
$$

where

$$
\begin{aligned}
\widetilde{F}_{(1,1)} &= \overline{q}\,{}^{(c)}\nabla_4\ddot{F}_{(1)} + O(ar^{-2})\,\not{\mathfrak{d}}\ddot{\Psi}_{(1)} + O(r^{-1})\ddot{\Psi}_{(1)} + O(r^{-1})\mathfrak{d}^{\leq 1}\ddot{\Psi}_{(2)} + \Gamma_b\cdot\mathfrak{d}^{\leq 1}(\ddot{\Psi}_{(1)},\ddot{\Psi}_{(2)}),\\
\widetilde{F}_{(1,2)} &= \overline{q}\,{}^{(c)}\nabla_4\ddot{F}_{(2)} + O(r^{-1})\ddot{\Psi}_{(2)} + O(r^{-1})\mathfrak{d}^{\leq 1}\ddot{\Psi}_{(1)} + r^{-1}\Gamma_b\cdot\mathfrak{d}^{\leq 1}(\ddot{\Psi}_{(1)},\ddot{\Psi}_{(2)}).
\end{aligned}
$$

Thus, in view of formulas (15.4.11) for $\ddot{F}_{(1)}, \ddot{F}_{(2)}$ and the definition of $\dot{\Psi}_{(1)}, \dot{\Psi}_{(2)}$ we deduce

$$\begin{aligned}
\widetilde{F}_{(1,1)} &= \overline{q}\,{}^{(c)}\nabla_4\Big\{ {}^{(c)}\nabla^2_{\widehat{R}} F_{(1)} - 2\omega\,{}^{(c)}\nabla_3\dot{\Psi}_{(1)} + O(r^{-2})\mathfrak{d}^{\leq 1}\dot{\Psi}_{(2)} + O(ar^{-3})\,\mathfrak{d}\!\!\!/\,\dot{\Psi}_{(1)}\Big\}\\
&+\overline{q}\,{}^{(c)}\nabla_4\Big\{O(r^{-2})\mathfrak{d}^{\leq 1}\big(\Psi_{(1)},\Psi_{(2)}\big) + \mathfrak{d}^{\leq 2}\big(\Gamma_b\cdot\Psi_{(2)}\big) + r^{-1}\mathfrak{d}^{\leq 2}\big(\Gamma_b\cdot\Psi_{(1)}\big)\Big\}\\
&+O(ar^{-2})\,\mathfrak{d}\!\!\!/\,\ddot{\Psi}_{(1)} + O(r^{-1})\ddot{\Psi}_{(1)} + O(r^{-1})\mathfrak{d}^{\leq 1}\ddot{\Psi}_{(2)} + \Gamma_b\cdot\mathfrak{d}^{\leq 1}(\ddot{\Psi}_{(1)},\ddot{\Psi}_{(2)})\\
&= \overline{q}\,{}^{(c)}\nabla_4\,{}^{(c)}\nabla^2_{\widehat{R}}F_{(1)} - 2\omega\overline{q}\,{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_{\widehat{R}}\Psi_{(1)} + O(r^{-2})\mathfrak{d}^{\leq 2}\,{}^{(c)}\nabla_{\widehat{R}}\Psi_{(2)}\\
&+O(ar^{-3})\mathfrak{d}\,\mathfrak{d}\!\!\!/\,{}^{(c)}\nabla_{\widehat{R}}\Psi_{(1)} + O(r^{-2})\mathfrak{d}^{\leq 2}\big(\Psi_{(1)},\Psi_{(2)}\big) + \mathfrak{d}^{\leq 3}\big(\Gamma_b\cdot\Psi_{(2)}\big)\\
&+r^{-1}\mathfrak{d}^{\leq 3}\big(\Gamma_b\cdot\Psi_{(1)}\big) + O(ar^{-2})\,\mathfrak{d}\!\!\!/\,{}^{(c)}\nabla^2_{\widehat{R}}\Psi_{(1)} + O(r^{-1})\,{}^{(c)}\nabla^2_{\widehat{R}}\Psi_{(1)}\\
&+O(r^{-1})\mathfrak{d}^{\leq 1}\,{}^{(c)}\nabla^2_{\widehat{R}}\Psi_{(2)} + \Gamma_b\cdot\mathfrak{d}^{\leq 1}\big({}^{(c)}\nabla^2_{\widehat{R}}\Psi_{(1)},\,{}^{(c)}\nabla^2_{\widehat{R}}\Psi_{(2)}\big).
\end{aligned}$$

Thus, in simplified form,

$$\begin{aligned}
\widetilde{F}_{(1,1)} &= \overline{q}\,{}^{(c)}\nabla_4\,{}^{(c)}\nabla^2_{\widehat{R}}F_{(1)} - 2\omega\overline{q}\,{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_{\widehat{R}}\Psi_{(1)} + O(r^{-1})\mathfrak{d}^{\leq 2}\,{}^{(c)}\nabla_{\widehat{R}}\Psi_{(2)}\\
&+O(ar^{-2})\mathfrak{d}\,\mathfrak{d}\!\!\!/\,{}^{(c)}\nabla_{\widehat{R}}\Psi_{(1)} + O(r^{-1})\mathfrak{d}^{\leq 2}\big(\Psi_{(1)},\Psi_{(2)}\big) + \mathfrak{d}^{\leq 3}\Big(\Gamma_b\cdot(\Psi_{(1)},\Psi_{(2)})\Big)
\end{aligned}$$

which corresponds to the first equation of (15.4.2) in the case $k=1$.

In the same fashion we find

$$\begin{aligned}
\widetilde{F}_{(2,1)} &= \overline{q}\,{}^{(c)}\nabla_4\,{}^{(c)}\nabla^2_{\widehat{R}}F_{(2)} + 2\frac{\Delta}{|q|^2}\omega\overline{q}\,{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_{\widehat{R}}\Psi_{(2)} + O(r^{-1})\mathfrak{d}^{\leq 2}\,{}^{(c)}\nabla_{\widehat{R}}\Psi_{(1)}\\
&+O(ar^{-2})\mathfrak{d}\,\mathfrak{d}\!\!\!/\,{}^{(c)}\nabla_{\widehat{R}}\Psi_{(2)} + O(r^{-1})\mathfrak{d}^{\leq 2}\big(\Psi_{(1)},\Psi_{(2)}\big) + \mathfrak{d}^{\leq 3}\Big(\Gamma_b\cdot(\Psi_{(1)},\Psi_{(2)})\Big)
\end{aligned}$$

which corresponds to the second equation of (15.4.2) in the case $k=1$. The general case, for all k, can be easily derived in the same manner by induction on k, hence concluding the proof of Proposition 15.4.1. The proof of Proposition 15.4.2 is similar.

15.5 Estimates for B and $\underline{B}$

The goal of this section is to prove Proposition 15.1.1 providing energy-Morawetz estimates for $(B, \underline{B})$ assuming corresponding energy-Morawetz estimates for $\check{P}$.

15.5.1 Estimates for $\nabla_3 B$, ∇B, $\nabla_4\underline{B}$, $\nabla\underline{B}$

In this section, we obtain the following lemma.

Lemma 15.5.1. *Recall the notation* $\delta_{J+1}[\check{P}] = BEF^J_\delta[r^2\check{P}]$. *The following hold true in* $\mathcal{M}$*:*

1. *We have*

$$BEF^{J-1}_\delta[r^2(\nabla_3 B, r\nabla B)] \lesssim \delta_{J+1}[\check{P}] + \mathfrak{G}_J\mathfrak{G}_{J+1} + \mathfrak{R}_J\mathfrak{R}_{J+1} + \epsilon_0^2 \\ + O(a^2, \epsilon^2) BEF_\delta[r^{J+1}\nabla_4^J B]. \tag{15.5.1}$$

2. *We have*

$$BEF^{J-1}_\delta[r(\nabla_4\underline{B}, \nabla\underline{B})] \lesssim \delta_{J+1}[\check{P}] + \mathfrak{G}_J\mathfrak{G}_{J+1} + \mathfrak{R}_J\mathfrak{R}_{J+1} + \epsilon_0^2 \\ + O(a^2, \epsilon^2) BEF_\delta[\nabla_3^J\underline{B}]. \tag{15.5.2}$$

Proof. We rely on the standard linearization of the second Bianchi pair which will be stated in (16.2.1), i.e.

$$\begin{aligned} {}^{(c)}\nabla_3 B + \mathrm{tr}\underline{X} B &= \mathcal{D}\overline{\check{P}} + O(r^{-2})\check{P} + O(r^{-4})\Gamma_b + r^{-2}\Gamma_b\cdot\check{R}_b, \\ {}^{(c)}\nabla_4\check{P} + \frac{3}{2}\mathrm{tr}X\check{P} &= \frac{1}{2}\mathcal{D}\cdot\overline{B} + O(r^{-2})B + O(r^{-4})\Gamma_b + r^{-2}\Gamma_b\cdot\check{R}_b. \end{aligned} \tag{15.5.3}$$

From the first equation we deduce

$$\begin{aligned} BEF^{J-1}_\delta[r^2\, {}^{(c)}\nabla_3 B] &\lesssim BEF^{J-1}_\delta[r^2\mathcal{D}\check{P}] + BEF^{J-1}_\delta[rB] + BEF^{J-1}_\delta[r^{-2}\Gamma_b] + \epsilon_0^2 \\ &\lesssim BEF^J_\delta[r\check{P}] + BEF^{J-1}_\delta[r^{-1}\check{R}_b] + \mathfrak{G}_J\mathfrak{G}_{J+1} + \epsilon_0^2 \\ &\lesssim \delta_{J+1}[\check{P}] + \mathfrak{G}_J\mathfrak{G}_{J+1} + \mathfrak{R}_J\mathfrak{R}_{J+1} + \epsilon_0^2, \end{aligned}$$

where we used Lemmas 14.1.1 and 14.1.2 to control $BEF^{J-1}_\delta[r^{-1}\check{R}_b]$ and $BEF^{J-1}_\delta[r^{-2}\Gamma_b]$. Similarly, from the second equation,

$$BEF^J_\delta[r^3\overline{\mathcal{D}}\cdot B] \lesssim \delta_{J+1}[\check{P}] + \mathfrak{G}_J\mathfrak{G}_{J+1} + \mathfrak{R}_J\mathfrak{R}_{J+1} + \epsilon_0^2.$$

Thus,

$$BEF^{J-1}_\delta[r^2\, {}^{(c)}\nabla_3 B] + BEF^{J-1}_\delta[r^3\overline{\mathcal{D}}\cdot B] \lesssim \delta_{J+1}[\check{P}] + \mathfrak{G}_J\mathfrak{G}_{J+1} + \mathfrak{R}_J\mathfrak{R}_{J+1} + \epsilon_0^2. \tag{15.5.4}$$

Remark 15.5.2. *Note that the terms* $\mathfrak{G}_J\mathfrak{G}_{J+1}$ *and* $\mathfrak{R}_J\mathfrak{R}_{J+1}$ *on the RHS of* (15.5.4) *are only needed to estimate the energy flux terms* $EF^{J-1}_\delta[r^{-1}\Gamma_b]$ *and* $EF^{J-1}_\delta[\check{R}_b]$ *thanks to Lemmas 14.1.1 and 14.1.2. In particular, in view of the control of the norms* $B^{J-1}_\delta[r^{-1}\Gamma_b]$ *and* $B^{J-1}_\delta[\check{R}_b]$ *provided by Lemmas 14.1.1 and 14.1.2, we infer the following stronger analog of* (15.5.4) *for the flux*

$$B^{J-1}_\delta[r^2\, {}^{(c)}\nabla_3 B] + B^{J-1}_\delta[r^3\overline{\mathcal{D}}\cdot B] \lesssim \delta_{J+1}[\check{P}] + \epsilon_J^2 + \epsilon_0^2. \tag{15.5.5}$$

Next, we apply Corollary 13.4.1 to derive

$$\begin{aligned} BEF_\delta^{J-1}[r^3\nabla B] &\lesssim BEF_\delta^{J-1}[r^3\overline{\mathcal{D}}\cdot B] + BEF_\delta^{J-1}[r^2 B] + O(a^2,\epsilon^2)BEF_\delta^{J-1}[r^2(\nabla_3 B, \nabla_4 B)] \\ &\lesssim BEF_\delta^{J-1}[r^3\overline{\mathcal{D}}\cdot B] + \mathfrak{R}_J\mathfrak{R}_{J+1} + O(a^2,\epsilon^2)BEF_\delta^{J-1}[r^2(\nabla_3 B, \nabla_4 B)]. \end{aligned}$$

We deduce

$$\begin{aligned} BEF_\delta^{J-1}[r^2\nabla_3 B] + BEF_\delta^{J-1}[r^3\nabla B] \lesssim& \delta_{J+1}[\check{P}] + \mathfrak{G}_J\mathfrak{G}_{J+1} + \mathfrak{R}_J\mathfrak{R}_{J+1} + \epsilon_0^2 \\ &+ O(a^2,\epsilon^2)BEF_\delta^{J-1}[r^2(\nabla_3 B, \nabla_4 B)]. \end{aligned}$$

Thus, for small a and ϵ,

$$\begin{aligned} BEF_\delta^{J-1}[r^2\nabla_3 B] + BEF_\delta^{J-1}[r^3\nabla B] &\lesssim \delta_{J+1}[\check{P}] + \mathfrak{G}_J\mathfrak{G}_{J+1} + \mathfrak{R}_J\mathfrak{R}_{J+1} + \epsilon_0^2 \\ &\quad +O(a^2,\epsilon^2)BEF_\delta^{J-1}[r^2\nabla_4 B]. \end{aligned}$$

Since

$$\begin{aligned} BEF_\delta^{J-1}[r^2\nabla_4 B] &\lesssim BEF_\delta[r^2(r\nabla_4)^{J-1}\nabla_4 B] + BEF_\delta^{J-1}[r(\nabla_3 B, \nabla B)] + BEF_\delta^{J-2}[rB] \\ &\lesssim BEF_\delta[r^{J+1}\nabla_4^J B] + BEF_\delta^{J-1}[r(\nabla_3 B, \nabla B)] + \epsilon_J^2, \end{aligned}$$

we infer, for small a and ϵ,

$$BEF_\delta^{J-1}[r^2(\nabla_3 B, r\nabla B)] \lesssim \delta_{J+1}[\check{P}] + \mathfrak{G}_J\mathfrak{G}_{J+1} + \mathfrak{R}_J\mathfrak{R}_{J+1} + \epsilon_0^2 + O(a^2,\epsilon^2)BEF_\delta[r^{J+1}\nabla_4^J B]$$

as stated in (15.5.1). Also, in view of Remark 15.5.2, we have the following stronger analog of (15.5.1) for the flux

$$B_\delta^{J-1}[r^2(\nabla_3 B, r\nabla B)] \lesssim \delta_{J+1}[\check{P}] + \epsilon_J^2 + \epsilon_0^2 + O(a^2,\epsilon^2)B_\delta[r^{J+1}\nabla_4^J B],$$

and hence, since $\epsilon^2 B_\delta[r^{J+1}\nabla_4^J B] \lesssim \epsilon^2 B_\delta^J[rB] \lesssim \epsilon^2\mathfrak{R}_{J+1}^2 \lesssim \epsilon_0^2$ in view of Lemma 14.1.1, we obtain

$$B_\delta^{J-1}[r^2(\nabla_3 B, r\nabla B)] \lesssim \delta_{J+1}[\check{P}] + \epsilon_J^2 + \epsilon_0^2 + O(a^2)B_\delta[r^{J+1}\nabla_4^J B]. \tag{15.5.6}$$

The estimate (15.5.2) follows in the same fashion from the standard linearization of the third Bianchi pair which will be stated in (16.2.2), i.e.

$$\begin{aligned} {}^{(c)}\nabla_3\check{P} + \frac{3}{2}\overline{\mathrm{tr}\underline{X}}\check{P} &= -\frac{1}{2}\overline{\mathcal{D}}\cdot\underline{B} + O(ar^{-2})\underline{B} + O(r^{-3})\Gamma_b + r^{-1}\Gamma_b\cdot\check{R}_b, \\ {}^{(c)}\nabla_4\underline{B} + \mathrm{tr}X\,\underline{B} &= -\mathcal{D}\check{P} + O(ar^{-2})\check{P} + O(r^{-3})\Gamma_b + r^{-1}\Gamma_b\cdot\check{R}_b. \end{aligned}$$

This concludes the proof of Lemma 15.5.1. □

Remark 15.5.3. *Note that we have the following analog of* (15.5.6)

$$B_\delta^{J-1}[r(\nabla_4\underline{B}, \nabla\underline{B})] \lesssim \delta_{J+1}[\check{P}] + \epsilon_J^2 + \epsilon_0^2 + O(a^2)B_\delta[\nabla_3^J\underline{B}]. \tag{15.5.7}$$

Remark 15.5.4. *It only remains to provide estimates for the top ∇_4 derivatives of B and top ∇_3 derivatives of $\underline{B}$. To this end, we need to include derivatives with respect to $\widehat{R}$ which leads us to introduce the following quantities*[8]

$$\begin{aligned} \widetilde{B} &:= (\overline{q}\,{}^{(c)}\nabla_4)^{\leq J-2}\,{}^{(c)}\nabla^2_{\widehat{R}}\dot{B}, & \widetilde{P}_+ &:= (\overline{q}\,{}^{(c)}\nabla_4)^{\leq J-2}\,{}^{(c)}\nabla^2_{\widehat{R}}\overline{\dot{P}}, \\ \widetilde{P}_- &:= (\,{}^{(c)}\nabla_3)^{\leq J-2}\,{}^{(c)}\nabla^2_{\widehat{R}}\dot{P}, & \widetilde{\underline{B}} &:= (\,{}^{(c)}\nabla_3)^{\leq J-2}\,{}^{(c)}\nabla^2_{\widehat{R}}\dot{\underline{B}}. \end{aligned} \tag{15.5.8}$$

15.5.2 Bianchi equations for the quantities $\widetilde{B}, \widetilde{P}_+, \widetilde{P}_-, \widetilde{\underline{B}}$

The following lemma provides Bianchi equations for the quantities $\widetilde{B}, \widetilde{P}_+, \widetilde{P}_-, \widetilde{\underline{B}}$.

Lemma 15.5.5. *The following equations hold true for the quantities $\widetilde{B}, \widetilde{P}_+, \widetilde{P}_-, \widetilde{\underline{B}}$ introduced in Remark 15.5.4:*

1. *The quantities $\widetilde{B}, \widetilde{P}_+$ verify the Bianchi pair*

$$\begin{aligned} {}^{(c)}\nabla_3\widetilde{B} + \left(1 - \frac{J-2}{2}\right) tr\underline{X}\,\widetilde{B} &= -\,{}^{(c)}\mathcal{D}_1^*\widetilde{P}_+ + \widetilde{F}_{(1,J-2)}, \\ {}^{(c)}\nabla_4\widetilde{P}_+ + \left(\frac{3}{2} - \frac{J-2}{2}\right) \overline{trX}\,\widetilde{P}_+ &= \,{}^{(c)}\mathcal{D}_1\widetilde{B} + \widetilde{F}_{(2,J-2)}, \end{aligned} \tag{15.5.9}$$

with

$$\begin{aligned} \widetilde{F}_{(1,J-2)} &= O(r^{-1})\mathfrak{d}^{J-2}\nabla_{\widehat{R}}\big({}^{(c)}\nabla_3\dot{B}, \,{}^{(c)}\mathcal{D}\dot{B}\big) + O(r^{-1})\mathfrak{d}^{J-1}\nabla_{\widehat{R}}\dot{P} \\ &\quad + O(r^{-3})(\nabla_3, \nabla)\mathfrak{d}^{\leq J}\Gamma_b + O(r^{-1})\mathfrak{d}^{\leq J-1}\dot{P} + O(r^{-1})\mathfrak{d}^{\leq J}B \\ &\quad + O(r^{-3})\mathfrak{d}^{\leq J}\Gamma_b + r^{-2}\mathfrak{d}^{\leq J+1}(\Gamma_b\cdot\check{R}_b) + r^{-2}\mathfrak{d}^{\leq J}\big(\Gamma_b\cdot\Gamma_b\big), \\ \widetilde{F}_{(2,J-2)} &= O(r^{-1})\mathfrak{d}^{J-1}\nabla_{\widehat{R}}(\dot{P}, \dot{B}) + O(r^{-3})\mathfrak{d}^{\leq J+1}\Gamma_b + O(r^{-1})\mathfrak{d}^{\leq J-1}(\dot{P}, \dot{B}) \\ &\quad + r^{-2}\mathfrak{d}^{\leq J+1}(\Gamma_b\cdot\check{R}_b). \end{aligned} \tag{15.5.10}$$

2. *The quantities $\widetilde{\underline{B}}, \widetilde{P}_-$ verify the Bianchi pair*

$$\begin{aligned} {}^{(c)}\nabla_3\widetilde{P}_- + \frac{3}{2}\overline{tr\underline{X}}\,\widetilde{P}_- &= -\,{}^{(c)}\mathcal{D}_1\widetilde{\underline{B}} + \widetilde{F}_{1,J-2}, \\ {}^{(c)}\nabla_4\widetilde{\underline{B}} + trX\,\widetilde{\underline{B}} &= \,{}^{(c)}\mathcal{D}_1^*\widetilde{P}_- + \widetilde{F}_{2,J-2}, \end{aligned} \tag{15.5.11}$$

[8] Note that the signatures of $\widetilde{B}, \widetilde{P}_+, \widetilde{P}_-, \widetilde{\underline{B}}$ are, respectively, $J-1$, $J-2$, $-J+2$, and $-J+1$. Also, recall that $\dot{B} = \mathcal{L}_{\mathbf{T}}B$, $\dot{P} = \mathbf{T}(P)$ and $\dot{\underline{B}} = \mathcal{L}_{\mathbf{T}}\underline{B}$.

with

$$
\begin{aligned}
\widetilde{F}_{(1,J-2)} &= O(r^{-1})\mathfrak{d}^{J-1}\nabla_{\widehat{R}}(\dot{P}, \dot{\underline{B}}) + O(r^{-3})\mathfrak{d}^{\leq J+1}\Gamma_b + O(r^{-1})\mathfrak{d}^{\leq J-1}(\dot{P}, \dot{\underline{B}}) \\
&\quad + r^{-1}\mathfrak{d}^{\leq J+1}(\Gamma_b \cdot \check{R}_b), \\
\widetilde{F}_{(2,J-2)} &= -4\omega\widetilde{\underline{B}} + O(r^{-3})\mathfrak{d}^{\leq J}(\nabla_4, \nabla)\Gamma_b + O(r^{-1})\mathfrak{d}^{\leq J-1}\nabla_{\widehat{R}}\dot{P} \\
&\quad + O(ar^{-2})\mathfrak{d}^{\leq J-2}\,\mathfrak{d}\!\!\!/\nabla_{\widehat{R}}\dot{\underline{B}} + O(r^{-1})\mathfrak{d}^{\leq J-1}\dot{P} + O(r^{-1})\mathfrak{d}^{\leq J}\underline{B} \\
&\quad + O(r^{-3})\mathfrak{d}^{\leq J}\Gamma_b + \mathfrak{d}^{\leq J+1}\Big(\Gamma_b \cdot \check{R}_b\Big).
\end{aligned}
\tag{15.5.12}
$$

Remark 15.5.6. *Note that the error terms for the equations which contain $\widetilde{B}$ or $\widetilde{\underline{B}}$ on the left hand side are more structured than those corresponding to the equations for $\widetilde{P}_+$ and $\widetilde{P}_-$. The reason, as it will become apparent in the next section, is that we already control the quantities $\widetilde{P}_+$ and $\widetilde{P}_-$.*

Proof. To prove the first statement, we apply Proposition 15.4.1 with $k = J-2$ to the Bianchi pair (15.2.1)

$$
\begin{aligned}
{}^{(c)}\nabla_3\dot{B} + \mathrm{tr}\underline{X}\dot{B} &= -\mathcal{D}\!\!\!/_1^*\overline{\dot{P}} + O(ar^{-2})\overline{\dot{P}} + O(r^{-3})\mathcal{L}\!\!\!/_{\mathbf{T}}\check{H} + O(r^{-4})\Gamma_b + r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_b \cdot \check{R}_b), \\
{}^{(c)}\nabla_4\overline{\dot{P}} + \frac{3}{2}\overline{\mathrm{tr}X}\,\overline{\dot{P}} &= \mathcal{D}\!\!\!/_1\dot{B} + O(ar^{-2})\dot{B} + O(r^{-3})\mathfrak{d}^{\leq 1}\Gamma_b + r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_b \cdot \check{R}_b).
\end{aligned}
$$

We deduce, with $\Psi_{(1)} = \dot{B}$, $\Psi_{(2)} = \overline{\dot{P}}$,

$$
\begin{aligned}
{}^{(c)}\nabla_3\widetilde{B} + \left(1 - \frac{J-2}{2}\right)\mathrm{tr}\underline{X}\widetilde{B} &= -\mathcal{D}\!\!\!/_1^*\overline{\widetilde{P}_+} + \widetilde{F}_{(1,J-2)}, \\
{}^{(c)}\nabla_4\widetilde{P}_+ + \left(\frac{3}{2} - \frac{J-2}{2}\right)\overline{\mathrm{tr}X}\widetilde{P}_+ &= \mathcal{D}\!\!\!/_1\overline{\widetilde{B}} + \widetilde{F}_{(2,J-2)},
\end{aligned}
$$

with, see (15.4.2),

$$
\begin{aligned}
\widetilde{F}_{(1,k)} &= \big(\overline{q}\,{}^{(c)}\nabla_4\big)^k\,{}^{(c)}\nabla_{\widehat{R}}^2 F_{(1)} - 2\omega\big(\overline{q}\,{}^{(c)}\nabla_4\big)^k\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_{\widehat{R}}\Psi_{(1)} \\
&\quad + O(r^{-1})\mathfrak{d}^{\leq k+1}\,{}^{(c)}\nabla_{\widehat{R}}\Psi_{(2)} + O(ar^{-2})\mathfrak{d}^{\leq k}\,\mathfrak{d}\!\!\!/\,{}^{(c)}\nabla_{\widehat{R}}\Psi_{(1)} \\
&\quad + O(r^{-1})\mathfrak{d}^{\leq k+1}\big(\Psi_{(1)}, \Psi_{(2)}\big) + \mathfrak{d}^{\leq k+2}\Big(\Gamma_b \cdot (\Psi_{(1)}, \Psi_{(2)})\Big), \\
\widetilde{F}_{(2,k)} &= \big(\overline{q}\,{}^{(c)}\nabla_4\big)^k\,{}^{(c)}\nabla_{\widehat{R}}^2 F_{(2)} + 2\frac{\Delta}{|q|^2}\omega\big(\overline{q}\,{}^{(c)}\nabla_4\big)^k\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_{\widehat{R}}\Psi_{(2)} \\
&\quad + O(r^{-1})\mathfrak{d}^{\leq k+1}\,{}^{(c)}\nabla_{\widehat{R}}\Psi_{(1)} + O(ar^{-2})\mathfrak{d}^{\leq k}\,\mathfrak{d}\!\!\!/\,{}^{(c)}\nabla_{\widehat{R}}\Psi_{(2)} \\
&\quad + O(r^{-1})\mathfrak{d}^{\leq k+1}\big(\Psi_{(1)}, \Psi_{(2)}\big) + \mathfrak{d}^{\leq k+2}\Big(\Gamma_b \cdot (\Psi_{(1)}, \Psi_{(2)})\Big),
\end{aligned}
$$

where

$$F_{(1)} = O(ar^{-2})\overline{\dot{P}} + O(r^{-3})\nabla\!\!\!\!/_{\mathbf{T}}\check{H} + O(r^{-4})\Gamma_b + r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_b \cdot \check{R}_b),$$
$$F_{(2)} = O(ar^{-2})\dot{B} + O(r^{-3})\mathfrak{d}^{\leq 1}\Gamma_b + r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_b \cdot \check{R}_b).$$

Setting $k = J-2$ and since we have chosen $\Psi_{(1)} = \dot{B}$, $\Psi_{(2)} = \overline{\dot{P}}$, we deduce

$$\begin{aligned}\widetilde{F}_{(1,J-2)} &= -2\omega\big(\overline{q}\,{}^{(c)}\nabla_4\big)^{J-2}\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_{\widehat{R}}\dot{B} + O(r^{-1})\mathfrak{d}^{\leq J-1}\,{}^{(c)}\nabla_{\widehat{R}}\dot{P} + O(r^{-3})\mathfrak{d}^{\leq J}\nabla\!\!\!\!/_{\mathbf{T}}\check{H}\\ &+O(r^{-4})\mathfrak{d}^{\leq J}\Gamma_b + O(ar^{-2})\mathfrak{d}^{\leq J-2}\,\mathfrak{d}\!\!\!/\,{}^{(c)}\nabla_{\widehat{R}}\dot{B} + O(r^{-1})\mathfrak{d}^{\leq J-1}(\dot{P}, \dot{B})\\ &+r^{-2}\mathfrak{d}^{\leq J+1}(\Gamma_b \cdot \check{R}_b).\end{aligned}$$

We rewrite in the simplified form, using in particular $2\nabla\!\!\!\!/_{\mathbf{T}}\check{H} = \nabla_4\check{H} + O(1)(\nabla_3, \nabla)\Gamma_b + O(1)\Gamma_b$,

$$\begin{aligned}\widetilde{F}_{(1,J-2)} &= O(r^{-1})\mathfrak{d}^{J-2}\nabla_{\widehat{R}}\big({}^{(c)}\nabla_3\dot{B},\, \mathfrak{d}\!\!\!/\dot{B}\big) + O(r^{-1})\mathfrak{d}^{J-1}\nabla_{\widehat{R}}\dot{P} + O(r^{-3})\mathfrak{d}^{\leq J}\nabla_4\check{H}\\ &+O(r^{-3})\mathfrak{d}^{\leq J}(\nabla_3, \nabla)\Gamma_b + O(r^{-3})\mathfrak{d}^{\leq J}\Gamma_b + O(r^{-1})\mathfrak{d}^{\leq J-1}(\dot{P}, \dot{B})\\ &+r^{-2}\mathfrak{d}^{\leq J+1}(\Gamma_b \cdot \check{R}_b).\end{aligned}$$

We next rewrite the dangerous term $O(r^{-3})\mathfrak{d}^{\leq J}\nabla_4\check{H}$ using the equation, see (13.2.1),

$${}^{(c)}\nabla_4\check{H} - {}^{(c)}\nabla_3\Xi = -B + O(r^{-1})\Gamma_b + \Gamma_b \cdot \Gamma_g,$$

Thus

$$O(r^{-3})\mathfrak{d}^{\leq J}\nabla_4\check{H} = O(r^{-3})\,{}^{(c)}\nabla_3\mathfrak{d}^{\leq J}\Xi + O(r^{-3})\mathfrak{d}^{\leq J}B + O(r^{-4})\mathfrak{d}^{\leq J}\Gamma_b + r^{-3}\mathfrak{d}^{\leq J}\big(\Gamma_b \cdot \Gamma_b\big).$$

Therefore,

$$\begin{aligned}\widetilde{F}_{(1,J-2)} &= O(r^{-1})\mathfrak{d}^{J-2}\nabla_{\widehat{R}}\big({}^{(c)}\nabla_3\dot{B},\, \mathfrak{d}\!\!\!/\dot{B}\big) + O(r^{-1})\mathfrak{d}^{J-1}\nabla_{\widehat{R}}\dot{P} + O(r^{-3})(\nabla_3, \nabla)\mathfrak{d}^{\leq J}\Gamma_b\\ &+O(r^{-1})\mathfrak{d}^{\leq J-1}\dot{P} + O(r^{-1})\mathfrak{d}^{\leq J}B + O(r^{-3})\mathfrak{d}^{\leq J}\Gamma_b + r^{-2}\mathfrak{d}^{\leq J+1}(\Gamma_b \cdot \check{R}_b)\\ &+r^{-2}\mathfrak{d}^{\leq J}\big(\Gamma_b \cdot \Gamma_b\big)\end{aligned}$$

as stated in the first equation of (15.5.10). The second equation of (15.5.10) is derived in the same manner and is in fact simpler.

To prove the second statement we apply the second part of Proposition 15.4.2 to the Bianchi pair (15.2.2)

$$\begin{aligned}{}^{(c)}\nabla_3\dot{P} + \frac{3}{2}\overline{\mathrm{tr}\underline{X}}\dot{P} &= -\mathcal{D}\!\!\!\!/_1^*\underline{\dot{B}} + O(ar^{-2})\underline{\dot{B}} + O(r^{-3})\mathfrak{d}^{\leq 1}\Gamma_b + r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b \cdot \check{R}_b),\\ {}^{(c)}\nabla_4\underline{\dot{B}} + \mathrm{tr}X\underline{\dot{B}} &= \mathcal{D}\!\!\!\!/_1^*\dot{P} + O(ar^{-2})\dot{P} + O(r^{-3})\nabla\!\!\!\!/_{\mathbf{T}}\,\underline{\check{H}} + O(r^{-4})\Gamma_b + r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b \cdot \check{R}_b).\end{aligned}$$

We deduce, for $\widetilde{P}_- = ({}^{(c)}\nabla_3)^{\leq J-2}\, {}^{(c)}\nabla^2_{\widehat{R}}\dot{P}$, $\widetilde{\underline{B}} = ({}^{(c)}\nabla_3)^{\leq J-2}\, {}^{(c)}\nabla^2_{\widehat{R}}\dot{B}$,

$$
\begin{aligned}
{}^{(c)}\nabla_3\widetilde{P}_- + \frac{3}{2}\overline{\mathrm{tr}\underline{X}}\widetilde{P}_- &= -\,\mathcal{D}_1\widetilde{\underline{B}} + \widetilde{F}_{1,J-2},\\
{}^{(c)}\nabla_4\widetilde{\underline{B}} + \mathrm{tr}X\widetilde{\underline{B}} &= \mathcal{D}_1^*\widetilde{P}_- + \widetilde{F}_{2,J-2},
\end{aligned}
$$

with, see (15.4.5),

$$
\begin{aligned}
\widetilde{F}_{(1,k)} &= \big({}^{(c)}\nabla_3\big)^k\, {}^{(c)}\nabla^2_{\widehat{R}}F_{(1)} - 2\omega\big({}^{(c)}\nabla_3\big)^k\, {}^{(c)}\nabla_3\, {}^{(c)}\nabla_{\widehat{R}}\Psi_{(1)}\\
&\quad +O(r^{-1})\mathfrak{d}^{\leq k+1}\, {}^{(c)}\nabla_{\widehat{R}}\Psi_{(2)} + O(ar^{-2})\mathfrak{d}^{\leq k}\, \not{\mathfrak{d}}\, {}^{(c)}\nabla_{\widehat{R}}\Psi_{(1)}\\
&\quad +O(r^{-1})\mathfrak{d}^{\leq k+1}\big(\Psi_{(1)}, \Psi_{(2)}\big) + \mathfrak{d}^{\leq k+2}\Big(\Gamma_b\cdot(\Psi_{(1)}, \Psi_{(2)})\Big),\\
\widetilde{F}_{(2,k)} &= \big({}^{(c)}\nabla_3\big)^k\, {}^{(c)}\nabla^2_{\widehat{R}}F_{(2)} + 2\frac{\Delta}{|q|^2}\omega\big({}^{(c)}\nabla_3\big)^k\, {}^{(c)}\nabla_3\nabla_{\widehat{R}}\Psi_{(2)}\\
&\quad +O(r^{-1})\mathfrak{d}^{\leq k+1}\, {}^{(c)}\nabla_{\widehat{R}}\Psi_{(1)} + O(ar^{-2})\mathfrak{d}^{\leq k}\, \not{\mathfrak{d}}\, {}^{(c)}\nabla_{\widehat{R}}\Psi_{(2)}\\
&\quad +O(r^{-1})\mathfrak{d}^{\leq k+1}\big(\Psi_{(1)}, \Psi_{(2)}\big) + \mathfrak{d}^{\leq k+2}\Big(\Gamma_b\cdot(\Psi_{(1)}, \Psi_{(2)})\Big),
\end{aligned}
$$

where $\Psi_{(1)} = \dot{P}, \Psi_{(2)} = \dot{\underline{B}}$ and

$$
\begin{aligned}
F_{(1)} &= O(ar^{-2})\dot{\underline{B}} + O(r^{-3})\mathfrak{d}^{\leq 1}\Gamma_b + r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\check{R}_b),\\
F_{(2)} &= O(ar^{-2})\dot{P} + O(r^{-3})\not{\mathcal{L}}_{\mathbf{T}}\,\widecheck{\underline{H}} + O(r^{-4})\Gamma_b + r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\check{R}_b).
\end{aligned}
$$

We deduce

$$
\begin{aligned}
\widetilde{F}_{(1,J-2)} &= O(r^{-1})\mathfrak{d}^{J-1}\nabla_{\widehat{R}}(\dot{P}, \dot{\underline{B}}) + O(r^{-3})\mathfrak{d}^{\leq J+1}\Gamma_b + O(r^{-1})\mathfrak{d}^{\leq J-1}(\dot{P}, \dot{B})\\
&\quad +r^{-1}\mathfrak{d}^{\leq J+1}(\Gamma_b\cdot\check{R}_b)
\end{aligned}
$$

as stated in (15.5.12). Also

$$
\begin{aligned}
\widetilde{F}_{(2,J-2)} &= \big({}^{(c)}\nabla_3\big)^{J-2}\, {}^{(c)}\nabla^2_{\widehat{R}}\Big(O(ar^{-2})\dot{P} + O(r^{-3})\not{\mathcal{L}}_{\mathbf{T}}\,\widecheck{\underline{H}} + O(r^{-4})\Gamma_b + r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\check{R}_b)\Big)\\
&\quad +2\omega\big({}^{(c)}\nabla_3\big)^{J-2}\left(\frac{\Delta}{|q|^2}\, {}^{(c)}\nabla_3\, {}^{(c)}\nabla_{\widehat{R}}\dot{\underline{B}}\right) + O(r^{-1})\mathfrak{d}^{\leq J-1}\nabla_{\widehat{R}}\dot{P}\\
&\quad +O(ar^{-2})\mathfrak{d}^{\leq J-2}\,\not{\mathfrak{d}}\nabla_{\widehat{R}}\dot{\underline{B}} + O(r^{-1})\mathfrak{d}^{\leq J-1}\big(\dot{P}, \dot{\underline{B}}\big)\\
&\quad +\mathfrak{d}^{\leq J}\Big(\Gamma_b\cdot(\dot{P}, \dot{\underline{B}})\Big).
\end{aligned}
$$

Recalling that $\widehat{R} = \frac{1}{2}(e_4 - \frac{\Delta}{|q|^2}e_3)$, we write

$$
\frac{\Delta}{|q|^2}\, {}^{(c)}\nabla_3\, {}^{(c)}\nabla_{\widehat{R}}\dot{\underline{B}} = {}^{(c)}\nabla_4\, {}^{(c)}\nabla_{\widehat{R}}\dot{\underline{B}} - 2\, {}^{(c)}\nabla^2_{\widehat{R}}\dot{\underline{B}}.
$$

We deduce

$$\left({}^{(c)}\nabla_3\right)^{J-2}\left(\frac{\Delta}{|q|^2}{}^{(c)}\nabla_3{}^{(c)}\nabla_{\widehat{R}}\dot{\underline{B}}\right) = -2\widetilde{\underline{B}} + \mathfrak{d}^{\leq J-2}\,{}^{(c)}\nabla_{\widehat{R}}{}^{(c)}\nabla_4\dot{\underline{B}} + O(r^{-1})\mathfrak{d}^{\leq J-1}\dot{\underline{B}}$$

and hence, using also $2\not{\mathcal{L}}_{\mathbf{T}}\widecheck{\underline{H}} = \frac{\Delta}{|q|^2}\nabla_3\widecheck{\underline{H}} + O(1)(\nabla_4,\nabla)\Gamma_g + O(1)\Gamma_g$, we obtain

$$\begin{aligned}\widetilde{F}_{(2,J-2)} &= O(r^{-3})\mathfrak{d}^{\leq J}\,{}^{(c)}\nabla_3\widecheck{\underline{H}} + O(r^{-3})\mathfrak{d}^{\leq J}(\nabla_4,\nabla)\Gamma_b + O(r^{-4})\mathfrak{d}^{\leq J}\Gamma_b - 4\omega\widetilde{\underline{B}}\\ &+O(r^{-1})\mathfrak{d}^{\leq J-1}\nabla_{\widehat{R}}\dot{P} + O(ar^{-2})\mathfrak{d}^{\leq J-2}\,\not{\mathfrak{d}}\nabla_{\widehat{R}}\dot{\underline{B}} + O(r^{-1})\mathfrak{d}^{\leq J-1}\left(\dot{P},\dot{\underline{B}}\right)\\ &+\mathfrak{d}^{\leq J+1}\left(\Gamma_b\cdot\check{R}_b\right).\end{aligned}$$

In view of the equation (13.2.1) for ${}^{(c)}\nabla_3\widecheck{\underline{H}}$, we have

$${}^{(c)}\nabla_3\widecheck{\underline{H}} - {}^{(c)}\nabla_4\Xi = \underline{B} + O(r^{-1})\Gamma_b + \Gamma_b\cdot\Gamma_b.$$

Therefore,

$$\begin{aligned}\widetilde{F}_{(2,J-2)} &= -4\omega\widetilde{\underline{B}} + O(r^{-3})\mathfrak{d}^{\leq J}(\nabla_4,\nabla)\Gamma_b + O(r^{-1})\mathfrak{d}^{\leq J-1}\nabla_{\widehat{R}}\dot{P} + O(ar^{-2})\mathfrak{d}^{\leq J-2}\,\not{\mathfrak{d}}\nabla_{\widehat{R}}\dot{\underline{B}}\\ &+O(r^{-1})\mathfrak{d}^{\leq J-1}\dot{P} + O(r^{-1})\mathfrak{d}^{\leq J}\underline{B} + O(r^{-3})\mathfrak{d}^{\leq J}\Gamma_b + \mathfrak{d}^{\leq J+1}\left(\Gamma_b\cdot\check{R}_b\right)\end{aligned}$$

as stated in (15.5.12). This concludes the proof of Lemma 15.5.5. □

15.5.3 Estimates for $\widetilde{B}$

We provide estimates for $\widetilde{B}$ using (15.5.9) and Proposition 15.3.12.

Proposition 15.5.7. *The following estimate holds true for $\widetilde{B}$, with $b = 2+\delta$ and for a sufficiently small,*

$$\begin{aligned}\int_{\mathcal{M}} r^{b-1}|\widetilde{B}|^2 + \int_{\partial^+\mathcal{M}} r^b|\widetilde{B}|^2 &\lesssim \delta_{J+1}[\check{P}] + \epsilon_J^2 + \epsilon_0^2 + \sqrt{\delta_{J+1}[\check{P}]}\left(B^J_\delta[r^2B] + \mathfrak{G}^2_{J+1}\right)^{\frac{1}{2}}\\ &+ O(a^2)B_\delta[r^{J+1}\nabla_4^J B] + \epsilon_J\left(B^J_\delta[r^2B]\right)^{\frac{1}{2}} \qquad (15.5.13)\\ &+ \mathfrak{G}_{J+1}\left(\delta_{J+1}[\check{P}] + \epsilon_J^2 + \epsilon_0^2 + O(a^2)B_\delta[r^{J+1}\nabla_4^J B]\right)^{\frac{1}{2}}.\end{aligned}$$

Proof. Note that the system (15.5.9) is of the form (15.3.8) with

$$\Psi_{(1)} = \widetilde{B},\qquad \Psi_{(2)} = \widetilde{P}_+,\qquad c_{(1)} = 1 - \frac{J-2}{2},\qquad c_{(2)} = \frac{3}{2} - \frac{J-2}{2}.$$

Recall also that the signatures of $\widetilde{P}_+$ and $\widetilde{B}$ are respectively $J-2$ and $J-1$. We are thus in the case corresponding to $2k-1>0$. Observe that the condition $-2c_{(1)}+1+\frac{b}{2}>0$ is verified for $b=2+\delta$ and therefore we can apply (15.3.14) in $\mathcal{M}=\mathcal{M}(1,\tau)$, $\tau\le\tau_*$, to derive

$$
\begin{aligned}
&\int_{\mathcal{M}} r^{b-1}|\Psi_{(1)}|^2+\int_{\partial^+\mathcal{M}} r^b|\Psi_{(1)}|^2\\
\lesssim &\int_{\mathcal{M}} r^{b-1}|\Psi_{(2)}|^2+\left|\int_{\mathcal{M}}|q|^b\Re(\widetilde{F}_{(1,J-2)}\cdot\overline{\Psi_{(1)}})\right|+\int_{\mathcal{M}} r^b\big|\widetilde{F}_{(2,J-2)}\big|\,\big|\Psi_{(2)}\big|\\
&+\int_{\partial^-\mathcal{M}}\big(r^b|\Psi_{(1)}|^2+r^{b-2}|\Psi_{(2)}|^2\big),
\end{aligned}
$$

with $\widetilde{F}_{(1,J-2)}, \widetilde{F}_{(2,J-2)}$ given by formula (15.5.10). We decompose

$$\widetilde{F}_{(1,J-2)}=O(r^{-3})(\nabla_3,\nabla)\mathfrak{d}^{\le J}\Gamma_b+\widetilde{F}'_{(1,J-2)}.$$

Thus

$$
\begin{aligned}
\int_{\mathcal{M}} r^{b-1}|\widetilde{B}|^2+\int_{\partial^+\mathcal{M}} r^b|\widetilde{B}|^2 \;\lesssim\; &\int_{\mathcal{M}} r^{b-1}|\widetilde{P}_+|^2+\int_{\mathcal{M}} r^b\Big(\big|\widetilde{F}'_{(1,J-2)}\big|\,\big|\widetilde{B}\big|+\big|\widetilde{F}_{(2,J-2)}\big|\,\big|\widetilde{P}_+\big|\Big)\\
&+\big|I\big|+\epsilon_0^2
\end{aligned}
$$

where

$$I:=\int_{\mathcal{M}} O(r^{-3})|q|^b\Re((\nabla_3,\nabla)\mathfrak{d}^{\le J}\Gamma_b\cdot\overline{\widetilde{B}})+\int_{\mathcal{M}} O(r^{-1})|q|^b\Re(\mathfrak{d}^{\le J}B\cdot\overline{\widetilde{B}}). \tag{15.5.14}$$

Recall that

$$\delta_{J+1}[\check{P}]=BEF^J_\delta[r^2\check{P}]\ge\int_{\mathcal{M}_{trap}}|\nabla_{\widehat{R}}\mathfrak{d}^{\le J}\check{P}|^2+|\mathfrak{d}^{\le J}\check{P}|^2+\int_{\mathcal{M}_{\not{trap}}} r^{\delta+1}|\mathfrak{d}^{\le J+1}\check{P}|^2.$$

Therefore, with $b=2+\delta$,

$$\int_{\mathcal{M}} r^{b-1}|\widetilde{P}_+|^2=\int_{\mathcal{M}} r^{1+\delta}\big|(\overline{q}\,{}^{(c)}\nabla_4)^{\le J-2}\,{}^{(c)}\nabla^2_{\widehat{R}}\overline{\dot{P}}\big|^2\lesssim\int_{\mathcal{M}} r^{1+\delta}\big|\mathfrak{d}^{J-1}\,{}^{(c)}\nabla_{\widehat{R}}\overline{\dot{P}}\big|\lesssim\delta_{J+1}[\check{P}].$$

By Cauchy-Schwartz and absorbing the term in $\widetilde{B}$ to the left we easily deduce

$$
\begin{aligned}
\int_{\mathcal{M}} r^{b-1}|\widetilde{B}|^2+\int_{\partial^+\mathcal{M}} r^b|\widetilde{B}|^2\lesssim\;&\delta_{J+1}[\check{P}]+\int_{\mathcal{M}} r^{b+1}\big|\widetilde{F}'_{(1,J-2)}\big|^2\\
&+\sqrt{\delta_{J+1}[\check{P}]}\left(\int_{\mathcal{M}} r^{b+1}\big|\widetilde{F}_{(2,J-2)}\big|^2\right)^{\frac12}+\big|I\big|+\epsilon_0^2.
\end{aligned}
\tag{15.5.15}
$$

In view of (15.5.10) and the definition of I, we have

$$\begin{aligned}
\widetilde{F}'_{(1,J-2)} &= O(r^{-1})\mathfrak{d}^{J-2}\nabla_{\widehat{R}}\big({}^{(c)}\nabla_3\dot{B},\, \not{\mathfrak{d}}\dot{B}\big) + O(r^{-1})\mathfrak{d}^{J-1}\nabla_{\widehat{R}}\dot{P} + O(r^{-1})\mathfrak{d}^{\leq J-1}\dot{P}\\
&+ O(r^{-3})\mathfrak{d}^{\leq J}\Gamma_b + r^{-2}\mathfrak{d}^{\leq J+1}(\Gamma_b\cdot \check{R}_b) + r^{-2}\mathfrak{d}^{\leq J}\big(\Gamma_b\cdot\Gamma_b\big),\\
\widetilde{F}_{(2,J-2)} &= O(r^{-1})\mathfrak{d}^{J-1}\nabla_{\widehat{R}}(\dot{P},\dot{B}) + O(r^{-3})\mathfrak{d}^{\leq J+1}\Gamma_b + O(r^{-1})\mathfrak{d}^{\leq J-1}(\dot{P},\dot{B})\\
&+ r^{-2}\mathfrak{d}^{\leq J+1}(\Gamma_b\cdot \check{R}_b).
\end{aligned}$$

We deduce, using the bootstrap assumptions, as well as the choice $b = 2+\delta$,

$$\int_{\mathcal{M}} r^{b+1}\big|\widetilde{F}'_{(1,J-2)}\big|^2 \lesssim B^{J-2}_\delta[r^2(\nabla_3\dot{B},\nabla\dot{B})] + B^{J-1}_\delta[r^2\dot{P}] + \int_{\mathcal{M}} r^{b-5}|\mathfrak{d}^{\leq J}\Gamma_b|^2 + \epsilon_0^2,$$

and

$$\int_{\mathcal{M}} r^{b+1}\big|\widetilde{F}_{(2,J-2)}\big|^2 \lesssim B^{J-1}_\delta[r^2(\dot{B},\dot{P})] + \int_{\mathcal{M}} r^{b-5}|\mathfrak{d}^{\leq J+1}\Gamma_b|^2 + \epsilon_0^2.$$

According to Lemma 13.5.5 we have, since $b = 2+\delta$,

$$\int_{\mathcal{M}} r^{b-5}|\mathfrak{d}^{\leq J+1}\Gamma_b|^2 \lesssim \mathfrak{G}^2_{J+1}, \qquad \int_{\mathcal{M}} r^{b-5}|\mathfrak{d}^{J}\Gamma_b|^2 \lesssim \mathfrak{G}^2_{J} \lesssim \epsilon_J^2,$$

where we have used the induction hypothesis. Also, in view of the proof of Lemma 14.2.2, we have $\dot{P} = \mathbf{T}\check{P} + r^{-3}\Gamma_g$ so that

$$B^{J-1}_\delta[r^2\dot{P}] \lesssim B^{J-1}_\delta[r^2\mathbf{T}\check{P}] + B^{J-1}_\delta[r^{-1}\Gamma_b] \lesssim \delta_{J+1}[\check{P}] + \mathfrak{G}^2_J \lesssim \delta_{J+1}[\check{P}] + \epsilon_J^2.$$

Therefore,

$$\begin{aligned}
&\int_{\mathcal{M}} r^{b+1}\big|\widetilde{F}'_{(1,J-2)}\big|^2 + \sqrt{\delta_{J+1}[\check{P}]}\left(\int_{\mathcal{M}} r^{b+1}\big|\widetilde{F}_{(2,J-2)}\big|^2\right)^{\frac{1}{2}}\\
\lesssim\ & B^{J-2}_\delta[r^2(\nabla_3\dot{B},\nabla\dot{B})] + \delta_{J+1}[\check{P}] + \epsilon_J^2 + \epsilon_0^2\\
&+\sqrt{\delta_{J+1}[\check{P}]}\left(B^{J-1}_\delta[r^2\dot{B}] + \delta_{J+1}[\check{P}] + \mathfrak{G}^2_{J+1} + \epsilon_J^2 + \epsilon_0^2\right)^{\frac{1}{2}}\\
\lesssim\ & B^{J-2}_\delta[r^2(\nabla_3\dot{B},\nabla\dot{B})] + \delta_{J+1}[\check{P}] + \epsilon_J^2 + \epsilon_0^2 + \sqrt{\delta_{J+1}[\check{P}]}\left(B^{J}_\delta[r^2 B] + \mathfrak{G}^2_{J+1}\right)^{\frac{1}{2}}\\
\lesssim\ & B^{J-1}_\delta[r^2(\nabla_3 B, r\nabla B] + \delta_{J+1}[\check{P}] + \epsilon_J^2 + \epsilon_0^2 + \sqrt{\delta_{J+1}[\check{P}]}\left(B^{J}_\delta[r^2 B] + \mathfrak{G}^2_{J+1}\right)^{\frac{1}{2}}.
\end{aligned}$$

Using (15.5.6), i.e.

$$B^{J-1}_\delta[r^2(\nabla_3 B, r\nabla B)] \lesssim \delta_{J+1}[\check{P}] + \epsilon_J^2 + \epsilon_0^2 + O(a^2)B_\delta[r^{J+1}\nabla_4^J B],$$

we infer

$$\int_{\mathcal{M}} r^{b+1}\big|\widetilde{F}'_{(1,J-2)}\big|^2 + \sqrt{\delta_{J+1}[\check{P}]}\left(\int_{\mathcal{M}} r^{b+1}\big|\widetilde{F}_{(2,J-2)}\big|^2\right)^{\frac{1}{2}}$$
$$\lesssim \delta_{J+1}[\check{P}] + \epsilon_J^2 + \epsilon_0^2 + \sqrt{\delta_{J+1}[\check{P}]}\left(B^J_\delta[r^2 B] + \mathfrak{G}^2_{J+1}\right)^{\frac{1}{2}} + O(a^2)B_\delta[r^{J+1}\nabla_4^J B].$$

Back to (15.5.15) we infer that

$$\int_{\mathcal{M}} r^{b-1}|\widetilde{B}|^2 + \int_{\partial^+\mathcal{M}} r^b|\widetilde{B}|^2 \lesssim \big|I\big| + \delta_{J+1}[\check{P}] + \epsilon_J^2 + \epsilon_0^2 + \sqrt{\delta_{J+1}[\check{P}]}\left(B^J_\delta[r^2B] + \mathfrak{G}^2_{J+1}\right)^{\frac{1}{2}}$$
$$+O(a^2)B_\delta[r^{J+1}\nabla_4^J B]. \tag{15.5.16}$$

It remains to estimate the term I. We decompose it as

$$\begin{aligned} I &= I_1 + I_2, \\ I_1 &:= \int_{\mathcal{M}} O(r^{-3})|q|^b \Re((\nabla_3, \nabla)\mathfrak{d}^{\leq J}\Gamma_b \cdot \overline{\widetilde{B}}), \\ I_2 &:= \int_{\mathcal{M}} O(r^{-1})|q|^b \Re(\mathfrak{d}^{\leq J} B \cdot \overline{\widetilde{B}}), \end{aligned}$$

and estimate I_1 and I_2 separately starting with I_1. Integrating by parts, we have

$$\begin{aligned} I_1 &= \int_{\mathcal{M}} O(r^{-3})|q|^b \Re((\nabla_3, \nabla)\mathfrak{d}^{\leq J}\Gamma_b \cdot \overline{\widetilde{B}}) \\ &= -\int_{\mathcal{M}} O(r^{-3})|q|^b \Re\left(\mathfrak{d}^{\leq J}\Gamma_b \cdot \overline{(\nabla_3, \nabla)\widetilde{B}}\right) + \int_{\partial^+\mathcal{M}} O(r^{b-3})|\mathfrak{d}^{\leq J}\Gamma_b||\widetilde{B}| \\ &\quad + \int_{\mathcal{M}} O(r^{b-4})|\mathfrak{d}^{\leq J}\Gamma_b||\widetilde{B}| + \epsilon_0^2. \end{aligned}$$

Since $\widetilde{B} = (\overline{q}\,{}^{(c)}\nabla_4)^{\leq J-2}\,{}^{(c)}\nabla^2_{\widehat{R}}\dot{B}$, we may integrate the first term on the RHS by parts again and obtain

$$\begin{aligned} I_1 &= -\int_{\mathcal{M}} O(r^{b-3})|\mathfrak{d}^{\leq J+1}\Gamma_b||\nabla_{\widehat{R}}(\nabla_3, \nabla)\mathfrak{d}^{\leq J-1}B| + \int_{\partial^+\mathcal{M}} O(r^{b-3})|\mathfrak{d}^{\leq J}\Gamma_b||\mathfrak{d}^{\leq J+1}B| \\ &\quad + \int_{\mathcal{M}} O(r^{b-4})|\mathfrak{d}^{\leq J}\Gamma_b||\nabla_{\widehat{R}}\mathfrak{d}^{\leq J}B| + \epsilon_0^2. \end{aligned}$$

Hence, using Lemma 13.5.5, Lemma 14.1.1, and the fact that $b = 2 + \delta$, we infer

$$\begin{aligned} |I_1| &\lesssim \left(\int_{\mathcal{M}} r^{b-5}|\mathfrak{d}^{\leq J+1}\Gamma_b|^2\right)^{\frac{1}{2}}\left(B^{J-1}_\delta[r^2(\nabla_3 B, r\nabla B)]\right)^{\frac{1}{2}} \\ &\quad + \left(\int_{\mathcal{M}} r^{b-5}|\mathfrak{d}^{\leq J}\Gamma_b|^2\right)^{\frac{1}{2}}\left(B^J_\delta[r^2 B]\right)^{\frac{1}{2}} + \epsilon_0^2 \\ &\lesssim \mathfrak{G}_{J+1}\left(B^{J-1}_\delta[r^2(\nabla_3 B, r\nabla B)]\right)^{\frac{1}{2}} + \epsilon_J\left(B^J_\delta[r^2B]\right)^{\frac{1}{2}} + \epsilon_0^2. \end{aligned}$$

Together with (15.5.6), we deduce

$$|I_1| \lesssim \mathfrak{G}_{J+1}\Big(\delta_{J+1}[\check{P}] + \epsilon_J^2 + \epsilon_0^2 + O(a^2)B_\delta[r^{J+1}\nabla_4^J B]\Big)^{\frac{1}{2}} + \epsilon_J\Big(B_\delta^J[r^2 B]\Big)^{\frac{1}{2}} + \epsilon_0^2.$$

Next, we estimate I_2. Using the fact that $\widetilde{B} = (\overline{q}\,^{(c)}\nabla_4)^{\leq J-2}\,^{(c)}\nabla_{\widehat{R}}^2\dot{B}$, we have

$$\widetilde{B} = \nabla_{\widehat{R}}^2\mathfrak{d}^{\leq J-1}B + O(1)\nabla_{\widehat{R}}\mathfrak{d}^{\leq J-1}B + O(1)\mathfrak{d}^{\leq J-1}B$$

and hence, introducing the notation

$$I_{2,1} := \int_{\mathcal{M}} O(r^{-1})|q|^b\Re(\mathfrak{d}^{\leq J}B\cdot\overline{\nabla_{\widehat{R}}^2\mathfrak{d}^{\leq J-1}B}),$$

we have, using Lemma 14.1.1 and the fact that $b = 2+\delta$,

$$\begin{aligned}
|I_2| &\lesssim |I_{2,1}| + \int_{\mathcal{M}} r^{b-1}|\mathfrak{d}^{\leq J}B||\nabla_{\widehat{R}}\mathfrak{d}^{\leq J-1}B| + \int_{\mathcal{M}} r^{b-1}|\mathfrak{d}^{\leq J}B||\mathfrak{d}^{\leq J-1}B|\\
&\lesssim |I_{2,1}| + \Big(B_\delta^J[r^2 B]\Big)^{\frac{1}{2}}\Big(B_\delta^{J-1}[r^2B]\Big)^{\frac{1}{2}}\\
&\lesssim |I_{2,1}| + \epsilon_J\Big(B_\delta^J[r^2B]\Big)^{\frac{1}{2}}.
\end{aligned}$$

Also, integrating by parts, we have

$$\begin{aligned}
I_{2,1} &= \int_{\mathcal{M}} O(r^{-1})|q|^b\Re(\mathfrak{d}^{\leq J}B\cdot\overline{\nabla_{\widehat{R}}^2\mathfrak{d}^{\leq J-1}B})\\
&= -\int_{\mathcal{M}} O(r^{-1})|q|^b\Re(\nabla_{\widehat{R}}\mathfrak{d}^{\leq J}B\cdot\overline{\nabla_{\widehat{R}}\mathfrak{d}^{\leq J-1}B})\\
&\quad + \int_{\partial^+\mathcal{M}} O(r^{-1})|q|^b\Re(\mathfrak{d}^{\leq J}B\cdot\overline{\nabla_{\widehat{R}}^2\mathfrak{d}^{\leq J-1}B})\\
&\quad + \int_{\mathcal{M}} O(r^{-2})|q|^b\Re(\mathfrak{d}^{\leq J}B\cdot\overline{\nabla_{\widehat{R}}\mathfrak{d}^{\leq J-1}B}) + \epsilon_0^2
\end{aligned}$$

and hence

$$\begin{aligned}
|I_{2,1}| &\lesssim \Big(B_\delta^J[r^2B]\Big)^{\frac{1}{2}}\Big(B_\delta^{J-1}[r^2B]\Big)^{\frac{1}{2}} + \epsilon_0^2\\
&\lesssim \epsilon_J\Big(B_\delta^J[r^2B]\Big)^{\frac{1}{2}} + \epsilon_0^2.
\end{aligned}$$

We deduce

$$\begin{aligned}
|I_2| &\lesssim |I_{2,1}| + \epsilon_J\Big(B_\delta^J[r^2B]\Big)^{\frac{1}{2}}\\
&\lesssim \epsilon_J\Big(B_\delta^J[r^2B]\Big)^{\frac{1}{2}} + \epsilon_0^2.
\end{aligned}$$

Together with the above estimate for I_1, this yields

$$\begin{aligned}|I| &\lesssim |I_1| + |I_2| \\ &\lesssim \mathfrak{G}_{J+1}\Big(\delta_{J+1}[\check{P}] + \epsilon_J^2 + \epsilon_0^2 + O(a^2)B_\delta[r^{J+1}\nabla_4^J B]\Big)^{\frac{1}{2}} + \epsilon_J\Big(B_\delta^J[r^2 B]\Big)^{\frac{1}{2}} + \epsilon_0^2.\end{aligned}$$

Finally, plugging the above estimate for I in (15.5.16), we infer

$$\begin{aligned}\int_{\mathcal{M}} r^{b-1}|\widetilde{B}|^2 + \int_{\partial^+\mathcal{M}} r^b|\widetilde{B}|^2 &\lesssim \delta_{J+1}[\check{P}] + \epsilon_J^2 + \epsilon_0^2 + \sqrt{\delta_{J+1}[\check{P}]}\,\big(B_\delta^J[r^2B] + \mathfrak{G}_{J+1}^2\big)^{\frac{1}{2}} \\ &\quad + O(a^2)B_\delta[r^{J+1}\nabla_4^J B] + \epsilon_J\Big(B_\delta^J[r^2B]\Big)^{\frac{1}{2}} \\ &\quad + \mathfrak{G}_{J+1}\Big(\delta_{J+1}[\check{P}] + \epsilon_J^2 + \epsilon_0^2 + O(a^2)B_\delta[r^{J+1}\nabla_4^J B]\Big)^{\frac{1}{2}}\end{aligned}$$

as stated. This concludes the proof of Proposition 15.5.7. □

15.5.4 Estimates for $\widetilde{\underline{B}}$

We provide estimates for $\widetilde{\underline{B}}$ using (15.5.11) and Proposition 15.3.12.

Proposition 15.5.8. *The following estimates hold true for $\widetilde{B}$, with $b = -\delta$ and a sufficiently small,*

$$\begin{aligned}\int_{\mathcal{M}} r^{b-1}|\widetilde{\underline{B}}|^2 + \int_{\partial^+\mathcal{M}} r^{b-2}|\widetilde{\underline{B}}|^2 &\lesssim \delta_{J+1}[\check{P}] + \epsilon_J^2 + \epsilon_0^2 + \sqrt{\delta_{J+1}[\check{P}]}\,\big(B_\delta^J[\underline{B}] + \mathfrak{G}_{J+1}^2\big)^{\frac{1}{2}} \\ &\quad + O(a^2)B_\delta[\nabla_3^J\underline{B}] + \epsilon_J\Big(B_\delta^J[\underline{B}]\Big)^{\frac{1}{2}} \\ &\quad + \mathfrak{G}_{J+1}\Big(\delta_{J+1}[\check{P}] + \epsilon_J^2 + \epsilon_0^2 + O(a^2)B_\delta[\nabla_3^J\underline{B}]\Big)^{\frac{1}{2}}.\end{aligned} \tag{15.5.17}$$

Proof. The proof is similar to the one of Proposition 15.5.7. We start with the Bianchi pair (15.5.11)

$$\begin{aligned}{}^{(c)}\nabla_3\widetilde{P}_- + \frac{3}{2}\overline{\mathrm{tr}\underline{X}}\widetilde{P}_- &= -\,\mathcal{D}_1\widetilde{\underline{B}} + \widetilde{F}_{1,J-2}, \\ {}^{(c)}\nabla_4\widetilde{\underline{B}} + \mathrm{tr}X\widetilde{\underline{B}} &= \mathcal{D}_1^*\widetilde{P}_- + \widetilde{F}_{2,J-2},\end{aligned}$$

with

$$\widetilde{F}_{(1,J-2)} = O(r^{-1})\mathfrak{d}^{J-1}\nabla_{\widehat{R}}(\dot{P}, \dot{\underline{B}}) + O(r^{-3})\mathfrak{d}^{\leq J+1}\Gamma_b + O(r^{-1})\mathfrak{d}^{\leq J-1}(\dot{P}, \dot{\underline{B}})$$
$$+ r^{-1}\mathfrak{d}^{\leq J+1}(\Gamma_b \cdot \check{R}_b),$$
$$\widetilde{F}_{(2,J-2)} = -4\omega\widetilde{\underline{B}} + O(r^{-3})\mathfrak{d}^{\leq J}(\nabla_4, \nabla)\Gamma_b + \widetilde{F}'_{(2,J-2)},$$
$$\widetilde{F}'_{(2,J-2)} = O(r^{-1})\mathfrak{d}^{\leq J-1}\nabla_{\widehat{R}}\dot{P} + O(ar^{-2})\mathfrak{d}^{\leq J-2}\not{\mathfrak{d}}\nabla_{\widehat{R}}\dot{\underline{B}} + O(r^{-1})\mathfrak{d}^{\leq J-1}\dot{P} + O(r^{-1})\mathfrak{d}^{\leq J}\underline{B}$$
$$+ O(r^{-3})\mathfrak{d}^{\leq J}\Gamma_b + \mathfrak{d}^{\leq J+1}\Big(\Gamma_b \cdot \check{R}_b\Big).$$

which can be written in the form (15.3.9) with

$$\Psi_{(1)} = \widetilde{P}_-, \qquad \Psi_{(2)} = \widetilde{\underline{B}}, \qquad c_{(1)} = \frac{3}{2}, \qquad c_{(2)} = 1,$$

where the signature of $\widetilde{P}_-$ is equal to $-J+2$ and that of $\widetilde{\underline{B}}$ is equal to $-J+1$. This corresponds to the case when $2k-1 = 2(-J+2)-1 = -2J+3 < 0$. To apply the integral estimate (15.3.15), we need $\Lambda_{(2)} = -2c_{(2)} + 1 + \frac{b}{2} < 0$ to be satisfied which holds true for the choice $b = -\delta$. Therefore

$$\begin{aligned}
&\int_{\mathcal{M}} r^{b-1}\left(1 + |2J-3|\frac{m}{r}\right)|\widetilde{\underline{B}}|^2 + \int_{\partial^+\mathcal{M}} r^{b-2}|\widetilde{\underline{B}}|^2 \\
&\lesssim \int_{\mathcal{M}} r^{b-1}|\widetilde{P}_-|^2 + \big|I_1\big| + \big|I_2\big| + \int_{\mathcal{M}} r^b\Big(\big|F_{(1,J-2)}\big|\,\big|\widetilde{P}_-\big| + \big|F'_{(2,J-2)}\big|\,\big|\widetilde{\underline{B}}\big|\Big) + \epsilon_0^2,
\end{aligned} \tag{15.5.18}$$

where

$$I_1 = \int_{\mathcal{M}} |q|^b\Re\Big(4\omega\widetilde{\underline{B}}\Big)\cdot\overline{\widetilde{\underline{B}}}\Big) = \int_{\mathcal{M}} 4|q|^b\omega\big|\widetilde{\underline{B}}\big|^2,$$
$$I_2 = \int_{\mathcal{M}} |q|^b\Re\Big(O(r^{-3})\mathfrak{d}^{\leq J}(\nabla_4, \nabla)\Gamma_b \cdot \overline{\widetilde{\underline{B}}}\Big).$$

Since $\omega = O(mr^{-2})$, we may absorb the term I_1 from the LHS for a sufficiently large[9] choice of J. We deduce

$$\begin{aligned}
\int_{\mathcal{M}} r^{b-1}|\widetilde{\underline{B}}|^2 + \int_{\partial^+\mathcal{M}} r^{b-2}|\widetilde{\underline{B}}|^2 \;\lesssim\; & \int_{\mathcal{M}} r^{b-1}|\widetilde{P}_-|^2 + \left(\int_{\mathcal{M}} r^{b-1}|\widetilde{P}_-|^2\right)^{\frac{1}{2}}\left(\int_{\mathcal{M}} r^{b+1}|F_{(1,J-2)}|^2\right)^{\frac{1}{2}} \\
&+ \int_{\mathcal{M}} r^{b+1}|F'_{(2,J-2)}|^2 + \big|I_2\big| + \epsilon_0^2
\end{aligned}$$

and hence

$$\begin{aligned}
\int_{\mathcal{M}} r^{b-1}|\widetilde{\underline{B}}|^2 + \int_{\partial^+\mathcal{M}} r^{b-2}|\widetilde{\underline{B}}|^2 \;\lesssim\; & \big|I_2\big| + \delta_{J+1}[\check{P}] + \sqrt{\delta_{J+1}[\check{P}]}\left(\int_{\mathcal{M}} r^{b+1}|F_{(1,J-2)}|^2\right)^{\frac{1}{2}} \\
&+ \int_{\mathcal{M}} r^{b+1}|F'_{(2,J-2)}|^2 + \epsilon_0^2.
\end{aligned}$$

[9]Recall that the iteration assumption (13.6.4) holds for $J \geq \frac{k_L}{2}$ and that k_L is chosen large enough.

We then estimate the integrals $\int_{\mathcal{M}} r^{b+1}|F_{(1,J-2)}|^2$ and $\int_{\mathcal{M}} r^{b+1}|F'_{(2,J-2)}|^2$. Taking advantage of the structure of $\widetilde{F}_{(1,J-2)}$ and $F'_{(2,J-2)}$, proceeding as for the corresponding estimate in the proof of Proposition 15.5.7, and using in particular (15.5.7) to control $B_\delta^{J-1}[r(\nabla_4\underline{B}, \nabla\underline{B})]$, we obtain

$$\begin{aligned}&\sqrt{\delta_{J+1}[\check{P}]}\left(\int_{\mathcal{M}} r^{b+1}|F_{(1,J-2)}|^2\right)^{\frac{1}{2}} + \int_{\mathcal{M}} r^{b+1}|F'_{(2,J-2)}|^2\\ \lesssim\;& \delta_{J+1}[\check{P}] + \epsilon_J^2 + \epsilon_0^2 + \sqrt{\delta_{J+1}[\check{P}]}\left(B_\delta^J[\underline{B}] + \mathfrak{G}_{J+1}^2\right)^{\frac{1}{2}} + O(a^2)B_\delta[\nabla_3^J\underline{B}].\end{aligned}$$

We deduce

$$\begin{aligned}\int_{\mathcal{M}} r^{b-1}|\widetilde{\underline{B}}|^2 + \int_{\partial^+\mathcal{M}} r^{b-2}|\widetilde{\underline{B}}|^2 \;\lesssim\;& \big|I_2\big| + \delta_{J+1}[\check{P}] + \epsilon_J^2 + \epsilon_0^2 + \sqrt{\delta_{J+1}[\check{P}]}\left(B_\delta^J[\underline{B}] + \mathfrak{G}_{J+1}^2\right)^{\frac{1}{2}}\\ &+O(a^2)B_\delta[\nabla_3^J\underline{B}].\end{aligned}$$

Then, the term I_2 can be integrated by parts twice, as the corresponding estimate in the proof of Proposition 15.5.7, to obtain

$$|I_2| \;\lesssim\; \mathfrak{G}_{J+1}\Big(\delta_{J+1}[\check{P}] + \epsilon_J^2 + \epsilon_0^2 + O(a^2)B_\delta[\nabla_3^J\underline{B}]\Big)^{\frac{1}{2}} + \epsilon_J\Big(B_\delta^J[\underline{B}]\Big)^{\frac{1}{2}} + \epsilon_0^2.$$

We infer

$$\begin{aligned}\int_{\mathcal{M}} r^{b-1}|\widetilde{\underline{B}}|^2 + \int_{\partial^+\mathcal{M}} r^{b-2}|\widetilde{\underline{B}}|^2 \;\lesssim\;& \delta_{J+1}[\check{P}] + \epsilon_J^2 + \epsilon_0^2 + \sqrt{\delta_{J+1}[\check{P}]}\left(B_\delta^J[\underline{B}] + \mathfrak{G}_{J+1}^2\right)^{\frac{1}{2}}\\ &+O(a^2)B_\delta[\nabla_3^J\underline{B}] + \epsilon_J\Big(B_\delta^J[\underline{B}]\Big)^{\frac{1}{2}}\\ &+\mathfrak{G}_{J+1}\Big(\delta_{J+1}[\check{P}] + \epsilon_J^2 + \epsilon_0^2 + O(a^2)B_\delta[\nabla_3^J\underline{B}]\Big)^{\frac{1}{2}}\end{aligned}$$

as stated in (15.5.17). This concludes the proof of Proposition 15.5.8. □

15.5.5 Proof of the estimates for B in Proposition 15.1.1

First, we have in view of Lemma 14.1.1

$$BEF_\delta^{J-1}[r^2B] \lesssim \mathfrak{R}_J\mathfrak{R}_{J+1}$$

and from (15.5.1)

$$\begin{aligned}BEF_\delta^{J-1}[r^2(\nabla_3B, r\nabla B)] \lesssim& \delta_{J+1}[\check{P}] + \mathfrak{G}_J\mathfrak{G}_{J+1} + \mathfrak{R}_J\mathfrak{R}_{J+1} + \epsilon_0^2\\ &+ O(a^2, \epsilon^2)BEF_\delta[r^{J+1}\nabla_4^JB].\end{aligned}$$

Also, since

$$2\widehat{R} = e_4 + O(1)e_3 + O(1)\nabla, \qquad 2\mathbf{T} = e_4 + O(1)e_3 + O(1)\nabla,$$

and since $\widetilde{B} = (\overline{q}\,{}^{(c)}\nabla_4)^{\leq J-2}\,{}^{(c)}\nabla^2_{\widehat{R}}\dot{B}$, we have

$$\begin{aligned} BEF_\delta[r^{J-1}\nabla_4^J B] \lesssim & \int_{\mathcal{M}} r^{\delta+1}|\widetilde{B}|^2 + \sup_{\tau\leq\tau_*}\int_{\partial^+\mathcal{M}(1,\tau)} r^{\delta+2}|\widetilde{B}|^2 + BEF_\delta^{J-1}[r^2(\nabla_3, r\nabla)B] \\ & + BEF_\delta^{J-1}[r^2 B]. \end{aligned}$$

Together with the above bounds for $BEF_\delta^{J-1}[r^2B]$ and $BEF_\delta^{J-1}[r^2(\nabla_3 B, r\nabla B)]$, and using the control of $\widetilde{B}$ in (15.5.13), i.e.

$$\begin{aligned} \int_{\mathcal{M}} r^{1+\delta}|\widetilde{B}|^2 + \sup_{\tau\leq\tau_*}\int_{\partial^+\mathcal{M}(1,\tau)} r^{2+\delta}|\widetilde{B}|^2 \lesssim\ & \delta_{J+1}[\check{P}] + \epsilon_J^2 + \epsilon_0^2 + \sqrt{\delta_{J+1}[\check{P}]}\left(B_\delta^J[r^2B] + \mathfrak{G}_{J+1}^2\right)^{\frac{1}{2}} \\ & + O(a^2)B_\delta[r^{J+1}\nabla_4^J B] + \epsilon_J\Big(B_\delta^J[r^2B]\Big)^{\frac{1}{2}} \\ & + \mathfrak{G}_{J+1}\Big(\delta_{J+1}[\check{P}] + \epsilon_J^2 + \epsilon_0^2 + O(a^2)B_\delta[r^{J+1}\nabla_4^J B]\Big)^{\frac{1}{2}}, \end{aligned}$$

we infer

$$\begin{aligned} & BEF_\delta^{J-1}[r^2(\nabla_3 B, r\nabla B)] + BEF_\delta[r^{J-1}\nabla_4^J B] + BEF_\delta^{J-1}[r^2 B] \\ \lesssim\ & \delta_{J+1}[\check{P}] + \mathfrak{G}_J\mathfrak{G}_{J+1} + \mathfrak{R}_J\mathfrak{R}_{J+1} + \epsilon_0^2 + O(a^2,\epsilon^2)BEF_\delta[r^{J+1}\nabla_4^J B] \\ & + \sqrt{\delta_{J+1}[\check{P}]}\left(B_\delta^J[r^2B] + \mathfrak{G}_{J+1}^2\right)^{\frac{1}{2}} + \epsilon_J\Big(B_\delta^J[r^2B]\Big)^{\frac{1}{2}} \\ & + \mathfrak{G}_{J+1}\Big(\delta_{J+1}[\check{P}] + \epsilon_J^2 + \epsilon_0^2 + O(a^2)B_\delta[r^{J+1}\nabla_4^J B]\Big)^{\frac{1}{2}}. \end{aligned} \tag{15.5.19}$$

Note that $BEF_\delta[r^{J-1}\nabla_4^J B]$ appearing on the LHS of (15.5.19) is not consistent in terms of powers of r compared to the other terms of the LHS. We thus need to upgrade this estimate. To this end, we introduce a smooth function $\chi_{far}(r)$ such that $\chi_{far}(r) = 0$ on $\mathcal{M}_{trap}$ and $\chi_{far}(r) = 1$ for $r \geq 5m$, and we derive the following bound

$$\begin{aligned} & \int_{\mathcal{M}} r^{1+\delta}|\chi_{far}(\overline{q}\nabla_4)^{J+1}B|^2 + \sup_{\tau\leq\tau_*}\int_{\partial^+\mathcal{M}(1,\tau)} r^{2+\delta}|\chi_{far}(\overline{q}\nabla_4)^{J+1}B|^2 \\ \lesssim\ & \delta_{J+1}[\check{P}] + \mathfrak{G}_J\mathfrak{G}_{J+1} + \mathfrak{R}_J\mathfrak{R}_{J+1} + \epsilon_0^2 + O(a^2,\epsilon^2)BEF_\delta[r^{J+1}\nabla_4^J B] \\ & + \sqrt{\delta_{J+1}[\check{P}]}\left(B_\delta^J[r^2B] + \mathfrak{G}_{J+1}^2\right)^{\frac{1}{2}} + \epsilon_J\Big(B_\delta^J[r^2B]\Big)^{\frac{1}{2}} \\ & + \mathfrak{G}_{J+1}\Big(\delta_{J+1}[\check{P}] + \epsilon_J^2 + \epsilon_0^2 + O(a^2)B_\delta[r^{J+1}\nabla_4^J B]\Big)^{\frac{1}{2}}. \end{aligned} \tag{15.5.20}$$

Assuming (15.5.20), and since $\chi_{far}(r) = 1$ for $r \geq 5m$, we have

$$\begin{aligned} BEF^J_\delta[r^2 B] &\lesssim BEF^{J-1}_\delta[r^2(\nabla_3, r\nabla)B] + BEF_\delta[r^{J+2}\nabla_4^J B] + BEF^{J-1}_\delta[r^2 B] \\ &\lesssim BEF^{J-1}_\delta[r^2(\nabla_3 B, r\nabla B)] + BEF_\delta[r^{J-1}\nabla_4^J B] + BEF^{J-1}_\delta[r^2 B] \\ &\quad + \int_{\mathcal{M}} r^{1+\delta}|\chi_{far}(\overline{q}\nabla_4)^{J+1}B|^2 + \sup_{\tau\leq\tau_*}\int_{\partial^+\mathcal{M}(1,\tau)} r^{2+\delta}|\chi_{far}(\overline{q}\nabla_4)^{J+1}B|^2 \end{aligned}$$

which together with (15.5.19) and (15.5.20) yields

$$\begin{aligned} BEF^J_\delta[r^2 B] &\lesssim \delta_{J+1}[\check{P}] + \mathfrak{G}_J\mathfrak{G}_{J+1} + \mathfrak{R}_J\mathfrak{R}_{J+1} + \epsilon_0^2 + O(a^2, \epsilon^2)BEF_\delta[r^{J+1}\nabla_4^J B] \\ &\quad + \sqrt{\delta_{J+1}[\check{P}]}\left(B^J_\delta[r^2 B] + \mathfrak{G}^2_{J+1}\right)^{\frac{1}{2}} + \epsilon_J\Big(B^J_\delta[r^2 B]\Big)^{\frac{1}{2}} \\ &\quad + \mathfrak{G}_{J+1}\Big(\delta_{J+1}[\check{P}] + \epsilon_J^2 + \epsilon_0^2 + O(a^2)B_\delta[r^{J+1}\nabla_4^J B]\Big)^{\frac{1}{2}}. \end{aligned}$$

For a and ϵ small enough, we infer

$$BEF^J_\delta[r^2 B] \lesssim \delta_{J+1}[\check{P}] + \epsilon_0^2 + \epsilon_J^2 + \epsilon_J\mathfrak{R}_{J+1} + \left(\sqrt{\delta_{J+1}[\check{P}]} + \epsilon_0 + \epsilon_J\right)\mathfrak{G}_{J+1} + |a|\mathfrak{G}^2_{J+1}$$

which is the stated estimate for $BEF^J_\delta[r^2 B]$ in (15.1.1).

It thus only remains to derive the estimate (15.5.20) relying in particular on (15.5.19). We sketch below the main steps.

Step 1. We commute the second Bianchi pair, see (15.3.5),

$$\begin{aligned} {}^{(c)}\nabla_3 B + \mathrm{tr}\underline{X} B &= -\,{\mathcal{D}\mkern-11mu /}_1^*\overline{P} + 3\overline{P}H + r^{-2}\Gamma_b\cdot\check{R}_b, \\ {}^{(c)}\nabla_4\overline{P} + \frac{3}{2}\overline{\mathrm{tr}X}\,\overline{P} &= {\mathcal{D}\mkern-11mu /}_1 B + O(ar^{-2})\overline{B} + r^{-2}\Gamma_b\cdot\check{R}_b. \end{aligned}$$

with $\overline{q}\,{}^{(c)}\nabla_4$ and then linearize the quantity $\nabla_4\overline{P}$ by subtracting its Kerr value[10], i.e.

$$\widecheck{\nabla_4\overline{P}} := \nabla_4\overline{P} - \frac{\Delta}{|q|^2}\frac{6m}{\overline{q}^4}, \qquad \widecheck{\nabla_4\overline{P}} = \nabla_4\check{P} + O(r^{-4})\Gamma_g.$$

We then commute with $\chi_{far}(\overline{q}\,{}^{(c)}\nabla_4)^J$ where we recall that $\chi_{far}(r) = 0$ on $\mathcal{M}_{trap}$ and $\chi_{far}(r) = 1$ for $r \geq 5m$. Setting $\Psi_{(1)} = \chi_{far}(\overline{q}\,{}^{(c)}\nabla_4)^{J+1}B$, $\Psi_{(2)} = \chi_{far}(\overline{q}\,{}^{(c)}\nabla_4)^J(\overline{q}\widecheck{\nabla_4\overline{P}})$, we deduce, in view of Lemma 15.4.5,

$$\begin{aligned} {}^{(c)}\nabla_3\Psi_{(1)} + \left(1 - \frac{J+1}{2}\right)\mathrm{tr}\underline{X}\Psi_{(1)} &= -\,{\mathcal{D}\mkern-11mu /}_1^*\Psi_{(2)} + F_{(1)}, \\ {}^{(c)}\nabla_4\Psi_{(2)} + \left(\frac{3}{2} - \frac{J+1}{2}\right)\overline{\mathrm{tr}X}\,\Psi_{(2)} &= {\mathcal{D}\mkern-11mu /}_1\Psi_{(1)} + F_{(2)}, \end{aligned} \tag{15.5.21}$$

[10]It is crucial to first commute with $\overline{q}\,{}^{(c)}\nabla_4$ and then linearize the quantity $\nabla_4\overline{P}$. Indeed, linearizing first $\overline{P}$ and then commuting with $\overline{q}\,{}^{(c)}\nabla_4$ would lead to a dangerous term $\mathfrak{d}^{\leq J+1}\Gamma_b$ in $F_{(1)}$.

with $F_{(1)}, F_{(2)}$ of the form

$$\begin{aligned}
F_{(1)} =& \chi_{far}(r)\Big[O(r^{-1})\mathfrak{d}^J\big({}^{(c)}\nabla_3 B,\, \not{\mathfrak{d}} B\big) + O(r^{-1})\mathfrak{d}^{\leq J+1}\check{P} + O(r^{-3})(\nabla_3, \nabla)\mathfrak{d}^{\leq J}\Gamma_b \\
&+ O(r^{-1})\mathfrak{d}^{\leq J}B + O(r^{-3})\mathfrak{d}^{\leq J}\Gamma_b + r^{-2}\mathfrak{d}^{\leq J+1}(\Gamma_b \cdot \check{R}_b)\Big] + \chi'_{far}(r)\mathfrak{d}^{\leq J+1}(\check{P}, B),\\
F_{(2)} =& \chi_{far}(r)\Big[O(r^{-1}) + \mathfrak{d}^{\leq J+1}(\check{P}, B) + O(r^{-3})\mathfrak{d}^{\leq J+1}\Gamma_b + r^{-2}\mathfrak{d}^{\leq J+1}(\Gamma_b \cdot \check{R}_b)\Big]\\
&+ \chi'_{far}(r)\mathfrak{d}^{\leq J+1}(\check{P}, B).
\end{aligned}$$

Step 2. We apply the integral estimate (15.3.14) of Proposition 15.3.12 to the system (15.5.21). Note that the signature of $\Psi_{(1)}$ is given by $J+1$, and that we have in this case $b = 2+\delta$ so that $\Lambda_{(1)} = -2 + (J+1) + 1 + \frac{b}{2} > 0$. We deduce

$$\begin{aligned}
&\int_{\mathcal{M}} r^{b-1}|\Psi_{(1)}|^2 + \int_{\partial^+\mathcal{M}} r^b|\Psi_{(1)}|^2\\
\lesssim& \int_{\mathcal{M}} r^{b-1}|\Psi_{(2)}|^2 + \left|\int_{\mathcal{M}} |q|^b\Re(F_{(1)}\cdot\overline{\Psi_{(1)}})\right| + \int_{\mathcal{M}} r^b|F_{(2)}||\Psi_{(2)}|\\
&+ \int_{\partial^-\mathcal{M}} \big(r^b|\Psi_{(1)}|^2 + r^{b-2}|\Psi_{(2)}|^2\big).
\end{aligned}$$

Step 3. We then proceed as in the proof of Proposition 15.5.7, see section 15.5.3, with $(\widetilde{B}, \widetilde{P}_+)$ being replaced by $(\Psi_{(1)}, \Psi_{(2)})$ where

$$\Psi_{(1)} = \chi_{far}(\overline{q}\,{}^{(c)}\nabla_4)^{J+1}B, \qquad \Psi_{(2)} = \chi_{far}(\overline{q}\,{}^{(c)}\nabla_4)^J(\widecheck{\overline{q}\nabla_4\overline{P}}).$$

The main differences are the fact that the argument is now simpler since $F_{(1)}$ and $F_{(2)}$ provided by Step 1 are supported away from $\mathcal{M}_{trap}$, and the fact that the new term, generated by $\chi'_{far}(r)\mathfrak{d}^{\leq J+1}B$ in $F_{(1)}$, can be controlled using (15.5.19) since it is compactly supported in r. This finally leads to (15.5.20), hence concluding the proof of the control of $BEF^J_\delta[r^2B]$ in (15.1.1).

15.5.6 Proof of the estimates for $\underline{B}$ in Proposition 15.1.1

Using the estimates for $\widetilde{\underline{B}}$ derived in Proposition 15.5.8 and the estimates for $\nabla_4\underline{B}, \nabla\underline{B}$ derived in Lemma 15.5.1, we derive estimates for $\nabla_{\widehat{R}}(\nabla_3)^k\underline{B}$. To this end, we make use of the following lemma.

Lemma 15.5.9. *We have the identity*

$$\left(\frac{\Delta}{|q|^2}\right)^2 \nabla_{\widehat{R}}\nabla_3^J\underline{B} = -2\widetilde{\underline{B}} + O(1)\nabla_{\widehat{R}}\mathfrak{d}^{\leq J-1}(\nabla_4, \nabla)\underline{B} + O(r^{-1})\nabla_{\widehat{R}}\mathfrak{d}^{\leq J-1}\underline{B} + O(r^{-1})\mathfrak{d}^{\leq J-1}\underline{B}.$$

Proof. We write

$$
\begin{aligned}
\widetilde{\underline{B}} &= {}^{(c)}\nabla_3^{J-2}\,{}^{(c)}\nabla_{\widehat{R}}^2\dot{\underline{B}} = {}^{(c)}\nabla_{\widehat{R}}^2\,{}^{(c)}\nabla_3^{J-2}\not{\mathcal{L}}_{\mathbf{T}}\underline{B} + O(r^{-1})\nabla_{\widehat{R}}\mathfrak{d}^{\leq J-1}\underline{B} + O(r^{-1})\mathfrak{d}^{\leq J-1}\underline{B}\\
&= \frac{1}{2}\nabla_{\widehat{R}}^2\nabla_3^{J-2}\left(\nabla_4 + \frac{\Delta}{|q|^2}\nabla_3 - 2a\Re(\mathfrak{J})^b\nabla_b\right)\underline{B} + O(r^{-1})\nabla_{\widehat{R}}\mathfrak{d}^{\leq J-1}\underline{B} + O(r^{-1})\mathfrak{d}^{\leq J-1}\underline{B}\\
&= \frac{1}{2}\frac{\Delta}{|q|^2}\nabla_{\widehat{R}}^2\nabla_3^{J-1}\underline{B} + \nabla_{\widehat{R}}^2\nabla_3^{J-2}\nabla_4\underline{B} + O(ar^{-2})\nabla_{\widehat{R}}^2\nabla_3^{J-2}\nabla\underline{B}\\
&\quad + O(r^{-1})\nabla_{\widehat{R}}\mathfrak{d}^{\leq J-1}\underline{B} + O(r^{-1})\mathfrak{d}^{\leq J-1}\underline{B}\\
&= \frac{1}{2}\frac{\Delta}{|q|^2}\,\nabla_{\widehat{R}}^2\nabla_3^{J-1}\underline{B} + O(1)\nabla_{\widehat{R}}\mathfrak{d}^{\leq J-1}(\nabla_4,\nabla)\underline{B} + O(r^{-1})\nabla_{\widehat{R}}\mathfrak{d}^{\leq J-1}\underline{B} + O(r^{-1})\mathfrak{d}^{\leq J-1}\underline{B}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
\nabla_{\widehat{R}}^2\nabla_3^{J-1}\underline{B} &= \frac{1}{2}\nabla_{\widehat{R}}\left(\nabla_4 - \frac{\Delta}{|q|^2}\nabla_3\right)\nabla_3^{J-1}\underline{B} + O(r^{-1})\nabla_{\widehat{R}}\mathfrak{d}^{\leq J-1}\underline{B} + O(r^{-1})\mathfrak{d}^{\leq J-1}\underline{B}\\
&= -\frac{1}{2}\frac{\Delta}{|q|^2}\nabla_{\widehat{R}}\nabla_3^J\underline{B} + \nabla_{\widehat{R}}\nabla_3^{J-1}\nabla_4\underline{B} + O(r^{-1})\nabla_{\widehat{R}}\mathfrak{d}^{\leq J-1}\underline{B} + O(r^{-1})\mathfrak{d}^{\leq J-1}\underline{B}.
\end{aligned}
$$

We deduce

$$
\widetilde{\underline{B}} = -\frac{1}{2}\left(\frac{\Delta}{|q|^2}\right)^2\nabla_{\widehat{R}}\nabla_3^J\underline{B} + O(1)\nabla_{\widehat{R}}\mathfrak{d}^{\leq J-1}(\nabla_4,\nabla)\underline{B} + O(r^{-1})\nabla_{\widehat{R}}\mathfrak{d}^{\leq J-1}\underline{B} + O(r^{-1})\mathfrak{d}^{\leq J-1}\underline{B}
$$

as stated. □

We then make use of the integral estimate of Proposition 15.5.8 for $\widetilde{\underline{B}}$, the estimates for $\nabla_4\underline{B}, \nabla\underline{B}$ already derived, see (15.5.2), the induction hypothesis, and Lemma 15.5.9, to deduce, with $b = -\delta$,

$$
\begin{aligned}
&\int_{\mathcal{M}} r^{b-1}\left(\frac{\Delta}{|q|^2}\right)^4\left|\nabla_{\widehat{R}}\nabla_3^J\underline{B}\right|^2 + \sup_{\tau\leq\tau_*}\int_{\partial^+\mathcal{M}(1,\tau)} r^{b-2}\left(\frac{\Delta}{|q|^2}\right)^4\left|\nabla_{\widehat{R}}\nabla_3^J\underline{B}\right|^2\\
\lesssim\ &\delta_{J+1}[\check{P}] + \mathfrak{G}_J\mathfrak{G}_{J+1} + \mathfrak{R}_J\mathfrak{R}_{J+1} + \epsilon_0^2 + O(a^2,\epsilon^2)B_\delta[\nabla_3^J\underline{B}]\\
&+ \sqrt{\delta_{J+1}[\check{P}]}\left(B_\delta^J[\underline{B}] + \mathfrak{G}_{J+1}^2\right)^{\frac{1}{2}} + \epsilon_J\Big(B_\delta^J[\underline{B}]\Big)^{\frac{1}{2}}\\
&+ \mathfrak{G}_{J+1}\Big(\delta_{J+1}[\check{P}] + \epsilon_J^2 + \epsilon_0^2 + O(a^2)B_\delta[\nabla_3^J\underline{B}]\Big)^{\frac{1}{2}}.
\end{aligned}
\tag{15.5.22}
$$

Note that (15.5.22) is degenerate in the redshift region $r \leq r_+(1+\delta_{\mathcal{H}})$. To get rid of this degeneracy, we introduce a smooth cut-off function $\chi_{red}(r)$ such that $\chi_{red}(r) = 1$ for

$r \leq r_+(1+\delta_{\mathcal{H}})$ and $\chi_{red}(r) = 0$ for $r \geq r_+(1+2\delta_{\mathcal{H}})$, and we derive in Steps 1–4 below the following bound

$$\begin{aligned}
&\int_{\mathcal{M}} \left|\chi_{red}\nabla_3^{J+1}\underline{B}\right|^2 + \sup_{\tau\leq\tau_*}\int_{\partial^+\mathcal{M}(1,\tau)} \left|\chi_{red}\nabla_3^{J+1}\underline{B}\right|^2 \\
\lesssim&\delta_{J+1}[\check{P}] + \mathfrak{G}_J\mathfrak{G}_{J+1} + \mathfrak{R}_J\mathfrak{R}_{J+1} + \epsilon_0^2 + O(a^2,\epsilon^2)B_\delta[\nabla_3^J\underline{B}] \\
&+ \sqrt{\delta_{J+1}[\check{P}]}\left(B_\delta^J[\underline{B}] + \mathfrak{G}_{J+1}^2\right)^{\frac{1}{2}} + \epsilon_J\Big(B_\delta^J[\underline{B}]\Big)^{\frac{1}{2}} \\
&+ \mathfrak{G}_{J+1}\Big(\delta_{J+1}[\check{P}] + \epsilon_J^2 + \epsilon_0^2 + O(a^2)B_\delta[\nabla_3^J\underline{B}]\Big)^{\frac{1}{2}}.
\end{aligned} \tag{15.5.23}$$

Since $\chi_{red}(r) = 1$ for $r \geq r_+(1+\delta_{\mathcal{H}})$, we have

$$\begin{aligned}
BEF_\delta^J[\underline{B}] &\lesssim BEF_\delta^{J-1}[r(\nabla_4,\nabla)\underline{B}] + BEF_\delta[\nabla_3^J\underline{B}] + BEF_\delta^{J-1}[\underline{B}] \\
&\lesssim BEF_\delta^{J-1}[r(\nabla_4,\nabla)\underline{B}] + BEF_\delta^{J-1}[\underline{B}] \\
&\quad + \int_{\mathcal{M}} r^{-\delta-1}\left(\frac{\Delta}{|q|^2}\right)^4 \left|\nabla_{\widehat{R}}\nabla_3^J\underline{B}\right|^2 + \sup_{\tau\leq\tau_*}\int_{\partial^+\mathcal{M}(1,\tau)} r^{-\delta-2}\left(\frac{\Delta}{|q|^2}\right)^4 \left|\nabla_{\widehat{R}}\nabla_3^J\underline{B}\right|^2 \\
&\quad + \int_{\mathcal{M}} \left|\chi_{red}\nabla_3^{J+1}\underline{B}\right|^2 + \sup_{\tau\leq\tau_*}\int_{\partial^+\mathcal{M}(1,\tau)} \left|\chi_{red}\nabla_3^{J+1}\underline{B}\right|^2
\end{aligned}$$

which together with (15.5.22), (15.5.23), and the estimates for $\nabla_4\underline{B}, \nabla\underline{B}$ derived in Lemma 15.5.1, yields

$$\begin{aligned}
BEF_\delta^J[\underline{B}] \lesssim&\delta_{J+1}[\check{P}] + \mathfrak{G}_J\mathfrak{G}_{J+1} + \mathfrak{R}_J\mathfrak{R}_{J+1} + \epsilon_0^2 + O(a^2,\epsilon^2)B_\delta[\nabla_3^J\underline{B}] \\
&+ \sqrt{\delta_{J+1}[\check{P}]}\left(B_\delta^J[\underline{B}] + \mathfrak{G}_{J+1}^2\right)^{\frac{1}{2}} + \epsilon_J\Big(B_\delta^J[\underline{B}]\Big)^{\frac{1}{2}} \\
&+ \mathfrak{G}_{J+1}\Big(\delta_{J+1}[\check{P}] + \epsilon_J^2 + \epsilon_0^2 + O(a^2)B_\delta[\nabla_3^J\underline{B}]\Big)^{\frac{1}{2}}.
\end{aligned}$$

For a and ϵ small enough, we infer

$$BEF_\delta^J[\underline{B}] \lesssim \delta_{J+1}[\check{P}] + \epsilon_0^2 + \epsilon_J^2 + \epsilon_J\mathfrak{R}_{J+1} + \left(\sqrt{\delta_{J+1}[\check{P}]} + \epsilon_0 + \epsilon_J\right)\mathfrak{G}_{J+1} + |a|^2\mathfrak{G}_{J+1}^2$$

which is the stated estimate for $BEF_\delta^J[\underline{B}]$ in (15.1.1).

It thus only remains to derive the non-degenerate estimate (15.5.23) by relying in particular on (15.5.22). We sketch below the main steps in the proof of this red shift type argument.

Step 1. Using the fact that $\widehat{R} = e_4 + \frac{\Delta}{|q|^2} e_3$, we obtain, as a consequence of (15.5.22) and the estimates for $(\nabla_4, \nabla)\underline{B}$ of Lemma 15.5.1,

$$\begin{aligned}
&\int_{\mathcal{M}(r\le r_+(1+2\delta_{\mathcal{H}}))} \left(\frac{\Delta}{|q|^2}\right)^6 \left|\nabla_3^{J+1}\underline{B}\right|^2 + \int_{\partial\mathcal{M}_+(r\le r_+(1+2\delta_{\mathcal{H}}))} \left(\frac{\Delta}{|q|^2}\right)^6 \left|\nabla_3^{J+1}\underline{B}\right|^2 \\
\lesssim& \delta_{J+1}[\check{P}] + \mathfrak{G}_J \mathfrak{G}_{J+1} + \mathfrak{R}_J \mathfrak{R}_{J+1} + \epsilon_0^2 + O(a^2, \epsilon^2) B_\delta[\nabla_3^J \underline{B}] \\
&+ \sqrt{\delta_{J+1}[\check{P}]} \left(B_\delta^J[\underline{B}] + \mathfrak{G}_{J+1}^2\right)^{\frac{1}{2}} + \epsilon_J \Big(B_\delta^J[\underline{B}]\Big)^{\frac{1}{2}} \\
&+ \mathfrak{G}_{J+1}\Big(\delta_{J+1}[\check{P}] + \epsilon_J^2 + \epsilon_0^2 + O(a^2) B_\delta[\nabla_3^J \underline{B}]\Big)^{\frac{1}{2}}.
\end{aligned} \tag{15.5.24}$$

Step 2. We commute the second Bianchi pair, see (15.3.6),

$$\begin{aligned}
{}^{(c)}\nabla_3 P + \frac{3}{2}\overline{\mathrm{tr}\underline{X}} P &= -\,\mathcal{D}_1 \underline{B} + O(ar^{-2})\underline{B} + r^{-1}\Gamma_b \cdot \check{R}_b, \\
{}^{(c)}\nabla_4 \underline{B} + \mathrm{tr}X\, \underline{B} &= \mathcal{D}_1^* P - 3P\, \underline{H} + r^{-1}\Gamma_b \cdot \check{R}_b,
\end{aligned}$$

with ${}^{(c)}\nabla_3$ and then linearize the quantity $\nabla_3 P$ by subtracting its Kerr value[11], i.e.

$$\widetilde{\nabla_3 P} := \nabla_3 P + \frac{6m}{\overline{q}^4}, \qquad \widetilde{\nabla_3 P} = \nabla_3 \check{P} + O(r^{-3})\Gamma_b.$$

We then commute with $\chi_{red}\, {}^{(c)}\nabla_3^J$ where χ_{red} is a smooth cut-off function equal to 1 in the red shift region[12] $\mathcal{M}_{red}$ and 0 for $r \ge r_+(1+2\delta_{red})$. Setting $\Psi_{(1)} = \chi_{red}\, {}^{(c)}\nabla_3^J\, {}^{(c)}\widetilde{\nabla_3 P}$, $\Psi_{(2)} = \chi_{red}\, {}^{(c)}\nabla_3^{J+1}\underline{B}$, we deduce, in view of Lemma 15.4.6,

$$\begin{aligned}
{}^{(c)}\nabla_3 \Psi_{(1)} + \frac{3}{2}\overline{\mathrm{tr}\underline{X}}\Psi_{(1)} &= -\,\mathcal{D}_1 \Psi_{(2)} + F_{(1)}, \\
{}^{(c)}\nabla_4 \Psi_{(2)} + \mathrm{tr}X\, \Psi_{(2)} &= \mathcal{D}_1^* \Psi_{(1)} + F_{(2)},
\end{aligned} \tag{15.5.25}$$

with $F_{(1)}, F_{(2)}$ supported in the region $r \le r_+(1+2\delta_{red})$ and of the form

$$\begin{aligned}
F_{(1)} &= \chi_{red}(r)\Big[O(1)\mathfrak{d}^{\le J+1}(\check{P}, \underline{B}) + O(1)\mathfrak{d}^{\le J+1}\Gamma_b + \mathfrak{d}^{\le J+1}\big(\Gamma_b \cdot \check{R}_b\big)\Big] \\
&\quad + \chi'_{red}(r)\mathfrak{d}^{\le J+1}(\check{P}, \underline{B}), \\
F_{(2)} &= \chi_{red}(r)\Big[O(1)\mathfrak{d}^{\le J}\nabla B + O(1)\mathfrak{d}^{\le J}(\nabla_4, \nabla)\Gamma_b + O(1)\mathfrak{d}^{\le J+1}\check{P} \\
&\quad + O(1)\mathfrak{d}^{\le J}\underline{B} + O(1)\mathfrak{d}^{\le J}\Gamma_b + \mathfrak{d}^{\le J+1}\big(\Gamma_b \cdot \check{R}_b\big)\Big] + \chi'_{red}(r)\mathfrak{d}^{\le J+1}(\check{P}, \underline{B}).
\end{aligned}$$

[11]It is crucial to first commute with ${}^{(c)}\nabla_3$ and then linearize the quantity $\nabla_3 P$. Indeed, linearizing first P and then commuting with ${}^{(c)}\nabla_3$ would lead to a dangerous term $\mathfrak{d}^{\le J+1}\Gamma_b$ in $F_{(2)}$.

[12]Recall that the red shift region $\mathcal{M}_{red}$ is defined by $\mathcal{M}(r \le r_+(1+\delta_{red}))$ with $\delta_{red} \ge \delta_{\mathcal{H}}$.

Step 3. We apply the integral estimate (15.3.15) of Proposition 15.3.12 to the system (15.5.25). Note that the signature of $\Psi_{(1)}$ is given by $-1-J$, and that we have in this case $b=-\delta$ so that $\Lambda_{(2)}=-2+1+\frac{b}{2}<0$. We deduce

$$\begin{aligned}&\int_{\mathcal{M}} r^{b-1}\left(1+|2k+1|\frac{m}{r}\right)|\Psi_{(2)}|^2+\int_{\partial^+\mathcal{M}} r^{b-2}|\Psi_{(2)}|^2\\ \lesssim &\int_{\mathcal{M}} r^{b-1}|\Psi_{(1)}|^2+\left|\int_{\mathcal{M}}|q|^b\Re(F_{(2)}\cdot\overline{\Psi_{(2)}})\right|+\int_{\mathcal{M}} r^b|F_{(1)}||\Psi_{(1)}|\\ &+\int_{\partial^-\mathcal{M}}\left(r^b|\Psi_{(1)}|^2+r^{b-2}|\Psi_{(2)}|^2\right).\end{aligned}$$

Step 4. We then proceed as in the proof of Proposition 15.5.8, see section 15.5.4, with $(\widetilde{P}_-,\widetilde{\underline{B}})$ being replaced by $(\Psi_{(1)},\Psi_{(2)})$ where

$$\Psi_{(1)}=\chi_{red}\,{}^{(c)}\nabla_3^J\,\widetilde{{}^{(c)}\nabla_3 P},\qquad \Psi_{(2)}=\chi_{red}\,{}^{(c)}\nabla_3^{J+1}\underline{B}.$$

The main differences are the fact that the argument is now simpler since $F_{(1)}$ and $F_{(2)}$ provided by Step 2 are supported away from $\mathcal{M}_{trap}$, and the fact that the new term, generated by $\chi'_{red}(r)\mathfrak{d}^{\leq J+1}\underline{B}$ in $F_{(2)}$, can be controlled using (15.5.24) and the estimates for $(\nabla_4,\nabla)\underline{B}$ of Lemma 15.5.1 since it is supported in $r_+(1+\delta_{\mathcal{H}})\leq r\leq r_+(1+2\delta_{\mathcal{H}})$. This finally leads to (15.5.23), hence concluding the proof of the control of $BEF^J_\delta[\underline{B}]$ in (15.1.1).

15.6 Estimates for A and $\underline{A}$

The goal of this section is to prove Proposition 15.1.2 providing energy-Morawetz estimates for $(A,\underline{A})$ assuming corresponding energy-Morawetz estimates for $(B,\underline{B})$.

15.6.1 Estimates for $\nabla_3 A$, ∇A, $\nabla_4\underline{A}$, $\nabla\underline{A}$

We derive the following lemma.

Lemma 15.6.1. *The following estimates hold true in $\mathcal{M}$:*

1. *We have*

$$BEF^{J-1}_\delta[r^2(\nabla_3 A, r\nabla A)]\lesssim \delta_{J+1}[B]+\epsilon_0^2+\epsilon_J^2+O(a^2,\epsilon^2)BEF_\delta[r^{J+1}\nabla_4^J A].\quad (15.6.1)$$

2. We have

$$BEF_\delta^{J-1}[r(\nabla_4\underline{A}, \nabla\underline{A})] \lesssim \delta_{J+1}[\underline{B}] + \epsilon_0^2 + \epsilon_J^2 + O(a^2, \epsilon^2)BEF_\delta[\nabla_3^J\underline{A}]. \qquad (15.6.2)$$

Proof. To prove (15.6.2), we rely on the fourth Bianchi pair (15.3.7), i.e.

$$\begin{aligned} {}^{(c)}\nabla_3\underline{B} + 2\overline{\mathrm{tr}\underline{X}}\,\underline{B} &= -\frac{1}{2}\overline{\mathcal{D}}\cdot\underline{A} + O(ar^{-2})\underline{A} - O(r^{-3})\,\underline{\Xi} + \Gamma_b\cdot\check{R}_b, \\ {}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A} &= -\frac{1}{2}\mathcal{D}\widehat{\otimes}\underline{B} + O(ar^{-2})\underline{B} + O(r^{-3})\widehat{\underline{X}} + \Gamma_b\cdot\check{R}_b. \end{aligned}$$

From second equation we deduce

$$BEF_\delta^{J-1}[r\nabla_4\underline{A}] \lesssim BEF_\delta^J[\underline{B}] + \epsilon_J^2 + \epsilon_0^2,$$

while from the first equation we deduce

$$BEF_\delta^{J-1}[r\overline{\mathcal{D}}\cdot\underline{A}] \lesssim BEF_\delta^J[\underline{B}] + \epsilon_J^2 + \epsilon_0^2.$$

Also, making use of the Hodge type estimates of Corollary 13.4.1, we have

$$BEF_\delta^{J-1}[r\nabla\underline{A}] \lesssim BEF_\delta^{J-1}[r\overline{\mathcal{D}}\cdot\underline{A}] + O(a^2, \epsilon^2)BEF_\delta^{J-1}[(\nabla_3\underline{A}, \nabla_4\underline{A})].$$

Combining the above estimates, we infer, for a and ϵ small enough,

$$BEF_\delta^{J-1}[r\nabla_4\underline{A}] + BEF_\delta^{J-1}[r\nabla\underline{A}] \lesssim BEF_\delta^J[\underline{B}] + \epsilon_J^2 + \epsilon_0^2 + O(a^2, \epsilon^2)BEF_\delta[\nabla_3^J\underline{A}]$$

as desired.

The estimates (15.6.1) for A are derived in the same manner, by relying on the first Bianchi pair (15.3.4). This concludes the proof of Lemma 15.6.1. □

In view of Lemma 15.6.1, it remains to estimate the top ∇_4 derivatives of A and the top ∇_3 derivatives of $\underline{A}$. We thus introduce the quantities

$$\begin{aligned} \widetilde{\underline{B}} &:= {}^{(c)}\nabla_3^{J-1}\,{}^{(c)}\nabla_{\widehat{R}}^2\underline{B}, & \widetilde{\underline{A}} &:= {}^{(c)}\nabla_3^{J-1}\,{}^{(c)}\nabla_{\widehat{R}}^2\underline{A}, \\ \widetilde{B} &:= (\overline{q}\,{}^{(c)}\nabla_4)^{J-1}\,{}^{(c)}\nabla_{\widehat{R}}^2 B, & \widetilde{A} &:= (\overline{q}\,{}^{(c)}\nabla_4)^{J-1}\,{}^{(c)}\nabla_{\widehat{R}}^2 A. \end{aligned} \qquad (15.6.3)$$

15.6.2 Estimates for $\widetilde{\underline{A}}$

We first derive equations for the Bianchi pair $\widetilde{\underline{A}}, \widetilde{\underline{B}}$. Commuting the Bianchi pair equation (15.3.7) with ${}^{(c)}\nabla_3^{J-1}\,{}^{(c)}\nabla_{\widehat{R}}^2$, we derive the following lemma, analogous to the second part of Lemma 15.5.5.

Lemma 15.6.2. *We have*

$$\begin{aligned} {}^{(c)}\nabla_3\widetilde{\underline{B}} + 2\overline{tr\underline{X}}\,\widetilde{\underline{B}} &= -\,\not{\mathcal{D}}_2\widetilde{\underline{A}} + \widetilde{F}_{(1)},\\ {}^{(c)}\nabla_4\widetilde{\underline{A}} + \frac{1}{2}trX\widetilde{\underline{A}} &= \not{\mathcal{D}}_2^*\widetilde{\underline{B}} + \widetilde{F}_{(2)}, \end{aligned} \tag{15.6.4}$$

with

$$\begin{aligned} \widetilde{F}_{(1)} &= O(r^{-1})\mathfrak{d}^J\nabla_{\widehat{R}}(\underline{B},\underline{A}) + O(r^{-3})\mathfrak{d}^{\leq J+1}\Gamma_b + O(r^{-2})\mathfrak{d}^{\leq J}(\underline{B},\underline{A}) + \mathfrak{d}^{\leq J+1}(\Gamma_b\cdot\check{R}_b),\\ \widetilde{F}_{(2)} &= -4\omega\widetilde{\underline{A}} + O(r^{-1})\nabla_{\widehat{R}}\mathfrak{d}^{J-1}\big(\nabla_4\underline{A},\nabla\underline{A}\big) + O(r^{-1})\nabla_{\widehat{R}}\mathfrak{d}^{\leq J}\underline{B} + O(r^{-3})\mathfrak{d}^J\nabla\Gamma_b\\ &\quad + O(r^{-3})\mathfrak{d}^{\leq J}\Gamma_b + O(r^{-1})\mathfrak{d}^{\leq J}(\underline{B},\underline{A}) + \mathfrak{d}^{\leq J+1}(\Gamma_b\cdot\check{R}_b). \end{aligned} \tag{15.6.5}$$

Proof. We apply the second part of Proposition 15.4.2 to the fourth Bianchi pair (15.3.7), i.e. to

$$\begin{aligned} {}^{(c)}\nabla_3\underline{B} + 2\overline{\mathrm{tr}\underline{X}}\,\underline{B} &= -\,\not{\mathcal{D}}_2\underline{A} + O(ar^{-2})\underline{A} - O(r^{-3})\,\underline{\Xi} + \Gamma_b\cdot\check{R}_b,\\ {}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A} &= \not{\mathcal{D}}_2^*\underline{B} + O(ar^{-2})\underline{B} + O(r^{-3})\widehat{\underline{X}} + \Gamma_b\cdot\check{R}_b. \end{aligned}$$

This yields the system (15.6.4) for $(\widetilde{\underline{B}},\widetilde{\underline{A}})$ with $\widetilde{F}_{(1)}$ as in (15.6.5) and $\widetilde{F}_{(2)}$ given by

$$\begin{aligned} \widetilde{F}_{(2)} &= -4\omega\widetilde{\underline{A}} + O(r^{-1})\nabla_{\widehat{R}}\mathfrak{d}^{J-1}\big(\nabla_4\underline{A},\nabla\underline{A}\big) + O(r^{-1})\nabla_{\widehat{R}}\mathfrak{d}^{\leq J}\underline{B} + O(r^{-3})\mathfrak{d}^J\nabla_3\widehat{\underline{X}}\\ &\quad +O(r^{-3})\mathfrak{d}^{\leq J}\Gamma_b + O(r^{-1})\mathfrak{d}^{\leq J}(\underline{B},\underline{A}) + \mathfrak{d}^{\leq J+1}(\Gamma_b\cdot\check{R}_b). \end{aligned}$$

Finally, we make use of the following consequence of the null structure equations, see (13.2.1),

$${}^{(c)}\nabla_3\widehat{\underline{X}} + \Re(\mathrm{tr}\underline{X})\widehat{\underline{X}} = \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\underline{\Xi} + O(ar^{-2})\Gamma_b - \underline{A} + \Gamma_b\cdot\Gamma_b$$

and obtain the desired expression for $F_{(2)}$. □

Based on the Bianchi pair equation in Lemma 15.6.2 for $(\widetilde{\underline{A}},\widetilde{\underline{B}})$, we derive the following estimate for $\widetilde{\underline{A}}$.

Lemma 15.6.3. *The following estimates hold true for $\widetilde{\underline{A}}$, with $b=-\delta$,*

$$\begin{aligned} &\int_{\mathcal{M}} r^{b-1}|\widetilde{\underline{A}}|^2 + \int_{\partial^+\mathcal{M}} r^{b-2}|\widetilde{\underline{A}}|^2\\ \lesssim&\,\delta_{J+1}[\underline{B}] + \epsilon_0^2 + \epsilon_J^2 + O(a^2,\epsilon^2)BEF_\delta[\nabla_3^J\underline{A}] + \sqrt{\delta_{J+1}[\underline{B}]}\Big(B^J_\delta[\underline{A}] + \mathfrak{G}^2_{J+1}\Big)^{\frac{1}{2}}\\ &+\epsilon_J\Big(B^J_\delta[\underline{A}]\Big)^{\frac{1}{2}} + \mathfrak{G}_{J+1}\Big(\delta_{J+1}[\underline{B}] + \epsilon_0^2 + \epsilon_J^2 + O(a^2,\epsilon^2)BEF_\delta[\nabla_3^J\underline{A}]\Big)^{\frac{1}{2}}. \end{aligned} \tag{15.6.6}$$

Proof. We proceed as in the proof of (15.5.17) starting with the Bianchi pair (15.6.4), which can be written in the form (15.3.9) with

$$\Psi_{(1)} = \widetilde{\underline{B}}, \qquad \Psi_{(2)} = \widetilde{\underline{A}}, \qquad c_{(1)} = 2, \qquad c_{(2)} = \frac{1}{2},$$

and $F_{(1)} = \widetilde{F}_{(1)}, F_{(2)} = \widetilde{F}_{(2)}$. Note that the signature of $\Psi_{(1)} = \widetilde{\underline{B}}$ is $k = -J$ and that of $\Psi_{(2)} = \widetilde{\underline{A}}$ equal to $k - 1 = -J - 1$. This corresponds to the case when $2k - 1 < 0$ in Proposition 15.3.12. To apply the integral estimate (15.3.15), we need $\Lambda_{(2)} = -2c_{(2)} + 1 + \frac{b}{2} < 0$ to be satisfied which holds true for the choice $b = -\delta$. Therefore

$$\begin{aligned}
&\int_{\mathcal{M}} r^{b-1}\left(1 + |2J-1|\frac{m}{r}\right)|\widetilde{\underline{A}}|^2 + \int_{\partial^+\mathcal{M}} r^{b-2}|\widetilde{\underline{A}}|^2 \\
&\lesssim \int_{\mathcal{M}} r^{b-1}|\widetilde{\underline{B}}|^2 + \int_{\mathcal{M}} |\omega||\widetilde{\underline{A}}|^2 + \left|\int_{\mathcal{M}} O(r^{-3})\Re\Big(\widetilde{\underline{A}} \cdot \overline{\nabla \mathfrak{d}^{\leq J}\Gamma_b}\Big)\right| \\
&\quad + \int_{\mathcal{M}} r^b\Big(|F_{(1)}|\,|\widetilde{\underline{B}}| + |F'_{(2)}|\,|\widetilde{\underline{A}}|\Big) + \epsilon_0^2,
\end{aligned}$$

where

$$\begin{aligned}
F_{(2)} &= F'_{(2)} - 4\omega\widetilde{\underline{A}} + O(r^{-3})\nabla\mathfrak{d}^J\Gamma_b, \\
F'_{(2)} &:= O(r^{-1})\nabla_{\widehat{R}}\mathfrak{d}^{J-1}(\nabla_4\underline{A}, \nabla\underline{A}) + O(r^{-1})\nabla_{\widehat{R}}\mathfrak{d}^{\leq J}\underline{B} + O(r^{-3})\mathfrak{d}^{\leq J}\Gamma_b \\
&\quad + O(r^{-1})\mathfrak{d}^{\leq J}(\underline{B}, \underline{A}) + \mathfrak{d}^{\leq J+1}(\Gamma_b \cdot \check{R}_b).
\end{aligned}$$

Since $\omega = O(mr^{-2})$, the term $\omega|\widetilde{\underline{A}}|^2$ can be absorbed from the LHS for J large enough (recalling that $J \geq \frac{k_L}{2} \ll 1$), and we infer, using also Cauchy Schwartz,

$$\begin{aligned}
&\int_{\mathcal{M}} r^{b-1}|\widetilde{\underline{A}}|^2 + \int_{\partial^+\mathcal{M}} r^{b-2}|\widetilde{\underline{A}}|^2 \\
&\lesssim \delta_{J+1}[\underline{B}] + \left|\int_{\mathcal{M}} O(r^{-3})\Re\Big(\widetilde{\underline{A}} \cdot \overline{\nabla\mathfrak{d}^{\leq J}\Gamma_b}\Big)\right| + \sqrt{\delta_{J+1}[\underline{B}]}\left(\int_{\mathcal{M}} r^{b+1}|F_{(1)}|^2\right)^{\frac{1}{2}} \\
&\quad + \int_{\mathcal{M}} r^{b+1}|F'_{(2)}|^2 + \epsilon_0^2.
\end{aligned}$$

Since

$$\begin{aligned}
&\sqrt{\delta_{J+1}[\underline{B}]}\left(\int_{\mathcal{M}} r^{b+1}|F_{(1)}|^2\right)^{\frac{1}{2}} + \int_{\mathcal{M}} r^{b+1}|F'_{(2)}|^2 \\
&\lesssim \delta_{J+1}[\underline{B}] + B_\delta^{J-1}[r(\nabla_4, \nabla)\underline{A}] + \epsilon_J^2 + \epsilon_0^2 + \sqrt{\delta_{J+1}[\underline{B}]}\Big(B_\delta^J[\underline{A}] + \mathfrak{G}_{J+1}^2\Big)^{\frac{1}{2}},
\end{aligned}$$

we deduce

$$\begin{aligned}
&\int_{\mathcal{M}} r^{b-1}|\widetilde{\underline{A}}|^2 + \int_{\partial^+\mathcal{M}} r^{b-2}|\widetilde{\underline{A}}|^2 \\
\lesssim & \left|\int_{\mathcal{M}} O(r^{-3})\Re\Big(\widetilde{\underline{A}}\cdot\overline{\nabla\mathfrak{d}^{\leq J}\Gamma_b}\Big)\right| + \delta_{J+1}[\underline{B}] + B_\delta^{J-1}[r(\nabla_4,\nabla)\underline{A}] + \epsilon_J^2 + \epsilon_0^2 \\
& + \sqrt{\delta_{J+1}[\underline{B}]}\Big(B^J_\delta[\underline{A}] + \mathfrak{G}^2_{J+1}\Big)^{\frac{1}{2}}.
\end{aligned}$$

Also, by integration by parts, proceeding as for the control of the term I_2 in the proof of Proposition 15.5.8, see section 15.5.4, we have

$$\left|\int_{\mathcal{M}} O(r^{-3})\Re\Big(\widetilde{\underline{A}}\cdot\overline{\nabla\mathfrak{d}^{\leq J}\Gamma_b}\Big)\right| \lesssim \mathfrak{G}_{J+1}\Big(BEF_\delta^{J-1}[r(\nabla_4,\nabla)\underline{A}]\Big)^{\frac{1}{2}} + \epsilon_J\Big(B_\delta^J[\underline{A}]\Big)^{\frac{1}{2}} + \epsilon_0^2.$$

We deduce

$$\begin{aligned}
&\int_{\mathcal{M}} r^{b-1}|\widetilde{\underline{A}}|^2 + \int_{\partial^+\mathcal{M}} r^{b-2}|\widetilde{\underline{A}}|^2 \\
\lesssim & \delta_{J+1}[\underline{B}] + B_\delta^{J-1}[r(\nabla_4,\nabla)\underline{A}] + \epsilon_J^2 + \epsilon_0^2 + \sqrt{\delta_{J+1}[\underline{B}]}\Big(B^J_\delta[\underline{A}] + \mathfrak{G}^2_{J+1}\Big)^{\frac{1}{2}} + \epsilon_J\Big(B_\delta^J[\underline{A}]\Big)^{\frac{1}{2}} \\
& + \mathfrak{G}_{J+1}\Big(BEF_\delta^{J-1}[r(\nabla_4,\nabla)\underline{A}]\Big)^{\frac{1}{2}}.
\end{aligned}$$

Together with the control of $BEF_\delta^{J-1}[r(\nabla_4,\nabla)\underline{A}]$ provided by (15.6.2), this implies

$$\begin{aligned}
&\int_{\mathcal{M}} r^{b-1}|\widetilde{\underline{A}}|^2 + \int_{\partial^+\mathcal{M}} r^{b-2}|\widetilde{\underline{A}}|^2 \\
\lesssim & \delta_{J+1}[\underline{B}] + \epsilon_0^2 + \epsilon_J^2 + O(a^2,\epsilon^2)BEF_\delta[\nabla_3^J\underline{A}] + \sqrt{\delta_{J+1}[\underline{B}]}\Big(B^J_\delta[\underline{A}] + \mathfrak{G}^2_{J+1}\Big)^{\frac{1}{2}} \\
& + \epsilon_J\Big(B_\delta^J[\underline{A}]\Big)^{\frac{1}{2}} + \mathfrak{G}_{J+1}\Big(\delta_{J+1}[\underline{B}] + \epsilon_0^2 + \epsilon_J^2 + O(a^2,\epsilon^2)BEF_\delta[\nabla_3^J\underline{A}]\Big)^{\frac{1}{2}}.
\end{aligned}$$

as stated in (15.6.6). This concludes the proof of Lemma 15.6.3. □

15.6.3 Estimates for $\underline{A}$ in Proposition 15.1.2

We proceed as in section 15.5.6. Using the estimates for $\widetilde{\underline{A}}$ derived in Lemma 15.6.3 and the estimates for $\nabla_4\underline{A}, \nabla\underline{A}$ derived in Lemma 15.6.1, we derive estimates for $\nabla_{\widehat{R}}(\nabla_3)^k\underline{A}$. To this end, we make use of the following identity

$$\left(\frac{\Delta}{|q|^2}\right)\nabla_{\widehat{R}}\nabla_3^J\underline{A} = -\widetilde{\underline{A}} + O(1)\nabla_{\widehat{R}}\mathfrak{d}^{\leq J-1}(\nabla_4,\nabla)\underline{A} + O(r^{-1})\nabla_{\widehat{R}}\mathfrak{d}^{\leq J-1}\underline{A} + O(r^{-1})\mathfrak{d}^{\leq J-1}\underline{A}$$

which is derived as the corresponding one in Lemma 15.5.9. We then make use of the estimates for $\widetilde{\underline{A}}$ derived in Lemma 15.6.3 and the estimates for $\nabla_4\underline{A}, \nabla\underline{A}$ derived in Lemma 15.6.1, the induction hypothesis, and the above identity, to deduce the following analog of (15.5.22), with $b = -\delta$,

$$\begin{aligned}&\int_{\mathcal{M}} r^{b-1}\left(\frac{\Delta}{|q|^2}\right)^2 \big|\nabla_{\widehat{R}}(\nabla_3)^J\underline{A}\big|^2 + \int_{\partial^+\mathcal{M}} r^{b-2}\left(\frac{\Delta}{|q|^2}\right)^2 \big|\nabla_{\widehat{R}}(\nabla_3)^J\underline{A}\big|^2\\ &\lesssim \delta_{J+1}[\underline{B}] + \epsilon_0^2 + \epsilon_J^2 + O(a^2,\epsilon^2)BEF_\delta[\nabla_3^J\underline{A}] + \sqrt{\delta_{J+1}[\underline{B}]}\Big(B_\delta^J[\underline{A}] + \mathfrak{G}_{J+1}^2\Big)^{\frac{1}{2}} \qquad (15.6.7)\\ &+ \epsilon_J\Big(B_\delta^J[\underline{A}]\Big)^{\frac{1}{2}} + \mathfrak{G}_{J+1}\Big(\delta_{J+1}[\underline{B}] + \epsilon_0^2 + \epsilon_J^2 + O(a^2,\epsilon^2)BEF_\delta[\nabla_3^J\underline{A}]\Big)^{\frac{1}{2}}.\end{aligned}$$

Finally we can repeat the procedure[13] of Steps 1–4 of section 15.5.6 to derive the following analog of (15.5.23)

$$\begin{aligned}&\int_{\mathcal{M}} r^{b-1}\big|\chi_{red}\nabla_3^{J+1}\underline{A}\big|^2 + \int_{\partial^+\mathcal{M}} r^{b-2}\big|\chi_{red}\nabla_3^{J+1}\underline{A}\big|^2\\ &\lesssim \delta_{J+1}[\underline{B}] + \epsilon_0^2 + \epsilon_J^2 + O(a^2,\epsilon^2)BEF_\delta[\nabla_3^J\underline{A}] + \sqrt{\delta_{J+1}[\underline{B}]}\Big(B_\delta^J[\underline{A}] + \mathfrak{G}_{J+1}^2\Big)^{\frac{1}{2}} \qquad (15.6.8)\\ &+ \epsilon_J\Big(B_\delta^J[\underline{A}]\Big)^{\frac{1}{2}} + \mathfrak{G}_{J+1}\Big(\delta_{J+1}[\underline{B}] + \epsilon_0^2 + \epsilon_J^2 + O(a^2,\epsilon^2)BEF_\delta[\nabla_3^J\underline{A}]\Big)^{\frac{1}{2}},\end{aligned}$$

where the smooth cut-off function $\chi_{red}(r)$ is such that $\chi_{red}(r) = 1$ for $r \le r_+(1+\delta_{\mathcal{H}})$ and $\chi_{red}(r) = 0$ for $r \ge r_+(1+2\delta_{\mathcal{H}})$.

Since $\chi_{red}(r) = 1$ for $r \ge r_+(1+\delta_{\mathcal{H}})$, we have

$$\begin{aligned}BEF_\delta^J[\underline{A}] &\lesssim BEF_\delta^{J-1}[r(\nabla_4,\nabla)\underline{A}] + BEF_\delta[\nabla_3^J\underline{A}] + BEF_\delta^{J-1}[\underline{A}]\\ &\lesssim BEF_\delta^{J-1}[r(\nabla_4,\nabla)\underline{A}] + BEF_\delta^{J-1}[\underline{A}]\\ &\quad + \int_{\mathcal{M}} r^{-\delta-1}\left(\frac{\Delta}{|q|^2}\right)^2\big|\nabla_{\widehat{R}}\nabla_3^J\underline{A}\big|^2 + \sup_{\tau\le\tau_*}\int_{\partial^+\mathcal{M}(1,\tau)} r^{-\delta-2}\left(\frac{\Delta}{|q|^2}\right)^2\big|\nabla_{\widehat{R}}\nabla_3^J\underline{A}\big|^2\\ &\quad + \int_{\mathcal{M}}\big|\chi_{red}\nabla_3^{J+1}\underline{A}\big|^2 + \sup_{\tau\le\tau_*}\int_{\partial^+\mathcal{M}(1,\tau)}\big|\chi_{red}\nabla_3^{J+1}\underline{A}\big|^2\end{aligned}$$

which together with (15.6.7), (15.6.8), and the estimates for $\nabla_4\underline{A}, \nabla\underline{A}$ derived in Lemma 15.6.1, yields

$$\begin{aligned}BEF_\delta^J[\underline{A}] \lesssim&\ \delta_{J+1}[\underline{B}] + \epsilon_0^2 + \epsilon_J^2 + O(a^2,\epsilon^2)BEF_\delta[\nabla_3^J\underline{A}] + \sqrt{\delta_{J+1}[\underline{B}]}\Big(B_\delta^J[\underline{A}] + \mathfrak{G}_{J+1}^2\Big)^{\frac{1}{2}}\\ &+ \epsilon_J\Big(B_\delta^J[\underline{A}]\Big)^{\frac{1}{2}} + \mathfrak{G}_{J+1}\Big(\delta_{J+1}[\underline{B}] + \epsilon_0^2 + \epsilon_J^2 + O(a^2,\epsilon^2)BEF_\delta[\nabla_3^J\underline{A}]\Big)^{\frac{1}{2}}.\end{aligned}$$

[13] Note that no additional linearization is needed in this case.

For a and ϵ small enough, we infer

$$BEF_\delta^J[\underline{A}] \lesssim \delta_{J+1}[\underline{B}] + \epsilon_0^2 + \epsilon_J^2 + \left(\sqrt{\delta_{J+1}[\underline{B}]} + \epsilon_0 + \epsilon_J\right)\mathfrak{G}_{J+1} + |a|^2\mathfrak{G}_{J+1}^2$$

which is the estimate for $BEF_\delta^J[\underline{A}]$ in (15.1.2) stated in Proposition 15.1.2.

15.6.4 Estimates for A in Proposition 15.1.2

We start by deriving equations for $(\widetilde{A}, \widetilde{B})$ where we recall that

$$\widetilde{A} = (\overline{q}\,{}^{(c)}\nabla_4)^{J-1}\,{}^{(c)}\nabla^2_{\widehat{R}}A, \qquad \widetilde{B} = (\overline{q}\,{}^{(c)}\nabla_4)^{J-1}\,{}^{(c)}\nabla^2_{\widehat{R}}B.$$

This is done by commuting the first Bianchi pair (15.3.4)

$$\begin{aligned}
{}^{(c)}\nabla_3 A + \frac{1}{2}\mathrm{tr}\underline{X} A &= -\mathcal{D}_2^* B + O(ar^{-2})B + O(r^{-3})\widehat{X} + r^{-2}\Gamma_b\cdot\check{R}_b,\\
{}^{(c)}\nabla_4 B + 2\overline{\mathrm{tr}X} B &= \mathcal{D}_2 A + O(ar^{-2})A + O(r^{-3})\Xi + r^{-2}\Gamma_b\cdot\check{R}_b,
\end{aligned}$$

with $(\overline{q}\,{}^{(c)}\nabla_4)^{J-1}\,{}^{(c)}\nabla^2_{\widehat{R}}$ to derive the following analog of the first part of Lemma 15.5.5.

Lemma 15.6.4. *The quantities $\widetilde{A}, \widetilde{B}$ verify the following system*

$$\begin{aligned}
{}^{(c)}\nabla_3\widetilde{A} + \left(\frac{1}{2} - \frac{J-1}{2}\right) tr\underline{X}\widetilde{A} &= -\mathcal{D}_2^*\widetilde{B} + \widetilde{F}_{(1)},\\
{}^{(c)}\nabla_4\widetilde{B} + \left(2 - \frac{J-1}{2}\right)\overline{trX}\widetilde{B} &= \mathcal{D}_2\widetilde{A} + \widetilde{F}_{(2)},
\end{aligned} \tag{15.6.9}$$

with

$$\begin{aligned}
\widetilde{F}_{(1)} &= O(r^{-1})\mathfrak{d}^{J-1}\nabla_{\widehat{R}}\big({}^{(c)}\nabla_3 A, \not{\mathfrak{d}}A\big) + O(r^{-1})\mathfrak{d}^J\nabla_{\widehat{R}}B + O(r^{-3})\nabla\mathfrak{d}^{\leq J}\Gamma_b\\
&\quad + O(r^{-1})\mathfrak{d}^{\leq J}(A,B) + O(r^{-3})\mathfrak{d}^{\leq J}\Gamma_b + r^{-2}\mathfrak{d}^{\leq J+1}(\Gamma_b\cdot\check{R}_b),\\
\widetilde{F}_{(2)} &= O(r^{-1})\mathfrak{d}^J\nabla_{\widehat{R}}(A,B) + O(r^{-3})\mathfrak{d}^{\leq J+1}\Gamma_b + O(r^{-1})\mathfrak{d}^{\leq J}(A,B) + r^{-2}\mathfrak{d}^{\leq J+1}(\Gamma_b\cdot\check{R}_b).
\end{aligned}$$

Remark 15.6.5. *A priori, $\widetilde{F}_{(1)}$ contains the dangerous term $O(r^{-3})(\overline{q}\,{}^{(c)}\nabla_4)^{J-1}\,{}^{(c)}\nabla^2_{\widehat{R}}\widehat{X}$ and we make use of the following null structure equation, see* (13.2.1),

$${}^{(c)}\nabla_4\widehat{X} + \Re(trX)\widehat{X} = \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\Xi + O(ar^{-2})\Xi - A + r^{-2}\Gamma_b\cdot\Gamma_b,$$

to trade it for suitable contributions to $\widetilde{F}_{(1)}$ such as $O(r^{-3})\nabla\mathfrak{d}^{\leq J}\Gamma_b$.

We are now ready to derive spacetime estimates by appealing to the first part of Proposition 15.3.12 with $\Psi_{(1)} = \widetilde{A}$, $\Psi_{(2)} = \widetilde{B}$. In this case $2k-1>0$, $c_{(1)} = \frac{1}{2}(2-J)$ and we need $\Lambda_{(1)} = -2c_{(1)} + 1 + \frac{b}{2} = J - 1 + \frac{b}{2} > 0$. We can thus choose $b = 2+\delta$ and proceed as in the derivation of the estimates for $\widetilde{B}$ in Proposition 15.5.7 to derive, for small a and $b = 2+\delta$,

$$\begin{aligned}
&\int_{\mathcal{M}} r^{b-1}|\widetilde{A}|^2 + \int_{\partial^+\mathcal{M}} r^b|\widetilde{A}|^2 \\
\lesssim&\delta_{J+1}[B] + \epsilon_0^2 + \epsilon_J^2 + O(a^2,\epsilon^2) BEF_\delta[r^{J+1}\nabla_4^J A] + \sqrt{\delta_{J+1}[B]}\Big(B_\delta^J[A] + \mathfrak{G}_{J+1}^2\Big)^{\frac{1}{2}} \\
&+ \epsilon_J\Big(B_\delta^J[A]\Big)^{\frac{1}{2}} + \mathfrak{G}_{J+1}\Big(\delta_{J+1}[B] + \epsilon_0^2 + \epsilon_J^2 + O(a^2,\epsilon^2) BEF_\delta[r^{J+1}\nabla_4^J A]\Big)^{\frac{1}{2}}.
\end{aligned}$$

We can then proceed as in section 15.5.5 to derive the following estimate for A

$$BEF_\delta^J[r^2 A] \lesssim \delta_{J+1}[B] + \epsilon_0^2 + \epsilon_J^2 + \Big(\sqrt{\delta_{J+1}[B]} + \epsilon_0 + \epsilon_J\Big)\mathfrak{G}_{J+1} + |a|^2\mathfrak{G}_{J+1}^2$$

which is the estimate for $BEF_\delta^J[r^2 A]$ in (15.1.2) stated in Proposition 15.1.2.

Chapter 16

Curvature estimates in ${}^{(ext)}\mathcal{M}$

16.1 Statement of the main result of Chapter 16

The goal of this chapter is to prove the following theorem.

Theorem 16.1.1. *The following estimate holds*

$$
{}^{(ext)}\mathfrak{R}^2_{J+1} \lesssim r_0^{3+\delta_B}\, {}^{(int)}\mathfrak{R}^2_{J+1} + r_0^{-\delta_B}\, {}^{(ext)}\mathfrak{G}^2_{J+1} + \epsilon_J^2 + \epsilon_0^2, \tag{16.1.1}
$$

with ${}^{(ext)}\mathfrak{R}_k$, ${}^{(int)}\mathfrak{R}_k$ *and* ${}^{(ext)}\mathfrak{G}_k$ *defined as in section 13.5.*

Remark 16.1.2. *Theorem 16.1.1 implies* (13.6.7)*. Together with the proof of* (13.6.6) *in section 15.1.3, this concludes the proof of Theorem 13.6.3.*

Remark 16.1.3. *The proof of Theorem 16.1.1 follows the main steps in the proof of the corresponding result in [50], see the second estimate in Proposition 8.10 of [50] for the corresponding statement, and section 8.7 in [50] for the corresponding proof.*

The rest of Chapter 16 focuses on the proof of Theorem 16.1.1. Before moving to $J+1$ derivatives, we first control ${}^{(ext)}\mathfrak{R}_0$ in section 16.4, by relying on the linearization of the Bianchi pairs of section 16.2 and on the estimates of section 16.3. Then, we prove Theorem 16.1.1 in section 16.7 by relying on the Bianchi pairs for higher order derivatives derived first for angular derivatives in section 16.5 and then for general derivatives in section 16.6.

16.2 Standard linearization of the Bianchi pairs

In order to prove Theorem 16.1.1, it suffices to make use of the standard linearization of the second and third Bianchi pairs[1] in Proposition 2.4.11.

Lemma 16.2.1. *The standard linearization of the second Bianchi pair* (15.3.5) *has the following form*[2]

$$\begin{aligned} {}^{(c)}\nabla_3 B + tr\underline{X} B &= -\,\not{\mathcal{D}}_1^*\overline{\check{P}} + O(ar^{-2})\check{P} + O(r^{-3})\Gamma_b' + r^{-2}\Gamma_b'\cdot\check{R}_b + \frac{1}{2}\Xi\cdot A,\\ {}^{(c)}\nabla_4\overline{\check{P}} + \frac{3}{2}\overline{trX}\,\overline{\check{P}} &= \not{\mathcal{D}}_1 B + O(ar^{-2})B + O(r^{-4})\Gamma_b + r^{-2}\Gamma_b'\cdot\check{R}_b. \end{aligned} \tag{16.2.1}$$

The standard linearization of the third Bianchi pair (15.3.6) *has the following form*

$$\begin{aligned} {}^{(c)}\nabla_3\check{P} + \frac{3}{2}\overline{tr\underline{X}}\check{P} &= -\,\not{\mathcal{D}}_1\underline{B} + O(ar^{-2})\underline{B} + O(r^{-3})\Gamma_b + r^{-1}\Gamma_b\cdot\check{R}_b,\\ {}^{(c)}\nabla_4\underline{B} + trX\underline{B} &= \not{\mathcal{D}}_1^*\check{P} + O(ar^{-2})\check{P} + O(r^{-3})\Gamma_b + r^{-1}\Gamma_b\cdot\check{R}_b. \end{aligned} \tag{16.2.2}$$

Proof. The proof follows easily from the form of the second and third Bianchi pairs in (15.3.5) (15.3.6) by writing $P = \check{P} - \frac{2m}{q^3}$, $H = \frac{aq}{|q|^2}\mathfrak{J} + \Gamma_b'$, $\underline{H} = -\frac{a\overline{q}}{|q|^2}\mathfrak{J} + \Gamma_g$, $\widecheck{e_4(q)} = \Gamma_g$, $\widecheck{e_3(q)} = r\Gamma_b$, and $\mathcal{D}q = -a\mathfrak{J} + r\Gamma_g$. □

Remark 16.2.2. *We note that the smallness of a plays no role in this chapter and we may thus ignore it in what follows. We will keep it however whenever it is convenient to take the scaling into account.*

We recall below Definition 15.3.5 exhibiting the general form of Bianchi pairs.

Definition 16.2.3. *We consider the following general Bianchi pairs in $\mathcal{M}$:*

- *For $\Psi_{(1)} \in \mathfrak{s}_p$, $\Psi_{(2)} \in \mathfrak{s}_{p-1}$, and $F_{(1)} \in \mathfrak{s}_p$, $F_{(2)} \in \mathfrak{s}_{p-1}$,*

$$\begin{aligned} {}^{(c)}\nabla_3(\Psi_{(1)}) + c_{(1)}tr\underline{X}\Psi_{(1)} &= -\,\not{\mathcal{D}}_p^*\Psi_{(2)} + F_{(1)},\\ {}^{(c)}\nabla_4(\Psi_{(2)}) + c_{(2)}\overline{trX}\Psi_{(2)} &= \not{\mathcal{D}}_p\,\Psi_{(1)} + F_{(2)}. \end{aligned} \tag{16.2.3}$$

[1]See also Definition 15.3.4, but note that we need a more precise form of the the error terms here.

[2]Recall that we split Γ_b into $\Gamma_b' = \Gamma_b \setminus \{\Xi\}$ and Ξ as the latter behaves somewhat worse in powers of r, see Definition 13.5.1, and that we similarly also split Γ_g into $\Gamma_g' = \Gamma_g \setminus \{\widecheck{\mathrm{tr}\underline{X}}\}$ and $\widecheck{\mathrm{tr}\underline{X}}$. Finally, recall that, in part III, we often identity $\check{R}_g$ with $r^{-2}\check{R}_b$, Γ_g with $r^{-1}\Gamma_b$ and Γ_g' with $r^{-1}\Gamma_b'$, see Remark 13.1.2.

- *For* $\Psi_{(1)} \in \mathfrak{s}_{p-1}$, $\Psi_{(2)} \in \mathfrak{s}_p$, *and* $F_{(1)} \in \mathfrak{s}_{p-1}$, $F_{(2)} \in \mathfrak{s}_p$,

$$\begin{aligned} {}^{(c)}\nabla_3(\Psi_{(1)}) + c_{(1)}\overline{tr\underline{X}}\Psi_{(1)} &= -\,\not{\mathcal{D}}_p\,\Psi_{(2)} + F_{(1)}, \\ {}^{(c)}\nabla_4(\Psi_{(2)}) + c_{(2)}trX\Psi_{(2)} &= \not{\mathcal{D}}_p^*\Psi_{(1)} + F_{(2)}. \end{aligned} \tag{16.2.4}$$

We rewrite below all the linearized Bianchi identities, see Proposition 2.4.11 and Lemma 16.2.1, in the form of Definition 16.2.3.

Proposition 16.2.4. *The linearized Bianchi identities have the following structure:*

- *The first Bianchi pair for* $(\Psi_{(1)} = A, \Psi_{(2)} = B)$ *can be written in the form* (16.2.3)

$$\begin{aligned} {}^{(c)}\nabla_3(\Psi_{(1)}) + c_{(1)}tr\underline{X}\Psi_{(1)} &= -\,\not{\mathcal{D}}_2^*\Psi_{(2)} + F_{(1)}, \\ {}^{(c)}\nabla_4(\Psi_{(2)}) + c_{(2)}\overline{trX}\Psi_{(2)} &= \not{\mathcal{D}}_2\,\Psi_{(1)} + F_{(2)}, \end{aligned}$$

  with $c_{(1)} = \frac{1}{2}$, $c_{(2)} = 2$ *and*

$$\begin{aligned} F_{(1)} &= O(ar^{-2})B + O(r^{-3})\Gamma_g + \Gamma_b \cdot B, \\ F_{(2)} &= O(ar^{-2})A + O(r^{-3})\Gamma_g + \Gamma_b \cdot A. \end{aligned}$$

- *The second Bianchi pair for*[3] $(\Psi_{(1)} = B, \Psi_{(2)} = \overline{\check{P}})$ *can be written in the form* (16.2.3)

$$\begin{aligned} {}^{(c)}\nabla_3(\Psi_{(1)}) + c_{(1)}tr\underline{X}\Psi_{(1)} &= -\,\not{\mathcal{D}}_1^*\Psi_{(2)} + F_{(1)}, \\ {}^{(c)}\nabla_4(\Psi_{(2)}) + c_{(2)}\overline{trX}\Psi_{(2)} &= \not{\mathcal{D}}_1\,\Psi_{(1)} + F_{(2)}, \end{aligned}$$

  with $c_{(1)} = 1$, $c_{(2)} = \frac{3}{2}$ *and*

$$\begin{aligned} F_{(1)} &= O(ar^{-2})\check{P} + O(r^{-3})\Gamma_b' + r^{-2}\Gamma_b' \cdot \check{R}_b + \Gamma_b \cdot A, \\ F_{(2)} &= O(ar^{-2})B + O(r^{-4})\Gamma_b + \Gamma_b' \cdot (A, B) + r^{-1}\Xi \cdot \check{R}_b, \end{aligned}$$

  where, see (13.5.2), $\Gamma_b' = \Gamma_b \setminus \Xi$.

- *The third Bianchi pair for* $(\Psi_{(1)} = \check{P}, \Psi_{(2)} = \underline{B})$ *can be written in the form* (16.2.4)

$$\begin{aligned} {}^{(c)}\nabla_3(\Psi_{(1)}) + c_{(1)}\overline{tr\underline{X}}\Psi_{(1)} &= -\,\not{\mathcal{D}}_1\,\Psi_{(2)} + F_{(1)}, \\ {}^{(c)}\nabla_4(\Psi_{(2)}) + c_{(2)}trX\Psi_{(2)} &= \not{\mathcal{D}}_1^*\Psi_{(1)} + F_{(2)}, \end{aligned}$$

[3] See Remark 15.3.6.

with $c_{(1)} = \frac{3}{2}$, $c_{(2)} = 1$ *and*

$$\begin{aligned} F_{(1)} &= O(ar^{-2})\underline{B} + O(r^{-3})\Gamma_b + r^{-1}\Gamma_b' \cdot \check{R}_b, \\ F_{(2)} &= O(ar^{-2})\check{P} + O(r^{-3})\Gamma_b + r^{-2}\Gamma_b \cdot \check{R}_b + \Xi \cdot \check{R}_b. \end{aligned}$$

- *The fourth Bianchi pair for* $(\Psi_{(1)} = \underline{B}, \Psi_{(2)} = \underline{A})$ *can be written in the form* (16.2.4)

$$\begin{aligned} {}^{(c)}\nabla_3(\Psi_{(1)}) + c_{(1)}\overline{tr\underline{X}}\Psi_{(1)} &= -\not{\mathcal{D}}_2\,\Psi_{(2)} + O(ar^{-2})\Psi_{(2)} + E_{(1)}, \\ {}^{(c)}\nabla_4(\Psi_{(2)}) + c_{(2)}trX\Psi_{(2)} &= \not{\mathcal{D}}_2^*\Psi_{(1)} + O(ar^{-2})\Psi_{(1)} + E_{(2)}, \end{aligned}$$

with $c_{(1)} = 2$, $c_{(2)} = \frac{1}{2}$ *and*

$$\begin{aligned} F_{(1)} &= O(ar^{-2})\underline{A} + O(r^{-3})\Gamma_b + \Gamma_b' \cdot \check{R}_b, \\ F_{(2)} &= O(ar^{-2})\underline{B} + O(r^{-3})\Gamma_b + r^{-1}\Gamma_b \cdot \check{R}_b. \end{aligned}$$

16.3 Exterior estimates for generalized Bianchi pairs

We now state the main integral exterior estimates for generalized Bianchi pairs.

Proposition 16.3.1. *For a given* b *real we define, as in Lemma 15.3.8,*

$$\Lambda_{(1)} := -2c_{(1)} + 1 + \frac{b}{2}, \qquad \Lambda_{(2)} := -2c_{(2)} + 1 + \frac{b}{2}.$$

Then in both cases (16.2.3) *and* (16.2.4)*, the following holds*[4]*:*

Case 1. *If* $\Lambda_{(1)} > 0$ *and* $\Lambda_{(2)} < 0$*, we have*

$$\begin{aligned} &\int_{{}^{(ext)}\mathcal{M}} r^{b-1}\Big(|\Psi_{(1)}|^2 + |\Psi_{(2)}|^2\Big) + \sup_\tau \int_{{}^{(ext)}\Sigma(\tau)} r^b\big(|\Psi_{(1)}|^2 + r^{-2}|\Psi_{(2)}|^2\big) \\ &+ \int_{\Sigma_*} r^b\Big(|\Psi_{(1)}|^2 + |\Psi_{(2)}|^2\Big) \\ \lesssim\;& \int_{{}^{(ext)}\Sigma(1)} r^b\big(|\Psi_{(1)}|^2 + r^{-2}|\Psi_{(2)}|^2\big) + r_0^b\, Int(\Psi_{(1)}, \Psi_{(2)}) \\ &+ \int_{\mathcal{M}(r\geq r_0/2)} r^{b+1}\Big(|F_{(1)}|^2 + |F_{(2)}|^2\Big), \end{aligned} \tag{16.3.1}$$

[4] Recall that ${}^{(ext)}\mathcal{M}$ is defined by $r \geq r_0$.

where the quantity

$$Int(\Psi_{(1)},\Psi_{(2)}) \;\;:=\;\; r_0^{-1}\int_{\mathcal{M}(r_0/2,r_0)}\big(|\Psi_{(1)}|^2+|\Psi_{(2)}|^2\big)$$

depends only on the control of $\Psi_{(1)},\Psi_{(2)}$ *in* ${}^{(int)}\mathcal{M}$, *i.e. in* $r\le r_0$.

Case 2. *If* $\Lambda_{(1)}\le 0$ *and* $\Lambda_{(2)}<0$, *we have*

$$\begin{aligned}
&\int_{{}^{(ext)}\mathcal{M}} r^{b-1}|\Psi_{(2)}|^2+\sup_\tau\int_{{}^{(ext)}\Sigma(\tau)} r^b\big(|\Psi_{(1)}|^2+r^{-2}|\Psi_{(2)}|^2\big)\\
&+\int_{\Sigma_*} r^b\Big(|\Psi_{(1)}|^2+|\Psi_{(2)}|^2\Big)\\
\lesssim\;& \int_{{}^{(ext)}\Sigma(1)} r^b\big(|\Psi_{(1)}|^2+r^{-2}|\Psi_{(2)}|^2\big)+r_0^b\,Int(\Psi_{(1)},\Psi_{(2)})+\int_{{}^{(ext)}\mathcal{M}} r^{b-1}|\Psi_{(1)}|^2\\
&+\int_{\mathcal{M}(r\ge r_0/2)} r^{b+1}\Big(|F_{(1)}|^2+|F_{(2)}|^2\Big).
\end{aligned}\tag{16.3.2}$$

Case 3. *If* $\Lambda_{(2)}=0$, *we have*

$$\begin{aligned}
&\sup_\tau\int_{{}^{(ext)}\Sigma(\tau)} r^b\big(|\Psi_{(1)}|^2+r^{-2}|\Psi_{(2)}|^2\big)+\int_{\Sigma_*} r^b\Big(|\Psi_{(1)}|^2+|\Psi_{(2)}|^2\Big)\\
\lesssim\;& \int_{{}^{(ext)}\Sigma(1)} r^b\big(|\Psi_{(1)}|^2+r^{-2}|\Psi_{(2)}|^2\big)+r_0^b\,Int(\Psi_{(1)},\Psi_{(2)})+\int_{{}^{(ext)}\mathcal{M}} r^{b-1+\delta_B}|\Psi_{(1)}|^2\\
&+\int_{{}^{(ext)}\mathcal{M}} r^{b-1-\delta_B}|\Psi_{(2)}|^2+\int_{\mathcal{M}(r\ge r_0/2)} r^{b+1}\Big(r^{-\delta_B}|F_{(1)}|^2+r^{\delta_B}|F_{(2)}|^2\Big).
\end{aligned}\tag{16.3.3}$$

Case 4. *If* $\Lambda_{(1)}>0$, *we have*

$$\begin{aligned}
&\int_{{}^{(ext)}\mathcal{M}} r^{b-1}|\Psi_{(1)}|^2+\sup_\tau\int_{{}^{(ext)}\Sigma(\tau)} r^b\big(|\Psi_{(1)}|^2+r^{-2}|\Psi_{(2)}|^2\big)\\
&+\int_{\Sigma_*} r^b\Big(|\Psi_{(1)}|^2+|\Psi_{(2)}|^2\Big)\\
\lesssim\;& \int_{{}^{(ext)}\Sigma(1)} r^b\big(|\Psi_{(1)}|^2+r^{-2}|\Psi_{(2)}|^2\big)+r_0^b\,Int(\Psi_{(1)},\Psi_{(2)})+\int_{{}^{(ext)}\mathcal{M}} r^{b-1}|\Psi_{(2)}|^2\\
&+\int_{\mathcal{M}(r\ge r_0/2)} r^{b+1}\Big(|F_{(1)}|^2+|F_{(2)}|^2\Big).
\end{aligned}\tag{16.3.4}$$

Proof. The proof, similar to the proof of Proposition 15.3.12, is based on the following steps.

Step 1. The following is an immediate corollary of Lemma 15.3.8 and the fact that, in ${}^{(ext)}\mathcal{M}$, we have $\omega = O(r^{-2})$, $\underline{\omega} = \Gamma_b$, and $\Gamma_b = O(\epsilon r^{-1})$.

Corollary 16.3.2. *Let $\Psi_{(1)}, \Psi_{(2)}$, verifying either one of the equations* (16.2.3) *and* (16.2.4), *for positive real numbers $c_{(1)}$ and $c_{(2)}$, with $\Psi_{(1)}$ of signature k and $\Psi_{(2)}$ of signature $k-1$. Then denoting*

$$\Lambda_{(1)} = -2c_{(1)} + 1 + \frac{b}{2}, \qquad \Lambda_{(2)} = -2c_{(2)} + 1 + \frac{b}{2},$$

the following pointwise identity holds true for any real b in ${}^{(ext)}\mathcal{M}$:

1. *If $\Psi_{(1)}, \Psi_{(2)}$, verify equation* (16.2.3), *then*

$$\begin{aligned}&Div\left(\frac{1}{2}|q|^b|\Psi_{(1)}|^2 e_3 + |q|^b|\Psi_{(2)}|^2 e_4 - 2|q|^b\Re(\Psi_{(1)}\cdot\overline{\Psi_{(2)}})\right)\\ =&\frac{1}{2}|q|^b\Lambda_{(1)} tr\underline{\chi}|\Psi_{(1)}|^2 + |q|^b\Lambda_{(2)} tr\,\chi|\Psi_{(2)}|^2 + O(r^b)\big(|F_{(1)}||\Psi_{(1)}| + |F_{(2)}||\Psi_{(2)}|\big)\\ &+ O(r^{b-2})\big(|\Psi_{(1)}|^2 + |\Psi_{(2)}|^2\big) + O(\epsilon r^{b-1})\big(|\Psi_{(1)}|^2 + |\Psi_{(1)}||\Psi_{(2)}|\big).\end{aligned} \tag{16.3.5}$$

2. *If $\Psi_{(1)}, \Psi_{(2)}$, verify equation* (16.2.4), *then*

$$\begin{aligned}&Div\left(|q|^b|\Psi_{(1)}|^2 e_3 + \frac{1}{2}|q|^b|\Psi_{(2)}|^2 e_4 + 2|q|^b\Re(\Psi_{(1)}\cdot\overline{\Psi_{(2)}})\right)\\ =&|q|^b\Lambda_{(1)} tr\underline{\chi}|\Psi_{(1)}|^2 + \frac{1}{2}|q|^b\Lambda_{(2)} tr\,\chi|\Psi_{(2)}|^2 + O(r^b)\big(|F_{(1)}||\Psi_{(1)}| + |F_{(2)}||\Psi_{(2)}|\big)\\ &+ O(r^{b-2})\big(|\Psi_{(1)}|^2 + |\Psi_{(2)}|^2\big) + O(\epsilon r^{b-1})\big(|\Psi_{(1)}|^2 + |\Psi_{(1)}||\Psi_{(2)}|\big).\end{aligned} \tag{16.3.6}$$

Step 2. We multiply the divergence identities of Corollary 16.3.2 by a smooth cut-off function $\chi_0(r)$ supported for $r \geq r_0/2$ and identically 1 for $r \geq r_0$. Integrating, applying the divergence Lemma 15.3.11, and using Lemma 15.3.13 to treat the boundary terms, we deduce

$$\begin{aligned}&\int_{{}^{(ext)}\mathcal{M}} |q|^b\left(-\Lambda_{(1)}\mathrm{tr}\,\underline{\chi}|\Psi_{(1)}|^2 - \frac{1}{2}|q|^b\Lambda_{(2)}\mathrm{tr}\,\chi|\Psi_{(2)}|^2\right) + \int_{{}^{(ext)}\Sigma} r^b\big(|\Psi_{(1)}|^2 + r^{-2}|\Psi_{(2)}|^2\big)\\ &+\int_{\Sigma_*} r^b\big(|\Psi_{(1)}|^2 + |\Psi_{(2)}|^2\big)\\ \lesssim\ &\int_{{}^{(ext)}\Sigma(1)} r^b\big(|\Psi_{(1)}|^2 + r^{-2}|\Psi_{(2)}|^2\big) + r_0^b\,\mathrm{Int}(\Psi_{(1)},\Psi_{(2)})\\ &+\int_{\mathcal{M}(\geq r_0/2)} r^b\Big(|F_{(1)}||\Psi_{(1)}| + |F_{(2)}||\Psi_{(2)}|\Big) + \int_{{}^{(ext)}\mathcal{M}} r^{b-2}\big(|\Psi_{(1)}|^2 + |\Psi_{(2)}|^2\big)\\ &+\epsilon\int_{{}^{(ext)}\mathcal{M}} r^{b-1}\big(|\Psi_{(1)}|^2 + |\Psi_{(1)}||\Psi_{(2)}|\big),\end{aligned}$$

where, from now on, we carry out the proof in the case (16.2.4), the case (16.2.3) being completely analogous.

Step 3. Since $\mathrm{tr}\,\chi = \frac{2}{r} + O(r^{-2})$ and $\mathrm{tr}\,\underline{\chi} = -\frac{2}{r} + O(r^{-2})$, we deduce

$$\begin{aligned}
&\int_{{}^{(ext)}\mathcal{M}} |q|^b r^{-1}\left(\Lambda_{(1)}|\Psi_{(1)}|^2 - \frac{1}{2}|q|^b\Lambda_{(2)}|\Psi_{(2)}|^2\right) + \int_{{}^{(ext)}\Sigma} r^b\big(|\Psi_{(1)}|^2 + r^{-2}|\Psi_{(2)}|^2\big)\\
&+\int_{\Sigma_*} r^b\big(|\Psi_{(1)}|^2 + |\Psi_{(2)}|^2\big)\\
\lesssim\ &\int_{{}^{(ext)}\Sigma(1)} r^b\big(|\Psi_{(1)}|^2 + r^{-2}|\Psi_{(2)}|^2\big) + r_0^b\,\mathrm{Int}(\Psi_{(1)},\Psi_{(2)})\\
&+\int_{\mathcal{M}(\geq r_0/2)} r^b\Big(|F_{(1)}||\Psi_{(1)}| + |F_{(2)}||\Psi_{(2)}|\Big) + \int_{{}^{(ext)}\mathcal{M}} r^{b-2}\big(|\Psi_{(1)}|^2 + |\Psi_{(2)}|^2\big)\\
&+\epsilon\int_{{}^{(ext)}\mathcal{M}} r^{b-1}\big(|\Psi_{(1)}|^2 + |\Psi_{(1)}||\Psi_{(2)}|\big).
\end{aligned}$$

Step 4. We now conclude the proof of Proposition 16.3.1 in the cases 1, 2 and 4, i.e. we prove (16.3.1), (16.3.2) and (16.3.4). By Cauchy- Schwartz, for any $\lambda > 0$, we have

$$\begin{aligned}
\int_{\mathcal{M}(\geq r_0/2)} \Big(|F_{(1)}||\Psi_{(1)}| + |F_{(2)}||\Psi_{(2)}|\Big) \ \lesssim\ & \lambda^{-1}\int_{\mathcal{M}(\geq r_0/2)} r^{b-1}\big(|\Psi_{(1)}|^2 + |\Psi_{(2)}|^2\big)\\
&+\lambda\int_{\mathcal{M}(\geq r_0/2)} r^{b+1}\big(|F_{(1)}|^2 + |F_{(2)}|^2\big).
\end{aligned}$$

Together with Step 3, we infer, using in particular the fact that $r \geq r_0$ on ${}^{(ext)}\mathcal{M}$,

$$\begin{aligned}
&\int_{{}^{(ext)}\mathcal{M}} |q|^b r^{-1}\left(\Lambda_{(1)}|\Psi_{(1)}|^2 - \frac{1}{2}|q|^b\Lambda_{(2)}|\Psi_{(2)}|^2\right) + \int_{{}^{(ext)}\Sigma} r^b\big(|\Psi_{(1)}|^2 + r^{-2}|\Psi_{(2)}|^2\big)\\
&+\int_{\Sigma_*} r^b\big(|\Psi_{(1)}|^2 + |\Psi_{(2)}|^2\big)\\
\lesssim\ &\int_{{}^{(ext)}\Sigma(1)} r^b\big(|\Psi_{(1)}|^2 + r^{-2}|\Psi_{(2)}|^2\big) + r_0^b\,\mathrm{Int}(\Psi_{(1)},\Psi_{(2)}) + \lambda\int_{\mathcal{M}(\geq r_0/2)} r^{b+1}\big(|F_{(1)}|^2 + |F_{(2)}|^2\big)\\
&+\Big(r_0^{-1} + \lambda^{-1} + \epsilon\Big)\int_{{}^{(ext)}\mathcal{M}} r^{b-1}\big(|\Psi_{(1)}|^2 + |\Psi_{(2)}|^2\big).
\end{aligned}$$

Choosing $\lambda > 0$ large enough, and for r_0 sufficiently large and ϵ sufficiently small, since $r \leq |q| \leq 2r$, we immediately conclude the proof of the stated estimates (16.3.1), (16.3.2) and (16.3.4).

Step 5. It remains to treat case 3, i.e. to prove (16.3.3). In this case, by Cauchy-Schwartz, we have

$$\begin{aligned}\int_{\mathcal{M}(\geq r_0/2)}\Big(|F_{(1)}||\Psi_{(1)}|+|F_{(2)}||\Psi_{(2)}|\Big) \;\lesssim\; & \int_{\mathcal{M}(\geq r_0/2)} r^{b-1}\big(r^{\delta_B}|\Psi_{(1)}|^2+r^{-\delta_B}|\Psi_{(2)}|^2\big)\\ &+\int_{\mathcal{M}(\geq r_0/2)} r^{b+1}\big(r^{-\delta_B}|F_{(1)}|^2+r^{\delta_B}|F_{(2)}|^2\big).\end{aligned}$$

Together with Step 3, since $\Lambda_{(2)}=0$ in this case, we immediately infer the stated estimate (16.3.3). This concludes the proof of Proposition 16.3.1. □

16.4 Basic curvature estimate in ${}^{(ext)}\mathcal{M}$

In this section, we provide basic curvature estimate in ${}^{(ext)}\mathcal{M}$ on the control of ${}^{(ext)}\mathfrak{R}_0$, i.e. we prove the following estimate

$$ {}^{(ext)}\mathfrak{R}_0^2 \;\lesssim r_0^{-\delta_B}\,{}^{(ext)}\mathfrak{G}_0^2+r_0^{3+\delta_B}\,{}^{(int)}\mathfrak{R}_0^2+\epsilon_0^2. \tag{16.4.1}$$

The procedure leading to (16.4.1) will be repeated in section 16.7 in order to prove Theorem 16.1.1 on the control of ${}^{(ext)}\mathfrak{R}_{J+1}$.

In order to prove (16.4.1), we consider successively each of the four Bianchi pairs starting with the first one.

16.4.1 First Bianchi pair

In the case of the first Bianchi pair we have $c_{(1)}=\frac{1}{2}$, $c_{(2)}=2$. We choose $b=4+\delta_B$ so that we have in this case

$$\Lambda_{(1)}=2+\frac{\delta_B}{2}>0,\qquad \Lambda_{(2)}=-1+\frac{\delta_B}{2}<0.$$

We may thus apply case 1 of Proposition 16.3.1, i.e. estimate (16.3.1). We infer

$$\begin{aligned}&\int_{{}^{(ext)}\mathcal{M}} r^{3+\delta_B}|(A,B)|^2+\sup_\tau\int_{\Sigma(\tau)\cap{}^{(ext)}\mathcal{M}} r^{4+\delta_B}\Big(|A|^2+r^{-2}|B|^2\Big)\\ &+\int_{\Sigma_*} r^{4+\delta_B}\Big(|A|^2+|B|^2\Big)\\ \lesssim\;&\int_{\Sigma(1)} r^{4+\delta_B}\Big(|A|^2+r^{-2}|B|^2\Big)+r_0^{4+\delta_B}\,\mathrm{Int}(A,B)+\int_{\mathcal{M}(r\geq r_0/2)} r^{5+\delta_B}\Big(|F_{(1)}|^2+|F_{(2)}|^2\Big).\end{aligned}$$

In view of the control of our initial conditions $\int_{\Sigma(1)} r^{4+\delta_B}(|A|^2 + r^{-2}|B|^2) \lesssim \epsilon_0^2$, the interior estimates $\text{Int}(A, B) \lesssim r_0^{-1}\, {}^{(int)}\mathfrak{R}_0^2[A, B]$, and the definition of the ${}^{(ext)}\mathfrak{R}$ norms in Definition 13.5.2, we deduce

$$ {}^{(ext)}\mathfrak{R}_0^2[A] + {}^{(ext)}\mathfrak{R}_0'[B]^2 \lesssim r_0^{3+\delta_B}\, {}^{(int)}\mathfrak{R}_0^2[A, B] + \int_{\mathcal{M}(r\geq r_0/2)} r^{5+\delta_B}\Big(|F_{(1)}|^2 + |F_{(2)}|^2\Big) + \epsilon_0^2 $$

where

$$ {}^{(ext)}\mathfrak{R}_0'[B]^2 := \int_{{}^{(ext)}\mathcal{M}} r^{3+\delta_B}|B|^2 + \sup_\tau \int_{\Sigma(\tau)\cap\, {}^{(ext)}\mathcal{M}} r^{2+\delta_B}|B|^2 + \int_{\Sigma_*} r^{4+\delta_B}|B|^2 \quad (16.4.2) $$

agrees with the norm $\mathfrak{R}_0[B]$ in Definition 13.5.2 on ${}^{(ext)}\mathcal{M}$ and Σ_*, but has a weaker weight in r on $\Sigma(\tau)$.

Now,

$$ \begin{aligned} \int_{\mathcal{M}(r\geq r_0/2)} r^{5+\delta_B}|F_{(1)}|^2 &\lesssim \int_{\mathcal{M}(r\geq r_0/2)} r^{5+\delta_B}\Big(r^{-4}|B|^2 + r^{-6}|\Gamma_g|^2 + |\Gamma_b|^2|B|^2\Big) \\ &\lesssim \int_{\mathcal{M}(r\geq r_0/2)} r^{1+\delta_B}|B|^2 + \int_{\mathcal{M}(r\geq r_0/2)} r^{-1+\delta_B}|\Gamma_g|^2 + \epsilon^4 \\ &\lesssim r_0^{-2}\, {}^{(ext)}\mathfrak{R}_0[B]^2 + \int_{\mathcal{M}(r\geq r_0/2)} r^{-1+\delta_B}|\Gamma_g|^2 + \epsilon_0^2. \end{aligned} $$

Hence, in view of Lemma 13.5.5,

$$ \int_{\mathcal{M}(r\geq r_0/2)} r^{5+\delta_B}|F_{(1)}|^2 \lesssim r_0^{-2}\, {}^{(ext)}\mathfrak{R}_0[B]^2 + r_0^{-2+2\delta_B}\, {}^{(ext)}\mathfrak{G}_0^2 + \epsilon_0^2. $$

Similarly

$$ \int_{\mathcal{M}(r\geq r_0/2)} r^{5+\delta_B}|F_{(2)}|^2 \lesssim r_0^{-2}\, {}^{(ext)}\mathfrak{R}_0[A]^2 + r_0^{-2+2\delta_B}\, {}^{(ext)}\mathfrak{G}_0^2 + \epsilon_0^2 $$

and we deduce

$$ {}^{(ext)}\mathfrak{R}_0^2[A] + {}^{(ext)}\mathfrak{R}_0'[B]^2 \lesssim r_0^{-2}\, {}^{(ext)}\mathfrak{R}_0^2 + r_0^{-2+2\delta_B}\, {}^{(ext)}\mathfrak{G}_0^2 + r_0^{3+\delta_B}\, {}^{(int)}\mathfrak{R}^2 + \epsilon_0^2. \quad (16.4.3) $$

16.4.2 Second Bianchi pair

First estimate for the second Bianchi pair

In the case of the second Bianchi pair, we have $c_{(1)} = 1$, $c_{(2)} = \frac{3}{2}$. We choose $b = 4 - \delta_B$ so that we have in this case

$$ \Lambda_{(1)} = 1 + \frac{\delta_B}{2} > 0, \qquad \Lambda_{(2)} = -\frac{\delta_B}{2} < 0. $$

We may thus apply case 1 of Proposition 16.3.1, i.e. estimate (16.3.1). We infer

$$\begin{aligned}&\int_{{}^{(ext)}\mathcal{M}} r^{3-\delta_B}|(B,\check{P})|^2 + \sup_\tau \int_{\Sigma(\tau)\cap {}^{(ext)}\mathcal{M}} r^{4-\delta_B}\Big(|B|^2 + r^{-2}|\check{P}|^2\Big)\\ &+\int_{\Sigma_*} r^{4-\delta_B}\Big(|B|^2+|\check{P}|^2\Big)\\ \lesssim\ &\int_{\Sigma(1)} r^{4-\delta_B}\Big(|B|^2+r^{-2}|\check{P}|^2\Big) + r_0^{4-\delta_B}\,\mathrm{Int}(B,\check{P}) + \int_{\mathcal{M}(r\geq r_0/2)} r^{5-\delta_B}\Big(|F_{(1)}|^2+|F_{(2)}|^2\Big).\end{aligned}$$

In view of the control of our initial conditions $\int_{\Sigma(1)} r^{4-\delta_B}(|B|^2+r^{-2}|\check{P}|^2)\lesssim \epsilon_0^2$, the interior estimates $\mathrm{Int}(B,\check{P})\lesssim r_0^{-1}\,{}^{(int)}\mathfrak{R}_0^2[B,\check{P}]$, and the definition of the ${}^{(ext)}\mathfrak{R}$ norms in Definition 13.5.2, we deduce

$$\begin{aligned}{}^{(ext)}\mathfrak{R}_0''[B]^2 + {}^{(ext)}\mathfrak{R}_0'[\check{P}]^2 &\lesssim r_0^{3-\delta_B}\,{}^{(int)}\mathfrak{R}_0^2[B,\check{P}] + \int_{\mathcal{M}(r\geq r_0/2)} r^{5-\delta_B}\Big(|F_{(1)}|^2+|F_{(2)}|^2\Big)+\epsilon_0^2,\\ {}^{(ext)}\mathfrak{R}_0''[B]^2 &:= \int_{{}^{(ext)}\mathcal{M}} r^{3-\delta_B}|B|^2 + \sup_\tau\int_{\Sigma(\tau)\cap{}^{(ext)}\mathcal{M}} r^{4-\delta_B}|B|^2 + \int_{\Sigma_*} r^{4-\delta_B}|B|^2,\\ {}^{(ext)}\mathfrak{R}_0'[\check{P}]^2 &:= \int_{{}^{(ext)}\mathcal{M}} r^{3-\delta_B}|\check{P}|^2 + \sup_\tau\int_{\Sigma(\tau)\cap{}^{(ext)}\mathcal{M}} r^{2-\delta_B}|\check{P}|^2 + \int_{\Sigma_*} r^{4-\delta_B}|\check{P}|^2.\end{aligned}$$

In this case

$$\begin{aligned}F_{(1)} &= O(ar^{-2})\check{P} + O(r^{-3})\Gamma_b' + r^{-2}\Gamma_b'\cdot\check{R}_b + \Gamma_b\cdot A,\\ F_{(2)} &= O(ar^{-2})B + O(r^{-4})\Gamma_b + \Gamma_b'\cdot(A,B) + r^{-1}\Xi\cdot\check{R}_b.\end{aligned}$$

Now, writing $\check{P}, B = r^{-2}\check{R}_b$,

$$\begin{aligned}\int_{\mathcal{M}(r\geq r_0/2)} r^{5-\delta_B}\Big(|F_{(1)}|^2+|F_{(2)}|^2\Big) &\lesssim \int_{\mathcal{M}(r\geq r_0/2)} r^{-3-\delta_B}|\check{R}_b|^2 + \int_{\mathcal{M}(r\geq r_0/2)} r^{-1-\delta_B}|\Gamma_b'|^2\\ &+\int_{\mathcal{M}(r\geq r_0/2)} r^{1-\delta_B}\Big(|\Gamma_b'|^2|\check{R}_b|^2 + r^4|\Gamma_b|^2|(A,B)|^2\Big).\end{aligned}$$

In view of (13.5.7), we have

$$\int_{\mathcal{M}(r\geq r_0/2)} r^{-1-\delta_B}|\Gamma_b'|^2 \lesssim r_0^{-\delta_B}\,{}^{(ext)}\mathfrak{G}_0^2.$$

Also, we have

$$\begin{aligned}\int_{\mathcal{M}(r\geq r_0/2)} r^{1-\delta_B}|\Gamma_b'|^2|\check{R}_b|^2 + \int_{\mathcal{M}(r\geq r_0/2)} r^{5-\delta_B}|\Gamma_b|^2|(A,B)|^2 &\lesssim\\ ep^4 \lesssim \epsilon_0^2.\end{aligned}$$

Hence

$${}^{(ext)}\mathfrak{R}_0''[B]^2 + {}^{(ext)}\mathfrak{R}_0'[\check{P}]^2 \lesssim r_0^{3-\delta_B}\,{}^{(int)}\mathfrak{R}^2 + r_0^{-\delta_B}\,{}^{(ext)}\mathfrak{G}_0^2 + r_0^{-2}\,{}^{(ext)}\mathfrak{R}_0^2+\epsilon_0^2. \quad (16.4.4)$$

Second estimate for the second Bianchi pair

In order to control the norm $\mathfrak{R}_0[B]^2$, we still need to recover the correct weight in r for the part of the norm on $\Sigma(\tau)$. To do this, we choose $b=4$ so that we have $\Lambda_{(2)}=0$ in this case. We may thus apply case 3 of Proposition 16.3.1, i.e. estimate (16.3.3). We infer

$$\begin{aligned}
&\sup_\tau\int_{\Sigma(\tau)\cap{}^{(ext)}\mathcal{M}} r^4\Big(|B|^2+r^{-2}|\check{P}|^2\Big)+\int_{\Sigma_*}r^4\Big(|B|^2+|\check{P}|^2\Big)\\
\lesssim\ & \epsilon_0^2+r_0^3\,{}^{(int)}\mathfrak{R}_0^2[B,\check{P}]+\int_{\mathcal{M}(\geq r_0/2)}r^5\Big(r^{-\delta_B}|F_{(1)}|^2+r^{\delta_B}|F_{(2)}|^2\Big)\\
&+\int_{{}^{(ext)}\mathcal{M}}r^{3+\delta_B}|B|^2+\int_{{}^{(ext)}\mathcal{M}}r^{3-\delta_B}|\check{P}|^2.
\end{aligned}$$

In view of the definition of ${}^{(ext)}\mathfrak{R}_0'[B]$ and ${}^{(ext)}\mathfrak{R}_0'[\check{P}]$, we infer

$$\begin{aligned}
&\sup_\tau\int_{\Sigma(\tau)\cap{}^{(ext)}\mathcal{M}}\Big(r^4|B|^2+r^2|\check{P}|^2\Big)+\int_{\Sigma_*}r^4|\check{P}|^2\\
\lesssim\ &\epsilon_0^2+r_0^3\,{}^{(int)}\mathfrak{R}_0^2[B,\check{P}]+{}^{(ext)}\mathfrak{R}_0'[B]^2+{}^{(ext)}\mathfrak{R}_0'[\check{P}]^2\\
&+\int_{\mathcal{M}(\geq r_0/2)}r^5\Big(r^{-\delta_B}|F_{(1)}|^2+r^{\delta_B}|F_{(2)}|^2\Big).
\end{aligned}$$

Together with (16.4.3) and (16.4.4), this yields

$$\begin{aligned}
&\sup_\tau\int_{\Sigma(\tau)\cap{}^{(ext)}\mathcal{M}}\Big(r^4|B|^2+r^2|\check{P}|^2\Big)+\int_{\Sigma_*}r^4|\check{P}|^2\\
\lesssim\ & r_0^{3+\delta_B}\,{}^{(int)}\mathfrak{R}_0^2+r_0^{-\delta_B}\,{}^{(ext)}\mathfrak{G}_0^2+r_0^{-2}\,{}^{(ext)}\mathfrak{R}_0^2+\epsilon_0^2+\int_{\mathcal{M}(\geq r_0/2)}r^5\Big(r^{-\delta_B}|F_{(1)}|^2+r^{\delta_B}|F_{(2)}|^2\Big).
\end{aligned}$$

Also, in the proof of (16.4.4), we have obtained the following estimate

$$\int_{\mathcal{M}(r\geq r_0/2)}r^{5-\delta_B}|F_{(1)}|^2\ \lesssim\ r_0^{-\delta_B}\,{}^{(ext)}\mathfrak{G}_0^2+r_0^{-2}\,{}^{(ext)}\mathfrak{R}_0^2+\epsilon_0^2$$

and hence

$$\begin{aligned}
&\sup_\tau\int_{\Sigma(\tau)\cap{}^{(ext)}\mathcal{M}}\Big(r^4|B|^2+r^2|\check{P}|^2\Big)+\int_{\Sigma_*}r^4|\check{P}|^2\\
\lesssim\ & r_0^{3+\delta_B}\,{}^{(int)}\mathfrak{R}_0^2+r_0^{-\delta_B}\,{}^{(ext)}\mathfrak{G}_0^2+r_0^{-2}\,{}^{(ext)}\mathfrak{R}_0^2+\epsilon_0^2+\int_{\mathcal{M}(\geq r_0/2)}r^{5+\delta_B}|F_{(2)}|^2.
\end{aligned}$$

It thus remain to control the last term on the RHS. Recalling that

$$F_{(2)}\ =\ O(ar^{-2})B+O(r^{-4})\Gamma_b+\Gamma_b'\cdot(A,B)+r^{-1}\Xi\cdot\check{R}_b,$$

we have

$$\begin{aligned}\int_{\mathcal{M}(r\geq r_0/2)} r^{5+\delta_B}|F_{(2)}|^2 &\lesssim \int_{\mathcal{M}(r\geq r_0/2)} r^{1+\delta_B}|B|^2+\int_{\mathcal{M}(r\geq r_0/2)} r^{-3+\delta_B}|\Gamma_b|^2\\ &\quad +\int_{\mathcal{M}(r\geq r_0/2)}\Big(r^{5+\delta_B}|\Gamma_b'|^2|(A,B)|^2+r^{3+\delta_B}|\Xi|^2|\check{R}_b|^2\Big)\\ &\lesssim r_0^{-2}\,{}^{(ext)}\mathfrak{R}_0^2+r_0^{-2+2\delta_B}\,{}^{(ext)}\mathfrak{G}_0^2+\epsilon^4\\ &\lesssim r_0^{-2}\,{}^{(ext)}\mathfrak{R}_0^2+r_0^{-2+2\delta_B}\,{}^{(ext)}\mathfrak{G}_0^2+\epsilon_0^2,\end{aligned}$$

where we have used in particular the fact that $\Xi\in r^{-1}\Gamma_g$ in Part III in view of (13.6.5). Hence, we infer

$$\begin{aligned}&\sup_\tau\int_{\Sigma(\tau)\cap{}^{(ext)}\mathcal{M}}\Big(r^4|B|^2+r^2|\check{P}|^2\Big)+\int_{\Sigma_*} r^4|\check{P}|^2\\ \lesssim\;& r_0^{3+\delta_B}\,{}^{(int)}\mathfrak{R}_0^2+r_0^{-\delta_B}\,{}^{(ext)}\mathfrak{G}_0^2+r_0^{-2}\,{}^{(ext)}\mathfrak{R}_0^2+\epsilon_0^2.\end{aligned}\tag{16.4.5}$$

Together with (16.4.3) and (16.4.4), we deduce

$${}^{(ext)}\mathfrak{R}_0[B,\check{P}]^2\lesssim r_0^{3+\delta_B}\,{}^{(int)}\mathfrak{R}_0^2+r_0^{-\delta_B}\,{}^{(ext)}\mathfrak{G}_0^2+r_0^{-2}\,{}^{(ext)}\mathfrak{R}_0^2+\epsilon_0^2.\tag{16.4.6}$$

16.4.3 Third Bianchi pair

First estimate for the third Bianchi pair

In the case of the third Bianchi pair, we have $c_{(1)}=3/2$, $c_{(2)}=1$. We choose $b=2-\delta_B$ so that we have in this case

$$\Lambda_{(1)} = -1-\frac{\delta_B}{2}<0,\qquad \Lambda_{(2)}=-\frac{\delta_B}{2}<0.$$

We may thus apply case 2 of Proposition 16.3.1, i.e. estimate (16.3.2). We infer

$$\begin{aligned}&\int_{{}^{(ext)}\mathcal{M}} r^{1-\delta_B}|\underline{B}|^2+\sup_\tau\int_{\Sigma(\tau)\cap{}^{(ext)}\mathcal{M}} r^{2-\delta_B}\Big(|\check{P}|^2+r^{-2}|\underline{B}|^2\Big)+\int_{\Sigma_*} r^{2-\delta_B}\Big(|\check{P}|^2+|\underline{B}|^2\Big)\\ \lesssim\;&\int_{\Sigma(1)} r^{2-\delta_B}\Big(|\check{P}|^2+r^{-2}|\underline{B}|^2\Big)+r_0^{2-\delta_B}\,\mathrm{Int}(\check{P},\underline{B})+\int_{\mathcal{M}(r\geq r_0/2)} r^{3-\delta_B}\Big(|F_{(1)}|^2+|F_{(2)}|^2\Big)\\ &+\int_{{}^{(ext)}\mathcal{M}} r^{1-\delta_B}|\check{P}|^2\\ \lesssim\;& r_0^{3+\delta_B}\,{}^{(int)}\mathfrak{R}_0^2+r_0^{-\delta_B}\,{}^{(ext)}\mathfrak{G}_0^2+r_0^{-2}\,{}^{(ext)}\mathfrak{R}_0^2+\epsilon_0^2+\int_{\mathcal{M}(r\geq r_0/2)} r^{3-\delta_B}\Big(|F_{(1)}|^2+|F_{(2)}|^2\Big),\end{aligned}$$

where we used in particular the control of ${}^{(ext)}\mathfrak{R}_0[\check{P}]$ in (16.4.6). Given that

$$\begin{aligned} F_{(1)} &= O(ar^{-2})\underline{B} + O(r^{-3})\Gamma_b + r^{-1}\Gamma_b' \cdot \check{R}_b, \\ F_{(2)} &= O(ar^{-2})\check{P} + O(r^{-3})\Gamma_b + r^{-2}\Gamma_b \cdot \check{R}_b + \Xi \cdot \check{R}_b, \end{aligned}$$

we easily deduce, using in particular the fact that $\Xi \in r^{-1}\Gamma_g$ in Part III in view of (13.6.5),

$$\int_{\mathcal{M}(r\geq r_0/2)} r^{3-\delta_B}\Big(|F_{(1)}|^2 + |F_{(2)}|^2\Big) \lesssim r_0^{-2}\,{}^{(ext)}\mathfrak{R}_0^2 + r_0^{-2-\delta_B}\,{}^{(ext)}\mathfrak{G}_0 + \epsilon_0^2,$$

and hence

$$\begin{aligned} &{}^{(ext)}\mathfrak{R}_0'[\underline{B}]^2 \lesssim r_0^{3+\delta_B}\,{}^{(int)}\mathfrak{R}_0^2 + r_0^{-\delta_B}\,{}^{(ext)}\mathfrak{G}_0^2 + r_0^{-2}\,{}^{(ext)}\mathfrak{R}_0^2 + \epsilon_0^2, \\ &{}^{(ext)}\mathfrak{R}_0'[\underline{B}]^2 := \int_{{}^{(ext)}\mathcal{M}} r^{1-\delta_B}|\underline{B}|^2 + \sup_\tau \int_{\Sigma(\tau)\cap{}^{(ext)}\mathcal{M}} r^{-\delta_B}|\underline{B}|^2 + \int_{\Sigma_*} r^{2-\delta_B}|\underline{B}|^2. \end{aligned} \tag{16.4.7}$$

Second estimate for the third Bianchi pair

In order to control the norm $\mathfrak{R}_0[\underline{B}]^2$, we still need to recover the correct weight in r for the part of the norm on $\Sigma(\tau)$ and on Σ_*. To do this, we choose $b = 2$ so that we have $\Lambda_{(2)} = 0$ in this case. We may thus apply case 3 of Proposition 16.3.1, i.e. estimate (16.3.3). We infer

$$\begin{aligned} &\sup_\tau \int_{\Sigma(\tau)\cap{}^{(ext)}\mathcal{M}} r^2\Big(|\check{P}|^2 + r^{-2}|\underline{B}|^2\Big) + \int_{\Sigma_*} r^2\Big(|\check{P}|^2 + |\underline{B}|^2\Big) \\ \lesssim\ & \int_{\Sigma(1)} r^2\Big(|\check{P}|^2 + r^{-2}|\underline{B}|^2\Big) + r_0^2\,\mathrm{Int}(\check{P},\underline{B}) + \int_{\mathcal{M}(r\geq r_0/2)} r^3\Big(|F_{(1)}|^2 + |F_{(2)}|^2\Big) \\ & + \int_{{}^{(ext)}\mathcal{M}} r^{1+\delta_B}|\check{P}|^2 + \int_{{}^{(ext)}\mathcal{M}} r^{1-\delta_B}|\underline{B}|^2 \\ \lesssim\ & r_0^{3+\delta_B}\,{}^{(int)}\mathfrak{R}_0^2 + r_0^{-\delta_B}\,{}^{(ext)}\mathfrak{G}_0^2 + r_0^{-2}\,{}^{(ext)}\mathfrak{R}_0^2 + \epsilon_0^2 \\ & + \int_{\mathcal{M}(r\geq r_0/2)} r^3\Big(r^{-\delta_B}|F_{(1)}|^2 + r^{\delta_B}|F_{(2)}|^2\Big), \end{aligned}$$

where we used in particular the control of ${}^{(ext)}\mathfrak{R}_0[\check{P}]$ and ${}^{(ext)}\mathfrak{R}_0'[\underline{B}]$ provided respectively by (16.4.6) and (16.4.7).

Next, recalling that we have in this case

$$\begin{aligned} F_{(1)} &= O(ar^{-2})\underline{B} + O(r^{-3})\Gamma_b + r^{-1}\Gamma_b' \cdot \check{R}_b, \\ F_{(2)} &= O(ar^{-2})\check{P} + O(r^{-3})\Gamma_b + r^{-2}\Gamma_b \cdot \check{R}_b + \Xi \cdot \check{R}_b, \end{aligned}$$

we derive

$$\int_{\mathcal{M}(r\geq r_0/2)} r^3\Big(r^{-\delta_B}|F_{(1)}|^2+r^{\delta_B}|F_{(2)}|^2\Big) \lesssim r_0^{-2}\,{}^{(ext)}\mathfrak{R}_0^2+r_0^{-2+2\delta_B}\,{}^{(ext)}\mathfrak{G}_0^2+\epsilon_0^2,$$

where we used again the fact that $\Xi\in r^{-1}\Gamma_g$ in Part III in view of (13.6.5). We infer

$$\begin{aligned}\sup_\tau\int_{\Sigma(\tau)\cap\,{}^{(ext)}\mathcal{M}}|\underline{B}|^2+\int_{\Sigma_*}r^2|\underline{B}|^2 \;&\lesssim\; r_0^{3+\delta_B}\,{}^{(int)}\mathfrak{R}_0^2+r_0^{-\delta_B}\,{}^{(ext)}\mathfrak{G}_0^2\\ &\quad +r_0^{-2}\,{}^{(ext)}\mathfrak{R}_0^2+\epsilon_0^2. \end{aligned}\tag{16.4.8}$$

Combining with (16.4.7), we thus deduce

$${}^{(ext)}\mathfrak{R}_0[\underline{B}]^2 \;\lesssim r_0^{3+\delta_B}\,{}^{(int)}\mathfrak{R}_0^2+r_0^{-\delta_B}\,{}^{(ext)}\mathfrak{G}_0^2+r_0^{-2}\,{}^{(ext)}\mathfrak{R}_0^2+\epsilon_0^2. \tag{16.4.9}$$

16.4.4 Fourth Bianchi pair

First estimate for the fourth Bianchi pair

In the case of the third Bianchi pair, we have $c_{(1)}=2$, $c_{(2)}=1/2$. We choose $b=-\delta_B$ so that we have in this case

$$\Lambda_{(1)} \;=\; -3-\frac{\delta_B}{2}<0, \qquad \Lambda_{(2)}=-\frac{\delta_B}{2}<0.$$

We may thus apply case 2 of Proposition 16.3.1, i.e. estimate (16.3.2). We infer

$$\begin{aligned}&\int_{{}^{(ext)}\mathcal{M}}r^{-1-\delta_B}|\underline{A}|^2+\sup_\tau\int_{\Sigma(\tau)\cap\,{}^{(ext)}\mathcal{M}}r^{-\delta_B}\Big(|\underline{B}|^2+r^{-2}|\underline{A}|^2\Big)+\int_{\Sigma_*}r^{-\delta_B}\Big(|\underline{B}|^2+|\underline{A}|^2\Big)\\ \lesssim\;& \int_{\Sigma(1)}r^{-\delta_B}\Big(|\underline{B}|^2+r^{-2}|\underline{A}|^2\Big)+r_0^{-\delta_B}\,\mathrm{Int}(\underline{B},\underline{A})+\int_{\mathcal{M}(r\geq r_0/2)}r^{1-\delta_B}\Big(|F_{(1)}|^2+|F_{(2)}|^2\Big)\\ &+\int_{{}^{(ext)}\mathcal{M}}r^{-1-\delta_B}|\underline{B}|^2\\ \lesssim\;& r_0^{3+\delta_B}\,{}^{(int)}\mathfrak{R}_0^2+r_0^{-\delta_B}\,{}^{(ext)}\mathfrak{G}_0^2+r_0^{-2}\,{}^{(ext)}\mathfrak{R}_0^2+\epsilon_0^2+\int_{\mathcal{M}(r\geq r_0/2)}r^{1-\delta_B}\Big(|F_{(1)}|^2+|F_{(2)}|^2\Big)\end{aligned}$$

where we used in particular the control of ${}^{(ext)}\mathfrak{R}_0[\underline{B}]$ in (16.4.9). Given that

$$\begin{aligned}F_{(1)} &= O(ar^{-2})\underline{A}+O(r^{-3})\Gamma_b+\Gamma_b'\cdot\check{R}_b,\\ F_{(2)} &= O(ar^{-2})\underline{B}+O(r^{-3})\Gamma_b+r^{-1}\Gamma_b\cdot\check{R}_b,\end{aligned}$$

and proceeding as before, we have

$$\int_{\mathcal{M}(r\geq r_0/2)} r^{1-\delta_B}\Big(|F_{(1)}|^2+|F_{(2)}|^2\Big) \lesssim r_0^{-2}\,{}^{(ext)}\mathfrak{R}_0^2 + r_0^{-4}\,{}^{(ext)}\mathfrak{G}_0+\epsilon_0^2.$$

We deduce

$$\begin{aligned}
&\mathfrak{R}_0'[\underline{A}] \lesssim r_0^{3+\delta_B}\,{}^{(int)}\mathfrak{R}_0^2 + r_0^{-\delta_B}\,{}^{(ext)}\mathfrak{G}_0^2 + r_0^{-2}\,{}^{(ext)}\mathfrak{R}_0^2+\epsilon_0^2,\\
&\mathfrak{R}_0'[\underline{A}] := \int_{{}^{(ext)}\mathcal{M}} r^{-1-\delta_B}|\underline{A}|^2 + \sup_\tau\int_{\Sigma(\tau)\cap{}^{(ext)}\mathcal{M}} r^{-2-\delta_B}|\underline{A}|^2 + \int_{\Sigma_*} r^{-\delta_B}|\underline{A}|^2.
\end{aligned} \tag{16.4.10}$$

Second estimate for the fourth Bianchi pair

In order to control the norm $\mathfrak{R}_0[\underline{A}]^2$, we still need to recover the correct weight in r for the part of the norm on $\Sigma(\tau)$ and on Σ_*. To do this, we choose $b=0$ so that we have $\Lambda_{(2)}=0$ in this case. We may thus apply case 3 of Proposition 16.3.1, i.e. estimate (16.3.3). We infer

$$\begin{aligned}
&\sup_\tau\int_{\Sigma(\tau)\cap{}^{(ext)}\mathcal{M}}\Big(|\underline{B}|^2+r^{-2}|\underline{A}|^2\Big) + \int_{\Sigma_*}\Big(|\underline{B}|^2+|\underline{A}|^2\Big)\\
\lesssim\;& \int_{\Sigma(1)} r^2\Big(|\underline{B}|^2+r^{-2}|\underline{A}|^2\Big) + \mathrm{Int}(\underline{B},\underline{A}) + \int_{\mathcal{M}(r\geq r_0/2)} r\Big(r^{-\delta_B}|F_{(1)}|^2+r^{\delta_B}|F_{(2)}|^2\Big)\\
&+\int_{{}^{(ext)}\mathcal{M}} r^{-1+\delta_B}|\underline{B}|^2 + \int_{{}^{(ext)}\mathcal{M}} r^{-1-\delta_B}|\underline{A}|^2\\
\lesssim\;& r_0^{3+\delta_B}\,{}^{(int)}\mathfrak{R}_0^2 + r_0^{-\delta_B}\,{}^{(ext)}\mathfrak{G}_0^2 + r_0^{-2}\,{}^{(ext)}\mathfrak{R}_0^2+\epsilon_0^2 + \int_{\mathcal{M}(r\geq r_0/2)} r\Big(r^{-\delta_B}|F_{(1)}|^2+r^{\delta_B}|F_{(2)}|^2\Big),
\end{aligned}$$

where we used in particular the control of ${}^{(ext)}\mathfrak{R}_0[\underline{B}]$ and ${}^{(ext)}\mathfrak{R}_0'[\underline{A}]$ provided respectively by (16.4.9) and (16.4.10).

Next, recalling that we have in this case

$$\begin{aligned}
F_{(1)} &= O(ar^{-2})\underline{A} + O(r^{-3})\Gamma_b + \Gamma_b'\cdot\check{R}_b,\\
F_{(2)} &= O(ar^{-2})\underline{B} + O(r^{-3})\Gamma_b + r^{-1}\Gamma_b\cdot\check{R}_b,
\end{aligned}$$

we derive

$$\int_{\mathcal{M}} r\Big(r^{-\delta_B}|F_{(1)}|^2+r^{\delta_B}|F_{(2)}|^2\Big) \lesssim r_0^{-2}\,{}^{(ext)}\mathfrak{R}_0^2 + r_0^{-4+2\delta_B}\,{}^{(ext)}\mathfrak{G}_0+\epsilon_0^2.$$

We thus derive

$$\begin{aligned}
&\sup_\tau\int_{\Sigma(\tau)\cap{}^{(ext)}\mathcal{M}} r^{-2}|\underline{A}|^2 + \int_{\Sigma_*}|\underline{A}|^2\\
\lesssim\;& r_0^{3+\delta_B}\,{}^{(int)}\mathfrak{R}_0^2 + r_0^{-\delta_B}\,{}^{(ext)}\mathfrak{G}_0^2 + r_0^{-2}\,{}^{(ext)}\mathfrak{R}_0^2+\epsilon_0^2.
\end{aligned} \tag{16.4.11}$$

Combining with (16.4.10), we thus deduce

$$ {}^{(ext)}\mathfrak{R}_0[\underline{A}] \lesssim r_0^{3+\delta_B}\, {}^{(int)}\mathfrak{R}_0^2 + r_0^{-\delta_B}\, {}^{(ext)}\mathfrak{G}_0^2 + r_0^{-2}\, {}^{(ext)}\mathfrak{R}_0^2 + \epsilon_0^2. \tag{16.4.12} $$

Conclusion

Finally, combining (16.4.3), (16.4.6), (16.4.9) and (16.4.12), we deduce

$$ {}^{(ext)}\mathfrak{R}_0^2 \lesssim r_0^{-2}\, {}^{(ext)}\mathfrak{R}_0^2 + r_0^{-\delta_B}\, {}^{(ext)}\mathfrak{G}_0^2 + r_0^{3+\delta_B}\, {}^{(int)}\mathfrak{R}_0^2 + \epsilon_0^2. $$

Thus, for r_0 sufficiently large,

$$ {}^{(ext)}\mathfrak{R}_0^2 \lesssim r_0^{-\delta_B}\, {}^{(ext)}\mathfrak{G}_0^2 + r_0^{3+\delta_B}\, {}^{(int)}\mathfrak{R}_0^2 + \epsilon_0^2, $$

which concludes the proof of (16.4.1).

Remark 16.4.1. *For convenience, we summarize the main steps in the proof of the basic curvature estimates* (16.4.1)*:*

1. *First, we control the first Bianchi pair by applying case 1 of Proposition 16.3.1 with the choice $b = 4 + \delta_B$.*
2. *Next, we control the second Bianchi pair in two steps:*
   (a) *First, we apply case 1 of Proposition 16.3.1 with the choice $b = 4 - \delta_B$.*
   (b) *Then, we apply case 3 of Proposition 16.3.1 with the choice $b = 4$.*
3. *Next, we control the third Bianchi pair in two steps:*
   (a) *First, we apply case 2 of Proposition 16.3.1 with the choice $b = 2 - \delta_B$.*
   (b) *Then, we apply case 3 of Proposition 16.3.1 with the choice $b = 2$.*
4. *Finally, we control the fourth Bianchi pair in two steps:*
   (a) *First, we apply case 2 of Proposition 16.3.1 with the choice $b = -\delta_B$.*
   (b) *Then, we apply case 3 of Proposition 16.3.1 with the choice $b = 0$.*

16.5 Hodge systems for angular derivatives

Having outlined the procedure for deriving basic curvature estimates in ${}^{(ext)}\mathcal{M}$ in section 16.4, we focus in the rest of this chapter on higher order derivatives curvature estimates in order to prove Theorem 16.1.1 on the control of ${}^{(ext)}\mathfrak{R}_{J+1}$. To this end, we first commute in this section the Bianchi pairs with weighted angular derivatives.

Definition 16.5.1. *Let Λ_p the operator acting on $\mathfrak{s}_p$ and defined by*

$$\Lambda_p := r^2\Delta_p. \tag{16.5.1}$$

We define weighted angular derivatives $\not{\mathfrak{d}}^j$ as follows:

1. *If $f \in \mathfrak{s}_p$ is a complex p-tensor, $p = 1, 2$, and j is a positive integer, we define*

$$\not{\mathfrak{d}}^j f = \begin{cases} \Lambda_p^{\frac{j}{2}} f, & \text{if } j \text{ is even,} \\ r\,\not{\mathcal{D}}_p \Lambda_p^{\frac{j-1}{2}} f, & \text{if } j \text{ is odd.} \end{cases} \tag{16.5.2}$$

2. *If $f \in \mathfrak{s}_0$ is a complex scalar and j is a positive integer, we define*

$$\not{\mathfrak{d}}^j f = \begin{cases} \Lambda_0^{\frac{j}{2}} f, & \text{if } j \text{ is even,} \\ r\,\not{\mathcal{D}}_1^* \Lambda_0^{\frac{j-1}{2}} f, & \text{if } j \text{ is odd.} \end{cases} \tag{16.5.3}$$

Remark 16.5.2. *Note that if $f \in \mathfrak{s}_p$, for $p = 1, 2$ and j is even, then $\not{\mathfrak{d}}^j f \in \mathfrak{s}_p$, and if j is odd then $\not{\mathfrak{d}}^j f \in \mathfrak{s}_{p-1}$. Also, if $f \in \mathfrak{s}_0$ is a complex scalar and j is even, then $\not{\mathfrak{d}}^j f \in \mathfrak{s}_0$, and if j is odd then $\not{\mathfrak{d}}^j f \in \mathfrak{s}_1$.*

16.5.1 Commutation formulas with ${}^{(c)}\nabla_3$, ${}^{(c)}\nabla_4$

Lemma 16.5.3. *The following holds true for any $\psi \in \mathfrak{s}_p(\mathbb{C})$*

$$\begin{aligned} [{}^{(c)}\nabla_4, r\,\not{\mathcal{D}}_p]\psi &= O(r^{-2})\mathfrak{d}^{\leq 1}\psi + \Gamma_g \cdot \mathfrak{d}^{\leq 1}\psi, \\ [{}^{(c)}\nabla_3, r\,\not{\mathcal{D}}_p]\psi &= O(ar^{-1})\nabla_3\psi + O(r^{-2})\mathfrak{d}^{\leq 1}\psi + r\Gamma_b' \cdot \nabla_3\psi + \Gamma_b \cdot \mathfrak{d}^{\leq 1}\psi. \end{aligned}$$

Also, The following holds true for any $\psi \in \mathfrak{s}_{p-1}(\mathbb{C})$

$$\begin{aligned} [{}^{(c)}\nabla_4, r\,\not{\mathcal{D}}_p^*]\psi &= O(r^{-2})\mathfrak{d}^{\leq 1}\psi + \Gamma_g \cdot \mathfrak{d}^{\leq 1}\psi, \\ [{}^{(c)}\nabla_3, r\,\not{\mathcal{D}}_p^*]\psi &= O(ar^{-1})\nabla_3\psi + O(r^{-2})\mathfrak{d}^{\leq 1}\psi + r\Gamma_b' \cdot \nabla_3\psi + \Gamma_b \cdot \mathfrak{d}^{\leq 1}\psi. \end{aligned}$$

Finally, the following holds true for any $\psi\in\mathfrak{s}_p(\mathbb{C})$

$$\begin{aligned}
[\,{}^{(c)}\nabla_4, \Lambda_p]\psi &= O(r^{-2})\mathfrak{d}^{\leq 2}\psi+\mathfrak{d}^{\leq 1}\big(\Gamma_g\cdot\mathfrak{d}^{\leq 1}\psi\big),\\
[\,{}^{(c)}\nabla_3, r^2\Lambda_p]\psi &= O(ar^{-1})\mathfrak{d}\nabla_3\psi+O(r^{-2})\mathfrak{d}^{\leq 2}\psi+\mathfrak{d}^{\leq 1}\big(r\Gamma_b'\cdot\nabla_3\psi\big)+\mathfrak{d}^{\leq 1}\big(\Gamma_b\cdot\mathfrak{d}^{\leq 1}\psi\big),
\end{aligned}$$

where $\Lambda_p=r^2\Delta_p$.

Proof. Straightforward verification based on the commutation formulas obtained in Lemma 4.2.2, and using the fact that $\Xi\in r^{-1}\Gamma_g$ in Part III in view of (13.6.5) so that the error terms $r\Xi\cdot\nabla_3\psi$ in commutations with ${}^{(c)}\nabla_4$ satisfy $r\Xi\cdot\nabla_3\psi=\Gamma_g\cdot\mathfrak{d}\psi$. □

The following corollary follows from Lemma 16.5.3 and Definition 16.5.1.

Corollary 16.5.4. *We have, for* $j\geq 1$,

$$\begin{aligned}
[\,{}^{(c)}\nabla_4, \not{\mathfrak{d}}^j]\psi &= O(r^{-2})\mathfrak{d}^{\leq j}\psi+\mathfrak{d}^{\leq j-1}\big(\Gamma_g\cdot\mathfrak{d}^{\leq 1}\psi\big),\\
[\,{}^{(c)}\nabla_3, \not{\mathfrak{d}}^j]\psi &= O(ar^{-1})\mathfrak{d}^{\leq j-1}\nabla_3\psi+O(r^{-2})\mathfrak{d}^{\leq j}\psi+\mathfrak{d}^{\leq j-1}\big(r\Gamma_b'\cdot\nabla_3\psi\big)\\
&\quad+\mathfrak{d}^{\leq j-1}\big(\Gamma_b\cdot\mathfrak{d}^{\leq 1}\psi\big).
\end{aligned}$$

16.5.2 Commutation formulas with Hodge operators

We start with the following lemma.

Lemma 16.5.5. *The following identities hold true for complex anti selfadjoint tensors* $\psi\in\mathfrak{s}_2(\mathbb{C})$

$$\not{\mathcal{D}}_2\not{\mathcal{D}}_2^*\psi=\not{\mathcal{D}}_1^*\not{\mathcal{D}}_1\psi-\frac{4ai\cos\theta}{|q|^2}\not{\mathcal{L}}_{\mathbf{T}}\psi-\frac{2}{r^2}+O(ar^{-3})\mathfrak{d}^{\leq 1}\psi+r^{-1}\Gamma_b'\cdot\mathfrak{d}^{\leq 1}\psi.$$

Proof. In view of Lemma 15.3.3, we have

$$\not{\mathcal{D}}_2\not{\mathcal{D}}_2^*\psi=\not{\mathcal{D}}_1^*\not{\mathcal{D}}_1\psi-i(\,{}^{(a)}\mathrm{tr}\chi\nabla_3+\,{}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4)\psi-2\,{}^{(h)}K\psi.$$

On the other hand, in view of (13.1.1),

$${}^{(a)}\mathrm{tr}\chi e_3+\,{}^{(a)}\mathrm{tr}\underline{\chi}e_4+2(\eta+\underline{\eta})\cdot\,{}^*\nabla \;=\; \frac{4a\cos\theta}{|q|^2}\mathbf{T}+r^{-1}\Gamma_b'\cdot\mathfrak{d},$$

where we used in particular the fact that $\check{H} \in \Gamma'_b$, $\widetilde{\mathrm{tr} X} \in \Gamma'_g$, and the identification $\Gamma'_g = r^{-1}\Gamma'_b$ valid in Part III. Thus,

$$ {}^{(a)}\mathrm{tr}\chi e_3 + {}^{(a)}\mathrm{tr}\underline{\chi} e_4 = \frac{4a\cos\theta}{|q|^2}\mathbf{T} + O(ar^{-3})\,\mathfrak{d}\!\!/ + r^{-1}\Gamma'_b \cdot \mathfrak{d}. $$

Also, in view of (2.1.30), we have

$$ \begin{aligned} {}^{(h)}K &= -\frac{1}{4}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} - \frac{1}{4}{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} + \frac{1}{2}\widehat{\chi}\cdot\widehat{\underline{\chi}} - \rho \\ &= \frac{1}{r^2} + O(a^2r^{-4}) + r^{-1}\Gamma_g + \Gamma_b\cdot\Gamma_g. \end{aligned} $$

Hence

$$ \mathcal{D}\!\!\!/_2\,\mathcal{D}\!\!\!/_2^*\psi = \mathcal{D}\!\!\!/_1^*\,\mathcal{D}\!\!\!/_1\psi - \frac{4ai\cos\theta}{|q|^2}\mathcal{L}\!\!\!/_{\mathbf{T}}\psi - \frac{2}{r^2} + O(ar^{-3})\mathfrak{d}^{\leq 1}\psi + r^{-1}\Gamma'_b\cdot\mathfrak{d}^{\leq 1}\psi $$

as stated. □

Lemma 16.5.6. *The following commutation formulas hold true*

$$ \begin{aligned} -\mathcal{D}\!\!\!/_p\triangle_p + \triangle_{p-1}\mathcal{D}\!\!\!/_p &= O(ar^{-3})\,\mathfrak{d}\!\!/^{\leq 1}\mathcal{L}\!\!\!/_{\mathbf{T}} + O(r^{-4})\,\mathfrak{d}\!\!/^2 + O(r^{-3})\,\mathfrak{d}\!\!/^{\leq 1} + r^{-2}\mathfrak{d}^{\leq 1}\big(\Gamma'_b\cdot\mathfrak{d}\big), \\ -\mathcal{D}\!\!\!/_p^*\triangle_{p-1} + \triangle_{p-1}\mathcal{D}\!\!\!/_p^* &= O(ar^{-3})\,\mathfrak{d}\!\!/^{\leq 1}\mathcal{L}\!\!\!/_{\mathbf{T}} + O(r^{-4})\,\mathfrak{d}\!\!/^2 + O(r^{-3})\,\mathfrak{d}\!\!/^{\leq 1} + r^{-2}\mathfrak{d}^{\leq 1}\big(\Gamma'_b\cdot\mathfrak{d}\big). \end{aligned} $$

Proof. The proof can be derived in the same manner as the proof of Lemma 16.5.5. □

We deduce the following corollary.

Corollary 16.5.7. *We have, for $p = 0, 1, 2$, and for any positive integer k,*

$$ \begin{aligned} \Lambda_{p-1}^k\,\mathcal{D}\!\!\!/_p =& \mathcal{D}\!\!\!/_p\Lambda_p^k + O(ar^{-1})\,\mathfrak{d}\!\!/^{\leq 2k-1}\mathcal{L}\!\!\!/_{\mathbf{T}} + O(r^{-2})\,\mathfrak{d}\!\!/^{2k} + O(r^{-1})\,\mathfrak{d}\!\!/^{\leq 2k-1} \\ &+ \mathfrak{d}^{\leq 2k-1}\big(\Gamma'_b\cdot\mathfrak{d}\big), \\ \Lambda_p^k\,\mathcal{D}\!\!\!/_p^* =& \mathcal{D}\!\!\!/_p^*\Lambda_{p-1}^k + O(ar^{-1})\,\mathfrak{d}\!\!/^{\leq 2k-1}\mathcal{L}\!\!\!/_{\mathbf{T}} + O(r^{-2})\,\mathfrak{d}\!\!/^{2k} + O(r^{-1})\,\mathfrak{d}\!\!/^{\leq 2k-1} \\ &+ \mathfrak{d}^{\leq 2k-1}\big(\Gamma'_b\cdot\mathfrak{d}\big). \end{aligned} \tag{16.5.4} $$

Proof. According to Lemma 16.5.6 and the fact that $\nabla(r) \in r\Gamma_g$, we have

$$ \begin{aligned} (r^2\triangle_{p-1})\,\mathcal{D}\!\!\!/_p &= \mathcal{D}\!\!\!/_p(r^2\triangle_p) + O(ar^{-1})\,\mathfrak{d}\!\!/^{\leq 1}\mathcal{L}\!\!\!/_{\mathbf{T}} + O(r^{-2})\,\mathfrak{d}\!\!/^2 + O(r^{-1})\,\mathfrak{d}\!\!/^{\leq 1} + \mathfrak{d}^{\leq 1}\big(\Gamma'_b\cdot\mathfrak{d}\big), \\ (r^2\triangle_{p-1})\,\mathcal{D}\!\!\!/_p^* &= \mathcal{D}\!\!\!/_p^*(r^2\triangle_p) + O(ar^{-1})\,\mathfrak{d}\!\!/^{\leq 1}\mathcal{L}\!\!\!/_{\mathbf{T}} + O(r^{-2})\,\mathfrak{d}\!\!/^2 + O(r^{-1})\,\mathfrak{d}\!\!/^{\leq 1} + \mathfrak{d}^{\leq 1}\big(\Gamma'_b\cdot\mathfrak{d}\big), \end{aligned} $$

which corresponds to the case $k = 1$ since $\Lambda_p = r^2\Delta_p$. The general case can be proved recursively by taking also into account the commutation formula $[\mathcal{L}\!\!\!/_{\mathbf{T}}, \mathfrak{d}\!\!/] = \Gamma_b\cdot\mathfrak{d}$ and the fact that $\mathbf{T}(r) \in r\Gamma_b$. □

Lemma 16.5.8. *For an integer $j \geq 1$ and $\psi \in \mathfrak{s}_p$, $p = 0, 1, 2$, let $Err[\mathfrak{d}\mkern-9mu /^j, \psi]$ denote a schematic expression of the following type*

$$Err[\mathfrak{d}\mkern-9mu /^j, \psi] \;=\; O(ar^{-1})\,\mathfrak{d}\mkern-9mu /^{\leq j-1}\,\mathcal{L}\mkern-9mu /_{\mathbf{T}}\psi + O(r^{-2})\,\mathfrak{d}\mkern-9mu /^j\psi + O(r^{-1})\,\mathfrak{d}\mkern-9mu /^{\leq j-1}\psi + \mathfrak{d}^{\leq j-1}\big(\Gamma_b' \cdot \mathfrak{d}\psi\big).$$

Then, the following commutation identities hold true:

1. *If $\psi \in \mathfrak{s}_2$ and $j = 2k$, we have $\mathfrak{d}\mkern-9mu /^j\psi \in \mathfrak{s}_2$ and*
$$\mathfrak{d}\mkern-9mu /^j(\mathcal{D}\mkern-11mu /_2\psi) = \mathcal{D}\mkern-11mu /_2(\mathfrak{d}\mkern-9mu /^j\psi) + Err[\mathfrak{d}\mkern-9mu /^j, \psi].$$

2. *If $\psi \in \mathfrak{s}_2$ and $j = 2k+1$, we have $\mathfrak{d}\mkern-9mu /^j\psi \in \mathfrak{s}_1$ and*
$$\mathfrak{d}\mkern-9mu /^j(\mathcal{D}\mkern-11mu /_2\psi) \;=\; \mathcal{D}\mkern-11mu /_1(\mathfrak{d}\mkern-9mu /^j\psi) + Err[\mathfrak{d}\mkern-9mu /^j, \psi].$$

3. *If $\psi \in \mathfrak{s}_1$ and $j = 2k$, we have $\mathfrak{d}\mkern-9mu /^j\psi \in \mathfrak{s}_1$ and*
$$\mathfrak{d}\mkern-9mu /^j(\mathcal{D}\mkern-11mu /_1\psi) = \mathcal{D}\mkern-11mu /_1(\mathfrak{d}\mkern-9mu /^j\psi) + Err[\mathfrak{d}\mkern-9mu /^j, \psi],$$
$$\mathfrak{d}\mkern-9mu /^j(\mathcal{D}\mkern-11mu /_2^*\psi) = \mathcal{D}\mkern-11mu /_2^*(\mathfrak{d}\mkern-9mu /^j\psi) + Err[\mathfrak{d}\mkern-9mu /^j, \psi].$$

4. *If $\psi \in \mathfrak{s}_1$ and $j = 2k+1$, we have $\mathfrak{d}\mkern-9mu /^j\psi \in \mathfrak{s}_0$ and*
$$\mathfrak{d}\mkern-9mu /^j(\mathcal{D}\mkern-11mu /_1\psi) = \mathcal{D}\mkern-11mu /_1^*(\mathfrak{d}\mkern-9mu /^j\psi) + Err[\mathfrak{d}\mkern-9mu /^j, \psi],$$
$$\mathfrak{d}\mkern-9mu /^j(\mathcal{D}\mkern-11mu /_2^*\psi) = \mathcal{D}\mkern-11mu /_1^*(\mathfrak{d}\mkern-9mu /^j\psi) + Err[\mathfrak{d}\mkern-9mu /^j, \psi].$$

5. *If $\psi \in \mathfrak{s}_0$ and $j = 2k$, we have $\mathfrak{d}\mkern-9mu /^j\psi \in \mathfrak{s}_0$ and*
$$\mathfrak{d}\mkern-9mu /^j(\mathcal{D}\mkern-11mu /_1^*\psi) = \mathcal{D}\mkern-11mu /_1^*(\mathfrak{d}\mkern-9mu /^j\psi) + Err[\mathfrak{d}\mkern-9mu /^j, \psi].$$

6. *If $\psi \in \mathfrak{s}_0$ and $j = 2k+1$, we have $\mathfrak{d}\mkern-9mu /^j\psi \in \mathfrak{s}_1$ and*
$$\mathfrak{d}\mkern-9mu /^j(\mathcal{D}\mkern-11mu /_1^*\psi) \;=\; \mathcal{D}\mkern-11mu /_1(\mathfrak{d}\mkern-9mu /^j\psi) + Err[\mathfrak{d}\mkern-9mu /^j, \psi].$$

Proof. The case of even j is an immediate consequence of Corollary 16.5.7. The case $j = 2k+1$ can be checked as follows.

Case 1. Let $\psi \in \mathfrak{s}_p(\mathbb{C})$ with $p = 2$ and consider the expression $\mathfrak{d}\mkern-9mu /^j(\mathcal{D}\mkern-11mu /_2\psi) = r\,\mathcal{D}\mkern-11mu /_1\Lambda_{p-1}^k(\mathcal{D}\mkern-11mu /_2\psi)$ with $j = 2k+1$. Commuting with the help of Corollary 16.5.7, we find

$$\begin{aligned}
\mathfrak{d}\mkern-9mu /^j(\mathcal{D}\mkern-11mu /_2\psi) &= r\,\mathcal{D}\mkern-11mu /_1\Lambda_{p-1}^k\,\mathcal{D}\mkern-11mu /_2\psi\\
&= r\,\mathcal{D}\mkern-11mu /_1\Big(\mathcal{D}\mkern-11mu /_2\Lambda_p^k\psi + O(ar^{-1})\,\mathfrak{d}\mkern-9mu /^{\leq 2k-1}\,\mathcal{L}\mkern-9mu /_{\mathbf{T}}\psi + O(r^{-2})\mathfrak{d}^{2k}\psi + O(r^{-1})\mathfrak{d}^{\leq 2k-1}\psi\\
&\qquad + \mathfrak{d}^{\leq 2k-1}\big(\Gamma_b' \cdot \mathfrak{d}\psi\big)\Big)\\
&= r\,\mathcal{D}\mkern-11mu /_1(r^{-1}\,\mathfrak{d}\mkern-9mu /^j\psi) + \mathrm{Err}[\mathfrak{d}\mkern-9mu /^j, \psi]\\
&= \mathcal{D}\mkern-11mu /_1\,\mathfrak{d}\mkern-9mu /^j\psi + \mathrm{Err}[\mathfrak{d}\mkern-9mu /^j, \psi]
\end{aligned}$$

as stated, where we used the fact that $\nabla(r) \in r\Gamma_g$.

Case 2. If $\psi \in \mathfrak{s}_p(\mathbb{C})$ with $p = 1$, we derive, using Corollary 16.5.7 and Lemma 16.5.5,

$$\begin{aligned} \not{\mathfrak{d}}^j(\,\not{\mathcal{D}}_2^*\psi) &= (r\,\not{\mathcal{D}}_2)\Lambda^k_{p+1}\,\not{\mathcal{D}}_2^*\psi = r\,\not{\mathcal{D}}_2\,\not{\mathcal{D}}_2^*\Lambda^k_p\psi + \mathrm{Err}[\,\not{\mathfrak{d}}^j,\psi] \\ &= r\,\not{\mathcal{D}}_1^*\not{\mathcal{D}}_1\Lambda^k_p\psi + \mathrm{Err}[\,\not{\mathfrak{d}}^j,\psi] = \not{\mathcal{D}}_1^*\big(r\,\not{\mathcal{D}}_1\Lambda^k_p\psi\big) + \mathrm{Err}[\,\not{\mathfrak{d}}^j,\psi] \\ &= \not{\mathcal{D}}_1^*(\,\not{\mathfrak{d}}^j\psi) + \mathrm{Err}[\,\not{\mathfrak{d}}^j,\psi] \end{aligned}$$

as stated. Also, using again Corollary 16.5.7,

$$\begin{aligned} \not{\mathfrak{d}}^j(\,\not{\mathcal{D}}_1\,\psi) &= r\,\not{\mathcal{D}}_1^*\Lambda^k_0(\,\not{\mathcal{D}}_1\,\psi) = r\,\not{\mathcal{D}}_1^*(\,\not{\mathcal{D}}_1\,\Lambda^k_1\psi) + \mathrm{Err}[\,\not{\mathfrak{d}}^j,\psi] \\ &= \not{\mathcal{D}}_1^*(r\,\not{\mathcal{D}}_1\,\Lambda^k_1\psi) + \mathrm{Err}[\,\not{\mathfrak{d}}^j,\psi] = \not{\mathcal{D}}_1^*(\,\not{\mathfrak{d}}^j\psi) + \mathrm{Err}[\,\not{\mathfrak{d}}^j,\psi] \end{aligned}$$

as stated.

Case 3. If $\psi \in \mathfrak{s}_0(\mathbb{C})$, we derive, using again Corollary 16.5.7,

$$\begin{aligned} \not{\mathfrak{d}}^j(\,\not{\mathcal{D}}_1^*\psi) &= r\,\not{\mathcal{D}}_1\Lambda^k_1\,\not{\mathcal{D}}_1^*\psi = r\,\not{\mathcal{D}}_1(\,\not{\mathcal{D}}_1^*\Lambda^k_0\psi) + \mathrm{Err}[\,\not{\mathfrak{d}}^j,\psi] \\ &= \not{\mathcal{D}}_1(r\,\not{\mathcal{D}}_1^*\Lambda^k_0\psi) + \mathrm{Err}[\,\not{\mathfrak{d}}^j,\psi] = \not{\mathcal{D}}_1(\,\not{\mathfrak{d}}^j\psi) + \mathrm{Err}[\,\not{\mathfrak{d}}^j,\psi] \end{aligned}$$

as stated. This concludes the proof of Lemma 16.5.8. □

16.5.3 Bianchi pairs for higher angular derivatives

We apply the results obtained in section 16.5.2 to the main Bianchi pairs verified by the curvature components $A, B, \check{P}, \underline{B}, \underline{A}$.

Proposition 16.5.9. *The following assertions hold true for*[5] $j \geq 1$*:*

First Bianchi Pair. *Consider the first Bianchi pair* (15.3.4) *verified by A, B and define the new quantities $\widetilde{A} = \not{\mathfrak{d}}^j A$, $\widetilde{B} = \not{\mathfrak{d}}^j B$.*

- *If j is even, then $\widetilde{A} \in \mathfrak{s}_2$, $\widetilde{B} \in \mathfrak{s}_1$, and we have*

$$\begin{aligned} {}^{(c)}\nabla_3\widetilde{A} + \frac{1}{2}tr\underline{X}\widetilde{A} &= -\,\not{\mathcal{D}}_2^*\widetilde{B} + O(ar^{-1})\,\not{\mathfrak{d}}^{j-1}\not{\mathcal{L}}_{\mathbf{T}}B + \widetilde{F}_{(1)}, \\ {}^{(c)}\nabla_4\widetilde{B} + 2\overline{trX}\widetilde{B} &= \not{\mathcal{D}}_2\,\widetilde{A} + O(ar^{-1})\,\not{\mathfrak{d}}^{j-1}\not{\mathcal{L}}_{\mathbf{T}}A + \widetilde{F}_{(2)}. \end{aligned}$$

  Note that this can be written in the form of the generalized Bianchi pair (16.2.3) *with $\Psi_{(1)} = \widetilde{A}, \Psi_{(2)} = \widetilde{B}$.*

[5]Note that for $j = 0$ the terms $O(ar^{-1})\,\not{\mathfrak{d}}^{j-1}\not{\mathcal{L}}_{\mathbf{T}}$ are not present. By convention we set $\not{\mathfrak{d}}^{-1} = 0$.

- *If j is odd, $\widetilde{A}\in\mathfrak{s}_1$, $\widetilde{B}\in\mathfrak{s}_0$, and we have*

$$
\begin{aligned}
{}^{(c)}\nabla_3\widetilde{A}+\frac{1}{2}tr\underline{X}\widetilde{A} &= -\not{\mathcal{D}}_1^*\widetilde{B}+O(ar^{-1})\,\not{\mathfrak{d}}^{j-1}\not{\mathcal{L}}_{\mathbf{T}}B+\ \widetilde{F}_{(1)},\\
{}^{(c)}\nabla_4\widetilde{B}+2\overline{trX}\widetilde{B} &= \not{\mathcal{D}}_1\widetilde{A}+O(ar^{-1})\,\not{\mathfrak{d}}^{j-1}\not{\mathcal{L}}_{\mathbf{T}}A+\widetilde{F}_{(2)}.
\end{aligned}
$$

Note that this can also be written in the form of the generalized Bianchi pair (16.2.3) *with $\Psi_{(1)}=\widetilde{A},\Psi_{(2)}=\widetilde{B}$.*

In both cases we have

$$
\begin{aligned}
\widetilde{F}_{(1)} &= O(r^{-3})\mathfrak{d}^{\le j}\Gamma_g+O(r^{-2})\mathfrak{d}^{\le j}\big(A,B\big)+O(r^{-1})\mathfrak{d}^{\le j-1}B+\mathfrak{d}^{\le j}\big(\Gamma_b\cdot(A,B)\big),\\
\widetilde{F}_{(2)} &= O(r^{-3})\mathfrak{d}^{\le j}\Gamma_g+O(r^{-2})\mathfrak{d}^{\le j}\big(A,B\big)+O(r^{-1})\mathfrak{d}^{\le j-1}A+\mathfrak{d}^{\le j}\big(\Gamma_b\cdot(A,B)\big).
\end{aligned}
$$

Second Bianchi Pair. *Consider the second Bianchi pair* (16.2.1) *verified by $B,\check{P}$, and define the new quantities*[6] $\widetilde{B}=\not{\mathfrak{d}}^jB$, $\widetilde{P}=\not{\mathfrak{d}}^j\overline{\check{P}}$.

- *If j is even, $\widetilde{B}\in\mathfrak{s}_1$, $\widetilde{P}\in\mathfrak{s}_0$, and we have*

$$
\begin{aligned}
{}^{(c)}\nabla_3\widetilde{B}+tr\underline{X}\widetilde{B} &= -\not{\mathcal{D}}_1^*\widetilde{P}+O(ar^{-1})\,\not{\mathfrak{d}}^{j-1}\not{\mathcal{L}}_{\mathbf{T}}\check{P}+\widetilde{F}_{(1)},\\
{}^{(c)}\nabla_4\widetilde{P}+\frac{3}{2}\overline{trX}\widetilde{P} &= \not{\mathcal{D}}_1\widetilde{B}+O(ar^{-1})\,\not{\mathfrak{d}}^{j-1}\not{\mathcal{L}}_{\mathbf{T}}B+\widetilde{F}_{(2)}.
\end{aligned}
$$

Note that this can be written in the form of the generalized Bianchi pair (16.2.3) *with $\Psi_{(1)}=\widetilde{B},\Psi_{(2)}=\widetilde{P}$.*

- *If j is odd, $\widetilde{B}\in\mathfrak{s}_0$, $\widetilde{P}\in\mathfrak{s}_1$, and we have*

$$
\begin{aligned}
{}^{(c)}\nabla_3\widetilde{B}+\overline{tr\underline{X}}\widetilde{B} &= -\not{\mathcal{D}}_1\widetilde{P}+O(ar^{-1})\,\not{\mathfrak{d}}^{j-1}\not{\mathcal{L}}_{\mathbf{T}}\check{P}+\ \widetilde{F}_{(1)},\\
{}^{(c)}\nabla_4\widetilde{P}+\frac{3}{2}trX\widetilde{P} &= \not{\mathcal{D}}_1^*\widetilde{B}+O(ar^{-1})\,\not{\mathfrak{d}}^{j-1}\not{\mathcal{L}}_{\mathbf{T}}B+\widetilde{F}_{(2)}.
\end{aligned}
$$

Note that this can be written in the form of the generalized Bianchi pair (16.2.4) *with $\Psi_{(1)}=\widetilde{B},\Psi_{(2)}=\widetilde{P}$.*

In both cases we have

$$
\begin{aligned}
\widetilde{F}_{(1)} &= O(r^{-3})\mathfrak{d}^{\le j}\Gamma_b'+O(r^{-2})\mathfrak{d}^{\le j}\big(B,\check{P}\big)+O(r^{-1})\mathfrak{d}^{\le j-1}\check{P}+r^{-2}\mathfrak{d}^{\le j}(\Gamma_b'\cdot\check{R}_b)\\
&\quad+\mathfrak{d}^{\le j}\big(\Gamma_b\cdot(A,B)\big),\\
\widetilde{F}_{(2)} &= O(r^{-4})\mathfrak{d}^{\le j}\Gamma_b+O(r^{-2})\mathfrak{d}^{\le j}\big(B,\check{P}\big)+O(r^{-1})\mathfrak{d}^{\le j-1}B+\mathfrak{d}^{\le j}\big(\Gamma_b'\cdot(A,B)\big)\\
&\quad+r^{-1}\mathfrak{d}^{\le j}(\Gamma_b\cdot\check{P})+r^{-1}\mathfrak{d}^{\le j}(\Xi\cdot\check{R}_b).
\end{aligned}
$$

[6]See remark 15.3.6 for the definition of $\widetilde{P}$ here.

Third Bianchi Pair. *Consider the third Bianchi pair* (16.2.2) *verified by* $\check{P}, \underline{B}$, *and define the new quantities* $\widetilde{P} = \not{\mathfrak{d}}^j \check{P}$, $\widetilde{\underline{B}} = \not{\mathfrak{d}}^j \underline{B}$.

- *If j is even,* $\widetilde{P} \in \mathfrak{s}_0$, $\widetilde{\underline{B}} \in \mathfrak{s}_1$, *and we have*

$$\begin{aligned} {}^{(c)}\nabla_3 \widetilde{P} + \frac{3}{2}\overline{tr\underline{X}}\,\widetilde{P} &= -\not{\mathcal{D}}_1\,\widetilde{\underline{B}} + O(ar^{-1})\,\not{\mathfrak{d}}^{j-1}\not{\mathcal{L}}_{\mathbf{T}}\underline{B} + \widetilde{F}_{(1)},\\ {}^{(c)}\nabla_4 \widetilde{\underline{B}} + trX\,\widetilde{\underline{B}} &= \not{\mathcal{D}}_1^*\widetilde{P} + O(ar^{-1})\,\not{\mathfrak{d}}^{j-1}\not{\mathcal{L}}_{\mathbf{T}}\check{P} + \widetilde{F}_{(2)}. \end{aligned}$$

  Note that this can be written in the form of the generalized Bianchi pair (16.2.4) *with* $\Psi_{(1)} = \widetilde{P}, \Psi_{(2)} = \widetilde{\underline{B}}$.

- *If j is odd,* $\widetilde{P} \in \mathfrak{s}_1$, $\widetilde{\underline{B}} \in \mathfrak{s}_0$, *and we have*

$$\begin{aligned} {}^{(c)}\nabla_3 \widetilde{P} + \frac{3}{2}tr\underline{X}\widetilde{P} &= -\not{\mathcal{D}}_1^*\widetilde{\underline{B}} + O(ar^{-1})\,\not{\mathfrak{d}}^{j-1}\not{\mathcal{L}}_{\mathbf{T}}\underline{B} + \ \widetilde{F}_{(1)},\\ {}^{(c)}\nabla_4 \widetilde{\underline{B}} + \overline{trX\widetilde{\underline{B}}} &= \not{\mathcal{D}}_1\,\widetilde{P} + O(ar^{-1})\,\not{\mathfrak{d}}^{j-1}\not{\mathcal{L}}_{\mathbf{T}}\check{P} + \widetilde{F}_{(2)}. \end{aligned}$$

  Note that this can be written in the form of the generalized Bianchi pair (16.2.3) *with* $\Psi_{(1)} = \widetilde{P}, \Psi_{(2)} = \widetilde{\underline{B}}$.

In both cases we have

$$\begin{aligned} \widetilde{F}_{(1)} &= O(r^{-3})\mathfrak{d}^{\leq j}\Gamma_b + O(r^{-2})\mathfrak{d}^{\leq j}\big(\check{P}, \underline{B}\big) + O(r^{-1})\mathfrak{d}^{\leq j-1}\underline{B} + r^{-1}\mathfrak{d}^{\leq j}\big(\Gamma_b' \cdot \check{R}_b\big),\\ \widetilde{F}_{(2)} &= O(r^{-3})\mathfrak{d}^{\leq j}\Gamma_b + O(r^{-2})\mathfrak{d}^{\leq j}\big(\check{P}, \underline{B}\big) + O(r^{-1})\mathfrak{d}^{\leq j-1}\check{P} + r^{-2}\mathfrak{d}^{\leq j}\big(\Gamma_b \cdot \check{R}_b\big)\\ &\quad +\mathfrak{d}^{\leq j}\big(\Xi \cdot \check{R}_b\big). \end{aligned}$$

Fourth Bianchi Pair. *Consider the fourth Bianchi pair* (15.3.7) *verified by* $\underline{B}, \underline{A}$, *and define the new quantities* $\widetilde{\underline{B}} = \not{\mathfrak{d}}^j \underline{B}$, $\widetilde{\underline{A}} = \not{\mathfrak{d}}^j \underline{A}$.

- *If j is even,* $\widetilde{\underline{B}} \in \mathfrak{s}_1$, $\widetilde{\underline{A}} \in \mathfrak{s}_2$, *and we have*

$$\begin{aligned} {}^{(c)}\nabla_3 \widetilde{\underline{B}} + 2\overline{tr\underline{X}}\,\widetilde{\underline{B}} &= -\not{\mathcal{D}}_2\widetilde{\underline{A}} + O(ar^{-1})\,\not{\mathfrak{d}}^{j-1}\not{\mathcal{L}}_{\mathbf{T}}\underline{A} + F_{(1)},\\ {}^{(c)}\nabla_4 \widetilde{\underline{A}} + \frac{1}{2}trX\widetilde{\underline{A}} &= \not{\mathcal{D}}_2^*\widetilde{\underline{B}} + O(ar^{-1})\,\not{\mathfrak{d}}^{j-1}\not{\mathcal{L}}_{\mathbf{T}}\underline{B} + F_{(2)}. \end{aligned}$$

  Note that this can be written in the form of the generalized Bianchi pair (16.2.4) *with* $\Psi_{(1)} = \widetilde{\underline{B}}, \Psi_{(2)} = \widetilde{\underline{A}}$.

- *If j is even, $\widetilde{\underline{B}} \in \mathfrak{s}_0$, $\widetilde{\underline{A}} \in \mathfrak{s}_1$, and we have*

$$\begin{aligned}
{}^{(c)}\nabla_3\widetilde{\underline{B}} + 2\overline{tr\underline{X}}\,\widetilde{\underline{B}} &= -\,\mathcal{D}_1\widetilde{\underline{A}} + O(ar^{-1})\,\mathfrak{d}\!\!/^{j-1}\mathcal{L}\!\!\!/_{\mathbf{T}}\underline{A} + F_{(1)},\\
{}^{(c)}\nabla_4\widetilde{\underline{A}} + \frac{1}{2}trX\widetilde{\underline{A}} &= \mathcal{D}_1^*\widetilde{\underline{B}} + O(ar^{-1})\,\mathfrak{d}\!\!/^{j-1}\mathcal{L}\!\!\!/_{\mathbf{T}}\underline{B} + F_{(2)}.
\end{aligned}$$

Note that this can also be written in the form of the generalized Bianchi pair (16.2.4) *with $\Psi_{(1)} = \widetilde{\underline{B}}, \Psi_{(2)} = \widetilde{\underline{A}}$.*

In both cases

$$\begin{aligned}
\widetilde{F}_{(1)} &= O(r^{-3})\mathfrak{d}^{\leq j}\Gamma_b + O(r^{-2})\mathfrak{d}^{\leq j}\big(\underline{B}, \underline{A}\big) + O(r^{-1})\mathfrak{d}^{\leq j-1}\underline{B} + \mathfrak{d}^{\leq j}(\Gamma_b' \cdot \check{R}_b),\\
\widetilde{F}_{(2)} &= O(r^{-3})\mathfrak{d}^{\leq j}\Gamma_b + O(r^{-2})\mathfrak{d}^{\leq j}\big(\underline{B}, \underline{A}\big) + O(r^{-1})\mathfrak{d}^{\leq j-1}\underline{B} + r^{-1}\mathfrak{d}^{\leq j}(\Gamma_b \cdot \check{R}_b).
\end{aligned}$$

Proof. We start with the first and the second Bianchi pairs, which, in view of Proposition 16.2.4, are of the general form (16.2.3), i.e.

$$\begin{aligned}
{}^{(c)}\nabla_3(\Psi_{(1)}) + c_{(1)}\mathrm{tr}\underline{X}\Psi_{(1)} &= -\,\mathcal{D}_p^*\Psi_{(2)} + F_{(1)},\\
{}^{(c)}\nabla_4(\Psi_{(2)}) + c_{(2)}\overline{\mathrm{tr}X}\Psi_{(2)} &= \mathcal{D}_p\,\Psi_{(1)} + F_{(2)}.
\end{aligned}$$

We consider the case j even and commute with $\mathfrak{d}\!\!/^j$. We obtain

$$\begin{aligned}
{}^{(c)}\nabla_3(\mathfrak{d}\!\!/^j\Psi_{(1)}) + c_{(1)}\mathrm{tr}\underline{X}\,\mathfrak{d}\!\!/^j\Psi_{(1)} &= -\,\mathcal{D}_p^*\,\mathfrak{d}\!\!/^j\Psi_{(2)} + \widetilde{F}'_{(1)},\\
{}^{(c)}\nabla_4(\mathfrak{d}\!\!/^j\Psi_{(2)}) + c_{(2)}\overline{\mathrm{tr}X}\,\mathfrak{d}\!\!/^j\Psi_{(2)} &= \mathcal{D}_p\,\mathfrak{d}\!\!/^j\Psi_{(1)} + \widetilde{F}'_{(2)},
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{F}'_{(1)} &= \mathfrak{d}\!\!/^jF_{(1)} + [{}^{(c)}\nabla_3, \mathfrak{d}\!\!/^j]\Psi_{(1)} - [\mathfrak{d}\!\!/^j, \mathcal{D}_p^*]\Psi_{(2)} + O(r^{-2})\,\mathfrak{d}\!\!/^{\leq j-1}\Psi_{(1)} + \mathfrak{d}\!\!/^{\leq j}(\Gamma_g \cdot \Psi_{(1)}),\\
\widetilde{F}'_{(2)} &= \mathfrak{d}\!\!/^jF_{(2)} + [{}^{(c)}\nabla_4, \mathfrak{d}\!\!/^j]\Psi_{(2)} - [\mathfrak{d}\!\!/^j, \mathcal{D}_p]\Psi_{(1)} + O(r^{-2})\,\mathfrak{d}\!\!/^{\leq j-1}\Psi_{(2)} + \mathfrak{d}\!\!/^{\leq j}(\Gamma_g \cdot \Psi_{(2)}).
\end{aligned}$$

Using Corollary 16.5.4 and Lemma 16.5.8, we infer

$$\begin{aligned}
\widetilde{F}'_{(1)} &= \mathfrak{d}\!\!/^jF_{(1)} + O(ar^{-1})\mathfrak{d}^{\leq j-1}\nabla_3\Psi_{(1)} + O(r^{-2})\mathfrak{d}^{\leq j}\Psi_{(1)} + \mathfrak{d}^{\leq j-1}\big(r\Gamma_b' \cdot \nabla_3\Psi_{(1)}\big)\\
&\quad+\mathfrak{d}^{\leq j-1}\big(\Gamma_b \cdot \mathfrak{d}^{\leq 1}\Psi_{(1)}\big) + O(ar^{-1})\,\mathfrak{d}\!\!/^{\leq j-1}\mathcal{L}\!\!\!/_{\mathbf{T}}\Psi_{(2)} + O(r^{-2})\,\mathfrak{d}\!\!/^j\Psi_{(2)}\\
&\quad+O(r^{-1})\,\mathfrak{d}\!\!/^{\leq j-1}\Psi_{(2)} + \mathfrak{d}^{\leq j-1}\big(\Gamma_b' \cdot \mathfrak{d}\Psi_{(2)}\big) + O(r^{-2})\,\mathfrak{d}\!\!/^{\leq j-1}\Psi_{(1)} + \mathfrak{d}\!\!/^{\leq j}(\Gamma_g \cdot \Psi_{(1)}),\\
\widetilde{F}'_{(2)} &= \mathfrak{d}\!\!/^jF_{(2)} + O(r^{-2})\mathfrak{d}^{\leq j}\Psi_{(2)} + \mathfrak{d}^{\leq j-1}\big(\Gamma_g \cdot \mathfrak{d}^{\leq 1}\Psi_{(2)}\big) + O(ar^{-1})\,\mathfrak{d}\!\!/^{\leq j-1}\mathcal{L}\!\!\!/_{\mathbf{T}}\Psi_{(1)}\\
&\quad+O(r^{-2})\,\mathfrak{d}\!\!/^j\Psi_{(1)} + O(r^{-1})\,\mathfrak{d}\!\!/^{\leq j-1}\Psi_{(1)} + \mathfrak{d}^{\leq j-1}\big(\Gamma_b' \cdot \mathfrak{d}\Psi_{(1)}\big) + O(r^{-2})\,\mathfrak{d}\!\!/^{\leq j-1}\Psi_{(2)}\\
&\quad+\mathfrak{d}\!\!/^{\leq j}(\Gamma_g \cdot \Psi_{(2)}).
\end{aligned}$$

We obtain

$$
\begin{aligned}
{}^{(c)}\nabla_3(\not{\mathfrak{d}}^j\Psi_{(1)}) + c_{(1)}\mathrm{tr}\underline{X}\,\not{\mathfrak{d}}^j\Psi_{(1)} &= -\,\not{\mathcal{D}}_p^*\not{\mathfrak{d}}^j\Psi_{(2)} + O(ar^{-1})\,\not{\mathfrak{d}}^{\leq j-1}\not{\mathcal{L}}_{\mathbf{T}}\Psi_{(2)} + \widetilde{F}_{(1)},\\
{}^{(c)}\nabla_4(\not{\mathfrak{d}}^j\Psi_{(2)}) + c_{(2)}\overline{\mathrm{tr}X}\,\not{\mathfrak{d}}^j\Psi_{(2)} &= \not{\mathcal{D}}_p\,\not{\mathfrak{d}}^j\Psi_{(1)} + O(ar^{-1})\,\not{\mathfrak{d}}^{\leq j-1}\not{\mathcal{L}}_{\mathbf{T}}\Psi_{(1)} + \widetilde{F}_{(2)},
\end{aligned}
$$

where

$$
\begin{aligned}
\widetilde{F}_{(1)} &= \not{\mathfrak{d}}^jF_{(1)} + O(ar^{-1})\mathfrak{d}^{\leq j-1}\nabla_3\Psi_{(1)} + O(r^{-2})\mathfrak{d}^{\leq j}\big(\Psi_{(1)},\Psi_{(2)}\big) + O(r^{-1})\mathfrak{d}^{\leq j-1}\Psi_{(2)}\\
&\quad +\mathfrak{d}^{\leq j-1}\big(r\Gamma_b'\cdot\nabla_3\Psi_{(1)}\big) + \mathfrak{d}^{\leq j}\big(\Gamma_b\cdot\Psi_{(1)}\big) + \mathfrak{d}^{\leq j}\big(\Gamma_b'\cdot\Psi_{(2)}\big),\\
\widetilde{F}_{(2)} &= \not{\mathfrak{d}}^jF_{(2)} + O(r^{-2})\mathfrak{d}^{\leq j}\big(\Psi_{(1)},\Psi_{(2)}\big) + O(r^{-1})\mathfrak{d}^{\leq j-1}\Psi_{(1)} + \mathfrak{d}^{\leq j}\big(\Gamma_b'\cdot\Psi_{(1)}\big)\\
&\quad +r^{-1}\mathfrak{d}^{\leq j}(\Gamma_b\cdot\Psi_{(2)}).
\end{aligned}
$$

In view of ${}^{(c)}\nabla_3\Psi_{(1)} = -c_{(1)}\mathrm{tr}\underline{X}\Psi_{(1)} - \not{\mathcal{D}}_p^*\Psi_{(2)} + F_{(1)}$, we have

$$
\nabla_3\Psi_{(1)} = O(r^{-1})\Psi_{(1)} + O(r^{-1})\mathfrak{d}^{\leq 1}\Psi_{(2)} + \Gamma_b\cdot\Psi_{(1)} + F_{(1)}
$$

which we use to remove the $\nabla_3\Psi_{(1)}$ terms from $\widetilde{F}_{(1)}$. This yields

$$
\begin{aligned}
\widetilde{F}_{(1)} &= \not{\mathfrak{d}}^jF_{(1)} + O(ar^{-1})\mathfrak{d}^{\leq j-1}F_{(1)} + O(r^{-2})\mathfrak{d}^{\leq j}\big(\Psi_{(1)},\Psi_{(2)}\big) + O(r^{-1})\mathfrak{d}^{\leq j-1}\Psi_{(2)}\\
&\quad +\mathfrak{d}^{\leq j-1}\big(r\Gamma_b\cdot F_{(1)}\big) + \mathfrak{d}^{\leq j}\big(\Gamma_b\cdot\Psi_{(1)}\big) + \mathfrak{d}^{\leq j}(\Gamma_b'\cdot\Psi_{(2)}),\\
\widetilde{F}_{(2)} &= \not{\mathfrak{d}}^jF_{(2)} + O(r^{-2})\mathfrak{d}^{\leq j}\big(\Psi_{(1)},\Psi_{(2)}\big) + O(r^{-1})\mathfrak{d}^{\leq j-1}\Psi_{(1)} + \mathfrak{d}^{\leq j}\big(\Gamma_b'\cdot\Psi_{(1)}\big)\\
&\quad +r^{-1}\mathfrak{d}^{\leq j}(\Gamma_b\cdot\Psi_{(2)}).
\end{aligned}
$$

We proceed in the same manner for j odd, and for the other Bianchi pairs, and we obtain in all cases that the Bianchi equations have the stated structure, with $\widetilde{F}_{(1)}$, $\widetilde{F}_{(2)}$ given as above by

$$
\begin{aligned}
\widetilde{F}_{(1)} &= \not{\mathfrak{d}}^jF_{(1)} + O(ar^{-1})\mathfrak{d}^{\leq j-1}F_{(1)} + O(r^{-2})\mathfrak{d}^{\leq j}\big(\Psi_{(1)},\Psi_{(2)}\big) + O(r^{-1})\mathfrak{d}^{\leq j-1}\Psi_{(2)}\\
&\quad +\mathfrak{d}^{\leq j-1}\big(r\Gamma_b\cdot F_{(1)}\big) + \mathfrak{d}^{\leq j}\big(\Gamma_b\cdot\Psi_{(1)}\big) + \mathfrak{d}^{\leq j}(\Gamma_b'\cdot\Psi_{(2)}),\\
\widetilde{F}_{(2)} &= \not{\mathfrak{d}}^jF_{(2)} + O(r^{-2})\mathfrak{d}^{\leq j}\big(\Psi_{(1)},\Psi_{(2)}\big) + O(r^{-1})\mathfrak{d}^{\leq j-1}\Psi_{(1)} + \mathfrak{d}^{\leq j}\big(\Gamma_b'\cdot\Psi_{(1)}\big)\\
&\quad +r^{-1}\mathfrak{d}^{\leq j}(\Gamma_b\cdot\Psi_{(2)}).
\end{aligned}
$$

Remark 16.5.10. *In the case where j is odd, for the second and third Bianchi pairs, when implementing the above procedure, we need in addition to use the fact that*

$$
\begin{aligned}
\overline{trX} &= trX + O(r^{-2}) + \Gamma_g = trX + O(r^{-2}) + r^{-1}\Gamma_b,\\
\overline{tr\underline{X}} &= tr\underline{X} + O(r^{-2}) + \Gamma_g = tr\underline{X} + O(r^{-2}) + r^{-1}\Gamma_b,
\end{aligned}
$$

in order to put the LHS in the general form of Definition 16.2.3. Note that this generates extra terms of the type $O(r^{-2})\mathfrak{d}^{\leq j}(\Psi_{(1)},\Psi_{(2)})$ and $r^{-1}\mathfrak{d}^{\leq j}\big(\Gamma_b\cdot(\Psi_{(1)},\Psi_{(2)})\big)$ which are incorporated in $\widetilde{F}_{(1)}$, $\widetilde{F}_{(2)}$.

To conclude, it remains to plug the structure of $F_{(1)}$, $F_{(2)}$ in the above formula for $\widetilde{F}_{(1)}$ and $\widetilde{F}_{(2)}$. We do this for all Bianchi pairs starting with the first one for which we have, according to Proposition 16.2.4,

$$\begin{aligned} F_{(1)} &= O(ar^{-2})B + O(r^{-3})\Gamma_g + \Gamma_b \cdot B, \\ F_{(2)} &= O(ar^{-2})A + O(r^{-3})\Gamma_g + \Gamma_b \cdot A. \end{aligned}$$

In this case, we infer, using also $\Psi_{(1)} = A$, $\Psi_{(2)} = B$, we obtain

$$\begin{aligned} \widetilde{F}_{(1)} &= O(r^{-3})\mathfrak{d}^{\leq j}\Gamma_g + O(r^{-2})\mathfrak{d}^{\leq j}\big(A, B\big) + O(r^{-1})\mathfrak{d}^{\leq j-1}B + \mathfrak{d}^{\leq j}\big(\Gamma_b \cdot (A, B)\big), \\ \widetilde{F}_{(2)} &= O(r^{-3})\mathfrak{d}^{\leq j}\Gamma_g + O(r^{-2})\mathfrak{d}^{\leq j}\big(A, B\big) + O(r^{-1})\mathfrak{d}^{\leq j-1}A + \mathfrak{d}^{\leq j}\big(\Gamma_b \cdot (A, B)\big), \end{aligned}$$

as stated.

Next, we consider the second Bianchi pair for which we have, according to Proposition 16.2.4,

$$\begin{aligned} F_{(1)} &= O(ar^{-2})\check{P} + O(r^{-3})\Gamma_b' + r^{-2}\Gamma_b' \cdot \check{R}_b + \Gamma_b \cdot A, \\ F_{(2)} &= O(ar^{-2})B + O(r^{-4})\Gamma_b + \Gamma_b' \cdot (A, B) + r^{-1}\Xi \cdot \check{R}_b. \end{aligned}$$

In this case, we infer, using also $\Psi_{(1)} = B$, $\Psi_{(2)} = \overline{\check{P}}$, we obtain

$$\begin{aligned} \widetilde{F}_{(1)} &= O(r^{-3})\mathfrak{d}^{\leq j}\Gamma_b' + O(r^{-2})\mathfrak{d}^{\leq j}\big(B, \check{P}\big) + O(r^{-1})\mathfrak{d}^{\leq j-1}\check{P} + r^{-2}\mathfrak{d}^{\leq j}(\Gamma_b' \cdot \check{R}_b) \\ &\quad + \mathfrak{d}^{\leq j}\big(\Gamma_b \cdot (A, B)\big), \\ \widetilde{F}_{(2)} &= O(r^{-4})\mathfrak{d}^{\leq j}\Gamma_b + O(r^{-2})\mathfrak{d}^{\leq j}\big(B, \check{P}\big) + O(r^{-1})\mathfrak{d}^{\leq j-1}B + \mathfrak{d}^{\leq j}\big(\Gamma_b' \cdot (A, B)\big) \\ &\quad + r^{-1}\mathfrak{d}^{\leq j}(\Gamma_b \cdot \check{P}) + r^{-1}\mathfrak{d}^{\leq j}(\Xi \cdot \check{R}_b) \end{aligned}$$

as stated.

Next, we consider the third Bianchi pair for which we have, according to Proposition 16.2.4,

$$\begin{aligned} F_{(1)} &= O(ar^{-2})\underline{B} + O(r^{-3})\Gamma_b + r^{-1}\Gamma_b' \cdot \check{R}_b, \\ F_{(2)} &= O(ar^{-2})\check{P} + O(r^{-3})\Gamma_b + r^{-2}\Gamma_b \cdot \check{R}_b + \Xi \cdot \check{R}_b. \end{aligned}$$

In this case, we infer, using also $\Psi_{(1)} = \check{P}$, $\Psi_{(2)} = \underline{B}$, we obtain

$$\begin{aligned} \widetilde{F}_{(1)} &= O(r^{-3})\mathfrak{d}^{\leq j}\Gamma_b + O(r^{-2})\mathfrak{d}^{\leq j}\big(\check{P}, \underline{B}\big) + O(r^{-1})\mathfrak{d}^{\leq j-1}\underline{B} + r^{-1}\mathfrak{d}^{\leq j}\big(\Gamma_b' \cdot \check{R}_b\big), \\ \widetilde{F}_{(2)} &= O(r^{-3})\mathfrak{d}^{\leq j}\Gamma_b + O(r^{-2})\mathfrak{d}^{\leq j}\big(\check{P}, \underline{B}\big) + O(r^{-1})\mathfrak{d}^{\leq j-1}\check{P} + r^{-2}\mathfrak{d}^{\leq j}\big(\Gamma_b \cdot \check{R}_b\big) \\ &\quad + \mathfrak{d}^{\leq j}\big(\Xi \cdot \check{R}_b\big) \end{aligned}$$

as stated.

Finally, we consider the fourth Bianchi pair for which we have, according to Proposition 16.2.4,

$$\begin{aligned} F_{(1)} &= O(ar^{-2})\underline{A} + O(r^{-3})\Gamma_b + \Gamma'_b \cdot \check{R}_b, \\ F_{(2)} &= O(ar^{-2})\underline{B} + O(r^{-3})\Gamma_b + r^{-1}\Gamma_b \cdot \check{R}_b. \end{aligned}$$

In this case, we infer, using also $\Psi_{(1)} = \underline{B}$, $\Psi_{(2)} = \underline{A}$, we obtain

$$\begin{aligned} \widetilde{F}_{(1)} &= O(r^{-3})\mathfrak{d}^{\leq j}\Gamma_b + O(r^{-2})\mathfrak{d}^{\leq j}\big(\underline{B}, \underline{A}\big) + O(r^{-1})\mathfrak{d}^{\leq j-1}\underline{B} + \mathfrak{d}^{\leq j}(\Gamma'_b \cdot \check{R}_b), \\ \widetilde{F}_{(2)} &= O(r^{-3})\mathfrak{d}^{\leq j}\Gamma_b + O(r^{-2})\mathfrak{d}^{\leq j}\big(\underline{B}, \underline{A}\big) + O(r^{-1})\mathfrak{d}^{\leq j-1}\underline{B} + r^{-1}\mathfrak{d}^{\leq j}(\Gamma_b \cdot \check{R}_b) \end{aligned}$$

as stated. This concludes the proof of Proposition 16.5.9. □

16.6 Bianchi pairs for general higher derivatives

To derive estimates for all derivatives $\leq J+1$, we commute the Bianchi equations of Proposition 16.2.4 with $(\overline{q}\,{}^{(c)}\nabla_4)^j\, \not{\mathfrak{d}}^{j_2} \not{\mathcal{L}}_{\mathbf{T}}^{j_1}$ or[7] $(q\,{}^{(c)}\nabla_4)^j\, \not{\mathfrak{d}}^{j_2} \not{\mathcal{L}}_{\mathbf{T}}^{j_1}$ for all multi-indices (j, j_1, j_2) with $j + j_1 + j_2 = J+1$. The result is stated in the following proposition. We also make the convention $\not{\mathfrak{d}}^{-1} = 0$, see footnote 5.

Proposition 16.6.1. *The following higher derivatives Bianchi pairs hold:*

1. *Consider the first Bianchi pair in A, B, and set, for $j + j_1 + j_2 = J+1$,*
$$\widetilde{A} = (\overline{q}\,{}^{(c)}\nabla_4)^j\, \not{\mathfrak{d}}^{j_2} \not{\mathcal{L}}_{\mathbf{T}}^{j_1} A, \qquad \widetilde{B} = (\overline{q}\,{}^{(c)}\nabla_4)^j\, \not{\mathfrak{d}}^{j_2} \not{\mathcal{L}}_{\mathbf{T}}^{j_1} B.$$
Then, with $c_{(1)} = \frac{1}{2}$, $c_{(2)} = 2$:
   - *If j_2 is even, $\widetilde{A} \in \mathfrak{s}_2$, $\widetilde{B} \in \mathfrak{s}_1$, and we have*[8]
$$\begin{aligned} {}^{(c)}\nabla_3\widetilde{A} + \left(c_{(1)} - \frac{j}{2}\right) tr\underline{X}\widetilde{A} &= -\not{\mathcal{D}}_2^*\widetilde{B} + O(ar^{-1})(\overline{q}\,{}^{(c)}\nabla_4)^j\, \not{\mathfrak{d}}^{j_2-1} \not{\mathcal{L}}_{\mathbf{T}}^{j_1+1} B \\ &\quad + O(r^{-1})(\overline{q}\,{}^{(c)}\nabla_4)^{j-1}\, \not{\mathfrak{d}}^{j_2+1} \not{\mathcal{L}}_{\mathbf{T}}^{j_1}(A, B) + \widetilde{F}_{(1)}, \\ {}^{(c)}\nabla_4\widetilde{B} + \left(c_{(2)} - \frac{j}{2}\right) \overline{trX}\widetilde{B} &= \not{\mathcal{D}}_2\widetilde{A} + O(ar^{-1})(\overline{q}\,{}^{(c)}\nabla_4)^j\, \not{\mathfrak{d}}^{j_2-1} \not{\mathcal{L}}_{\mathbf{T}}^{j_1+1} A \\ &\quad + O(r^{-1})(\overline{q}\,{}^{(c)}\nabla_4)^{j-1}\, \not{\mathfrak{d}}^{j_2+1} \not{\mathcal{L}}_{\mathbf{T}}^{j_1} A + \widetilde{F}_{(2)}. \end{aligned}$$

[7]Depending on the type of Bianchi pair in Definition 16.2.3.

[8]Note that for $j_2 = 0$, the terms in $O(ar^{-1})(\overline{q}\,{}^{(c)}\nabla_4)\, \not{\mathfrak{d}}^{j_2-1} \not{\mathcal{L}}_{\mathbf{T}}^{j_1+1}$ do not appear at all. Also, for $j = 0$, the terms $O(r^{-1})(\overline{q}\,{}^{(c)}\nabla_4)^{j-1}\, \not{\mathfrak{d}}^{j_2+1} \not{\mathcal{L}}_{\mathbf{T}}^{j_1}$ do not appear at all. This holds true for all Bianchi pairs below.

Note that this can be written in the form of the generalized Bianchi pair (16.2.3) *with* $\Psi_{(1)} = \widetilde{A}, \Psi_{(2)} = \widetilde{B}$.

- *If j_2 is odd, $\widetilde{A} \in \mathfrak{s}_1$, $\widetilde{B} \in \mathfrak{s}_0$, and we have*

$$\begin{aligned}
{}^{(c)}\nabla_3\widetilde{A} + \left(c_{(1)} - \frac{j}{2}\right) tr\underline{X}\widetilde{A} &= -\not{\mathcal{D}}_1^*\widetilde{B} + O(ar^{-1})(\overline{q}\,{}^{(c)}\nabla_4)^j \not{\mathfrak{d}}^{j_2-1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1+1}B \\
&\quad + O(r^{-1})(\overline{q}\,{}^{(c)}\nabla_4)^{j-1}\not{\mathfrak{d}}^{j_2+1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1}(A, B) + \widetilde{F}_{(1)}, \\
{}^{(c)}\nabla_4\widetilde{B} + \left(c_{(2)} - \frac{j}{2}\right) \overline{trX}\widetilde{B} &= \not{\mathcal{D}}_1\widetilde{A} + O(ar^{-1})(\overline{q}\,{}^{(c)}\nabla_4)^j \not{\mathfrak{d}}^{j_2-1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1+1}A \\
&\quad + O(r^{-1})(\overline{q}\,{}^{(c)}\nabla_4)^{j-1}\not{\mathfrak{d}}^{j_2+1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1}A + \widetilde{F}_{(2)}.
\end{aligned}$$

Note that this can also be written in the form of the generalized Bianchi pair (16.2.3) *with* $\Psi_{(1)} = \widetilde{A}, \Psi_{(2)} = \widetilde{B}$.

In both cases we have

$$\begin{aligned}
\widetilde{F}_{(1)} &= O(r^{-3})\mathfrak{d}^{\leq J+1}\Gamma_g + O(r^{-2})\mathfrak{d}^{\leq J+1}\big(A, B\big) + O(r^{-1})\mathfrak{d}^{\leq J}B + \mathfrak{d}^{\leq J+1}\big(\Gamma_b \cdot (A, B)\big), \\
\widetilde{F}_{(2)} &= O(r^{-3})\mathfrak{d}^{\leq J+1}\Gamma_g + O(r^{-2})\mathfrak{d}^{\leq J+1}\big(A, B\big) + O(r^{-1})\mathfrak{d}^{\leq J}A + \mathfrak{d}^{\leq J+1}\big(\Gamma_b \cdot (A, B)\big).
\end{aligned}$$

Second Bianchi Pair. *Consider the second Bianchi pair in $B, \check{P}$, and set*[9], *for* $j + j_1 + j_2 = J + 1$,

$$\widetilde{B} = (\overline{q}\,{}^{(c)}\nabla_4)^j \not{\mathfrak{d}}^{j_2}\not{\mathcal{L}}_{\mathbf{T}}^{j_1}B, \qquad \widetilde{P} = (\overline{q}\,{}^{(c)}\nabla_4)^j \not{\mathfrak{d}}^{j_2}\not{\mathcal{L}}_{\mathbf{T}}^{j_1}\overline{\check{P}}.$$

Then, with $c_{(1)} = 1$, $c_{(2)} = 3/2$*:*

- *If j_2 is even, $\widetilde{B} \in \mathfrak{s}_1$, $\widetilde{P} \in \mathfrak{s}_0$, and we have*

$$\begin{aligned}
{}^{(c)}\nabla_3\widetilde{B} + \left(c_{(1)} - \frac{j}{2}\right) tr\underline{X}\widetilde{B} &= -\not{\mathcal{D}}_1^*\widetilde{P} + O(ar^{-1})(\overline{q}\,{}^{(c)}\nabla_4)^j \not{\mathfrak{d}}^{j_2-1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1+1}\check{P} \\
&\quad + O(r^{-1})(\overline{q}\,{}^{(c)}\nabla_4)^{j-1}\not{\mathfrak{d}}^{j_2+1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1}(B, \check{P}) + \widetilde{F}_{(1)}, \\
{}^{(c)}\nabla_4\widetilde{P} + \left(c_{(2)} - \frac{j}{2}\right) \overline{trX}\widetilde{P} &= \not{\mathcal{D}}_1\widetilde{B} + O(ar^{-1})(\overline{q}\,{}^{(c)}\nabla_4^j)\not{\mathfrak{d}}^{j_2-1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1+1}B \\
&\quad + O(r^{-1})(\overline{q}\,{}^{(c)}\nabla_4)^{j-1}\not{\mathfrak{d}}^{j_2+1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1}B + \widetilde{F}_{(2)}.
\end{aligned}$$

Note that this can be written in the form of the generalized Bianchi pair (16.2.3) *with* $\Psi_{(1)} = \widetilde{B}, \Psi_{(2)} = \widetilde{P}$.

[9]See remark 15.3.6 for the definition of $\widetilde{P}$ here.

- *If j_2 is odd, $\widetilde{B}\in\mathfrak{s}_0$, $\widetilde{P}\in\mathfrak{s}_1$, and we have*

$$\begin{aligned}
{}^{(c)}\nabla_3\widetilde{B}+\left(c_{(1)}-\frac{j}{2}\right)\overline{tr\underline{X}}\widetilde{B} &= -\not{\mathcal{D}}_1\widetilde{P}+O(ar^{-1})(q\,{}^{(c)}\nabla_4)^j\,\not{\mathfrak{d}}^{j_2-1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1+1}\check{P}\\
&\quad+O(r^{-1})(\overline{q}\,{}^{(c)}\nabla_4)^{j-1}\,\not{\mathfrak{d}}^{j_2+1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1}(B,\check{P})+\widetilde{F}_{(1)},\\
{}^{(c)}\nabla_4\widetilde{P}+\left(c_{(2)}-\frac{j}{2}\right)trX\widetilde{P} &= \not{\mathcal{D}}_1^*\widetilde{B}+O(ar^{-1})(q\,{}^{(c)}\nabla_4)^j\,\not{\mathfrak{d}}^{j_2-1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1+1}B\\
&\quad+O(r^{-1})(\overline{q}\,{}^{(c)}\nabla_4)^{j-1}\,\not{\mathfrak{d}}^{j_2+1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1}B+\widetilde{F}_{(2)}.
\end{aligned}$$

Note that this can be written in the form of the generalized Bianchi pair (16.2.4) *with $\Psi_{(1)}=\widetilde{B}$, $\Psi_{(2)}=\widetilde{P}$.*

In both cases we have

$$\begin{aligned}
\widetilde{F}_{(1)} &= O(r^{-3})\mathfrak{d}^{\leq J+1}\Gamma'_b+O(r^{-2})\mathfrak{d}^{\leq J+1}\big(B,\check{P}\big)+O(r^{-1})\mathfrak{d}^{\leq J}\check{P}+r^{-2}\mathfrak{d}^{\leq J+1}(\Gamma'_b\cdot\check{R}_b)\\
&\quad+\mathfrak{d}^{\leq J+1}\big(\Gamma_b\cdot(A,B)\big),\\
\widetilde{F}_{(2)} &= O(r^{-4})\mathfrak{d}^{\leq J+1}\Gamma_b+O(r^{-2})\mathfrak{d}^{\leq J+1}\big(B,\check{P}\big)+O(r^{-1})\mathfrak{d}^{\leq J}B+\mathfrak{d}^{\leq J+1}\big(\Gamma'_b\cdot(A,B)\big)\\
&\quad+r^{-1}\mathfrak{d}^{\leq J+1}(\Gamma_b\cdot\check{P})+r^{-1}\mathfrak{d}^{\leq J+1}(\Xi\cdot\check{R}_b).
\end{aligned}$$

Third Bianchi Pair. *Consider the third Bianchi pair in $\check{P},\underline{B}$, and set, for $j+j_1+j_2=J+1$,*

$$\widetilde{P}=(q\,{}^{(c)}\nabla_4)^j\,\not{\mathfrak{d}}^{j_2}\not{\mathcal{L}}_{\mathbf{T}}^{j_1}\check{P},\qquad \widetilde{\underline{B}}=(q\,{}^{(c)}\nabla_4)^j\,\not{\mathfrak{d}}^{j_2}\not{\mathcal{L}}_{\mathbf{T}}^{j_1}\underline{B}.$$

Then, with $c_{(1)}=3/2$, $c_{(2)}=1$:

- *If j_2 is even, $\widetilde{P}\in\mathfrak{s}_0$, $\widetilde{\underline{B}}\in\mathfrak{s}_1$, and we have*

$$\begin{aligned}
{}^{(c)}\nabla_3\widetilde{P}+\left(c_{(1)}-\frac{j}{2}\right)\overline{tr\underline{X}}\,\widetilde{\underline{B}} &= -\not{\mathcal{D}}_1\widetilde{\underline{B}}+O(ar^{-1})(q\,{}^{(c)}\nabla_4)^j\,\not{\mathfrak{d}}^{j_2-1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1+1}\widetilde{\underline{B}}\\
&\quad+O(r^{-1})(\overline{q}\,{}^{(c)}\nabla_4)^{j-1}\,\not{\mathfrak{d}}^{j_2+1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1}(\check{P},\underline{B})+\widetilde{F}_{(1)},\\
{}^{(c)}\nabla_4\widetilde{\underline{B}}+\left(c_{(2)}-\frac{j}{2}\right)trX\,\widetilde{\underline{B}} &= \not{\mathcal{D}}_1^*\widetilde{P}+O(ar^{-1})(q\,{}^{(c)}\nabla_4)^j\,\not{\mathfrak{d}}^{j_2-1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1+1}\check{P}\\
&\quad+O(r^{-1})(\overline{q}\,{}^{(c)}\nabla_4)^{j-1}\,\not{\mathfrak{d}}^{j_2+1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1}\check{P}+\widetilde{F}_{(2)}.
\end{aligned}$$

Note that this can be written in the form of the generalized Bianchi pair (16.2.4) *with $\Psi_{(1)}=\widetilde{P}$, $\Psi_{(2)}=\widetilde{\underline{B}}$.*

- *If j_2 is odd, $\widetilde{P} \in \mathfrak{s}_1$, $\widetilde{\underline{B}} \in \mathfrak{s}_0$, and we have*

$$\begin{aligned}
{}^{(c)}\nabla_3\widetilde{P} + \left(c_{(1)} - \frac{j}{2}\right)\overline{tr\underline{X}}\widetilde{P} &= -\not{\mathcal{D}}_1^*\widetilde{\underline{B}} + O(ar^{-1})(q\,{}^{(c)}\nabla_4)^j\,\not{\mathfrak{d}}^{j_2-1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1+1}\underline{B}\\
&\quad +O(r^{-1})(\overline{q}\,{}^{(c)}\nabla_4)^{j-1}\,\not{\mathfrak{d}}^{j_2+1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1}(\check{P},\underline{B}) + \widetilde{F}_{(1)},\\
{}^{(c)}\nabla_4\underline{B} + \left(c_{(2)} - \frac{j}{2}\right)\overline{trX}\underline{B} &= \not{\mathcal{D}}_1\widetilde{P} + O(ar^{-1})(q\,{}^{(c)}\nabla_4)^j\,\not{\mathfrak{d}}^{j_2-1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1+1}\check{P}\\
&\quad +O(r^{-1})(\overline{q}\,{}^{(c)}\nabla_4)^{j-1}\,\not{\mathfrak{d}}^{j_2+1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1}\check{P} + \widetilde{F}_{(2)}.
\end{aligned}$$

  Note that this can also be written in the form of the generalized Bianchi pair (16.2.3) *with $\Psi_{(1)} = \widetilde{P}, \Psi_{(2)} = \widetilde{\underline{B}}$.*

In both cases we have

$$\begin{aligned}
\widetilde{F}_{(1)} &= O(r^{-3})\mathfrak{d}^{\leq J+1}\Gamma_b + O(r^{-2})\mathfrak{d}^{\leq J+1}\big(\check{P},\underline{B}\big) + O(r^{-1})\mathfrak{d}^{\leq J}\underline{B} + r^{-1}\mathfrak{d}^{\leq J+1}\big(\Gamma_b'\cdot\check{R}_b\big),\\
\widetilde{F}_{(2)} &= O(r^{-3})\mathfrak{d}^{\leq J+1}\Gamma_b + O(r^{-2})\mathfrak{d}^{\leq J+1}\big(\check{P},\underline{B}\big) + O(r^{-1})\mathfrak{d}^{\leq J}\check{P} + r^{-2}\mathfrak{d}^{\leq J+1}\big(\Gamma_b\cdot\check{R}_b\big)\\
&\quad +\mathfrak{d}^{\leq J+1}\big(\Xi\cdot\check{R}_b\big).
\end{aligned}$$

Fourth Bianchi Pair. *Consider the fourth Bianchi pair in $\underline{B}, \underline{A}$, and set, for $j + j_1 + j_2 = J + 1$,*

$$\widetilde{\underline{B}} = (q\,{}^{(c)}\nabla_4)^j\,\not{\mathfrak{d}}^{j_2}\not{\mathcal{L}}_{\mathbf{T}}^{j_1}\underline{B}, \qquad \widetilde{\underline{A}} = (q\,{}^{(c)}\nabla_4)^j\,\not{\mathfrak{d}}^{j_2}\not{\mathcal{L}}_{\mathbf{T}}^{j_1}\underline{A}.$$

Then, with $c_{(1)} = 2$, $c_{(2)} = 1/2$:

- *If j_2 is even, $\widetilde{\underline{B}} \in \mathfrak{s}_1$, $\widetilde{\underline{A}} \in \mathfrak{s}_2$, and we have*

$$\begin{aligned}
{}^{(c)}\nabla_3\widetilde{\underline{B}} + \left(c_{(1)} - \frac{j}{2}\right)\overline{tr\underline{X}}\,\widetilde{\underline{B}} &= -\not{\mathcal{D}}_2\widetilde{\underline{A}} + O(ar^{-1})(q\,{}^{(c)}\nabla_4)^j\,\not{\mathfrak{d}}^{j_2-1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1+1}\underline{A}\\
&\quad +O(r^{-1})(\overline{q}\,{}^{(c)}\nabla_4)^{j-1}\,\not{\mathfrak{d}}^{j_2+1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1}(\underline{B},\underline{A}) + \widetilde{F}_{(1)},\\
{}^{(c)}\nabla_4\widetilde{\underline{A}} + \left(c_{(2)} - \frac{j}{2}\right)trX\widetilde{\underline{A}} &= \not{\mathcal{D}}_2^*\widetilde{\underline{B}} + O(ar^{-1})(q\,{}^{(c)}\nabla_4)^j\,\not{\mathfrak{d}}^{j_2-1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1+1}\underline{B}\\
&\quad +O(r^{-1})(\overline{q}\,{}^{(c)}\nabla_4)^{j-1}\,\not{\mathfrak{d}}^{j_2+1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1}\underline{B} + \widetilde{F}_{(2)}.
\end{aligned}$$

  Note that this can be written in the form of the generalized Bianchi pair (16.2.4) *with $\Psi_{(1)} = \widetilde{\underline{B}}, \Psi_{(2)} = \widetilde{\underline{A}}$.*

- *If j_2 is even, $\widetilde{\underline{B}} \in \mathfrak{s}_0$, $\widetilde{\underline{A}} \in \mathfrak{s}_1$, and we have*

$$\begin{aligned}
{}^{(c)}\nabla_3\widetilde{\underline{B}} + \left(c_{(1)} - \frac{j}{2}\right)\overline{tr\underline{X}}\,\widetilde{\underline{B}} &= -\,\not{\mathcal{D}}_1\widetilde{\underline{A}} + O(ar^{-1})(q\,{}^{(c)}\nabla_4)^j\,\not{\mathfrak{d}}^{j_2-1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1+1}\underline{A}\\
&\quad + O(r^{-1})(\overline{q}\,{}^{(c)}\nabla_4)^{j-1}\,\not{\mathfrak{d}}^{j_2+1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1}(\underline{B},\underline{A}) + \widetilde{F}_{(1)},\\
{}^{(c)}\nabla_4\widetilde{\underline{A}} + \left(c_{(2)} - \frac{j}{2}\right)trX\widetilde{\underline{A}} &= \not{\mathcal{D}}_1^*\widetilde{\underline{B}} + O(ar^{-1})(q\,{}^{(c)}\nabla_4)^j\,\not{\mathfrak{d}}^{j_2-1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1+1}\underline{B}\\
&\quad + O(r^{-1})(\overline{q}\,{}^{(c)}\nabla_4)^{j-1}\,\not{\mathfrak{d}}^{j_2+1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1}\underline{B} + \widetilde{F}_{(2)}.
\end{aligned}$$

  Note that this can also be written in the form of the generalized Bianchi pair (16.2.4) *with* $\Psi_{(1)} = \widetilde{\underline{B}}, \Psi_{(2)} = \widetilde{\underline{A}}$.

In both cases

$$\begin{aligned}
\widetilde{F}_{(1)} &= O(r^{-3})\mathfrak{d}^{\leq J+1}\Gamma_b + O(r^{-2})\mathfrak{d}^{\leq J+1}\big(\underline{B},\underline{A}\big) + O(r^{-1})\mathfrak{d}^{\leq J}\underline{B} + \mathfrak{d}^{\leq J+1}(\Gamma_b'\cdot\check{R}_b),\\
\widetilde{F}_{(2)} &= O(r^{-3})\mathfrak{d}^{\leq J+1}\Gamma_b + O(r^{-2})\mathfrak{d}^{\leq J+1}\big(\underline{B},\underline{A}\big) + O(r^{-1})\mathfrak{d}^{\leq J}\underline{B} + r^{-1}\mathfrak{d}^{\leq J+1}(\Gamma_b\cdot\check{R}_b).
\end{aligned}$$

Proof. The case $j = j_2 = 0$ can be easily be derived by commuting the Bianchi pairs in Proposition 16.2.4 with $\not{\mathcal{L}}_{\mathbf{T}}$, in view of Lemma 13.3.3. The case $j = 0$ can then be derived by proceeding exactly as Proposition 16.5.9, starting with the results already derived by commutation with $\not{\mathcal{L}}_{\mathbf{T}}$. Finally we derive the results for all (j, j_1, j_2) by making use of the commutators in Lemma 13.3.4. □

16.7 Proof of Theorem 16.1.1

We are now in position to prove Theorem 16.1.1 on the control of ${}^{(ext)}\mathfrak{R}_{J+1}$. To this end, we rely on the procedure for deriving basic curvature estimates in ${}^{(ext)}\mathcal{M}$ outlined in section 16.4 and on the structure of Bianchi pairs for higher derivatives in Proposition 16.6.1. The proof proceeds in the following steps.

Step 1. We first derive estimates for $\not{\mathcal{L}}_{\mathbf{T}}^{J+1}$ derivatives of curvature in ${}^{(ext)}\mathcal{M}$. To this end, we consider the Bianchi pairs in Proposition 16.6.1 in the particular case

$$j = 0, \qquad j_2 = 0, \qquad j_1 = J + 1.$$

In that case, the terms $O(ar^{-1})(q\,{}^{(c)}\nabla_4)^j\,\not{\mathfrak{d}}^{j_2-1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1+1}$ and $O(r^{-1})(q\,{}^{(c)}\nabla_4)^{j-1}\,\not{\mathfrak{d}}^{j_2+1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1}$ are not present, see footnote 8, and hence the structure of the Bianchi pairs in Proposition 16.6.1 are the exact analog for $J+1$ derivatives of the ones in Proposition 16.2.4 used for the proof of the basic curvature estimates of section 16.4, with the exception of

- the new terms of type $O(r^{-1})\mathfrak{d}^{\leq J}\Psi_{(1)}$ in $\widetilde{F}_{(1)}$,
- the new terms of type $O(r^{-1})\mathfrak{d}^{\leq J}\Psi_{(2)}$ in $\widetilde{F}_{(2)}$.

Following the same steps as in section section 16.4, that are summarized in Remark 16.4.1, we thus obtain the following analog of (16.4.1) for $\not{\mathcal{L}}_{\mathbf{T}}^{J+1}$ derivatives of curvature

$$ {}^{(ext)}\mathfrak{R}_0^2[\not{\mathcal{L}}_{\mathbf{T}}^{J+1}(A,B,\check{P},\underline{B},\underline{A})] \lesssim r_0^{-\delta_B}\,{}^{(ext)}\mathfrak{G}_0^2 + r_0^{3+\delta_B}\,{}^{(int)}\mathfrak{R}_0^2 + r_0^{-2}\,{}^{(ext)}\mathfrak{R}_{J+1}^2 + \epsilon_J^2 + \epsilon_0^2, \tag{16.7.1} $$

where the extra terms $r_0^{-2}\,{}^{(ext)}\mathfrak{R}_{J+1}^2$ and ϵ_J^2 are respectively due to the fact that $r_0^{-2}\,{}^{(ext)}\mathfrak{R}_{J+1}^2$ cannot yet be absorbed by the LHS and from the new terms of type $O(r^{-1})\mathfrak{d}^{\leq J}\Psi_{(1)}$ and $O(r^{-1})\mathfrak{d}^{\leq J}\Psi_{(2)}$ in $\widetilde{F}_{(1)}$ and $\widetilde{F}_{(2)}$ mentioned above.

Step 2. Next, we derive estimates for $(\not{\mathfrak{d}},\not{\mathcal{L}}_{\mathbf{T}})^{J+1}$ derivatives of curvature in ${}^{(ext)}\mathcal{M}$. To this end, we consider the Bianchi pairs in Proposition 16.6.1 in the particular case

$$ j=0, \qquad j_2\geq 1, \qquad j_1+j_2=J+1. $$

In that case, the terms $O(r^{-1})(q\,{}^{(c)}\nabla_4)^{j-1}\not{\mathfrak{d}}^{j_2+1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1}$ are not present, see footnote 8, and hence the structure of the Bianchi pairs in Proposition 16.6.1 are the exact analog for $J+1$ derivatives of the ones in Proposition 16.2.4 used for the proof of the basic curvature estimates of section 16.4, with the exception of

- the new terms of type $O(r^{-1})\mathfrak{d}^{\leq J}\Psi_{(1)}$ and $O(ar^{-1})\not{\mathfrak{d}}^{j_2-1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1+1}\Psi_{(2)}$ in $\widetilde{F}_{(1)}$,
- the new terms of type $O(r^{-1})\mathfrak{d}^{\leq J}\Psi_{(2)}$ and $O(ar^{-1})\not{\mathfrak{d}}^{j_2-1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1+1}\Psi_{(1)}$ in $\widetilde{F}_{(2)}$.

Following the same steps as in section section 16.4, that are summarized in Remark 16.4.1, we thus obtain the following analog of (16.4.1) for $\not{\mathfrak{d}}^{j_2}\not{\mathcal{L}}_{\mathbf{T}}^{j_1}$ derivatives of curvature

$$ {}^{(ext)}\mathfrak{R}_0^2[\not{\mathfrak{d}}^{j_2}\not{\mathcal{L}}_{\mathbf{T}}^{j_1}(A,B,\check{P},\underline{B},\underline{A})] \lesssim r_0^{-\delta_B}\,{}^{(ext)}\mathfrak{G}_0^2 + r_0^{3+\delta_B}\,{}^{(int)}\mathfrak{R}_0^2 + r_0^{-2}\,{}^{(ext)}\mathfrak{R}_{J+1}^2 + {}^{(ext)}\mathfrak{R}_0^2[\not{\mathfrak{d}}^{j_2-1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1+1}(A,B,\check{P},\underline{B},\underline{A})] + \epsilon_J^2 + \epsilon_0^2, $$

where the new extra term ${}^{(ext)}\mathfrak{R}_0^2[\not{\mathfrak{d}}^{j_2-1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1+1}(A,B,\check{P},\underline{B},\underline{A})]$ is due to the new terms of type $O(ar^{-1})\not{\mathfrak{d}}^{j_2-1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1+1}\Psi_{(2)}$ in $\widetilde{F}_{(1)}$, and of type $O(ar^{-1})\not{\mathfrak{d}}^{j_2-1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1+1}\Psi_{(1)}$ in $\widetilde{F}_{(2)}$. Together with the estimate in the case $j_2=0$, i.e. (16.7.1), we immediately infer, by iteration on j_2,

$$ {}^{(ext)}\mathfrak{R}_0^2[(\not{\mathfrak{d}},\not{\mathcal{L}}_{\mathbf{T}})^{J+1}(A,B,\check{P},\underline{B},\underline{A})] \lesssim r_0^{-\delta_B}\,{}^{(ext)}\mathfrak{G}_0^2 + r_0^{3+\delta_B}\,{}^{(int)}\mathfrak{R}_0^2 + r_0^{-2}\,{}^{(ext)}\mathfrak{R}_{J+1}^2 + \epsilon_J^2 + \epsilon_0^2. \tag{16.7.2} $$

Step 3. We now recover weighted e_4 derivatives and start with the ones for $(B, \check{P}, \underline{B}, \underline{A})$. To this end, we use the following simple consequence of the second Bianchi identity of a given general Bianchi pair as in Definition 16.2.3

$$\nabla_4\Psi_{(2)} = O(r^{-1})\,\not{\mathfrak{d}}\Psi_{(1)} + O(r^{-1})\Psi_{(2)} + F_{(2)},$$

which implies

$${}^{(ext)}\mathfrak{R}_0[r\nabla_4\Psi_{(2)}] \lesssim {}^{(ext)}\mathfrak{R}_0[\not{\mathfrak{d}}\Psi_{(1)}] + {}^{(ext)}\mathfrak{R}_0[\Psi_{(2)}] + {}^{(ext)}\mathfrak{R}_0[F_{(2)}].$$

Applying this estimate to the second Bianchi identity in each of the four Bianchi pairs in Proposition 16.6.1 in the particular case

$$j \geq 1, \qquad j + j_1 + j_2 = J + 1,$$

we immediately obtain

$$\begin{aligned}
{}^{(ext)}\mathfrak{R}_0^2[(r\nabla_4)^j\,\not{\mathfrak{d}}^{j_2}\not{\mathcal{L}}_{\mathbf{T}}^{j_1}(B, \check{P}, \underline{B}, \underline{A})] \lesssim\; & {}^{(ext)}\mathfrak{R}_0^2[(r\nabla_4)^{j-1}\,\not{\mathfrak{d}}^{j_2+1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1}(A, B, \check{P}, \underline{B})] \\
& + {}^{(ext)}\mathfrak{R}_0^2[(r\nabla_4)^j\,\not{\mathfrak{d}}^{j_2-1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1+1}(A, B, \check{P}, \underline{B})] \\
& + r_0^{-\delta_B}\,{}^{(ext)}\mathfrak{G}_0^2 + r_0^{3+\delta_B}\,{}^{(int)}\mathfrak{R}_0^2 + r_0^{-2}\,{}^{(ext)}\mathfrak{R}_{J+1}^2 \\
& + \epsilon_J^2 + \epsilon_0^2, \qquad (16.7.3)
\end{aligned}$$

where we recall the convention $\not{\mathfrak{d}}^{-1} = 0$.

Step 4. We now recover weighted e_4 derivatives for A. To this end, we consider again the particular case

$$j \geq 1, \qquad j + j_1 + j_2 = J + 1$$

and focus on the first Bianchi pair in Proposition 16.6.1. In that case, we choose $b = 4+\delta_B$, and we have then

$$\Lambda_{(1)} = j + 2 + \frac{\delta_B}{2} > 0.$$

We may thus apply case 4 in Proposition 16.3.1 to the first Bianchi pair in Proposition 16.6.1 which yields

$$\begin{aligned}
{}^{(ext)}\mathfrak{R}_0^2[(r\nabla_4)^j\,\not{\mathfrak{d}}^{j_2}\not{\mathcal{L}}_{\mathbf{T}}^{j_1}A] \lesssim\; & {}^{(ext)}\mathfrak{R}_0^2[(r\nabla_4)^j\,\not{\mathfrak{d}}^{j_2}\not{\mathcal{L}}_{\mathbf{T}}^{j_1}B] + {}^{(ext)}\mathfrak{R}_0^2[(r\nabla_4)^{j-1}\,\not{\mathfrak{d}}^{j_2+1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1}(A, B)] \\
& + {}^{(ext)}\mathfrak{R}_0^2[(r\nabla_4)^j\,\not{\mathfrak{d}}^{j_2-1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1+1}(A, B)] \\
& + r_0^{-\delta_B}\,{}^{(ext)}\mathfrak{G}_0^2 + r_0^{3+\delta_B}\,{}^{(int)}\mathfrak{R}_0^2 + r_0^{-2}\,{}^{(ext)}\mathfrak{R}_{J+1}^2 + \epsilon_J^2 + \epsilon_0^2.
\end{aligned}$$

Together with (16.7.3), we infer

$$\begin{aligned}{}^{(ext)}\mathfrak{R}_0^2[(r\nabla_4)^j \,\not{\mathfrak{d}}^{j_2}\not{\mathcal{L}}_{\mathbf{T}}^{j_1}(A, B, \check{P}, \underline{B}, \underline{A})] \;\lesssim\;& {}^{(ext)}\mathfrak{R}_0^2[(r\nabla_4)^{j-1}\,\not{\mathfrak{d}}^{j_2+1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1}(A, B, \check{P}, \underline{B})]\\ &+{}^{(ext)}\mathfrak{R}_0^2[(r\nabla_4)^j\,\not{\mathfrak{d}}^{j_2-1}\not{\mathcal{L}}_{\mathbf{T}}^{j_1+1}(A, B, \check{P}, \underline{B})]\\ &+r_0^{-\delta_B}\,{}^{(ext)}\mathfrak{G}_0^2 + r_0^{3+\delta_B}\,{}^{(int)}\mathfrak{R}_0^2 + r_0^{-2}\,{}^{(ext)}\mathfrak{R}_{J+1}^2\\ &+\epsilon_J^2+\epsilon_0^2. \end{aligned} \tag{16.7.4}$$

In view of (16.7.2) and (16.7.4), and arguing by iteration on j, we infer[10]

$$\begin{aligned}{}^{(ext)}\mathfrak{R}_0^2[(r\nabla_4,\,\not{\mathfrak{d}},\not{\mathcal{L}}_{\mathbf{T}})^{J+1}(A, B, \check{P}, \underline{B}, \underline{A})] \;\lesssim\;& r_0^{-\delta_B}\,{}^{(ext)}\mathfrak{G}_0^2 + r_0^{3+\delta_B}\,{}^{(int)}\mathfrak{R}_0^2 + r_0^{-2}\,{}^{(ext)}\mathfrak{R}_{J+1}^2\\ &+\epsilon_J^2+\epsilon_0^2.\end{aligned}$$

Comparing $\not{\mathcal{L}}_{\mathbf{T}}$ and $\nabla_{\mathbf{T}}$, and using ϵ_J to absorb the corresponding lower term, we infer

$$\begin{aligned}{}^{(ext)}\mathfrak{R}_0^2[(r\nabla_4,\,\not{\mathfrak{d}},\nabla_{\mathbf{T}})^{J+1}(A, B, \check{P}, \underline{B}, \underline{A})] \;\lesssim\;& r_0^{-\delta_B}\,{}^{(ext)}\mathfrak{G}_0^2 + r_0^{3+\delta_B}\,{}^{(int)}\mathfrak{R}_0^2 + r_0^{-2}\,{}^{(ext)}\mathfrak{R}_{J+1}^2\\ &+\epsilon_J^2+\epsilon_0^2\end{aligned}$$

and hence

$${}^{(ext)}\mathfrak{R}_{J+1}^2 \;\lesssim\; r_0^{-\delta_B}\,{}^{(ext)}\mathfrak{G}_0^2 + r_0^{3+\delta_B}\,{}^{(int)}\mathfrak{R}_0^2 + r_0^{-2}\,{}^{(ext)}\mathfrak{R}_{J+1}^2 + \epsilon_J^2+\epsilon_0^2.$$

For r_0 large enough, we deduce

$${}^{(ext)}\mathfrak{R}_{J+1}^2 \;\lesssim\; r_0^{-\delta_B}\,{}^{(ext)}\mathfrak{G}_0^2 + r_0^{3+\delta_B}\,{}^{(int)}\mathfrak{R}_0^2 + \epsilon_J^2+\epsilon_0^2$$

as stated. This ends the proof of Theorem 16.1.1.

[10]Notice that $j = 0$ holds in view of (16.7.2), and for each j with $1 \leq j \leq J+1$, start with the fact that $\not{\mathfrak{d}}^{-1} = 0$ by convention and argue by iteration on j_2 from $j_2 = 0$ to $j_2 = J+1-j$.

Appendix A

Complement for Chapter 2

A.1 Corollary to Lemma 2.2.7

Corollary A.1.1. *Let $U_A = U_{a_1\ldots a_k}$ be a k-horizontal tensorfield symmetric traceless in all indices, i.e. $U \in \mathfrak{s}_k$.*

1. *We have*

$$\begin{aligned}
[\nabla_3, \nabla_b]U_A &= -\frac{1}{2}\big(tr\underline{\chi}\nabla_b U_A + {}^{(a)}tr\underline{\chi}\ {}^*\nabla_b U_A\big) + (\eta_b - \zeta_b)\nabla_3 U_A \\
&+ \frac{1}{2}\sum_{i=1}^{k}\big(\delta_{a_i b} tr\underline{\chi} + \in_{b a_i}\ {}^{(a)}tr\underline{\chi}\big)\eta_c U_{a_1\ldots}{}^{c}{}_{\ldots a_k} \\
&- \frac{1}{2}\sum_{i=1}^{k}\eta_{a_i}\big(tr\underline{\chi} U_{a_1\ldots b\ldots a_k} + {}^{(a)}tr\underline{\chi}\ {}^*U_{a_1\ldots b\ldots a_k}\big) \\
&+ Err_{3bA}[U],
\end{aligned} \tag{A.1.1}$$

$$\begin{aligned}
Err_{3bA}[U] &= \sum_{i=1}^{k}\Big(-\in_{a_i c}\ {}^*\underline{\beta}_b + \underline{\widehat{\chi}}_{b a_i}\eta_c - \underline{\widehat{\chi}}_{bc}\eta_{a_i} + \chi_{b a_i}\underline{\xi}_c - \chi_{bc}\underline{\xi}_{a_i}\Big) U_{a_1\ldots}{}^{c}{}_{\ldots a_k} \\
&- \underline{\widehat{\chi}}_{bc}\nabla_c U_A + \underline{\xi}_b \nabla_4 U_A.
\end{aligned}$$

2. *We have*

$$
\begin{aligned}
[\nabla_4, \nabla_b] U_A = & -\frac{1}{2}\big(tr\,\chi \nabla_b U_A + {}^{(a)}tr\chi\ {}^*\nabla_b U_A\big) + (\underline{\eta}_b + \zeta_b)\nabla_4 U_a \\
& + \frac{1}{2}\sum_{i=1}^k \big(\delta_{a_i b} tr\,\chi + \in_{b a_i}\ {}^{(a)}tr\chi\big)\underline{\eta}_c U_{a_1 \ldots}{}^c{}_{\ldots a_k} \\
& - \frac{1}{2}\sum_{i=1}^k \underline{\eta}_{a_i}\big(tr\,\chi U_{a_1 \ldots b \ldots a_k} + {}^{(a)}tr\chi\ {}^* U_{a_1 \ldots b \ldots a_k}\big) \\
& + Err_{4bA}[U], \\
Err_{4bA}[U] = & \sum_{i=1}^k \Big(\in_{a_i c}\ {}^*\beta_b + \widehat{\chi}_{b a_i}\underline{\eta}_c - \widehat{\chi}_{bc}\underline{\eta}_{a_i} + \underline{\chi}_{b a_i}\xi_c - \underline{\chi}_{bc}\xi_{a_i}\Big) U_{a_1 \ldots}{}^c{}_{\ldots a_k} \\
& - \widehat{\chi}_{bc}\nabla_c U_A + \xi_b \nabla_3 U_A.
\end{aligned} \tag{A.1.2}
$$

3. *We have,*

$$
\begin{aligned}
[\nabla_4, \nabla_3] U_A = & \, 2(\underline{\eta}_b - \eta_b)\nabla_b U_A + 2\omega\nabla_3 U_A - 2\underline{\omega}\nabla_4 U_A \\
& + 2\sum_{i=1}^k \big(\eta_{a_i}\underline{\eta}_b - \underline{\eta}_{a_i}\eta_b - \in_{a_i b}\ {}^*\rho\big) U_{a_1 \ldots}{}^b{}_{\ldots a_k} + Err_{43A}, \\
Err_{43A} = & \, 2\sum_{i=1}^k \big(\underline{\xi}_{a_i}\xi_b - \xi_{a_i}\underline{\xi}_b\big) U_{a_1 \ldots}{}^b{}_{\ldots a_k}.
\end{aligned} \tag{A.1.3}
$$

A.2 Proof of Lemma 2.3.3

We write

$$
\begin{aligned}
& \dot{\square}(X^\beta \dot{\mathbf{D}}_\beta U_a) = \mathbf{g}^{\mu\nu}\dot{\mathbf{D}}_\mu \dot{\mathbf{D}}_\nu (X^\beta \dot{\mathbf{D}}_\beta U_a) \\
= \ & (\mathbf{g}^{\mu\nu}\dot{\mathbf{D}}_\mu \dot{\mathbf{D}}_\nu X^\beta)\dot{\mathbf{D}}_\beta U_a + \mathbf{g}^{\mu\nu}\big(\dot{\mathbf{D}}_\mu X^\beta \dot{\mathbf{D}}_\nu \dot{\mathbf{D}}_\beta U_a + \dot{\mathbf{D}}_\nu X^\beta \dot{\mathbf{D}}_\mu \dot{\mathbf{D}}_\beta U_a\big) + X^\beta \mathbf{g}^{\mu\nu}\dot{\mathbf{D}}_\mu \dot{\mathbf{D}}_\nu \dot{\mathbf{D}}_\beta U_a \\
= \ & (\mathbf{g}^{\mu\nu}\dot{\mathbf{D}}_\mu \dot{\mathbf{D}}_\nu X^\beta)\dot{\mathbf{D}}_\beta U_a + X^\beta \dot{\mathbf{D}}_\beta\big(\mathbf{g}^{\mu\nu}\dot{\mathbf{D}}_\mu \dot{\mathbf{D}}_\nu\big)U_a \\
& + \mathbf{g}^{\mu\nu}\big(\dot{\mathbf{D}}_\mu X^\beta \dot{\mathbf{D}}_\nu \dot{\mathbf{D}}_\beta U_a + \dot{\mathbf{D}}_\nu X^\beta \dot{\mathbf{D}}_\mu \dot{\mathbf{D}}_\beta U_a\big) + X^\beta \mathbf{g}^{\mu\nu}\Big(\dot{\mathbf{D}}_\mu \dot{\mathbf{D}}_\nu \dot{\mathbf{D}}_\beta - \dot{\mathbf{D}}_\beta \dot{\mathbf{D}}_\mu \dot{\mathbf{D}}_\nu\Big)U_a.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\dot{\square}(X^\beta \dot{\mathbf{D}}_\beta U_a) \ = \ & X^\beta \dot{\mathbf{D}}_\beta \dot{\square} U_a + (\dot{\square} X^\beta)\dot{\mathbf{D}}_\beta U_a + \mathbf{g}^{\mu\nu}\big(\dot{\mathbf{D}}_\mu X^\beta \dot{\mathbf{D}}_\nu \dot{\mathbf{D}}_\beta U_a + \dot{\mathbf{D}}_\nu X^\beta \dot{\mathbf{D}}_\mu \dot{\mathbf{D}}_\beta U_a\big) \\
& + X^\beta \mathbf{g}^{\mu\nu}\Big(\dot{\mathbf{D}}_\mu\big(\dot{\mathbf{D}}_\nu \dot{\mathbf{D}}_\beta - \dot{\mathbf{D}}_\beta \dot{\mathbf{D}}_\nu\big)U_a + \big(\dot{\mathbf{D}}_\mu \dot{\mathbf{D}}_\beta - \dot{\mathbf{D}}_\beta \dot{\mathbf{D}}_\mu\big)\dot{\mathbf{D}}_\nu U_a\Big).
\end{aligned}
$$

Using Lemma 2.2.11, (2.2.26) and the fact that $\mathbf{R}_{\mu\nu} = 0$, we obtain

$$\begin{aligned}\mathbf{g}^{\mu\nu}\mathbf{D}_\mu\mathbf{D}_\nu X_\beta &= \mathbf{g}^{\mu\nu}\mathbf{R}_{\beta\mu\nu\gamma}X^\gamma + \mathbf{g}^{\mu\nu}\,{}^{(X)}\Gamma_{\mu\nu\beta}\\ &= \frac{1}{2}\mathbf{g}^{\mu\nu}(\mathbf{D}_\mu\,{}^{(X)}\pi_{\nu\beta} + \mathbf{D}_\nu\,{}^{(X)}\pi_{\mu\beta} - \mathbf{D}_\beta\,{}^{(X)}\pi_{\mu\nu}) = \mathbf{D}^\mu\pi_{\mu\beta} - \frac{1}{2}\mathbf{D}_\beta\mathrm{tr}\pi.\end{aligned}$$

Consider the term

$$\begin{aligned}\mathbf{g}^{\mu\nu}\big(\dot{\mathbf{D}}_\mu X^\beta\dot{\mathbf{D}}_\nu\dot{\mathbf{D}}_\beta U_a + \dot{\mathbf{D}}_\nu X^\beta\dot{\mathbf{D}}_\mu\dot{\mathbf{D}}_\beta U_a\big) &= \dot{\mathbf{D}}^\nu X^\beta\dot{\mathbf{D}}_\nu\dot{\mathbf{D}}_\beta U_a + \dot{\mathbf{D}}^\mu X^\beta\dot{\mathbf{D}}_\mu\dot{\mathbf{D}}_\beta U_a\\ &= \dot{\mathbf{D}}^\nu X^\beta\dot{\mathbf{D}}_\nu\dot{\mathbf{D}}_\beta U_a + \dot{\mathbf{D}}^\beta X^\nu\dot{\mathbf{D}}_\beta\dot{\mathbf{D}}_\nu U_a\\ &= (\dot{\mathbf{D}}^\nu X^\beta + \dot{\mathbf{D}}^\beta X^\nu)\dot{\mathbf{D}}_\nu\dot{\mathbf{D}}_\beta U_a\\ &\quad +\dot{\mathbf{D}}^\beta X^\nu(\dot{\mathbf{D}}_\beta\dot{\mathbf{D}}_\nu - \dot{\mathbf{D}}_\nu\dot{\mathbf{D}}_\beta)U_a.\end{aligned}$$

Using the commutator formula of Proposition 2.1.27 and definition of π we deduce

$$\mathbf{g}^{\mu\nu}\big(\dot{\mathbf{D}}_\mu X^\beta\dot{\mathbf{D}}_\nu\dot{\mathbf{D}}_\beta U_a + \dot{\mathbf{D}}_\nu X^\beta\dot{\mathbf{D}}_\mu\dot{\mathbf{D}}_\beta U_a\big) = \pi^{\mu\nu}\dot{\mathbf{D}}_\mu\dot{\mathbf{D}}_\nu U_a + \dot{\mathbf{R}}_{ac\beta\nu}U^c\mathbf{D}^\beta X^\nu.$$

Therefore,

$$\begin{aligned}&\dot{\Box}(X^\beta\dot{\mathbf{D}}_\beta U_a) - X^\beta\dot{\mathbf{D}}_\beta\dot{\Box}U_a\\ =\ &\pi^{\mu\nu}\dot{\mathbf{D}}_\mu\dot{\mathbf{D}}_\nu U_a + \big(\mathbf{D}^\mu\pi_\mu{}^\beta - \frac{1}{2}\mathbf{D}^\beta\mathrm{tr}\pi)\dot{\mathbf{D}}_\beta U_a + \dot{\mathbf{R}}_{ac\beta\nu}U^c\mathbf{D}^\beta X^\nu\\ &+X^\beta\mathbf{g}^{\mu\nu}\Big(\dot{\mathbf{D}}_\mu\big(\dot{\mathbf{D}}_\nu\dot{\mathbf{D}}_\beta - \dot{\mathbf{D}}_\beta\dot{\mathbf{D}}_\nu\big)U_a + \big(\dot{\mathbf{D}}_\mu\dot{\mathbf{D}}_\beta - \dot{\mathbf{D}}_\beta\dot{\mathbf{D}}_\mu\big)\dot{\mathbf{D}}_\nu U_a\Big).\end{aligned}$$

Now,

$$\begin{aligned}\mathbf{g}^{\mu\nu}\dot{\mathbf{D}}_\mu\big(\dot{\mathbf{D}}_\nu\dot{\mathbf{D}}_\beta - \dot{\mathbf{D}}_\beta\dot{\mathbf{D}}_\nu\big)U_a &= \mathbf{g}^{\mu\nu}\dot{\mathbf{D}}_\mu\big(\dot{\mathbf{R}}_{ac\nu\beta}U^c\big) = (\mathbf{D}^\mu\dot{\mathbf{R}}_{ac\mu\beta})U^c + \mathbf{g}^{\mu\nu}\dot{\mathbf{R}}_{ac\nu\beta}\dot{\mathbf{D}}_\mu U_c,\\ \mathbf{g}^{\mu\nu}\big(\dot{\mathbf{D}}_\mu\dot{\mathbf{D}}_\beta - \dot{\mathbf{D}}_\beta\dot{\mathbf{D}}_\mu\big)\dot{\mathbf{D}}_\nu U_a &= \mathbf{g}^{\mu\nu}\big(\mathbf{R}_{\nu\lambda\mu\beta}\dot{\mathbf{D}}_\lambda U_a + \dot{\mathbf{R}}_{ac\mu\beta}\dot{\mathbf{D}}_\nu U_c\big) = \mathbf{g}^{\mu\nu}\dot{\mathbf{R}}_{ac\mu\beta}\dot{\mathbf{D}}_\nu U_c.\end{aligned}$$

Using the fact that $\mathbf{D}^\mu\mathbf{R}_{ac\mu\beta} = 0$, we finally have

$$\begin{aligned}&\dot{\Box}(X^\beta\dot{\mathbf{D}}_\beta U_a) - X^\beta\dot{\mathbf{D}}_\beta\dot{\Box}U_a\\ =\ &\pi^{\mu\nu}\dot{\mathbf{D}}_\mu\dot{\mathbf{D}}_\nu U_a + \big(\mathbf{D}^\mu\pi_\mu{}^\beta - \frac{1}{2}\mathbf{D}^\beta\mathrm{tr}\pi)\dot{\mathbf{D}}_\beta U_a + \dot{\mathbf{R}}_{ac\beta\nu}U^c\mathbf{D}^\beta X^\nu\\ &+\frac{1}{2}X^\beta(\mathbf{D}^\mu\mathbf{B}_{ac\mu\beta})U^c + 2X^\beta\dot{\mathbf{R}}_{ac\mu\beta}\dot{\mathbf{D}}^\mu U^c.\end{aligned}$$

Writing that $\dot{\mathbf{R}} = \mathbf{R} + \frac{1}{2}\mathbf{B}$, we obtain the stated identity.

A.3 Proof of Proposition 2.3.7

We first prove the following.

Lemma A.3.1. *Let K be a symmetric tensor which satisfies (2.3.2). Then*

$$\begin{aligned} \mathbf{D}_\mu K^{\mu\nu} &= -\frac{1}{2}\mathbf{D}^\nu(trK) + \frac{3}{2}\Pi_\mu{}^{\mu\nu}, && \text{(A.3.1)}\\ \mathbf{D}^\mu\mathbf{D}_\alpha K^{\alpha\nu} - \mathbf{D}^\nu\mathbf{D}_\alpha K^{\alpha\mu} &= \frac{3}{2}\mathbf{D}^\mu\Pi_\alpha{}^{\alpha\nu} - \frac{3}{2}\mathbf{D}^\nu\Pi_\alpha{}^{\alpha\mu}, && \text{(A.3.2)}\\ \mathbf{D}^\mu\mathbf{D}^\alpha\mathbf{D}_\alpha K_{\mu\nu} &= -\left(\frac{1}{3}\mathbf{R}^\alpha{}_{\nu\mu}{}^\epsilon - \mathbf{R}^\alpha{}_{\mu\nu}{}^\epsilon\right)\mathbf{D}^\mu K_{\alpha\epsilon} - \mathbf{D}^\mu\mathbf{D}^\alpha\Pi_{\alpha\nu\mu}\\ &\quad +\frac{1}{2}\mathbf{D}^\mu\mathbf{D}_\mu\Pi^\alpha{}_{\alpha\nu} - \frac{1}{2}\mathbf{D}^\mu\mathbf{D}_\nu\Pi^\alpha{}_{\alpha\mu}. && \text{(A.3.3)} \end{aligned}$$

Let ϕ be a scalar function. Then

$$\begin{aligned} (\mathbf{D}_\alpha K^{\mu\nu})\mathbf{D}^\alpha\mathbf{D}_\nu\phi &= -\frac{1}{2}\mathbf{D}^\mu K^\nu{}_\alpha\mathbf{D}^\alpha\mathbf{D}_\nu\phi + \frac{3}{2}\Pi_\alpha{}^{\mu\nu}\mathbf{D}^\alpha\mathbf{D}_\nu\phi, && \text{(A.3.4)}\\ (\mathbf{D}^\mu K^\nu{}_\alpha)\mathbf{D}_\mu\mathbf{D}^\alpha\mathbf{D}_\nu\phi &= -\frac{2}{3}\mathbf{D}^\alpha K^\nu{}_\mu\mathbf{R}^\mu{}_{\alpha\nu}{}^\delta\mathbf{D}_\delta\phi + \Pi_{\mu\alpha\nu}\mathbf{D}^\mu\mathbf{D}_\alpha\mathbf{D}_\nu\phi. && \text{(A.3.5)} \end{aligned}$$

Let $\Psi \in \mathfrak{s}_2$. Then

$$\begin{aligned} (\mathbf{D}_\alpha K^{\mu\nu})\dot{\mathbf{D}}^\alpha\dot{\mathbf{D}}_\nu\Psi_{ab} = &-\frac{1}{2}\mathbf{D}^\mu K^\nu{}_\alpha\dot{\mathbf{D}}^\alpha\dot{\mathbf{D}}_\nu\Psi_{ab}\\ &+\frac{1}{2}\mathbf{D}^\nu K^\mu{}_\alpha(\mathbf{R}_\nu{}^\alpha{}_a{}^c\Psi_{cb} + \mathbf{R}_\nu{}^\alpha{}_b{}^c\Psi_{ac}) + \frac{3}{2}\Pi_\alpha{}^{\mu\nu}\dot{\mathbf{D}}^\alpha\dot{\mathbf{D}}_\nu\Psi_{ab}, \end{aligned} \quad \text{(A.3.6)}$$

and

$$\begin{aligned} (\mathbf{D}^\mu K^\nu{}_\alpha)\dot{\mathbf{D}}_\mu\dot{\mathbf{D}}^\alpha\dot{\mathbf{D}}_\nu\Psi_{ab} = &-\frac{2}{3}\mathbf{D}^\alpha K^\nu{}_\mu\mathbf{R}^\mu{}_{\alpha\nu}{}^\delta\dot{\mathbf{D}}_\delta\Psi_{ab} + \Pi_{\alpha\mu\nu}\dot{\mathbf{D}}^\alpha\dot{\mathbf{D}}_\mu\dot{\mathbf{D}}_\nu\Psi_{ab}\\ &-\frac{2}{3}\mathbf{D}^\mu K^\nu{}_\alpha(\mathbf{R}^\alpha{}_{\mu a}{}^c\dot{\mathbf{D}}_\nu\Psi_{cb} + \mathbf{R}^\alpha{}_{\mu b}{}^c\dot{\mathbf{D}}_\nu\Psi_{ac})\\ &-\frac{1}{3}\mathbf{D}^\nu K^\mu{}_\alpha\dot{\mathbf{D}}^\alpha(\mathbf{R}_{\nu\mu a}{}^c\Psi_{cb} + \mathbf{R}_{\nu\mu b}{}^c\Psi_{ac}). \end{aligned} \quad \text{(A.3.7)}$$

Proof. From contracting $3\Pi_{\mu\nu\rho} = \mathbf{D}_\mu K_{\nu\rho} + \mathbf{D}_\nu K_{\rho\mu} + \mathbf{D}_\rho K_{\mu\nu}$ with $\mathbf{g}^{\mu\nu}$ we obtain (A.3.1). Using (A.3.1), we can write

$$\begin{aligned} \mathbf{D}^\mu\mathbf{D}_\alpha K^{\alpha\nu} - \mathbf{D}^\nu\mathbf{D}_\alpha K^{\alpha\mu} &= -\frac{1}{2}\mathbf{D}^\mu\mathbf{D}^\nu(\mathrm{tr}K) + \frac{1}{2}\mathbf{D}^\nu\mathbf{D}^\mu(\mathrm{tr}K) + \frac{3}{2}\mathbf{D}^\mu\Pi_\alpha{}^{\alpha\nu} - \frac{3}{2}\mathbf{D}^\nu\Pi_\alpha{}^{\alpha\mu}\\ &= \frac{3}{2}\mathbf{D}^\mu\Pi_\alpha{}^{\alpha\nu} - \frac{3}{2}\mathbf{D}^\nu\Pi_\alpha{}^{\alpha\mu}, \end{aligned}$$

which proves (A.3.2). Applying $\mathbf{D}^\mu$ to $3\Pi_{\mu\nu\rho} = \mathbf{D}_\mu K_{\nu\rho} + \mathbf{D}_\nu K_{\rho\mu} + \mathbf{D}_\rho K_{\mu\nu}$, we have

$$\begin{aligned} 3\mathbf{D}^\mu\Pi_{\mu\nu\rho} &= \mathbf{D}^\mu\mathbf{D}_\mu K_{\nu\rho} + \mathbf{D}^\mu\mathbf{D}_\nu K_{\rho\mu} + \mathbf{D}^\mu\mathbf{D}_\rho K_{\mu\nu}\\ &= \mathbf{D}^\mu\mathbf{D}_\mu K_{\nu\rho} + \mathbf{D}_\nu\mathbf{D}^\mu K_{\rho\mu} + \mathbf{D}_\rho\mathbf{D}^\mu K_{\mu\nu} + \mathbf{R}^\mu{}_{\nu\rho}{}^\epsilon K_{\epsilon\mu} + \mathbf{R}^\mu{}_{\rho\nu}{}^\epsilon K_{\mu\epsilon}, \end{aligned}$$

which gives

$$\mathbf{D}^\alpha \mathbf{D}_\alpha K_{\mu\nu} = -\mathbf{D}_\nu \mathbf{D}^\alpha K_{\mu\alpha} - \mathbf{D}_\mu \mathbf{D}^\alpha K_{\alpha\nu} - \mathbf{R}^\alpha{}_{\nu\mu}{}^\epsilon K_{\epsilon\alpha} - \mathbf{R}^\alpha{}_{\mu\nu}{}^\epsilon K_{\alpha\epsilon} - 3\mathbf{D}^\alpha \Pi_{\alpha\nu\mu}.$$

Using (A.3.2), this gives

$$\begin{aligned}\mathbf{D}^\alpha \mathbf{D}_\alpha K_{\mu\nu} &= -2\mathbf{D}_\mu \mathbf{D}^\alpha K_{\alpha\nu} - (\mathbf{R}^\alpha{}_{\nu\mu}{}^\epsilon + \mathbf{R}^\alpha{}_{\mu\nu}{}^\epsilon)\, K_{\alpha\epsilon} - 3\mathbf{D}^\alpha \Pi_{\alpha\nu\mu} \\ &\quad + \frac{3}{2}\mathbf{D}_\mu \Pi^\alpha{}_{\alpha\nu} - \frac{3}{2}\mathbf{D}_\nu \Pi^\alpha{}_{\alpha\mu}.\end{aligned}$$

Applying $\mathbf{D}^\mu$ to the above we have

$$\begin{aligned}\mathbf{D}^\mu \mathbf{D}^\alpha \mathbf{D}_\alpha K_{\mu\nu} + 2\mathbf{D}^\mu \mathbf{D}_\mu \mathbf{D}^\alpha K_{\alpha\nu} &= -(\mathbf{R}^\alpha{}_{\nu\mu}{}^\epsilon + \mathbf{R}^\alpha{}_{\mu\nu}{}^\epsilon)\, \mathbf{D}^\mu K_{\alpha\epsilon} \\ &\quad -3\mathbf{D}^\mu \mathbf{D}^\alpha \Pi_{\alpha\nu\mu} + \frac{3}{2}\mathbf{D}^\mu \mathbf{D}_\mu \Pi^\alpha{}_{\alpha\nu} - \frac{3}{2}\mathbf{D}^\mu \mathbf{D}_\nu \Pi^\alpha{}_{\alpha\mu}.\end{aligned}$$

The left hand side is given by

$$\begin{aligned}\mathbf{D}^\mu \mathbf{D}^\alpha \mathbf{D}_\alpha K_{\mu\nu} + 2\mathbf{D}^\mu \mathbf{D}_\mu \mathbf{D}^\alpha K_{\alpha\nu} &= 3\mathbf{D}^\alpha \mathbf{D}_\alpha \mathbf{D}^\mu K_{\mu\nu} + [\mathbf{D}^\mu, \mathbf{D}^\alpha \mathbf{D}_\alpha] K_{\mu\nu} \\ &= 3\mathbf{D}^\alpha \mathbf{D}_\alpha \mathbf{D}^\mu K_{\mu\nu} + 2\mathbf{R}_\alpha{}^\mu{}_{\nu\epsilon} \mathbf{D}_\mu K^{\alpha\epsilon},\end{aligned}$$

which gives

$$\begin{aligned}\mathbf{D}^\alpha \mathbf{D}_\alpha \mathbf{D}^\mu K_{\mu\nu} &= -\left(\frac{1}{3}\mathbf{R}^\alpha{}_{\nu\mu}{}^\epsilon + \mathbf{R}^\alpha{}_{\mu\nu}{}^\epsilon\right) \mathbf{D}^\mu K_{\alpha\epsilon} \\ &\quad -\mathbf{D}^\mu \mathbf{D}^\alpha \Pi_{\alpha\nu\mu} + \frac{1}{2}\mathbf{D}^\mu \mathbf{D}_\mu \Pi^\alpha{}_{\alpha\nu} - \frac{1}{2}\mathbf{D}^\mu \mathbf{D}_\nu \Pi^\alpha{}_{\alpha\mu}.\end{aligned}$$

Writing again $\mathbf{D}^\mu \mathbf{D}^\alpha \mathbf{D}_\alpha K_{\mu\nu} = \mathbf{D}^\alpha \mathbf{D}_\alpha \mathbf{D}^\mu K_{\mu\nu} + 2\mathbf{R}_\alpha{}^\mu{}_{\nu\epsilon} \mathbf{D}_\mu K^{\alpha\epsilon}$, we obtain (A.3.3).

From $3\Pi_\alpha{}^{\mu\nu} = \mathbf{D}_\alpha K^{\mu\nu} + \mathbf{D}^\mu K^\nu{}_\alpha + \mathbf{D}^\nu K^\mu{}_\alpha$, we have

$$\begin{aligned}(\mathbf{D}_\alpha K^{\mu\nu})\mathbf{D}^\alpha \mathbf{D}_\nu \phi &= (-\mathbf{D}^\mu K^\nu{}_\alpha - \mathbf{D}^\nu K^\mu{}_\alpha + 3\Pi_\alpha{}^{\mu\nu})\mathbf{D}^\alpha \mathbf{D}_\nu \phi \\ &= -\mathbf{D}^\mu K^\nu{}_\alpha \mathbf{D}^\alpha \mathbf{D}_\nu \phi - \mathbf{D}^\nu K^\mu{}_\alpha \mathbf{D}^\alpha \mathbf{D}_\nu \phi + 3\Pi_\alpha{}^{\mu\nu} \mathbf{D}^\alpha \mathbf{D}_\nu \phi \\ &= -\mathbf{D}^\mu K^\nu{}_\alpha \mathbf{D}^\alpha \mathbf{D}_\nu \phi - \mathbf{D}_\alpha K^{\mu\nu} \mathbf{D}^\alpha \mathbf{D}_\nu \phi + 3\Pi_\alpha{}^{\mu\nu} \mathbf{D}^\alpha \mathbf{D}_\nu \phi,\end{aligned}$$

which implies (A.3.4).

We similarly compute

$$\begin{aligned}(\mathbf{D}_\alpha K^{\mu\nu})\mathbf{D}^\alpha \mathbf{D}_\mu \mathbf{D}_\nu \phi &= (-\mathbf{D}^\mu K^\nu{}_\alpha - \mathbf{D}^\nu K^\mu{}_\alpha + 3\Pi_{\alpha\mu\nu})\mathbf{D}^\alpha \mathbf{D}_\mu \mathbf{D}_\nu \phi \\ &= -\mathbf{D}^\mu K^\nu{}_\alpha \mathbf{D}^\alpha \mathbf{D}_\mu \mathbf{D}_\nu \phi - \mathbf{D}^\nu K^\mu{}_\alpha \mathbf{D}^\alpha \mathbf{D}_\nu \mathbf{D}_\mu \phi + 3\Pi_{\alpha\mu\nu} \mathbf{D}^\alpha \mathbf{D}_\mu \mathbf{D}_\nu \phi \\ &= -2\mathbf{D}^\mu K^\nu{}_\alpha \mathbf{D}^\alpha \mathbf{D}_\mu \mathbf{D}_\nu \phi + 3\Pi_{\alpha\mu\nu} \mathbf{D}^\alpha \mathbf{D}_\mu \mathbf{D}_\nu \phi \\ &= -2\mathbf{D}^\mu K^\nu{}_\alpha (\mathbf{D}_\mu \mathbf{D}^\alpha \mathbf{D}_\nu \phi + [\mathbf{D}^\alpha, \mathbf{D}_\mu]\mathbf{D}_\nu \phi) + 3\Pi_{\alpha\mu\nu} \mathbf{D}^\alpha \mathbf{D}_\mu \mathbf{D}_\nu \phi \\ &= -2\mathbf{D}^\mu K^\nu{}_\alpha (\mathbf{D}_\mu \mathbf{D}^\alpha \mathbf{D}_\nu \phi + \mathbf{R}^\alpha{}_{\mu\nu}{}^\delta \mathbf{D}_\delta \phi) + 3\Pi_{\alpha\mu\nu} \mathbf{D}^\alpha \mathbf{D}_\mu \mathbf{D}_\nu \phi.\end{aligned}$$

Observe that the first term on the right hand side is the same as the term on the left hand side. We therefore obtain (A.3.5).

From $3\Pi_\alpha{}^{\mu\nu} = \mathbf{D}_\alpha K^{\mu\nu} + \mathbf{D}^\mu K^\nu{}_\alpha + \mathbf{D}^\nu K^\mu{}_\alpha$, we have

$$\begin{aligned}
(\mathbf{D}_\alpha K^{\mu\nu})\mathbf{D}^\alpha\mathbf{D}_\nu\Psi_{ab} &= (-\mathbf{D}^\mu K^\nu{}_\alpha - \mathbf{D}^\nu K^\mu{}_\alpha + 3\Pi_\alpha{}^{\mu\nu})\mathbf{D}^\alpha\mathbf{D}_\nu\Psi_{ab} \\
&= -\mathbf{D}^\mu K^\nu{}_\alpha\mathbf{D}^\alpha\mathbf{D}_\nu\Psi_{ab} - \mathbf{D}^\nu K^\mu{}_\alpha\mathbf{D}^\alpha\mathbf{D}_\nu\Psi_{ab} + 3\Pi_\alpha{}^{\mu\nu}\mathbf{D}^\alpha\mathbf{D}_\nu\Psi_{ab} \\
&= -\mathbf{D}^\mu K^\nu{}_\alpha\mathbf{D}^\alpha\mathbf{D}_\nu\Psi_{ab} - \mathbf{D}^\nu K^\mu{}_\alpha\mathbf{D}_\nu\mathbf{D}^\alpha\Psi_{ab} \\
&\quad +\mathbf{D}^\nu K^\mu{}_\alpha(\mathbf{R}_\nu{}^\alpha{}_a{}^\epsilon\Psi_{\epsilon b} + \mathbf{R}_\nu{}^\alpha{}_b{}^\epsilon\Psi_{a\epsilon}) + 3\Pi_\alpha{}^{\mu\nu}\mathbf{D}^\alpha\mathbf{D}_\nu\Psi_{ab} \\
&= -\mathbf{D}^\mu K^\nu{}_\alpha\mathbf{D}^\alpha\mathbf{D}_\nu\Psi_{ab} - \mathbf{D}_\alpha K^{\mu\nu}\mathbf{D}^\alpha\mathbf{D}_\nu\Psi_{ab} \\
&\quad +\mathbf{D}^\nu K^\mu{}_\alpha(\mathbf{R}_\nu{}^\alpha{}_a{}^\epsilon\Psi_{\epsilon b} + \mathbf{R}_\nu{}^\alpha{}_b{}^\epsilon\Psi_{a\epsilon}) + 3\Pi_\alpha{}^{\mu\nu}\mathbf{D}^\alpha\mathbf{D}_\nu\Psi_{ab},
\end{aligned}$$

which implies (A.3.6).

We similarly compute

$$\begin{aligned}
&(\mathbf{D}_\alpha K^{\mu\nu})\mathbf{D}^\alpha\mathbf{D}_\mu\mathbf{D}_\nu\Psi_{ab} \\
=\;& (-\mathbf{D}^\mu K^\nu{}_\alpha - \mathbf{D}^\nu K^\mu{}_\alpha + 3\Pi_{\alpha\mu\nu})\mathbf{D}^\alpha\mathbf{D}_\mu\mathbf{D}_\nu\Psi_{ab} \\
=\;& -\mathbf{D}^\mu K^\nu{}_\alpha\mathbf{D}^\alpha\mathbf{D}_\mu\mathbf{D}_\nu\Psi_{ab} - \mathbf{D}^\nu K^\mu{}_\alpha\mathbf{D}^\alpha\mathbf{D}_\nu\mathbf{D}_\mu\Psi_{ab} \\
& -\mathbf{D}^\nu K^\mu{}_\alpha\mathbf{D}^\alpha(\mathbf{R}_{\nu\mu a}{}^c\Psi_{cb} + \mathbf{R}_{\nu\mu b}{}^c\Psi_{ac}) + 3\Pi_{\alpha\mu\nu}\mathbf{D}^\alpha\mathbf{D}_\mu\mathbf{D}_\nu\Psi_{ab} \\
=\;& -2\mathbf{D}^\mu K^\nu{}_\alpha\mathbf{D}^\alpha\mathbf{D}_\mu\mathbf{D}_\nu\Psi_{ab} - \mathbf{D}^\nu K^\mu{}_\alpha\mathbf{D}^\alpha(\mathbf{R}_{\nu\mu a}{}^c\Psi_{cb} + \mathbf{R}_{\nu\mu b}{}^c\Psi_{ac}) + 3\Pi_{\alpha\mu\nu}\mathbf{D}^\alpha\mathbf{D}_\mu\mathbf{D}_\nu\Psi_{ab} \\
=\;& -2\mathbf{D}^\mu K^\nu{}_\alpha(\mathbf{D}_\mu\mathbf{D}^\alpha\mathbf{D}_\nu\Psi_{ab} + [\mathbf{D}^\alpha, \mathbf{D}_\mu]\mathbf{D}_\nu\Psi_{ab}) \\
& -\mathbf{D}^\nu K^\mu{}_\alpha\mathbf{D}^\alpha(\mathbf{R}_{\nu\mu a}{}^c\Psi_{cb} + \mathbf{R}_{\nu\mu b}{}^c\Psi_{ac}) + 3\Pi_{\alpha\mu\nu}\mathbf{D}^\alpha\mathbf{D}_\mu\mathbf{D}_\nu\Psi_{ab} \\
=\;& -2\mathbf{D}^\mu K^\nu{}_\alpha(\mathbf{D}_\mu\mathbf{D}^\alpha\mathbf{D}_\nu\Psi_{ab} + \mathbf{R}^\alpha{}_{\mu\nu}{}^\delta\mathbf{D}_\delta\Psi_{ab} + \mathbf{R}^\alpha{}_{\mu a}{}^c\mathbf{D}_\nu\Psi_{cb} + \mathbf{R}^\alpha{}_{\mu b}{}^c\mathbf{D}_\nu\Psi_{ac}) \\
& -\mathbf{D}^\nu K^\mu{}_\alpha\mathbf{D}^\alpha(\mathbf{R}_{\nu\mu a}{}^c\Psi_{cb} + \mathbf{R}_{\nu\mu b}{}^c\Psi_{ac}) + 3\Pi_{\alpha\mu\nu}\mathbf{D}^\alpha\mathbf{D}_\mu\mathbf{D}_\nu\Psi_{ab}.
\end{aligned}$$

Observe that the first term on the right hand side is the same as the term on the left hand side. We therefore obtain (A.3.7). □

We now prove the following.

Lemma A.3.2. *In a vacuum spacetime, the following commutation formulas hold for a scalar ϕ, a horizontal 1-tensor ψ and a horizontal 2-tensor Ψ:*

$$[\mathbf{D}_\nu, \square_{\mathbf{g}}]\phi = 0, \qquad [\dot{\mathbf{D}}_\mu, \dot{\square}_1]\psi^\mu = 0 \qquad [\dot{\mathbf{D}}_\nu, \dot{\square}_2]\Psi^\nu{}_\delta = 2\mathbf{R}_{\nu\alpha\delta\epsilon}\dot{\mathbf{D}}^\alpha\Psi^{\nu\epsilon}.$$

Also, for Φ a 3-tensor which is symmetric in the last two indices, we have,

$$[\dot{\mathbf{D}}_\nu, \square]\Phi^\nu{}_{\delta\lambda} = 2\mathbf{R}_{\nu\alpha\delta\epsilon}\dot{\mathbf{D}}^\alpha\Phi^{\nu\epsilon}{}_\lambda + 2\mathbf{R}_{\nu\alpha\lambda\epsilon}\dot{\mathbf{D}}^\alpha\Phi^{\nu\epsilon}{}_\delta.$$

Proof. We have for a scalar function ϕ:

$$[\mathbf{D}_\alpha, \mathbf{D}_\beta]\phi = 0, \qquad [\mathbf{D}_\alpha, \mathbf{D}_\beta]\mathbf{D}_\gamma\phi = \mathbf{R}_{\alpha\beta\gamma}{}^{\delta}\mathbf{D}_\delta\phi.$$

We therefore compute

$$\begin{aligned}
[\mathbf{D}_\nu, \square_{\mathbf{g}}]\phi &= [\mathbf{D}_\nu, \mathbf{D}^\alpha\mathbf{D}_\alpha]\phi = [\mathbf{D}_\nu, \mathbf{D}^\alpha]\mathbf{D}_\alpha\phi + \mathbf{D}^\alpha[\mathbf{D}_\nu, \mathbf{D}_\alpha]\phi \\
&= \mathbf{g}^{\alpha\mu}\mathbf{R}_{\nu\mu\alpha}{}^{\delta}\mathbf{D}_\delta\phi = -\mathbf{R}_\nu{}^{\delta}\mathbf{D}_\delta\phi = 0.
\end{aligned}$$

For a 1-tensor X we have

$$[\mathbf{D}_\alpha, \mathbf{D}_\beta]X_\gamma = \mathbf{R}_{\alpha\beta\gamma\epsilon}X^\epsilon, \qquad [\mathbf{D}_\alpha, \mathbf{D}_\beta]\mathbf{D}_\delta X_\gamma = \mathbf{R}_{\alpha\beta\delta\epsilon}\mathbf{D}^\epsilon X_\gamma + \mathbf{R}_{\alpha\beta\gamma\epsilon}\mathbf{D}_\delta X^\epsilon.$$

This gives

$$\begin{aligned}
[\mathbf{D}_\mu, \square_{\mathbf{g}}]X_\beta &= [\mathbf{D}_\mu, \mathbf{D}^\alpha\mathbf{D}_\alpha]X_\beta = [\mathbf{D}_\mu, \mathbf{D}^\alpha]\mathbf{D}_\alpha X_\beta + \mathbf{D}^\alpha([\mathbf{D}_\mu, \mathbf{D}_\alpha]X_\beta) \\
&= \mathbf{g}^{\alpha\zeta}(\mathbf{R}_{\mu\zeta\alpha}{}^{\epsilon}\mathbf{D}_\epsilon X_\beta + \mathbf{R}_{\mu\zeta\beta}{}^{\epsilon}\mathbf{D}_\alpha X_\epsilon) + \mathbf{D}^\alpha(\mathbf{R}_{\mu\alpha\beta}{}^{\epsilon}X_\epsilon) \\
&= -\mathbf{R}_\mu{}^{\epsilon}\mathbf{D}_\epsilon X_\beta + \mathbf{R}_\mu{}^{\alpha}{}_{\beta}{}^{\epsilon}\mathbf{D}_\alpha X_\epsilon + \mathbf{D}^\alpha\mathbf{R}_{\mu\alpha\beta}{}^{\epsilon}X_\epsilon + \mathbf{R}_{\mu\alpha\beta}{}^{\epsilon}\mathbf{D}^\alpha X_\epsilon.
\end{aligned}$$

Since by the Bianchi identities $\mathbf{D}^\alpha\mathbf{R}_{\alpha\mu\nu\gamma} = \mathbf{D}_\nu\mathbf{R}_{\mu\gamma} - \mathbf{D}_\gamma\mathbf{R}_{\mu\nu}$, the divergence of the Riemann tensor vanishes in the case of a vacuum spacetime. We therefore have

$$[\mathbf{D}_\mu, \square_{\mathbf{g}}]X_\beta = 2\mathbf{R}_{\mu\alpha\beta}{}^{\epsilon}\mathbf{D}^\alpha X_\epsilon, \qquad [\mathbf{D}_\mu, \square_{\mathbf{g}}]X^\mu = 2\mathbf{R}_{\mu\alpha}{}^{\mu\epsilon}\mathbf{D}^\alpha X_\epsilon = 0.$$

For a 2-tensor Ψ we have

$$\begin{aligned}
[\mathbf{D}_\nu, \mathbf{D}_\alpha]\Psi_{\gamma\delta} &= \mathbf{R}_{\nu\alpha\gamma}{}^{\epsilon}\Psi_{\epsilon\delta} + \mathbf{R}_{\nu\alpha\delta}{}^{\epsilon}\Psi_{\gamma\epsilon}, \\
[\mathbf{D}_\nu, \mathbf{D}_\zeta]\mathbf{D}_\alpha\Psi_{\gamma\delta} &= \mathbf{R}_{\nu\zeta\alpha}{}^{\epsilon}\mathbf{D}_\epsilon\Psi_{\gamma\delta} + \mathbf{R}_{\nu\zeta\gamma}{}^{\epsilon}\mathbf{D}_\alpha\Psi_{\epsilon\delta} + \mathbf{R}_{\nu\zeta\delta}{}^{\epsilon}\mathbf{D}_\alpha\Psi_{\gamma\epsilon}.
\end{aligned}$$

In the case of a horizontal 2-tensor

$$\begin{aligned}
[\dot{\mathbf{D}}_\nu, \dot{\mathbf{D}}_\mu]\Psi_{ab} &= \mathbf{R}_{\nu\mu a}{}^{c}\Psi_{cb} + \mathbf{R}_{\nu\mu b}{}^{c}\Psi_{ac}, \\
[\dot{\mathbf{D}}_\nu, \dot{\mathbf{D}}_\zeta]\mathbf{D}_\alpha\Psi_{ab} &= \mathbf{R}_{\nu\zeta\alpha}{}^{\epsilon}\dot{\mathbf{D}}_\epsilon\Psi_{ab} + \mathbf{R}_{\nu\zeta a}{}^{c}\dot{\mathbf{D}}_\alpha\Psi_{cb} + \mathbf{R}_{\nu\zeta b}{}^{c}\dot{\mathbf{D}}_\alpha\Psi_{ac}.
\end{aligned}$$

We therefore compute

$$\begin{aligned}
[\mathbf{D}_\nu, \square_{\mathbf{g}}]\Psi_{\gamma\delta} &= [\mathbf{D}_\nu, \mathbf{D}^\alpha\mathbf{D}_\alpha]\Psi_{\gamma\delta} = [\mathbf{D}_\nu, \mathbf{D}^\alpha]\mathbf{D}_\alpha\Psi_{\gamma\delta} + \mathbf{D}^\alpha[\mathbf{D}_\nu, \mathbf{D}_\alpha]\Psi_{\gamma\delta} \\
&= \mathbf{g}^{\alpha\zeta}\left(\mathbf{R}_{\nu\zeta\alpha}{}^{\epsilon}\mathbf{D}_\epsilon\Psi_{\gamma\delta} + \mathbf{R}_{\nu\zeta\gamma}{}^{\epsilon}\mathbf{D}_\alpha\Psi_{\epsilon\delta} + \mathbf{R}_{\nu\zeta\delta}{}^{\epsilon}\mathbf{D}_\alpha\Psi_{\gamma\epsilon}\right) \\
&\quad + \mathbf{D}^\alpha(\mathbf{R}_{\nu\alpha\gamma}{}^{\epsilon}\Psi_{\epsilon\delta} + \mathbf{R}_{\nu\alpha\delta}{}^{\epsilon}\Psi_{\gamma\epsilon}) \\
&= -\mathbf{R}_\nu{}^{\epsilon}\mathbf{D}_\epsilon\Psi_{\gamma\delta} + \mathbf{R}_\nu{}^{\alpha}{}_{\gamma}{}^{\epsilon}\mathbf{D}_\alpha\Psi_{\epsilon\delta} + \mathbf{R}_\nu{}^{\alpha}{}_{\delta}{}^{\epsilon}\mathbf{D}_\alpha\Psi_{\gamma\epsilon} \\
&\quad + \mathbf{D}^\alpha\mathbf{R}_{\nu\alpha\gamma}{}^{\epsilon}\Psi_{\epsilon\delta} + \mathbf{R}_{\nu\alpha\gamma}{}^{\epsilon}\mathbf{D}^\alpha\Psi_{\epsilon\delta} + \mathbf{D}^\alpha\mathbf{R}_{\nu\alpha\delta}{}^{\epsilon}\Psi_{\gamma\epsilon} + \mathbf{R}_{\nu\alpha\delta}{}^{\epsilon}\mathbf{D}^\alpha\Psi_{\gamma\epsilon} \\
&= 2\mathbf{R}_\nu{}^{\alpha}{}_{\gamma}{}^{\epsilon}\mathbf{D}_\alpha\Psi_{\epsilon\delta} + 2\mathbf{R}_\nu{}^{\alpha}{}_{\delta}{}^{\epsilon}\mathbf{D}_\alpha\Psi_{\gamma\epsilon}.
\end{aligned}$$

This gives

$$[\mathbf{D}_\nu, \Box_{\mathbf{g}}]\Psi^\nu{}_\delta = 2\mathbf{R}_\nu{}^{\alpha\nu\epsilon}\mathbf{D}_\alpha\Psi_{\epsilon\delta} + 2\mathbf{R}_\nu{}^\alpha{}_\delta{}^\epsilon\mathbf{D}_\alpha\Psi^\nu{}_\epsilon = 2\mathbf{R}_\nu{}^\alpha{}_\delta{}^\epsilon\mathbf{D}_\alpha\Psi^\nu{}_\epsilon.$$

For a 3-tensor Φ we have

$$\begin{aligned}
[\mathbf{D}_\nu, \mathbf{D}_\alpha]\Phi_{\gamma\delta\lambda} &= \mathbf{R}_{\nu\alpha\gamma}{}^\epsilon\Phi_{\epsilon\delta\lambda} + \mathbf{R}_{\nu\alpha\delta}{}^\epsilon\Phi_{\gamma\epsilon\lambda} + \mathbf{R}_{\nu\alpha\lambda}{}^\epsilon\Phi_{\gamma\delta\epsilon},\\
[\mathbf{D}_\nu, \mathbf{D}_\zeta]\mathbf{D}_\alpha\Phi_{\gamma\delta\lambda} &= \mathbf{R}_{\nu\zeta\alpha}{}^\epsilon\mathbf{D}_\epsilon\Phi_{\gamma\delta\lambda} + \mathbf{R}_{\nu\zeta\gamma}{}^\epsilon\mathbf{D}_\alpha\Phi_{\epsilon\delta\lambda} + \mathbf{R}_{\nu\zeta\delta}{}^\epsilon\mathbf{D}_\alpha\Phi_{\gamma\epsilon\lambda} + \mathbf{R}_{\nu\zeta\lambda}{}^\epsilon\mathbf{D}_\alpha\Phi_{\gamma\delta\epsilon}.
\end{aligned}$$

We therefore compute

$$\begin{aligned}
[\mathbf{D}_\nu, \Box_{\mathbf{g}}]\Phi_{\gamma\delta\lambda} &= [\mathbf{D}_\nu, \mathbf{D}^\alpha\mathbf{D}_\alpha]\Phi_{\gamma\delta\lambda} = [\mathbf{D}_\nu, \mathbf{D}^\alpha]\mathbf{D}_\alpha\Phi_{\gamma\delta\lambda} + \mathbf{D}^\alpha[\mathbf{D}_\nu, \mathbf{D}_\alpha]\Phi_{\gamma\delta\lambda}\\
&= \mathbf{g}^{\alpha\zeta}\left(\mathbf{R}_{\nu\zeta\alpha}{}^\epsilon\mathbf{D}_\epsilon\Phi_{\gamma\delta\lambda} + \mathbf{R}_{\nu\zeta\gamma}{}^\epsilon\mathbf{D}_\alpha\Phi_{\epsilon\delta\lambda} + \mathbf{R}_{\nu\zeta\delta}{}^\epsilon\mathbf{D}_\alpha\Phi_{\gamma\epsilon\lambda} + \mathbf{R}_{\nu\zeta\lambda}{}^\epsilon\mathbf{D}_\alpha\Phi_{\gamma\delta\epsilon}\right)\\
&\quad +\mathbf{D}^\alpha(\mathbf{R}_{\nu\alpha\gamma}{}^\epsilon\Phi_{\epsilon\delta\lambda} + \mathbf{R}_{\nu\alpha\delta}{}^\epsilon\Phi_{\gamma\epsilon\lambda} + \mathbf{R}_{\nu\alpha\lambda}{}^\epsilon\Phi_{\gamma\delta\epsilon})\\
&= \mathbf{R}_\nu{}^\alpha{}_\gamma{}^\epsilon\mathbf{D}_\alpha\Phi_{\epsilon\delta\lambda} + \mathbf{R}_\nu{}^\alpha{}_\delta{}^\epsilon\mathbf{D}_\alpha\Phi_{\gamma\epsilon\lambda} + \mathbf{R}_\nu{}^\alpha{}_\lambda{}^\epsilon\mathbf{D}_\alpha\Phi_{\gamma\delta\epsilon}\\
&\quad +\mathbf{R}_{\nu\alpha\gamma}{}^\epsilon\mathbf{D}^\alpha\Phi_{\epsilon\delta\lambda} + \mathbf{R}_{\nu\alpha\delta}{}^\epsilon\mathbf{D}^\alpha\Phi_{\gamma\epsilon\lambda} + \mathbf{R}_{\nu\alpha\lambda}{}^\epsilon\mathbf{D}^\alpha\Phi_{\gamma\delta\epsilon}\\
&= 2\mathbf{R}_\nu{}^\alpha{}_\gamma{}^\epsilon\mathbf{D}_\alpha\Phi_{\epsilon\delta\lambda} + 2\mathbf{R}_\nu{}^\alpha{}_\delta{}^\epsilon\mathbf{D}_\alpha\Phi_{\gamma\epsilon\lambda} + 2\mathbf{R}_\nu{}^\alpha{}_\lambda{}^\epsilon\mathbf{D}_\alpha\Phi_{\gamma\delta\epsilon}.
\end{aligned}$$

This gives

$$[\mathbf{D}_\nu, \Box]\Phi^\nu{}_{\delta\lambda} = 2\mathbf{R}_{\nu\alpha\delta\epsilon}\mathbf{D}^\alpha\Phi^{\nu\epsilon}{}_\lambda + 2\mathbf{R}_{\nu\alpha\lambda\epsilon}\mathbf{D}^\alpha\Phi^{\nu\epsilon}{}_\delta.$$

We can similarly adapt the above computations to the case of horizontal tensors. $\Box$

We can finally prove Proposition 2.3.7. We first consider the case of a k-tensor Ψ:

$$\begin{aligned}
[\mathcal{K}, \Box_{\mathbf{g}}]\Psi &= [\mathbf{D}_\mu K^{\mu\nu}\mathbf{D}_\nu, \Box_{\mathbf{g}}]\Psi = \mathbf{D}_\mu[K^{\mu\nu}\mathbf{D}_\nu, \Box_{\mathbf{g}}]\Psi + [\mathbf{D}_\mu, \Box_{\mathbf{g}}]K^{\mu\nu}\mathbf{D}_\nu\Psi\\
&= \mathbf{D}_\mu(K^{\mu\nu}\mathbf{D}_\nu\Box_{\mathbf{g}}\Psi - \Box_{\mathbf{g}}(K^{\mu\nu}\mathbf{D}_\nu\Psi)) + [\mathbf{D}_\mu, \Box_{\mathbf{g}}]K^{\mu\nu}\mathbf{D}_\nu\Psi.
\end{aligned}$$

Writing $\Box_{\mathbf{g}}(K^{\mu\nu}\mathbf{D}_\nu\Psi) = \Box_{\mathbf{g}}K^{\mu\nu}\mathbf{D}_\nu\Psi + 2\mathbf{D}^\alpha K^{\mu\nu}\mathbf{D}_\alpha\mathbf{D}_\nu\Psi + K^{\mu\nu}\Box_{\mathbf{g}}\mathbf{D}_\nu\Psi$, we have

$$[\mathcal{K}, \Box_{\mathbf{g}}]\Psi = \mathbf{D}_\mu(K^{\mu\nu}[\mathbf{D}_\nu, \Box_{\mathbf{g}}]\Psi - \Box_{\mathbf{g}}K^{\mu\nu}\mathbf{D}_\nu\Psi - 2\mathbf{D}^\alpha K^{\mu\nu}\mathbf{D}_\alpha\mathbf{D}_\nu\Psi) + [\mathbf{D}_\mu, \Box_{\mathbf{g}}]K^{\mu\nu}\mathbf{D}_\nu\Psi.$$

We now specialize to the case of a scalar function ϕ. By Lemma A.3.2, we have $[\mathbf{D}_\nu, \Box_{\mathbf{g}}]\phi = 0$, and $[\mathbf{D}_\mu, \Box]X^\mu = 0$ applied to $X^\mu = K^{\mu\nu}\mathbf{D}_\nu\phi$ gives $[\mathbf{D}_\mu, \Box_k]K^{\mu\nu}\mathbf{D}_\nu\phi = 0$. We are left with

$$\begin{aligned}
[\mathcal{K}, \Box_{\mathbf{g}}]\phi &= \mathbf{D}_\mu(K^{\mu\nu}[\mathbf{D}_\nu, \Box_{\mathbf{g}}]\phi - \Box_{\mathbf{g}}K^{\mu\nu}\mathbf{D}_\nu\phi - 2\mathbf{D}^\alpha K^{\mu\nu}\mathbf{D}_\alpha\mathbf{D}_\nu\phi) + [\mathbf{D}_\mu, \Box_{\mathbf{g}}]K^{\mu\nu}\mathbf{D}_\nu\phi\\
&= \mathbf{D}_\mu(-\Box_{\mathbf{g}}K^{\mu\nu}\mathbf{D}_\nu\phi - 2\mathbf{D}_\alpha K^{\mu\nu}\mathbf{D}^\alpha\mathbf{D}_\nu\phi).
\end{aligned}$$

Using (A.3.4), we obtain

$$[\mathcal{K}, \Box_{\mathbf{g}}]\phi = \mathbf{D}_\mu(-\Box_2 K^{\mu\nu}\mathbf{D}_\nu\phi + \mathbf{D}^\mu K^\nu{}_\alpha\mathbf{D}^\alpha\mathbf{D}_\nu\phi) - 3\mathbf{D}_\mu(\Pi_\alpha{}^{\mu\nu}\mathbf{D}^\alpha\mathbf{D}_\nu\phi).$$

Observe that in expanding the derivative there is a cancellation of the term $(\Box_{\mathbf{g}}K^{\mu\nu})\mathbf{D}_\mu\mathbf{D}_\nu\phi$. Indeed,

$$\begin{aligned}
[\mathcal{K},\Box_{\mathbf{g}}]\phi &= -\mathbf{D}_\mu(\Box_{\mathbf{g}}K^{\mu\nu})\mathbf{D}_\nu\phi - \Box_{\mathbf{g}}K^{\mu\nu}\mathbf{D}_\mu\mathbf{D}_\nu\phi + \mathbf{D}_\mu\mathbf{D}^\mu K^\nu{}_\alpha\mathbf{D}_\nu\mathbf{D}^\alpha\phi \\
&\quad +\mathbf{D}^\mu K^\nu{}_\alpha\mathbf{D}_\mu\mathbf{D}^\alpha\mathbf{D}_\nu\phi - 3\mathbf{D}_\mu(\Pi_\alpha{}^{\mu\nu}\mathbf{D}^\alpha\mathbf{D}_\nu\phi) \\
&= -\mathbf{D}_\mu(\Box_{\mathbf{g}}K^{\mu\nu})\mathbf{D}_\nu\phi + \mathbf{D}^\mu K^\nu{}_\alpha\mathbf{D}_\mu\mathbf{D}^\alpha\mathbf{D}_\nu\phi - 3\mathbf{D}_\mu(\Pi_\alpha{}^{\mu\nu}\mathbf{D}^\alpha\mathbf{D}_\nu\phi).
\end{aligned}$$

Using (A.3.3) and (A.3.5), we obtain

$$\begin{aligned}
[\mathcal{K},\Box_{\mathbf{g}}]\phi &= \mathbf{D}^\mu K_{\alpha\epsilon}\left(\frac{1}{3}\mathbf{R}^{\alpha\delta}{}_\mu{}^\epsilon - \mathbf{R}^\alpha{}_\mu{}^{\delta\epsilon}\right)\mathbf{D}_\delta\phi - \frac{2}{3}\mathbf{D}^\mu K^\epsilon{}_\alpha\mathbf{R}^\alpha{}_{\mu\epsilon}{}^\delta\mathbf{D}_\delta\phi \\
&\quad +(\mathbf{D}^\mu\mathbf{D}^\alpha\Pi_{\alpha\nu\mu} - \frac{1}{2}\mathbf{D}^\mu\mathbf{D}_\mu\Pi^\alpha{}_{\alpha\nu} + \frac{1}{2}\mathbf{D}^\mu\mathbf{D}_\nu\Pi^\alpha{}_{\alpha\mu})\mathbf{D}_\nu\phi \\
&\quad +\Pi_{\mu\alpha\nu}\mathbf{D}^\mu\mathbf{D}_\alpha\mathbf{D}_\nu\phi - 3\mathbf{D}_\mu(\Pi_\alpha{}^{\mu\nu}\mathbf{D}^\alpha\mathbf{D}_\nu\phi).
\end{aligned}$$

Using that $\mathbf{R}^\alpha{}_{\mu\epsilon}{}^\delta = -\mathbf{R}^\alpha{}_\mu{}^\delta{}_\epsilon$ we have

$$\begin{aligned}
[\mathcal{K},\Box_{\mathbf{g}}]\phi &= \frac{1}{3}\mathbf{D}^\mu K_{\alpha\epsilon}\left(\mathbf{R}^{\alpha\delta}{}_\mu{}^\epsilon - \mathbf{R}^\alpha{}_\mu{}^{\delta\epsilon}\right)\mathbf{D}_\delta\phi \\
&\quad +\left(\mathbf{D}^\mu\mathbf{D}^\alpha\Pi_{\alpha\nu\mu} - \frac{1}{2}\mathbf{D}^\mu\mathbf{D}_\mu\Pi^\alpha{}_{\alpha\nu} + \frac{1}{2}\mathbf{D}^\mu\mathbf{D}_\nu\Pi^\alpha{}_{\alpha\mu}\right)\mathbf{D}_\nu\phi \\
&\quad +\Pi_{\mu\alpha\nu}\mathbf{D}^\mu\mathbf{D}_\alpha\mathbf{D}_\nu\phi - 3\mathbf{D}_\mu(\Pi_\alpha{}^{\mu\nu}\mathbf{D}^\alpha\mathbf{D}_\nu\phi).
\end{aligned}$$

Writing

$$\mathbf{D}_\mu K_{\epsilon\alpha} = -\mathbf{D}_\epsilon K_{\alpha\mu} - \mathbf{D}_\alpha K_{\mu\epsilon} + 3\Pi_{\mu\epsilon\alpha}$$

we can observe that the first term is symmetric in $\alpha\mu$ while the second Riemann tensor term in antisymmetric in $\alpha\mu$, and the second term in symmetric in $\mu\epsilon$ while the first Riemann tensor is antisymmetric in $\mu\nu$. We therefore are left with

$$\begin{aligned}
[\mathcal{K},\Box_{\mathbf{g}}]\phi &= \frac{1}{3}(-\mathbf{D}_\epsilon K_\alpha{}^\mu\mathbf{R}^{\alpha\delta}{}_\mu{}^\epsilon + \mathbf{D}_\alpha K^\mu{}_\epsilon\mathbf{R}^\alpha{}_\mu{}^{\delta\epsilon})\mathbf{D}_\delta\phi \\
&\quad +\left(\mathbf{D}^\mu\mathbf{D}^\alpha\Pi_{\alpha\nu\mu} - \frac{1}{2}\mathbf{D}^\mu\mathbf{D}_\mu\Pi^\alpha{}_{\alpha\nu} + \frac{1}{2}\mathbf{D}^\mu\mathbf{D}_\nu\Pi^\alpha{}_{\alpha\mu}\right)\mathbf{D}_\nu\phi \\
&\quad +\Pi_{\mu\alpha\nu}\mathbf{D}^\mu\mathbf{D}_\alpha\mathbf{D}_\nu\phi - 3\mathbf{D}_\mu(\Pi_\alpha{}^{\mu\nu}\mathbf{D}^\alpha\mathbf{D}_\nu\phi).
\end{aligned}$$

Writing $\mathbf{D}^\nu K_\alpha{}^\mu\mathbf{R}^{\alpha\delta}{}_{\mu\nu} = \mathbf{D}^\nu K_\alpha{}^\mu\mathbf{R}_{\mu\nu}{}^{\alpha\delta} = \mathbf{D}^\alpha K_\nu{}^\mu\mathbf{R}_{\mu\alpha}{}^{\nu\delta} = -\mathbf{D}^\alpha K_\nu{}^\mu\mathbf{R}_{\alpha\mu}{}^{\nu\delta}$ we obtain the cancellation of the first line, which gives

$$[\mathcal{K},\Box_{\mathbf{g}}]\phi = \mathrm{Err}[\Pi](\phi),$$

with

$$\begin{aligned}\mathrm{Err}[\Pi](\phi) &= \left(\mathbf{D}^\mu\mathbf{D}^\alpha\Pi_{\alpha\nu\mu} - \frac{1}{2}\mathbf{D}^\mu\mathbf{D}_\mu\Pi^\alpha{}_{\alpha\nu} + \frac{1}{2}\mathbf{D}^\mu\mathbf{D}_\nu\Pi^\alpha{}_{\alpha\mu}\right)\mathbf{D}_\nu\phi \\ &\quad +\Pi_{\mu\alpha\nu}\mathbf{D}^\mu\mathbf{D}_\alpha\mathbf{D}_\nu\phi - 3\mathbf{D}_\mu(\Pi_\alpha{}^{\mu\nu}\mathbf{D}^\alpha\mathbf{D}_\nu\phi).\end{aligned}$$

We now simplify $\mathrm{Err}[\Pi](\phi)$. By writing

$$\Pi_{\mu\alpha\nu}\mathbf{D}^\mu\mathbf{D}^\alpha\mathbf{D}^\nu\phi = \mathbf{D}^\mu(\Pi_{\mu\alpha\nu}\mathbf{D}^\alpha\mathbf{D}^\nu\phi) - \mathbf{D}^\mu\Pi_{\mu\alpha\nu}\mathbf{D}^\alpha\mathbf{D}^\nu\phi,$$

we obtain

$$\begin{aligned}\mathrm{Err}[\Pi](\phi) &= \mathbf{D}^\mu\left(\mathbf{D}^\alpha\Pi_{\alpha\nu\mu} - \frac{1}{2}\mathbf{D}_\mu\Pi^\alpha{}_{\alpha\nu} + \frac{1}{2}\mathbf{D}_\nu\Pi^\alpha{}_{\alpha\mu}\right)\mathbf{D}^\nu\phi \\ &\quad -\mathbf{D}^\mu\Pi_{\mu\alpha\nu}\mathbf{D}^\alpha\mathbf{D}^\nu\phi - 2\mathbf{D}^\mu(\Pi_{\mu\alpha\nu}\mathbf{D}^\alpha\mathbf{D}^\nu\phi).\end{aligned}$$

By writing

$$\begin{aligned}&\mathbf{D}^\mu\Big[\left(\mathbf{D}^\alpha\Pi_{\alpha\nu\mu} - \frac{1}{2}\mathbf{D}_\mu\Pi^\alpha{}_{\alpha\nu} + \frac{1}{2}\mathbf{D}_\nu\Pi^\alpha{}_{\alpha\mu}\right)\mathbf{D}^\nu\phi\Big] \\ = \;&\mathbf{D}^\mu\left(\mathbf{D}^\alpha\Pi_{\alpha\nu\mu} - \frac{1}{2}\mathbf{D}_\mu\Pi^\alpha{}_{\alpha\nu} + \frac{1}{2}\mathbf{D}_\nu\Pi^\alpha{}_{\alpha\mu}\right)\mathbf{D}^\nu\phi \\ &+\left(\mathbf{D}^\alpha\Pi_{\alpha\nu\mu} - \frac{1}{2}\mathbf{D}_\mu\Pi^\alpha{}_{\alpha\nu} + \frac{1}{2}\mathbf{D}_\nu\Pi^\alpha{}_{\alpha\mu}\right)\mathbf{D}^\mu\mathbf{D}^\nu\phi.\end{aligned}$$

Observe that

$$\left(-\frac{1}{2}\mathbf{D}_\mu\Pi^\alpha{}_{\alpha\nu} + \frac{1}{2}\mathbf{D}_\nu\Pi^\alpha{}_{\alpha\mu}\right)\mathbf{D}^\mu\mathbf{D}^\nu\phi = 0$$

since the first term is antisymmetric in $\mu\nu$ and the second term is symmetric in $\mu\nu$. We can therefore write

$$\begin{aligned}&\mathbf{D}^\mu\left(\mathbf{D}^\alpha\Pi_{\alpha\nu\mu} - \frac{1}{2}\mathbf{D}_\mu\Pi^\alpha{}_{\alpha\nu} + \frac{1}{2}\mathbf{D}_\nu\Pi^\alpha{}_{\alpha\mu}\right)\mathbf{D}^\nu\phi \\ = \;&\mathbf{D}^\mu\Big[\left(\mathbf{D}^\alpha\Pi_{\alpha\nu\mu} - \frac{1}{2}\mathbf{D}_\mu\Pi^\alpha{}_{\alpha\nu} + \frac{1}{2}\mathbf{D}_\nu\Pi^\alpha{}_{\alpha\mu}\right)\mathbf{D}^\nu\phi\Big] - (\mathbf{D}^\alpha\Pi_{\alpha\nu\mu})\mathbf{D}^\mu\mathbf{D}^\nu\phi\end{aligned}$$

and we obtain

$$\begin{aligned}\mathrm{Err}[\Pi](\phi) &= \mathbf{D}^\mu\Big[(\mathbf{D}^\alpha\Pi_{\alpha\nu\mu} - \frac{1}{2}\mathbf{D}_\mu\Pi^\alpha{}_{\alpha\nu} + \frac{1}{2}\mathbf{D}_\nu\Pi^\alpha{}_{\alpha\mu})\mathbf{D}^\nu\phi - 2\Pi_{\mu\alpha\nu}\mathbf{D}^\alpha\mathbf{D}^\nu\phi\Big] \\ &\quad -2(\mathbf{D}^\alpha\Pi_{\alpha\nu\mu})\mathbf{D}^\mu\mathbf{D}^\nu\phi\end{aligned}$$

which proves the Proposition.

A.4 Proof of Proposition 2.1.48

We write, for any $\psi \in \mathbf{O}_k$

$$\begin{aligned}
\nabla_a\nabla_b\psi\cdot\nabla_c\nabla_d\psi &= \nabla_a\Big(\nabla_b\psi\cdot\nabla_c\nabla_d\psi\Big)-\nabla_b\psi\cdot\nabla_a\nabla_c\nabla_d\psi\\
&= -\nabla_b\psi\cdot\nabla_c\nabla_a\nabla_d\psi-\nabla_b\psi\cdot[\nabla_a,\nabla_c]\nabla_d\psi+\nabla_a\Big(\nabla_b\psi\cdot\nabla_c\nabla_d\psi\Big)\\
&= \nabla_c\nabla_b\psi\cdot\nabla_a\nabla_d\psi-\nabla_b\psi\cdot[\nabla_a,\nabla_c]\nabla_d\psi+\nabla_a\Big(\nabla_b\psi\cdot\nabla_c\nabla_d\psi\Big)\\
&\quad -\nabla_c\Big(\nabla_b\psi\cdot\nabla_a\nabla_d\psi\Big).
\end{aligned}$$

We deduce, for $\triangle=\triangle_k$,

$$\begin{aligned}
\big|\triangle\psi|^2 &= g^{ab}g^{cd}\nabla_a\nabla_b\psi\cdot\nabla_c\nabla_d\psi\\
&= \nabla^a\nabla^c\psi\cdot\nabla_c\nabla_a\psi-g^{ab}g^{cd}\nabla_b\psi\cdot[\nabla_a,\nabla_c]\nabla_d\psi\\
&\quad +\nabla_a\Big(\nabla^a\psi\cdot\triangle\psi\Big)-\nabla_c\Big(\nabla_a\psi\cdot\nabla^a\nabla^c\psi\Big)\\
&= |\nabla^2\psi|^2-\nabla^a\nabla^c\psi\cdot[\nabla_a,\nabla_c]\psi-g^{ab}g^{cd}\nabla_b\psi\cdot[\nabla_a,\nabla_c]\nabla_d\psi+\mathrm{d}iv_{\,k}[\triangle\psi]\\
&= |\nabla^2\psi|^2-\frac12[\nabla^a,\nabla^c]\psi\cdot[\nabla_a,\nabla_c]\psi-g^{ab}g^{cd}\nabla_b\psi\cdot[\nabla_a,\nabla_c]\nabla_d\psi+\mathrm{d}iv_{\,k}[\triangle\psi],
\end{aligned}$$

with divergence term of the form

$$\mathrm{d}iv_{\,k}[\triangle\psi]:\ =\ \nabla_a\Big(\nabla^a\triangle\psi\Big)-\nabla_c\Big(\nabla_a\cdot\nabla^a\nabla^c\psi\Big)=\nabla_a\Big(\nabla^a\psi\cdot\triangle\psi\Big)-\nabla_a\Big(\nabla_c\psi\cdot\nabla^c\nabla^a\psi\Big).$$

Hence

$$\begin{aligned}
\big|\triangle\psi|^2 &= \big|\nabla^2\psi\big|^2-A-\frac12 B+\mathrm{d}iv_{\,k}[\Delta\psi]\\
A &:= g^{ab}g^{cd}\nabla_b\psi\cdot[\nabla_a,\nabla_c]\nabla_d\psi\\
B &:= [\nabla^a,\nabla^c]\psi\cdot[\nabla_a,\nabla_c]\psi\\
\mathrm{d}iv_{\,k}[\triangle\psi] &:= \nabla_a\Big(\nabla^a\psi\cdot\triangle\psi\Big)-\nabla_a\Big(\nabla_c\psi\cdot\nabla^c\nabla^a\psi\Big).
\end{aligned}\tag{A.4.1}$$

Now, for horizontal indices $I=i_1\dots i_k$, we deduce using Proposition 2.1.45

$$\begin{aligned}
[\nabla_a,\nabla_c]\nabla_d\psi_I &= \frac12\big({}^{(a)}\mathrm{tr}\chi\nabla_3+{}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4\big)\nabla_d\psi_I\in_{ac}+{}^{(h)}K\big(g_{da}g_{tc}-g_{dc}g_{ta}\big)\nabla^t\psi_I\\
&\quad +{}^{(h)}K\Big[\big(g_{i_1a}g_{tc}-g_{i_1c}g_{ta}\big)\nabla_dU^t{}_{i_2\dots i_k}+\cdots\big(g_{i_ka}g_{tc}-g_{i_kc}g_{ta}\big)\nabla_dU_{i_1\dots i_{k-1}}{}^t\Big].
\end{aligned}$$

We deduce, assuming that $\psi\in\mathbf{O}_k$ is symmetric

$$\begin{aligned}
A &= \frac12\nabla\psi\cdot\big({}^{(a)}\mathrm{tr}\chi\nabla_3+{}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4\big)\ {}^*\nabla\psi-{}^{(h)}K|\nabla\psi|^2\\
&\quad +k\,{}^{(h)}K\Big(\nabla_a\psi^{a\dots}\nabla^a\psi_{a\dots}-\nabla_a\psi_{c\dots}\nabla^c\psi^{a\dots}\Big).
\end{aligned}\tag{A.4.2}$$

Similarly,

$$
\begin{aligned}
[\nabla_a, \nabla_c]\psi_I &= \frac{1}{2}\big({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4\big)\psi_I \in_{ac} \\
&+ {}^{(h)}K\Big[\big(g_{i_1 a}g_{tc} - g_{i_1 c}g_{ta}\big)U^t{}_{i_2\ldots i_k} + \cdots \big(g_{i_k a}g_{tc} - g_{i_k c}g_{ta}\big)U_{i_1\ldots i_{k-1}}{}^t\Big],
\end{aligned}
$$

and

$$
B = \frac{1}{2}\Big|\big({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4\big)\psi\Big|^2 + 2k\,{}^{(h)}K\big({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4\big)\psi\cdot\ {}^*\psi + 2k\,{}^{(h)}K^2|\psi|^2.
$$

We thus deduce,

$$
\begin{aligned}
\big|\triangle\psi|^2 = |\nabla^2\psi|^2 &+ {}^{(h)}K\Big(|\nabla\psi|^2 - k\,{}^{(h)}K|\psi|^2\Big) \\
&+ k\,{}^{(h)}K\Big(\nabla_a\psi_{c\cdots}\nabla^c\psi^{a\cdots} - \nabla_a\psi^{a\cdots}\nabla^a\psi_{a\ldots}\Big) \\
&+ \mathrm{Err}_{k,1}[\triangle\psi] + \mathrm{d}iv\,_k[\psi],
\end{aligned}
\tag{A.4.3}
$$

with error and divergence terms

$$
\begin{aligned}
\mathrm{Err}_{k,1}[\triangle\psi] &= -\frac{1}{2}\nabla\psi\cdot\big({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4\big)\ {}^*\nabla\psi + \frac{1}{2}\Big|\big({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4\big)\psi\Big|^2 \\
&\quad - k\,{}^{(h)}K\big({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4\big)\psi\cdot\ {}^*\psi \\
\mathrm{d}iv\,_k[\triangle\psi] &= \nabla_a\Big(\nabla^a\psi\cdot\triangle\psi - \nabla_c\psi\cdot\nabla^c\nabla^a\psi\Big).
\end{aligned}
$$

It remains to calculate the term

$$
J_k \;:=\; k\Big(\nabla_a\psi_{c\cdots}\nabla^c\psi^{a\cdots} - \nabla_a\psi^{a\cdots}\nabla^a\psi_{a\ldots}\Big).
$$

Scalar case. In the particular case when ψ is a scalar, $k = 0$, we deduce

$$
\begin{aligned}
\big|\triangle\psi|^2 &= \big|\nabla^2\psi\big|^2 + {}^{(h)}K|\nabla\psi|^2 + \mathrm{Err}_0[\triangle\psi] + \mathrm{d}iv\,_0[\triangle\psi] \\
\mathrm{Err}_0[\psi] &= -\frac{1}{2}\nabla\psi\cdot\big({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4\big)\ {}^*\nabla\psi + \frac{1}{2}\Big|\big({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4\big)\psi\Big|^2 \\
\mathrm{d}iv\,[\triangle\psi] &= \nabla_a\Big(\nabla^a\psi\cdot\triangle\psi - \frac{1}{2}\nabla^a|\nabla\psi|^2\Big),
\end{aligned}
$$

as stated.

Case $k = 1$. We consider now the case $k = 1$ in which case the term $J_1[\psi]$ takes the form

$$
\begin{aligned}
J_1 &= \Big(\nabla_a\psi_c\nabla^c\psi^a - \nabla_a\psi^a\nabla^a\psi_a\Big) = |\nabla\psi|^2 + \nabla_a\psi_c\big(\nabla^c\psi^a - \nabla^a\psi^c\big) - |\mathrm{d}iv\,\psi|^2 \\
&= |\nabla\psi|^2 + \frac{1}{2}\big(\nabla_a\psi_c - \nabla_c\psi_a\big)\big(\nabla^c\psi^a - \nabla^a\psi^c\big) - |\mathrm{d}iv\,\psi|^2 \\
&= |\nabla\psi|^2 - |curl\,\psi|^2 - |\mathrm{d}iv\,\psi|^2 = |\nabla\psi|^2 - |\,\mathcal{D}_1\psi|^2.
\end{aligned}
$$

We now recall (2.1.33) in Proposition 2.1.47. We deduce

$$\begin{aligned}J_1 &= |\nabla\psi|^2 - |\not{\mathcal{D}}_1\psi|^2\\ &= -{}^{(h)}K|\psi|^2 + \frac{1}{2}\Big(\Big({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4\Big)\,{}^*\psi\Big)\cdot\psi + \mathrm{d}iv\,[\not{\mathcal{D}}_1\psi]\\ div[\not{\mathcal{D}}_1\psi] &= \nabla_a\Big(\nabla^a\psi\cdot\psi - (\mathrm{d}iv\,\psi)\psi^a - (curl\,\psi)(\,{}^*\psi)^a\Big).\end{aligned}$$

Hence, (A.4.3) becomes

$$\begin{aligned}\big|\triangle\psi|^2 &= |\nabla^2\psi|^2 + {}^{(h)}K\Big(|\nabla\psi|^2 - {}^{(h)}K|\psi|^2\Big) + {}^{(h)}KJ_1 + \mathrm{Err}_{1,1}[\triangle\psi] + \mathrm{d}iv\,{}_1[\psi]\\ &= |\nabla^2\psi|^2 + {}^{(h)}K\big(|\nabla\psi|^2 - 2\,{}^{(h)}K|\psi|^2\big) + \mathrm{Err}_1[\triangle\psi] + \mathrm{d}iv\,{}_1[\triangle\psi],\end{aligned}$$

where

$$\begin{aligned}\mathrm{Err}_1[\triangle\psi] &= \mathrm{Err}_{1,1}[\triangle\psi] + \frac{1}{2}\,{}^{(h)}K\Big(\Big({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4\Big)\,{}^*\psi\Big)\cdot\psi + {}^{(h)}K\mathrm{d}iv\,[\not{\mathcal{D}}_1\psi]\\ &= -\frac{1}{2}\nabla\psi\cdot\big({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4\big)\,{}^*\nabla\psi + \frac{1}{2}\Big|\big({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4\big)\psi\Big|^2\\ &\quad -{}^{(h)}K\big({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4\big)\psi\cdot\,{}^*\psi\\ &\quad +\frac{1}{2}\,{}^{(h)}K\Big(\Big({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4\Big)\,{}^*\psi\Big)\cdot\psi + {}^{(h)}K\mathrm{d}iv\,[\not{\mathcal{D}}_1\psi]\\ &= -\frac{1}{2}\nabla\psi\cdot\big({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4\big)\,{}^*\nabla\psi + \frac{1}{2}\Big|\big({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4\big)\psi\Big|^2\\ &\quad -\frac{3}{2}\,{}^{(h)}K\big({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4\big)\psi\cdot\,{}^*\psi + {}^{(h)}K\mathrm{d}iv\,[\not{\mathcal{D}}_1\psi],\end{aligned}$$

and

$$\begin{aligned}\mathrm{d}iv\,{}_1[\triangle\psi] &= \mathrm{d}iv\,{}_1[\triangle\psi] + \mathrm{d}iv\,[\not{\mathcal{D}}_1\psi]\\ &= \nabla_a\Big(\nabla^a\psi\cdot\triangle\psi - \frac{1}{2}\nabla^a|\nabla\psi|^2\Big) + \nabla_a\Big(\nabla^a\psi\cdot\psi - (\mathrm{d}iv\,\psi)\psi^a - (curl\,\psi)(\,{}^*\psi)^a\Big),\end{aligned}$$

as stated in part 2 of the proposition.

Case $k = 2$. It remains to consider the case $k = 2$ in which case the term

$$J_2 = 2\Big(\nabla_a\psi_{ci}\nabla^c\psi^{ai} - \nabla_a\psi^i\nabla^a\psi_{ai}\Big)$$

in (A.4.3) becomes

$$\begin{aligned}\frac{1}{2}J_2 &= \nabla_a\psi_{ci}\nabla^c\psi^{ai} - |\mathrm{d}iv\,\psi|^2 = |\nabla\psi|^2 + \nabla_a\psi_{ci}\big(\nabla^c\psi^{ai} - \nabla^a\psi^{ci}\big) - |\mathrm{d}iv\,\psi|^2\\ &= |\nabla\psi|^2 + \frac{1}{2}\big(\nabla_a\psi_{ci} - \nabla_c\psi_{ai}\big)\big(\nabla^c\psi^{ai} - \nabla^a\psi^{ci}\big) - |\mathrm{d}iv\,\psi|^2.\end{aligned}$$

Note that $\nabla_a\psi_{ci} - \nabla_c\psi_{ai} = \in_{ac} ({}^*\text{d}iv\,\psi)_i$. Hence

$$J_2 = 2|\nabla\psi|^2 - 4\text{d}iv\,\psi = 2|\nabla\psi|^2 - 4\,\not{\mathcal{D}}_2\psi.$$

Recalling the identity 2.1.34 of Proposition 2.1.47 we deduce

$$J_2 = 2\big(|\nabla\psi|^2 - 2\,\not{\mathcal{D}}_2\psi\big) = -4\,{}^{(h)}K|\psi|^2 + \left(\left({}^{(a)}\text{tr}\chi\nabla_3 + {}^{(a)}\text{tr}\underline{\chi}\nabla_4\right)\,{}^*f\psi\right)\cdot\psi + 2\text{d}iv\,[\not{\mathcal{D}}_2\psi]$$

and, in view of (A.4.3),

$$\begin{aligned}\big|\triangle\psi|^2 &= |\nabla^2\psi|^2 + {}^{(h)}K\Big(|\nabla\psi|^2 - 2\,{}^{(h)}K|\psi|^2\Big) + {}^{(h)}K\,J_2 + \text{Err}_{2,1}[\triangle\psi] + \text{d}iv\,{}_2[\psi]\\ &= |\nabla^2\psi|^2 + {}^{(h)}K\Big(|\nabla\psi|^2 - 6\,{}^{(h)}K|\psi|^2\Big) + {}^{(h)}K\left(\left({}^{(a)}\text{tr}\chi\nabla_3 + {}^{(a)}\text{tr}\underline{\chi}\nabla_4\right)\,{}^*\psi\right)\cdot\psi\\ &\quad +2\,{}^{(h)}K\text{d}iv\,[\not{\mathcal{D}}_2\psi] + \text{Err}_{2,1}[\triangle\psi] + \text{d}iv\,{}_2[\psi]\end{aligned}$$

or,

$$\big|\triangle\psi|^2 = |\nabla^2\psi|^2 + {}^{(h)}K\Big(|\nabla\psi|^2 - 6\,{}^{(h)}K|\psi|^2\Big) + \text{Err}_2[\psi] + \text{d}iv\,{}_2\psi,$$

where

$$\begin{aligned}\text{Err}_2[\triangle\psi] &= -\frac{1}{2}\nabla\psi\cdot\big({}^{(a)}\text{tr}\chi\nabla_3 + {}^{(a)}\text{tr}\underline{\chi}\nabla_4\big)\,{}^*\nabla\psi + \frac{1}{2}\Big|\big({}^{(a)}\text{tr}\chi\nabla_3 + {}^{(a)}\text{tr}\underline{\chi}\nabla_4\big)\psi\Big|^2\\ &\quad -3\,{}^{(h)}K\big({}^{(a)}\text{tr}\chi\nabla_3 + {}^{(a)}\text{tr}\underline{\chi}\nabla_4\big)\psi\cdot\,{}^*\psi + 2\,{}^{(h)}K\text{d}iv\,[\not{\mathcal{D}}_2\psi]\\ \text{d}iv\,{}_k[\triangle\psi] &= \nabla_a\Big(\nabla^a\psi\cdot\triangle\psi - \nabla_c\psi\cdot\nabla^c\nabla^a\psi\Big).\end{aligned}$$

This ends the proof of the Proposition.

A.5 Proof of Lemma 2.4.6

The identities in (2.4.4) are proved by straightforward computation. We show here some of the computations.

To prove the fourth formula in (2.4.4), we write

$$\begin{aligned}\mathcal{D}\widehat{\otimes}(\overline{F}\cdot U)_{ab} &= \mathcal{D}_a(\overline{F}\cdot U)_b + \mathcal{D}_b(\overline{F}\cdot U)_a - \delta_{ab}\mathcal{D}^c(\overline{F}\cdot U)_c\\ &= \mathcal{D}_a(\overline{F}^c U_{cb}) + \mathcal{D}_b(\overline{F}^c U_{ca}) - \delta_{ab}\mathcal{D}^d(\overline{F}^c U_{cd})\\ &= \mathcal{D}_a\overline{F}^c U_{cb} + \mathcal{D}_b\overline{F}^c U_{ca} - \delta_{ab}\mathcal{D}^d\overline{F}^c U_{cd}\\ &+ \overline{F}^c\big(\mathcal{D}_a U_{cb} + \mathcal{D}_b U_{ca} - \delta_{ab}\mathcal{D}^d U_{cd}\big).\end{aligned}$$

Now, in view of Lemma 2.1.18,

$$\begin{aligned}\mathcal{D}_a\overline{F}^c U_{cb} + \mathcal{D}_b\overline{F}^c U_{ca} &= \delta_{ab}(\mathcal{D}^d\overline{F}^c)U_{cd} + (\mathcal{D}\cdot\overline{F})U_{ab}\\ &+\frac{1}{2}\Big(\big(\mathcal{D}_a\overline{F}_c - \mathcal{D}_c\overline{F}_a\big)U_{cb} + \big(\mathcal{D}_b\overline{F}_c - \mathcal{D}_c\overline{F}_b\big)U_{ca}\Big).\end{aligned}$$

Hence

$$\begin{aligned}\mathcal{D}_a\overline{F}^c U_{cb} + \mathcal{D}_b\overline{F}^c U_{ca} - \delta_{ab}\mathcal{D}^d\overline{F}^c U_{cd} &= (\mathcal{D}\cdot\overline{F})U_{ab}\\ &+\frac{1}{2}\Big(\big(\mathcal{D}_a\overline{F}_c - \mathcal{D}_c\overline{F}_a\big)U_{cb} + \big(\mathcal{D}_b\overline{F}_c - \mathcal{D}_c\overline{F}_b\big)U_{ca}\Big).\end{aligned}$$

Recall that ${}^*F = -iF$, ${}^*U = -iU$, ${}^*\mathcal{D} = -i\mathcal{D}$. We deduce,

$$\mathcal{D}_a\overline{F}_b - \mathcal{D}_b\overline{F}_a = i \in_{ab} (\mathcal{D}\cdot\overline{F}).$$

To check note that

$$\begin{aligned}\mathcal{D}_1\overline{F}_2 - \mathcal{D}_2\overline{F}_1 &= 2\Big[(\nabla_1 f_2 - \nabla_2 f_1) + i(\nabla_1 f_1 + \nabla_2 f_2)\Big],\\ (\mathcal{D}\cdot\overline{F}) &= 2\Big[(\nabla_1 f_1 + \nabla_2 f_2) - i(\nabla_1 f_2 - \nabla_2 f_1)\Big].\end{aligned}$$

We deduce,

$$\begin{aligned}\mathcal{D}_a\overline{F}^c U_{cb} + \mathcal{D}_b\overline{F}^c U_{ca} - \delta_{ab}\mathcal{D}^d\overline{F}^c U_{cd} &= (\mathcal{D}\cdot\overline{F})U_{ab} + \frac{1}{2}i(\mathcal{D}\cdot\overline{F})\big(\in_{ac} U_{cb} + \in_{bc} U_{ca}\big)\\ &= (\mathcal{D}\cdot\overline{F})U_{ab} + \frac{1}{2}i(\mathcal{D}\cdot\overline{F})\big(-2iU_{ab}\big)\\ &= 2(\mathcal{D}\cdot\overline{F})U_{ab}.\end{aligned}$$

Therefore,

$$\mathcal{D}\widehat{\otimes}(\overline{F}\cdot U)_{ab} = 2(\mathcal{D}\cdot\overline{F})U_{ab} + \overline{F}^c\big(\mathcal{D}_a U_{cb} + \mathcal{D}_b U_{ca} - \delta_{ab}\mathcal{D}^d U_{cd}\big).$$

It remains to re-express the tensor

$$\mathcal{D}_a U_{cb} + \mathcal{D}_b U_{ca} - \delta_{ab}\mathcal{D}^d U_{cd}.$$

Note also that $\mathcal{D}^d U_{cd} = 0$. We claim

$$\mathcal{D}_a U_{cb} + \mathcal{D}_b U_{ca} = 2\mathcal{D}_c U_{ab}.$$

Indeed, for $a = b = 1$, $c = 2$,

$$\begin{aligned}2\mathcal{D}_1 U_{21} &= 2(\nabla_1 + i\,{}^*\nabla_1)U_{21} = 2(\nabla_1 + i\nabla_2)U_{12} = -2i(\nabla_1 + i\nabla_2)U_{11} = 2(\nabla_2 - i\nabla_1)U_{11},\\ 2\mathcal{D}_2 U_{11} &= 2(\nabla_2 + i\,{}^*\nabla_2)U_{11} = 2(\nabla_2 - i\nabla_1)U_{11}.\end{aligned}$$

For $a = c = 1$, $b = 2$,

$$\begin{aligned}\mathcal{D}_1 U_{12} + \mathcal{D}_2 U_{11} &= (\nabla_1 + i\nabla_2)U_{12} + (\nabla_2 - i\nabla_1)U_{11} = (\nabla_1 + i\nabla_2)U_{12} + i(\nabla_2 - i\nabla_1)U_{12} \\ &= 2(\nabla_1 + i\nabla_2)U_{12} = 2\mathcal{D}_1 U_{12}.\end{aligned}$$

We deduce,

$$\mathcal{D}\widehat{\otimes}(\overline{F}\cdot U)_{ab} = 2(\mathcal{D}\cdot\overline{F})U_{ab} + 2\overline{F}^c\mathcal{D}_c U_{ab},$$

as stated.

To prove the last formula in (2.4.4), we write

$$\begin{aligned}(\overline{\mathcal{D}}\cdot F) &= \overline{\mathcal{D}}^a F_a = (\nabla - i\,{}^*\nabla)^a F_a = (\nabla_1 - i\,{}^*\nabla_1)F_1 + (\nabla_2 - i\,{}^*\nabla_2)F_2 \\ &= (\nabla_1 - i\nabla_2)F_1 + (\nabla_2 + i\nabla_1)(-iF_1) = 2(\nabla_1 - i\nabla_2)F_1,\end{aligned}$$

which gives

$$(U(\overline{\mathcal{D}}\cdot F))_{ab} = 2(\nabla_1 - i\nabla_2)F_1 U_{ab}.$$

On the other hand,

$$\begin{aligned}(U\cdot\overline{\mathcal{D}}F)_{ab} &= U_{ad}\overline{\mathcal{D}}_d F_b = U_{a1}\overline{\mathcal{D}}_1 F_b + U_{a2}\overline{\mathcal{D}}_2 F_b \\ &= (u_{a1} + i\,{}^*u_{a1})(\nabla_1 - i\,{}^*\nabla_1)F_b + (u_{a2} + i\,{}^*u_{a2})(\nabla_2 - i\,{}^*\nabla_2)F_b \\ &= (u_{a1} + i\,{}^*u_{a1})(\nabla_1 - i\nabla_2)F_b + (u_{a2} + i\,{}^*u_{a2})(\nabla_2 + i\nabla_1)F_b,\end{aligned}$$

and therefore

$$\begin{aligned}(\overline{U}\cdot\mathcal{D}F)_{11} &= (u_{11} + iu_{12})(\nabla_1 - i\nabla_2)F_1 + (u_{12} - iu_{11})(\nabla_2 + i\nabla_1)F_1 \\ &= 2(u_{11}\nabla_1 - iu_{11}\nabla_2 + u_{12}\nabla_2 + iu_{12}\nabla_1)F_1 \\ &= 2(\nabla_1 - i\nabla_2)F_1 U_{ab}.\end{aligned}$$

Similarly for the other components. We obtain the stated identity.

To prove the first formula in (2.4.5), we write

$$\begin{aligned}(\overline{\mathcal{D}}\cdot U)_1 &= \overline{\mathcal{D}}^a U_{1a} = (\nabla - i\,{}^*\nabla)^a U_{1a} \\ &= (\nabla_1 - i\,{}^*\nabla_1)U_{11} + (\nabla_2 - i\,{}^*\nabla_2)U_{12} \\ &= (\nabla_1 - i\nabla_2)U_{11} + (\nabla_2 + i\nabla_1)(-iU_{11}) \\ &= 2(\nabla_1 - i\nabla_2)U_{11},\end{aligned}$$

and

$$\begin{aligned}(\overline{\mathcal{D}}\cdot U)_2 &= \overline{\mathcal{D}}^a U_{2a} = (\nabla - i\,{}^*\nabla)^a U_{2a}\\ &= (\nabla_1 - i\,{}^*\nabla_1)U_{21} + (\nabla_2 - i\,{}^*\nabla_2)U_{22}\\ &= (\nabla_1 - i\nabla_2)(-iU_{11}) + (\nabla_2 + i\nabla_1)(-U_{11})\\ &= -2i(\nabla_1 - i\nabla_2)U_{11}.\end{aligned}$$

Therefore

$$\begin{aligned}(F\widehat{\otimes}(\overline{\mathcal{D}}\cdot U))_{11} &= 2F_1(\overline{\mathcal{D}}\cdot U)_1 - \delta_{11}F\cdot(\overline{\mathcal{D}}\cdot U)\\ &= F_1(\overline{\mathcal{D}}\cdot U)_1 - F_2(\overline{\mathcal{D}}\cdot U)_2\\ &= (f_1 + i\,{}^*f_1)2(\nabla_1 - i\nabla_2)U_{11} - (f_2 + i\,{}^*f_2)(-2i(\nabla_1 - i\nabla_2)U_{11})\\ &= 4f_1\nabla_1U_{11} - 4if_1\nabla_2U_{11} + 4if_2\nabla_1U_{11} + 4f_2\nabla_2U_{11}.\end{aligned}$$

On the other hand,

$$\begin{aligned}(F\widehat{\otimes}(\overline{\mathcal{D}}\cdot U))_{12} &= F_1(\overline{\mathcal{D}}\cdot U)_2 + F_2(\overline{\mathcal{D}}\cdot U)_1\\ &= (f_1 + i\,{}^*f_1)(2(\nabla_1 - i\nabla_2)U_{12}) + (f_2 + i\,{}^*f_2)2(\nabla_1 - i\nabla_2)iU_{12}\\ &= (f_1 + if_2)(2(\nabla_1 - i\nabla_2)U_{12}) + (f_2 - if_1)2(\nabla_1 - i\nabla_2)iU_{12}\\ &= 4f_1\nabla_1U_{12} - 4if_1\nabla_2U_{12} + 4if_2\nabla_1U_{12} + 4f_2\nabla_2U_{12},\end{aligned}$$

which therefore gives

$$\frac{1}{2}(F\widehat{\otimes}(\overline{\mathcal{D}}\cdot U))_{ab} = 2f_1\nabla_1U_{ab} - 2if_1\nabla_2U_{ab} + 2if_2\nabla_1U_{ab} + 2f_2\nabla_2U_{ab}.$$

On the other hand,

$$\begin{aligned}(F\cdot\overline{\mathcal{D}}U)_{ab} &= F^c\overline{\mathcal{D}}_cU_{ab} = F_1\overline{\mathcal{D}}_1U_{ab} + F_2\overline{\mathcal{D}}_2U_{ab}\\ &= (f_1 + i\,{}^*f_1)(\nabla_1 - i\,{}^*\nabla_1)U_{ab} + (f_2 + i\,{}^*f_2)(\nabla_2 - i\,{}^*\nabla_2)U_{ab}\\ &= (f_1 + if_2)(\nabla_1 - i\nabla_2)U_{ab} + (f_2 - if_1)(\nabla_2 + i\nabla_1)U_{ab}\\ &= 2f_1\nabla_1U_{ab} - 2if_1\nabla_2U_{ab} + 2if_2\nabla_1U_{ab} + 2f_2\nabla_2U_{ab}.\end{aligned}$$

Therefore $\frac{1}{2}(F\widehat{\otimes}(\overline{\mathcal{D}}\cdot U))_{ab} = (F\cdot\overline{\mathcal{D}}U)_{ab}$ as stated. Finally,

$$F^c\overline{\mathcal{D}_c}U = f^c\overline{\mathcal{D}_c}U + i({}^*f^c)\overline{\mathcal{D}_c}U = f^c\overline{\mathcal{D}_c}U - if^c({}^*\overline{\mathcal{D}_c}U) = 2f^c\overline{\mathcal{D}_c}U = 2F^c\nabla_cU.$$

To prove the second formula in (2.4.5), we write from the above

$$(\overline{F}\cdot\mathcal{D}U)_{ab} = 2f_1\nabla_1U_{ab} + 2if_1\nabla_2U_{ab} - 2if_2\nabla_1U_{ab} + 2f_2\nabla_2U_{ab}$$

which implies

$$(F\cdot\overline{\mathcal{D}}U)_{ab} + (\overline{F}\cdot\mathcal{D}U)_{ab} = 4f_1\nabla_1U_{ab} + 4f_2\nabla_2U_{ab} = 4f\cdot\nabla U$$

as stated.

Appendix B

Complement for Chapter 3

B.1 Proof of Proposition 3.7.2

We decompose the symmetric 3-tensor $\mathbf{D}_{(\mu}K_{\nu\rho)}$ in the null frame. We have

$$3\Pi_{\mu\nu\rho} = \mathbf{D}_\mu K_{\nu\rho} + \mathbf{D}_\nu K_{\rho\mu} + \mathbf{D}_\rho K_{\mu\nu}.$$

We have

$$\begin{aligned} 3\Pi_{abc} &= \mathbf{D}_a K_{bc} + \mathbf{D}_b K_{ca} + \mathbf{D}_c K_{ab} \\ &= \nabla_a K_{bc} - K_{\mathbf{D}_a bc} - K_{b\mathbf{D}_a c} + \nabla_b K_{ca} - K_{\mathbf{D}_b ca} - K_{c\mathbf{D}_b a} + \nabla_c K_{ab} - K_{\mathbf{D}_c ab} - K_{a\mathbf{D}_c b}. \end{aligned}$$

Since $\mathbf{D}_a e_b = \nabla_a e_b + \frac{1}{2}\chi_{ab}e_3 + \frac{1}{2}\underline{\chi}_{ab}e_4$ and $K_{a3} = K_{a4} = 0$, we have

$$\begin{aligned} 3\Pi_{abc} &= \nabla_a K_{bc} + \nabla_b K_{ca} + \nabla_c K_{ab} \\ &= \nabla_a(r^2\delta_{bc}) + \nabla_b(r^2\delta_{ca}) + \nabla_c(r^2\delta_{ab}) = 3\nabla_{(a}r^2\delta_{bc)}. \end{aligned}$$

Since in Kerr $\nabla_a(r) = 0$, $\Pi_{abc} = 0$. We have

$$\begin{aligned} 3\Pi_{ab3} &= \mathbf{D}_a K_{b3} + \mathbf{D}_b K_{3a} + \mathbf{D}_3 K_{ab} \\ &= \nabla_a K_{b3} - K_{\mathbf{D}_a b3} - K_{b\mathbf{D}_a 3} + \nabla_b K_{3a} - K_{\mathbf{D}_b 3a} - K_{3\mathbf{D}_b a} + \nabla_3 K_{ab} - K_{\mathbf{D}_3 ab} - K_{a\mathbf{D}_3 b} \\ &= -\frac{1}{2}\underline{\chi}_{ab}K_{43} - \underline{\chi}_{ac}K_{bc} - \underline{\chi}_{bc}K_{ca} - \frac{1}{2}\underline{\chi}_{ba}K_{34} + \nabla_3 K_{ab} \\ &= -(\underline{\chi}_{ab} + \underline{\chi}_{ba})(a^2\cos^2\theta) - r^2\underline{\chi}_{ac}\delta_{bc} - r^2\underline{\chi}_{bc}\delta_{ca} + \nabla_3(r^2\delta_{ab}) \\ &= -(\underline{\chi}_{ab} + \underline{\chi}_{ba})(r^2 + a^2\cos^2\theta) + \nabla_3(r^2)\delta_{ab} \\ &= (2r\nabla_3(r) - |q|^2\mathrm{tr}\,\underline{\chi})\delta_{ab} - 2\underline{\widehat{\chi}}_{ab}|q|^2, \end{aligned}$$

which can also be written as

$$3\Pi_{ab3} = (\nabla_3(r^2) - |q|^2 \mathrm{tr}\,\underline{\chi})\delta_{ab} - 2\widehat{\underline{\chi}}_{ab}|q|^2,$$

and similarly for $3\Pi_{ab4}$. Using Lemma 3.4.1, we see that $\Pi_{ab3} = \Pi_{ab4} = 0$ in Kerr. We have

$$\begin{aligned}
3\Pi_{a34} &= \mathbf{D}_a K_{34} + \mathbf{D}_3 K_{4a} + \mathbf{D}_4 K_{a3}\\
&= \nabla_3 K_{4a} - K_{\mathbf{D}_3 4a} - K_{4\mathbf{D}_3 a} + \nabla_4 K_{a3} - K_{\mathbf{D}_4 a3} - K_{a\mathbf{D}_4 3} + \nabla_a K_{34} - K_{\mathbf{D}_a 34} - K_{3\mathbf{D}_a 4}\\
&= -2\eta_b K_{ab} - \eta_a K_{34} - \underline{\eta}_a K_{34} - 2\underline{\eta}_b K_{ab} + \nabla_a K_{34}\\
&= \nabla_a K_{34} - 2(\eta_b + \underline{\eta}_b)K_{ab} - (\eta_a + \underline{\eta}_a)K_{34}\\
&= \nabla_a 2(a^2\cos^2\theta) - 2(\eta_b + \underline{\eta}_b)r^2\delta_{ab} - (\eta_a + \underline{\eta}_a)2(a^2\cos^2\theta)\\
&= 2\nabla_a(a^2\cos^2\theta) - 2(\eta_a + \underline{\eta}_a)|q|^2.
\end{aligned}$$

Using Lemma 3.4.1, we see that $\Pi_{a34} = 0$ in Kerr. We have

$$\begin{aligned}
3\Pi_{a33} &= \mathbf{D}_a K_{33} + 2\mathbf{D}_3 K_{3a}\\
&= \nabla_a K_{33} - 2K_{\mathbf{D}_a 33} + 2\nabla_3 K_{3a} - 2K_{\mathbf{D}_3 3a} - 2K_{3\mathbf{D}_3 a}\\
&= -4\underline{\xi}_b K_{ba} - 2\underline{\xi}_a K_{34}\\
&= -4\underline{\xi}_b r^2\delta_{ba} - 4\underline{\xi}_a(a^2\cos^2\theta) = -4|q|^2\underline{\xi}_a,
\end{aligned}$$

and similarly for $3\Pi_{a44}$. We have

$$\begin{aligned}
3\Pi_{343} &= 2\mathbf{D}_3 K_{43} + \mathbf{D}_4 K_{33}\\
&= 2\nabla_3 K_{43} - 2K_{\mathbf{D}_3 43} - 2K_{4\mathbf{D}_3 3} + \nabla_4 K_{33} - 2K_{\mathbf{D}_4 33}\\
&= 2\nabla_3 K_{43} - 4\underline{\omega} K_{43} + 4\underline{\omega} K_{43}\\
&= 4\nabla_3(a^2\cos^2\theta),
\end{aligned}$$

and similarly for Π_{434}. Consider the component Π_{333}. We have

$$\Pi_{333} = \mathbf{D}_3 K_{33} = \nabla_3 K_{33} - 2K_{\mathbf{D}_3 33} = 0,$$

and similarly for Π_{444}.

B.2 Proof of Proposition 3.7.3

The operator $\mathcal{K}$ applied to $\psi \in \mathfrak{s}_k$ is given by

$$\mathcal{K}(\psi) = \dot{\mathbf{D}}_\mu(K^{\mu\nu}\dot{\mathbf{D}}_\nu(\psi)) = \dot{\mathbf{D}}_\mu K^{\mu\nu}\dot{\mathbf{D}}_\nu(\psi) + K^{\mu\nu}\dot{\mathbf{D}}_\mu\dot{\mathbf{D}}_\nu(\psi).$$

Using the general computations in (A.3.1) for $\Pi = 0$, we have

$$\begin{aligned}\mathcal{K}(\psi) &= -\frac{1}{2}\dot{\mathbf{D}}^\nu(\mathrm{tr}K)\dot{\mathbf{D}}_\nu(\psi) + K^{\mu\nu}\dot{\mathbf{D}}_\mu\dot{\mathbf{D}}_\nu(\psi)\\ &= -\frac{1}{2}\delta^{ab}\dot{\mathbf{D}}_a(\mathrm{tr}K)\dot{\mathbf{D}}_b(\psi) - \frac{1}{2}\mathbf{g}^{34}\dot{\mathbf{D}}_4(\mathrm{tr}K)\dot{\mathbf{D}}_3(\psi) - \frac{1}{2}\mathbf{g}^{43}\dot{\mathbf{D}}_3(\mathrm{tr}K)\dot{\mathbf{D}}_4(\psi)\\ &\quad +(-(a^2\cos^2\theta)\mathbf{g}^{\mu\nu} + |q|^2\gamma^{ab}e_a^\mu e_b^\nu)\dot{\mathbf{D}}_\mu\dot{\mathbf{D}}_\nu(\psi),\end{aligned}$$

which gives

$$\begin{aligned}\mathcal{K}(\psi) &= -(a^2\cos^2\theta)\dot{\square}_k\psi - \frac{1}{2}\nabla^c(\mathrm{tr}K)\nabla_c(\psi) + \frac{1}{4}\nabla_4(\mathrm{tr}K)\nabla_3(\psi) + \frac{1}{4}\nabla_3(\mathrm{tr}K)\nabla_4(\psi)\\ &\quad +|q|^2\gamma^{ab}\dot{\mathbf{D}}_a\dot{\mathbf{D}}_b\psi.\end{aligned}$$

Using that $\mathrm{tr}K = 2(r^2 - a^2\cos^2\theta)$, we compute

$$\nabla^c(\mathrm{tr}K) = 2\nabla^c(r^2) - 2\nabla^c(a^2\cos^2\theta) = -2(\eta^c + \underline{\eta}^c)|q|^2.$$

Similarly we compute

$$\begin{aligned}\nabla_3(\mathrm{tr}K) &= 4r\nabla_3 r - 2\nabla_3(a^2\cos^2\theta) = 2|q|^2\mathrm{tr}\,\underline{\chi},\\ \nabla_4(\mathrm{tr}K) &= 2|q|^2\mathrm{tr}\,\chi.\end{aligned}$$

The above gives

$$\begin{aligned}\mathcal{K}(\psi) &= -(a^2\cos^2\theta)\dot{\square}_k\psi - \frac{1}{2}\left(-2(\eta^c + \underline{\eta}^c)|q|^2 - 3\Pi^c{}_{34}\right)\nabla_c\psi\\ &\quad +\frac{1}{4}\left(2|q|^2\mathrm{tr}\,\chi\right)\nabla_3\psi + \frac{1}{4}\left(2|q|^2\mathrm{tr}\,\underline{\chi}\right)\nabla_4\psi + |q|^2\delta^{ab}(\nabla_b\nabla_a\psi - \nabla_{\mathbf{D}_b e_a}\psi)\\ &= -(a^2\cos^2\theta)\dot{\square}_k\psi + |q|^2(\eta + \underline{\eta})\cdot\nabla\psi + \frac{1}{2}|q|^2\mathrm{tr}\,\chi\nabla_3\psi + \frac{1}{2}|q|^2\mathrm{tr}\,\underline{\chi}\nabla_4\psi\\ &\quad +|q|^2\delta^{ab}(\nabla_b\nabla_a\psi - \frac{1}{2}\chi_{ba}\nabla_3\psi - \frac{1}{2}\underline{\chi}_{ba}\nabla_4\psi)\\ &= -(a^2\cos^2\theta)\dot{\square}_k\psi + |q|^2\triangle_k\psi + |q|^2(\eta + \underline{\eta})\cdot\nabla\psi,\end{aligned}$$

as stated.

B.3 Proof of Proposition 3.7.6

Recall from Lemma 4.7.4 that we have

$$\begin{aligned}|q|^2\dot{\square}_2\psi &= -\frac{1}{2}|q|^2\big(\nabla_3\nabla_4\psi + \nabla_4\nabla_3\psi\big) + |q|^2\left(\underline{\omega} - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\right)\nabla_4\psi + |q|^2\left(\omega - \frac{1}{2}\mathrm{tr}\,\chi\right)\nabla_3\psi\\ &\quad +\mathcal{O}(\psi).\end{aligned}$$

Using the ingoing frame to write $e_4^{(in)} = \frac{r^2+a^2}{|q|^2}\big(\widehat{T}+\widehat{R}\big)$, see (3.3.5), we have

$$
\begin{aligned}
\nabla_3\nabla_4+\nabla_4\nabla_3 &= \nabla_3\Big(\frac{r^2+a^2}{|q|^2}(\nabla_{\widehat{T}}+\nabla_{\widehat{R}})\Big)+\frac{r^2+a^2}{|q|^2}(\nabla_{\widehat{T}}+\nabla_{\widehat{R}})\nabla_3\\
&= \frac{r^2+a^2}{|q|^2}\Big(\nabla_{\widehat{T}}\nabla_3+\nabla_3\nabla_{\widehat{T}}+\nabla_{\widehat{R}}\nabla_3+\nabla_3\nabla_{\widehat{R}}\Big)\\
&\quad +e_3^{(in)}\big(\frac{r^2+a^2}{|q|^2}\big)\big(\nabla_{\widehat{T}}+\nabla_{\widehat{R}}\big)\\
&= \frac{r^2+a^2}{|q|^2}\Big(\nabla_{\widehat{T}}\nabla_3+\nabla_3\nabla_{\widehat{T}}+\nabla_{\widehat{R}}\nabla_3+\nabla_3\nabla_{\widehat{R}}\Big)+O(a^2r^{-3})\big(\nabla_{\widehat{T}}+\nabla_{\widehat{R}}\big).
\end{aligned}
$$

We can also write in the ingoing frame, using (3.3.6),

$$
\begin{aligned}
& |q|^2\left(\underline{\omega}-\frac{1}{2}\mathrm{tr}\,\underline{\chi}\right)\nabla_4+|q|^2\left(\omega-\frac{1}{2}\mathrm{tr}\,\chi\right)\nabla_3\\
&= |q|^2\left(\frac{r}{|q|^2}\right)\nabla_4+|q|^2\left(-\frac{1}{2}\partial_r\big(\frac{\Delta}{|q|^2}\big)-\frac{r\Delta}{|q|^4}\right)\nabla_3\\
&= r\big(\nabla_4-\frac{\Delta}{|q|^2}\nabla_3\big)-\frac{1}{2}|q|^2\partial_r\big(\frac{\Delta}{|q|^2}\big)\nabla_3\\
&= 2\frac{r(r^2+a^2)}{|q|^2}\nabla_{\widehat{R}}-\frac{1}{2}|q|^2\partial_r\big(\frac{\Delta}{|q|^2}\big)\nabla_3.
\end{aligned}
$$

We deduce in the ingoing frame

$$
\begin{aligned}
|q|^2\dot{\square}_2\psi &= -\frac{1}{2}(r^2+a^2)\Big(\nabla_{\widehat{T}}\nabla_3+\nabla_3\nabla_{\widehat{T}}+\nabla_{\widehat{R}}\nabla_3+\nabla_3\nabla_{\widehat{R}}\Big)\psi\\
&\quad +2\frac{r(r^2+a^2)}{|q|^2}\nabla_{\widehat{R}}\psi-\frac{1}{2}|q|^2\partial_r\big(\frac{\Delta}{|q|^2}\big)\nabla_3\psi+O(a^2r^{-1})\big(\nabla_{\widehat{T}}+\nabla_{\widehat{R}}\big)\psi+\mathcal{O}(\psi),
\end{aligned}
$$

and therefore

$$
\begin{aligned}
[\mathcal{O},|q|^2\dot{\square}_2]\psi &= -\frac{1}{2}(r^2+a^2)[\mathcal{O},\big(\nabla_{\widehat{T}}\nabla_3+\nabla_3\nabla_{\widehat{T}}+\nabla_{\widehat{R}}\nabla_3+\nabla_3\nabla_{\widehat{R}}\big)]\psi\\
&\quad +\frac{2r(r^2+a^2)}{|q|^2}[\mathcal{O},\nabla_{\widehat{R}}]\psi-\frac{1}{2}|q|^2\partial_r\big(\frac{\Delta}{|q|^2}\big)[\mathcal{O},\nabla_3]\psi\\
&\quad +O(a^2r^{-1})[\mathcal{O},\big(\nabla_{\widehat{T}}+\nabla_{\widehat{R}}\big)]\psi+O(a^2r^{-2})\mathfrak{d}^{\leq 1}\psi.
\end{aligned}
$$

Using Lemma 4.7.10, Lemma 9.2.3 and Corollary 9.2.6, i.e.

$$
\begin{aligned}
[\mathcal{O},\nabla_3]\psi &= O(ar^{-2})\mathfrak{d}^{\leq 1}\psi,\\
[\mathcal{O},\nabla_{\widehat{R}}]\psi &= O(ar^{-2})\mathfrak{d}^{\leq 1}\psi,\\
[\mathcal{O},\nabla_{\widehat{T}}] &= O(ar^{-2})\,\mathfrak{d}\!\!\!/^{\leq 1}\psi,
\end{aligned}
$$

we obtain

$$\begin{aligned}
[\mathcal{O}, |q|^2\dot{\square}_2]\psi &= -\frac{1}{2}(r^2+a^2)\Big([\mathcal{O}, \nabla_{\widehat{T}}]\nabla_3 + \nabla_{\widehat{T}}[\mathcal{O}, \nabla_3] + [\mathcal{O}, \nabla_3]\nabla_{\widehat{T}} + \nabla_3[\mathcal{O}, \nabla_{\widehat{T}}])\psi\Big)\\
&\quad -\frac{1}{2}(r^2+a^2)\Big([\mathcal{O}, \nabla_{\widehat{R}}]\nabla_3 + \nabla_{\widehat{R}}[\mathcal{O}, \nabla_3] + [\mathcal{O}, \nabla_3]\nabla_{\widehat{R}} + \nabla_3[\mathcal{O}, \nabla_{\widehat{R}}])\psi\Big)\\
&\quad +O(ar^{-1})\mathfrak{d}^{\leq 1}\psi\\
&= -\frac{1}{2}(r^2+a^2)\Big([\mathcal{O}, \nabla_{\widehat{T}}]\nabla_3 + \nabla_{\widehat{T}}[\mathcal{O}, \nabla_3] + [\mathcal{O}, \nabla_3]\nabla_{\widehat{T}} + \nabla_3[\mathcal{O}, \nabla_{\widehat{T}}]\Big)\psi\\
&\quad -\frac{1}{2}(r^2+a^2)\Big([\mathcal{O}, \nabla_{\widehat{R}}]\nabla_3 + \nabla_3[\mathcal{O}, \nabla_{\widehat{R}}]\Big)\psi\\
&\quad +O(a)\nabla_{\widehat{R}}\mathfrak{d}^{\leq 1}\psi + O(ar^{-1})\mathfrak{d}^{\leq 1}\psi.
\end{aligned}$$

Observe that, since the commutators $[\mathcal{O}, \nabla_3]$, $[\mathcal{O}, \nabla_{\widehat{R}}]$, $[\mathcal{O}, \nabla_{\widehat{T}}]$ involve only one derivative, then

$$\nabla_3[\mathcal{O}, \nabla_{\widehat{T}}] = [\mathcal{O}, \nabla_{\widehat{T}}]\nabla_3 + O(ar^{-3})\mathfrak{d}^{\leq 1}$$

and similarly for the others. We then have

$$\begin{aligned}
[\mathcal{O}, |q|^2\dot{\square}_2]\psi &= -(r^2+a^2)\Big([\mathcal{O}, \nabla_{\widehat{T}}]\nabla_3 + [\mathcal{O}, \nabla_3]\nabla_{\widehat{T}} + [\mathcal{O}, \nabla_{\widehat{R}}]\nabla_3\Big)\psi\\
&\quad +O(a)\nabla_{\widehat{R}}\mathfrak{d}^{\leq 1}\psi + O(ar^{-1})\mathfrak{d}^{\leq 1}\psi.
\end{aligned}$$

Now using (3.3.5) to write $\widehat{R} = \widehat{T} - \frac{\Delta}{r^2+a^2}e_3^{(in)}$, we obtain

$$\begin{aligned}
[\mathcal{O}, |q|^2\dot{\square}_2]\psi &= -(r^2+a^2)\Big([\mathcal{O}, \nabla_{\widehat{T}}]\nabla_3 + [\mathcal{O}, \nabla_3]\nabla_{\widehat{T}} + [\mathcal{O}, \nabla_{\widehat{T}} - \frac{\Delta}{r^2+a^2}\nabla_3]\nabla_3\Big)\psi\\
&\quad +O(a)\nabla_{\widehat{R}}\mathfrak{d}^{\leq 1}\psi + O(ar^{-1})\mathfrak{d}^{\leq 1}\psi\\
&= -(r^2+a^2)\Big(2[\mathcal{O}, \nabla_{\widehat{T}}]\nabla_3 + [\mathcal{O}, \nabla_3]\big(\nabla_{\widehat{T}} - \frac{\Delta}{r^2+a^2}\nabla_3)\Big)\psi\\
&\quad +O(a)\nabla_{\widehat{R}}\mathfrak{d}^{\leq 1}\psi + O(ar^{-1})\mathfrak{d}^{\leq 1}\psi\\
&= -2(r^2+a^2)[\mathcal{O}, \nabla_{\widehat{T}}]\nabla_3\psi + O(a)\nabla_{\widehat{R}}\mathfrak{d}^{\leq 1}\psi + O(ar^{-1})\mathfrak{d}^{\leq 1}\psi.
\end{aligned}$$

Next we compute $[\mathcal{O}, \nabla_{\widehat{T}}]$. Recall that we have from Lemma 9.2.1,

$$\nabla_{\mathbf{T}}\psi = \not{\mathcal{L}}_{\mathbf{T}}\psi + \frac{4amr\cos\theta}{|q|^4}\,{}^*\psi.$$

We infer

$$\begin{aligned}
[\nabla_{\mathbf{T}}, \mathcal{O}]\psi &= \left[\not{\mathcal{L}}_{\mathbf{T}}\psi + \frac{4amr\cos\theta}{|q|^4}\ {}^*, \mathcal{O}\right]\psi \\
&= 4amr\left[\frac{\cos\theta}{|q|^4}, \mathcal{O}\right]\ {}^*\psi \\
&= 4amr|q|^2\left[\frac{\cos\theta}{|q|^4}, \Delta + (\eta+\underline{\eta})\cdot\nabla\right]\ {}^*\psi \\
&= -8amr|q|^2\nabla\left(\frac{\cos\theta}{|q|^4}\right)\cdot\nabla\ {}^*\psi + O(ar^{-3})\psi.
\end{aligned}$$

Also, recall that we have

$$\nabla_{\mathbf{Z}}\psi = \not{\mathcal{L}}_{\mathbf{Z}}\psi - \frac{2\cos\theta((r^2+a^2)^2 - a^2(\sin\theta)^2\Delta)}{|q|^4}\ {}^*\psi.$$

We infer

$$\begin{aligned}
[\nabla_{\mathbf{Z}}, \mathcal{O}]\psi &= \left[\not{\mathcal{L}}_{\mathbf{Z}}\psi - \frac{2\cos\theta((r^2+a^2)^2 - a^2(\sin\theta)^2\Delta)}{|q|^4}\ {}^*, \mathcal{O}\right]\psi \\
&= -2\left[\frac{\cos\theta((r^2+a^2)^2 - a^2(\sin\theta)^2\Delta)}{|q|^4}, \mathcal{O}\right]\ {}^*\psi \\
&= -2|q|^2\left[\frac{\cos\theta((r^2+a^2)^2 - a^2(\sin\theta)^2\Delta)}{|q|^4}, \Delta + (\eta+\underline{\eta})\cdot\nabla\right]\ {}^*\psi \\
&= 4|q|^2\nabla\left(\frac{\cos\theta((r^2+a^2)^2 - a^2(\sin\theta)^2\Delta)}{|q|^4}\right)\cdot\nabla\ {}^*\psi + O(1)\psi.
\end{aligned}$$

Next, recall that $\widehat{T} = \mathbf{T} + \frac{a}{r^2+a^2}\mathbf{Z}$ which together with the above yields

$$\begin{aligned}
[\nabla_{\widehat{T}}, \mathcal{O}]\psi &= [\nabla_{\mathbf{T}}, \mathcal{O}]\psi + \frac{a}{r^2+a^2}[\nabla_{\mathbf{Z}}, \mathcal{O}]\psi \\
&= -8amr|q|^2\nabla\left(\frac{\cos\theta}{|q|^4}\right)\cdot\nabla\ {}^*\psi \\
&\quad + \frac{4a|q|^2}{r^2+a^2}\nabla\left(\frac{\cos\theta((r^2+a^2)^2 - a^2(\sin\theta)^2\Delta)}{|q|^4}\right)\cdot\nabla\ {}^*\psi + O(ar^{-2})\psi.
\end{aligned}$$

Observe that we can simplify

$$
\begin{aligned}
f(r,\cos\theta) &:= -8amr\frac{\cos\theta}{|q|^4} + \frac{4a}{r^2+a^2}\frac{\cos\theta((r^2+a^2)^2 - a^2(\sin\theta)^2\Delta)}{|q|^4} \\
&= \frac{4a\cos\theta}{|q|^4(r^2+a^2)}\Big(-2mr(r^2+a^2) + (r^2+a^2)^2 - a^2(\sin\theta)^2\Delta\Big) \\
&= \frac{4a\cos\theta}{|q|^4(r^2+a^2)}\Big((r^2+a^2)\Delta - a^2(\sin\theta)^2\Delta\Big) \\
&= \frac{4a\cos\theta\Delta}{|q|^2(r^2+a^2)}.
\end{aligned}
$$

We deduce

$$
[\nabla_{\widehat{T}}, \mathcal{O}]\psi = \frac{|q|^2\Delta}{r^2+a^2}\nabla\left(\frac{4a\cos\theta}{|q|^2}\right)\cdot\nabla\ {}^*\psi + O(ar^{-2})\psi.
$$

This gives

$$
[\mathcal{O}, |q|^2\dot{\square}_2]\psi = |q|^2\nabla\left(\frac{8a\cos\theta}{|q|^2}\right)\cdot\nabla(\Delta\nabla_3)\ {}^*\psi + O(a)\nabla_{\widehat{R}}^{\leq 1}\mathfrak{d}^{\leq 1}\psi.
$$

By writing $\Delta e_3^{(in)} = (r^2+a^2)(\widehat{T} - \widehat{R})$, we finally obtain

$$
[\mathcal{O}, |q|^2\dot{\square}_2]\psi = |q|^2\nabla\left(\frac{8a(r^2+a^2)\cos\theta}{|q|^2}\right)\cdot\nabla\nabla_{\widehat{T}}\ {}^*\psi + O(a)\nabla_{\widehat{R}}^{\leq 1}\mathfrak{d}^{\leq 1}\psi,
$$

as stated. This concludes the proof of the lemma.

Appendix C

Complement for Chapter 4

C.1 Proof of Lemma 4.2.1

Formula (4.2.1) is straightforward from (2.2.15).

According to Corollary 2.2.9, from (2.2.23) we have

$$
\begin{aligned}
[\nabla_4, \nabla\widehat{\otimes}]f &= -\frac{1}{2}\mathrm{tr}\,\chi\left(\nabla\widehat{\otimes}f + \underline{\eta}\widehat{\otimes}f\right) - \frac{1}{2}\,{}^{(a)}\mathrm{tr}\chi\ {}^*\left(\nabla\widehat{\otimes}f + \underline{\eta}\widehat{\otimes}f\right) + (\underline{\eta}+\zeta)\widehat{\otimes}\nabla_4 f,\\
&\quad + {}^*\beta\widehat{\otimes}\ {}^*f + \xi\widehat{\otimes}\nabla_3 f - \xi\widehat{\otimes}(\underline{\chi}\cdot f) + \widehat{\underline{\chi}}\,(\xi\cdot f) - \widehat{\chi}\cdot\nabla f\\
&\quad -\underline{\eta}\widehat{\otimes}(\widehat{\chi}\cdot f) + \widehat{\chi}(\underline{\eta}\cdot f).
\end{aligned}
$$

Hence for $F = f + i\,{}^*f$,

$$
\begin{aligned}
[\nabla_4, \nabla\widehat{\otimes}]F &= -\frac{1}{2}\mathrm{tr}\,\chi\left(\nabla\widehat{\otimes}F + \underline{\eta}\widehat{\otimes}F\right) - \frac{1}{2}\,{}^{(a)}\mathrm{tr}\chi\ {}^*\left(\nabla\widehat{\otimes}F + \underline{\eta}\widehat{\otimes}F\right) + (\underline{\eta}+\zeta)\widehat{\otimes}\nabla_4 F,\\
&\quad + {}^*\beta\widehat{\otimes}\ {}^*F + \xi\widehat{\otimes}\nabla_3 F - \xi\widehat{\otimes}(\underline{\chi}\cdot F) + \widehat{\underline{\chi}}\,(\xi\cdot F) - \widehat{\chi}\cdot\nabla F\\
&\quad -\underline{\eta}\widehat{\otimes}(\widehat{\chi}\cdot F) + \widehat{\chi}(\underline{\eta}\cdot F).
\end{aligned}
$$

Recalling that ${}^*F=-iF$ and ${}^*(\nabla\widehat{\otimes}f)= {}^*\nabla\widehat{\otimes}f=\nabla\widehat{\otimes}\,{}^*f$ we deduce,

$$
\begin{aligned}
[\nabla_4,\nabla\widehat{\otimes}]F &= -\frac{1}{2}\mathrm{tr}\,\chi\left(\nabla\widehat{\otimes}F+\underline{\eta}\widehat{\otimes}F\right)+i\frac{1}{2}{}^{(a)}\mathrm{tr}\chi\left(\nabla\widehat{\otimes}F+\underline{\eta}\widehat{\otimes}F\right)+(\underline{\eta}+\zeta)\widehat{\otimes}\nabla_4F\\
&\quad -i\,{}^*\beta\widehat{\otimes}F+\xi\widehat{\otimes}\nabla_3F-\xi\widehat{\otimes}(\underline{\chi}\cdot F)+\underline{\widehat{\chi}}\,(\xi\cdot F)-\widehat{\chi}\cdot\nabla F\\
&\quad -\underline{\eta}\widehat{\otimes}(\widehat{\chi}\cdot F)+\widehat{\chi}(\underline{\eta}\cdot F)\\
&= -\frac{1}{2}\mathrm{tr}X\left(\nabla\widehat{\otimes}F+\underline{\eta}\widehat{\otimes}F\right)+(\underline{\eta}+\zeta)\widehat{\otimes}\nabla_4F\\
&\quad -i\,{}^*\beta\widehat{\otimes}F+\xi\widehat{\otimes}\nabla_3F-\xi\widehat{\otimes}(\underline{\chi}\cdot F)+\underline{\widehat{\chi}}\,(\xi\cdot F)-\widehat{\chi}\cdot\nabla F\\
&\quad -\underline{\eta}\widehat{\otimes}(\widehat{\chi}\cdot F)+\widehat{\chi}(\underline{\eta}\cdot F).
\end{aligned}
$$

Taking the dual

$$
\begin{aligned}
[\nabla_4,\,{}^*\nabla\widehat{\otimes}]F &= -\frac{1}{2}\mathrm{tr}X\left({}^*\nabla\widehat{\otimes}F+{}^*\underline{\eta}\widehat{\otimes}F\right)+{}^*(\underline{\eta}+\zeta)\widehat{\otimes}\nabla_4F+i\beta\widehat{\otimes}F+{}^*\xi\widehat{\otimes}\nabla_3F\\
&\quad -{}^*\xi\widehat{\otimes}(\underline{\chi}\cdot F)+{}^*\underline{\widehat{\chi}}\,(\xi\cdot F)+\widehat{\chi}\cdot{}^*\nabla F-{}^*\underline{\eta}\widehat{\otimes}(\widehat{\chi}\cdot F)+{}^*\widehat{\chi}(\underline{\eta}\cdot F).
\end{aligned}
$$

Finally adding the above we derive

$$
\begin{aligned}
[\nabla_4,\mathcal{D}\widehat{\otimes}]F &= -\frac{1}{2}\mathrm{tr}X\left(\mathcal{D}\widehat{\otimes}F+\underline{H}\widehat{\otimes}F\right)+(\underline{H}+Z)\widehat{\otimes}\nabla_4F+\Xi\widehat{\otimes}\nabla_3F\\
&\quad -B\widehat{\otimes}F-\Xi\widehat{\otimes}(\underline{\chi}\cdot F)-\widehat{\chi}\cdot\overline{\mathcal{D}}F-\underline{H}\widehat{\otimes}(\widehat{\chi}\cdot F)+\widehat{X}(\underline{\eta}\cdot F)+\underline{\widehat{X}}\,(\xi\cdot F)\\
&= -\frac{1}{2}\mathrm{tr}X\left(\mathcal{D}\widehat{\otimes}F+\underline{H}\widehat{\otimes}F\right)+(\underline{H}+Z)\widehat{\otimes}\nabla_4F+\Xi\widehat{\otimes}\nabla_3F\\
&\quad -B\widehat{\otimes}F-\frac{1}{2}\mathrm{tr}\underline{X}\Xi\widehat{\otimes}F-\frac{1}{2}\widehat{X}\cdot\overline{\mathcal{D}}F+\frac{1}{2}\widehat{X}(\overline{\underline{H}}\cdot F)+(\Gamma_b\cdot\Gamma_g)F,
\end{aligned}
$$

where observe that $\underline{H}\widehat{\otimes}(\widehat{\chi}\cdot F)=0$. By symmetry we also have

$$
\begin{aligned}
[\nabla_3,\mathcal{D}\widehat{\otimes}]F &= -\frac{1}{2}\mathrm{tr}\underline{X}\left(\mathcal{D}\widehat{\otimes}F+H\widehat{\otimes}F\right)+(H-Z)\widehat{\otimes}\nabla_3F+\underline{\Xi}\widehat{\otimes}\nabla_4F\\
&\quad +\underline{B}\widehat{\otimes}F-\frac{1}{2}\mathrm{tr}X\underline{\Xi}\widehat{\otimes}F-\frac{1}{2}\underline{\widehat{X}}\cdot\overline{\mathcal{D}}F+\frac{1}{2}\underline{\widehat{X}}(\overline{H}\cdot F)+(\Gamma_b\cdot\Gamma_g)F.
\end{aligned}
$$

This proves (4.2.2).

According to Corollary 2.2.9, from (2.2.24) we have

$$
\begin{aligned}
[\nabla_4,\mathrm{d}iv]u &= -\frac{1}{2}\mathrm{tr}\,\chi\big(\mathrm{d}iv\,u-2\underline{\eta}\cdot u\big)+\frac{1}{2}{}^{(a)}\mathrm{tr}\chi\big(\mathrm{d}iv\,{}^*u-2\underline{\eta}\cdot{}^*u\big)+(\underline{\eta}+\zeta)\cdot\nabla_4u\\
&\quad +2\,{}^*\beta\cdot{}^*u+\xi\cdot\nabla_3u-\xi\cdot\underline{\chi}\cdot u-(\underline{\chi}\cdot u)\xi+\xi\cdot u\cdot\underline{\chi}-\widehat{\chi}\cdot\nabla u\\
&\quad -\underline{\eta}\cdot\widehat{\chi}\cdot u-(\widehat{\chi}\cdot u)\underline{\eta}+\underline{\eta}\cdot u\cdot\widehat{\chi}.
\end{aligned}
$$

Hence for $U = u + i\,{}^*u$,

$$
\begin{aligned}
[\nabla_4, \mathrm{d}iv\,]U &= -\frac{1}{2}\mathrm{tr}\,\chi\big(\mathrm{d}iv\,U - 2\underline{\eta}\cdot U\big) + \frac{1}{2}\,{}^{(a)}\mathrm{tr}\chi\big(\mathrm{d}iv\ {}^*U - 2\underline{\eta}\cdot\ {}^*U\big) + (\underline{\eta}+\zeta)\cdot\nabla_4 U\\
&\quad +2\,{}^*\beta\cdot\ {}^*U + \xi\cdot\nabla_3 U - \xi\cdot\underline{\chi}\cdot U - (\underline{\chi}\cdot U)\xi + \xi\cdot U\cdot\underline{\chi} - \widehat{\chi}\cdot\nabla U\\
&\quad -\underline{\eta}\cdot\widehat{\chi}\cdot U - (\widehat{\chi}\cdot U)\underline{\eta} + \underline{\eta}\cdot U\cdot\widehat{\chi}.
\end{aligned}
$$

Recalling that ${}^*U = -iU$ and ${}^*(\mathrm{d}iv\,U) = \mathrm{d}iv\ {}^*U$ we deduce

$$
\begin{aligned}
[\nabla_4, \mathrm{d}iv\,]U &= -\frac{1}{2}\mathrm{tr}\,\chi\big(\mathrm{d}iv\,U - 2\underline{\eta}\cdot U\big) - \frac{1}{2}i\,{}^{(a)}\mathrm{tr}\chi\big(\mathrm{d}iv\,U - 2\underline{\eta}\cdot U\big) + (\underline{\eta}+\zeta)\cdot\nabla_4 U\\
&\quad -2i\,{}^*\beta\cdot U + \xi\cdot\nabla_3 U - \xi\cdot\underline{\chi}\cdot U - (\underline{\chi}\cdot U)\xi + \xi\cdot U\cdot\underline{\chi} - \widehat{\chi}\cdot\nabla U\\
&\quad -\underline{\eta}\cdot\widehat{\chi}\cdot U - (\widehat{\chi}\cdot U)\underline{\eta} + \underline{\eta}\cdot U\cdot\widehat{\chi}\\
&= -\frac{1}{2}\overline{\mathrm{tr}X}\big(\mathrm{d}iv\,U - 2\underline{\eta}\cdot U\big) + (\underline{\eta}+\zeta)\cdot\nabla_4 U\\
&\quad -2i\,{}^*\beta\cdot U + \xi\cdot\nabla_3 U - \xi\cdot\underline{\chi}\cdot U - (\underline{\chi}\cdot U)\xi + \xi\cdot U\cdot\underline{\chi} - \widehat{\chi}\cdot\nabla U\\
&\quad -\underline{\eta}\cdot\widehat{\chi}\cdot U - (\widehat{\chi}\cdot U)\underline{\eta} + \underline{\eta}\cdot U\cdot\widehat{\chi}.
\end{aligned}
$$

Taking the dual

$$
\begin{aligned}
[\nabla_4, {}^*\mathrm{d}iv\,]U &= -\frac{1}{2}\overline{\mathrm{tr}X}\big({}^*\mathrm{d}iv\,U - 2\,{}^*\underline{\eta}\cdot U\big) + {}^*(\underline{\eta}+\zeta)\cdot\nabla_4 U + 2i\beta\cdot U + {}^*\xi\cdot\nabla_3 U\\
&\quad - {}^*\xi\cdot\underline{\chi}\cdot U - {}^*(\underline{\chi}\cdot U)\xi + {}^*\xi\cdot U\cdot\underline{\chi} - \widehat{\chi}\cdot\ {}^*\nabla U\\
&\quad - {}^*\underline{\eta}\cdot\widehat{\chi}\cdot U - {}^*(\widehat{\chi}\cdot U)\underline{\eta} + {}^*\underline{\eta}\cdot U\cdot\widehat{\chi}.
\end{aligned}
$$

Finally we derive for $\overline{\mathcal{D}}\cdot = \mathrm{d}iv\, - i\,{}^*\mathrm{d}iv\,$,

$$
\begin{aligned}
[\nabla_4, \overline{\mathcal{D}}\cdot]U &= -\frac{1}{2}\overline{\mathrm{tr}X}\big(\overline{\mathcal{D}}\cdot U - 2\overline{\underline{H}}\cdot U\big) + (\,\underline{H}+Z)\cdot\nabla_4 U + \overline{\Xi}\cdot\nabla_3 U\\
&\quad +2\overline{B}\cdot U - \overline{\Xi}\cdot\underline{\chi}\cdot U - (\underline{\chi}\cdot U)\overline{\Xi} + \overline{\Xi}\cdot U\cdot\underline{\chi} - \widehat{\chi}\cdot\overline{\mathcal{D}}U\\
&\quad -\overline{\underline{H}}\cdot\widehat{\chi}\cdot U - (\widehat{\chi}\cdot U)\,\overline{\underline{H}} + \overline{\underline{H}}\cdot U\cdot\widehat{\chi}\\
&= -\frac{1}{2}\overline{\mathrm{tr}X}\big(\overline{\mathcal{D}}\cdot U - 2\overline{\underline{H}}\cdot U\big) + (\,\underline{H}+Z)\cdot\nabla_4 U + \overline{\Xi}\cdot\nabla_3 U\\
&\quad +2\overline{B}\cdot U - \frac{1}{2}\overline{\mathrm{tr}\underline{X}\Xi}\cdot U - \frac{1}{2}\widehat{X}\cdot\overline{\mathcal{D}}U - \frac{1}{2}(\overline{\widehat{X}}\cdot U)\,\overline{\underline{H}} + (\Gamma_b\cdot\Gamma_g)U.
\end{aligned}
$$

By symmetry we also have

$$
\begin{aligned}
[\nabla_3, \overline{\mathcal{D}}\cdot]U &= -\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\big(\overline{\mathcal{D}}\cdot U - 2\overline{H}\cdot U\big) + (H-Z)\cdot\nabla_3 U + \overline{\underline{\Xi}}\cdot\nabla_4 U\\
&\quad -2\overline{\underline{B}}\cdot U - \frac{1}{2}\overline{\mathrm{tr}X\underline{\Xi}}\cdot U - \frac{1}{2}\widehat{\underline{X}}\cdot\overline{\mathcal{D}}U - \frac{1}{2}(\overline{\widehat{\underline{X}}}\cdot U)\overline{H} + (\Gamma_b\cdot\Gamma_g)U.
\end{aligned}
$$

This proves (4.2.4).

Recall from (2.2.21),

$$
\begin{aligned}
[\nabla_3, \nabla_4] U_{ab} &= -2\omega \nabla_3 U_{ab} + 2\underline{\omega} \nabla_4 U_{ab} + 2(\eta_c - \underline{\eta}_c) \nabla_c U_{ab} \\
&\quad +2\underline{\eta}_a \eta_c U_{bc} + 2\underline{\eta}_b \eta_c U_{ac} - 2\eta_a \underline{\eta}_c U_{bc} - 2\eta_b \underline{\eta}_c U_{ac} + 4 \ {}^*\rho \ {}^*U_{ab} + \mathrm{Err}_{34ab}[U].
\end{aligned}
$$

Applying it to $U \in \mathfrak{s}_k(\mathbb{C})$ for which ${}^*U = -iU$, we have

$$
\begin{aligned}
[\nabla_3, \nabla_4] U_{ab} &= -2\omega \nabla_3 U_{ab} + 2\underline{\omega} \nabla_4 U_{ab} + 2(\eta_c - \underline{\eta}_c) \nabla_c U_{ab} - 4i \ {}^*\rho U_{ab} + C_{3,4}^{\eta,\underline{\eta}}(U)_{ab} \\
&\quad +\mathrm{Err}_{34ab}[U],
\end{aligned}
$$

where

$$
C_{3,4}^{\eta,\underline{\eta}}(U)_{ab} = 2\underline{\eta}_a \eta_c U_{bc} + 2\underline{\eta}_b \eta_c U_{ac} - 2\eta_a \underline{\eta}_c U_{bc} - 2\eta_b \underline{\eta}_c U_{ac}.
$$

We have that

$$
C_{3,4}^{\eta,\underline{\eta}}(U) = 4i(\eta \wedge \underline{\eta})U.
$$

Indeed, we compute

$$
\begin{aligned}
C_{3,4}^{\eta,\underline{\eta}}(U)_{11} &= 2\underline{\eta}_1 \eta_c U_{1c} + 2\underline{\eta}_1 \eta_c U_{1c} - 2\eta_1 \underline{\eta}_c U_{1c} - 2\eta_1 \underline{\eta}_c U_{1c} \\
&= 4\underline{\eta}_1 \eta_c U_{1c} - 4\eta_1 \underline{\eta}_c U_{1c} \\
&= 4\underline{\eta}_1 (\eta_1 U_{11} + \eta_2 U_{12}) - 4\eta_1 (\underline{\eta}_1 U_{11} + \underline{\eta}_2 U_{12}) \\
&= 4(\underline{\eta}_1 \eta_2 - \eta_1 \underline{\eta}_2) U_{12} = 4i(\eta_1 \underline{\eta}_2 - \underline{\eta}_1 \eta_2) U_{11}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
C_{3,4}^{\eta,\underline{\eta}}(U)_{12} &= 2\underline{\eta}_1 \eta_c U_{2c} + 2\underline{\eta}_2 \eta_c U_{1c} - 2\eta_1 \underline{\eta}_c U_{2c} - 2\eta_2 \underline{\eta}_c U_{1c} \\
&= 2\underline{\eta}_1 (\eta_1 U_{21} + \eta_2 U_{22}) + 2\underline{\eta}_2 (\eta_1 U_{11} + \eta_2 U_{12}) \\
&\quad -2\eta_1 (\underline{\eta}_1 U_{21} + \underline{\eta}_2 U_{22}) - 2\eta_2 (\underline{\eta}_1 U_{11} + \underline{\eta}_2 U_{12}) \\
&= 2\underline{\eta}_1 \eta_2 U_{22} + 2\underline{\eta}_2 \eta_1 U_{11} - 2\eta_1 \underline{\eta}_2 U_{22} - 2\eta_2 \underline{\eta}_1 U_{11} \\
&= -4\underline{\eta}_1 \eta_2 U_{11} + 4\eta_1 \underline{\eta}_2 U_{11} = 4i(\eta_1 \underline{\eta}_2 - \underline{\eta}_1 \eta_2) U_{12}.
\end{aligned}
$$

We conclude by writing $\eta_1 \underline{\eta}_2 - \underline{\eta}_1 \eta_2 = \eta \wedge \underline{\eta}$.

Starting with (2.2.21)

$$
\begin{aligned}
[\nabla_3, \nabla_a] u_{bc} &= -\frac{1}{2} \mathrm{tr}\, \underline{\chi} \, (\nabla_a u_{bc} + \eta_b u_{ac} + \eta_c u_{ab} - \delta_{ab} (\eta \cdot u)_c - \delta_{ac} (\eta \cdot u)_b) \\
&\quad -\frac{1}{2} {}^{(a)}\mathrm{tr}\underline{\chi} \, (\ {}^*\nabla_a u_{bc} + \eta_b \ {}^*u_{ac} + \eta_c \ {}^*u_{ab} - \in_{ab} (\eta \cdot u)_c - \in_{ac} (\eta \cdot u)_b) \\
&\quad +(\eta_a - \zeta_a) \nabla_3 u_{bc} + \mathrm{Err}_{3abc}[u],
\end{aligned}
$$

and adding the same expression for u replaced with $i\ {}^*u$ we derive

$$\begin{aligned}
[\nabla_3, \nabla_a]U_{bc} &= -\frac{1}{2}\mathrm{tr}\,\underline{\chi}\Big(\nabla_a U_{bc} + \eta_b U_{ac} + \eta_c U_{ab} - \delta_{ab}(\eta\cdot U)_c - \delta_{ac}(\eta\cdot U)_b\Big)\\
&- \frac{1}{2}{}^{(a)}\mathrm{tr}\underline{\chi}\Big({}^*\nabla_a U_{bc} + \eta_b\ {}^*U_{ac} + \eta_c\ {}^*U_{ab} - \in_{ab}(\eta\cdot U)_c - \in_{ac}(\eta\cdot U)_b\Big)\\
&+ (\eta_a - \zeta_a)\nabla_3 U_{bc} + \mathrm{Err}_{3abc}[U]
\end{aligned}$$

as stated.

C.2 Proof of Lemma 4.2.2

Using (4.2.1), we have

$$\begin{aligned}
[\,{}^{(c)}\nabla_4,\ {}^{(c)}\mathcal{D}]h &= (\nabla_4 + 2s\omega)(\mathcal{D} + sZ)h - (\mathcal{D} + (s+1)Z)(\nabla_4 + 2s\omega)h\\
&= [\nabla_4, \mathcal{D}]h + 2s\omega\,{}^{(c)}\mathcal{D}h + s\nabla_4(Zh) - (s+1)Z\,{}^{(c)}\nabla_4 h - 2s\mathcal{D}(\omega h)\\
&= -\frac{1}{2}\mathrm{tr}X\mathcal{D}h + (\,\underline{H} + Z)\nabla_4 h - \widehat{\chi}\cdot\overline{\mathcal{D}}h + \Xi\nabla_3 h + 2s\omega\,{}^{(c)}\mathcal{D}h + s(\nabla_4 Z)h\\
&\quad +sZ\nabla_4 h - (s+1)Z\,{}^{(c)}\nabla_4 h - 2s\mathcal{D}(\omega)h - 2s\omega\mathcal{D}h.
\end{aligned}$$

Using the null structure equation for $\nabla_4 Z$, i.e.

$$\begin{aligned}
\nabla_4 Z + \frac{1}{2}\mathrm{tr}X(Z - \,\underline{H}) - 2\omega(Z + \,\underline{H}) &= 2\mathcal{D}\omega + \frac{1}{2}\widehat{X}\cdot(-\overline{Z} + \overline{\underline{H}})\\
&\quad -\frac{1}{2}\mathrm{tr}\underline{X}\Xi - 2\underline{\omega}\Xi - B - \frac{1}{2}\overline{\Xi}\cdot\widehat{\underline{X}},
\end{aligned}$$

we infer

$$\begin{aligned}
[\,{}^{(c)}\nabla_4,\ {}^{(c)}\mathcal{D}]h &= -\frac{1}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}h + (\,\underline{H} + Z)\nabla_4 h - \frac{1}{2}\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}h + \Xi\nabla_3 h\\
&\quad +2s\omega\,{}^{(c)}\mathcal{D}h + s\left(\frac{1}{2}\mathrm{tr}X\,\underline{H} + 2\omega\,\underline{H} + \frac{1}{2}\widehat{X}\cdot\overline{\underline{H}} - \frac{1}{2}\mathrm{tr}\underline{X}\Xi - 2\underline{\omega}\Xi - B\right)h\\
&\quad +sZ\,{}^{(c)}\nabla_4 h - (s+1)Z\,{}^{(c)}\nabla_4 h - 2s\omega\mathcal{D}h + (\Gamma_b\cdot\Gamma_g)h\\
&= -\frac{1}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}h + (\,\underline{H} + Z)\nabla_4 h - \frac{1}{2}\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}h + \Xi\nabla_3 h\\
&\quad +2s\omega\,{}^{(c)}\mathcal{D}h + s\left(\frac{1}{2}\mathrm{tr}X\,\underline{H} + 2\omega\,\underline{H} + \frac{1}{2}\widehat{X}\cdot\overline{\underline{H}} - \frac{1}{2}\mathrm{tr}\underline{X}\Xi - 2\underline{\omega}\Xi - B\right)h\\
&\quad -Z(\nabla_4 + 2s\omega)h - 2s\omega\mathcal{D}h + (\Gamma_b\cdot\Gamma_g)h\\
&= -\frac{1}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}h + \,\underline{H}\,{}^{(c)}\nabla_4 h - \frac{1}{2}\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}h + \Xi\,{}^{(c)}\nabla_3 h\\
&\quad +s\left(\frac{1}{2}\mathrm{tr}X\,\underline{H} + \frac{1}{2}\widehat{X}\cdot\overline{\underline{H}} - \frac{1}{2}\mathrm{tr}\underline{X}\Xi - B\right)h + (\Gamma_b\cdot\Gamma_g)h,
\end{aligned}$$

as stated. Similarly for $[\,^{(c)}\nabla_3,\,^{(c)}\mathcal{D}]h$. This proves (4.2.9).

For a s-conformally invariant scalar h, we have

$$
\begin{array}{rcl}
[\,^{(c)}\nabla_3,\,^{(c)}\nabla_4]h &=& ^{(c)}\nabla_3\,^{(c)}\nabla_4 h - \,^{(c)}\nabla_4\,^{(c)}\nabla_3 h\\
&=& \big(e_3-2(s+1)\underline{\omega}\big)\big(e_4h+2s\omega h\big)-\big(e_4+2(s-1)\omega\big)\big(e_3h-2s\underline{\omega}h\big)\\
&=& e_3e_4h-2(s+1)\underline{\omega}e_4h+2se_3(\omega h)-4s(s+1)\underline{\omega}\omega h\\
&-& e_4e_3h-2(s-1)\omega e_3h+2se_4(\underline{\omega}h)+4s(s-1)\omega\underline{\omega}h\\
&=& [e_3,e_4]h-2\underline{\omega}e_4h+2\omega e_3h+2s\Big(e_3\omega+e_4\underline{\omega}-4\omega\underline{\omega}\Big)h.
\end{array}
$$

Using (2.2.15) and the null structure equation $\nabla_3\omega+\nabla_4\underline{\omega}=\rho+4\omega\underline{\omega}+\xi\cdot\underline{\xi}+(\eta-\underline{\eta})\cdot\zeta-\eta\cdot\underline{\eta}$, we obtain

$$
\begin{array}{rcl}
[\,^{(c)}\nabla_3,\,^{(c)}\nabla_4]h &=& 2(\eta-\underline{\eta})\cdot\nabla h+2s\Big(\rho+(\eta-\underline{\eta})\cdot\zeta-\eta\cdot\underline{\eta}+\xi\cdot\underline{\xi}\Big)h\\
&=& 2(\eta-\underline{\eta})\cdot\,^{(c)}\nabla h+2s\Big(\rho-\eta\cdot\underline{\eta}+\xi\cdot\underline{\xi}\Big)h,
\end{array}
$$

as stated. This proves (4.2.11).

For $F\in\mathfrak{s}_1(\mathbb{C})$ s-conformally invariant we have

$$
\begin{array}{rcl}
^{(c)}\nabla_4(\,^{(c)}\mathcal{D}\widehat{\otimes}F) &=& ^{(c)}\nabla_4(\mathcal{D}\widehat{\otimes}F+sZ\widehat{\otimes}F)\\
&=& \nabla_4(\mathcal{D}\widehat{\otimes}F+sZ\widehat{\otimes}F)+2s\omega(\mathcal{D}\widehat{\otimes}F+sZ\widehat{\otimes}F)\\
&=& \nabla_4(\mathcal{D}\widehat{\otimes}F)+s\nabla_4(Z)\widehat{\otimes}F+sZ\widehat{\otimes}\nabla_4F+2s\omega(\mathcal{D}\widehat{\otimes}F+sZ\widehat{\otimes}F).
\end{array}
$$

Using (4.2.2) and the null structure equation for $\nabla_4 Z$, we obtain

$$
\begin{array}{rcl}
^{(c)}\nabla_4(\,^{(c)}\mathcal{D}\widehat{\otimes}F) &=& \mathcal{D}\widehat{\otimes}(\nabla_4F)-\frac{1}{2}\mathrm{tr}X\,\big(\mathcal{D}\widehat{\otimes}F+\underline{H}\widehat{\otimes}F\big)+(\underline{H}+Z)\widehat{\otimes}\nabla_4F+\Xi\widehat{\otimes}\nabla_3F\\
&& -B\widehat{\otimes}F-\frac{1}{2}\mathrm{tr}\underline{X}\Xi\widehat{\otimes}F-\frac{1}{2}\widehat{X}\cdot\overline{\mathcal{D}}F+\frac{1}{2}\widehat{X}(\overline{\underline{H}}\cdot F)\\
&& +s\Big[-\frac{1}{2}\mathrm{tr}X(Z-\underline{H})+2\omega(Z+\underline{H})+2\mathcal{D}\omega\\
&& +\frac{1}{2}\widehat{X}\cdot(-\overline{Z}+\overline{\underline{H}})-\frac{1}{2}\mathrm{tr}\underline{X}\Xi-2\underline{\omega}\Xi-B\Big]\widehat{\otimes}F\\
&& +sZ\widehat{\otimes}\nabla_4F+2s\omega(\mathcal{D}\widehat{\otimes}F+sZ\widehat{\otimes}F)+(\Gamma_b\cdot\Gamma_g)F.
\end{array}
$$

We can rewrite the above as

$$
\begin{aligned}
{}^{(c)}\nabla_4({}^{(c)}\mathcal{D}\widehat{\otimes}F) \;=\;& \mathcal{D}\widehat{\otimes}(\nabla_4F)+sZ\widehat{\otimes}\nabla_4F+(\,\underline{H}+Z)\widehat{\otimes}\nabla_4F+2s\omega(Z+\,\underline{H})\widehat{\otimes}F\\
&+\Xi\widehat{\otimes}\nabla_3F-2s\underline{\omega}\Xi\widehat{\otimes}F-\frac{1}{2}\mathrm{tr}X\left(\mathcal{D}\widehat{\otimes}F+\,\underline{H}\widehat{\otimes}F+s(Z-\,\underline{H})\widehat{\otimes}F\right)\\
&+2s\omega\mathcal{D}\widehat{\otimes}F+2s^2\omega Z\widehat{\otimes}F+2s\mathcal{D}\omega\widehat{\otimes}F\\
&-(s+1)B\widehat{\otimes}F-(s+1)\frac{1}{2}\mathrm{tr}\underline{X}\Xi\widehat{\otimes}F-\frac{1}{2}\widehat{X}\cdot\overline{\mathcal{D}}F-\frac{1}{2}s(\widehat{X}\cdot\overline{Z})\widehat{\otimes}F\\
&+\frac{1}{2}\widehat{X}(\overline{\,\underline{H}}\cdot F)+s\frac{1}{2}(\widehat{X}\cdot\overline{\,\underline{H}})\widehat{\otimes}F+(\Gamma_b\cdot\Gamma_g)F,
\end{aligned}
$$

which gives

$$
\begin{aligned}
{}^{(c)}\nabla_4({}^{(c)}\mathcal{D}\widehat{\otimes}F) \;=\;& \mathcal{D}\widehat{\otimes}(\nabla_4F)+sZ\widehat{\otimes}\nabla_4F+(\,\underline{H}+Z)\widehat{\otimes}\nabla_4F+2s\omega(Z+\,\underline{H})\widehat{\otimes}F\\
&+\Xi\widehat{\otimes}\nabla_3F-2s\underline{\omega}\Xi\widehat{\otimes}F-\frac{1}{2}\mathrm{tr}X\left(\mathcal{D}\widehat{\otimes}F+\,\underline{H}\widehat{\otimes}F+s(Z-\,\underline{H})\widehat{\otimes}F\right)\\
&+2s\omega\mathcal{D}\widehat{\otimes}F+2s^2\omega Z\widehat{\otimes}F+2s\mathcal{D}\omega\widehat{\otimes}F\\
&-(s+1)B\widehat{\otimes}F-(s+1)\frac{1}{2}\mathrm{tr}\underline{X}\Xi\widehat{\otimes}F-\frac{1}{2}\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}F\\
&+\frac{1}{2}\widehat{X}(\overline{\,\underline{H}}\cdot F)+s\frac{1}{2}(\widehat{X}\cdot\overline{\,\underline{H}})\widehat{\otimes}F+(\Gamma_b\cdot\Gamma_g)F.
\end{aligned}
$$

Recall that ${}^{(c)}\nabla_4F$ is conformal of type $s+1$, therefore

$$
{}^{(c)}\mathcal{D}\widehat{\otimes}({}^{(c)}\nabla_4F)=\mathcal{D}\widehat{\otimes}({}^{(c)}\nabla_4F)+(s+1)Z\widehat{\otimes}\,{}^{(c)}\nabla_4F.
$$

We finally obtain

$$
\begin{aligned}
[{}^{(c)}\nabla_4,\,{}^{(c)}\mathcal{D}\widehat{\otimes}]F \;=\;& -\frac{1}{2}\mathrm{tr}X\left({}^{(c)}\mathcal{D}\widehat{\otimes}F+(1-s)\,\underline{H}\widehat{\otimes}F\right)+\,\underline{H}\widehat{\otimes}\,{}^{(c)}\nabla_4F+\Xi\widehat{\otimes}\,{}^{(c)}\nabla_3F\\
&-(s+1)B\widehat{\otimes}F-(s+1)\frac{1}{2}\mathrm{tr}\underline{X}\Xi\widehat{\otimes}F-\frac{1}{2}\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}F\\
&+\frac{1}{2}\widehat{X}(\overline{\,\underline{H}}\cdot F)+s\frac{1}{2}(\widehat{X}\cdot\overline{\,\underline{H}})\widehat{\otimes}F+(\Gamma_b\cdot\Gamma_g)F,
\end{aligned}
$$

as stated. Similarly for $[{}^{(c)}\nabla_3,\,{}^{(c)}\mathcal{D}\widehat{\otimes}]F$. This proves (4.2.12).

For $U\in\mathfrak{s}_2(\mathbb{C})$ s-conformally invariant, we have

$$
\begin{aligned}
{}^{(c)}\nabla_4\overline{{}^{(c)}\mathcal{D}}\cdot U \;&=\; {}^{(c)}\nabla_4(\overline{\mathcal{D}}\cdot U+s\overline{Z}\cdot U)\\
&=\; \nabla_4(\overline{\mathcal{D}}\cdot U+s\overline{Z}\cdot U)+2s\omega(\overline{\mathcal{D}}\cdot U+s\overline{Z}\cdot U)\\
&=\; \nabla_4(\overline{\mathcal{D}}\cdot U)+s\nabla_4(\overline{Z})\cdot U+s\overline{Z}\cdot\nabla_4U+2s\omega(\overline{\mathcal{D}}\cdot U+s\overline{Z}\cdot U).
\end{aligned}
$$

Using the null structure equation for $\nabla_4\overline{Z}$ and (4.2.4), we have

$$\begin{aligned}
{}^{(c)}\nabla_4\overline{{}^{(c)}\mathcal{D}}\cdot U &= \overline{\mathcal{D}}\cdot\nabla_4 U - \frac{1}{2}\overline{\mathrm{tr}X}\big(\overline{\mathcal{D}}\cdot U - 2\underline{\overline{H}}\cdot U\big) + \overline{(\,\underline{H}+Z)}\cdot\nabla_4 U + \Xi\cdot\nabla_3 U\\
&+2\overline{B}\cdot U - \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\Xi\cdot U - \frac{1}{2}\widehat{X}\cdot\overline{\mathcal{D}}U - \frac{1}{2}(\overline{\widehat{X}}\cdot U)\underline{\overline{H}}\\
&+s\Big[-\frac{1}{2}\overline{\mathrm{tr}X}(\overline{Z-\underline{H}}) + 2\omega(\overline{Z+\underline{H}}) + 2\overline{\mathcal{D}}\omega\\
&+\frac{1}{2}\overline{\widehat{X}}\cdot(-Z+\underline{H}) - \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\Xi - 2\underline{\omega}\Xi - \overline{B}\Big]\cdot U\\
&+s\overline{Z}\cdot\nabla_4 U + 2s\omega(\overline{\mathcal{D}}\cdot U + s\overline{Z}\cdot U) + (\Gamma_b\cdot\Gamma_g)U.
\end{aligned}$$

We can rewrite the above as

$$\begin{aligned}
{}^{(c)}\nabla_4\overline{{}^{(c)}\mathcal{D}}\cdot U &= \overline{{}^{(c)}\mathcal{D}}\cdot{}^{(c)}\nabla_4 U - \frac{1}{2}\overline{\mathrm{tr}X}\big(\overline{{}^{(c)}\mathcal{D}}\cdot U - (s+2)\underline{\overline{H}}\cdot U\big) + \overline{\underline{H}}\cdot{}^{(c)}\nabla_4 U\\
&+\Xi\cdot{}^{(c)}\nabla_3 U - (s-2)\overline{B}\cdot U - (s+1)\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\Xi\cdot U - \frac{1}{2}\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}U\\
&-\frac{1}{2}(\overline{\widehat{X}}\cdot U)\underline{\overline{H}} + s\frac{1}{2}(\overline{\widehat{X}}\cdot(\,\underline{H}))\cdot U + (\Gamma_b\cdot\Gamma_g)U.
\end{aligned}$$

Similarly for $[\,{}^{(c)}\nabla_3, \overline{{}^{(c)}\mathcal{D}}\cdot]$. This proves (4.2.14).

Using (4.2.6), we write

$$\begin{aligned}
{}^{(c)}\nabla_4(\overline{{}^{(c)}\mathcal{D}}\cdot F) &= {}^{(c)}\nabla_4(\overline{\mathcal{D}}\cdot F + s\overline{Z}\cdot F)\\
&= \nabla_4(\overline{\mathcal{D}}\cdot F + s\overline{Z}\cdot F) + 2s\omega(\overline{\mathcal{D}}\cdot F + s\overline{Z}\cdot F)\\
&= \nabla_4(\overline{\mathcal{D}}\cdot F) + s\nabla_4(\overline{Z})\cdot F + s\overline{Z}\cdot\nabla_4 F + 2s\omega(\overline{\mathcal{D}}\cdot F + s\overline{Z}\cdot F)\\
&= \overline{\mathcal{D}}\cdot(\nabla_4 F) - \frac{1}{2}\overline{\mathrm{tr}X}\,(\overline{\mathcal{D}}\cdot F - \overline{\underline{H}}\cdot F) + \overline{(\,\underline{H}+Z)}\cdot\nabla_4 F\\
&\quad+s(-\frac{1}{2}\overline{\mathrm{tr}X}(\overline{Z-\underline{H}}) + 2\omega(\overline{Z+\underline{H}}) + 2\overline{\mathcal{D}}\omega)\cdot F\\
&\quad+s\overline{Z}\cdot\nabla_4 F + 2s\omega(\overline{\mathcal{D}}\cdot F + s\overline{Z}\cdot F) + r^{-1}\Gamma_g\cdot\mathfrak{d}^{\leq 1}F.
\end{aligned}$$

We rewrite the above as

$$\begin{aligned}
{}^{(c)}\nabla_4(\overline{{}^{(c)}\mathcal{D}}\cdot F) &= \overline{\mathcal{D}}\cdot(\,{}^{(c)}\nabla_4 F) + s\overline{Z}\cdot{}^{(c)}\nabla_4 F + \overline{(\,\underline{H}+Z)}\cdot{}^{(c)}\nabla_4 F\\
&\quad-\frac{1}{2}\overline{\mathrm{tr}X}\,(\overline{\mathcal{D}}\cdot F - \overline{\underline{H}}\cdot F + s(\overline{Z-\underline{H}})\cdot F) + r^{-1}\Gamma_g\cdot\mathfrak{d}^{\leq 1}F.
\end{aligned}$$

Since $\overline{{}^{(c)}\mathcal{D}}\cdot(\,{}^{(c)}\nabla_4 F) = \overline{\mathcal{D}}\cdot(\,{}^{(c)}\nabla_4 F) + (s+1)\overline{Z}\cdot{}^{(c)}\nabla_4 F$, we obtain the stated identity.

We have

$$\begin{aligned}
[\,{}^{(c)}\nabla_3,\,{}^{(c)}\nabla_4]U &= {}^{(c)}\nabla_3\,{}^{(c)}\nabla_4 U - {}^{(c)}\nabla_4\,{}^{(c)}\nabla_3 U\\
&= (\nabla_3 - 2(s+1)\underline{\omega})(\nabla_4 U + 2s\omega U) - (\nabla_4 + 2(s-1)\omega)(\nabla_3 U - 2s\underline{\omega}U)\\
&= \nabla_3\nabla_4 U + 2s\nabla_3(\omega U) - 2(s+1)\underline{\omega}(\nabla_4 U) - 4s(s+1)\omega\underline{\omega}U\\
&\quad -\nabla_4\nabla_3 U + 2s\nabla_4(\underline{\omega}U) - 2(s-1)\omega\nabla_3 U + 4s(s+1)\omega\underline{\omega}U\\
&= [\nabla_3,\nabla_4]U - 2\underline{\omega}\nabla_4 U + 2\omega\nabla_3 U + 2s(\nabla_3\omega + \nabla_4\underline{\omega} - 4\omega\underline{\omega})U.
\end{aligned}$$

Using (4.2.7) and

$$\nabla_3\omega + \nabla_4\underline{\omega} - 4\omega\underline{\omega} = \rho + (\eta - \underline{\eta})\cdot\zeta - \eta\cdot\underline{\eta} + \xi\cdot\underline{\xi},$$

we obtain

$$\begin{aligned}
[\,{}^{(c)}\nabla_3,\,{}^{(c)}\nabla_4]U &= -2\omega\nabla_3 U + 2\underline{\omega}\nabla_4 U + 2(\eta_c - \underline{\eta}_c)\nabla_c U + 4i\left(-\,{}^*\rho + \eta\wedge\underline{\eta}\right)U\\
&\quad -2\underline{\omega}\nabla_4 U + 2\omega\nabla_3 U + 2s(\rho + (\eta-\underline{\eta})\cdot\zeta - \eta\cdot\underline{\eta})U + \mathrm{Err}_{34}[U]\\
&= 2(\eta_c - \underline{\eta}_c)\,{}^{(c)}\nabla_c U + 2s(\rho - \eta\cdot\underline{\eta})U + 4i\left(-\,{}^*\rho + \eta\wedge\underline{\eta}\right)U + \mathrm{Err}_{34}[U]
\end{aligned}$$

as stated.

We have

$$\begin{aligned}
{}^{(c)}\nabla_3\,{}^{(c)}\nabla_a U_{bc} &= {}^{(c)}\nabla_3(\nabla_a U_{bc} + s\zeta_a U_{bc})\\
&= \nabla_3(\nabla_a U_{bc} + s\zeta_a U_{bc}) - 2s\underline{\omega}(\nabla_a U_{bc} + s\zeta_a U_{bc})\\
&= \nabla_3\nabla_a U_{bc} + s\nabla_3\zeta_a U_{bc} + s\zeta_a\nabla_3 U_{bc} - 2s\underline{\omega}(\nabla_a U_{bc} + s\zeta_a U_{bc}).
\end{aligned}$$

Using (4.2.8) and the null structure equation

$$\begin{aligned}
\nabla_3\zeta + 2\nabla\underline{\omega} &= -\frac{1}{2}\mathrm{tr}\,\underline{\chi}(\zeta+\eta) - \frac{1}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi}(\,{}^*\zeta + \,{}^*\eta) + 2\underline{\omega}(\zeta-\eta)\\
&\quad +\widehat{\chi}\cdot\underline{\xi} + \frac{1}{2}tr\,\chi\,\underline{\xi} + \frac{1}{2}\,{}^{(a)}\mathrm{tr}\chi\,{}^*\underline{\xi} + 2\omega\underline{\xi} - \underline{\widehat{\chi}}\cdot(\zeta+\eta) - \underline{\beta}
\end{aligned}$$

we obtain

$$\begin{aligned}
{}^{(c)}\nabla_3\,{}^{(c)}\nabla_a U_{bc} &= \nabla_a\nabla_3 U_{bc} - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\Big(\nabla_a U_{bc} + \eta_b U_{ac} + \eta_c U_{ab} - \delta_{ab}(\eta\cdot U)_c - \delta_{ac}(\eta\cdot U)_b\Big)\\
&\quad -\frac{1}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi}\Big(\,{}^*\nabla_a U_{bc} + \eta_b\,{}^*U_{ac} + \eta_c\,{}^*U_{ab} - \in_{ab}(\eta\cdot U)_c - \in_{ac}(\eta\cdot U)_b\Big)\\
&\quad +(\eta_a - \zeta_a)\nabla_3 U_{bc}\\
&\quad +s(-2\nabla_a\underline{\omega} - \frac{1}{2}\mathrm{tr}\,\underline{\chi}(\zeta_a+\eta_a) - \frac{1}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi}(\,{}^*\zeta_a + \,{}^*\eta_a) + 2\underline{\omega}(\zeta_a - \eta_a))U_{bc}\\
&\quad +s\zeta_a\nabla_3 U_{bc} - 2s\underline{\omega}(\nabla_a U_{bc} + s\zeta_a U_{bc}) + \Gamma_g\cdot\mathfrak{d}^{\le 1}U.
\end{aligned}$$

We can rewrite the above as

$$
\begin{aligned}
& {}^{(c)}\nabla_3\, {}^{(c)}\nabla_a U_{bc} \\
= \;& \nabla_a \nabla_3 U_{bc} + s\zeta_a \nabla_3 U_{bc} + (\eta_a - \zeta_a)\nabla_3 U_{bc} + 2s\underline{\omega}(\zeta_a - \eta_a) U_{bc} \\
& -\frac{1}{2}\mathrm{tr}\,\underline{\chi}\left(\nabla_a U_{bc} + s(\zeta_a + \eta_a) U_{bc} + \eta_b U_{ac} + \eta_c U_{ab} - \delta_{ab}(\eta\cdot U)_c - \delta_{ac}(\eta\cdot U)_b\right) \\
& -\frac{1}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi}\Big({}^*\nabla_a U_{bc} + s({}^*\zeta_a + {}^*\eta_a) U_{bc} + \eta_b\, {}^*U_{ac} + \eta_c\, {}^*U_{ab} \\
& - \in_{ab} (\eta\cdot U)_c - \in_{ac} (\eta\cdot U)_b\Big) \\
& -2s\underline{\omega}\nabla_a U_{bc} - 2s^2\underline{\omega}\zeta_a U_{bc} - 2s\nabla_a\underline{\omega} U_{bc} + \Gamma_g\cdot \mathfrak{d}^{\leq 1} U,
\end{aligned}
$$

which gives

$$
\begin{aligned}
& {}^{(c)}\nabla_3\, {}^{(c)}\nabla_a U_{bc} \\
= \;& \nabla_a\, {}^{(c)}\nabla_3 U_{bc} + s\zeta_a\, {}^{(c)}\nabla_3 U_{bc} + (\eta_a - \zeta_a)\, {}^{(c)}\nabla_3 U_{bc} \\
& -\frac{1}{2}\mathrm{tr}\,\underline{\chi}\left({}^{(c)}\nabla_a U_{bc} + s(\eta_a) U_{bc} + \eta_b U_{ac} + \eta_c U_{ab} - \delta_{ab}(\eta\cdot U)_c - \delta_{ac}(\eta\cdot U)_b\right) \\
& -\frac{1}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi}\Big({}^*\,{}^{(c)}\nabla_a U_{bc} + s({}^*\eta_a) U_{bc} + \eta_b\, {}^*U_{ac} + \eta_c\, {}^*U_{ab} \\
& - \in_{ab} (\eta\cdot U)_c - \in_{ac} (\eta\cdot U)_b\Big) + \Gamma_g\cdot \mathfrak{d}^{\leq 1} U.
\end{aligned}
$$

Recall that ${}^{(c)}\nabla_3 U$ is of conformal type $s-1$, therefore

$$
{}^{(c)}\nabla_a\, {}^{(c)}\nabla_3 U_{bc} \;=\; \nabla_a\, {}^{(c)}\nabla_3 U_{bc} + (s-1)\zeta_a\, {}^{(c)}\nabla_3 U_{bc}.
$$

We finally obtain

$$
\begin{aligned}
& {}^{(c)}\nabla_3\, {}^{(c)}\nabla_a U_{bc} \\
= \;& {}^{(c)}\nabla_a\, {}^{(c)}\nabla_3 U_{bc} + \eta_a\, {}^{(c)}\nabla_3 U_{bc} \\
& -\frac{1}{2}\mathrm{tr}\,\underline{\chi}\left({}^{(c)}\nabla_a U_{bc} + s(\eta_a) U_{bc} + \eta_b U_{ac} + \eta_c U_{ab} - \delta_{ab}(\eta\cdot U)_c - \delta_{ac}(\eta\cdot U)_b\right) \\
& -\frac{1}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi}\Big({}^*\,{}^{(c)}\nabla_a U_{bc} + s({}^*\eta_a) U_{bc} + \eta_b\, {}^*U_{ac} + \eta_c\, {}^*U_{ab} \\
& - \in_{ab} (\eta\cdot U)_c - \in_{ac} (\eta\cdot U)_b\Big) + \Gamma_g\cdot \mathfrak{d}^{\leq 1} U,
\end{aligned}
$$

as stated.

C.3 Proof of Lemma 4.3.2

For ${}^{(\mathbf{T})}\pi$, one can easily adapt the proof of Proposition 2.6.10 in [53], which is done in the particular case of outgoing geodesic frame. Also,

$$
\begin{aligned}
(\mathrm{Div}{}^{(\mathbf{T})}\pi)_a &= -\frac{1}{2}\mathbf{D}_3{}^{(\mathbf{T})}\pi_{4a} - \frac{1}{2}\mathbf{D}_4{}^{(\mathbf{T})}\pi_{3a} + \mathbf{g}^{bc}\mathbf{D}_b{}^{(\mathbf{T})}\pi_{ca} = \mathfrak{d}^{\leq 1}\Gamma_g\\
(\mathrm{Div}{}^{(\mathbf{T})}\pi)_4 &= -\frac{1}{2}\mathbf{D}_3{}^{(\mathbf{T})}\pi_{44} - \frac{1}{2}\mathbf{D}_4{}^{(\mathbf{T})}\pi_{34} + \mathbf{g}^{ab}\mathbf{D}_a{}^{(\mathbf{T})}\pi_{b4}\\
&= -\frac{1}{2}\mathbf{D}_3{}^{(\mathbf{T})}\pi_{44} + \mathbf{g}^{ab}\mathbf{D}_a{}^{(\mathbf{T})}\pi_{b4} = \mathfrak{d}^{\leq 1}\Gamma_g,\\
(\mathrm{Div}{}^{(\mathbf{T})}\pi)_3 &= -\frac{1}{2}\mathbf{D}_3{}^{(\mathbf{T})}\pi_{43} - \frac{1}{2}\mathbf{D}_4{}^{(\mathbf{T})}\pi_{33} + \mathbf{g}^{ab}\mathbf{D}_a{}^{(\mathbf{T})}\pi_{b3}\\
&= -\frac{1}{2}\mathbf{D}_4{}^{(\mathbf{T})}\pi_{33} + \mathbf{g}^{ab}\mathbf{D}_a{}^{(\mathbf{T})}\pi_{b3} = r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b.
\end{aligned}
$$

For ${}^{(\mathbf{Z})}\pi$, since the only difficulty is to track the weights in r, we provide the proof in the case where the normalization of (e_3, e_4) is outgoing, in which case we have

$$
\mathbf{Z} = \frac{1}{2}\left(2(r^2+a^2)\Re(\mathfrak{J})^b e_b - a(\sin\theta)^2 e_3 - \frac{a(\sin\theta)^2\Delta}{|q|^2}e_4\right).
$$

We write

$$
\mathbf{Z} = \mathbf{Z}^{(1)} + \mathbf{Z}^{(2)}, \qquad \mathbf{Z}^{(1)} := (r^2+a^2)\Re(\mathfrak{J})^b e_b, \qquad \mathbf{Z}^{(2)} := \frac{1}{2}\left(-a(\sin\theta)^2 e_3 - \frac{a(\sin\theta)^2\Delta}{|q|^2}e_4\right).
$$

Using $\widecheck{\nabla\mathfrak{J}} \in r^{-1}\Gamma_b$, we compute

$$
\begin{aligned}
\widecheck{{}^{(\mathbf{Z}^{(1)})}\pi}_{44} &= 0,\\
\widecheck{{}^{(\mathbf{Z}^{(1)})}\pi}_{43} &= r\widecheck{e_3(r)}\Re(\mathfrak{J})^b\mathbf{g}(e_b,e_4) + r^2\nabla_3\widecheck{\Re(\mathfrak{J})}^b = r^2\nabla_3\widecheck{\Re(\mathfrak{J})}^b = r\Gamma_b,\\
\widecheck{{}^{(\mathbf{Z}^{(1)})}\pi}_{4a} &= r\widecheck{\nabla(r)}\Re(\mathfrak{J})^b\mathbf{g}(e_b,e_4) = r\Gamma_g,\\
\widecheck{{}^{(\mathbf{Z}^{(1)})}\pi}_{ab} &= r\widecheck{\nabla(r)}\Re(\mathfrak{J})^b\mathbf{g}(e_b,e_a) + r^2\nabla\widecheck{\Re(\mathfrak{J})}^b = r\Gamma_b,\\
\widecheck{{}^{(\mathbf{Z}^{(1)})}\pi}_{3a} &= r\widecheck{e_3(r)}\Re(\mathfrak{J})^b\mathbf{g}(e_b,e_a) + r\widecheck{\nabla(r)}\Re(\mathfrak{J})^b\mathbf{g}(e_b,e_3) + r^2\widecheck{\nabla_3\Re(\mathfrak{J})}^b + r^2\widecheck{\nabla\Re(\mathfrak{J})}^b = r\Gamma_b,\\
\widecheck{{}^{(\mathbf{Z}^{(1)})}\pi}_{33} &= r\widecheck{e_3(r)}\Re(\mathfrak{J})^b\mathbf{g}(e_b,e_3) + r\nabla_3\widecheck{\Re(\mathfrak{J})}^b = r^2\nabla_3\widecheck{\Re(\mathfrak{J})}^b = r\Gamma_b.
\end{aligned}
$$

Since $\mathbf{Z}^{(2)}$ is at the same level as $\mathbf{T}$, then $\widecheck{{}^{(\mathbf{Z}^{(1)})}\pi}$ has the same decay properties as ${}^{(\mathbf{T})}\pi$. We deduce

$$
{}^{(\mathbf{Z})}\pi_{44} \in \Gamma_g, \qquad {}^{(\mathbf{Z})}\pi_{4a} \in r\Gamma_g, \qquad {}^{(\mathbf{Z})}\pi_{ab},\ {}^{(\mathbf{Z})}\pi_{43},\ {}^{(\mathbf{Z})}\pi_{33},\ {}^{(\mathbf{Z})}\pi_{3a} \in r\Gamma_b,
$$

as stated. In particular, we have

$$\mathrm{tr}({}^{(\mathbf{Z})}\pi) \;=\; -{}^{(\mathbf{Z})}\pi_{43} + \mathbf{g}^{ab}{}^{(\mathbf{Z})}\pi_{ab} \in r\Gamma_b.$$

Also we compute

$$\begin{aligned}
(\mathrm{Div}{}^{(\mathbf{Z})}\pi)_a &= -\frac{1}{2}\mathbf{D}_3({}^{(\mathbf{Z})}\pi)_{4a} - \frac{1}{2}\mathbf{D}_4({}^{(\mathbf{Z})}\pi)_{3a} + \mathbf{g}^{bc}\mathbf{D}_b({}^{(\mathbf{Z})}\pi)_{ca} = r\mathfrak{d}^{\leq 1}\Gamma_g,\\
(\mathrm{Div}\,{}^{(\mathbf{Z})}\pi)_4 &= -\frac{1}{2}\mathbf{D}_3({}^{(\mathbf{Z})}\pi)_{44} + \mathbf{g}^{ab}\mathbf{D}_a({}^{(\mathbf{Z})}\pi)_{b4} = \mathfrak{d}^{\leq 1}(\Gamma_g),\\
(\mathrm{Div}\,{}^{(\mathbf{Z})}\pi)_3 &= -\frac{1}{2}\mathbf{D}_4({}^{(\mathbf{Z})}\pi)_{33} + \mathbf{g}^{ab}\mathbf{D}_a({}^{(\mathbf{Z})}\pi)_{b3} = \mathfrak{d}^{\leq 1}(\Gamma_b),
\end{aligned}$$

as stated.

C.4 Proof of Proposition 4.3.3

Recall the following general commutation formula for a scalar ψ and vectorfield X in a Ricci flat manifold, as implied by Lemma 2.3.3:

$$[X, \square_{\mathbf{g}}]\psi \;=\; -2\,{}^{(X)}\pi^{\mu\alpha}\mathbf{D}_\mu\mathbf{D}_\alpha\psi + \big(\mathbf{D}^\alpha(\mathrm{tr}\,{}^{(X)}\pi) - 2(\mathrm{Div}\,{}^{(X)}\pi)^\alpha\big)\mathbf{D}_\alpha\psi. \quad \text{(C.4.1)}$$

From (C.4.1), we infer

$$\begin{aligned}
[X, \square_{\mathbf{g}}]\psi \;=\;& r^{-2}\Big({}^{(X)}\pi_{33},\ {}^{(X)}\pi_{3a},\ {}^{(X)}\pi_{ab}\Big)\mathfrak{d}^{\leq 2}\psi + r^{-1}\,{}^{(X)}\pi_{4a}\mathfrak{d}^{\leq 2}\psi + {}^{(X)}\pi_{44}\mathfrak{d}^{\leq 2}\psi\\
&+ {}^{(X)}\pi_{34}\mathbf{D}_3\mathbf{D}_4\psi + \Big(\mathbf{D}_4(\mathrm{tr}\,{}^{(X)}\pi) - 2(\mathrm{Div}\,{}^{(X)}\pi)_4\Big)\mathfrak{d}^{\leq 1}\psi\\
&+ r^{-1}\Big(\mathbf{D}_3(\mathrm{tr}\,{}^{(X)}\pi) - 2(\mathrm{Div}\,{}^{(X)}\pi)_3,\ \mathbf{D}_a(\mathrm{tr}\,{}^{(X)}\pi) - 2(\mathrm{Div}\,{}^{(X)}\pi)_a\Big)\mathfrak{d}^{\leq 1}\psi.
\end{aligned}$$

Writing

$$\mathbf{D}_3\mathbf{D}_4\psi \;=\; -\square_{\mathbf{g}}\psi + \mathbf{g}^{ab}\mathbf{D}_a\mathbf{D}_b\psi,$$

we obtain

$$\begin{aligned}
[X, \square_{\mathbf{g}}]\psi =\;& r^{-2}\Big({}^{(X)}\pi_{33},\ {}^{(X)}\pi_{3a},\ {}^{(X)}\pi_{ab},\ {}^{(X)}\pi_{34}\Big)\mathfrak{d}^{\leq 2}\psi + r^{-1}\,{}^{(X)}\pi_{4a}\mathfrak{d}^{\leq 2}\psi + {}^{(X)}\pi_{44}\mathfrak{d}^{\leq 2}\psi\\
&- {}^{(X)}\pi_{34}\square_{\mathbf{g}}\psi + \Big(\mathbf{D}_4(\mathrm{tr}\,{}^{(X)}\pi) - 2(\mathrm{Div}\,{}^{(X)}\pi)_4\Big)\mathfrak{d}^{\leq 1}\psi \qquad \text{(C.4.2)}\\
&+ r^{-1}\Big(\mathbf{D}_3(\mathrm{tr}\,{}^{(X)}\pi) - 2(\mathrm{Div}\,{}^{(X)}\pi)_3,\ \mathbf{D}_a(\mathrm{tr}\,{}^{(X)}\pi) - 2(\mathrm{Div}\,{}^{(X)}\pi)_a\Big)\mathfrak{d}^{\leq 1}\psi.
\end{aligned}$$

Using Lemma 4.3.2 and (C.4.2), we obtain for $X = \mathbf{T}$:

$$\begin{aligned}
[\mathbf{T}, \square_{\mathbf{g}}]\psi &= r^{-2}\Gamma_b \cdot \mathfrak{d}^{\leq 2}\psi + r^{-1}\Gamma_g \cdot \mathfrak{d}^{\leq 2}\psi + \Gamma_g \cdot \mathfrak{d}^{\leq 2}\psi \\
&\quad + \Gamma_b \cdot \square_{\mathbf{g}}\psi + \Big(\mathfrak{d}^{\leq 1}\Gamma_g\Big)\mathfrak{d}^{\leq 1}\psi + r^{-1}\Big(\mathfrak{d}^{\leq 1}\Gamma_g + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b\Big)\mathfrak{d}^{\leq 1}\psi \\
&= \Gamma_g \cdot \mathfrak{d}^{\leq 2}\psi + \mathfrak{d}^{\leq 1}\Gamma_g \cdot \mathfrak{d}^{\leq 1}\psi + \Gamma_b \cdot \square_{\mathbf{g}}\psi \\
&= \mathfrak{d}\big(\Gamma_g \cdot \mathfrak{d}\psi\big) + \Gamma_b \cdot \square_{\mathbf{g}}\psi,
\end{aligned}$$

as stated.

Similarly, using Lemma 4.3.2 and applying (C.4.2) to $X = \mathbf{Z}$, we obtain

$$\begin{aligned}
[\mathbf{Z}, \square_{\mathbf{g}}]\psi &= r^{-1}\Gamma_b \cdot \mathfrak{d}^{\leq 2}\psi + \Gamma_g \cdot \mathfrak{d}^{\leq 2}\psi + \Gamma_g \cdot \mathfrak{d}^{\leq 2}\psi \\
&\quad + r\Gamma_b \cdot \square_{\mathbf{g}}\psi + \Big(\mathbf{D}_4(\mathrm{tr}\,{}^{(\mathbf{Z})}\pi) + \mathfrak{d}^{\leq 1}\Gamma_g\Big)\mathfrak{d}^{\leq 1}\psi \\
&\quad + r^{-1}\Big(\mathbf{D}_3(\mathrm{tr}\,{}^{(\mathbf{Z})}\pi) + \mathfrak{d}^{\leq 1}\Gamma_b,\ \Gamma_b + r\mathfrak{d}^{\leq 1}\Gamma_g\Big)\mathfrak{d}^{\leq 1}\psi \\
&= \Gamma_g \cdot \mathfrak{d}^{\leq 2}\psi + r\Gamma_b \cdot \square_{\mathbf{g}}\psi \\
&\quad + \Big(\mathbf{D}_4(\mathrm{tr}\,{}^{(\mathbf{Z})}\pi) + r^{-1}\mathbf{D}_3(\mathrm{tr}\,{}^{(\mathbf{Z})}\pi) + \mathfrak{d}^{\leq 1}\Gamma_g\Big)\mathfrak{d}^{\leq 1}\psi.
\end{aligned}$$

We now show that

$$\mathbf{D}_4(\mathrm{tr}\,{}^{(\mathbf{Z})}\pi) + r^{-1}\mathbf{D}_3(\mathrm{tr}\,{}^{(\mathbf{Z})}\pi) = \mathfrak{d}^{\leq 1}\Gamma_g. \tag{C.4.3}$$

We have

$$\begin{aligned}
\mathbf{D}_4(\mathrm{tr}\,{}^{(\mathbf{Z})}\pi) &= \mathbf{D}_4\Big(-{}^{(\mathbf{Z})}\pi_{43} + \mathbf{g}^{ab}{}^{(\mathbf{Z})}\pi_{ab}\Big) = \mathbf{D}_4\Big(\mathbf{g}^{ab}{}^{(\mathbf{Z})}\pi_{ab}\Big) \\
&= \nabla_4(r^2 \widecheck{\mathrm{d}iv\,(\Re(\mathfrak{J}))}) = \mathfrak{d}^{\leq 1}\Gamma_g,
\end{aligned}$$

where we used the good transport equation in e_4 for $\widecheck{\mathrm{d}iv\,(\Re(\mathfrak{J}))}$. Also,

$$\begin{aligned}
\mathbf{D}_3(\mathrm{tr}\,{}^{(\mathbf{Z})}\pi) &= \mathbf{D}_3\Big(-{}^{(\mathbf{Z})}\pi_{43} + \mathbf{g}^{ab}{}^{(\mathbf{Z})}\pi_{ab}\Big) = \mathbf{D}_3\Big(\mathbf{g}^{ab}{}^{(\mathbf{Z})}\pi_{ab}\Big) \\
&= \nabla_3(r^2 \widecheck{\nabla\mathfrak{J}}) = \mathfrak{d}^{\leq 1}\Gamma_b,
\end{aligned}$$

where we use Lemma C.4.1 below. Since $r^{-1}\Gamma_b$ decays faster than Γ_g, this proves (C.4.3), and therefore the Proposition.

Lemma C.4.1. *We have the following improved estimate:*

$$\nabla_3(r^2 \widecheck{\nabla\mathfrak{J}}) \in \mathfrak{d}^{\leq 1}\Gamma_b.$$

Proof. Note first from Proposition 5.6.16 in [53] that

$$\nabla_\nu(r^2\widecheck{\nabla\mathfrak{J}}) \in \mathfrak{d}^{\leq 1}\Gamma_b \text{ on } \Sigma_*.$$

Also, we have in view of Lemma 6.1.15 in [53]:

$$\begin{aligned}
\nabla_4(\mathcal{D}\widehat{\otimes}\mathfrak{J}) + \frac{2}{q}\mathcal{D}\widehat{\otimes}\mathfrak{J} &= O(r^{-1})B + O(r^{-2})\widecheck{\mathrm{tr}X} + O(r^{-2})\widehat{X} \\
&\quad + O(r^{-2})\widecheck{Z} + O(r^{-3})\widecheck{\mathcal{D}(\cos\theta)} + r^{-1}\Gamma_b\cdot\Gamma_g, \\
\nabla_4\big(\widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}}\big) + \Re\left(\frac{2}{q}\right)\widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}} &= O(r^{-1})B + O(r^{-2})\widecheck{\mathrm{tr}X} + O(r^{-2})\widehat{X} + O(r^{-2})\widecheck{Z} \\
&\quad + O(r^{-3})\widecheck{\mathcal{D}(\cos\theta)} + r^{-1}\Gamma_b\cdot\Gamma_g.
\end{aligned}$$

In particular, we infer

$$\begin{aligned}
\nabla_4\Big(q^2\mathcal{D}\widehat{\otimes}\mathfrak{J}\Big) &= \Big(rB,\, \widecheck{\mathrm{tr}X},\, \widehat{X},\, \widecheck{Z}\Big) + r^{-1}\Gamma_b, \\
\nabla_4\Big(|q|^2\widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}}\Big) &= \Big(rB,\, \widecheck{\mathrm{tr}X},\, \widehat{X},\, \widecheck{Z}\Big) + r^{-1}\Gamma_b.
\end{aligned}$$

Commuting with ∇_3, we infer

$$\begin{aligned}
\nabla_4\nabla_3\Big(q^2\mathcal{D}\widehat{\otimes}\mathfrak{J}\Big) &= r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b, \\
\nabla_4\nabla_3\Big(|q|^2\widecheck{\overline{\mathcal{D}}\cdot\mathfrak{J}}\Big) &= r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b,
\end{aligned}$$

which immediately implies by integration from Σ_* the estimate $\nabla_3(r^2\widecheck{\nabla\mathfrak{J}}) \in \mathfrak{d}^{\leq 1}\Gamma_b$ as desired. $\square$

C.5 Proof of Proposition 4.5.3

Using Lemma 4.7.4, we have

$$\begin{aligned}
|q|^2\dot{\square}_2\psi &= -\frac{1}{2}|q|^2\big(\nabla_3\nabla_4\psi + \nabla_4\nabla_3\psi\big) + |q|^2\left(\underline{\omega} - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\right)\nabla_4\psi + |q|^2\left(\omega - \frac{1}{2}\mathrm{tr}\,\chi\right)\nabla_3\psi \\
&\quad + |q|^2(\triangle\psi + (\eta + \underline{\eta})\cdot\nabla\psi) \\
&= -\frac{1}{2}|q|^2\big(\nabla_3\nabla_4\psi + \nabla_4\nabla_3\psi\big) + |q|^2\left(\underline{\omega} - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\right)\nabla_4\psi + |q|^2\left(\omega - \frac{1}{2}\mathrm{tr}\,\chi\right)\nabla_3\psi \\
&\quad + \mathcal{O}(\psi) + r\Gamma_b\cdot\mathfrak{d}\psi,
\end{aligned}$$

and therefore

$$
\begin{aligned}
[\mathcal{O}, |q|^2\dot{\square}_2]\psi &= -\frac{1}{2}[\mathcal{O}, |q|^2\big(\nabla_3\nabla_4+\nabla_4\nabla_3\big)]\psi \\
&+\big[\mathcal{O}, |q|^2\left(\underline{\omega}-\frac{1}{2}\mathrm{tr}\,\underline{\chi}\right)\nabla_4\big]\psi+\big[\mathcal{O}, |q|^2\left(\omega-\frac{1}{2}\mathrm{tr}\,\chi\right)\nabla_3\big]\psi+r\mathfrak{d}^2\big(\Gamma_b\cdot\mathfrak{d}\psi\big).
\end{aligned}
$$

We compute $[\mathcal{O}, |q|^2\big(\nabla_3\nabla_4+\nabla_4\nabla_3\big)]\psi$. We write,

$$
\begin{aligned}
[\mathcal{O}, |q|^2\nabla_3\nabla_4]\psi &= |q|^2[\mathcal{O}, \nabla_3\nabla_4]\psi+2|q|^2\nabla(|q|^2)\cdot\nabla\nabla_3\nabla_4\psi+\mathcal{O}(|q|^2)\nabla_3\nabla_4\psi \\
&= |q|^2[\mathcal{O}, \nabla_3]\nabla_4\psi+|q|^2\nabla_3([\mathcal{O}, \nabla_4]\psi) \\
&\quad +2|q|^2\nabla(|q|^2)\cdot\nabla\nabla_3\nabla_4\psi+\mathcal{O}(|q|^2)\nabla_3\nabla_4\psi.
\end{aligned}
$$

Using the explicit expressions for the commutators $[\mathcal{O}, \nabla_3]$ and $[\mathcal{O}, \nabla_4]$ in Lemma 4.7.10, we obtain

$$
\begin{aligned}
[\mathcal{O}, \nabla_3]\nabla_4\psi &= -(\eta-\zeta)\cdot|q|^2\nabla_3\nabla\nabla_4\psi-(\eta-\zeta)\cdot|q|^2\nabla\nabla_3\nabla_4\psi \\
&\quad -\big(\mathrm{d}iv\,(\eta-\zeta)+(\eta+\underline{\eta})\cdot(\eta-\zeta)\big)|q|^2\nabla_3\nabla_4\psi \\
&\quad +O(ar^{-3})\mathfrak{d}^{\leq 2}\psi+r^{-1}\mathfrak{d}\big(\Gamma_b\cdot\mathfrak{d}^2\psi\big),
\end{aligned}
$$

and

$$
\begin{aligned}
\nabla_3([\mathcal{O}, \nabla_4]\psi) &= -(\underline{\eta}+\zeta)\cdot|q|^2\nabla_3\nabla_4\nabla\psi-(\underline{\eta}+\zeta)\cdot|q|^2\nabla_3\nabla\nabla_4\psi \\
&\quad -\nabla_3(|q|^2(\underline{\eta}+\zeta))\cdot\nabla_4\nabla\psi-\nabla_3(|q|^2(\underline{\eta}+\zeta))\cdot\nabla\nabla_4\psi \\
&\quad -\big(\mathrm{d}iv\,(\underline{\eta}+\zeta)+(\eta+\underline{\eta})\cdot(\underline{\eta}+\zeta)\big)|q|^2\nabla_3\nabla_4\psi \\
&\quad -\nabla_3\Big(\big(\mathrm{d}iv\,(\underline{\eta}+\zeta)+(\eta+\underline{\eta})\cdot(\underline{\eta}+\zeta)\big)|q|^2\Big)\nabla_4\psi \\
&\quad +O(ar^{-2})\mathfrak{d}^{\leq 2}\psi+\dot{\mathbf{D}}_3\dot{\mathbf{D}}_3\big(|q|^2\xi\cdot\dot{\mathbf{D}}_a\psi\big)+\mathfrak{d}^2\big(\Gamma_g\cdot\mathfrak{d}\psi\big).
\end{aligned}
$$

By considering for each term in the above its value in Kerr plus the error terms we obtain

$$
\begin{aligned}
|q|^{-2}[\mathcal{O}, |q|^2\big(\nabla_3\nabla_4+\nabla_4\nabla_3\big)]\psi &= |q|^{-2}[\mathcal{O}, |q|^2\big(\nabla_3\nabla_4+\nabla_4\nabla_3\big)]_{Kerr}\psi \\
&\quad +\dot{\mathbf{D}}_3\mathfrak{d}\big(|q|^2\xi\cdot\dot{\mathbf{D}}_a\psi\big)+\mathfrak{d}^2\big(\Gamma_g\cdot\mathfrak{d}\psi\big),
\end{aligned}
$$

We compute $\Big[\mathcal{O}, |q|^2\left(\underline{\omega}-\frac{1}{2}\mathrm{tr}\,\underline{\chi}\right)\nabla_4\Big]\psi+\Big[\mathcal{O}, |q|^2\left(\omega-\frac{1}{2}\mathrm{tr}\,\chi\right)\nabla_3\Big]\psi$.

We write

$$
\begin{aligned}
&\Big[\mathcal{O}, |q|^2\left(\underline{\omega}-\frac{1}{2}\mathrm{tr}\,\underline{\chi}\right)\nabla_4\Big]\psi \\
= \;& |q|^2\left(\underline{\omega}-\frac{1}{2}\mathrm{tr}\,\underline{\chi}\right)[\mathcal{O}, \nabla_4]\psi \\
&+2|q|^2\nabla\big(|q|^2\big(\underline{\omega}-\frac{1}{2}\mathrm{tr}\,\underline{\chi}\big)\big)\cdot\nabla\nabla_4\psi+\mathcal{O}\Big(|q|^2\big(\underline{\omega}-\frac{1}{2}\mathrm{tr}\,\underline{\chi}\big)\Big)\nabla_4\psi,
\end{aligned}
$$

which gives

$$
\begin{aligned}
& |q|^{-2}\Big[\mathcal{O}, |q|^2\left(\underline{\omega}-\frac{1}{2}\mathrm{tr}\,\underline{\chi}\right)\nabla_4\Big]\psi \\
= \ & \big(\underline{\omega}-\frac{1}{2}\mathrm{tr}\,\underline{\chi}\big)\Big(-(\underline{\eta}+\zeta)\cdot|q|^2\nabla_4\nabla\psi-(\underline{\eta}+\zeta)\cdot|q|^2\nabla\nabla_4\psi\Big) \\
& +2\nabla\left(|q|^2\big(\underline{\omega}-\frac{1}{2}\mathrm{tr}\,\underline{\chi}\big)\right)\cdot\nabla\nabla_4\psi+O(ar^{-3})\mathfrak{d}\psi+\mathfrak{d}\big(\Gamma_g\cdot\mathfrak{d}\psi\big).
\end{aligned}
$$

By symmetry we obtain

$$
\begin{aligned}
& |q|^{-2}\Big[\mathcal{O}, |q|^2\left(\omega-\frac{1}{2}\mathrm{tr}\,\chi\right)\nabla_3\Big]\psi \\
= \ & \big(\omega-\frac{1}{2}\mathrm{tr}\,\chi\big)\Big(-(\eta-\zeta)\cdot|q|^2\nabla_3\nabla\psi-(\eta-\zeta)\cdot|q|^2\nabla\nabla_3\psi\Big) \\
& +2\nabla\left(|q|^2\big(\omega-\frac{1}{2}\mathrm{tr}\,\chi\big)\right)\cdot\nabla\nabla_3\psi+O(ar^{-3})\mathfrak{d}\psi+r^{-1}\mathfrak{d}\big(\Gamma_b\cdot\mathfrak{d}\psi\big).
\end{aligned}
$$

We therefore deduce

$$
\begin{aligned}
& |q|^{-2}\Big([\mathcal{O}, |q|^2\big(\underline{\omega}-\frac{1}{2}\mathrm{tr}\,\underline{\chi}\big)\nabla_4]\psi+[\mathcal{O}, |q|^2\big(\omega-\frac{1}{2}\mathrm{tr}\,\chi\big)\nabla_3]\psi\Big) \\
= \ & |q|^{-2}\Big([\mathcal{O}, |q|^2\big(\underline{\omega}-\frac{1}{2}\mathrm{tr}\,\underline{\chi}\big)\nabla_4]\psi+[\mathcal{O}, |q|^2\big(\omega-\frac{1}{2}\mathrm{tr}\,\chi\big)\nabla_3]\psi\Big)_{Kerr}+\mathfrak{d}\big(\Gamma_g\cdot\mathfrak{d}\psi\big).
\end{aligned}
$$

We sum the above two contributions, and obtain

$$
|q|^{-2}[\mathcal{O}, |q|^2\square_{\mathbf{g}}]\psi \ = \ |q|^{-2}[\mathcal{O}, |q|^2\square_{\mathbf{g}}]_{Kerr}\psi+\dot{\mathbf{D}}_3\mathfrak{d}\big(|q|^2\xi\cdot\dot{\mathbf{D}}_a\psi\big)+\mathfrak{d}^2\big(\Gamma_g\cdot\mathfrak{d}\psi\big).
$$

Using Proposition 3.7.6 to write

$$
[\mathcal{O}, |q|^2\square_{\mathbf{g}}]_{Kerr}\psi \ = \ |q|^2\Big[\nabla\left(\frac{8a(r^2+a^2)\cos\theta}{|q|^2}\right)\cdot\nabla\nabla_{\widehat{T}}\,{}^*\psi+O(ar^{-2})\nabla_{\widehat{R}}^{\leq 1}\mathfrak{d}^{\leq 1}\psi\Big],
$$

we prove the Proposition.

C.6 Proof of Lemma 4.6.2

We write, for two vectorfields X and Y:

$$
\begin{aligned}
& |q|^2\dot{\mathbf{D}}_\alpha(|q|^{-2}X^{(\alpha}Y^{\beta)}\dot{\mathbf{D}}_\beta\psi) \\
= \ & \nabla_{(X}\nabla_{Y)}\psi+\frac{1}{2}\Big[|q|^2\nabla_X(|q|^{-2})\nabla_Y\psi+|q|^2\nabla_Y(|q|^{-2})\nabla_X\psi \\
& +\mathrm{tr}\,{}^{(X)}\pi\nabla_Y\psi+\mathrm{tr}^{(Y)}\pi\nabla_X\psi+{}^{(Y)}\pi_{\alpha\beta}X^\alpha\dot{\mathbf{D}}^\beta\psi+\,{}^{(X)}\pi_{\alpha\beta}Y^\alpha\dot{\mathbf{D}}^\beta\psi\Big].
\end{aligned}
$$

For $X = Y = \mathbf{T}$, we obtain

$$\mathcal{S}_1(\psi) = \nabla_{\mathbf{T}}\nabla_{\mathbf{T}}\psi + |q|^2\nabla_{\mathbf{T}}(|q|^{-2})\nabla_{\mathbf{T}}\psi + \mathrm{tr}^{(\mathbf{T})}\pi\nabla_{\mathbf{T}}\psi + {}^{(\mathbf{T})}\pi_{\alpha\beta}\mathbf{T}^\alpha\dot{\mathbf{D}}^\beta\psi.$$

Writing $\nabla_{\mathbf{T}} = \mathfrak{d}$ and using that $\nabla_{\mathbf{T}}(|q|) \in r\Gamma_b$, $\mathrm{tr}^{(\mathbf{T})}\pi \in \Gamma_g$, and

$$\begin{aligned} {}^{(\mathbf{T})}\pi_{\alpha\beta}\mathbf{T}^\alpha\dot{\mathbf{D}}^\beta\psi &= {}^{(\mathbf{T})}\pi_{3\beta}\mathbf{T}^3\dot{\mathbf{D}}^\beta\psi + {}^{(\mathbf{T})}\pi_{4\beta}\mathbf{T}^4\dot{\mathbf{D}}^\beta\psi + {}^{(\mathbf{T})}\pi_{a\beta}\mathbf{T}^a\dot{\mathbf{D}}^\beta\psi \\ &= {}^{(\mathbf{T})}\pi_{34}\mathbf{T}^3\dot{\mathbf{D}}_3\psi + \Gamma_g\cdot\mathfrak{d}\psi = \Gamma_b\cdot\mathfrak{d}\psi, \end{aligned}$$

we have

$$\mathcal{S}_1(\psi) = \nabla_{\mathbf{T}}\nabla_{\mathbf{T}}\psi + \Gamma_b\cdot\mathfrak{d}\psi.$$

For $X = Y = \mathbf{Z}$, we obtain

$$\mathcal{S}_3(\psi) = a^2\nabla_{\mathbf{Z}}\nabla_{\mathbf{Z}}\psi + |q|^2\nabla_{\mathbf{Z}}(|q|^{-2})\nabla_{\mathbf{Z}}\psi + \mathrm{tr}\,^{(\mathbf{Z})}\pi\nabla_{\mathbf{Z}}\psi + {}^{(\mathbf{Z})}\pi_{\alpha\beta}\mathbf{Z}^\alpha\dot{\mathbf{D}}^\beta\psi.$$

Writing $\nabla_{\mathbf{Z}} = \mathfrak{d}$ and using that $\nabla_{\mathbf{Z}}(|q|) \in r\Gamma_b$, $\mathrm{tr}\,^{(\mathbf{Z})}\pi \in r\Gamma_b$, and

$$\begin{aligned} {}^{(\mathbf{Z})}\pi_{\alpha\beta}\mathbf{Z}^\alpha\dot{\mathbf{D}}^\beta\psi &= {}^{(\mathbf{Z})}\pi_{3\beta}\mathbf{Z}^3\dot{\mathbf{D}}^\beta\psi + {}^{(\mathbf{Z})}\pi_{4\beta}\mathbf{Z}^4\dot{\mathbf{D}}^\beta\psi + {}^{(\mathbf{Z})}\pi_{a\beta}\mathbf{Z}^a\dot{\mathbf{D}}^\beta\psi \\ &= {}^{(\mathbf{Z})}\pi_{34}\mathbf{Z}^3\dot{\mathbf{D}}_3\psi + \Gamma_b\cdot\mathfrak{d}\psi = r\Gamma_b\cdot\mathfrak{d}\psi, \end{aligned}$$

we have

$$\mathcal{S}_3(\psi) = a^2\nabla_{\mathbf{Z}}\nabla_{\mathbf{Z}}\psi + r\Gamma_b\cdot\mathfrak{d}\psi.$$

Combining the above, for $X = \mathbf{T}$ and $Y = \mathbf{Z}$ we have

$$\mathcal{S}_2(\psi) = a\nabla_{\mathbf{T}}\nabla_{\mathbf{Z}}\psi + r\Gamma_b\cdot\mathfrak{d}\psi + {}^{(\mathbf{T})}\pi_{\alpha\beta}\mathbf{Z}^\alpha\dot{\mathbf{D}}^\beta\psi + {}^{(\mathbf{Z})}\pi_{\alpha\beta}\mathbf{T}^\alpha\dot{\mathbf{D}}^\beta\psi = a\nabla_{\mathbf{T}}\nabla_{\mathbf{Z}}\psi + r\Gamma_b\cdot\mathfrak{d}\psi.$$

Finally, we write

$$\begin{aligned} \mathcal{S}_4(\psi) &= |q|^2\dot{\mathbf{D}}_\beta(|q|^{-2}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\psi) = |q|^2\dot{\mathbf{D}}_\beta(\gamma^{ab}e_a^\alpha e_b^\beta\dot{\mathbf{D}}_\alpha\psi) \\ &= |q|^2\gamma^{ab}e_a^\alpha e_b^\beta\dot{\mathbf{D}}_\alpha\dot{\mathbf{D}}_\beta\psi + \gamma^{ab}|q|^2\dot{\mathbf{D}}_\beta(e_a^\alpha)e_b^\beta\dot{\mathbf{D}}_\alpha\psi + \gamma^{ab}|q|^2e_a^\alpha\dot{\mathbf{D}}_\beta(e_b^\beta)\dot{\mathbf{D}}_\alpha\psi \\ &= |q|^2\gamma^{ab}\dot{\mathbf{D}}_a\dot{\mathbf{D}}_b\psi + \gamma^{ab}|q|^2\dot{\mathbf{D}}_b(e_a^\alpha)\dot{\mathbf{D}}_\alpha\psi + \gamma^{ab}|q|^2\dot{\mathbf{D}}_\beta(e_b^\beta)\dot{\mathbf{D}}_a\psi \\ &= |q|^2\gamma^{ab}\nabla_a\nabla_b\psi - \frac{1}{2}|q|^2\mathrm{tr}\,\chi\nabla_3\psi - \frac{1}{2}|q|^2\mathrm{tr}\,\underline{\chi}\nabla_4\psi \\ &\quad +\gamma^{ab}|q|^2\big(\frac{1}{2}\chi_{ba}e_3^\alpha + \frac{1}{2}\underline{\chi}_{ba}e_4^\alpha\big)\dot{\mathbf{D}}_\alpha\psi + \gamma^{ab}|q|^2\big((\dot{\mathbf{D}}_3e_b)^3 + (\dot{\mathbf{D}}_4e_b)^4\big)\dot{\mathbf{D}}_a\psi \\ &= |q|^2\gamma^{ab}\nabla_a\nabla_b\psi + \gamma^{ab}|q|^2(\eta_b + \underline{\eta}_b)\nabla_a\psi = \mathcal{O}(\psi) + r\Gamma_b\cdot\mathfrak{d}, \end{aligned}$$

as stated.

C.7 Proof of Lemma 4.7.6

Using the expression for the metric given by (4.4.5), we have

$$\begin{aligned}
|q|^2\dot{\square}_k\psi &= |q|^2\mathbf{g}^{\alpha\beta}\dot{\mathbf{D}}_\alpha\dot{\mathbf{D}}_\beta\psi = \left(\frac{(r^2+a^2)^2}{\Delta}\big(-\widehat{T}^\alpha\widehat{T}^\beta+\widehat{R}^\alpha\widehat{R}^\beta\big)+O^{\alpha\beta}\right)\dot{\mathbf{D}}_\alpha\dot{\mathbf{D}}_\beta\psi\\
&= \frac{(r^2+a^2)^2}{\Delta}\big(-\dot{\mathbf{D}}_{\widehat{T}}\dot{\mathbf{D}}_{\widehat{T}}\psi+\dot{\mathbf{D}}_{\widehat{R}}\dot{\mathbf{D}}_{\widehat{R}}\psi\big)+O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\dot{\mathbf{D}}_\beta\psi.
\end{aligned}$$

We write

$$\dot{\mathbf{D}}_{\widehat{T}}\dot{\mathbf{D}}_{\widehat{T}}\psi = \nabla_{\widehat{T}}\nabla_{\widehat{T}}\psi-\dot{\mathbf{D}}_{\mathbf{D}_{\widehat{T}}\widehat{T}}\psi, \qquad \dot{\mathbf{D}}_{\widehat{R}}\dot{\mathbf{D}}_{\widehat{R}}\psi=\nabla_{\widehat{R}}\nabla_{\widehat{R}}\psi-\dot{\mathbf{D}}_{\mathbf{D}_{\widehat{R}}\widehat{R}}\psi.$$

Using the definitions of $\widehat{T}$ and $\widehat{R}$ in the outgoing frame, we obtain

$$\begin{aligned}
2\mathbf{D}_{\widehat{T}}\widehat{T} &= \frac{\Delta}{r^2+a^2}\mathbf{D}_4\widehat{T}+\frac{|q|^2}{r^2+a^2}\mathbf{D}_3\widehat{T}\\
&= \frac{1}{2}\frac{\Delta}{r^2+a^2}\mathbf{D}_4\left(\frac{\Delta}{r^2+a^2}e_4+\frac{|q|^2}{r^2+a^2}e_3\right)+\frac{1}{2}\frac{|q|^2}{r^2+a^2}\mathbf{D}_3\left(\frac{\Delta}{r^2+a^2}e_4+\frac{|q|^2}{r^2+a^2}e_3\right)\\
&= \frac{1}{2}\frac{\Delta}{r^2+a^2}\left(\frac{\Delta}{r^2+a^2}\mathbf{D}_4e_4+\frac{|q|^2}{r^2+a^2}\mathbf{D}_4e_3\right)\\
&\quad+\frac{1}{2}\frac{|q|^2}{r^2+a^2}\left(\frac{\Delta}{r^2+a^2}\mathbf{D}_3e_4+\frac{|q|^2}{r^2+a^2}\mathbf{D}_3e_3\right)\\
&\quad+\frac{1}{2}\frac{\Delta}{r^2+a^2}\left(e_4\big(\frac{\Delta}{r^2+a^2}\big)e_4+e_4\big(\frac{|q|^2}{r^2+a^2}\big)e_3\right)\\
&\quad+\frac{1}{2}\frac{|q|^2}{r^2+a^2}\left(e_3\big(\frac{\Delta}{r^2+a^2}\big)e_4+e_3\big(\frac{|q|^2}{r^2+a^2}\big)e_3\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
2\mathbf{D}_{\widehat{R}}\widehat{R} &= \frac{1}{2}\frac{\Delta}{r^2+a^2}\left(\frac{\Delta}{r^2+a^2}\mathbf{D}_4e_4-\frac{|q|^2}{r^2+a^2}\mathbf{D}_4e_3\right)\\
&\quad-\frac{1}{2}\frac{|q|^2}{r^2+a^2}\left(\frac{\Delta}{r^2+a^2}\mathbf{D}_3e_4-\frac{|q|^2}{r^2+a^2}\mathbf{D}_3e_3\right)\\
&\quad+\frac{1}{2}\frac{\Delta}{r^2+a^2}\left(e_4\big(\frac{\Delta}{r^2+a^2}\big)e_4-e_4\big(\frac{|q|^2}{r^2+a^2}\big)e_3\right)\\
&\quad-\frac{1}{2}\frac{|q|^2}{r^2+a^2}\left(e_3\big(\frac{\Delta}{r^2+a^2}\big)e_4-e_3\big(\frac{|q|^2}{r^2+a^2}\big)e_3\right).
\end{aligned}$$

This gives

$$\begin{aligned}
2\big(\mathbf{D}_{\widehat{T}}\widehat{T}-\mathbf{D}_{\widehat{R}}\widehat{R}\big) &= \frac{\Delta|q|^2}{(r^2+a^2)^2}\left(\mathbf{D}_4 e_3+\mathbf{D}_3 e_4\right)\\
&+\frac{\Delta}{r^2+a^2}e_4\big(\frac{|q|^2}{r^2+a^2}\big)e_3+\frac{|q|^2}{r^2+a^2}e_3\big(\frac{\Delta}{r^2+a^2}\big)e_4.
\end{aligned}$$

Using (2.2.3), we obtain

$$\begin{aligned}
2\big(\mathbf{D}_{\widehat{T}}\widehat{T}-\mathbf{D}_{\widehat{R}}\widehat{R}\big) &= \Big(\frac{|q|^2}{r^2+a^2}2\omega+e_4\big(\frac{|q|^2}{r^2+a^2}\big)\Big)\frac{\Delta}{r^2+a^2}e_3\\
&+\Big(\frac{\Delta}{r^2+a^2}2\underline{\omega}+e_3\big(\frac{\Delta}{r^2+a^2}\big)\Big)\frac{|q|^2}{r^2+a^2}e_4\\
&+\frac{\Delta|q|^2}{(r^2+a^2)^2}2(\eta_b+\underline{\eta}_b)e_b.
\end{aligned}$$

Computing in the outgoing frame

$$\begin{aligned}
e_4\left(\frac{|q|^2}{r^2+a^2}\right) &= \frac{2a^2r\sin^2\theta}{(r^2+a^2)^2}+r^{-3}\Gamma_g\\
e_3\left(\frac{\Delta}{r^2+a^2}\right) &= -\frac{\Delta}{|q|^2}\frac{\partial_r(\Delta)}{r^2+a^2}+\frac{2r\Delta^2}{|q|^2(r^2+a^2)^2}+\Gamma_b,
\end{aligned} \tag{C.7.1}$$

we obtain

$$\begin{aligned}
\frac{|q|^2}{r^2+a^2}2\omega+e_4\big(\frac{|q|^2}{r^2+a^2}\big) &= \frac{2ra^2\sin^2\theta}{(r^2+a^2)^2}+\Gamma_g\\
\frac{\Delta}{r^2+a^2}2\underline{\omega}+e_3\big(\frac{\Delta}{r^2+a^2}\big) &= \frac{\Delta}{r^2+a^2}\partial_r\left(\frac{\Delta}{|q|^2}\right)-\frac{\Delta}{|q|^2}\frac{\partial_r(\Delta)}{r^2+a^2}+\frac{2r\Delta^2}{|q|^2(r^2+a^2)^2}+\Gamma_b\\
&= -\frac{2r\Delta^2}{|q|^4(r^2+a^2)}+\frac{2r\Delta^2}{|q|^2(r^2+a^2)^2}+\Gamma_b\\
&= -\frac{2ra^2\sin^2\theta\Delta^2}{|q|^4(r^2+a^2)^2}+\Gamma_b.
\end{aligned}$$

We write

$$\begin{aligned}
2\big(\mathbf{D}_{\widehat{T}}\widehat{T}-\mathbf{D}_{\widehat{R}}\widehat{R}\big) &= \frac{2ra^2\sin^2\theta\Delta}{|q|^2(r^2+a^2)^2}\big(\frac{|q|^2}{r^2+a^2}e_3-\frac{\Delta}{r^2+a^2}e_4\big)\\
&+\frac{\Delta|q|^2}{(r^2+a^2)^2}2(\eta_b+\underline{\eta}_b)e_b+\Gamma_g\cdot\mathfrak{d}\\
&= -\frac{4ra^2\sin^2\theta\Delta}{|q|^2(r^2+a^2)^2}\widehat{R}+\frac{\Delta|q|^2}{(r^2+a^2)^2}2(\eta_b+\underline{\eta}_b)e_b+\Gamma_g\cdot\mathfrak{d}.
\end{aligned}$$

We therefore obtain

$$\begin{aligned}|q|^2\dot{\square}_k\psi &= \frac{(r^2+a^2)^2}{\Delta}\big(-\nabla_{\widehat{T}}\nabla_{\widehat{T}}\psi+\nabla_{\widehat{R}}\nabla_{\widehat{R}}\psi+\dot{\mathbf{D}}_{\mathbf{D}_{\widehat{T}}\widehat{T}-\mathbf{D}_{\widehat{R}}\widehat{R}}\psi\big)+O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\dot{\mathbf{D}}_\beta\psi\\ &= \frac{(r^2+a^2)^2}{\Delta}\big(-\nabla_{\widehat{T}}\nabla_{\widehat{T}}\psi+\nabla_{\widehat{R}}\nabla_{\widehat{R}}\psi\big)-\frac{2ra^2\sin^2\theta}{|q|^2}\nabla_{\widehat{R}}\psi\\ &\quad +O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\dot{\mathbf{D}}_\beta\psi+|q|^2(\eta_b+\underline{\eta}_b)\nabla_b\psi+r^2\Gamma_g\cdot\mathfrak{d}\psi.\end{aligned}$$

Finally, using the computations in Lemma 4.7.4, i.e.

$$\dot{\mathbf{D}}_c\dot{\mathbf{D}}_d\psi = \nabla_c\nabla_d\psi-\frac{1}{2}\chi_{cd}\nabla_3\psi-\frac{1}{2}\underline{\chi}_{cd}\nabla_4\psi,$$

we write

$$\begin{aligned}O^{\alpha\beta}\dot{\mathbf{D}}_\alpha\dot{\mathbf{D}}_\beta\psi &= |q|^2\gamma^{ab}e_a^\alpha e_b^\beta\dot{\mathbf{D}}_\alpha\dot{\mathbf{D}}_\beta\psi\\ &= |q|^2\gamma^{ab}\nabla_a\nabla_b\psi-\frac{1}{2}|q|^2\mathrm{tr}\,\chi\nabla_3\psi-\frac{1}{2}|q|^2\mathrm{tr}\,\underline{\chi}\nabla_4\psi\\ &= |q|^2\triangle_k\psi-r\nabla_3\psi+\frac{r\Delta}{|q|^2}\nabla_4\psi+r^2\Gamma_g\cdot\mathfrak{d}\psi\\ &= |q|^2\triangle_k\psi+\frac{2r(r^2+a^2)}{|q|^2}\nabla_{\widehat{R}}\psi+r^2\Gamma_g\cdot\mathfrak{d}\psi.\end{aligned}$$

This finally gives

$$\begin{aligned}|q|^2\dot{\square}_k\psi &= \frac{(r^2+a^2)^2}{\Delta}\big(-\nabla_{\widehat{T}}\nabla_{\widehat{T}}\psi+\nabla_{\widehat{R}}\nabla_{\widehat{R}}\psi\big)-\frac{2ra^2\sin^2\theta}{|q|^2}\nabla_{\widehat{R}}\psi\\ &\quad +|q|^2\triangle_k\psi+\frac{2r(r^2+a^2)}{|q|^2}\nabla_{\widehat{R}}\psi+|q|^2(\eta_b+\underline{\eta}_b)\nabla_b\psi+r^2\Gamma_g\cdot\mathfrak{d}\psi\\ &= \frac{(r^2+a^2)^2}{\Delta}\big(-\nabla_{\widehat{T}}\nabla_{\widehat{T}}\psi+\nabla_{\widehat{R}}\nabla_{\widehat{R}}\psi\big)+2r\nabla_{\widehat{R}}\psi\\ &\quad +|q|^2\triangle_k\psi+|q|^2(\eta_b+\underline{\eta}_b)\nabla_b\psi+r^2\Gamma_g\cdot\mathfrak{d}\psi,\end{aligned}$$

as stated.

C.8 Proof of Lemma 4.7.7

We define $Y_{ab} := \big(\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot\psi)\big)_{ab}$, and express Y in frames. We have

$$Y_{ab} = \mathcal{D}_a\overline{\mathcal{D}}^c\psi_{cb}+\mathcal{D}_b\overline{\mathcal{D}}^c\psi_{ca}-\delta_{ab}\mathcal{D}^d\overline{\mathcal{D}}^c\psi_{dc}.$$

By construction, Y is symmetric and traceless. We calculate first $Y_{11} = -Y_{22}$. For $a = b = 1$ we derive

$$\begin{aligned} Y_{11} &= \mathcal{D}_1\overline{\mathcal{D}}^c\psi_{c1} + \mathcal{D}_1\overline{\mathcal{D}}^c\psi_{c1} - \delta_{11}\mathcal{D}^d\overline{\mathcal{D}}^c\psi_{dc} \\ &= 2\mathcal{D}_1\overline{\mathcal{D}}_c\psi_{c1} - \mathcal{D}^d\overline{\mathcal{D}}^c\psi_{dc} \\ &= 2\mathcal{D}_1\overline{\mathcal{D}}_1\psi_{11} + 2\mathcal{D}_1\overline{\mathcal{D}}_2\psi_{21} - \mathcal{D}^d\overline{\mathcal{D}}^1\psi_{d1} - \mathcal{D}^d\overline{\mathcal{D}}^2\psi_{d2} \\ &= 2\mathcal{D}_1\overline{\mathcal{D}}_1\psi_{11} + 2\mathcal{D}_1\overline{\mathcal{D}}_2\psi_{21} - \mathcal{D}_1\overline{\mathcal{D}}_1\psi_{11} - \mathcal{D}_2\overline{\mathcal{D}}_1\psi_{21} - \mathcal{D}_1\overline{\mathcal{D}}_2\psi_{12} - \mathcal{D}_2\overline{\mathcal{D}}_2\psi_{22} \\ &= \mathcal{D}_1\overline{\mathcal{D}}_1\psi_{11} + \mathcal{D}_1\overline{\mathcal{D}}_2\psi_{21} - \mathcal{D}_2\overline{\mathcal{D}}_1\psi_{21} - \mathcal{D}_2\overline{\mathcal{D}}_2\psi_{22}. \end{aligned}$$

Writing $\psi_{22} = -\psi_{11}$, we have

$$Y_{11} = (\mathcal{D}_1\overline{\mathcal{D}}_1 + \mathcal{D}_2\overline{\mathcal{D}}_2)\psi_{11} + (\mathcal{D}_1\overline{\mathcal{D}}_2 - \mathcal{D}_2\overline{\mathcal{D}}_1)\psi_{12}.$$

We compute

$$\begin{aligned} Y_{11} &= \Big(\big(\nabla_1 + i\,{}^*\nabla_1\big)\big(\nabla_1 - i\,{}^*\nabla_1\big) + \big(\nabla_2 + i\,{}^*\nabla_2\big)\big(\nabla_2 - i\,{}^*\nabla_2\big)\Big)\psi_{11} \\ &\quad + \Big(\big(\nabla_1 + i\,{}^*\nabla_1\big)\big(\nabla_2 - i\,{}^*\nabla_2\big) - \big(\nabla_2 + i\,{}^*\nabla_2\big)\big(\nabla_1 - i\,{}^*\nabla_1\big)\Big)\psi_{12} \\ &= \Big(\big(\nabla_1 + i\nabla_2\big)\big(\nabla_1 - i\nabla_2\big) + \big(\nabla_2 - i\nabla_1\big)\big(\nabla_2 + i\nabla_1\big)\Big)\psi_{11} \\ &\quad + \Big(\big(\nabla_1 + i\nabla_2\big)\big(\nabla_2 + i\nabla_1\big) - \big(\nabla_2 - i\nabla_1\big)\big(\nabla_1 - i\nabla_2\big)\Big)\psi_{12} \\ &= 2\triangle\Psi_{11} - 2i(\nabla_1\nabla_2 - \nabla_2\nabla_1)\Psi_{11} + 2\big(\nabla_1\nabla_2 - \nabla_2\nabla_1\big)\psi_{12} + 2i\triangle\psi_{12}. \end{aligned}$$

Using that $\psi_{12} = -i\psi_{11}$, we obtain

$$\begin{aligned} Y_{11} &= 2\triangle\psi_{11} - 2i(\nabla_1\nabla_2 - \nabla_2\nabla_1)\psi_{11} - 2i\big(\nabla_1\nabla_2 - \nabla_2\nabla_1\big)\psi_{11} + 2\triangle\psi_{11} \\ &= 4\triangle\psi_{11} - 4i(\nabla_1\nabla_2 - \nabla_2\nabla_1)\psi_{11}. \end{aligned}$$

Similarly we compute $Y_{12} = Y_{21}$. We therefore obtain

$$Y_{ab} = 4\triangle\psi_{ab} - 4i(\nabla_1\nabla_2 - \nabla_2\nabla_1)\psi_{ab}.$$

Using the Gauss formulas (2.4.7), we prove the desired formula.

C.9 Proof of Lemma 4.7.8

We have for a s-conformally invariant horizontal tensor ψ,

$$\begin{aligned} {}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot\psi) &= \mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot\psi) + sZ\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot\psi) \\ &= \mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot\psi + s\overline{Z}\cdot\psi) + sZ\widehat{\otimes}(\overline{\mathcal{D}}\cdot\psi + s\overline{Z}\cdot\psi) \\ &= \mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot\psi) + s\mathcal{D}\widehat{\otimes}(\overline{Z}\cdot\psi) + sZ\widehat{\otimes}(\overline{\mathcal{D}}\cdot\psi) + s^2Z\widehat{\otimes}(\overline{Z}\cdot\psi). \end{aligned}$$

Using (2.4.2), (2.4.4), and (2.4.5), we have

$$\begin{aligned} {}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot\psi) &= \mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot\psi)+2s(\mathcal{D}\cdot\overline{Z})\psi+2s(\overline{Z}\cdot\mathcal{D})\psi+2s(Z\cdot\overline{\mathcal{D}})\psi+2s^2(Z\cdot\overline{Z})\psi\\ &= \mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot\psi)+8s\zeta\cdot\nabla\psi+4s(div\,\zeta-icurl\,\zeta)\psi+4s^2|\zeta|^2\psi, \end{aligned}$$

where we wrote $Z\cdot\overline{Z}=2|\zeta|^2$ and $\mathcal{D}\cdot\overline{Z}=2(div\,\zeta-icurl\,\zeta)$. On the other hand,

$$\begin{aligned} {}^{(c)}\triangle_2\psi &= {}^{(c)}\nabla^a\,{}^{(c)}\nabla_a\psi=\nabla^a\,{}^{(c)}\nabla_a\psi+s\zeta^a\,{}^{(c)}\nabla_a\psi=\nabla^a(\nabla_a\psi+s\zeta_a\psi)+s\zeta^a(\nabla_a\psi+s\zeta_a\psi)\\ &= \nabla^a\nabla_a\psi+s\nabla^a\zeta_a\psi+s\zeta_a\nabla^a\psi+s\zeta^a\nabla_a\psi+s^2|\zeta|^2\psi\\ &= \triangle_2\psi+2s\zeta\cdot\nabla\psi+s\nabla^a\zeta_a\psi+s^2|\zeta|^2\psi. \end{aligned}$$

Using (4.7.9), we obtain

$$\begin{aligned} &{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot\psi)\\ =\;& \mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot\psi)+8s\zeta\cdot\nabla\psi+4s(div\,\zeta-icurl\,\zeta)\psi+4s^2|\zeta|^2\psi\\ =\;& 4\triangle_2\psi-2i({}^{(a)}\mathrm{tr}\chi\nabla_3+{}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4)\psi+2\left(tr\,\chi\mathrm{tr}\,\underline{\chi}+{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi}+4\rho\right)\psi\\ &+8s\zeta\cdot\nabla\psi+4s(div\,\zeta-icurl\,\zeta)\psi+4s^2|\zeta|^2\Psi+(\Gamma_g\cdot\Gamma_b)\cdot\psi\\ =\;& 4\,{}^{(c)}\triangle_2\psi-2i({}^{(a)}\mathrm{tr}\chi\nabla_3+{}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4)\psi+2\left(tr\,\chi\mathrm{tr}\,\underline{\chi}+{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi}+4\rho\right)\psi\\ &-4sicurl\,\zeta\psi+(\Gamma_g\cdot\Gamma_b)\cdot\psi. \end{aligned}$$

Writing $\nabla_3={}^{(c)}\nabla_3+2s\underline{\omega}$, $\nabla_4={}^{(c)}\nabla_4-2s\omega$, we have

$$\begin{aligned} &{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot\psi)\\ =\;& 4\,{}^{(c)}\triangle_2\psi-2i\left({}^{(a)}\mathrm{tr}\chi({}^{(c)}\nabla_3\psi+2s\underline{\omega}\psi)+{}^{(a)}\mathrm{tr}\underline{\chi}({}^{(c)}\nabla_4\psi-2s\omega\psi)\right)\\ &+2\left(tr\,\chi\mathrm{tr}\,\underline{\chi}+{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi}+4\rho\right)\psi-4sicurl\,\zeta\psi+(\Gamma_g\cdot\Gamma_b)\cdot\psi\\ =\;& 4\,{}^{(c)}\triangle_2\psi-2i({}^{(a)}\mathrm{tr}\chi\,{}^{(c)}\nabla_3+{}^{(a)}\mathrm{tr}\underline{\chi}\,{}^{(c)}\nabla_4)\psi+2\left(tr\,\chi\mathrm{tr}\,\underline{\chi}+{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi}+4\rho\right)\psi\\ &-4si\big(curl\,\zeta+\underline{\omega}\,{}^{(a)}\mathrm{tr}\chi-\omega\,{}^{(a)}\mathrm{tr}\underline{\chi}\big)\psi+(\Gamma_g\cdot\Gamma_b)\cdot\psi. \end{aligned}$$

Using the null structure equation

$$curl\,\zeta = -\frac{1}{2}\widehat{\chi}\wedge\widehat{\underline{\chi}}+\frac{1}{4}\big(\mathrm{tr}\,\chi\,{}^{(a)}\mathrm{tr}\underline{\chi}-\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\chi\big)+\omega\,{}^{(a)}\mathrm{tr}\underline{\chi}-\underline{\omega}\,{}^{(a)}\mathrm{tr}\chi+{}^*\rho,$$

we finally obtain the stated relation.

C.10 Proof of Lemma 4.7.10

Writing that $\triangle=\gamma^{ab}\nabla_a\nabla_b$, and using Lemma 2.2.8, i.e. for a scalar ψ,

$$[\nabla_3,\nabla_a]\psi = -\frac{1}{2}\left(\mathrm{tr}\,\underline{\chi}\nabla_a\psi+{}^{(a)}\mathrm{tr}\underline{\chi}\,{}^*\nabla_a\psi\right)+(\eta_a-\zeta_a)\nabla_3\psi-\widehat{\underline{\chi}}_{ab}\nabla_b\psi+\underline{\xi}_a\nabla_4\psi,$$

we obtain

$$
\begin{aligned}
[\nabla_3, \triangle]\psi &= \gamma^{ab}[\nabla_3, \nabla_a]\nabla_b\psi + \gamma^{ab}\nabla_a[\nabla_3, \nabla_b]\psi\\
&= -\frac{1}{2}\big(\mathrm{tr}\,\underline{\chi}\gamma^{ab}\nabla_a\nabla_b\psi + {}^{(a)}\mathrm{tr}\underline{\chi}\gamma^{ab}\,{}^*\nabla_a\nabla_b\psi\big) + \gamma^{ab}(\eta_a - \zeta_a)\nabla_3\nabla_b\psi\\
&\quad +\gamma^{ab}\nabla_a\big(-\frac{1}{2}\big(\mathrm{tr}\,\underline{\chi}\nabla_b\psi + {}^{(a)}\mathrm{tr}\underline{\chi}\,{}^*\nabla_b\psi\big) + (\eta_b - \zeta_b)\nabla_3\psi\big)\\
&\quad -\gamma^{ab}\widehat{\underline{\chi}}_{ac}\nabla_c\nabla_b\psi + \gamma^{ab}\underline{\xi}_a\nabla_4\nabla_b\psi + \gamma^{ab}\nabla_a\big(-\widehat{\underline{\chi}}_{bc}\nabla_c\psi + \underline{\xi}_b\nabla_4\psi\big)\\
&= -\frac{1}{2}\big(\mathrm{tr}\,\underline{\chi}\gamma^{ab}\nabla_a\nabla_b\psi + {}^{(a)}\mathrm{tr}\underline{\chi}\gamma^{ab}\,{}^*\nabla_a\nabla_b\psi\big) + \gamma^{ab}(\eta_a - \zeta_a)\nabla_3\nabla_b\psi\\
&\quad -\frac{1}{2}\big(\mathrm{tr}\,\underline{\chi}\gamma^{ab}\nabla_a\nabla_b\psi + {}^{(a)}\mathrm{tr}\underline{\chi}\gamma^{ab}\nabla_a\,{}^*\nabla_b\psi\big) + \gamma^{ab}(\eta_b - \zeta_b)\nabla_a\nabla_3\psi\\
&\quad -\frac{1}{2}\big(\gamma^{ab}\nabla_a\mathrm{tr}\,\underline{\chi}\nabla_b\psi + \gamma^{ab}\nabla_a\,{}^{(a)}\mathrm{tr}\underline{\chi}\,{}^*\nabla_b\psi\big) + \gamma^{ab}\nabla_a(\eta_b - \zeta_b)\nabla_3\psi + \mathrm{Err}_3,
\end{aligned}
$$

where $\mathrm{Err}_3 := -\gamma^{ab}\widehat{\underline{\chi}}_{ac}\nabla_c\nabla_b\psi + \gamma^{ab}\underline{\xi}_a\nabla_4\nabla_b\psi + \gamma^{ab}\nabla_a\big(-\widehat{\underline{\chi}}_{bc}\nabla_c\psi + \underline{\xi}_b\nabla_4\psi\big)$. Writing ${}^*\nabla_a = \in_{ac}\nabla_c$, we obtain

$$
\begin{aligned}
[\nabla_3, \triangle]\psi &= -\frac{1}{2}\big(\mathrm{tr}\,\underline{\chi}\triangle\psi + {}^{(a)}\mathrm{tr}\underline{\chi}\in_{ba}\nabla_a\nabla_b\psi\big) + \gamma^{ab}(\eta_a - \zeta_a)\nabla_3\nabla_b\psi\\
&\quad -\frac{1}{2}\big(\mathrm{tr}\,\underline{\chi}\triangle\psi + {}^{(a)}\mathrm{tr}\underline{\chi}\in_{ab}\nabla_a\nabla_b\psi\big) + \gamma^{ab}(\eta_b - \zeta_b)\nabla_a\nabla_3\psi\\
&\quad -\frac{1}{2}\big(\nabla\mathrm{tr}\,\underline{\chi}\cdot\nabla\psi + \nabla\,{}^{(a)}\mathrm{tr}\underline{\chi}\cdot{}^*\nabla\psi\big) + \mathrm{d}iv\,(\eta - \zeta)\nabla_3\psi + \mathrm{Err}_3\\
&= -\mathrm{tr}\,\underline{\chi}\triangle\psi + (\eta - \zeta)\cdot\nabla_3\nabla\psi + (\eta - \zeta)\cdot\nabla\nabla_3\psi + \mathrm{d}iv\,(\eta - \zeta)\nabla_3\psi\\
&\quad -\frac{1}{2}\big(\nabla\mathrm{tr}\,\underline{\chi}\cdot\nabla\psi + \nabla\,{}^{(a)}\mathrm{tr}\underline{\chi}\cdot{}^*\nabla\psi\big) + \mathrm{Err}_3.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
[\nabla_4, \triangle]\psi &= -\mathrm{t}r\,\chi\triangle\psi + (\underline{\eta} + \zeta)\cdot\nabla_4\nabla\psi + (\underline{\eta} + \zeta)\cdot\nabla\nabla_4\psi + \mathrm{d}iv\,(\underline{\eta} + \zeta)\nabla_4\psi\\
&\quad -\frac{1}{2}\big(\nabla\mathrm{t}r\,\chi\cdot\nabla\psi + \nabla\,{}^{(a)}\mathrm{tr}\chi\cdot{}^*\nabla\psi\big) + \mathrm{Err}_4,
\end{aligned}
$$

where $\mathrm{Err}_4 = -\gamma^{ab}\widehat{\chi}_{ac}\nabla_c\nabla_b\psi + \gamma^{ab}\xi_a\nabla_3\nabla_b\psi + \gamma^{ab}\nabla_a\big(-\widehat{\chi}_{bc}\nabla_c\psi + \xi_b\nabla_3\psi\big)$. In particular, $\mathrm{Err}_3 = r^{-2}\mathfrak{d}\big(\Gamma_b\cdot\mathfrak{d}\psi\big)$ and $\mathrm{Err}_4 = \dot{\mathbf{D}}_3\big(\xi\cdot\dot{\mathbf{D}}_a\psi\big) + r^{-2}\mathfrak{d}\big(\Gamma_g\cdot\mathfrak{d}\psi\big)$, as stated.

Writing $\nabla_3(|q|^2) = \mathrm{tr}\,\underline{\chi}|q|^2 + \big(\widetilde{re_3(r)} + \Gamma_b\big)$, we have

$$
\begin{aligned}
[\nabla_3, |q|^2\triangle]\psi &= |q|^2[\nabla_3, \triangle]\psi + \nabla_3(|q|^2)\triangle\psi\\
&= (\eta - \zeta)\cdot|q|^2\nabla_3\nabla\psi + (\eta - \zeta)\cdot|q|^2\nabla\nabla_3\psi + \mathrm{d}iv\,(\eta - \zeta)|q|^2\nabla_3\psi\\
&\quad -\frac{1}{2}|q|^2\big(\nabla\mathrm{tr}\,\underline{\chi}\cdot\nabla\psi + \nabla\,{}^{(a)}\mathrm{tr}\underline{\chi}\cdot{}^*\nabla\psi\big)\\
&\quad +r^2\mathrm{Err}_3 + \big(\widetilde{re_3(r)} + \Gamma_b\big)\triangle\psi.
\end{aligned}
$$

The error terms are given by

$$r^2\mathrm{Err}_3 + \big(\widetilde{re_3(r)} + \Gamma_b\big)\triangle\psi \;=\; \mathfrak{d}\big(\Gamma_b\cdot\mathfrak{d}\psi\big) + r^2\Gamma_b\cdot r^{-2}\mathfrak{d}^2\psi = \mathfrak{d}\big(\Gamma_b\cdot\mathfrak{d}\psi\big).$$

Similarly we obtain

$$\begin{aligned}[\nabla_4, |q|^2\triangle]\psi &= (\underline{\eta}+\zeta)\cdot|q|^2\nabla_4\nabla\psi + (\underline{\eta}+\zeta)\cdot|q|^2\nabla\nabla_4\psi + \mathrm{d}iv\,(\underline{\eta}+\zeta)|q|^2\nabla_4\psi\\ &\quad -\frac{1}{2}|q|^2\left(\nabla\mathrm{tr}\,\chi\cdot\nabla\psi + \nabla\,{}^{(a)}\mathrm{tr}\chi\cdot\;{}^*\nabla\psi\right)\\ &\quad + r^2\dot{\mathbf{D}}_3\big(\xi\cdot\dot{\mathbf{D}}_a\psi\big) + \mathfrak{d}\big(\Gamma_g\cdot\mathfrak{d}\psi\big),\end{aligned}$$

as stated.

Using the above, we deduce for $\mathcal{O} = |q|^2\triangle\psi - 2a^2\cos\theta\Im(\mathfrak{J})^b\nabla_b\psi$,

$$\begin{aligned}[\nabla_3, \mathcal{O}]\psi &= [\nabla_3, |q|^2\triangle]\psi - [\nabla_3, 2a^2\cos\theta\Im(\mathfrak{J})^b\nabla_b]\psi\\ &= (\eta-\zeta)\cdot|q|^2\nabla_3\nabla\psi + (\eta-\zeta)\cdot|q|^2\nabla\nabla_3\psi + \mathrm{d}iv\,(\eta-\zeta)|q|^2\nabla_3\psi\\ &\quad -\frac{1}{2}|q|^2\left(\nabla\mathrm{tr}\,\underline{\chi}\cdot\nabla\psi + \nabla\,{}^{(a)}\mathrm{tr}\underline{\chi}\cdot\;{}^*\nabla\psi\right)\\ &\quad -\nabla_3\big(2a^2\cos\theta\Im(\mathfrak{J})\big)\cdot\nabla\psi - 2a^2\cos\theta\Im(\mathfrak{J})\cdot[\nabla_3,\nabla]\psi + \mathfrak{d}\big(\Gamma_b\cdot\mathfrak{d}\psi\big)\\ &= (\eta-\zeta)\cdot|q|^2\nabla_3\nabla\psi + (\eta-\zeta)\cdot|q|^2\nabla\nabla_3\psi + \mathrm{d}iv\,(\eta-\zeta)|q|^2\nabla_3\psi\\ &\quad -\frac{1}{2}|q|^2\left(\nabla\mathrm{tr}\,\underline{\chi}\cdot\nabla\psi + \nabla\,{}^{(a)}\mathrm{tr}\underline{\chi}\cdot\;{}^*\nabla\psi\right)\\ &\quad -\nabla_3\big(2a^2\cos\theta\Im(\mathfrak{J})\big)\cdot\nabla\psi\\ &\quad -2a^2\cos\theta\Im(\mathfrak{J})\cdot\left(-\frac{1}{2}\left(\mathrm{tr}\,\underline{\chi}\nabla\psi + {}^{(a)}\mathrm{tr}\underline{\chi}\;{}^*\nabla\psi\right) + (\eta-\zeta)\nabla_3\psi\right) + \mathfrak{d}\big(\Gamma_b\cdot\mathfrak{d}\psi\big),\end{aligned}$$

which gives

$$\begin{aligned}[\nabla_3, \mathcal{O}]\psi &= (\eta-\zeta)\cdot|q|^2\nabla_3\nabla\psi + (\eta-\zeta)\cdot|q|^2\nabla\nabla_3\psi\\ &\quad +\big(\mathrm{d}iv\,(\eta-\zeta) + (\eta+\underline{\eta})\cdot(\eta-\zeta)\big)|q|^2\nabla_3\psi + O(ar^{-2})\mathfrak{d}^{\leq 1}\psi + \mathfrak{d}\big(\Gamma_b\cdot\mathfrak{d}\psi\big).\end{aligned}$$

Similarly,

$$\begin{aligned}[\nabla_4, \mathcal{O}]\psi &= (\underline{\eta}+\zeta)\cdot|q|^2\nabla_4\nabla\psi + (\underline{\eta}+\zeta)\cdot|q|^2\nabla\nabla_4\psi\\ &\quad +\big(\mathrm{d}iv\,(\underline{\eta}+\zeta) + (\eta+\underline{\eta})\cdot(\underline{\eta}+\zeta)\big)|q|^2\nabla_4\psi + O(ar^{-2})\mathfrak{d}^{\leq 1}\psi\\ &\quad + r^2\dot{\mathbf{D}}_3\big(\xi\cdot\dot{\mathbf{D}}_a\psi\big) + \mathfrak{d}\big(\Gamma_g\cdot\mathfrak{d}\psi\big),\end{aligned}$$

as stated.

In the case of a 2-tensor, we have to modify the above using the formulas (2.2.21)

$$\begin{aligned}[\nabla_3, \nabla_a]\psi &= -\frac{1}{2}\left(\mathrm{tr}\,\underline{\chi}\nabla_a\psi + {}^{(a)}\mathrm{tr}\underline{\chi}\;{}^*\nabla_a\psi\right) + (\eta_a-\zeta_a)\nabla_3\psi - \widehat{\underline{\chi}}_{ab}\nabla_b\psi + \underline{\xi}_a\nabla_4\psi\\ &\quad + O(ar^{-3})\psi + r^{-1}\Gamma_b\cdot\psi.\end{aligned}$$

Following the same steps as above, we obtain for $\psi \in \mathfrak{s}_2$

$$
\begin{aligned}
[\nabla_3, \triangle_2]\psi &= -\mathrm{tr}\,\underline{\chi}\triangle\psi + (\eta-\zeta)\cdot\nabla_3\nabla\psi + (\eta-\zeta)\cdot\nabla\nabla_3\psi + \mathrm{d}iv\,(\eta-\zeta)\nabla_3\psi\\
&\quad -\frac{1}{2}\left(\nabla\mathrm{tr}\,\underline{\chi}\cdot\nabla\psi + \nabla\,^{(a)}\mathrm{tr}\underline{\chi}\cdot\ {}^*\nabla\psi\right) + O(ar^{-4})\mathfrak{d}^{\leq 1}\psi + r^{-2}\mathfrak{d}\big(\Gamma_b\cdot\mathfrak{d}\psi\big),
\end{aligned}
$$

and therefore

$$
\begin{aligned}
[\nabla_3, |q|^2\triangle_2]\psi &= (\eta-\zeta)\cdot|q|^2\nabla_3\nabla\psi + (\eta-\zeta)\cdot|q|^2\nabla\nabla_3\psi + \mathrm{d}iv\,(\eta-\zeta)|q|^2\nabla_3\psi\\
&\quad -\frac{1}{2}|q|^2\left(\nabla\mathrm{tr}\,\underline{\chi}\cdot\nabla\psi + \nabla\,^{(a)}\mathrm{tr}\underline{\chi}\cdot\ {}^*\nabla\psi\right) + O(ar^{-4})\mathfrak{d}^{\leq 1}\psi + \mathfrak{d}\big(\Gamma_b\cdot\mathfrak{d}\psi\big),
\end{aligned}
$$

and

$$
\begin{aligned}
[\nabla_3, \mathcal{O}]\psi &= (\eta-\zeta)\cdot|q|^2\nabla_3\nabla\psi + (\eta-\zeta)\cdot|q|^2\nabla\nabla_3\psi\\
&\quad +\big(\mathrm{d}iv\,(\eta-\zeta) + (\eta+\underline{\eta})\cdot(\eta-\zeta)\big)|q|^2\nabla_3\psi + O(ar^{-2})\mathfrak{d}^{\leq 1}\psi + \mathfrak{d}\big(\Gamma_b\cdot\mathfrak{d}\psi\big).
\end{aligned}
$$

Similarly for ∇_4.

C.11 Proof of Lemma 4.7.11

Using Lemma 4.7.4 and the commutator in Lemma 2.2.8, we deduce for a scalar ψ,

$$
\square_{\mathbf{g}}\psi = -\nabla_3\nabla_4\psi - \frac{1}{2}\mathrm{tr}\,\chi\nabla_3\psi + \left(2\underline{\omega} - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\right)\nabla_4\psi + \triangle\psi + 2\eta\cdot\nabla\psi.
$$

We therefore compute

$$
\begin{aligned}
[\nabla_3, \square_{\mathbf{g}}]\psi &= -\nabla_3[\nabla_3,\nabla_4]\psi - \frac{1}{2}\nabla_3 tr\,\chi\nabla_3\psi\\
&\quad +\nabla_3\left(2\underline{\omega} - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\right)\nabla_4\psi + \left(2\underline{\omega} - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\right)[\nabla_3,\nabla_4]\psi\\
&\quad +[\nabla_3,\triangle]\psi + 2\nabla_3\eta\cdot\nabla\psi + 2\eta\cdot[\nabla_3,\nabla]\psi.
\end{aligned}
$$

Using Lemma 2.2.8 and Lemma 4.7.10, we obtain

$$\begin{aligned}
& [\nabla_3, \square_{\mathbf{g}}]\psi \\
= \ & -\nabla_3\big(2(\eta-\underline{\eta})\cdot\nabla\psi - 2\omega\nabla_3\psi + 2\underline{\omega}\nabla_4\psi\big) - \frac{1}{2}\nabla_3 tr\,\chi\nabla_3\psi \\
& + \left(2\nabla_3\underline{\omega} - \frac{1}{2}\nabla_3\mathrm{tr}\,\underline{\chi}\right)\nabla_4\psi + \left(2\underline{\omega} - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\right)\big(2(\eta-\underline{\eta})\cdot\nabla\psi - 2\omega\nabla_3\psi + 2\underline{\omega}\nabla_4\psi\big) \\
& -\mathrm{tr}\,\underline{\chi}\triangle\psi + 2(\eta-\zeta)\cdot\nabla\nabla_3\psi + \big(\mathrm{d}iv\,(\eta-\zeta) + |\eta-\zeta|^2\big)\nabla_3\psi \\
& -\frac{1}{2}\big(\nabla\mathrm{tr}\,\underline{\chi} + \mathrm{tr}\,\underline{\chi}(\eta-\zeta)\big)\cdot\nabla\psi - \frac{1}{2}\big(\nabla\,{}^{(a)}\mathrm{tr}\underline{\chi} + {}^{(a)}\mathrm{tr}\underline{\chi}(\eta-\zeta)\big)\cdot\ {}^*\nabla\psi \\
& +2\nabla_3\eta\cdot\nabla\psi + 2\eta\cdot\left(-\frac{1}{2}\left(\mathrm{tr}\,\underline{\chi}\nabla\psi + {}^{(a)}\mathrm{tr}\underline{\chi}\ {}^*\nabla\psi\right) + (\eta-\zeta)\nabla_3\psi\right) + \mathrm{Err}_3,
\end{aligned}$$

which gives

$$\begin{aligned}
& [\nabla_3, \square_{\mathbf{g}}]\psi \\
= \ & 2\omega\nabla_3\nabla_3\psi - 2\underline{\omega}\nabla_3\nabla_4\psi - \mathrm{tr}\,\underline{\chi}\triangle\psi \\
& + \left(2\nabla_3\omega - \frac{1}{2}\nabla_3 tr\,\chi - 2\omega\left(2\underline{\omega} - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\right) + \mathrm{d}iv\,(\eta-\zeta) + |\eta-\zeta|^2 + 2\eta\cdot(\eta-\zeta)\right)\nabla_3\psi \\
& + \left(-\frac{1}{2}\nabla_3\mathrm{tr}\,\underline{\chi} + 2\underline{\omega}(2\underline{\omega} - \frac{1}{2}\mathrm{tr}\,\underline{\chi})\right)\nabla_4\psi \\
& -2(\eta-\underline{\eta})\cdot\nabla_3\nabla\psi + 2(\eta-\zeta)\cdot\nabla\nabla_3\psi \\
& + \left(2\nabla_3\underline{\eta} + 2\left(2\underline{\omega} - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\right)(\eta-\underline{\eta}) - \frac{1}{2}\big(\nabla\mathrm{tr}\,\underline{\chi} + \mathrm{tr}\,\underline{\chi}(3\eta-\zeta)\big)\right)\cdot\nabla\psi \\
& -\frac{1}{2}\big(\nabla\,{}^{(a)}\mathrm{tr}\underline{\chi} + {}^{(a)}\mathrm{tr}\underline{\chi}(3\eta-\zeta)\big)\cdot\ {}^*\nabla\psi + \mathrm{Err}_3.
\end{aligned}$$

Using again the expression for $\square_{\mathbf{g}}$, writing

$$\triangle\psi \ = \ \square_{\mathbf{g}}\psi + \nabla_3\nabla_4\psi + \frac{1}{2}tr\,\chi\nabla_3\psi - \left(2\underline{\omega} - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\right)\nabla_4\psi - 2\eta\cdot\nabla\psi,$$

we finally obtain

$$
\begin{aligned}
& [\nabla_3, \square_{\mathbf{g}}]\psi \\
= \; & 2\omega\nabla_3\nabla_3\psi - (\mathrm{tr}\,\underline{\chi} + 2\underline{\omega})\nabla_3\nabla_4\psi - \mathrm{tr}\,\underline{\chi}\square_{\mathbf{g}}\psi \\
& + \Big[2\nabla_3\omega - \frac{1}{2}\nabla_3 tr\,\chi - \frac{1}{2}tr\,\chi\mathrm{tr}\,\underline{\chi} - 2\omega\left(2\underline{\omega} - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\right) + \mathrm{d}iv\,(\eta-\zeta) + |\eta-\zeta|^2 \\
& + 2\eta\cdot(\eta-\zeta)\Big]\nabla_3\psi + \left(-\frac{1}{2}\nabla_3\mathrm{tr}\,\underline{\chi} + 2\underline{\omega}\left(2\underline{\omega} - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\right) + \mathrm{tr}\,\underline{\chi}(2\underline{\omega} - \frac{1}{2}\mathrm{tr}\,\underline{\chi})\right)\nabla_4\psi \\
& - 2(\eta - \underline{\eta})\cdot\nabla_3\nabla\psi + 2(\eta-\zeta)\cdot\nabla\nabla_3\psi \\
& + \left(2\nabla_3\underline{\eta} + 2\left(2\underline{\omega} - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\right)(\eta-\underline{\eta}) - \frac{1}{2}\big(\nabla\mathrm{tr}\,\underline{\chi} + \mathrm{tr}\,\underline{\chi}(3\eta-\zeta)\big) + 2\mathrm{tr}\,\underline{\chi}\eta\right)\cdot\nabla\psi \\
& - \frac{1}{2}\big(\nabla\,{}^{(a)}\mathrm{tr}\underline{\chi} + {}^{(a)}\mathrm{tr}\underline{\chi}(3\eta-\zeta)\big)\cdot\,{}^*\nabla\psi + \mathrm{Err}_3,
\end{aligned}
$$

and similarly for ∇_4. Schematically, the commutator can be written as

$$
\begin{aligned}
[\nabla_3, \square_{\mathbf{g}}]\psi &= 2\omega\nabla_3\nabla_3\psi - (\mathrm{tr}\,\underline{\chi} + 2\underline{\omega})\nabla_3\nabla_4\psi - \mathrm{tr}\,\underline{\chi}\square_{\mathbf{g}}\psi \\
& \quad + r^{-2}\mathfrak{d}\psi + O(ar^{-2})\nabla_3\nabla\psi + r^{-2}\mathfrak{d}\big(\Gamma_b\cdot\mathfrak{d}\psi\big),
\end{aligned}
$$

and

$$
\begin{aligned}
[\nabla_4, \square_{\mathbf{g}}]\psi &= 2\underline{\omega}\nabla_4\nabla_4\psi - (\mathrm{tr}\,\chi + 2\omega)\nabla_4\nabla_3\psi - \mathrm{tr}\,\chi\square_{\mathbf{g}}\psi \\
& \quad + r^{-2}\mathfrak{d}\psi + O(ar^{-2})\nabla_4\nabla\psi + r^{-1}\mathfrak{d}\big(\Gamma_g\cdot\mathfrak{d}\psi\big),
\end{aligned}
$$

as stated.

For $\psi \in \mathfrak{s}_2$, using (4.7.5), we compute

$$
\begin{aligned}
[\nabla_4, \dot{\square}_2]\psi &= -\nabla_4[\nabla_4, \nabla_3]\psi - \frac{1}{2}\nabla_4\mathrm{tr}\,\underline{\chi}\nabla_4\psi \\
& \quad + \nabla_4\left(2\omega - \frac{1}{2}tr\,\chi\right)\nabla_3\psi + \left(2\omega - \frac{1}{2}tr\,\chi\right)[\nabla_4, \nabla_3]\psi \\
& \quad + [\nabla_4, \triangle_2]\psi + 2\nabla_4\underline{\eta}\cdot\nabla\psi + 2\underline{\eta}\cdot[\nabla_4, \nabla]\psi + 2i\nabla_4\big({}^*\rho - \eta\wedge\underline{\eta}\big)\psi.
\end{aligned}
$$

Writing that

$$
[\nabla_4, \nabla_3]\psi \; = 2\omega\nabla_3\psi - 2\underline{\omega}\nabla_4\psi + 2(\underline{\eta}_c - \eta_c)\nabla_c\psi + O(ar^{-4})\psi,
$$

and using Lemma 4.7.10, to write

$$
\begin{aligned}
[\nabla_4, \triangle_2]\psi &= -\mathrm{tr}\,\chi\triangle_2\psi + 2(\underline{\eta} + \zeta)\cdot\nabla_4\nabla\psi + O(ar^{-4})\mathfrak{d}^{\leq 1}\psi \\
& \quad + \dot{\mathbf{D}}_3\big(\xi\cdot\dot{\mathbf{D}}_a\psi\big) + r^{-2}\mathfrak{d}\big(\Gamma_g\cdot\mathfrak{d}\psi\big),
\end{aligned}
$$

we obtain

$$\begin{aligned}[\nabla_4, \dot{\square}_2]\psi &= -\nabla_4\big(2\omega\nabla_3\psi - 2\underline{\omega}\nabla_4\psi + 2(\underline{\eta}_c - \eta_c)\nabla_c\psi\big) + O(r^{-3})\mathfrak{d}^{\leq 1}\psi\\ &\quad -\frac{1}{2}\nabla_4 \mathrm{tr}\,\chi\nabla_3\psi - \mathrm{tr}\,\chi\triangle_2\psi + 2(\underline{\eta}+\zeta)\cdot\nabla_4\nabla\psi\\ &\quad +\dot{\mathbf{D}}_3\big(\xi\cdot\dot{\mathbf{D}}_a\psi\big) + r^{-2}\mathfrak{d}\big(\Gamma_g\cdot\mathfrak{d}\psi\big),\end{aligned}$$

which gives

$$\begin{aligned}[\nabla_4, \dot{\square}_2]\psi &= 2\underline{\omega}\nabla_4\nabla_4\psi - 2\omega\nabla_4\nabla_3\psi - \mathrm{tr}\,\chi\triangle_2\psi + 2(\eta+\zeta)\cdot\nabla_4\nabla\psi\\ &\quad -\frac{1}{2}\nabla_4 \mathrm{tr}\,\chi\nabla_3\psi + O(r^{-3})\mathfrak{d}^{\leq 1}\psi + \dot{\mathbf{D}}_3\big(\xi\cdot\dot{\mathbf{D}}_a\psi\big) + r^{-2}\mathfrak{d}\big(\Gamma_g\cdot\mathfrak{d}\psi\big).\end{aligned}$$

Using again the expression for $\dot{\square}_2$, writing

$$\triangle_2\psi = \dot{\square}_2\psi + \nabla_4\nabla_3\psi + \frac{1}{2}\mathrm{tr}\,\chi\nabla_3\psi + O(r^{-2})\mathfrak{d}^{\leq 1}\psi,$$

we finally obtain

$$\begin{aligned}[\nabla_4, \dot{\square}_2]\psi &= 2\underline{\omega}\nabla_4\nabla_4\psi - (\mathrm{tr}\,\chi + 2\omega)\nabla_4\nabla_3\psi + 2(\eta+\zeta)\cdot\nabla_4\nabla\psi - \mathrm{tr}\,\chi\dot{\square}_2\psi\\ &\quad -\frac{1}{2}\big(\nabla_4\mathrm{tr}\,\chi + (\mathrm{tr}\,\chi)^2\big)\nabla_3\psi + O(r^{-3})\mathfrak{d}^{\leq 1}\psi + \dot{\mathbf{D}}_3\big(\xi\cdot\dot{\mathbf{D}}_a\psi\big) + r^{-2}\mathfrak{d}\big(\Gamma_g\cdot\mathfrak{d}\psi\big).\end{aligned}$$

Using the null structure equation for $\nabla_4\mathrm{tr}\,\chi$ we obtain the stated.

Using the above commutator, we compute

$$\begin{aligned}[r\nabla_4, \dot{\square}_2]\psi &= r[\nabla_4, \dot{\square}_2]\psi + (\nabla_3 r)\nabla_4\nabla_4\psi + (\nabla_4 r)\nabla_3\nabla_4\psi - (\square_{\mathbf{g}} r)\nabla_4\psi\\ &= -r(\mathrm{tr}\,\chi + 2\omega)\nabla_4\nabla_3\psi - \nabla_4\nabla_4\psi - r\mathrm{tr}\,\chi\dot{\square}_2\psi - \frac{1}{4}r(\mathrm{tr}\,\chi)^2\nabla_3\psi\\ &\quad +\frac{1}{2}\mathrm{tr}\,\chi r\nabla_3\nabla_4\psi + O(r^{-2})\mathfrak{d}^{\leq 1}\psi + O(r^{-3})\mathfrak{d}^{\leq 2}\psi\\ &\quad +r\dot{\mathbf{D}}_3\big(\xi\cdot\dot{\mathbf{D}}_a\psi\big) + r^{-1}\mathfrak{d}\big(\Gamma_b\cdot\mathfrak{d}\psi\big),\end{aligned}$$

where we used $\nabla_4 r = \frac{1}{2}\mathrm{tr}\,\chi r + O(r^{-2}) + \Gamma_g$, and $e_3(r) = -\frac{\Delta}{|q|^2} + r\Gamma_b$ (and thus $e_3(r) = -1$ at main order). We therefore obtain

$$\begin{aligned}[r\nabla_4, \dot{\square}_2]\psi &= -r\left(\frac{1}{2}\mathrm{tr}\,\chi + 2\omega\right)\nabla_4\nabla_3\psi - \nabla_4\nabla_4\psi - \frac{1}{4}r(\mathrm{tr}\,\chi)^2\nabla_3\psi - r\mathrm{tr}\,\chi\dot{\square}_2\psi\\ &\quad +O(r^{-2})\mathfrak{d}^{\leq 1}\psi + O(r^{-3})\mathfrak{d}^{\leq 2}\psi + r\dot{\mathbf{D}}_3\big(\xi\cdot\dot{\mathbf{D}}_a\psi\big) + r^{-1}\mathfrak{d}\big(\Gamma_g\cdot\mathfrak{d}\psi\big).\end{aligned}$$

Using once again that

$$\nabla_4\nabla_3\psi = -\dot{\square}_2\psi + \triangle_2\psi - \frac{1}{2}\mathrm{tr}\,\chi\nabla_3\psi + O(r^{-2})\mathfrak{d}^{\leq 1}\psi,$$

we have

$$
\begin{aligned}
& [r\nabla_4, \dot{\square}_2]\psi \\
= \; & -r\left(\frac{1}{2}\mathrm{tr}\,\chi + 2\omega\right)\big(-\dot{\square}_2\psi + \triangle_2\psi - \frac{1}{2}\mathrm{tr}\,\chi\nabla_3\psi\big) - \nabla_4\nabla_4\psi - \frac{1}{4}r(\mathrm{tr}\,\chi)^2\nabla_3\psi \\
& -r\mathrm{tr}\,\chi\dot{\square}_2\psi + O(r^{-2})\mathfrak{d}^{\leq 1}\psi + O(r^{-3})\mathfrak{d}^{\leq 2}\psi + r\dot{\mathbf{D}}_3\big(\xi\cdot\dot{\mathbf{D}}_a\psi\big) + r^{-1}\mathfrak{d}\big(\Gamma_g\cdot\mathfrak{d}\psi\big) \\
= \; & r\left(-\frac{1}{2}\mathrm{tr}\,\chi + 2\omega\right)\dot{\square}_2\psi - r\left(\frac{1}{2}\mathrm{tr}\,\chi + 2\omega\right)\triangle_2\psi - \nabla_4\nabla_4\psi \\
& +O(r^{-2})\mathfrak{d}^{\leq 1}\psi + O(r^{-3})\mathfrak{d}^{\leq 2}\psi + r\dot{\mathbf{D}}_3\big(\xi\cdot\dot{\mathbf{D}}_a\psi\big) + r^{-1}\mathfrak{d}\big(\Gamma_g\cdot\mathfrak{d}\psi\big),
\end{aligned}
$$

as stated.

C.12 Proof of Lemma 4.7.12

As a consequence of Lemma 4.7.4, for scalar functions f and ψ, we have

$$
\square_{\mathbf{g}}(f\psi) \;=\; \square_{\mathbf{g}}(f)\psi + f\square_{\mathbf{g}}\psi - \nabla_3 f\nabla_4\psi - \nabla_4 f\nabla_3\psi + 2\nabla f\cdot\nabla\psi,
$$

which implies

$$
[fX, \square_{\mathbf{g}}]\psi \;=\; f[X, \square_{\mathbf{g}}]\psi + \nabla_3 f\nabla_4 X\psi + \nabla_4 f\nabla_3 X\psi - 2\nabla f\cdot\nabla X\psi - \square_{\mathbf{g}}(f)X\psi.
$$

Using the definition in the outgoing frame $2\widehat{R} = \frac{\Delta}{r^2+a^2}e_4 - \frac{|q|^2}{r^2+a^2}e_3$, we compute

$$
\begin{aligned}
2[\nabla_{\widehat{R}}, \square_{\mathbf{g}}]\psi \;=\; & \frac{\Delta}{r^2+a^2}[\nabla_4, \square_{\mathbf{g}}]\psi - \frac{|q|^2}{r^2+a^2}[\nabla_3, \square_{\mathbf{g}}]\psi \\
& +e_4\left(\frac{\Delta}{r^2+a^2}\right)\nabla_3\nabla_4\psi - e_3\left(\frac{|q|^2}{r^2+a^2}\right)\nabla_4\nabla_3\psi \\
& +e_3\left(\frac{\Delta}{r^2+a^2}\right)\nabla_4\nabla_4\psi - e_4\left(\frac{|q|^2}{r^2+a^2}\right)\nabla_3\nabla_3\psi - 2\nabla\left(\frac{\Delta}{r^2+a^2}\right)\cdot\nabla\nabla_4\psi \\
& +2\nabla\left(\frac{|q|^2}{r^2+a^2}\right)\cdot\nabla\nabla_3\psi - \square_{\mathbf{g}}\left(\frac{\Delta}{r^2+a^2}\right)\nabla_4\psi + \square_{\mathbf{g}}\left(\frac{|q|^2}{r^2+a^2}\right)\nabla_3\psi.
\end{aligned}
$$

Using Lemma 4.7.11 we deduce

$$\begin{aligned}
& 2[\nabla_{\widehat{R}}, \square_{\mathbf{g}}]\psi \\
= \; & \Big(-\mathrm{tr}\,\chi\frac{\Delta}{r^2+a^2} + \mathrm{tr}\,\underline{\chi}\frac{|q|^2}{r^2+a^2}\Big)\square_{\mathbf{g}}\psi \\
& +\Big(e_4\left(\frac{\Delta}{r^2+a^2}\right) + (\mathrm{tr}\,\underline{\chi}+2\underline{\omega})\frac{|q|^2}{r^2+a^2}\Big)\nabla_3\nabla_4\psi \\
& -\Big(e_3\left(\frac{|q|^2}{r^2+a^2}\right) + (\mathrm{tr}\,\chi+2\omega)\frac{\Delta}{r^2+a^2}\Big)\nabla_4\nabla_3\psi \\
& +\Big(e_3\left(\frac{\Delta}{r^2+a^2}\right) + 2\underline{\omega}\frac{\Delta}{r^2+a^2}\Big)\nabla_4\nabla_4\psi - \Big(e_4\left(\frac{|q|^2}{r^2+a^2}\right) + 2\omega\frac{|q|^2}{r^2+a^2}\Big)\nabla_3\nabla_3\psi \\
& +r^{-2}\mathfrak{d}^{\leq 1}\psi + O(ar^{-2})\big(\nabla_3\nabla\psi + \nabla_4\nabla\psi\big) + r^{-1}\mathfrak{d}\big(\Gamma_g\cdot\mathfrak{d}\psi\big).
\end{aligned}$$

Using (C.7.1) we write

$$\begin{aligned}
& \Big(e_3\left(\frac{\Delta}{r^2+a^2}\right) + 2\underline{\omega}\frac{\Delta}{r^2+a^2}\Big)\nabla_4\nabla_4\psi - \Big(e_4\left(\frac{|q|^2}{r^2+a^2}\right) + 2\omega\frac{|q|^2}{r^2+a^2}\Big)\nabla_3\nabla_3\psi \\
= \; & \Big(-\frac{2ra^2\sin^2\theta\Delta^2}{|q|^4(r^2+a^2)^2} + \Gamma_b\Big)\nabla_4\nabla_4\psi - \Big(\frac{2ra^2\sin^2\theta}{(r^2+a^2)^2} + \Gamma_g\Big)\nabla_3\nabla_3\psi \\
= \; & -\frac{2ra^2\sin^2\theta}{(r^2+a^2)^2}\Big(\frac{\Delta^2}{|q|^4}\nabla_4\nabla_4\psi + \nabla_3\nabla_3\psi\Big) + \Gamma_g\cdot\mathfrak{d}^2\psi.
\end{aligned}$$

Also, writing $\nabla_3\nabla_4\psi = -\square_{\mathbf{g}}\psi - \triangle\psi + r^{-1}\mathfrak{d}\psi$ and $\nabla_4\nabla_3\psi = -\square_{\mathbf{g}}\psi - \triangle\psi + r^{-1}\mathfrak{d}\psi$, we deduce

$$\begin{aligned}
[\nabla_{\widehat{R}}, \square_{\mathbf{g}}]\psi \;=\; & O(r^{-1})\square_{\mathbf{g}}\psi + O(r^{-1})\triangle\psi + r^{-2}\mathfrak{d}^{\leq 1}\psi \\
& +O(a^2r^{-3})\big(\nabla_4\nabla_4\psi + \nabla_3\nabla_3\psi\big) + O(ar^{-2})\big(\nabla_3\nabla\psi + \nabla_4\nabla\psi\big) \\
& +\mathfrak{d}\big(\Gamma_g\cdot\mathfrak{d}\psi\big),
\end{aligned}$$

as stated. The last commutations with $\triangle$ are obtained in a similar way as a consequence of Lemma 4.7.10.

C.13 Proof of Lemma 4.7.13

Combining Lemma 2.1.36 and Proposition 2.1.43, we have for $\xi \in \mathfrak{s}_1$, $u \in \mathfrak{s}_2$

$$\begin{aligned}
\not{\mathcal{D}}_2\not{\mathcal{D}}_2^\star\xi &= -\frac{1}{2}\triangle_1\xi - \frac{1}{2}\,{}^{(h)}K\xi + \frac{1}{4}\big({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4\big)\,{}^*\xi, \\
\not{\mathcal{D}}_2^\star\not{\mathcal{D}}_2 u &= -\frac{1}{2}\triangle_2 u + {}^{(h)}K u - \frac{1}{4}\big({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4\big)\,{}^*u.
\end{aligned}$$

We then have for $\psi \in \mathfrak{s}_2$:

$$
\begin{aligned}
{\not{\mathcal{D}}}_2\triangle_2\psi - \triangle_1 {\not{\mathcal{D}}}_2\psi &= {\not{\mathcal{D}}}_2\big(- 2\,{\not{\mathcal{D}}}_2^*{\not{\mathcal{D}}}_2\psi + 2\,{}^{(h)}K\psi - \frac{1}{2}({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4)\ {}^*\psi\big)\\
&\quad -\big(- 2\,{\not{\mathcal{D}}}_2\,{\not{\mathcal{D}}}_2^* - {}^{(h)}K + \frac{1}{2}({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4)\ {}^*\big)\,{\not{\mathcal{D}}}_2\psi\\
&= 2\,{}^{(h)}K\,{\not{\mathcal{D}}}_2\psi + 2\nabla\,{}^{(h)}K\cdot\psi - \frac{1}{2}\,{\not{\mathcal{D}}}_2(({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4)\ {}^*\psi)\\
&\quad + {}^{(h)}K\,{\not{\mathcal{D}}}_2\psi - \frac{1}{2}({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4)\ {}^*\,{\not{\mathcal{D}}}_2\psi,
\end{aligned}
$$

which gives

$$
\begin{aligned}
{\not{\mathcal{D}}}_2\triangle_2\psi - \triangle_1 {\not{\mathcal{D}}}_2\psi &= 3\,{}^{(h)}K\,{\not{\mathcal{D}}}_2\psi - ({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4)\ {}^*\,{\not{\mathcal{D}}}_2\psi\\
&\quad +O(ar^{-3})\mathfrak{d}^{\leq 1}\psi + r^{-1}\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\psi).
\end{aligned}
$$

From Corollary 2.2.9, i.e.

$$
\begin{aligned}
[\nabla_3, {\not{\mathcal{D}}}_2]u &= -\frac{1}{2}\mathrm{tr}\,\underline{\chi}\,\big({\not{\mathcal{D}}}_2 u - \eta\cdot u\big) + \frac{1}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi}\,\big({\not{\mathcal{D}}}_2\ {}^*u - \eta\cdot\ {}^*u\big) + (\eta-\zeta)\cdot\nabla_3 u\\
&\quad +r^{-1}\Gamma_b\cdot\mathfrak{d}^{\leq 1}u,\\
[\nabla_4, {\not{\mathcal{D}}}_2]u &= -\frac{1}{2}\mathrm{tr}\,\chi\,\big({\not{\mathcal{D}}}_2 u - \underline{\eta}\cdot u\big) + \frac{1}{2}\,{}^{(a)}\mathrm{tr}\chi\,\big({\not{\mathcal{D}}}_2\ {}^*u - \underline{\eta}\cdot\ {}^*u\big) + (\underline{\eta}+\zeta)\cdot\nabla_4 u\\
&\quad +\xi\cdot\ {}^{(c)}\nabla_3 f + r^{-1}\Gamma_g\cdot\mathfrak{d}^{\leq 1}u,
\end{aligned}
$$

we deduce, see also Corollary 13.3.2,

$$
\begin{aligned}
[\nabla_3, |q|\,{\not{\mathcal{D}}}_2]u &= \frac{1}{2}|q|\,{}^{(a)}\mathrm{tr}\underline{\chi}\,{\not{\mathcal{D}}}_2\ {}^*u + |q|(\eta-\zeta)\cdot\nabla_3 u + O(ar^{-3})u + r\check{H}\nabla_3 u + \Gamma_b\cdot\mathfrak{d}^{\leq 1}u,\\
[\nabla_4, |q|\,{\not{\mathcal{D}}}_2]u &= \frac{1}{2}|q|\,{}^{(a)}\mathrm{tr}\chi\,{\not{\mathcal{D}}}_2\ {}^*u + |q|(\underline{\eta}+\zeta)\cdot\nabla_4 u + O(ar^{-3})u + r\xi\cdot\nabla_3 u + \Gamma_g\cdot\mathfrak{d}^{\leq 1}u.
\end{aligned}
$$

We can therefore compute, using the null decomposition in Lemma 4.7.4,

$$
\begin{aligned}
& |q|\,{\not{\mathcal{D}}}_2\dot{\square}_2 - \dot{\square}_1|q|\,{\not{\mathcal{D}}}_2\\
={}& -\frac{1}{2}\big([|q|\,{\not{\mathcal{D}}}_2, \nabla_3]\nabla_4\psi + \nabla_3[|q|\,{\not{\mathcal{D}}}_2, \nabla_4]\psi + [|q|\,{\not{\mathcal{D}}}_2, \nabla_4]\nabla_3\psi + \nabla_4[|q|\,{\not{\mathcal{D}}}_2, \nabla_3]\psi\big)\\
& + \left(\underline{\omega} - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\right)[|q|\,{\not{\mathcal{D}}}_2, \nabla_4]\psi + \left(\omega - \frac{1}{2}\mathrm{tr}\,\chi\right)[|q|\,{\not{\mathcal{D}}}_2, \nabla_3]\psi\\
& +|q|\,{\not{\mathcal{D}}}_2\triangle_2\psi - \triangle_1|q|\,{\not{\mathcal{D}}}_2\psi + O(ar^{-2})\mathfrak{d}^{\leq 1}\psi,
\end{aligned}
$$

which gives

$$\begin{aligned}
&|q|\,\not{\mathcal{D}}_2\dot{\square}_2-\dot{\square}_1|q|\,\not{\mathcal{D}}_2\\
=\;&\frac{1}{2}\Big[\left(\frac{1}{2}|q|\,{}^{(a)}\mathrm{tr}\underline{\chi}\,\not{\mathcal{D}}_2{}^*+|q|(\eta-\zeta)\cdot\nabla_3\right)\nabla_4\psi+\nabla_3\left(\frac{1}{2}|q|\,{}^{(a)}\mathrm{tr}\chi\,\not{\mathcal{D}}_2{}^*\psi+|q|(\underline{\eta}+\zeta)\cdot\nabla_4\psi\right)\\
&+\left(\frac{1}{2}|q|\,{}^{(a)}\mathrm{tr}\chi\,\not{\mathcal{D}}_2{}^*+|q|(\underline{\eta}+\zeta)\cdot\nabla_4\right)\nabla_3\psi+\nabla_4\left(\frac{1}{2}|q|\,{}^{(a)}\mathrm{tr}\underline{\chi}\,\not{\mathcal{D}}_2{}^*\psi+|q|(\eta-\zeta)\cdot\nabla_3\psi\right)\Big]\\
&+3\,{}^{(h)}K|q|\,\not{\mathcal{D}}_2\psi-|q|({}^{(a)}\mathrm{tr}\chi\nabla_3+{}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4)\,{}^*\not{\mathcal{D}}_2\psi\\
&+O(ar^{-2})\mathfrak{d}^{\leq 1}\psi+\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\psi)+O(ar^{-2})\mathfrak{d}^{\leq 1}\psi\\
=\;&3\,{}^{(h)}K|q|\,\not{\mathcal{D}}_2\psi-\frac{1}{2}|q|\,{}^{(a)}\mathrm{tr}\underline{\chi}\,{}^*\nabla_4\,\not{\mathcal{D}}_2\psi-\frac{1}{2}|q|\,{}^{(a)}\mathrm{tr}\chi\,{}^*\nabla_3\,\not{\mathcal{D}}_2\psi+|q|(\eta+\underline{\eta})\cdot\nabla_3\nabla_4\psi\\
&+O(ar^{-2})\mathfrak{d}^{\leq 1}\psi+\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\psi)+O(ar^{-2})\mathfrak{d}^{\leq 1}\psi\\
&+r\check{H}\left(-\nabla_3\nabla_4\psi-\frac{1}{2}\mathrm{tr}\,\chi\nabla_3\psi\right)-\frac{1}{2}\nabla_4(r\check{H})\nabla_3\psi+\dot{\mathbf{D}}_3(r\xi\cdot\nabla_3 u).
\end{aligned}$$

Writing $\nabla_3\nabla_4=-\dot{\square}_2+\triangle_2+O(r^{-1})\mathfrak{d}^{\leq 1}=-\dot{\square}_2-2\,\not{\mathcal{D}}_2^*\not{\mathcal{D}}_2+O(r^{-1})\mathfrak{d}^{\leq 1}$, and recalling that, see (3.4.3),

$$ {}^{(a)}\mathrm{tr}\chi e_3+{}^{(a)}\mathrm{tr}\underline{\chi}e_4+2(\eta+\underline{\eta})\cdot\,{}^*\nabla \;=\; \frac{4a\cos\theta}{|q|^2}\mathbf{T}, $$

we have

$$\begin{aligned}
|q|\,\not{\mathcal{D}}_2\dot{\square}_2-\dot{\square}_1|q|\,\not{\mathcal{D}}_2 \;=\;& 3\,{}^{(h)}K|q|\,\not{\mathcal{D}}_2\psi-\frac{2a\cos\theta}{|q|}\,{}^*\nabla_{\mathbf{T}}\,\not{\mathcal{D}}_2\psi-|q|(\eta+\underline{\eta})\cdot\dot{\square}_2\psi\\
&+O(ar^{-2})\mathfrak{d}^{\leq 1}\psi+\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\psi)+O(ar^{-2})\mathfrak{d}^{\leq 1}\psi+O(ar^{-3})\mathfrak{d}^{\leq 2}\psi\\
&+r\check{H}\left(-\nabla_3\nabla_4\psi-\frac{1}{2}\mathrm{tr}\,\chi\nabla_3\psi\right)-\frac{1}{2}\nabla_4(r\check{H})\nabla_3\psi+\dot{\mathbf{D}}_3(r\xi\cdot\nabla_3 u).
\end{aligned}$$

Using that, see Lemma 9.2.1,

$$ \nabla_{\mathbf{T}}\psi \;=\; \not{\mathcal{L}}_{\mathbf{T}}\psi+\frac{4amr\cos\theta}{|q|^4}\,{}^*\psi, $$

we obtain

$$\begin{aligned}
|q|\,\not{\mathcal{D}}_2\dot{\square}_2-\dot{\square}_1|q|\,\not{\mathcal{D}}_2 \;=\;& 3\,{}^{(h)}K|q|\,\not{\mathcal{D}}_2\psi-\frac{2a\cos\theta}{|q|}\,{}^*\not{\mathcal{D}}_2\not{\mathcal{L}}_{\mathbf{T}}\psi-|q|(\eta+\underline{\eta})\cdot\dot{\square}_2\psi\\
&+O(ar^{-2})\mathfrak{d}^{\leq 1}\psi+\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\psi)+O(ar^{-2})\mathfrak{d}^{\leq 1}\psi+O(ar^{-3})\mathfrak{d}^{\leq 2}\psi\\
&+r\check{H}\left(-\nabla_3\nabla_4\psi-\frac{1}{2}\mathrm{tr}\,\chi\nabla_3\psi\right)-\frac{1}{2}\nabla_4(r\check{H})\nabla_3\psi+\dot{\mathbf{D}}_3(r\xi\cdot\nabla_3 u).
\end{aligned}$$

Finally, using the linearized null structure equation (D.5.3), we deduce that $\nabla_4(r\check{H}) = r^{-1}\Gamma_g$ and by writing $-\nabla_3\nabla_4\psi - \frac{1}{2}tr\,\chi\nabla_3\psi = \dot{\square}_2\psi + r^{-2}\mathfrak{d}^{\leq 2}\psi$, we obtain the stated identity.

Similarly, combining Lemma 2.1.36 and Proposition 2.1.43, we have for $\xi \in \mathfrak{s}_1$, $f \in \mathfrak{s}_0$

$$\begin{aligned}
{\mathcal{D}\mkern-11mu /\,}_1^*{\mathcal{D}\mkern-11mu /\,}_1\xi &= -\triangle_1\xi + {}^{(h)}K\xi - \frac{1}{2}\big({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4\big)\,{}^*\xi,\\
{\mathcal{D}\mkern-11mu /\,}_1\,{\mathcal{D}\mkern-11mu /\,}_1^* f &= -\triangle_0 f.
\end{aligned}$$

We then have for $\psi \in \mathfrak{s}_1$:

$$\begin{aligned}
& {\mathcal{D}\mkern-11mu /\,}_1\triangle_1\psi - \triangle_0\,{\mathcal{D}\mkern-11mu /\,}_1\psi\\
= {} & {\mathcal{D}\mkern-11mu /\,}_1\Big(-\,{\mathcal{D}\mkern-11mu /\,}_1^*{\mathcal{D}\mkern-11mu /\,}_1\psi + {}^{(h)}K\psi - \frac{1}{2}\big({}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4\big)\,{}^*\psi\Big) + {\mathcal{D}\mkern-11mu /\,}_1\,{\mathcal{D}\mkern-11mu /\,}_1^*{\mathcal{D}\mkern-11mu /\,}_1\psi\\
= {} & {}^{(h)}K\,{\mathcal{D}\mkern-11mu /\,}_1\psi + O(ar^{-3})\mathfrak{d}^{\leq 2}\psi + r^{-1}\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\psi),
\end{aligned}$$

from which we deduce the stated formula.

Appendix D

Complement for Chapter 5

D.1 Proof of Proposition 5.1.1

According to Proposition 2.4.11, we have the following Bianchi identity for A:

$$ {}^{(c)}\nabla_3 A - \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes} B = -\frac{1}{2}\mathrm{tr}\underline{X} A + 2H\widehat{\otimes} B - 3\overline{P}\widehat{X}. \tag{D.1.1} $$

We apply ${}^{(c)}\nabla_4$ to (D.1.1):

$$ \begin{aligned} {}^{(c)}\nabla_4\,{}^{(c)}\nabla_3 A &= \frac{1}{2}\,{}^{(c)}\nabla_4({}^{(c)}\mathcal{D}\widehat{\otimes} B) - \frac{1}{2}\,{}^{(c)}\nabla_4\mathrm{tr}\underline{X} A - \frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(c)}\nabla_4 A \\ &\quad +2\,{}^{(c)}\nabla_4(H)\widehat{\otimes} B + 2H\widehat{\otimes}\,{}^{(c)}\nabla_4(B) - 3\,{}^{(c)}\nabla_4\overline{P}\widehat{X} - 3\overline{P}\,{}^{(c)}\nabla_4\widehat{X}. \end{aligned} $$

According to Lemma 4.2.2 applied to B, which is 1-conformally invariant, we have

$$ [{}^{(c)}\nabla_4, {}^{(c)}\mathcal{D}\widehat{\otimes}]B = -\frac{1}{2}\mathrm{tr}X({}^{(c)}\mathcal{D}\widehat{\otimes} B) + \underline{H}\widehat{\otimes}\,{}^{(c)}\nabla_4 B + r^{-1}\Gamma_g\cdot\mathfrak{d}^{\leq 1}B. $$

We therefore obtain

$$ \begin{aligned} & {}^{(c)}\nabla_4({}^{(c)}\mathcal{D}\widehat{\otimes} B) + 4H\widehat{\otimes}\,{}^{(c)}\nabla_4(B) \\ = \;& {}^{(c)}\mathcal{D}\widehat{\otimes}({}^{(c)}\nabla_4 B) + 4H\widehat{\otimes}\,{}^{(c)}\nabla_4(B) - \frac{1}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes} B + \underline{H}\widehat{\otimes}\,{}^{(c)}\nabla_4 B + r^{-1}\Gamma_g\cdot\mathfrak{d}^{\leq 1}B \\ = \;& {}^{(c)}\mathcal{D}\widehat{\otimes}({}^{(c)}\nabla_4 B) + (4H+\underline{H})\,\widehat{\otimes}\,{}^{(c)}\nabla_4 B - \frac{1}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes} B + r^{-1}\Gamma_g\cdot\mathfrak{d}^{\leq 1}B. \end{aligned} $$

According to Proposition 2.4.11, we have the following Bianchi identity for B:

$$ {}^{(c)}\nabla_4 B - \frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot A = -2\overline{\mathrm{tr}X}B + \frac{1}{2}A\cdot\overline{\underline{H}} + 3\overline{P}\,\Xi. $$

This gives

$$
\begin{aligned}
&{}^{(c)}\nabla_4({}^{(c)}\mathcal{D}\widehat{\otimes}B)+4H\widehat{\otimes}\,{}^{(c)}\nabla_4(B)\\
&= {}^{(c)}\mathcal{D}\widehat{\otimes}\left(\frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot A-2\overline{\mathrm{tr}X}B+\frac{1}{2}A\cdot\overline{\underline{H}}+3\overline{P}\,\Xi\right)\\
&+(4H+\underline{H})\widehat{\otimes}\left(\frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot A-2\overline{\mathrm{tr}X}B+\frac{1}{2}A\cdot\overline{\underline{H}}+3\overline{P}\,\Xi\right)-\frac{1}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}B+r^{-1}\Gamma_g\cdot\mathfrak{d}^{\leq 1}B\\
&=\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A+\overline{\underline{H}}\cdot A)+\left(-\frac{1}{2}\mathrm{tr}X-2\overline{\mathrm{tr}X}\right){}^{(c)}\mathcal{D}\widehat{\otimes}B\\
&+\left(2H+\frac{1}{2}\underline{H}\right)\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A+\overline{\underline{H}}\cdot A)+2\left(-{}^{(c)}\mathcal{D}\overline{\mathrm{tr}X}-\overline{\mathrm{tr}X}\,(4H+\underline{H})\right)\widehat{\otimes}B\\
&+3\overline{P}\left({}^{(c)}\mathcal{D}\widehat{\otimes}\Xi+(4H+\underline{H})\widehat{\otimes}\Xi\right)+3\,{}^{(c)}\mathcal{D}\overline{P}\widehat{\otimes}\Xi+r^{-1}\Gamma_g\cdot\mathfrak{d}^{\leq 1}B.
\end{aligned}
\tag{D.1.2}
$$

According to Proposition 2.4.13, we have the following null structure equation:

$$
{}^{(c)}\nabla_4\mathrm{tr}\underline{X}+\frac{1}{2}\mathrm{tr}X\mathrm{tr}\underline{X} \;=\; {}^{(c)}\mathcal{D}\cdot\overline{\underline{H}}+\underline{H}\cdot\overline{\underline{H}}+2\overline{P}+\Gamma_b\cdot\Gamma_g,
$$

which gives

$$
\begin{aligned}
&-\frac{1}{2}\,{}^{(c)}\nabla_4\mathrm{tr}\underline{X}A-\frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(c)}\nabla_4A\\
&=\left(\frac{1}{4}\mathrm{tr}X\mathrm{tr}\underline{X}-\frac{1}{2}({}^{(c)}\mathcal{D}\cdot\overline{\underline{H}}+\underline{H}\cdot\overline{\underline{H}})-\overline{P}\right)A-\frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(c)}\nabla_4A+\Gamma_b\cdot\Gamma_g\cdot A.
\end{aligned}
\tag{D.1.3}
$$

According to Proposition 2.4.13, we have the following null structure equation:

$$
{}^{(c)}\nabla_4H-{}^{(c)}\nabla_3\Xi \;=\; -\frac{1}{2}\overline{\mathrm{tr}X}(H-\underline{H})-\frac{1}{2}\widehat{X}\cdot(\overline{H}-\overline{\underline{H}})-B,
$$

which gives

$$
4\,{}^{(c)}\nabla_4(H)\widehat{\otimes}B \;=\; -2\overline{\mathrm{tr}X}(H-\underline{H})\widehat{\otimes}B+\left({}^{(c)}\nabla_3\Xi+r^{-1}\Gamma_g\right)\cdot B. \tag{D.1.4}
$$

Finally using again Proposition 2.4.13 and Proposition 2.4.11,

$$
\begin{aligned}
{}^{(c)}\nabla_4\widehat{X}+\Re(\mathrm{tr}X)\widehat{X} &= \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\Xi+\frac{1}{2}\Xi\widehat{\otimes}(\underline{H}+H)-A,\\
{}^{(c)}\nabla_4P-\frac{1}{2}\,{}^{(c)}\mathcal{D}\cdot\overline{B} &= -\frac{3}{2}\mathrm{tr}XP+\underline{H}\cdot\overline{B}-\overline{\Xi}\cdot\underline{B}+\Gamma_b\cdot A,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&-3\,{}^{(c)}\nabla_4\overline{P}\widehat{X}-3\overline{P}\,{}^{(c)}\nabla_4\widehat{X}\\
&=-3\left(-\frac{3}{2}\overline{\mathrm{tr}X}\;\overline{P}\right)\widehat{X}-3\overline{P}\left(-\frac{1}{2}(\mathrm{tr}X+\overline{\mathrm{tr}X})\widehat{X}-A+\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\Xi+\frac{1}{2}\Xi\widehat{\otimes}(\underline{H}+H)\right)\\
&+r^{-1}\Gamma_g\cdot\mathfrak{d}^{\leq 1}B+\Gamma_b\cdot\Gamma_g\cdot A\\
&=\left(\frac{3}{2}\mathrm{tr}X+6\overline{\mathrm{tr}X}\right)\overline{P}\widehat{X}+3\overline{P}A-\frac{3}{2}\overline{P}\left({}^{(c)}\mathcal{D}\widehat{\otimes}\Xi+\Xi\widehat{\otimes}(\underline{H}+H)\right)+r^{-1}\Gamma_g\cdot\mathfrak{d}^{\leq 1}B.
\end{aligned}
\tag{D.1.5}
$$

Summing $\frac{1}{2}$ (D.1.2), (D.1.3), (D.1.4) and (D.1.5), we obtain

$$\begin{aligned}
{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3 A &= \left(-\frac{1}{2}\mathrm{tr}X - 2\overline{\mathrm{tr}X}\right)\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}B + \left(\frac{3}{2}\mathrm{tr}X + 6\overline{\mathrm{tr}X}\right)\overline{P}\widehat{X}\\
&+\left(-\,{}^{(c)}\mathcal{D}\overline{\mathrm{tr}X} - 5\overline{\mathrm{tr}X}H\right)\widehat{\otimes}B\\
&+\left(\frac{1}{4}\mathrm{tr}X\mathrm{tr}\underline{X} - \frac{1}{2}({}^{(c)}\mathcal{D}\cdot\overline{\underline{H}} + \underline{H}\cdot\overline{\underline{H}}) + 2\overline{P}\right)A - \frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(c)}\nabla_4 A\\
&+\frac{1}{4}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A + \overline{\underline{H}}\cdot A) + \left(H + \frac{1}{4}\underline{H}\right)\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A + \overline{\underline{H}}\cdot A)\\
&+r^{-1}\mathfrak{d}^{\leq 1}\big(\Gamma_g\cdot B\big) + {}^{(c)}\nabla_3\Xi\cdot B + \Gamma_b\cdot\Gamma_g\cdot A.
\end{aligned}$$

Using again (D.1.1) we have

$$\begin{aligned}
&\left(-\frac{1}{2}\mathrm{tr}X - 2\overline{\mathrm{tr}X}\right)\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}B + \left(\frac{3}{2}\mathrm{tr}X + 6\overline{\mathrm{tr}X}\right)\overline{P}\widehat{X}\\
=\;&\left(-\frac{1}{2}\mathrm{tr}X - 2\overline{\mathrm{tr}X}\right)\left(\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}B - 3\overline{P}\widehat{X}\right)\\
=\;&\left(-\frac{1}{2}\mathrm{tr}X - 2\overline{\mathrm{tr}X}\right)\left({}^{(c)}\nabla_3 A + \frac{1}{2}\mathrm{tr}\underline{X}A - 2H\widehat{\otimes}B\right).
\end{aligned}$$

This gives

$$\begin{aligned}
{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3 A &= \left(-\frac{1}{2}\mathrm{tr}X - 2\overline{\mathrm{tr}X}\right){}^{(c)}\nabla_3 A - \frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(c)}\nabla_4 A\\
&+\left(-\,{}^{(c)}\mathcal{D}\overline{\mathrm{tr}X} + (\mathrm{tr}X - \overline{\mathrm{tr}X})H\right)\widehat{\otimes}B\\
&+\left(-\overline{\mathrm{tr}X}\mathrm{tr}\underline{X} - \frac{1}{2}({}^{(c)}\mathcal{D}\cdot\overline{\underline{H}} + \underline{H}\cdot\overline{\underline{H}}) + 2\overline{P}\right)A\\
&+\frac{1}{4}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A + \overline{\underline{H}}\cdot A) + \left(H + \frac{1}{4}\underline{H}\right)\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A + \overline{\underline{H}}\cdot A)\\
&+r^{-1}\mathfrak{d}^{\leq 1}\big(\Gamma_g\cdot B\big) + {}^{(c)}\nabla_3\Xi\cdot B + \Gamma_b\cdot\Gamma_g\cdot A.
\end{aligned}$$

By the Codazzi equation

$$\frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot\widehat{X} = \frac{1}{2}\,{}^{(c)}\mathcal{D}\overline{\mathrm{tr}X} - i\Im(\mathrm{tr}X)(H+\Xi) - B,$$

the second line is absorbed by the quadratic terms in the last line. We also simplify

$$\begin{aligned}
&\frac{1}{4}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A + \overline{\underline{H}}\cdot A) + \left(H + \frac{1}{4}\underline{H}\right)\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A + \overline{\underline{H}}\cdot A)\\
=\;&\frac{1}{4}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A) + \frac{1}{4}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{\underline{H}}\cdot A) + \left(H + \frac{1}{4}\underline{H}\right)\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A)\\
&+\left(H + \frac{1}{4}\underline{H}\right)\widehat{\otimes}(\overline{\underline{H}}\cdot A).
\end{aligned}$$

Applying (2.4.2) and (2.4.4), we write

$$\begin{aligned} {}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{\underline{H}}\cdot A) &= 2({}^{(c)}\mathcal{D}\cdot\overline{\underline{H}})A + 2(\overline{\underline{H}}\cdot{}^{(c)}\mathcal{D})A, \\ \underline{H}\widehat{\otimes}(\overline{\underline{H}}\cdot A) &= 2(\underline{H}\cdot\overline{\underline{H}})\, A, \end{aligned}$$

which implies

$$\begin{aligned} {}^{(c)}\nabla_4\,{}^{(c)}\nabla_3 A &= \frac{1}{4}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A) + \left(-\frac{1}{2}\mathrm{tr}X - 2\overline{\mathrm{tr}X}\right){}^{(c)}\nabla_3 A - \frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(c)}\nabla_4 A \\ &+ \left(-\overline{\mathrm{tr}X}\mathrm{tr}\underline{X} - \frac{1}{2}({}^{(c)}\mathcal{D}\cdot\overline{\underline{H}} + \underline{H}\cdot\overline{\underline{H}}) + 2\overline{P}\right)A \\ &+ \frac{1}{2}\Big(({}^{(c)}\mathcal{D}\cdot\overline{\underline{H}})A + (\overline{\underline{H}}\cdot{}^{(c)}\mathcal{D})A\Big) + \left(H + \frac{1}{4}\underline{H}\right)\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A) \\ &+ H\widehat{\otimes}(\overline{\underline{H}}\cdot A) + \frac{1}{2}(\underline{H}\cdot\overline{\underline{H}})\, A + r^{-1}\mathfrak{d}^{\leq 1}\big(\Gamma_g\cdot B\big) + {}^{(c)}\nabla_3\Xi\cdot B + \Gamma_b\cdot\Gamma_g\cdot A \\ &= \frac{1}{4}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A) + \left(-\frac{1}{2}\mathrm{tr}X - 2\overline{\mathrm{tr}X}\right){}^{(c)}\nabla_3 A - \frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(c)}\nabla_4 A \\ &+ \frac{1}{2}(\overline{\underline{H}}\cdot{}^{(c)}\mathcal{D})A + \left(H + \frac{1}{4}\underline{H}\right)\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A) + \big(-\overline{\mathrm{tr}X}\mathrm{tr}\underline{X} + 2\overline{P}\big)A \\ &+ H\widehat{\otimes}(\overline{\underline{H}}\cdot A) + r^{-1}\mathfrak{d}^{\leq 1}\big(\Gamma_g\cdot B\big) + {}^{(c)}\nabla_3\Xi\cdot B + \Gamma_b\cdot\Gamma_g\cdot A. \end{aligned}$$

Using (2.4.5), we further simplify the angular part writing

$$\begin{aligned} \frac{1}{2}(\overline{\underline{H}}\cdot{}^{(c)}\mathcal{D})A + \left(H + \frac{1}{4}\underline{H}\right)\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A) &= \frac{1}{2}(\overline{\underline{H}}\cdot{}^{(c)}\mathcal{D})A + \left(2H + \frac{1}{2}\underline{H}\right)\cdot\overline{{}^{(c)}\mathcal{D}}A \\ &= \frac{1}{2}(\overline{\underline{H}}\cdot{}^{(c)}\mathcal{D} + \underline{H}\cdot\overline{{}^{(c)}\mathcal{D}})A + (2H\cdot\overline{{}^{(c)}\mathcal{D}})A \\ &= \big(4H + \underline{H} + \overline{\underline{H}}\big)\cdot{}^{(c)}\nabla A. \end{aligned}$$

This proves the proposition.

D.2 Proof of Corollary 5.1.2

Applying Lemma 4.7.8 to A, which is 2-conformally invariant, we obtain

$$\begin{aligned} {}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A) &= 4\,{}^{(c)}\triangle_2 A - 2i({}^{(a)}\mathrm{tr}\chi\,{}^{(c)}\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\,{}^{(c)}\nabla_4)A \\ &+ 2\left(\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} + {}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} + 4\rho\right)A \\ &- 2i\big(\mathrm{tr}\,\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} - \mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\chi + 4\,{}^*\rho\big)A. \end{aligned}$$

From (5.1.2), we then obtain

$$\begin{aligned}\mathcal{L}(A) &= -\,{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3 A + {}^{(c)}\triangle_2 A - \frac{1}{2}i\big({}^{(a)}\mathrm{tr}\chi\,{}^{(c)}\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\,{}^{(c)}\nabla_4\big)A\\ &+\frac{1}{2}\left(\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} + {}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} + 4\rho\right)A - i\Big(\frac{1}{2}\big(\mathrm{tr}\,\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} - \mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\chi\big) + 2\,{}^*\rho\Big)A\\ &+\left(-\frac{1}{2}\mathrm{tr}X - 2\overline{\mathrm{tr}X}\right){}^{(c)}\nabla_3 A - \frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(c)}\nabla_4 A\\ &+\left(4H + \underline{H} + \overline{\underline{H}}\right)\cdot{}^{(c)}\nabla A + \left(-\overline{\mathrm{tr}X}\mathrm{tr}\underline{X} + 2\overline{P}\right)A + H\widehat{\otimes}(\overline{\underline{H}}\cdot A).\end{aligned}$$

Writing

$$\begin{aligned}-\frac{1}{2}\mathrm{tr}\underline{X} &= -\frac{1}{2}\mathrm{tr}\,\underline{\chi} + \frac{1}{2}i\,{}^{(a)}\mathrm{tr}\underline{\chi}\\ -\frac{1}{2}\mathrm{tr}X - 2\overline{\mathrm{tr}X} &= \left(-\frac{5}{2}\mathrm{tr}\,\chi\right) + i\left(-\frac{3}{2}\,{}^{(a)}\mathrm{tr}\chi\right)\\ -\overline{\mathrm{tr}X}\mathrm{tr}\underline{X} + 2\overline{P} &= -\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} - {}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} + 2\rho + i\big(\mathrm{tr}\,\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} - \mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\chi - 2\,{}^*\rho\big),\end{aligned}$$

we obtain

$$\begin{aligned}\mathcal{L}(A) &= -\,{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3 A + {}^{(c)}\triangle_2 A + (4\eta + 2\underline{\eta})\cdot{}^{(c)}\nabla A - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\,{}^{(c)}\nabla_4 A - \frac{5}{2}\mathrm{tr}\,\chi\,{}^{(c)}\nabla_3 A\\ &+\left(-\frac{1}{2}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} - \frac{1}{2}\,{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} + 4\rho\right)A + H\widehat{\otimes}(\overline{\underline{H}}\cdot A)\\ &+i\Big[-2\,{}^{(a)}\mathrm{tr}\chi\,{}^{(c)}\nabla_3 A + 4\,{}^*\eta\cdot{}^{(c)}\nabla A + \left(\frac{1}{2}\mathrm{tr}\,\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\chi - 4\,{}^*\rho\right)A\Big]\end{aligned}$$

Using (2.4.3), we write

$$H\widehat{\otimes}(\overline{\underline{H}}\cdot A) = \big(4\eta\cdot\underline{\eta} - 4i\eta\wedge\underline{\eta}\big)A, \tag{D.2.1}$$

and this completes the proof.

D.3 Proof of Proposition 5.2.4

We first prove the relations in Kerr. For simplicity we choose the normalization $e_3 = e_3^{(in)}$ for which $\underline{\omega} = 0$ and thus ${}^{(c)}\nabla_3 = \nabla_3$. Let I be the expression

$$I := r\nabla_3\Big(r^2\big(\nabla_3(rfA)\big)\Big), \tag{D.3.1}$$

with

$$f = \frac{\overline{q}^4}{r^4} = f_1 f_2^2, \qquad f_1 = \frac{|q|^4}{r^4}, \qquad f_2 = \frac{\overline{q}}{q}.$$

We calculate

$$\begin{aligned}
I &= r\nabla_3\Big(r^3 f\nabla_3 A + r^2 e_3(rf)A\Big)\\
&= r^4 f\nabla_3\nabla_3 A + \big(re_3(r^3 f) + r^3 e_3(rf)\big)\nabla_3 A + r\nabla_3(r^2 e_3(rf))A\\
&= r^4 f\nabla_3\nabla_3 A + \big(2r^4 e_3 f - 4r^3 f\big)\nabla_3 A + \big(r^4\nabla_3 e_3(f) - 4r^3 e_3 f + 2r^2 f\big)A.
\end{aligned}$$

We deduce

$$\begin{aligned}
I &= r^4 f\Big(\nabla_3\nabla_3 A + I_1\nabla_3 A + I_2 A\Big),\\
I_1 &= 2f^{-1}(e_3 f - 2r^{-1} f),\\
I_2 &= f^{-1}\Big(\nabla_3 e_3(f) - 4r^{-1} e_3 f + 2r^{-2} f\Big).
\end{aligned} \tag{D.3.2}$$

We rewrite I_2 in the form

$$\begin{aligned}
I_2 &= \nabla_3\big(e_3(f) - 2r^{-1} f)\big) + 2r^{-1} e_3 f + 2r^{-2} f - 4r^{-1} e_3 f + 2r^{-2} f\\
&= \nabla_3\big(e_3(f) - 2r^{-1} f)\big) - 2r^{-1}\big(e_3 f - 2r^{-1} f\big) = \frac{1}{2}\nabla_3(fI_1) - r^{-1} fI_1.
\end{aligned}$$

Hence

$$\begin{aligned}
I &= r^4 f\Big(\nabla_3\nabla_3 A_{11} + I_1\nabla_3 A_{11} + I_2 A\Big),\\
I_1 &= 2f^{-1}(e_3 f - 2r^{-1} f),\\
I_2 &= \frac{1}{2}\nabla_3(fI_1) - r^{-1} fI_1.
\end{aligned}$$

Now, in view of our choices of the scalar functions f_1, f_2,

$$\begin{aligned}
e_3 f_1 &= \frac{r^4 e_3(|q|^4) - |q|^4 e_3(r^4)}{r^8} = \frac{-4r^5|q|^2 + 4|q|^4 r^3}{r^8} = -\frac{4|q|^2}{r^5}\Big(r^2 - |q|^2\Big)\\
&= \frac{4a^2\cos^2|q|^2}{r^5} = \frac{4a^2\cos^2\theta}{r|q|^2} f_1,\\
e_3 f_2 &= -i\,{}^{(a)}\mathrm{tr}\underline{\chi}\frac{\overline{q}}{q} = -i\,{}^{(a)}\mathrm{tr}\underline{\chi} f_2.
\end{aligned}$$

Hence

$$\begin{aligned}
e_3 f &= e_3(f_1) f_2^2 + 2f_1 f_2 e_3(f_2) = \frac{4a^2\cos^2\theta}{r|q|^2} f_1 f_2^2 - 2i\,{}^{(a)}\mathrm{tr}\underline{\chi} f_1 f_2^2\\
&= \Big(\frac{4a^2\cos^2\theta}{r|q|^2} - 2i\,{}^{(a)}\mathrm{tr}\underline{\chi}\Big) f
\end{aligned}$$

i.e.

$$e_3 f = \Big(\frac{4a^2\cos^2\theta}{r|q|^2} - i\,{}^{(a)}\mathrm{tr}\underline{\chi}\Big) f.$$

We deduce

$$\begin{aligned}
I_1 &= 2f^{-1}(e_3 f - 2r^{-1}f) = 2\Big(\frac{4a^2\cos^2\theta}{r|q|^2} - 2i\,{}^{(a)}\mathrm{tr}\underline{\chi} - \frac{2}{r}\Big)\\
&= 2\Big(\frac{2a^2\cos^2\theta}{r|q|^2} + \frac{2}{r} - \frac{2r}{|q|^2} - \frac{2}{r} - 2i\,{}^{(a)}\mathrm{tr}\underline{\chi}\Big) = 2\mathrm{tr}\,\underline{\chi} - 2\frac{{}^{(a)}\mathrm{tr}\underline{\chi}^2}{\mathrm{tr}\,\underline{\chi}} - 4i\,{}^{(a)}\mathrm{tr}\underline{\chi}.
\end{aligned}$$

Hence, recalling the definition of C_1 in (5.2.3), $I_1 = C_1$.

It remains to calculate I_2.

$$\begin{aligned}
I_2 &= \frac{1}{2}\nabla_3(fI_1) - r^{-1}fI_1 = \frac{1}{2}I_1\nabla_3 f + \frac{1}{2}f\nabla_3 I_1 - r^{-1}fI_1\\
&= \frac{1}{2}f\nabla_3 I_1 + \frac{1}{2}I_1\big(\nabla_3 f - 2r^{-1}f\big) = \frac{1}{2}f\nabla_3 I_1 + \frac{1}{4}fI_1^2.
\end{aligned}$$

Therefore,

$$I_2 = \frac{1}{2}f\big(\nabla_3 I_1 + \frac{1}{2}I_1^2\big).$$

Now,

$$\begin{aligned}
\nabla_3 I_1 &= \nabla_3\Big(2\mathrm{tr}\,\underline{\chi} - 2\frac{{}^{(a)}\mathrm{tr}\underline{\chi}^2}{\mathrm{tr}\,\underline{\chi}} - 4i\,{}^{(a)}\mathrm{tr}\underline{\chi}\Big)\\
&= 2\nabla_3\mathrm{tr}\,\underline{\chi} - 2\frac{\mathrm{tr}\,\underline{\chi}\nabla_3({}^{(a)}\mathrm{tr}\underline{\chi}^2) - {}^{(a)}\mathrm{tr}\underline{\chi}^2\nabla_3\mathrm{tr}\,\underline{\chi}}{\mathrm{tr}\,\underline{\chi}^2} - 4i\nabla_3\,{}^{(a)}\mathrm{tr}\underline{\chi}.
\end{aligned}$$

Recall, in Kerr,

$$\nabla_3\mathrm{tr}\,\underline{\chi} = -\frac{1}{2}\big(\mathrm{tr}\,\underline{\chi}^2 - {}^{(a)}\mathrm{tr}\underline{\chi}^2\big), \qquad \nabla_3\,{}^{(a)}\mathrm{tr}\underline{\chi} = -\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\underline{\chi}.$$

Therefore,

$$\begin{aligned}
{}^{(c)}\nabla_3 E_1 &= -\mathrm{tr}\,\underline{\chi}^2 + {}^{(a)}\mathrm{tr}\underline{\chi}^2 - 2\frac{-2\mathrm{tr}\,\underline{\chi}^2\,{}^{(a)}\mathrm{tr}\underline{\chi}^2 + \frac{1}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi}^2\big(\mathrm{tr}\,\underline{\chi}^2 - {}^{(a)}\mathrm{tr}\underline{\chi}^2\big)}{\mathrm{tr}\,\underline{\chi}^2} + 4i\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\underline{\chi}\\
&= -\mathrm{tr}\,\underline{\chi}^2 + {}^{(a)}\mathrm{tr}\underline{\chi}^2 + \frac{3\mathrm{tr}\,\underline{\chi}^2\,{}^{(a)}\mathrm{tr}\underline{\chi}^2 + {}^{(a)}\mathrm{tr}\underline{\chi}^4}{\mathrm{tr}\,\underline{\chi}^2} + 4i\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\underline{\chi}\\
&= -\mathrm{tr}\,\underline{\chi}^2 + {}^{(a)}\mathrm{tr}\underline{\chi}^2 + 3\,{}^{(a)}\mathrm{tr}\underline{\chi}^2 + \frac{{}^{(a)}\mathrm{tr}\underline{\chi}^4}{\mathrm{tr}\,\underline{\chi}^2} + 4i\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\underline{\chi}\\
&= -\mathrm{tr}\,\underline{\chi}^2 + 4\,{}^{(a)}\mathrm{tr}\underline{\chi}^2 + \frac{{}^{(a)}\mathrm{tr}\underline{\chi}^4}{\mathrm{tr}\,\underline{\chi}^2} + 4i\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\underline{\chi}.
\end{aligned}$$

Adding, simplifying and recalling the definition of C_2 we deduce

$$
\begin{aligned}
\nabla_3 I_1 + \frac{1}{2} I_1^2 &= -\mathrm{tr}\,\underline{\chi}^2 + 4\,{}^{(a)}\mathrm{tr}\underline{\chi}^2 + \frac{{}^{(a)}\mathrm{tr}\underline{\chi}^4}{\mathrm{tr}\,\underline{\chi}^2} + 4i\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\underline{\chi} \\
&+ 2\Big(\mathrm{tr}\,\underline{\chi} - \frac{{}^{(a)}\mathrm{tr}\underline{\chi}^2}{\mathrm{tr}\,\underline{\chi}} - 2i\,{}^{(a)}\mathrm{tr}\underline{\chi}\Big)^2 \\
&= -\mathrm{tr}\,\underline{\chi}^2 + 4\,{}^{(a)}\mathrm{tr}\underline{\chi}^2 + \frac{{}^{(a)}\mathrm{tr}\underline{\chi}^4}{\mathrm{tr}\,\underline{\chi}^2} + 4i\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\underline{\chi} \\
&+ 2\Big(\mathrm{tr}\,\underline{\chi}^2 + \frac{{}^{(a)}\mathrm{tr}\underline{\chi}^4}{\mathrm{tr}\,\underline{\chi}^2} - 4\,{}^{(a)}\mathrm{tr}\underline{\chi}^2 - 2\,{}^{(a)}\mathrm{tr}\chi^2 - 4i\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\underline{\chi} + 4i\frac{{}^{(a)}\mathrm{tr}\underline{\chi}^3}{\mathrm{tr}\,\underline{\chi}}\Big) \\
&= \mathrm{tr}\,\underline{\chi}^2 - 8\,{}^{(a)}\mathrm{tr}\underline{\chi}^2 + 3\frac{{}^{(a)}\mathrm{tr}\underline{\chi}^4}{\mathrm{tr}\,\underline{\chi}^2} - 4i\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\underline{\chi} + 8i\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\underline{\chi} \\
&= 2C_2.
\end{aligned}
$$

Hence $I_2 = C_2$ and, recalling the definition of $f = \frac{\overline{q}^4}{r^4}$,

$$
I = r^4 f\Big(\nabla_3\nabla_3 A + I_1\nabla_3 A + I_2 A\Big) = \frac{\overline{q}}{q} q\overline{q}^3\Big(\nabla_3\nabla_3 A + C_1\nabla_3 A + C_2 A\Big) = \frac{\overline{q}}{q}\mathfrak{q},
$$

as stated in (5.2.6).

It remains to check (5.2.7). To do this it helps to write the expression $E = r^2\nabla_3\nabla_3\big(\frac{\overline{q}^4}{r^2}A\big)$ using $f = \frac{\overline{q}^4}{r^4}$,

$$
\begin{aligned}
E &= r^2\nabla_3\nabla_3\big(r^2 f A\big) = r^4 f\Big(\nabla_3\nabla_3 A + \frac{2}{r^2 f} e_3(r^2 f)\nabla_3 A + \frac{1}{r^2 f} e_3 e_3(r^2 f) A\Big) \\
&= r^4 f\Big(\nabla_3\nabla_3 A_{11} + I_1\nabla_3 A_{11} + I_2 A\Big),
\end{aligned}
$$

with I_1, I_2 as in (D.3.2). Therefore

$$
r^2\nabla_3\nabla_3\big(r^2 f A\big) = r\nabla_3\Big(r^2\big(\nabla_3(rfA)\big)\Big) = \frac{\overline{q}}{q}\mathfrak{q},
$$

as stated.

Using that $\widetilde{e_3(r)} = r\Gamma_b$ and $e_3(\cos\theta) = \Gamma_b$ we deduce the relations in perturbations of Kerr.

D.4 Proof of Theorem 5.2.9

In this section we present the proof of Theorem 5.2.9. The computations are obtained in the outgoing frame.

D.4.1 Step 1. Compute the commutator $[Q, \mathcal{L}]$

Recall the Teukolsky equation as in Proposition 5.1.1, i.e.

$$\mathcal{L}(A) = \mathrm{Err}[\mathcal{L}(A)]. \tag{D.4.1}$$

We apply the Chandrasekhar transformation, i.e. the operator Q as defined in (5.2.1), to the above. We obtain

$$\mathcal{L}(Q(A)) + [Q, \mathcal{L}](A) = Q(\mathrm{Err}[\mathcal{L}(A)]). \tag{D.4.2}$$

We compute the commutator $[Q, \mathcal{L}]$ between $\mathcal{L}$ and the second order differential operator Q for any scalar functions C_1 and C_2. In order to obtain cancellations in the commutator, we impose conditions on the functions C_1 and C_2, which for $a = 0$ coincide with the ones in Schwarzschild as in [50]. We obtain the following.

Proposition D.4.1. *Let* $Q(A) = {}^{(c)}\nabla_3 {}^{(c)}\nabla_3 A + C_1 {}^{(c)}\nabla_3 A + C_2 A$ *with* C_1 *and* C_2 *given by*

$$C_1 = 2tr\underline{\chi} + \widetilde{C_1}, \qquad C_2 = \frac{1}{2} tr\underline{\chi}^2 + \widetilde{C_2} \tag{D.4.3}$$

where $\widetilde{C_1}$ *and* $\widetilde{C_2}$ *are complex functions satisfying* $\widetilde{C_1} = O(|a|r^{-2})$, $\widetilde{C_2} = O(|a|r^{-3})$. *Then the commutator between* Q *and* $\mathcal{L}$ *is given by*

$$[Q, \mathcal{L}](A) = 4\underline{\eta} \cdot {}^{(c)}\nabla Q(A) - 2tr\underline{\chi} {}^{(c)}\nabla_4 Q(A) + \hat{V} Q(A) + L_Q(A) + Err[[Q, \mathcal{L}]A], \tag{D.4.4}$$

where

- *the potential* $\hat{V}$ *is given by*

$$\hat{V} = I_{33} + J_{33} + K_{33} + M_{33} \tag{D.4.5}$$

  where I_{33}, J_{33}, K_{33}, M_{33} *are given in Section D.4.1,*

- *$L_Q[A]$ is a second order linear operator in A, given by*

$$\begin{aligned} L_Q[A] = {} & Z_{43}\ {}^{(c)}\nabla_4\,{}^{(c)}\nabla_3 A + Z_4\ {}^{(c)}\nabla_4 A + Z_{a3}\,{}^{(c)}\nabla_a\,{}^{(c)}\nabla_3 A \\ & + Z_3\,{}^{(c)}\nabla_3 A + Z_a\,{}^{(c)}\nabla_a A + Z_0 A, \end{aligned} \tag{D.4.6}$$

where Z_{a3}, Z_a are complex one-forms and Z_{43}, Z_4, Z_3 and Z_0 are complex functions of (r,θ), all of which vanish for zero angular momentum, having the following fall-off in r

$$Z_{43}, Z_{a3} = O\left(\frac{|a|}{r^3}\right), \qquad Z_4, Z_3, Z_a = O\left(\frac{|a|}{r^4}\right), \qquad Z_0 = O\left(\frac{|a|}{r^5}\right).$$

More precisely,

$$\begin{aligned} Z_{43} &= {}^{(c)}\nabla_3\widetilde{C_1} + tr\underline{\chi}\widetilde{C_1}, \\ Z_4 &= {}^{(c)}\nabla_3\widetilde{C_2} + 2tr\underline{\chi}\widetilde{C_2} - \frac{1}{4}(tr\underline{\chi}^2 + {}^{(a)}tr\underline{\chi}^2)\widetilde{C_1}, \end{aligned}$$

and

$$Z_{a3} = I_{a3} + J_{a3} + L_{a3} + M_{a3} - 4C_1\underline{\eta},$$

where I_{a3}, J_{a3}, L_{a3}, M_{a3} are given in Section D.4.1,

- *the error terms $Err[[Q,\mathcal{L}]A]$ are given by*

$$Err[[Q,\mathcal{L}]A] = r^{-2}\mathfrak{d}^{\leq 3}(\Gamma_g\cdot(A,B)) + \nabla_3\big(r^{-2}\Gamma_b\cdot\mathfrak{d}^{\leq 2}A\big) + r^{-1}\mathfrak{d}\big((\Gamma_b\cdot\Gamma_g)A\big).$$

The proof of Proposition D.4.1 is obtained as a result of the computations below. We first collect the following commutators for Q.

Proposition D.4.2. *Let $U \in \mathfrak{s}_2(\mathbb{C})$ s-conformally invariant. We have:*

- *the following commutators with ${}^{(c)}\nabla_3$ and ${}^{(c)}\nabla_4$:*

$$\begin{aligned} [Q, {}^{(c)}\nabla_3]U &= (-{}^{(c)}\nabla_3 C_1)\ {}^{(c)}\nabla_3 U + (-{}^{(c)}\nabla_3 C_2)\ U, \\ [Q, {}^{(c)}\nabla_4]U &= 4(\eta-\underline{\eta})\cdot{}^{(c)}\nabla\,{}^{(c)}\nabla_3 U \\ & \quad +\Big(2\,{}^{(c)}\nabla_3(\eta-\underline{\eta}) + (2C_1 - tr\underline{\chi})(\eta-\underline{\eta}) + {}^{(a)}tr\underline{\chi}\ {}^*(\eta-\underline{\eta})\Big)\cdot{}^{(c)}\nabla U \\ & \quad +\Big(\mathcal{V}^s_{[3,4]} + \mathcal{V}^{s-1}_{[3,4]} + 2\eta\cdot(\eta-\underline{\eta}) - {}^{(c)}\nabla_4 C_1\Big)\,{}^{(c)}\nabla_3 U \\ & \quad +\Big({}^{(c)}\nabla_3\mathcal{V}^s_{[3,4]} + C_1(\mathcal{V}^s_{[3,4]}) - {}^{(c)}\nabla_4 C_2\Big)U + 2(\eta-\underline{\eta})\cdot\mathcal{V}^s_{[3,a]}(U) \\ & \quad + r^{-3}\Gamma_b\cdot\mathfrak{d}^{\leq 1}U + \mathfrak{d}\big((\Gamma_b\cdot\Gamma_g)U\big), \end{aligned}$$

where $\mathcal{V}^s_{[3,4]}$ and $\mathcal{V}^s_{[3,a]}$ are given by (D.4.7) *and* (D.4.8) *respectively.*

- *the following commutator with* ${}^{(c)}\nabla_a$*:*

$$
\begin{aligned}
& [Q, {}^{(c)}\nabla_a]U \\
= \; & -tr\underline{\chi}\, {}^{(c)}\nabla_a {}^{(c)}\nabla_3 U - {}^{(a)}tr\underline{\chi}\, {}^{*(c)}\nabla_a {}^{(c)}\nabla_3 U + 2\eta_a {}^{(c)}\nabla_3 {}^{(c)}\nabla_3 U \\
& + \left(-\frac{1}{2}tr\underline{\chi}(C_1 - tr\underline{\chi}) - \frac{1}{2}{}^{(a)}tr\underline{\chi}^2\right) {}^{(c)}\nabla_a U - \frac{1}{2}{}^{(a)}tr\underline{\chi}\big(C_1 - 2tr\underline{\chi}\big)\, {}^{*(c)}\nabla_a U \\
& + \left({}^{(c)}\nabla_3\eta_a + (C_1 - \frac{1}{2}tr\underline{\chi})\eta_a - \frac{1}{2}{}^{(a)}tr\underline{\chi}\, {}^*\eta_a - {}^{(c)}\nabla_a C_1\right) {}^{(c)}\nabla_3 U \\
& + {}^{(c)}\nabla_3(\mathcal{V}^s_{[3,a]}(U)) + \mathcal{V}^{s-1}_{[3,a]}({}^{(c)}\nabla_3 U) - \frac{1}{2}tr\underline{\chi}\mathcal{V}^s_{[3,a]}(U) - \frac{1}{2}{}^{(a)}tr\underline{\chi}\, {}^*\mathcal{V}^s_{[3,a]}(U) \\
& - {}^{(c)}\nabla_a(C_2)U + C_1\mathcal{V}^s_{[3,a]}(U) + \nabla_3\big(r^{-1}\Gamma_b \cdot \mathfrak{d}^{\leq 1}U\big).
\end{aligned}
$$

- *the following commutator with* $\overline{{}^{(c)}\mathcal{D}}\cdot$*:*

$$
\begin{aligned}
[Q, \overline{{}^{(c)}\mathcal{D}}\cdot]U \;=\; & 2\overline{H}\cdot {}^{(c)}\nabla_3 {}^{(c)}\nabla_3 U - \overline{tr\underline{X}}\, \overline{{}^{(c)}\mathcal{D}}\cdot {}^{(c)}\nabla_3 U + \frac{1}{2}\overline{tr\underline{X}}(\overline{tr\underline{X}} - C_1)\, \overline{{}^{(c)}\mathcal{D}}\cdot U \\
& + \left({}^{(c)}\nabla_3\overline{H} + (-(s-2)\overline{tr\underline{X}} + C_1)\,\overline{H} - \overline{{}^{(c)}\mathcal{D}}(C_1)\right)\cdot {}^{(c)}\nabla_3 U \\
& + \left(\frac{1}{2}(s-2)\overline{tr\underline{X}}\left(-{}^{(c)}\nabla_3\overline{H} + (\overline{tr\underline{X}} - C_1)\,\overline{H}\right) - \overline{{}^{(c)}\mathcal{D}}(C_2)\right)\cdot U \\
& + \nabla_3\big(r^{-1}\Gamma_b \cdot \mathfrak{d}^{\leq 1}U\big).
\end{aligned}
$$

Let $F \in \mathfrak{s}_1(\mathbb{C})$ *of conformal type* s*. Then we have the following commutator with* ${}^{(c)}\mathcal{D}\widehat{\otimes}$*:*

$$
\begin{aligned}
[Q, {}^{(c)}\mathcal{D}\widehat{\otimes}]F \;=\; & 2H\widehat{\otimes}\, {}^{(c)}\nabla_3 {}^{(c)}\nabla_3 F - tr\underline{X}\, {}^{(c)}\mathcal{D}\widehat{\otimes}\, {}^{(c)}\nabla_3 F + \frac{1}{2}(tr\underline{X})(tr\underline{X} - C_1)\, {}^{(c)}\mathcal{D}\widehat{\otimes} F \\
& + \big({}^{(c)}\nabla_3 H + (-(s+1)tr\underline{X} + C_1)H - {}^{(c)}\mathcal{D}(C_1)\big)\,\widehat{\otimes}\, {}^{(c)}\nabla_3 F \\
& + \left(\frac{1}{2}(s+1)tr\underline{X}\left(-{}^{(c)}\nabla_3 H + (tr\underline{X} - C_1)H\right) - {}^{(c)}\mathcal{D}(C_2)\right)\widehat{\otimes} F \\
& + \nabla_3\big(r^{-1}\Gamma_b \cdot \mathfrak{d}^{\leq 1}U\big).
\end{aligned}
$$

Proof. We compute

$$
\begin{aligned}
[Q, {}^{(c)}\nabla_3]U \;=\; & \big({}^{(c)}\nabla_3 {}^{(c)}\nabla_3 + C_1 {}^{(c)}\nabla_3 + C_2\big)\, {}^{(c)}\nabla_3 U \\
& - {}^{(c)}\nabla_3\big({}^{(c)}\nabla_3 {}^{(c)}\nabla_3 U + C_1 {}^{(c)}\nabla_3 U + C_2 U\big) \\
=\; & (-{}^{(c)}\nabla_3 C_1)\, {}^{(c)}\nabla_3 U + (-{}^{(c)}\nabla_3 C_2)U.
\end{aligned}
$$

Similarly, we compute

$$
\begin{aligned}
[Q, {}^{(c)}\nabla_4]U \;=\; & {}^{(c)}\nabla_3([{}^{(c)}\nabla_3, {}^{(c)}\nabla_4]U) + [{}^{(c)}\nabla_3, {}^{(c)}\nabla_4]\, {}^{(c)}\nabla_3 U \\
& + C_1[{}^{(c)}\nabla_3, {}^{(c)}\nabla_4]U - {}^{(c)}\nabla_4(C_1)\, {}^{(c)}\nabla_3 U - {}^{(c)}\nabla_4(C_2)\, U.
\end{aligned}
$$

Recall from Lemma 4.2.2,

$$[\,^{(c)}\nabla_3,\,^{(c)}\nabla_4]U \;=\; 2(\eta-\underline{\eta})\cdot\,^{(c)}\nabla U+\mathcal{V}^s_{[3,4]}U+(\Gamma_b\cdot\Gamma_g)U,$$

where

$$\mathcal{V}^s_{[3,4]}=2s\left(\rho-\eta\cdot\underline{\eta}\right)+4i\left(-\,{}^*\rho+\eta\wedge\underline{\eta}\right). \tag{D.4.7}$$

In particular, since $^{(c)}\nabla_3U$ is conformal of type $s-1$, we have

$$[\,^{(c)}\nabla_3,\,^{(c)}\nabla_4]\,^{(c)}\nabla_3U \;=\; 2(\eta-\underline{\eta})\cdot\,^{(c)}\nabla\,^{(c)}\nabla_3U+\mathcal{V}^{s-1}_{[3,4]}\,^{(c)}\nabla_3U+(\Gamma_b\cdot\Gamma_g)\mathfrak{d}U.$$

On the other hand we compute

$$\begin{aligned}^{(c)}\nabla_3([\,^{(c)}\nabla_3,\,^{(c)}\nabla_4]U) \;=\;& 2(\eta-\underline{\eta})\cdot\,^{(c)}\nabla_3\,^{(c)}\nabla U+2\,^{(c)}\nabla_3(\eta-\underline{\eta})\cdot\,^{(c)}\nabla U\\ &+\mathcal{V}^s_{[3,4]}\,^{(c)}\nabla_3U+\,^{(c)}\nabla_3\mathcal{V}^s_{[3,4]}U+\mathfrak{d}\big((\Gamma_b\cdot\Gamma_g)U\big).\end{aligned}$$

Recall from Lemma 4.2.2,

$$\begin{aligned}[\,^{(c)}\nabla_3,\,^{(c)}\nabla_a]U_{bc} \;=\;& -\frac{1}{2}\mathrm{tr}\,\underline{\chi}\,^{(c)}\nabla_aU_{bc}-\frac{1}{2}\,^{(a)}\mathrm{tr}\underline{\chi}\,{}^*\,^{(c)}\nabla_aU_{bc}+\eta_a\,^{(c)}\nabla_3U_{bc}+\mathcal{V}^s_{[3,a]}(U)\\ &+r^{-1}\Gamma_b\cdot\mathfrak{d}^{\leq 1}U,\end{aligned}$$

where

$$\begin{aligned}\mathcal{V}^s_{[3,a]}(U)=&-\frac{1}{2}\mathrm{tr}\,\underline{\chi}\Big(s(\eta_a)U_{bc}+\eta_bU_{ac}+\eta_cU_{ab}-\delta_{ab}(\eta\cdot U)_c-\delta_{ac}(\eta\cdot U)_b\Big)\\ &-\frac{1}{2}\,^{(a)}\mathrm{tr}\underline{\chi}\Big(s({}^*\eta_a)U_{bc}+\eta_b\,{}^*U_{ac}+\eta_c\,{}^*U_{ab}-\in_{ab}(\eta\cdot U)_c-\in_{ac}(\eta\cdot U)_b\Big).\end{aligned} \tag{D.4.8}$$

We therefore obtain

$$\begin{aligned}&^{(c)}\nabla_3([\,^{(c)}\nabla_3,\,^{(c)}\nabla_4]U)\\ =\;& 2(\eta-\underline{\eta})\cdot\Big(\,^{(c)}\nabla\,^{(c)}\nabla_3U-\frac{1}{2}\mathrm{tr}\,\underline{\chi}\,^{(c)}\nabla U-\frac{1}{2}\,^{(a)}\mathrm{tr}\underline{\chi}\,{}^*\,^{(c)}\nabla U+\eta\,^{(c)}\nabla_3U+\mathcal{V}^s_{[3,a]}(U)\Big)\\ &+2\,^{(c)}\nabla_3(\eta-\underline{\eta})\cdot\,^{(c)}\nabla U+\mathcal{V}^s_{[3,4]}\,^{(c)}\nabla_3U+\,^{(c)}\nabla_3\mathcal{V}^s_{[3,4]}U\\ &+r^{-3}\Gamma_b\cdot\mathfrak{d}^{\leq 1}U+\mathfrak{d}\big((\Gamma_b\cdot\Gamma_g)U\big)\\ =\;& 2(\eta-\underline{\eta})\cdot\,^{(c)}\nabla\,^{(c)}\nabla_3U+\Big(2\,^{(c)}\nabla_3(\eta-\underline{\eta})-\mathrm{tr}\,\underline{\chi}(\eta-\underline{\eta})+\,^{(a)}\mathrm{tr}\underline{\chi}\,{}^*(\eta-\underline{\eta})\Big)\cdot\,^{(c)}\nabla U\\ &+\Big(\mathcal{V}^s_{[3,4]}+2\eta\cdot(\eta-\underline{\eta})\Big)\,^{(c)}\nabla_3U+\,^{(c)}\nabla_3\mathcal{V}^s_{[3,4]}U+2(\eta-\underline{\eta})\cdot\mathcal{V}^s_{[3,a]}(U)\\ &+r^{-3}\Gamma_b\cdot\mathfrak{d}^{\leq 1}U+\mathfrak{d}\big((\Gamma_b\cdot\Gamma_g)U\big).\end{aligned}$$

Putting the above together, we obtain the stated expression.

We compute

$$
\begin{aligned}
[Q, {}^{(c)}\nabla_a]U &= {}^{(c)}\nabla_3([{}^{(c)}\nabla_3, {}^{(c)}\nabla_a]U) + [{}^{(c)}\nabla_3, {}^{(c)}\nabla_a]{}^{(c)}\nabla_3 U + C_1[{}^{(c)}\nabla_3, {}^{(c)}\nabla_a]U \\
&\quad - {}^{(c)}\nabla_a(C_1){}^{(c)}\nabla_3 U - {}^{(c)}\nabla_a(C_2)U.
\end{aligned}
$$

We have, again from Lemma 4.2.2,

$$
\begin{aligned}
[{}^{(c)}\nabla_3, {}^{(c)}\nabla_a]{}^{(c)}\nabla_3 U &= -\frac{1}{2}\mathrm{tr}\,\underline{\chi}\,{}^{(c)}\nabla_a{}^{(c)}\nabla_3 U - \frac{1}{2}{}^{(a)}\mathrm{tr}\underline{\chi}\,{}^*{}^{(c)}\nabla_a{}^{(c)}\nabla_3 U + \eta_a{}^{(c)}\nabla_3{}^{(c)}\nabla_3 U \\
&\quad + \mathcal{V}^{s-1}_{[3,a]}({}^{(c)}\nabla_3 U) + r^{-1}\Gamma_b\cdot\mathfrak{d}^{\leq 2}U,
\end{aligned}
$$

and

$$
\begin{aligned}
{}^{(c)}\nabla_3([{}^{(c)}\nabla_3, {}^{(c)}\nabla_a]U) &= -\frac{1}{2}\mathrm{tr}\,\underline{\chi}\,{}^{(c)}\nabla_3{}^{(c)}\nabla_a U - \frac{1}{2}{}^{(a)}\mathrm{tr}\underline{\chi}\,{}^*{}^{(c)}\nabla_3{}^{(c)}\nabla_a U + \eta_a{}^{(c)}\nabla_3{}^{(c)}\nabla_3 U \\
&\quad - \frac{1}{2}({}^{(c)}\nabla_3\mathrm{tr}\,\underline{\chi})\,{}^{(c)}\nabla_a U - \frac{1}{2}({}^{(c)}\nabla_3{}^{(a)}\mathrm{tr}\underline{\chi})\,{}^*{}^{(c)}\nabla_a U \\
&\quad + ({}^{(c)}\nabla_3\eta_a){}^{(c)}\nabla_3 U + {}^{(c)}\nabla_3(\mathcal{V}^s_{[3,a]}(U)) + r^{-1}\mathfrak{d}(\Gamma_b\cdot\mathfrak{d}^{\leq 1}U).
\end{aligned}
$$

Applying once again the commutator, the above becomes

$$
\begin{aligned}
& {}^{(c)}\nabla_3([{}^{(c)}\nabla_3, {}^{(c)}\nabla_a]U_{bc}) \\
={}& -\frac{1}{2}\mathrm{tr}\,\underline{\chi}\Big({}^{(c)}\nabla_a{}^{(c)}\nabla_3 U_{bc} - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\,{}^{(c)}\nabla_a U_{bc} - \frac{1}{2}{}^{(a)}\mathrm{tr}\underline{\chi}\,{}^*{}^{(c)}\nabla_a U_{bc} + \eta_a{}^{(c)}\nabla_3 U_{bc} + \mathcal{V}^s_{[3,a]}(U)\Big) \\
& -\frac{1}{2}{}^{(a)}\mathrm{tr}\underline{\chi}\,{}^*\Big[{}^{(c)}\nabla_a{}^{(c)}\nabla_3 U_{bc} - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\,{}^{(c)}\nabla_a U_{bc} - \frac{1}{2}{}^{(a)}\mathrm{tr}\underline{\chi}\,{}^*{}^{(c)}\nabla_a U_{bc} \\
& + \eta_a{}^{(c)}\nabla_3 U_{bc} + \mathcal{V}^s_{[3,a]}(U)\Big] \\
& + \eta_a{}^{(c)}\nabla_3{}^{(c)}\nabla_3 U_{bc} - \frac{1}{2}({}^{(c)}\nabla_3\mathrm{tr}\,\underline{\chi})\,{}^{(c)}\nabla_a U_{bc} - \frac{1}{2}({}^{(c)}\nabla_3{}^{(a)}\mathrm{tr}\underline{\chi})\,{}^*{}^{(c)}\nabla_a U_{bc} \\
& + ({}^{(c)}\nabla_3\eta_a){}^{(c)}\nabla_3 U_{bc} + {}^{(c)}\nabla_3(\mathcal{V}^s_{[3,a]}(U)) + r^{-1}\mathfrak{d}(\Gamma_b\cdot\mathfrak{d}^{\leq 1}U) \\
={}& -\frac{1}{2}\mathrm{tr}\,\underline{\chi}\,{}^{(c)}\nabla_a{}^{(c)}\nabla_3 U_{bc} - \frac{1}{2}{}^{(a)}\mathrm{tr}\underline{\chi}\,{}^*{}^{(c)}\nabla_a{}^{(c)}\nabla_3 U_{bc} + \eta_a{}^{(c)}\nabla_3{}^{(c)}\nabla_3 U_{bc} \\
& + \Big(\frac{1}{2}\mathrm{tr}\,\underline{\chi}^2 - \frac{1}{2}{}^{(a)}\mathrm{tr}\underline{\chi}^2\Big)\,{}^{(c)}\nabla_a U_{bc} + \big(\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\underline{\chi}\big)\,{}^*{}^{(c)}\nabla_a U_{bc} \\
& + ({}^{(c)}\nabla_3\eta_a - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\eta_a - \frac{1}{2}{}^{(a)}\mathrm{tr}\underline{\chi}\,{}^*\eta_a){}^{(c)}\nabla_3 U_{bc} \\
& + {}^{(c)}\nabla_3(\mathcal{V}^s_{[3,a]}(U)) - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\mathcal{V}^s_{[3,a]}(U) - \frac{1}{2}{}^{(a)}\mathrm{tr}\underline{\chi}\,{}^*\mathcal{V}^s_{[3,a]}(U) + r^{-1}\mathfrak{d}(\Gamma_b\cdot\mathfrak{d}^{\leq 1}U).
\end{aligned}
$$

Putting the above together, we obtain the stated expression.

We compute

$$
\begin{aligned}
[Q, \overline{{}^{(c)}\mathcal{D}}\cdot]U &= {}^{(c)}\nabla_3([{}^{(c)}\nabla_3, \overline{{}^{(c)}\mathcal{D}}\cdot]U) + [{}^{(c)}\nabla_3, \overline{{}^{(c)}\mathcal{D}}\cdot]{}^{(c)}\nabla_3 U + C_1\,[{}^{(c)}\nabla_3, \overline{{}^{(c)}\mathcal{D}}\cdot]U \\
&\quad - \overline{{}^{(c)}\mathcal{D}}(C_1)\cdot{}^{(c)}\nabla_3 U - \overline{{}^{(c)}\mathcal{D}}(C_2)\cdot U.
\end{aligned}
$$

Recall from Lemma 4.2.2,

$$[\,^{(c)}\nabla_3, \overline{^{(c)}\mathcal{D}}\cdot]U = -\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\left(\overline{^{(c)}\mathcal{D}}\cdot U+(s-2)\overline{H}\cdot U\right)+\overline{H}\cdot\,^{(c)}\nabla_3 U+r^{-1}\Gamma_b\cdot\mathfrak{d}^{\leq 1}U.$$

In particular, since $^{(c)}\nabla_3 U$ is conformal of type $s-1$, we have

$$\begin{aligned}[\,^{(c)}\nabla_3, \overline{^{(c)}\mathcal{D}}\cdot]\,^{(c)}\nabla_3 U &= -\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\left(\overline{^{(c)}\mathcal{D}}\cdot\,^{(c)}\nabla_3 U+(s-3)\overline{H}\cdot\,^{(c)}\nabla_3 U\right)+\overline{H}\cdot\,^{(c)}\nabla_3\,^{(c)}\nabla_3 U\\ &\quad +r^{-1}\Gamma_b\cdot\mathfrak{d}^{\leq 2}U.\end{aligned}$$

On the other hand we compute

$$\begin{aligned}&\,^{(c)}\nabla_3([\,^{(c)}\nabla_3, \overline{^{(c)}\mathcal{D}}\cdot]U)\\
=&-\frac{1}{2}\,^{(c)}\nabla_3\overline{\mathrm{tr}\underline{X}}\left(\overline{^{(c)}\mathcal{D}}\cdot U+(s-2)\overline{H}\cdot U\right)\\
&-\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\left(\,^{(c)}\nabla_3\overline{^{(c)}\mathcal{D}}\cdot U+(s-2)\,^{(c)}\nabla_3\overline{H}\cdot U+(s-2)\overline{H}\cdot\,^{(c)}\nabla_3 U\right)\\
&+\,^{(c)}\nabla_3\overline{H}\cdot\,^{(c)}\nabla_3 U+\overline{H}\cdot\,^{(c)}\nabla_3\,^{(c)}\nabla_3 U+r^{-1}\mathfrak{d}(\Gamma_b\cdot\mathfrak{d}^{\leq 1}U)\\
=&\ \frac{1}{2}(\overline{\mathrm{tr}\underline{X}})^2\left(\overline{^{(c)}\mathcal{D}}\cdot U+(s-2)\overline{H}\cdot U\right)\\
&-\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\left(\overline{^{(c)}\mathcal{D}}\cdot\,^{(c)}\nabla_3 U+(s-2)\,^{(c)}\nabla_3\overline{H}\cdot U+(s-1)\overline{H}\cdot\,^{(c)}\nabla_3 U\right)\\
&+\,^{(c)}\nabla_3\overline{H}\cdot\,^{(c)}\nabla_3 U+\overline{H}\cdot\,^{(c)}\nabla_3\,^{(c)}\nabla_3 U+r^{-1}\mathfrak{d}(\Gamma_b\cdot\mathfrak{d}^{\leq 1}U).\end{aligned}$$

Putting the above together we obtain the desired formula.

We compute

$$\begin{aligned}[Q, \,^{(c)}\mathcal{D}\widehat{\otimes}]F &= \,^{(c)}\nabla_3([\,^{(c)}\nabla_3, \,^{(c)}\mathcal{D}\widehat{\otimes}]F)+[\,^{(c)}\nabla_3, \,^{(c)}\mathcal{D}\widehat{\otimes}]\,^{(c)}\nabla_3 F+C_1\,[\,^{(c)}\nabla_3, \,^{(c)}\mathcal{D}\widehat{\otimes}]F\\ &\quad -\,^{(c)}\mathcal{D}(C_1)\widehat{\otimes}\,^{(c)}\nabla_3 F-\,^{(c)}\mathcal{D}(C_2)\widehat{\otimes}F.\end{aligned}$$

Recall from Lemma 4.2.2,

$$[\,^{(c)}\nabla_3, \,^{(c)}\mathcal{D}\widehat{\otimes}]F = -\frac{1}{2}\mathrm{tr}\underline{X}\left(\,^{(c)}\mathcal{D}\widehat{\otimes}F+(s+1)H\widehat{\otimes}F\right)+H\widehat{\otimes}\,^{(c)}\nabla_3 F+r^{-1}\Gamma_b\cdot\mathfrak{d}^{\leq 1}F.$$

In particular, since $^{(c)}\nabla_3 F$ is conformal of type $s-1$, we have

$$\begin{aligned}[\,^{(c)}\nabla_3, \,^{(c)}\mathcal{D}\widehat{\otimes}]\,^{(c)}\nabla_3 F &= -\frac{1}{2}\mathrm{tr}\underline{X}\left(\,^{(c)}\mathcal{D}\widehat{\otimes}\,^{(c)}\nabla_3 F+(s)H\widehat{\otimes}\,^{(c)}\nabla_3 F\right)+H\widehat{\otimes}\,^{(c)}\nabla_3\,^{(c)}\nabla_3 F\\ &\quad +r^{-1}\Gamma_b\cdot\mathfrak{d}^{\leq 2}F.\end{aligned}$$

On the other hand we compute

$$
\begin{aligned}
& {}^{(c)}\nabla_3([{}^{(c)}\nabla_3, {}^{(c)}\mathcal{D}\widehat{\otimes}]F) \\
= \; & -\frac{1}{2} {}^{(c)}\nabla_3 \mathrm{tr}\underline{X} \left({}^{(c)}\mathcal{D}\widehat{\otimes}F + (s+1)H\widehat{\otimes}F \right) \\
& -\frac{1}{2}\mathrm{tr}\underline{X} \left({}^{(c)}\nabla_3 {}^{(c)}\mathcal{D}\widehat{\otimes}F + (s+1) {}^{(c)}\nabla_3 H\widehat{\otimes}F + (s+1)H\widehat{\otimes} {}^{(c)}\nabla_3 F \right) \\
& + {}^{(c)}\nabla_3 H\widehat{\otimes} {}^{(c)}\nabla_3 F + H\widehat{\otimes} {}^{(c)}\nabla_3 {}^{(c)}\nabla_3 F + r^{-1}\mathfrak{d}(\Gamma_b \cdot \mathfrak{d}^{\leq 1}F) \\
= \; & \frac{1}{2}(\mathrm{tr}\underline{X})^2 \left({}^{(c)}\mathcal{D}\widehat{\otimes}F + (s+1)H\widehat{\otimes}F \right) \\
& -\frac{1}{2}\mathrm{tr}\underline{X} \left({}^{(c)}\mathcal{D}\widehat{\otimes} {}^{(c)}\nabla_3 F + (s+1) {}^{(c)}\nabla_3 H\widehat{\otimes}F + (s+2)H\widehat{\otimes} {}^{(c)}\nabla_3 F \right) \\
& + {}^{(c)}\nabla_3 H\widehat{\otimes} {}^{(c)}\nabla_3 F + H\widehat{\otimes} {}^{(c)}\nabla_3 {}^{(c)}\nabla_3 F + r^{-1}\mathfrak{d}(\Gamma_b \cdot \mathfrak{d}^{\leq 1}F).
\end{aligned}
$$

Putting the above together we obtain the desired formula. □

We now compute the commutator between Q and $\mathcal{L}$. Using (5.1.2), we separate the commutator into

$$
[Q, \mathcal{L}]A \;=\; I + J + K + L + M + N
$$

where

$$
\begin{aligned}
I &= -[Q, {}^{(c)}\nabla_4 {}^{(c)}\nabla_3]A, & J &= \frac{1}{4}[Q, {}^{(c)}\mathcal{D}\widehat{\otimes}\overline{{}^{(c)}\mathcal{D}}\cdot]A, \\
K &= \left[Q, \left(-\frac{1}{2}\mathrm{tr}X - 2\overline{\mathrm{tr}X}\right) {}^{(c)}\nabla_3\right] A, & L &= [Q, -\frac{1}{2}\mathrm{tr}\underline{X} {}^{(c)}\nabla_4]A, \\
M &= [Q, \left(4H + \underline{H} + \overline{\underline{H}}\right) \cdot {}^{(c)}\nabla]A, & N &= [Q, \left(-\overline{\mathrm{tr}X}\mathrm{tr}\underline{X} + 2\overline{P}\right)]A + [Q, H\widehat{\otimes}\overline{\underline{H}}\cdot]A.
\end{aligned}
$$

Expression for I

We have

$$
I \;=\; -[Q, {}^{(c)}\nabla_4] {}^{(c)}\nabla_3 A - {}^{(c)}\nabla_4([Q, {}^{(c)}\nabla_3]A).
$$

Using Proposition D.4.2, applied to $U = {}^{(c)}\nabla_3 A$ of conformal type $s=1$, we obtain

$$\begin{aligned}
& [Q,\, {}^{(c)}\nabla_4]\, {}^{(c)}\nabla_3 A \\
= \;& 4(\eta - \underline{\eta}) \cdot {}^{(c)}\nabla\, {}^{(c)}\nabla_3\, {}^{(c)}\nabla_3 A \\
& + \Big(2\, {}^{(c)}\nabla_3(\eta - \underline{\eta}) + (2C_1 - \mathrm{tr}\,\underline{\chi})(\eta - \underline{\eta}) + {}^{(a)}\mathrm{tr}\underline{\chi}\; {}^*(\eta - \underline{\eta}) \Big) \cdot {}^{(c)}\nabla\, {}^{(c)}\nabla_3 A \\
& + \Big(\mathcal{V}^{s=1}_{[3,4]} + \mathcal{V}^{s-1=0}_{[3,4]} + 2\eta \cdot (\eta - \underline{\eta}) - {}^{(c)}\nabla_4(C_1) \Big)\, {}^{(c)}\nabla_3\, {}^{(c)}\nabla_3 A \\
& + \Big({}^{(c)}\nabla_3 \mathcal{V}^{s=1}_{[3,4]} + C_1(\mathcal{V}^{s=1}_{[3,4]}) - {}^{(c)}\nabla_4(C_2) \Big)\, {}^{(c)}\nabla_3 A + 2(\eta - \underline{\eta}) \cdot \mathcal{V}^{s=1}_{[3,a]}({}^{(c)}\nabla_3 A) \\
& + r^{-3}\Gamma_b \cdot \mathfrak{d}^{\leq 2} A + \mathfrak{d}\big((\Gamma_b \cdot \Gamma_g)\mathfrak{d}^{\leq 1} A \big).
\end{aligned}$$

We also deduce

$$\begin{aligned}
{}^{(c)}\nabla_4([Q,\, {}^{(c)}\nabla_3]A) &= {}^{(c)}\nabla_4((-\, {}^{(c)}\nabla_3 C_1)\;\, {}^{(c)}\nabla_3 A + (-\, {}^{(c)}\nabla_3 C_2)\; A) \\
&= (-\, {}^{(c)}\nabla_3 C_1)\;\, {}^{(c)}\nabla_4\, {}^{(c)}\nabla_3 A + (-\, {}^{(c)}\nabla_4\, {}^{(c)}\nabla_3 C_1)\;\, {}^{(c)}\nabla_3 A \\
&\quad + (-\, {}^{(c)}\nabla_3 C_2)\;\, {}^{(c)}\nabla_4 A + (-\, {}^{(c)}\nabla_4\, {}^{(c)}\nabla_3 C_2)\; A.
\end{aligned}$$

We therefore obtain

$$\begin{aligned}
I \;=\;& -4(\eta - \underline{\eta}) \cdot {}^{(c)}\nabla\, {}^{(c)}\nabla_3\, {}^{(c)}\nabla_3 A + I_{43}\, {}^{(c)}\nabla_4\, {}^{(c)}\nabla_3 A + I_{33}\, {}^{(c)}\nabla_3\, {}^{(c)}\nabla_3 A \\
& + I_{a3} \cdot {}^{(c)}\nabla\, {}^{(c)}\nabla_3 A + I_4\, {}^{(c)}\nabla_4 A + I_3\, {}^{(c)}\nabla_3 A + I_0 A \\
& + r^{-3}\Gamma_b \cdot \mathfrak{d}^{\leq 2} A + \mathfrak{d}\big((\Gamma_b \cdot \Gamma_g)\mathfrak{d}^{\leq 1} A \big),
\end{aligned}$$

where

$$\begin{aligned}
I_{43} &= {}^{(c)}\nabla_3 C_1 \\
I_{33} &= -\mathcal{V}^{s=1}_{[3,4]} - \mathcal{V}^{s-1=0}_{[3,4]} - 2\eta \cdot (\eta - \underline{\eta}) + {}^{(c)}\nabla_4(C_1) \\
I_{a3} &= -2\, {}^{(c)}\nabla_3(\eta - \underline{\eta}) - (2C_1 - \mathrm{tr}\,\underline{\chi})(\eta - \underline{\eta}) - {}^{(a)}\mathrm{tr}\underline{\chi}\; {}^*(\eta - \underline{\eta}) \\
I_4 &= {}^{(c)}\nabla_3 C_2 \\
I_3 &= -\, {}^{(c)}\nabla_3 \mathcal{V}^{s=1}_{[3,4]} - C_1(\mathcal{V}^{s=1}_{[3,4]}) + {}^{(c)}\nabla_4(C_2) + {}^{(c)}\nabla_4\, {}^{(c)}\nabla_3 C_1 - 2(\eta - \underline{\eta}) \cdot \mathcal{V}^{s=1}_{[3,a]} \\
I_0 &= {}^{(c)}\nabla_4\, {}^{(c)}\nabla_3 C_2.
\end{aligned}$$

Using (D.4.7), we have

$$\begin{aligned}
\mathcal{V}^{s=1}_{[3,4]} &= 2\rho - 2\eta \cdot \underline{\eta} + i\left(-4\; {}^*\rho + 4\eta \wedge \underline{\eta} \right), \\
\mathcal{V}^{s-1=0}_{[3,4]} &= i\left(-4\; {}^*\rho + 4\eta \wedge \underline{\eta} \right)
\end{aligned}$$

and therefore we can write for I_{33}

$$I_{33} \;=\; -2\rho - 2\eta \cdot (\eta - 2\underline{\eta}) + i\left(8\; {}^*\rho - 8\eta \wedge \underline{\eta} \right) + {}^{(c)}\nabla_4(C_1). \tag{D.4.9}$$

Expression for J

We have

$$J = \frac{1}{4}[Q, {}^{(c)}\mathcal{D}\widehat{\otimes}](\overline{{}^{(c)}\mathcal{D}}\cdot A) + \frac{1}{4}{}^{(c)}\mathcal{D}\widehat{\otimes}([Q, \overline{{}^{(c)}\mathcal{D}}\cdot]A).$$

Using Proposition D.4.2, applied to $F = \overline{{}^{(c)}\mathcal{D}}\cdot A$ of conformal type $s=2$, we obtain

$$\begin{aligned}
[Q, {}^{(c)}\mathcal{D}\widehat{\otimes}](\overline{{}^{(c)}\mathcal{D}}\cdot A) &= 2H\widehat{\otimes}\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_3\overline{{}^{(c)}\mathcal{D}}\cdot A - \mathrm{tr}\underline{X}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\nabla_3\overline{{}^{(c)}\mathcal{D}}\cdot A\\
&+\frac{1}{2}(\mathrm{tr}\underline{X})(\mathrm{tr}\underline{X} - C_1)\,{}^{(c)}\mathcal{D}\widehat{\otimes}\overline{{}^{(c)}\mathcal{D}}\cdot A\\
&+\big({}^{(c)}\nabla_3 H + (-3\mathrm{tr}\underline{X} + C_1)H - {}^{(c)}\mathcal{D}(C_1)\big)\widehat{\otimes}\,{}^{(c)}\nabla_3\overline{{}^{(c)}\mathcal{D}}\cdot A\\
&+\left(\frac{3}{2}\mathrm{tr}\underline{X}\big(-{}^{(c)}\nabla_3 H + (\mathrm{tr}\underline{X} - C_1)H\big) - {}^{(c)}\mathcal{D}(C_2)\right)\widehat{\otimes}\overline{{}^{(c)}\mathcal{D}}\cdot A\\
&+\nabla_3\big(r^{-2}\Gamma_b\cdot \mathfrak{d}^{\leq 2}A\big).
\end{aligned}$$

We write

$$\begin{aligned}
{}^{(c)}\nabla_3\overline{{}^{(c)}\mathcal{D}}\cdot A &= \overline{{}^{(c)}\mathcal{D}}\cdot\,{}^{(c)}\nabla_3 A - \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\left(\overline{{}^{(c)}\mathcal{D}}\cdot A\right) + \overline{H}\cdot\,{}^{(c)}\nabla_3 A + r^{-1}\Gamma_b\cdot\mathfrak{d}^{\leq 1}A,
\end{aligned}$$

$$\begin{aligned}
{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\nabla_3\overline{{}^{(c)}\mathcal{D}}\cdot A &= {}^{(c)}\nabla_3\,{}^{(c)}\mathcal{D}\widehat{\otimes}\overline{{}^{(c)}\mathcal{D}}\cdot A - \big[{}^{(c)}\nabla_3, {}^{(c)}\mathcal{D}\widehat{\otimes}\big]\overline{{}^{(c)}\mathcal{D}}\cdot A\\
&= {}^{(c)}\nabla_3\,{}^{(c)}\mathcal{D}\widehat{\otimes}\overline{{}^{(c)}\mathcal{D}}\cdot A + \frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\overline{{}^{(c)}\mathcal{D}}\cdot A\\
&-H\widehat{\otimes}\,{}^{(c)}\nabla_3\overline{{}^{(c)}\mathcal{D}}\cdot A + \frac{3}{2}\mathrm{tr}\underline{X}H\widehat{\otimes}\overline{{}^{(c)}\mathcal{D}}\cdot A + r^{-2}\Gamma_b\cdot\mathfrak{d}^{\leq 2}A\\
&= {}^{(c)}\nabla_3\,{}^{(c)}\mathcal{D}\widehat{\otimes}\overline{{}^{(c)}\mathcal{D}}\cdot A + \frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\overline{{}^{(c)}\mathcal{D}}\cdot A - H\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot\,{}^{(c)}\nabla_3 A)\\
&-H\widehat{\otimes}(\overline{H}\cdot\,{}^{(c)}\nabla_3 A) + \left(\frac{3}{2}\mathrm{tr}\underline{X} + \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\right)H\widehat{\otimes}\overline{{}^{(c)}\mathcal{D}}\cdot A + r^{-2}\Gamma_b\cdot\mathfrak{d}^{\leq 2}A,
\end{aligned}$$

and

$$\begin{aligned}
{}^{(c)}\nabla_3\,{}^{(c)}\nabla_3\overline{{}^{(c)}\mathcal{D}}\cdot A &= {}^{(c)}\nabla_3\overline{{}^{(c)}\mathcal{D}}\cdot\,{}^{(c)}\nabla_3 A + {}^{(c)}\nabla_3([{}^{(c)}\nabla_3, \overline{{}^{(c)}\mathcal{D}}\cdot]A)\\
&= \overline{{}^{(c)}\mathcal{D}}\cdot\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_3 A + [{}^{(c)}\nabla_3, \overline{{}^{(c)}\mathcal{D}}\cdot]\,{}^{(c)}\nabla_3 A + {}^{(c)}\nabla_3([{}^{(c)}\nabla_3, \overline{{}^{(c)}\mathcal{D}}\cdot]A)\\
&= \overline{{}^{(c)}\mathcal{D}}\cdot\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_3 A\\
&-\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\left(\overline{{}^{(c)}\mathcal{D}}\cdot\,{}^{(c)}\nabla_3 A - \overline{H}\cdot\,{}^{(c)}\nabla_3 A\right) + \overline{H}\cdot\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_3 A\\
&+\frac{1}{2}(\overline{\mathrm{tr}\underline{X}})^2\left(\overline{{}^{(c)}\mathcal{D}}\cdot A\right) - \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\left(\overline{{}^{(c)}\mathcal{D}}\cdot\,{}^{(c)}\nabla_3 A + \overline{H}\cdot\,{}^{(c)}\nabla_3 A\right)\\
&+{}^{(c)}\nabla_3\overline{H}\cdot\,{}^{(c)}\nabla_3 A + \overline{H}\cdot\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_3 A + r^{-1}\Gamma_b\cdot\mathfrak{d}^{\leq 2}A\\
&= \overline{{}^{(c)}\mathcal{D}}\cdot\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_3 A + 2\overline{H}\cdot\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_3 A - \overline{\mathrm{tr}\underline{X}}\,\overline{{}^{(c)}\mathcal{D}}\cdot\,{}^{(c)}\nabla_3 A\\
&+\frac{1}{2}(\overline{\mathrm{tr}\underline{X}})^2\,\overline{{}^{(c)}\mathcal{D}}\cdot A + {}^{(c)}\nabla_3\overline{H}\cdot\,{}^{(c)}\nabla_3 A + r^{-1}\Gamma_b\cdot\mathfrak{d}^{\leq 2}A,
\end{aligned}$$

where we used the intermediate computations in Proposition D.4.2. Putting the above together we obtain

$$\begin{aligned}
&[Q, {}^{(c)}\mathcal{D}\widehat{\otimes}](\overline{{}^{(c)}\mathcal{D}}\cdot A)\\
= \quad & -\mathrm{tr}\underline{X}\,{}^{(c)}\nabla_3\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,\overline{{}^{(c)}\mathcal{D}}\cdot A + 2H\widehat{\otimes}\,\overline{{}^{(c)}\mathcal{D}}\cdot {}^{(c)}\nabla_3\,{}^{(c)}\nabla_3 A - \frac{1}{2}(\mathrm{tr}\underline{X})C_1\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,\overline{{}^{(c)}\mathcal{D}}\cdot A\\
&+4H\widehat{\otimes}(\overline{H}\cdot {}^{(c)}\nabla_3\,{}^{(c)}\nabla_3 A) + 2H\widehat{\otimes}({}^{(c)}\nabla_3\overline{H}\cdot {}^{(c)}\nabla_3 A)\\
&+\Big({}^{(c)}\nabla_3 H + (-2\mathrm{tr}\underline{X} - 2\overline{\mathrm{tr}\underline{X}} + C_1)\,H - {}^{(c)}\mathcal{D}(C_1)\Big)\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot {}^{(c)}\nabla_3 A)\\
&+\Big({}^{(c)}\nabla_3 H + (-2\mathrm{tr}\underline{X} + C_1)H - {}^{(c)}\mathcal{D}(C_1)\Big)\widehat{\otimes}(\overline{H}\cdot {}^{(c)}\nabla_3 A)\\
&+\Bigg[\left(\frac{3}{2}\mathrm{tr}\underline{X} + \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\right)\Big(-{}^{(c)}\nabla_3 H - C_1 H\Big) + \Big(\mathrm{tr}\underline{X}\overline{\mathrm{tr}\underline{X}} + (\overline{\mathrm{tr}\underline{X}})^2\Big)\, H\\
&+\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\,{}^{(c)}\mathcal{D}(C_1) - {}^{(c)}\mathcal{D}(C_2)\Bigg]\widehat{\otimes}\,\overline{{}^{(c)}\mathcal{D}}\cdot A + \nabla_3\big(r^{-2}\Gamma_b\cdot\mathfrak{d}^{\leq 2}A\big).
\end{aligned}$$

Using Lemma 2.4.6, the above becomes

$$\begin{aligned}
&[Q, {}^{(c)}\mathcal{D}\widehat{\otimes}](\overline{{}^{(c)}\mathcal{D}}\cdot A)\\
= \quad & -\mathrm{tr}\underline{X}\,{}^{(c)}\nabla_3\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,\overline{{}^{(c)}\mathcal{D}}\cdot A + 8H\cdot {}^{(c)}\nabla\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_3 A\\
&-\frac{1}{2}(\mathrm{tr}\underline{X})C_1\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,\overline{{}^{(c)}\mathcal{D}}\cdot A + 8(H\cdot\overline{H})\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_3 A\\
&+4\Big({}^{(c)}\nabla_3 H + (-2\mathrm{tr}\underline{X} - 2\overline{\mathrm{tr}\underline{X}} + C_1)\,H - {}^{(c)}\mathcal{D}(C_1)\Big)\cdot {}^{(c)}\nabla\,{}^{(c)}\nabla_3 A\\
&+2H\widehat{\otimes}({}^{(c)}\nabla_3\overline{H}\cdot {}^{(c)}\nabla_3 A)\\
&+\Big({}^{(c)}\nabla_3 H + (-2\mathrm{tr}\underline{X} + C_1)H - {}^{(c)}\mathcal{D}(C_1)\Big)\widehat{\otimes}(\overline{H}\cdot {}^{(c)}\nabla_3 A)\\
&+4\Bigg[\left(\frac{3}{2}\mathrm{tr}\underline{X} + \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\right)\Big(-{}^{(c)}\nabla_3 H - C_1 H\Big) + \Big(\mathrm{tr}\underline{X}\overline{\mathrm{tr}\underline{X}} + (\overline{\mathrm{tr}\underline{X}})^2\Big)\, H\\
&+\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\,{}^{(c)}\mathcal{D}(C_1) - {}^{(c)}\mathcal{D}(C_2)\Bigg]\cdot {}^{(c)}\nabla A + \nabla_3\big(r^{-2}\Gamma_b\cdot\mathfrak{d}^{\leq 2}A\big).
\end{aligned}$$

Using Proposition D.4.2, we deduce

$$\begin{aligned}
&{}^{(c)}\mathcal{D}\widehat{\otimes}([Q, \overline{{}^{(c)}\mathcal{D}}\cdot]A)\\
= \quad & {}^{(c)}\mathcal{D}\widehat{\otimes}\Bigg[2\overline{H}\cdot {}^{(c)}\nabla_3\,{}^{(c)}\nabla_3 A - \overline{\mathrm{tr}\underline{X}}\,\overline{{}^{(c)}\mathcal{D}}\cdot {}^{(c)}\nabla_3 A\\
&+\frac{1}{2}\overline{\mathrm{tr}\underline{X}}(\mathrm{tr}\underline{X} - C_1)\,\overline{{}^{(c)}\mathcal{D}}\cdot A + \Big({}^{(c)}\nabla_3\overline{H} + C_1\overline{H} - \overline{{}^{(c)}\mathcal{D}}(C_1)\Big)\cdot {}^{(c)}\nabla_3 A\\
&+\Big(-\overline{{}^{(c)}\mathcal{D}}(C_2)\Big)\cdot A + \nabla_3\big(r^{-2}\Gamma_b\cdot\mathfrak{d}^{\leq 1}A\big)\Bigg],
\end{aligned}$$

which gives

$$\begin{aligned}
{}^{(c)}\mathcal{D}\widehat{\otimes}([Q,\overline{{}^{(c)}\mathcal{D}}\cdot]A) &= 2\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{H}\cdot{}^{(c)}\nabla_3{}^{(c)}\nabla_3 A)-\overline{\mathrm{tr}\underline{X}}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\overline{{}^{(c)}\mathcal{D}}\cdot{}^{(c)}\nabla_3 A\\
&-{}^{(c)}\mathcal{D}\overline{\mathrm{tr}\underline{X}}\,\widehat{\otimes}\,\overline{{}^{(c)}\mathcal{D}}\cdot{}^{(c)}\nabla_3 A+\frac{1}{2}\overline{\mathrm{tr}\underline{X}}(\mathrm{tr}\underline{X}-C_1)\,{}^{(c)}\mathcal{D}\widehat{\otimes}\overline{{}^{(c)}\mathcal{D}}\cdot A\\
&+{}^{(c)}\mathcal{D}(\frac{1}{2}\overline{\mathrm{tr}\underline{X}}(\mathrm{tr}\underline{X}-C_1))\widehat{\otimes}\,\overline{{}^{(c)}\mathcal{D}}\cdot A\\
&+{}^{(c)}\mathcal{D}\widehat{\otimes}\Big[\Big({}^{(c)}\nabla_3\overline{H}+C_1\overline{H}-\overline{{}^{(c)}\mathcal{D}}(C_1)\Big)\cdot{}^{(c)}\nabla_3 A\Big]\\
&-{}^{(c)}\mathcal{D}\widehat{\otimes}\Big(\overline{{}^{(c)}\mathcal{D}}(C_2)\cdot A\Big)+\nabla_3\big(r^{-2}\Gamma_b\cdot\mathfrak{d}^{\leq 2}A\big).
\end{aligned}$$

Writing

$$\begin{aligned}
{}^{(c)}\mathcal{D}\widehat{\otimes}\overline{{}^{(c)}\mathcal{D}}\cdot{}^{(c)}\nabla_3 A &= {}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\nabla_3\overline{{}^{(c)}\mathcal{D}}\cdot A-{}^{(c)}\mathcal{D}\widehat{\otimes}[{}^{(c)}\nabla_3,\overline{{}^{(c)}\mathcal{D}}\cdot]A\\
&= {}^{(c)}\nabla_3\,{}^{(c)}\mathcal{D}\widehat{\otimes}\overline{{}^{(c)}\mathcal{D}}\cdot A+\frac{1}{2}(\mathrm{tr}\underline{X}+\overline{\mathrm{tr}\underline{X}})\,{}^{(c)}\mathcal{D}\widehat{\otimes}\overline{{}^{(c)}\mathcal{D}}\cdot A\\
&-H\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot{}^{(c)}\nabla_3 A)-H\widehat{\otimes}(\overline{H}\cdot{}^{(c)}\nabla_3 A)\\
&+\left(\frac{3}{2}\mathrm{tr}\underline{X}+\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\right)H\widehat{\otimes}\overline{{}^{(c)}\mathcal{D}}\cdot A+\frac{1}{2}\,{}^{(c)}\mathcal{D}(\overline{\mathrm{tr}\underline{X}})\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A)\\
&-{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{H}\cdot{}^{(c)}\nabla_3 A)+\nabla_3\big(r^{-2}\Gamma_b\cdot\mathfrak{d}^{\leq 2}A\big),
\end{aligned}$$

we have

$$\begin{aligned}
{}^{(c)}\mathcal{D}\widehat{\otimes}([Q,\overline{{}^{(c)}\mathcal{D}}\cdot]A) &= -\overline{\mathrm{tr}\underline{X}}\,{}^{(c)}\nabla_3\,{}^{(c)}\mathcal{D}\widehat{\otimes}\overline{{}^{(c)}\mathcal{D}}\cdot A+\frac{1}{2}\overline{\mathrm{tr}\underline{X}}(-\mathrm{tr}\underline{X}-C_1)\,{}^{(c)}\mathcal{D}\widehat{\otimes}\overline{{}^{(c)}\mathcal{D}}\cdot A\\
&+2\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{H}\cdot{}^{(c)}\nabla_3\,{}^{(c)}\nabla_3 A)+\overline{\mathrm{tr}\underline{X}}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{H}\cdot{}^{(c)}\nabla_3 A)\\
&+\overline{\mathrm{tr}\underline{X}}H\widehat{\otimes}(\overline{H}\cdot{}^{(c)}\nabla_3 A)+\big(\overline{\mathrm{tr}\underline{X}}H-{}^{(c)}\mathcal{D}\overline{\mathrm{tr}\underline{X}}\big)\,\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot{}^{(c)}\nabla_3 A)\\
+\quad&\Big[\left(-\frac{3}{2}\mathrm{tr}\underline{X}\overline{\mathrm{tr}\underline{X}}-\frac{1}{2}\overline{\mathrm{tr}\underline{X}}^2\right)H+\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\,{}^{(c)}\mathcal{D}(\overline{\mathrm{tr}\underline{X}})\\
&-\frac{1}{2}\,{}^{(c)}\mathcal{D}(\overline{\mathrm{tr}\underline{X}}C_1)\Big]\widehat{\otimes}\overline{{}^{(c)}\mathcal{D}}\cdot A\\
&+{}^{(c)}\mathcal{D}\widehat{\otimes}(({}^{(c)}\nabla_3\overline{H}+C_1\overline{H}-\overline{{}^{(c)}\mathcal{D}}(C_1))\cdot{}^{(c)}\nabla_3 A)\\
&-{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}(C_2)\cdot A)+\nabla_3\big(r^{-2}\Gamma_b\cdot\mathfrak{d}^{\leq 2}A\big).
\end{aligned}$$

Using Lemma 2.4.6, the above simplifies to

$$\begin{aligned}
& {}^{(c)}\mathcal{D}\widehat{\otimes}([Q, \overline{{}^{(c)}\mathcal{D}}\cdot]A) \\
= & -\overline{\mathrm{tr}\underline{X}}\, {}^{(c)}\nabla_3\, {}^{(c)}\mathcal{D}\widehat{\otimes}\,\overline{{}^{(c)}\mathcal{D}}\cdot A + 8\overline{H}\cdot {}^{(c)}\nabla\, {}^{(c)}\nabla_3\, {}^{(c)}\nabla_3 A \\
& +\frac{1}{2}\overline{\mathrm{tr}\underline{X}}(-\mathrm{tr}\underline{X} - C_1)\, {}^{(c)}\mathcal{D}\widehat{\otimes}\,\overline{{}^{(c)}\mathcal{D}}\cdot A + 4({}^{(c)}\mathcal{D}\cdot\overline{H})\, {}^{(c)}\nabla_3\, {}^{(c)}\nabla_3 A \\
& +2\left(4\overline{\mathrm{tr}\underline{X}}\eta - 2\, {}^{(c)}\mathcal{D}\overline{\mathrm{tr}\underline{X}} + 2({}^{(c)}\nabla_3\overline{H} + C_1\overline{H} - \overline{{}^{(c)}\mathcal{D}}(C_1))\right)\cdot {}^{(c)}\nabla\, {}^{(c)}\nabla_3 A \\
& +2\left(\overline{\mathrm{tr}\underline{X}}({}^{(c)}\mathcal{D}\cdot\overline{H} + H\cdot\overline{H} + {}^{(c)}\mathcal{D}\cdot({}^{(c)}\nabla_3\overline{H} + C_1\overline{H} - \overline{{}^{(c)}\mathcal{D}}(C_1))\right)\, {}^{(c)}\nabla_3 A \\
& +4\Bigg[\left(-\frac{3}{2}\mathrm{tr}\underline{X}\overline{\mathrm{tr}\underline{X}} - \frac{1}{2}\overline{\mathrm{tr}\underline{X}}^2\right)H + \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\, {}^{(c)}\mathcal{D}(\overline{\mathrm{tr}\underline{X}}) \\
& -\frac{1}{2}\, {}^{(c)}\mathcal{D}(\overline{\mathrm{tr}\underline{X}}C_1) - \overline{{}^{(c)}\mathcal{D}}(C_2)\Bigg]\cdot {}^{(c)}\nabla A - 2({}^{(c)}\mathcal{D}\cdot\overline{{}^{(c)}\mathcal{D}}(C_2))A + \nabla_3\big(r^{-2}\Gamma_b\cdot\mathfrak{d}^{\leq 2}A\big).
\end{aligned}$$

Putting the above together we finally obtain

$$\begin{aligned}
J = & -\big(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\big)\, {}^{(c)}\nabla_3\left(\frac{1}{4}\, {}^{(c)}\mathcal{D}\widehat{\otimes}\,\overline{{}^{(c)}\mathcal{D}}\cdot A\right) \\
& +\left(-\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\mathrm{tr}\underline{X} - \frac{1}{2}(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}})C_1\right)\left(\frac{1}{4}\, {}^{(c)}\mathcal{D}\widehat{\otimes}\,\overline{{}^{(c)}\mathcal{D}}\cdot A\right) \\
& +4\eta\cdot {}^{(c)}\nabla\, {}^{(c)}\nabla_3\, {}^{(c)}\nabla_3 A + \tilde{J}_{33}\, {}^{(c)}\nabla_3\, {}^{(c)}\nabla_3 A + \tilde{J}_{a3}\cdot {}^{(c)}\nabla\, {}^{(c)}\nabla_3 A \\
& +\tilde{J}_3\, {}^{(c)}\nabla_3 A + \tilde{J}_a\cdot {}^{(c)}\nabla A + \tilde{J}_0 A + \nabla_3\big(r^{-2}\Gamma_b\cdot\mathfrak{d}^{\leq 2}A\big),
\end{aligned}$$

where

$$\begin{aligned}
\tilde{J}_{33} = & {}^{(c)}\mathcal{D}\cdot\overline{H} + 2H\cdot\overline{H}, \\
\tilde{J}_{a3} = & {}^{(c)}\nabla_3 H + (-2\mathrm{tr}\underline{X} - 2\overline{\mathrm{tr}\underline{X}} + \mathbf{C})\,H - {}^{(c)}\mathcal{D}(C_1) \\
& +2\overline{\mathrm{tr}\underline{X}}\eta - {}^{(c)}\mathcal{D}\overline{\mathrm{tr}\underline{X}} + {}^{(c)}\nabla_3\overline{H} + C_1\overline{H} - \overline{{}^{(c)}\mathcal{D}}(C_1), \\
\tilde{J}_3 = & \frac{1}{2}\overline{\mathrm{tr}\underline{X}}({}^{(c)}\mathcal{D}\cdot\overline{H} + H\cdot\overline{H}) + \frac{1}{2}\, {}^{(c)}\mathcal{D}\cdot({}^{(c)}\nabla_3\overline{H} + C_1\overline{H} - \overline{{}^{(c)}\mathcal{D}}(C_1)) \\
& +H\widehat{\otimes}({}^{(c)}\nabla_3\overline{H}\cdot) + \frac{1}{2}\big({}^{(c)}\nabla_3 H + (-2\mathrm{tr}\underline{X} + C_1)H - {}^{(c)}\mathcal{D}(C_1)\big)\widehat{\otimes}(\overline{H}\cdot), \\
\tilde{J}_a = & \left(\frac{3}{2}\mathrm{tr}\underline{X} + \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\right)\big(-{}^{(c)}\nabla_3 H - C_1 H\big) + \big(\mathrm{tr}\underline{X}\overline{\mathrm{tr}\underline{X}} + (\overline{\mathrm{tr}\underline{X}})^2\big)\,H \\
& +\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\, {}^{(c)}\mathcal{D}(C_1) - {}^{(c)}\mathcal{D}(C_2) + \left(-\frac{3}{2}\mathrm{tr}\underline{X}\overline{\mathrm{tr}\underline{X}} - \frac{1}{2}\overline{\mathrm{tr}\underline{X}}^2\right)H \\
& +\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\, {}^{(c)}\mathcal{D}(\overline{\mathrm{tr}\underline{X}}) - \frac{1}{2}\, {}^{(c)}\mathcal{D}(\overline{\mathrm{tr}\underline{X}}C_1) - \overline{{}^{(c)}\mathcal{D}}(C_2), \\
\tilde{J}_0 = & -\frac{1}{2}({}^{(c)}\mathcal{D}\cdot\overline{{}^{(c)}\mathcal{D}}(C_2)).
\end{aligned}$$

We can simplify $\tilde{J}_{a3}$ as

$$\begin{aligned}
\tilde{J}_{a3} &= {}^{(c)}\nabla_3 H + (-2\mathrm{tr}\underline{X} - 2\overline{\mathrm{tr}\underline{X}} + C_1)\, H - {}^{(c)}\mathcal{D}(C_1) \\
&\quad +2\overline{\mathrm{tr}\underline{X}}\eta - {}^{(c)}\mathcal{D}\mathrm{tr}\underline{X} + {}^{(c)}\nabla_3 \overline{H} + C_1\overline{H} - \overline{{}^{(c)}\mathcal{D}}(C_1) \\
&= 2\,{}^{(c)}\nabla_3\eta - 2\,{}^{(c)}\nabla(C_1) + 2C_1\eta - {}^{(c)}\mathcal{D}\mathrm{tr}\underline{X} + (-2\mathrm{tr}\underline{X} - \overline{\mathrm{tr}\underline{X}})\, H + \overline{\mathrm{tr}\underline{X}H} \\
&= 2\,{}^{(c)}\nabla_3\eta - 2\,{}^{(c)}\nabla(C_1) + 2C_1\eta - (\mathrm{tr}\underline{X} - \overline{\mathrm{tr}\underline{X}})\,\underline{H} + (-2\mathrm{tr}\underline{X} - \overline{\mathrm{tr}\underline{X}})\, H + \overline{\mathrm{tr}\underline{X}H}.
\end{aligned}$$

Recalling the definition (5.1.2) of $\mathcal{L}(A)$, we write

$$\begin{aligned}
\frac{1}{4}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A) &= {}^{(c)}\nabla_4\,{}^{(c)}\nabla_3 A + \left(\frac{1}{2}\mathrm{tr}X + 2\overline{\mathrm{tr}X}\right){}^{(c)}\nabla_3 A + \frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(c)}\nabla_4 A \\
&\quad - \big(4H + \underline{H} + \overline{\underline{H}}\big)\cdot{}^{(c)}\nabla A + \big(\overline{\mathrm{tr}X}\mathrm{tr}\underline{X} - 2\overline{P} - 4\eta\cdot\underline{\eta} + 4i\eta\wedge\underline{\eta}\big)\, A
\end{aligned}$$

which gives

$$\begin{aligned}
&{}^{(c)}\nabla_3\left(\frac{1}{4}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A)\right) \\
&= {}^{(c)}\nabla_3\,{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3 A + \left(\frac{1}{2}\mathrm{tr}X + 2\overline{\mathrm{tr}X}\right){}^{(c)}\nabla_3\,{}^{(c)}\nabla_3 A \\
&\quad + {}^{(c)}\nabla_3\left(\frac{1}{2}\mathrm{tr}X + 2\overline{\mathrm{tr}X}\right){}^{(c)}\nabla_3 A + \frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_4 A + \frac{1}{2}\,{}^{(c)}\nabla_3\mathrm{tr}\underline{X}\,{}^{(c)}\nabla_4 A \\
&\quad - \big(4H + \underline{H} + \overline{\underline{H}}\big)\cdot{}^{(c)}\nabla_3\,{}^{(c)}\nabla A - {}^{(c)}\nabla_3\big(4H + \underline{H} + \overline{\underline{H}}\big)\cdot{}^{(c)}\nabla A \\
&\quad + \big(\overline{\mathrm{tr}X}\mathrm{tr}\underline{X} - 2\overline{P} - 4\eta\cdot\underline{\eta} + 4i\eta\wedge\underline{\eta}\big)\,{}^{(c)}\nabla_3 A \\
&\quad + {}^{(c)}\nabla_3\big(\overline{\mathrm{tr}X}\mathrm{tr}\underline{X} - 2\overline{P} - 4\eta\cdot\underline{\eta} + 4i\eta\wedge\underline{\eta}\big)\, A.
\end{aligned}$$

Writing

$$\begin{aligned}
{}^{(c)}\nabla_3\,{}^{(c)}\nabla_4 A &= {}^{(c)}\nabla_4\,{}^{(c)}\nabla_3 A + 2(\eta - \underline{\eta})\cdot{}^{(c)}\nabla A + (\mathcal{V}^{s=2}_{[3,4]})A, \\
{}^{(c)}\nabla_3\,{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3 A &= {}^{(c)}\nabla_4\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_3 A + 2(\eta - \underline{\eta})\cdot{}^{(c)}\nabla\,{}^{(c)}\nabla_3 A + (\mathcal{V}^{s=1}_{[3,4]})\,{}^{(c)}\nabla_3 A, \\
{}^{(c)}\nabla_3\,{}^{(c)}\nabla A &= {}^{(c)}\nabla\,{}^{(c)}\nabla_3 A - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\,{}^{(c)}\nabla A - \frac{1}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi}\;{}^{*\,(c)}\nabla A + \eta\,{}^{(c)}\nabla_3 A + \mathcal{V}^{s=2}_{[3,a]}(A),
\end{aligned}$$

we have

$$\begin{aligned}
{}^{(c)}\nabla_3\left(\frac{1}{4}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A)\right) &= {}^{(c)}\nabla_4\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_3 A + \frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3 A - \frac{1}{4}\mathrm{tr}\underline{X}^2\,{}^{(c)}\nabla_4 A \\
&\quad + \hat{J}_{a3}\cdot{}^{(c)}\nabla\,{}^{(c)}\nabla_3 A + \hat{J}_{33}\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_3 A + \hat{J}_3\,{}^{(c)}\nabla_3 A + \hat{J}^a_3(A) \\
&\quad + \hat{J}_a\cdot{}^{(c)}\nabla A + \hat{J}_{*a}\cdot{}^{*\,(c)}\nabla A + \hat{J}_0 A + \hat{J}^a_0(A)
\end{aligned}$$

where

$$
\begin{aligned}
\hat{J}_{a3} &= 2(\eta-\underline{\eta}) - \left(4H + \underline{H} + \overline{\underline{H}}\right),\\
\hat{J}_{33} &= \frac{1}{2}\mathrm{tr}X + 2\overline{\mathrm{tr}X},\\
\hat{J}_{3} &= {}^{(c)}\nabla_3\left(\frac{1}{2}\mathrm{tr}X + 2\overline{\mathrm{tr}X}\right) + \overline{\mathrm{tr}X}\mathrm{tr}\underline{X} - P + \overline{P} - 2\eta\cdot\underline{\eta} - \left(4H + \underline{H} + \overline{\underline{H}}\right)\cdot\eta,\\
\hat{J}^a_{3}(A) &= -4\eta\widehat{\otimes}(\underline{\eta}\cdot {}^{(c)}\nabla_3 A) + 4\underline{\eta}\widehat{\otimes}(\eta\cdot {}^{(c)}\nabla_3 A) - 2H\widehat{\otimes}(\overline{\underline{H}}\cdot {}^{(c)}\nabla_3 A),\\
\hat{J}_{a} &= \mathrm{tr}\underline{X}(\eta-\underline{\eta}) - {}^{(c)}\nabla_3\left(4H + \underline{H} + \overline{\underline{H}}\right) + \frac{1}{2}\mathrm{tr}\,\underline{\chi}\left(4H + \underline{H} + \overline{\underline{H}}\right),\\
\hat{J}_{*a} &= \frac{1}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi}\left(4H + \underline{H} + \overline{\underline{H}}\right),\\
\hat{J}_{0} &= {}^{(c)}\nabla_3\left(\overline{\mathrm{tr}X}\mathrm{tr}\underline{X} - 2\overline{P}\right) + \frac{1}{2}\mathrm{tr}\underline{X}(4\overline{P} - 4\eta\cdot\underline{\eta})\\
&\quad +(\mathrm{tr}\,\underline{\chi}\left(4H + \underline{H} + \overline{\underline{H}}\right) - {}^{(a)}\mathrm{tr}\underline{\chi}\,{}^*\left(4H + \underline{H} + \overline{\underline{H}}\right))\cdot\eta,\\
\hat{J}^a_{0}(A) &= \frac{1}{2}\mathrm{tr}\,\underline{\chi}\Big(\eta\widehat{\otimes}((4H + \underline{H} + \overline{\underline{H}})\cdot A) - \left(4H + \underline{H} + \overline{\underline{H}}\right)\widehat{\otimes}(\eta\cdot A)\Big)\\
&\quad +\frac{1}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi}\Big(-\eta\widehat{\otimes}({}^*\left(4H + \underline{H} + \overline{\underline{H}}\right)\cdot A) + {}^*\left(4H + \underline{H} + \overline{\underline{H}}\right)\widehat{\otimes}(\eta\cdot A)\Big)\\
&\quad -2\,{}^{(c)}\nabla_3 H\widehat{\otimes}(\overline{\underline{H}}\cdot A) - 2H\widehat{\otimes}({}^{(c)}\nabla_3\overline{\underline{H}}\cdot A) + \frac{1}{2}\mathrm{tr}\underline{X}(-4\eta\widehat{\otimes}(\underline{\eta}\cdot A) + 4\underline{\eta}\widehat{\otimes}(\eta\cdot A)).
\end{aligned}
$$

We therefore finally obtain

$$
\begin{aligned}
J &= -\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right)\,{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_3 A + 4\eta\cdot\,{}^{(c)}\nabla\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_3 A\\
&\quad +J_{43}\,{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3 A + J_4\,{}^{(c)}\nabla_4 A + J_{a3}\cdot\,{}^{(c)}\nabla\,{}^{(c)}\nabla_3 A + J_{33}\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_3 A\\
&\quad +J_3\,{}^{(c)}\nabla_3 A + J^a_3(A) + J_a\cdot\,{}^{(c)}\nabla A + J_{*a}\cdot\,{}^{*\,(c)}\nabla A + J_0 A + J^a_0(A)\\
&\quad +\nabla_3\big(r^{-2}\Gamma_b\cdot\mathfrak{d}^{\leq 2}A\big),
\end{aligned}
$$

where

$$
\begin{aligned}
J_{43} &= -\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\mathrm{tr}\underline{X} - \frac{C_1 + \mathrm{tr}\underline{X}}{2}(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}),\\
J_4 &= \frac{1}{4}(\mathrm{tr}\underline{X})^2\,(\mathrm{tr}\underline{X}) - \frac{1}{4}\mathrm{tr}\underline{X}(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}})C_1,
\end{aligned}
$$

and

$$
\begin{aligned}
J_{a3} &= -\left(\mathrm{tr}\underline{X}+\overline{\mathrm{tr}\underline{X}}\right)\hat{J}_{a3}+\tilde{J}_{a3},\\
J_{33} &= -\left(\mathrm{tr}\underline{X}+\overline{\mathrm{tr}\underline{X}}\right)\hat{J}_{33}+\tilde{J}_{33},\\
J_{3} &= -\left(\mathrm{tr}\underline{X}+\overline{\mathrm{tr}\underline{X}}\right)\hat{J}_{3}+\left(-\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\mathrm{tr}\underline{X}-\frac{1}{2}C_1(\mathrm{tr}\underline{X}+\overline{\mathrm{tr}\underline{X}})\right)\left(\frac{1}{2}\mathrm{tr}X+2\overline{\mathrm{tr}X}\right)+\tilde{J}_{3},\\
J_3^a(A) &= -\left(\mathrm{tr}\underline{X}+\overline{\mathrm{tr}\underline{X}}\right)(\hat{J}_3^a(A)+\tilde{J}_3^a(A),\\
J_{a} &= -\left(\mathrm{tr}\underline{X}+\overline{\mathrm{tr}\underline{X}}\right)\hat{J}_{a}-\left(-\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\mathrm{tr}\underline{X}-\frac{1}{2}C_1(\mathrm{tr}\underline{X}+\overline{\mathrm{tr}\underline{X}})\right)\left(4H+\underline{H}+\overline{\underline{H}}\right)+\tilde{J}_{a},\\
J_{\,{}^*a} &= -\left(\mathrm{tr}\underline{X}+\overline{\mathrm{tr}\underline{X}}\right)\hat{J}_{\,{}^*a},\\
J_{0} &= -\left(\mathrm{tr}\underline{X}+\overline{\mathrm{tr}\underline{X}}\right)\hat{J}_{0}+\left(-\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\mathrm{tr}\underline{X}-\frac{1}{2}C_1(\mathrm{tr}\underline{X}+\overline{\mathrm{tr}\underline{X}})\right)\left(\overline{\mathrm{tr}X}\mathrm{tr}\underline{X}-2\overline{P}\right)+\tilde{J}_{0},\\
J_0^a(A) &= -\left(\mathrm{tr}\underline{X}+\overline{\mathrm{tr}\underline{X}}\right)\hat{J}_0^a(A)-\left(-\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\mathrm{tr}\underline{X}-\frac{1}{2}C_1(\mathrm{tr}\underline{X}+\overline{\mathrm{tr}\underline{X}})\right)(2H\widehat{\otimes}(\overline{\underline{H}}\cdot A)).
\end{aligned}
$$

Using the above computation for $\tilde{J}_{a3}$, we simplify

$$
\begin{aligned}
J_{a3} &= -2\mathrm{tr}\,\underline{\chi}(2(\eta-\underline{\eta})-\left(4(\eta+i\,{}^*\eta)+2\underline{\eta}\right))\\
&\quad +2\,{}^{(c)}\nabla_3\eta-4\mathrm{tr}\,\underline{\chi}\,H-2\,{}^{(c)}\nabla(C_1)+2\overline{\mathrm{tr}\underline{X}}\eta-{}^{(c)}\mathcal{D}\overline{\mathrm{tr}\underline{X}}+2C_1\eta\\
&= -4\mathrm{tr}\,\underline{\chi}(\eta-\underline{\eta})+2\mathrm{tr}\,\underline{\chi}\left(4(\eta+i\,{}^*\eta)+2\underline{\eta}\right)\\
&\quad +2\,{}^{(c)}\nabla_3\eta-4\mathrm{tr}\,\underline{\chi}\,(\eta+i\,{}^*\eta)-2\,{}^{(c)}\nabla(C_1)\\
&\quad +2(\mathrm{tr}\,\underline{\chi}+i\,{}^{(a)}\mathrm{tr}\underline{\chi})\eta+2i\,{}^{(a)}\mathrm{tr}\underline{\chi}(\underline{\eta}+i\,{}^*\underline{\eta})+2C_1\eta\\
&= 2\,{}^{(c)}\nabla_3\eta-2\,{}^{(c)}\nabla(C_1)+2C_1\eta+\mathrm{tr}\,\underline{\chi}(8\underline{\eta}+2\eta)-2\,{}^{(a)}\mathrm{tr}\underline{\chi}\,{}^*\underline{\eta}\\
&\quad +i\left(4\mathrm{tr}\,\underline{\chi}\,{}^*\eta+2\,{}^{(a)}\mathrm{tr}\underline{\chi}\eta+2\,{}^{(a)}\mathrm{tr}\underline{\chi}\underline{\eta}\right)
\end{aligned}
$$

Also, we simplify

$$
J_{33} = -2\mathrm{tr}\,\underline{\chi}\left(\frac{1}{2}\mathrm{tr}X+2\overline{\mathrm{tr}X}\right)+{}^{(c)}\mathcal{D}\cdot\overline{\underline{H}}+2H\cdot\overline{\underline{H}}. \tag{D.4.10}
$$

Expression for K

Observe that

$$
\begin{aligned}
Q(fU) &= Q(f)U+fQ(U)+2\,{}^{(c)}\nabla_3 f\,{}^{(c)}\nabla_3 U-C_2 fU\\
&= fQ(U)+2\,{}^{(c)}\nabla_3 f\,{}^{(c)}\nabla_3 U+\left({}^{(c)}\nabla_3\,{}^{(c)}\nabla_3 f+C_1\,{}^{(c)}\nabla_3 f\right)U.
\end{aligned} \tag{D.4.11}
$$

Using Proposition D.4.2, we obtain

$$\begin{aligned}
& [Q, \mathcal{F}\ {}^{(c)}\nabla_3]A \\
= \ & \left({}^{(c)}\nabla_3\, {}^{(c)}\nabla_3 \mathcal{F} + C_1\, {}^{(c)}\nabla_3 \mathcal{F}\right)\, {}^{(c)}\nabla_3 A + \mathcal{F}[Q,\, {}^{(c)}\nabla_3]A + 2\, {}^{(c)}\nabla_3 \mathcal{F}\, {}^{(c)}\nabla_3\, {}^{(c)}\nabla_3 A \\
= \ & \left(2\, {}^{(c)}\nabla_3 \mathcal{F}\right)\, {}^{(c)}\nabla_3\, {}^{(c)}\nabla_3 A + \left({}^{(c)}\nabla_3\, {}^{(c)}\nabla_3 \mathcal{F} + C_1\, {}^{(c)}\nabla_3 \mathcal{F} - \mathcal{F}\, {}^{(c)}\nabla_3 C_1\right)\, {}^{(c)}\nabla_3 A \\
& +(-\mathcal{F}\, {}^{(c)}\nabla_3 C_2)A.
\end{aligned}$$

In particular,

$$K = [Q, \mathcal{F}\ \nabla_3^{(c)}]A \qquad \text{for} \qquad \mathcal{F} = -\frac{1}{2}\mathrm{tr}X - 2\overline{\mathrm{tr}X}.$$

We therefore obtain

$$K = K_{33}\, {}^{(c)}\nabla_3\, {}^{(c)}\nabla_3 A + K_3\, {}^{(c)}\nabla_3 A + K_0 A,$$

where

$$\begin{aligned}
K_{33} &= 2\, {}^{(c)}\nabla_3 \left(-\frac{1}{2}\mathrm{tr}X - 2\overline{\mathrm{tr}X}\right), \qquad \text{(D.4.12)}\\
K_3 &= {}^{(c)}\nabla_3\, {}^{(c)}\nabla_3 \left(-\frac{1}{2}\mathrm{tr}X - 2\overline{\mathrm{tr}X}\right) + C_1\, {}^{(c)}\nabla_3 \left(-\frac{1}{2}\mathrm{tr}X - 2\overline{\mathrm{tr}X}\right) \\
& \quad - \left(-\frac{1}{2}\mathrm{tr}X - 2\overline{\mathrm{tr}X}\right)\, {}^{(c)}\nabla_3 C_1, \\
K_0 &= -\left(-\frac{1}{2}\mathrm{tr}X - 2\overline{\mathrm{tr}X}\right)\, {}^{(c)}\nabla_3 C_2.
\end{aligned}$$

Expression for L

Using (D.4.11), we obtain for a scalar $\mathcal{E}$,

$$\begin{aligned}
& [Q, \mathcal{E}\ {}^{(c)}\nabla_4]A \\
= \ & \left({}^{(c)}\nabla_3\, {}^{(c)}\nabla_3 \mathcal{E} + C_1\, {}^{(c)}\nabla_3 \mathcal{E}\right)\, {}^{(c)}\nabla_4 A + \mathcal{E}[Q,\, {}^{(c)}\nabla_4]A + 2\, {}^{(c)}\nabla_3 \mathcal{E}\, {}^{(c)}\nabla_3\, {}^{(c)}\nabla_4 A \\
= \ & \left({}^{(c)}\nabla_3\, {}^{(c)}\nabla_3 \mathcal{E} + C_1\, {}^{(c)}\nabla_3 \mathcal{E}\right)\, {}^{(c)}\nabla_4 A + \mathcal{E}[Q,\, {}^{(c)}\nabla_4]A + 2\, {}^{(c)}\nabla_3 \mathcal{E}\, {}^{(c)}\nabla_4\, {}^{(c)}\nabla_3 A \\
& +2\, {}^{(c)}\nabla_3 \mathcal{E}[\, {}^{(c)}\nabla_3,\, {}^{(c)}\nabla_4]A.
\end{aligned}$$

Using Proposition D.4.2 applied to $U = A$ of conformal type $s = 2$, we obtain

$$\begin{aligned}
[Q, \mathcal{E}\ {}^{(c)}\nabla_4]A &= 2\,{}^{(c)}\nabla_3\mathcal{E}\,{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3 A + \big({}^{(c)}\nabla_3\,{}^{(c)}\nabla_3\mathcal{E} + C_1\,{}^{(c)}\nabla_3\mathcal{E}\big)\,{}^{(c)}\nabla_4 A\\
&+\mathcal{E}\Big[4(\eta-\underline{\eta})\cdot\,{}^{(c)}\nabla\,{}^{(c)}\nabla_3 A\\
&+\Big(2\,{}^{(c)}\nabla_3(\eta-\underline{\eta}) + (2C_1 - \mathrm{tr}\,\underline{\chi})(\eta-\underline{\eta}) + {}^{(a)}\mathrm{tr}\underline{\chi}\ {}^*(\eta-\underline{\eta})\Big)\cdot\,{}^{(c)}\nabla A\\
&+\Big(\mathcal{V}^{s=2}_{[3,4]} + \mathcal{V}^{s-1=1}_{[3,4]} + 2\eta\cdot(\eta-\underline{\eta}) - {}^{(c)}\nabla_4(C_1)\Big)\,{}^{(c)}\nabla_3 A\\
&+\Big({}^{(c)}\nabla_3\mathcal{V}^{s=2}_{[3,4]} + C_1(\mathcal{V}^{s=2}_{[3,4]}) - {}^{(c)}\nabla_4(C_2)\Big)A + 2(\eta-\underline{\eta})\cdot\mathcal{V}^{s=2}_{[3,a]}(A)\\
&+r^{-3}\Gamma_b\cdot\mathfrak{d}^{\leq 1}U + \mathfrak{d}\big((\Gamma_b\cdot\Gamma_g)U\big)\Big]\\
&+2\,{}^{(c)}\nabla_3\mathcal{E}\Big(2(\eta-\underline{\eta})\cdot\,{}^{(c)}\nabla A + (\mathcal{V}^{s=2}_{[3,4]})A\Big).
\end{aligned}$$

In particular,

$$L = [Q, \mathcal{E}\ {}^{(c)}\nabla_4]A \qquad \text{for} \qquad \mathcal{E} = -\frac{1}{2}\mathrm{tr}\underline{X}.$$

We therefore obtain

$$\begin{aligned}
L &= L_{43}\,{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3 A + L_{a3}\cdot\,{}^{(c)}\nabla\,{}^{(c)}\nabla_3 A + L_4\,{}^{(c)}\nabla_4 A + L_a\cdot\,{}^{(c)}\nabla A\\
&+r^{-4}\Gamma_b\cdot\mathfrak{d}^{\leq 1}A + r^{-1}\mathfrak{d}\big((\Gamma_b\cdot\Gamma_g)A\big),
\end{aligned}$$

where

$$\begin{aligned}
L_{43} &= 2\,{}^{(c)}\nabla_3\mathcal{E} = \frac{1}{2}(\mathrm{tr}\underline{X})^2\\
L_{a3} &= 4\mathcal{E}(\eta-\underline{\eta}) = -2\mathrm{tr}\underline{X}(\eta-\underline{\eta}),\\
L_4 &= {}^{(c)}\nabla_3\,{}^{(c)}\nabla_3\mathcal{E} + C_1\,{}^{(c)}\nabla_3\mathcal{E} = -\frac{1}{4}(\mathrm{tr}\underline{X})^3 + \frac{1}{4}(\mathrm{tr}\underline{X})^2 C_1,\\
L_a &= \mathcal{E}\Big(2\,{}^{(c)}\nabla_3(\eta-\underline{\eta}) + (2C_1-\mathrm{tr}\,\underline{\chi})(\eta-\underline{\eta}) + {}^{(a)}\mathrm{tr}\underline{\chi}\ {}^*(\eta-\underline{\eta})\Big) + 4\,{}^{(c)}\nabla_3\mathcal{E}(\eta-\underline{\eta})\\
L_3 &= \mathcal{E}\Big(\mathcal{V}^{s=2}_{[3,4]} + \mathcal{V}^{s-1=1}_{[3,4]} + 2\eta\cdot(\eta-\underline{\eta}) - {}^{(c)}\nabla_4(C_1)\Big)\\
L_0 &= \mathcal{E}\Big({}^{(c)}\nabla_3\mathcal{V}^{s=2}_{[3,4]} + C_1(\mathcal{V}^{s=2}_{[3,4]}) - {}^{(c)}\nabla_4(C_2)\Big) + 2\,{}^{(c)}\nabla_3\mathcal{E}(\mathcal{V}^{s=2}_{[3,4]}) + 2\mathcal{E}(\eta-\underline{\eta})\cdot\mathcal{V}^{s=2}_{[3,a]}.
\end{aligned}$$

Using (D.4.7), we compute

$$L_3 = -\frac{1}{2}\mathrm{tr}\underline{X}\Big(6\left(\rho-\eta\cdot\underline{\eta}\right) + 8i\left(-\ {}^*\rho + \eta\wedge\underline{\eta}\right) + 2\eta\cdot(\eta-\underline{\eta}) - {}^{(c)}\nabla_4(C_1)\Big),$$

and

$$
\begin{aligned}
L_0 &= -\frac{1}{2}\mathrm{tr}\underline{X}\Big[{}^{(c)}\nabla_3\big(4\left(\rho-\eta\cdot\underline{\eta}\right)+4i\left(-\ {}^*\rho+\eta\wedge\underline{\eta}\right)\big) \\
&+C_1(4\left(\rho-\eta\cdot\underline{\eta}\right)+4i\left(-\ {}^*\rho+\eta\wedge\underline{\eta}\right))-\ {}^{(c)}\nabla_4(C_2)\Big]+\frac{1}{2}(\mathrm{tr}\underline{X})^2(4\left(\rho-\eta\cdot\underline{\eta}\right) \\
&+4i\left(-\ {}^*\rho+\eta\wedge\underline{\eta}\right))+2\mathcal{E}(\eta-\underline{\eta})\cdot\mathcal{V}^{s=2}_{[3,a]}.
\end{aligned}
$$

Expression for M

Observe that

$$
Q(F\cdot U) = F\cdot Q(U)+2\,{}^{(c)}\nabla_3 F\cdot\ {}^{(c)}\nabla_3 U+\left({}^{(c)}\nabla_3\,{}^{(c)}\nabla_3 F+C_1\,{}^{(c)}\nabla_3 F\right)\cdot U.
$$

We therefore obtain

$$
\begin{aligned}
M &= \left(4H+\underline{H}+\overline{\underline{H}}\right)\cdot[Q,\ {}^{(c)}\nabla]A+2\,{}^{(c)}\nabla_3\left(4H+\underline{H}+\overline{\underline{H}}\right)\cdot\ {}^{(c)}\nabla_3\,{}^{(c)}\nabla A \\
&+\left({}^{(c)}\nabla_3\,{}^{(c)}\nabla_3(4H+\underline{H}+\overline{\underline{H}})+C_1\,{}^{(c)}\nabla_3(4H+\underline{H}+\overline{\underline{H}})\right)\cdot\ {}^{(c)}\nabla A \\
&= \left(4H+\underline{H}+\overline{\underline{H}}\right)\cdot[Q,\ {}^{(c)}\nabla]A+2\,{}^{(c)}\nabla_3\left(4H+\underline{H}+\overline{\underline{H}}\right)\cdot\ {}^{(c)}\nabla\,{}^{(c)}\nabla_3 A \\
&+2\,{}^{(c)}\nabla_3\left(4H+\underline{H}+\overline{\underline{H}}\right)\cdot[\,{}^{(c)}\nabla_3,\ {}^{(c)}\nabla]A \\
&+\left({}^{(c)}\nabla_3\,{}^{(c)}\nabla_3(4H+\underline{H}+\overline{\underline{H}})+C_1\,{}^{(c)}\nabla_3(4H+\underline{H}+\overline{\underline{H}})\right)\cdot\ {}^{(c)}\nabla A.
\end{aligned}
$$

Using Proposition D.4.2 applied to $U = A$ of conformal type $s = 2$, we obtain

$$\begin{aligned}
M &= \big(4H + \underline{H} + \overline{\underline{H}}\big)_a \Bigg[- \mathrm{tr}\,\underline{\chi}\ {}^{(c)}\nabla_a\, {}^{(c)}\nabla_3 A_{bc} - {}^{(a)}\mathrm{tr}\underline{\chi}\ {}^{*\,(c)}\nabla_a\, {}^{(c)}\nabla_3 A_{bc} \\
&\quad +2\eta_a\, {}^{(c)}\nabla_3\, {}^{(c)}\nabla_3 A_{bc} + \left(-\frac{1}{2}\mathrm{tr}\,\underline{\chi}(C_1 - \mathrm{tr}\,\underline{\chi}) - \frac{1}{2}\, {}^{(a)}\mathrm{tr}\underline{\chi}^2\right)\, {}^{(c)}\nabla_a A_{bc} \\
&\quad -\frac{1}{2}\, {}^{(a)}\mathrm{tr}\underline{\chi}\big(C_1 - 2\mathrm{tr}\,\underline{\chi}\big)\ {}^{*\,(c)}\nabla_a A_{bc} \\
&\quad + \left({}^{(c)}\nabla_3 \eta_a + (C_1 - \frac{1}{2}\mathrm{tr}\,\underline{\chi})\eta_a - \frac{1}{2}\, {}^{(a)}\mathrm{tr}\underline{\chi}\ {}^*\eta_a - {}^{(c)}\nabla_a C_1 \right)\, {}^{(c)}\nabla_3 A_{bc} \\
&\quad + {}^{(c)}\nabla_3(\mathcal{V}^{s=2}_{[3,a]}(A)) + \mathcal{V}^{s-1=1}_{[3,a]}({}^{(c)}\nabla_3 A) - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\mathcal{V}^{s=2}_{[3,a]}(A) - \frac{1}{2}\, {}^{(a)}\mathrm{tr}\underline{\chi}\ {}^*\mathcal{V}^{s=2}_{[3,a]}(A) \\
&\quad - {}^{(c)}\nabla_a(C_2) A_{bc} + C_1 \mathcal{V}^{s=2}_{[3,a]}(A) + r^{-1}\mathfrak{d}(\Gamma_b \cdot \mathfrak{d}^{\leq 1} A) \Bigg] \\
&\quad +2\, {}^{(c)}\nabla_3 \big(4H + \underline{H} + \overline{\underline{H}}\big) \cdot {}^{(c)}\nabla\, {}^{(c)}\nabla_3 A \\
&\quad +2\, {}^{(c)}\nabla_3 \big(4H + \underline{H} + \overline{\underline{H}}\big)_a \Big[-\frac{1}{2}\mathrm{tr}\,\underline{\chi}\ {}^{(c)}\nabla_a A_{bc} - \frac{1}{2}\, {}^{(a)}\mathrm{tr}\underline{\chi}\ {}^{*\,(c)}\nabla_a A_{bc} + \eta_a\, {}^{(c)}\nabla_3 A_{bc} \\
&\quad + \mathcal{V}^{s=2}_{[3,a]}(A) + r^{-1}\Gamma_b \cdot \mathfrak{d}^{\leq 1} A \Big] \\
&\quad + \big({}^{(c)}\nabla_3\, {}^{(c)}\nabla_3 (4H + \underline{H} + \overline{\underline{H}}) + C_1\, {}^{(c)}\nabla_3(4H + \underline{H} + \overline{\underline{H}}) \big) \cdot {}^{(c)}\nabla A,
\end{aligned}$$

which gives

$$\begin{aligned}
M &= M_{a3} \cdot {}^{(c)}\nabla\, {}^{(c)}\nabla_3 A + M_{33}\, {}^{(c)}\nabla_3\, {}^{(c)}\nabla_3 A + M_a \cdot {}^{(c)}\nabla A + M_3\, {}^{(c)}\nabla_3 A + M_0 A \\
&\quad + r^{-3}\mathfrak{d}(\Gamma_b \cdot \mathfrak{d}^{\leq 1} A),
\end{aligned}$$

where

$$
\begin{aligned}
M_{a3} &= 2\,{}^{(c)}\nabla_3\left(4H+\underline{H}+\overline{\underline{H}}\right)-\mathrm{tr}\,\underline{\chi}\left(4H+\underline{H}+\overline{\underline{H}}\right)+{}^{(a)}\mathrm{tr}\underline{\chi}\;{}^*\left(4H+\underline{H}+\overline{\underline{H}}\right)\\
M_{33} &= 2\eta\cdot\left(4H+\underline{H}+\overline{\underline{H}}\right) \qquad\qquad (\mathrm{D.4.13})\\
M_a &= -\mathrm{tr}\,\underline{\chi}\,{}^{(c)}\nabla_3\left(4H+\underline{H}+\overline{\underline{H}}\right)+{}^{(a)}\mathrm{tr}\underline{\chi}\;{}^*\,{}^{(c)}\nabla_3\left(4H+\underline{H}+\overline{\underline{H}}\right)\\
&\quad+\left(-\frac{1}{2}\mathrm{tr}\,\underline{\chi}(C_1-\mathrm{tr}\,\underline{\chi})-\frac{1}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi}^2\right)\left(4H+\underline{H}+\overline{\underline{H}}\right)\\
&\quad+\frac{1}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi}\left(C_1-2\mathrm{tr}\,\underline{\chi}\right)\,{}^*\left(4H+\underline{H}+\overline{\underline{H}}\right)\\
&\quad+{}^{(c)}\nabla_3\,{}^{(c)}\nabla_3(4H+\underline{H}+\overline{\underline{H}})+C_1\,{}^{(c)}\nabla_3(4H+\underline{H}+\overline{\underline{H}})\\
M_3 &= \left(4H+\underline{H}+\overline{\underline{H}}\right)_a\Bigg(\left({}^{(c)}\nabla_3\eta_a+(C_1-\frac{1}{2}\mathrm{tr}\,\underline{\chi})\eta_a-\frac{1}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi}\;{}^*\eta_a-{}^{(c)}\nabla_a C_1\right)\\
&\quad+{}^{(c)}\nabla_3(\mathcal{V}^{s=2}_{[3,a]})+\mathcal{V}^{s-1=1}_{[3,a]}\Bigg)+2\,{}^{(c)}\nabla_3\left(4H+\underline{H}+\overline{\underline{H}}\right)_a\left[\eta_a\,{}^{(c)}\nabla_3 A_{bc}\right]\\
M_0 &= \left(4H+\underline{H}+\overline{\underline{H}}\right)_a\Bigg[-\frac{1}{2}\mathrm{tr}\,\underline{\chi}\mathcal{V}^{s=2}_{[3,a]}(A)-\frac{1}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi}\;{}^*\mathcal{V}^{s=2}_{[3,a]}(A)\\
&\quad-{}^{(c)}\nabla_a(C_2)A_{bc}+C_1\mathcal{V}^{s=2}_{[3,a]}(A)\Bigg]+2\,{}^{(c)}\nabla_3\left(4H+\underline{H}+\overline{\underline{H}}\right)_a\left[\mathcal{V}^{s=2}_{[3,a]}(A)\right].
\end{aligned}
$$

Expression for N

Using (D.2.1) we write

$$
\begin{aligned}
N &= [Q,\left(-\overline{\mathrm{tr}X}\mathrm{tr}\underline{X}+2\overline{\underline{P}}\right)]A+[Q,H\widehat{\otimes}\overline{\underline{H}}\cdot]A\\
&= [Q,\left(-\overline{\mathrm{tr}X}\mathrm{tr}\underline{X}+2\overline{\underline{P}}+4\eta\cdot\underline{\eta}-4i\eta\wedge\underline{\eta}\right)]A.
\end{aligned}
$$

Using (D.4.11), we obtain

$$
N = N_3\,{}^{(c)}\nabla_3 A+N_0 A,
$$

where

$$
\begin{aligned}
N_3 &= 2\,{}^{(c)}\nabla_3\left(-\overline{\mathrm{tr}X}\mathrm{tr}\underline{X}+2\overline{\underline{P}}+4\eta\cdot\underline{\eta}-4i\eta\wedge\underline{\eta}\right)\\
N_0 &= {}^{(c)}\nabla_3\,{}^{(c)}\nabla_3\left(-\overline{\mathrm{tr}X}\mathrm{tr}\underline{X}+2\overline{\underline{P}}+4\eta\cdot\underline{\eta}-4i\eta\wedge\underline{\eta}\right)\\
&\quad+C_1\,{}^{(c)}\nabla_3\left(-\overline{\mathrm{tr}X}\mathrm{tr}\underline{X}+2\overline{\underline{P}}+4\eta\cdot\underline{\eta}-4i\eta\wedge\underline{\eta}\right).
\end{aligned}
$$

The sum

Putting the above expressions together we obtain

$$\begin{aligned}
[Q,\mathcal{L}]A &= I+J+K+L+M+N\\
&= -4(\eta-\underline{\eta})\cdot {}^{(c)}\nabla\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_3A+4\eta\cdot {}^{(c)}\nabla\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_3A\\
&\quad -\big(\mathrm{tr}\underline{X}+\overline{\mathrm{tr}\underline{X}}\big)\,{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_3A\\
&\quad +(I_{43}+J_{43}+L_{43})\;\;{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3A+(I_4+J_4+L_4)\;\;{}^{(c)}\nabla_4A\\
&\quad +(I_{33}+J_{33}+K_{33}+M_{33})\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_3A\\
&\quad +(I_{a3}+J_{a3}+L_{a3}+M_{a3})\cdot\,{}^{(c)}\nabla\,{}^{(c)}\nabla_3A\\
&\quad +(I_3+J_3+K_3+L_3+N_3)\,{}^{(c)}\nabla_3A+J^a_3(A)\\
&\quad +(J_a+L_a+M_a)\cdot\,{}^{(c)}\nabla A+(J_{*a})\cdot\;{}^{*\,(c)}\nabla A\\
&\quad +(I_0+J_0+K_0+L_0+M_0+N_0)\;\;A+J^a_0(A)\\
&\quad +\nabla_3\big(r^{-2}\Gamma_b\cdot\mathfrak{d}^{\leq 2}A\big)+r^{-3}\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\mathfrak{d}^{\leq 1}A)+r^{-1}\mathfrak{d}\big((\Gamma_b\cdot\Gamma_g)A\big),
\end{aligned}$$

which gives

$$\begin{aligned}
[Q,\mathcal{L}]A &= 4\underline{\eta}\cdot\,{}^{(c)}\nabla\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_3A-\big(\mathrm{tr}\underline{X}+\overline{\mathrm{tr}\underline{X}}\big)\,{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_3A\\
&\quad +(I_{43}+J_{43}+L_{43})\;\;{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3A+(I_4+J_4+L_4)\;\;{}^{(c)}\nabla_4A\\
&\quad +(I_{33}+J_{33}+K_{33}+M_{33})\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_3A\\
&\quad +(I_{a3}+J_{a3}+L_{a3}+M_{a3})\cdot\,{}^{(c)}\nabla\,{}^{(c)}\nabla_3A\\
&\quad +(I_3+J_3+K_3+L_3+N_3)\,{}^{(c)}\nabla_3A+J^a_3(A)\\
&\quad +(J_a+L_a+M_a)\cdot\,{}^{(c)}\nabla A+(J_{*a})\cdot\;{}^{*\,(c)}\nabla A\\
&\quad +(I_0+J_0+K_0+L_0+M_0+N_0)\;\;A+J^a_0(A)+\mathrm{Err},
\end{aligned}$$

with $\mathrm{Err}=\nabla_3\big(r^{-2}\Gamma_b\cdot\mathfrak{d}^{\leq 2}A\big)+r^{-3}\mathfrak{d}^{\leq 2}(\Gamma_b\cdot\mathfrak{d}^{\leq 1}A)+r^{-1}\mathfrak{d}\big((\Gamma_b\cdot\Gamma_g)A\big)$. Recalling the definition of $Q(A)$, we write

$$ {}^{(c)}\nabla_3\,{}^{(c)}\nabla_3A \;=\; Q(A)-C_1\,{}^{(c)}\nabla_3A-C_2A, $$

and therefore

$$\begin{aligned}
{}^{(c)}\nabla\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_3A &= {}^{(c)}\nabla Q(A)-C_1\,{}^{(c)}\nabla\,{}^{(c)}\nabla_3A-(\,{}^{(c)}\nabla C_1)\,{}^{(c)}\nabla_3A\\
&\quad -C_2\,{}^{(c)}\nabla A-(\,{}^{(c)}\nabla C_2)A,\\
{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_3A &= {}^{(c)}\nabla_4Q(A)-C_1\,{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3A-C_2\,{}^{(c)}\nabla_4A\\
&\quad -\,{}^{(c)}\nabla_4C_1\,{}^{(c)}\nabla_3A-(\,{}^{(c)}\nabla_4C_2)A.
\end{aligned}$$

Hence,

$$\begin{aligned}[Q,\mathcal{L}]A &= 4\underline{\eta}\cdot{}^{(c)}\nabla Q(A) - 2\mathrm{tr}\,\underline{\chi}\,{}^{(c)}\nabla_4 Q(A) + \hat{V}Q(A)\\ &\quad + Z_{43}\ {}^{(c)}\nabla_4\,{}^{(c)}\nabla_3 A + Z_4\ {}^{(c)}\nabla_4 A + Z_{a3}\,{}^{(c)}\nabla_a\,{}^{(c)}\nabla_3 A\\ &\quad + Z_3\,{}^{(c)}\nabla_3 A + Z_a\,{}^{(c)}\nabla_a A + Z_0 A + \mathrm{Err},\end{aligned}$$

where

$$\hat{V} = I_{33} + J_{33} + K_{33} + M_{33},$$

and

$$\begin{aligned} Z_{43} &= I_{43} + J_{43} + L_{43} + C_1\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right)\\ Z_4 &= I_4 + J_4 + L_4 + C_2\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right),\\ Z_{a3} &= I_{a3} + J_{a3} + L_{a3} + M_{a3} - 4C_1\underline{\eta},\\ Z_3 &= I_3 + J_3 + K_3 + L_3 + N_3 - 4\underline{\eta}\,{}^{(c)}\nabla C_1 + {}^{(c)}\nabla_4 C_1\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right)\\ &\quad - C_1\left(I_{33} + J_{33} + K_{33} + M_{33}\right) + J_3^a,\\ Z_a &= J_a + L_a + M_a - 4C_2\underline{\eta} - {}^*J_{*a},\\ Z_0 &= I_0 + J_0 + K_0 + L_0 + N_0 - 4\underline{\eta}\cdot{}^{(c)}\nabla C_2 + {}^{(c)}\nabla_4 C_2\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right)\\ &\quad - C_2\left(I_{33} + J_{33} + K_{33} + M_{33}\right) + J_0^a.\end{aligned}$$

The Z coefficients

We now show that with the choices of C_1 and C_2 given by (D.4.3), i.e. $C_1 = 2\mathrm{tr}\,\underline{\chi} + \widetilde{C_1}$, $C_2 = \frac{1}{2}\mathrm{tr}\,\underline{\chi}^2 + \widetilde{C_2}$ all the Z coefficients are $O(|a|)$. We denote by $O(|a|r^{-c})$ any function which vanishes in Schwarzschild, such as multiples of η, $\underline{\eta}$, ${}^{(a)}\mathrm{tr}\underline{\chi}$, ${}^*\rho$, and has a r^{-c} fall-off in r.

We start with Z_{43} and Z_4. We have

$$\begin{aligned} Z_{43} &= I_{43} + J_{43} + L_{43} + C_1\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right)\\ &= {}^{(c)}\nabla_3 C_1 + \frac{1}{2}(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}})C_1 - \overline{\mathrm{tr}\underline{X}}\mathrm{tr}\underline{X}\\ &= {}^{(c)}\nabla_3(2\mathrm{tr}\,\underline{\chi} + \widetilde{C_1}) + (2\mathrm{tr}\,\underline{\chi} + \widetilde{C_1})\mathrm{tr}\,\underline{\chi} - (\mathrm{tr}\,\underline{\chi}^2 + {}^{(a)}\mathrm{tr}\underline{\chi}^2)\\ &= -\left(\mathrm{tr}\,\underline{\chi}^2 - {}^{(a)}\mathrm{tr}\underline{\chi}^2\right) + {}^{(c)}\nabla_3\widetilde{C_1} + (2\mathrm{tr}\,\underline{\chi} + \widetilde{C_1})\mathrm{tr}\,\underline{\chi} - (\mathrm{tr}\,\underline{\chi}^2 + {}^{(a)}\mathrm{tr}\underline{\chi}^2) + r^{-1}\Gamma_b\\ &= {}^{(c)}\nabla_3\widetilde{C_1} + \mathrm{tr}\,\underline{\chi}\widetilde{C_1} + r^{-1}\Gamma_b,\end{aligned}$$

and

$$\begin{aligned}
Z_4 &= I_4 + J_4 + L_4 + C_2\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right)\\
&= {}^{(c)}\nabla_3 C_2 + \left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right)C_2 - \frac{1}{4}\mathrm{tr}\underline{X}\overline{\mathrm{tr}\underline{X}}C_1\\
&= {}^{(c)}\nabla_3\left(\frac{1}{2}\mathrm{tr}\,\underline{\chi}^2 + \widetilde{C_2}\right) + \left(\frac{1}{2}\mathrm{tr}\,\underline{\chi}^2 + \widetilde{C_2}\right)2\mathrm{tr}\,\underline{\chi} - \frac{1}{4}\big(2\mathrm{tr}\,\underline{\chi} + \widetilde{C_1}\big)(\mathrm{tr}\,\underline{\chi}^2 + {}^{(a)}\mathrm{tr}\underline{\chi}^2)\\
&= \mathrm{tr}\,\underline{\chi}\left(-\frac{1}{2}\big(\mathrm{tr}\,\underline{\chi}^2 - {}^{(a)}\mathrm{tr}\underline{\chi}^2\big) + r^{-1}\Gamma_b\right) + {}^{(c)}\nabla_3\widetilde{C_2}\\
&\quad + \left(\frac{1}{2}\mathrm{tr}\,\underline{\chi}^2 + \tilde{C}_2\right)2\mathrm{tr}\,\underline{\chi} - \frac{1}{4}(2\mathrm{tr}\,\underline{\chi} + \widetilde{C_1})(\mathrm{tr}\,\underline{\chi}^2 + {}^{(a)}\mathrm{tr}\underline{\chi}^2)\\
&= {}^{(c)}\nabla_3\widetilde{C_2} + 2\mathrm{tr}\,\underline{\chi}\widetilde{C_2} - \frac{1}{4}(\mathrm{tr}\,\underline{\chi}^2 + {}^{(a)}\mathrm{tr}\underline{\chi}^2)\widetilde{C_1} + r^{-2}\Gamma_b.
\end{aligned}$$

From the above expressions we clearly see that $Z_{43} = O(|a|r^{-3})$ and $Z_4 = O(|a|r^{-4})$, and the error terms are of the same form as Err. Indeed,

$$r^{-1}\Gamma_b \cdot {}^{(c)}\nabla_4\,{}^{(c)}\nabla_3 A + r^{-2}\Gamma_b \cdot {}^{(c)}\nabla_4 A = r^{-2}\Gamma_b \cdot \mathfrak{d}^{\leq 1}\,{}^{(c)}\nabla_3 A = r^{-3}\Gamma_b \cdot \mathfrak{d}^{\leq 2}(A, B),$$

where we used the Bianchi identity for A.

From the expressions above, we immediately have that Z_{a3} and Z_a are $O(|a|)$, and are given by

$$\begin{aligned}
Z_{a3} &= I_{a3} + J_{a3} + L_{a3} + M_{a3} - 4C_1\underline{\eta} = O(|a|r^{-3}) + \mathfrak{d}^{\leq 1}\Gamma_g,\\
Z_a &= J_a + L_a + M_a - 4C_2\underline{\eta} - {}^*J_{*a} = O(|a|r^{-4}) + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_g.
\end{aligned}$$

The error terms are then given by

$$\mathfrak{d}^{\leq 1}\Gamma_g \cdot {}^{(c)}\nabla_a\,{}^{(c)}\nabla_3 A + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_g \cdot {}^{(c)}\nabla_a A = r^{-1}\mathfrak{d}^{\leq 1}\Gamma_g \cdot \mathfrak{d}^{\leq 1}\,{}^{(c)}\nabla_3 A = r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g \cdot \mathfrak{d}^{\leq 2}(A, B),$$

as above.

Finally, we consider the coefficients Z_3 and Z_0. In particular observe that, according to (D.4.3), we can write

$$C_1 = 2\mathrm{tr}\,\underline{\chi} + O(|a|r^{-2}), \qquad C_2 = \frac{1}{2}\mathrm{tr}\,\underline{\chi}^2 + O(|a|r^{-3}).$$

We therefore have

$$
\begin{aligned}
{}^{(c)}\nabla_4 C_1 &= -\mathrm{tr}\,\chi \mathrm{tr}\,\underline{\chi} + 4\rho + O(|a|r^{-3}) + r^{-1}\Gamma_g,\\
{}^{(c)}\nabla_3 C_1 &= -\mathrm{tr}\,\underline{\chi}^2 + O(|a|r^{-3}) + r^{-1}\Gamma_b,\\
{}^{(c)}\nabla_4\, {}^{(c)}\nabla_3 C_1 &= \mathrm{tr}\,\chi \mathrm{tr}\,\underline{\chi}^2 - 4\mathrm{tr}\,\underline{\chi}\rho + O(|a|r^{-4}) + r^{-2}\Gamma_b,\\
{}^{(c)}\nabla_4 C_2 &= -\frac{1}{2}\mathrm{tr}\,\chi \mathrm{tr}\,\underline{\chi}^2 + 2\mathrm{tr}\,\underline{\chi}\rho + O(|a|r^{-4}) + r^{-2}\Gamma_g,\\
{}^{(c)}\nabla_3 C_2 &= -\frac{1}{2}\mathrm{tr}\,\underline{\chi}^3 + O(|a|r^{-4}) + r^{-2}\Gamma_b,\\
{}^{(c)}\nabla_4\, {}^{(c)}\nabla_3 C_2 &= \frac{3}{4}\mathrm{tr}\,\chi \mathrm{tr}\,\underline{\chi}^3 - 3\mathrm{tr}\,\underline{\chi}^2\rho + O(|a|r^{-5}) + r^{-3}\Gamma_b.
\end{aligned}
$$

In what follows we omit the error terms as they are of the form Err given above. We compute

$$
\begin{aligned}
I_3 &= -\,{}^{(c)}\nabla_3(2\rho) - 2\mathrm{tr}\,\underline{\chi}\,(2\rho) - \frac{1}{2}\mathrm{tr}\,\chi \mathrm{tr}\,\underline{\chi}^2 + 2\mathrm{tr}\,\underline{\chi}\rho + \mathrm{tr}\,\chi \mathrm{tr}\,\underline{\chi}^2 - 4\mathrm{tr}\,\underline{\chi}\rho + O(|a|r^{-4})\\
&= \frac{1}{2}\mathrm{tr}\,\chi \mathrm{tr}\,\underline{\chi}^2 - 3\mathrm{tr}\,\underline{\chi}\rho + O(|a|r^{-4}),\\
J_3 &= -2\mathrm{tr}\,\underline{\chi}\left({}^{(c)}\nabla_3\left(\frac{5}{2}\mathrm{tr}\,\chi\right) + (\mathrm{tr}\,\chi)(\mathrm{tr}\,\underline{\chi})\right) + \left(-\frac{5}{2}\mathrm{tr}\,\underline{\chi}^2\right)\left(\frac{5}{2}\mathrm{tr}\,\chi\right) + O(|a|r^{-4})\\
&= -\frac{23}{4}\mathrm{tr}\,\chi \mathrm{tr}\,\underline{\chi}^2 - 10\mathrm{tr}\,\underline{\chi}\rho + O(|a|r^{-4}),\\
K_3 &= {}^{(c)}\nabla_3\,{}^{(c)}\nabla_3\left(-\frac{5}{2}\mathrm{tr}\,\chi\right) + 2\mathrm{tr}\,\underline{\chi}\,{}^{(c)}\nabla_3\left(-\frac{5}{2}\mathrm{tr}\,\chi\right) + \frac{5}{2}\mathrm{tr}\,\chi(-\mathrm{tr}\,\underline{\chi}^2) + O(|a|r^{-4})\\
&= -\frac{5}{2}\,{}^{(c)}\nabla_3\left(-\frac{1}{2}\mathrm{tr}\,\underline{\chi}\mathrm{tr}\,\chi + 2\rho\right) - 5\mathrm{tr}\,\underline{\chi}\left(-\frac{1}{2}\mathrm{tr}\,\underline{\chi}\mathrm{tr}\,\chi + 2\rho\right) + \frac{5}{2}\mathrm{tr}\,\chi(-\mathrm{tr}\,\underline{\chi}^2)\\
&\quad +O(|a|r^{-4})\\
&= -\frac{5}{2}\left(\frac{1}{4}\mathrm{tr}\,\underline{\chi}^2\mathrm{tr}\,\chi - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\left(-\frac{1}{2}\mathrm{tr}\,\underline{\chi}\mathrm{tr}\,\chi + 2\rho\right) - 3\mathrm{tr}\,\underline{\chi}\rho\right) - 10\mathrm{tr}\,\underline{\chi}\rho + O\left(\frac{|a|}{r^4}\right)\\
&= -\frac{5}{4}\mathrm{tr}\,\underline{\chi}^2\mathrm{tr}\,\chi + O(|a|r^{-4}),
\end{aligned}
$$

$$
\begin{aligned}
L_3 &= -\frac{1}{2}\mathrm{tr}\,\underline{\chi}\big(6\rho - (-\mathrm{tr}\,\chi \mathrm{tr}\,\underline{\chi} + 4\rho)\big) + O(|a|r^{-4}) = -\frac{1}{2}\mathrm{tr}\,\chi \mathrm{tr}\,\underline{\chi}^2 - \mathrm{tr}\,\underline{\chi}\rho + O(|a|r^{-4}),\\
N_3 &= 2\big(-\,{}^{(c)}\nabla_3\mathrm{tr}\,\chi \mathrm{tr}\,\underline{\chi} - \mathrm{tr}\,\chi\,{}^{(c)}\nabla_3\mathrm{tr}\,\underline{\chi} + 2\,{}^{(c)}\nabla_3\rho\big) + O(|a|r^{-4})\\
&= 2\mathrm{tr}\,\chi \mathrm{tr}\,\underline{\chi}^2 - 10\rho\mathrm{tr}\,\underline{\chi} + O(|a|r^{-4}).
\end{aligned}
$$

We compute

$$
\begin{aligned}
I_{33} &= 3P-5\overline{P}+4\eta\cdot\underline{\eta}-2|\eta|^2+{}^{(c)}\nabla_4(C_1)=-\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}+2\rho+O(|a|r^{-3}),\\
J_{33} &= -\left(\mathrm{tr}\underline{X}+\overline{\mathrm{tr}\underline{X}}\right)(\frac{1}{2}\mathrm{tr}X+2\overline{\mathrm{tr}X})+{}^{(c)}\mathcal{D}\cdot\overline{H}+2H\cdot\overline{H}=-5\mathrm{tr}\,\underline{\chi}\mathrm{tr}\,\chi+O(|a|r^{-3}),\\
K_{33} &= 2\,{}^{(c)}\nabla_3\left(-\frac{1}{2}\mathrm{tr}X-2\overline{\mathrm{tr}X}\right)=\frac{5}{2}\mathrm{tr}\,\chi\mathrm{tr}\,\chi-10\rho+O(|a|r^{-3}),\\
M_{33} &= 2\left(4H+\underline{H}+\overline{\underline{H}}\right)\cdot\eta=O(|a|r^{-4}).
\end{aligned}
$$

We compute

$$
\begin{aligned}
I_0 &= {}^{(c)}\nabla_4\,{}^{(c)}\nabla_3C_2=\frac{3}{4}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}^3-3\mathrm{tr}\,\underline{\chi}^2\rho+O(|a|r^{-5}),\\
J_0 &= -2\mathrm{tr}\,\underline{\chi}\Big({}^{(c)}\nabla_3\left(\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}-2\rho\right)+\frac{1}{2}\mathrm{tr}\,\underline{\chi}(4\rho)\Big)+\left(-\frac{5}{2}\mathrm{tr}\,\underline{\chi}^2\right)\left(\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}-2\rho\right)\\
&\quad +O(|a|r^{-5})\\
&= -\frac{1}{2}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}^3-9\rho\mathrm{tr}\,\underline{\chi}^2+O(|a|r^{-5}),\\
K_0 &= -\left(-\frac{1}{2}\mathrm{tr}X-2\overline{\mathrm{tr}X}\right){}^{(c)}\nabla_3C_2=-\frac{5}{4}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}^3+O(|a|r^{-5}),\\
L_0 &= -2\mathrm{tr}\,\underline{\chi}^2\rho+\frac{1}{2}\mathrm{tr}\,\underline{\chi}(-\frac{1}{2}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}^2+2\mathrm{tr}\,\underline{\chi}\rho)\;-2\mathrm{tr}\,\underline{\chi}\,{}^{(c)}\nabla_3(\rho)+O(|a|r^{-5})\\
&= -\frac{1}{4}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}^3+2\mathrm{tr}\,\underline{\chi}^2\rho\;+O(|a|r^{-5})\\
N_0 &= {}^{(c)}\nabla_3\,{}^{(c)}\nabla_3\left(-\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}+2\rho\right)+2\mathrm{tr}\,\underline{\chi}\,{}^{(c)}\nabla_3\left(-\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}+2\rho\right)\\
&= (-\frac{1}{2}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}+2\rho)\mathrm{tr}\,\underline{\chi}^2+2\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}(-\frac{1}{2}\mathrm{tr}\,\underline{\chi}^2)-5(-\frac{3}{2}\mathrm{tr}\,\underline{\chi}\rho)\mathrm{tr}\,\underline{\chi}-5\rho(-\frac{1}{2}\mathrm{tr}\,\underline{\chi}^2)\\
&\quad +2\mathrm{tr}\,\underline{\chi}\left(\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}^2-5\rho\mathrm{tr}\,\underline{\chi}\right)+O(|a|r^{-5})=\frac{1}{2}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}^3+2\rho\mathrm{tr}\,\underline{\chi}^2+O(|a|r^{-5}).
\end{aligned}
$$

We finally obtain

$$
\begin{aligned}
Z_3 &= \frac{1}{2}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}^2-3\mathrm{tr}\,\underline{\chi}\rho-\frac{23}{4}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}^2-10\mathrm{tr}\,\underline{\chi}\rho-\frac{5}{4}\mathrm{tr}\,\underline{\chi}^2\mathrm{tr}\,\chi-\frac{1}{2}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}^2-\mathrm{tr}\,\underline{\chi}\rho\\
&\quad +2\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}^2-10\rho\mathrm{tr}\,\underline{\chi}+2\mathrm{tr}\,\underline{\chi}(-\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}+4\rho)\\
&\quad -2\mathrm{tr}\,\underline{\chi}\left(-\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}+2\rho-5\mathrm{tr}\,\underline{\chi}\mathrm{tr}\,\chi+\frac{5}{2}\mathrm{tr}\,\chi\mathrm{tr}\,\chi-10\rho\right)+O(|a|r^{-4})=O(|a|r^{-4}).
\end{aligned}
$$

and

$$
\begin{aligned}
Z_0 &= \frac{3}{4}tr\,\chi \mathrm{tr}\,\underline{\chi}^3 - 3\mathrm{tr}\,\underline{\chi}^2\rho - \frac{1}{2}tr\,\chi \mathrm{tr}\,\underline{\chi}^3 - 9\rho \mathrm{tr}\,\underline{\chi}^2 - \frac{5}{4}tr\,\chi \mathrm{tr}\,\underline{\chi}^3 + 2\mathrm{tr}\,\underline{\chi}^2\rho - \frac{1}{4}tr\,\chi \mathrm{tr}\,\underline{\chi}^3 \\
&\quad +2\rho \mathrm{tr}\,\underline{\chi}^2 + \frac{1}{2}tr\,\chi \mathrm{tr}\,\underline{\chi}^3 + 2\mathrm{tr}\,\underline{\chi}(-\frac{1}{2}tr\,\chi \mathrm{tr}\,\underline{\chi}^2 + 2\mathrm{tr}\,\underline{\chi}\rho) \\
&\quad -\frac{1}{2}\mathrm{tr}\,\underline{\chi}^2\left(-\mathrm{tr}\,\chi \mathrm{tr}\,\underline{\chi} + 2\rho - 5\mathrm{tr}\,\underline{\chi} tr\,\chi + \frac{5}{2}tr\,\chi tr\,\chi - 10\rho\right) + O(|a|r^{-5}) = O(|a|r^{-5}).
\end{aligned}
$$

as stated. This completes the proof of Proposition D.4.1.

D.4.2 Step 2. Derive the wave equation for $Q(A)$ and $\mathfrak{q}$

We use the previous two steps to derive the wave equation for $Q = Q(A)$ from the Teukolsky equation for A and the commutator $[Q, \mathcal{L}]$.

Proposition D.4.3. *Let $Q = Q(A) \in \mathfrak{s}_2$ as in Proposition D.4.1. Then Q satisfies the following wave equation:*

$$
\begin{aligned}
\dot{\square}_2 Q &= (2\overline{trX})\nabla_3 Q + \left(tr\underline{X} + \overline{tr\underline{X}}\right)\nabla_4 Q - \left(4H + 2\,\underline{H} + 2\,\overline{\underline{H}}\right)\cdot \nabla Q + \widetilde{V} Q \\
&\quad - L_Q(A) + Err[\square_2 Q],
\end{aligned} \tag{D.4.14}
$$

where

$$
\begin{aligned}
\widetilde{V} &= \frac{1}{2}tr\,\chi tr\underline{\chi} + \frac{1}{2}\,{}^{(a)}tr\chi\,{}^{(a)}tr\underline{\chi} - 4\rho - 4\eta\cdot\underline{\eta} \\
&\quad + i\left(-tr\,\chi\,{}^{(a)}tr\underline{\chi} + {}^{(a)}tr\chi tr\underline{\chi} + 4\,{}^*\rho + 2\eta\wedge\underline{\eta}\right) - \hat{V},
\end{aligned} \tag{D.4.15}
$$

and

$$
Err[\dot{\square}_2 Q] \quad = \quad Q(Err[\mathcal{L}(A)]) + Err[[Q,\mathcal{L}]A] + (\Gamma_b\cdot\Gamma_g)\cdot Q. \tag{D.4.16}
$$

Proof. Recall (D.4.2), i.e.

$$
\mathcal{L}(Q(A)) + [Q,\mathcal{L}](A) \quad = \quad Q(\mathrm{Err}[\mathcal{L}(A)]).
$$

Applying the definition of the operator $\mathcal{L}$ given by (5.1.2) to $Q = Q(A)$, we obtain[1]

$$
\begin{aligned}
\mathcal{L}(Q) &= -\nabla_4\nabla_3 Q + \frac{1}{4}\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot Q) + \left(-\frac{1}{2}\mathrm{tr}X - 2\overline{\mathrm{tr}X} - 2\omega\right)\nabla_3 Q - \frac{1}{2}\mathrm{tr}\underline{X}\nabla_4 Q \\
&\quad + \left(4H + \underline{H} + \overline{\underline{H}}\right)\cdot\nabla Q + \left(-\overline{\mathrm{tr}X}\mathrm{tr}\underline{X} + 2\overline{P} + 4\eta\cdot\underline{\eta} - 4i\eta\wedge\underline{\eta}\right)Q.
\end{aligned}
$$

[1]Recall that $Q(A)$ is of conformal type 0, therefore all conformal derivatives coincide with the non-conformal ones, see also (4.7.6).

Using (D.4.4) of Proposition D.4.1, equation (D.4.2) gives

$$\begin{aligned}
& -\nabla_4\nabla_3 Q + \frac{1}{4}\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot Q) + \left(-\frac{1}{2}\mathrm{tr} X - 2\overline{\mathrm{tr} X}\right)\nabla_3 Q - \left(\frac{3}{2}\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right)\nabla_4 Q \\
& + \left(4H + \underline{H} + \overline{\underline{H}} + 4\underline{\eta}\right)\cdot\nabla Q + \left(-\overline{\mathrm{tr} X}\mathrm{tr}\underline{X} + 2\overline{P} + \hat{V} + 4\eta\cdot\underline{\eta} - 4i\eta\wedge\underline{\eta}\right) Q \\
= \; & -L_Q(A) + Q(\mathrm{Err}[\mathcal{L}(A)]) + \mathrm{Err}[[Q,\mathcal{L}]A].
\end{aligned}$$

Using the formula for the wave equation (4.7.12) applied to $Q \in \mathfrak{s}_2(\mathbb{C})$, i.e.

$$\begin{aligned}
\dot{\square}_2 Q \;=\; & -\nabla_4\nabla_3 Q + \frac{1}{4}\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot Q) + \left(2\omega - \frac{1}{2}\mathrm{tr} X\right)\nabla_3 Q - \frac{1}{2}\mathrm{tr}\underline{X}\nabla_4 Q + 2\underline{\eta}\cdot\nabla Q \\
& + \left(-\frac{1}{2}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} - \frac{1}{2}{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} - 2\rho\right) Q + 2i\left({}^*\rho - \eta\wedge\underline{\eta}\right) Q + (\Gamma_b\cdot\Gamma_g)\cdot Q,
\end{aligned}$$

we obtain (D.4.14), with

$$\begin{aligned}
\widetilde{V} \;=\; & -\frac{1}{2}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} - \frac{1}{2}{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} - 2\rho + 2i\left({}^*\rho - \eta\wedge\underline{\eta}\right) \\
& - \left(-\overline{\mathrm{tr} X}\mathrm{tr}\underline{X} + 2\overline{P} + \hat{V} + 4\eta\cdot\underline{\eta} - 4i\eta\wedge\underline{\eta}\right),
\end{aligned}$$

as stated. □

We now want to rescale Q in order to absorb the real parts of the first order terms in (D.4.14) into the wave operator. The rescaling is obtained through a scalar function of q and $\overline{q}$, i.e.

$$\mathfrak{q} \;=\; fQ(A) = f\left({}^{(c)}\nabla_3\,{}^{(c)}\nabla_3 A + C_1\,{}^{(c)}\nabla_3 A + C_2 A\right) \in \mathfrak{s}_2(\mathbb{C}),$$

with C_1 and C_2 given as in Proposition D.4.1.

Proposition D.4.4. *Let f be given by*

$$f = q\overline{q}^3.$$

Then $\mathfrak{q} = fQ(A) \in \mathfrak{s}_2$ satisfies the following wave equation:

$$\dot{\square}_2\mathfrak{q} - i\frac{4a\cos\theta}{|q|^2}\nabla_{\mathbf{T}}\mathfrak{q} - V_1\mathfrak{q} \;=\; \widetilde{L_{\mathfrak{q}}[A]} + f\left(Err[\dot{\square}_2 Q] + \Gamma_g\cdot\mathfrak{d}^{\leq 1}Q\right), \quad \text{(D.4.17)}$$

where

$$\begin{aligned}
V_1 \;&:=\; f^{-1}\square_{\mathbf{g}}(f) + \widetilde{V}, \qquad \text{(D.4.18)}\\
\widetilde{L_{\mathfrak{q}}[A]} \;&:=\; -fL_Q(A),\\
Err[\dot{\square}_2\mathfrak{q}] \;&:=\; f\left(Err[\dot{\square}_2 Q] + \Gamma_g\cdot\mathfrak{d}^{\leq 1}Q\right).
\end{aligned}$$

Proof. Let f be given by

$$f = (q)^n(\overline{q})^m.$$

Using Lemma 3.4.1, we deduce

$$\begin{aligned}
\nabla_3(f) &= \Big(\frac{n}{2}\overline{\mathrm{tr}\underline{X}} + \frac{m}{2}\mathrm{tr}\underline{X}\Big) f + r^{n+m}\Gamma_b,\\
\nabla_4(f) &= \Big(\frac{n}{2}\mathrm{tr}X + \frac{m}{2}\overline{\mathrm{tr}X}\Big) f + r^{n+m-1}\Gamma_g,\\
\nabla f &= \Big(\frac{m}{2}H + \frac{n}{2}\overline{H} + \frac{n}{2}\,\underline{H} + \frac{m}{2}\overline{\underline{H}}\Big) f + r^{n+m}\Gamma_g.
\end{aligned}$$

We obtain for $\mathfrak{q} = fQ$ from (D.4.14),

$$\begin{aligned}
\dot{\square}_2\mathfrak{q} &= \square_{\mathbf{g}}(f)Q + f\dot{\square}_2(Q) - \nabla_3 f\nabla_4 Q - \nabla_4 f\nabla_3 Q + 2\nabla f\cdot\nabla Q\\
&= \square_{\mathbf{g}}(f)Q\\
&\quad +\Big(2\overline{\mathrm{tr}X} f\nabla_3 Q + \big(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\big) f\nabla_4 Q - \big(4H + 2\,\underline{H} + 2\overline{\underline{H}}\big)\cdot f\nabla Q + f\widetilde{V}Q\Big)\\
&\quad -\Big(\frac{n}{2}\overline{\mathrm{tr}\underline{X}} + \frac{m}{2}\mathrm{tr}\underline{X} + \Gamma_b\Big) f\nabla_4 Q - \Big(\frac{n}{2}\mathrm{tr}X + \frac{m}{2}\overline{\mathrm{tr}X} + r^{-1}\Gamma_g\Big) f\nabla_3 Q\\
&\quad +2\Big(\frac{m}{2}H + \frac{n}{2}\overline{H} + \frac{n}{2}\,\underline{H} + \frac{m}{2}\overline{\underline{H}} + \Gamma_g\Big) f\cdot\nabla Q + f\Big(-L_Q(A) + \mathrm{Err}[\square_2 Q]\Big),
\end{aligned}$$

which gives

$$\begin{aligned}
\dot{\square}_2\mathfrak{q} &= \Big((1-\frac{n}{2})\overline{\mathrm{tr}\underline{X}} + (1-\frac{m}{2})\mathrm{tr}\underline{X}\Big) f\nabla_4 Q + \Big(-\frac{n}{2}\mathrm{tr}X + (2-\frac{m}{2})\overline{\mathrm{tr}X}\Big) f\nabla_3 Q\\
&\quad + \big((m-4)H + n\overline{H} + (n-2)\,\underline{H} + (m-2)\overline{\underline{H}}\big) f\cdot\nabla Q + \Big(f^{-1}\square(f) + \widetilde{V}\Big)\,\mathfrak{q}\\
&\quad +f\Big(-L_Q(A) + \mathrm{Err}[\square_2 Q] + r^{-1}\Gamma_b\cdot\mathfrak{d}Q\Big).
\end{aligned}$$

Observe that the real part of the coefficients of all the first derivatives are multiple of $m+n-4$. To cancel the real part of those coefficients we then take $m = 4-n$, therefore for $f = (q)^n(\overline{q})^{4-n}$, we have

$$\begin{aligned}
\dot{\square}_2\mathfrak{q} &= \Big((1-\frac{n}{2})\overline{\mathrm{tr}\underline{X}} - (1-\frac{n}{2})\mathrm{tr}\underline{X}\Big) f\nabla_4 Q + \Big(-\frac{n}{2}\mathrm{tr}X + \frac{n}{2}\overline{\mathrm{tr}X}\Big) f\nabla_3 Q\\
&\quad + \big((-n)H + n\overline{H} + (n-2)\,\underline{H} + (2-n)\overline{\underline{H}}\big) f\cdot\nabla Q + \Big(f^{-1}\square_{\mathbf{g}}(f) + \widetilde{V}\Big)\,\mathfrak{q}\\
&\quad +f\Big(-L_Q(A) + \mathrm{Err}[\square_2 Q] + r^{-1}\Gamma_b\cdot\mathfrak{d}Q\Big)\\
&= if\Big((2-n)\,{}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4 + n\,{}^{(a)}\mathrm{tr}\chi\nabla_3 + \big((-2n)\,{}^*\eta + 2(n-2)\,{}^*\underline{\eta}\big)\cdot\nabla\Big)Q\\
&\quad + \Big(f^{-1}\square_{\mathbf{g}}(f) + \widetilde{V}\Big)\,\mathfrak{q} + f\Big(-L_Q(A) + \mathrm{Err}[\square_2 Q] + r^{-1}\Gamma_b\cdot\mathfrak{d}Q\Big).
\end{aligned}$$

Using, see (4.1.14), that

$$
{}^{(a)}\mathrm{tr}\chi e_3 + {}^{(a)}\mathrm{tr}\underline{\chi} e_4 + 2(\eta+\underline{\eta})\cdot {}^*\nabla = \frac{4a\cos\theta}{|q|^2}\mathbf{T} + \Gamma_g\cdot\mathfrak{d},
$$

we obtain for $n=1$,

$$
\begin{aligned}
\dot{\square}_2\mathfrak{q} &= if\frac{4a\cos\theta}{|q|^2}\nabla_{\mathbf{T}}Q + \Big(f^{-1}\square_{\mathbf{g}}(f) + \widetilde{V}\Big)\,\mathfrak{q}\\
&\quad + f\Big(-L_Q(A) + \mathrm{Err}[\square_2 Q] + \Gamma_g\cdot\mathfrak{d}^{\leq 1}Q\Big),
\end{aligned}
$$

as stated.

□

We now analyze the error terms. We have

$$
\begin{aligned}
\mathrm{Err}[\dot{\square}_2\mathfrak{q}] &:= f\Big(\mathrm{Err}[\dot{\square}_2 Q] + \Gamma_g\cdot\mathfrak{d}^{\leq 1}Q\Big)\\
&= f\Big(Q(\mathrm{Err}[\mathcal{L}(A)]) + \mathrm{Err}[[Q,\mathcal{L}]A] + (\Gamma_b\cdot\Gamma_g)\cdot Q + \Gamma_g\cdot\mathfrak{d}^{\leq 1}Q\Big)\\
&= f\Big(Q\big(r^{-1}\mathfrak{d}^{\leq 1}\big(\Gamma_g\cdot B\big) + {}^{(c)}\nabla_3\Xi\cdot B + r^{-1}\Gamma_b\cdot\Gamma_g\cdot\Gamma_g\big)\\
&\quad + r^{-2}\mathfrak{d}^{\leq 3}(\Gamma_g\cdot(A,B)) + \nabla_3\big(r^{-2}\Gamma_b\cdot\mathfrak{d}^{\leq 2}A\big) + r^{-1}\mathfrak{d}\big((\Gamma_b\cdot\Gamma_g)A\big)\\
&\quad + (\Gamma_b\cdot\Gamma_g)\cdot Q + \Gamma_g\cdot\mathfrak{d}^{\leq 1}Q\Big),
\end{aligned}
$$

which gives

$$
\begin{aligned}
\mathrm{Err}[\dot{\square}_2\mathfrak{q}] &= r^2\mathfrak{d}^{\leq 3}(\Gamma_g\cdot(A,B)) + \nabla_3(r^2\mathfrak{d}^{\leq 2}(\Gamma_b\cdot(A,B)))\\
&\quad + \mathfrak{d}^{\leq 1}(\Gamma_g\cdot\mathfrak{q}) + r^3\mathfrak{d}^{\leq 2}\big(\Gamma_b\cdot\Gamma_g\cdot\Gamma_g\big),
\end{aligned}
$$

as stated in Theorem 5.2.9.

D.4.3 Step 3. Reality of the potential and lower order terms

Observe that in order to obtain equation (D.4.17) for $\mathfrak{q}$ we only needed so far to impose that C_1 and C_2 are given by (D.4.3), i.e.

$$
C_1 = 2\mathrm{tr}\,\underline{\chi} + \widetilde{C_1}, \qquad C_2 = \frac{1}{2}\mathrm{tr}\,\underline{\chi}^2 + \widetilde{C_2},
$$

with $\widetilde{C_1}$ and $\widetilde{C_2}$ are complex functions satisfying $\widetilde{C_1} = O(|a|r^{-2})$, $\widetilde{C_2} = O(|a|r^{-3})$.

To complete the proof of Theorem 5.2.9, we need to show that there exists a choice of complex functions $\widetilde{C_1}$ and $\widetilde{C_2}$ for which the potential V is real, the scalar Z_{43} and the one-form Z_{a3} are real and the one-form Z_{13} vanishes.

We have the following.

Proposition D.4.5. *Let $\widetilde{C_1}$ be the complex function with $C_1 = 2tr\underline{\chi} + \widetilde{C_1}$. Then*

1. *if $\Im(\widetilde{C_1}) = -4\,{}^{(a)}tr\underline{\chi}$, then*
   - *the potential V_1 as given in* (D.4.18) *is real, and is given by*

$$V_1 = -tr\,\chi tr\underline{\chi} + O(\frac{|a|}{r^4});$$

   - *the scalar Z_{43} and the one-form Z_{a3} are real;*

2. *if $\Re(\widetilde{C_1}) = -2\frac{({}^{(a)}tr\underline{\chi})^2}{tr\underline{\chi}}$, then the one-form Z_{13} vanishes. Moreover, V_1, Z_{43} and Z_{23} are given (in the outgoing frame) by*

$$\begin{aligned}
V_1 &= \frac{4}{|q|^2}\frac{r^2-2mr+2a^2}{r^2}\\
&\quad -\frac{4a^2\cos^2\theta}{r^2|q|^6}(2r^4+4mr^3+a^2r^2+2a^2\cos^2\theta r^2-2mra^2\cos^2\theta+a^4\cos^2\theta),\\
Z_{43} &= {}^{(a)}tr\underline{\chi}^2\big(1+\frac{{}^{(a)}tr\underline{\chi}^2}{tr\underline{\chi}^2}\big) = {}^{(a)}tr\underline{\chi}^2\frac{|q|^2}{r^2} = \frac{4a^2\Delta^2\cos^2\theta}{r^2|q|^6},\\
Z_{23} &= \frac{8a\sin\theta\Delta}{|q|^5}.
\end{aligned}$$

Proof. Here we compute $\Im(V_1) = \Im(f^{-1}\square_{\mathbf{g}}(f) + \widetilde{V})$. We start with the following.

Lemma D.4.6. *Let $f = q\overline{q}^3$. Then*

$$\begin{aligned}
\Im(f^{-1}\square_{\mathbf{g}}(f)) &= -3tr\underline{\chi}\,{}^{(a)}tr\chi + 2tr\,\chi\,{}^{(a)}tr\underline{\chi} + 2\,{}^*\rho + div\,({}^*\eta + {}^*\underline{\eta}) - 10\eta\wedge\underline{\eta} + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b,\\
\Re(f^{-1}\square_{\mathbf{g}}f) &= -5tr\,\chi tr\underline{\chi} - 2\,{}^{(a)}tr\chi\,{}^{(a)}tr\underline{\chi} - 4\rho\\
&\quad +2div\,(\eta-\underline{\eta}) + 3(|\eta|^2+|\underline{\eta}|^2) + 14\eta\cdot\underline{\eta} + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_g.
\end{aligned}$$

Proof. Using that

$$\begin{aligned}
\nabla_3 f &= \left(\frac{1}{2}\overline{\mathrm{tr}\underline{X}}+\frac{3}{2}\mathrm{tr}\underline{X}\right) f + r^4\Gamma_b = \left(2\mathrm{tr}\,\underline{\chi} - i\,{}^{(a)}\mathrm{tr}\underline{\chi}\right) f + r^4\Gamma_b,\\
\nabla_4 f &= \left(\frac{1}{2}\mathrm{tr}X+\frac{3}{2}\overline{\mathrm{tr}X}\right) f + r^3\Gamma_g = \left(2\mathrm{tr}\,\chi + i\,{}^{(a)}\mathrm{tr}\chi\right) f + r^3\Gamma_g,\\
\nabla f &= \left(\frac{3}{2}H+\frac{1}{2}\overline{H}+\frac{1}{2}\,\underline{H}+\frac{3}{2}\overline{\underline{H}}\right) f + r^4\Gamma_g = \left(2(\eta+\underline{\eta}) + i({}^*\eta - {}^*\underline{\eta})\right) f + r^4\Gamma_g,
\end{aligned}$$

we deduce

$$\begin{aligned}
\nabla_4\nabla_3 f &= \left(2\nabla_4\mathrm{tr}\,\underline{\chi} - i\nabla_4\,{}^{(a)}\mathrm{tr}\underline{\chi}\right) f + \left(2\mathrm{tr}\,\underline{\chi} - i\,{}^{(a)}\mathrm{tr}\underline{\chi}\right)\nabla_4 f + r^3\mathfrak{d}\Gamma_b\\
&= \left(2\nabla_4\mathrm{tr}\,\underline{\chi} - i\nabla_4\,{}^{(a)}\mathrm{tr}\underline{\chi}\right) f + \left(2\mathrm{tr}\,\underline{\chi} - i\,{}^{(a)}\mathrm{tr}\underline{\chi}\right)\left(2\mathrm{tr}\,\chi + i\,{}^{(a)}\mathrm{tr}\chi\right) f + r^3\mathfrak{d}\Gamma_b\\
\triangle f &= \left(2\mathrm{d}iv\,(\eta+\underline{\eta}) + i\mathrm{d}iv\,({}^*\eta - {}^*\underline{\eta})\right) f + \left(2(\eta+\underline{\eta}) + i({}^*\eta - {}^*\underline{\eta})\right)\cdot\nabla f + r^3\mathfrak{d}\Gamma_g\\
&= \left(2\mathrm{d}iv\,(\eta+\underline{\eta}) + i\mathrm{d}iv\,({}^*\eta - {}^*\underline{\eta})\right) f\\
&\quad + \left(2(\eta+\underline{\eta}) + i({}^*\eta - {}^*\underline{\eta})\right)\cdot\left(2(\eta+\underline{\eta}) + i({}^*\eta - {}^*\underline{\eta})\right) f + r^3\mathfrak{d}\Gamma_g.
\end{aligned}$$

Using Lemma 4.7.5 applied to a scalar function, we obtain

$$\begin{aligned}
\Im(f^{-1}\square_{\mathbf{g}} f) &= -\Im(f^{-1}\nabla_4\nabla_3 f) - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\Im(f^{-1}\nabla_4 f) + \left(2\omega - \frac{1}{2}\mathrm{t}r\,\chi\right)\Im(f^{-1}\nabla_3 f)\\
&\quad +\Im(f^{-1}\triangle_2 f) + 2\underline{\eta}\cdot\Im(f^{-1}\nabla f)\\
&= \nabla_4\,{}^{(a)}\mathrm{tr}\underline{\chi} - 2\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\chi + 2\mathrm{t}r\,\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\chi\\
&\quad -\left(2\omega - \frac{1}{2}\mathrm{t}r\,\chi\right){}^{(a)}\mathrm{tr}\underline{\chi} + \mathrm{d}iv\,({}^*\eta - {}^*\underline{\eta})\\
&\quad +4(\eta+\underline{\eta})\cdot({}^*\eta - {}^*\underline{\eta}) + 2\underline{\eta}\cdot({}^*\eta - {}^*\underline{\eta}) + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b\\
&= -3\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\chi + 2\mathrm{t}r\,\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} + 2\,{}^*\rho + \mathrm{d}iv\,({}^*\eta + {}^*\underline{\eta}) - 10\eta\wedge\underline{\eta}\\
&\quad +r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b,
\end{aligned}$$

where we used the null structure equation for $\nabla_4\,{}^{(a)}\mathrm{tr}\underline{\chi}$. Similarly we have

$$\begin{aligned}
\Re(f^{-1}\square_{\mathbf{g}} f) &= -\Re(f^{-1}\nabla_4\nabla_3 f) - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\Re(f^{-1}\nabla_4 f) + \left(2\omega - \frac{1}{2}\mathrm{tr}\,\chi\right)\Re(f^{-1}\nabla_3 f)\\
&\quad + \Re(f^{-1}\triangle_2 f) + 2\underline{\eta}\cdot\Re(f^{-1}\nabla f)\\
&= -2\nabla_4\mathrm{tr}\,\underline{\chi} - 4\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} - {}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} - \frac{1}{2}\mathrm{tr}\,\underline{\chi}(2\mathrm{tr}\,\chi)\\
&\quad + \left(2\omega - \frac{1}{2}\mathrm{tr}\,\chi\right)(2\mathrm{tr}\,\underline{\chi}) + 2\mathrm{d}iv\,(\eta+\underline{\eta}) + 4|\eta+\underline{\eta}|^2 - |\,{}^*\eta - {}^*\underline{\eta}|^2\\
&\quad + 2\underline{\eta}\cdot(2(\eta+\underline{\eta}))\\
&= -5\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} - 2\,{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} - 4\rho\\
&\quad + 2\mathrm{d}iv\,(\eta-\underline{\eta}) + 3(|\eta|^2+|\underline{\eta}|^2) + 14\eta\cdot\underline{\eta} + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_g,
\end{aligned}$$

where we used the null structure equation for $\nabla_4\mathrm{tr}\,\underline{\chi}$. □

We now compute $\Im(\widetilde{V})$ using (D.4.15). We have

$$\Im(\widetilde{V}) = -\mathrm{tr}\,\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} + {}^{(a)}\mathrm{tr}\chi\mathrm{tr}\,\underline{\chi} + 4\,{}^*\rho + 2\eta\wedge\underline{\eta} - \Im(\hat{V}),$$

and

$$\Im(\hat{V}) = \Im(I_{33}) + \Im(J_{33}) + \Im(K_{33}) + \Im(M_{33}).$$

Using (D.4.9), (D.4.10), (D.4.12), and (D.4.13), we compute

$$\begin{aligned}
\Im(I_{33}) &= 8\,{}^*\rho - 8\eta\wedge\underline{\eta} + {}^{(c)}\nabla_4\Im(\widetilde{C_1}),\\
\Im(J_{33}) &= -3\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\chi - 2\mathrm{d}iv\,{}^*\eta,\\
\Im(K_{33}) &= -3\,{}^{(c)}\nabla_3\,{}^{(a)}\mathrm{tr}\chi = \frac{3}{2}({}^{(a)}\mathrm{tr}\underline{\chi}\mathrm{tr}\,\chi + \mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\chi) + 6\,{}^*\rho - 6\mathrm{d}iv\,{}^*\eta,\\
\Im(M_{33}) &= \Im(2\left(4(\eta + i\,{}^*\eta) + 2\underline{\eta}\right)\cdot\eta) = 0,
\end{aligned}$$

which gives

$$\Im(\hat{V}) = \frac{3}{2}({}^{(a)}\mathrm{tr}\underline{\chi}\mathrm{tr}\,\chi - \mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\chi) + 14\,{}^*\rho - 8\mathrm{d}iv\,{}^*\eta - 8\eta\wedge\underline{\eta} + {}^{(c)}\nabla_4\Im(\widetilde{C_1}),$$

and

$$\Im(\widetilde{V}) = -\frac{5}{2}(\mathrm{tr}\,\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} - \mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\chi) - 10\,{}^*\rho + 8\mathrm{d}iv\,{}^*\eta + 10\eta\wedge\underline{\eta} - {}^{(c)}\nabla_4\Im(\widetilde{C_1}).$$

Combining the above with Lemma D.4.6, we finally deduce

$$\begin{aligned}\Im(V_1) &= \Im(f^{-1}\square_{\mathbf{g}}(f)+\widetilde{V})\\ &= -\frac{1}{2}(\mathrm{tr}\,\chi\,{}^{(a)}\mathrm{tr}\underline{\chi}+\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\chi)-8\,{}^*\rho+\mathrm{d}iv\,({}^*\eta+{}^*\underline{\eta})+8\mathrm{d}iv\,{}^*\eta-{}^{(c)}\nabla_4\Im(\widetilde{C_1})\\ &\quad +r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b.\end{aligned}$$

Using that, see (3.4.2) and (3.4.2),

$$\begin{aligned}\mathrm{tr}\,\chi\,{}^{(a)}\mathrm{tr}\underline{\chi}+\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\chi &= r^{-1}\Gamma_g,\\ \mathrm{d}iv\,({}^*\eta+{}^*\underline{\eta}) &= r^{-1}\mathfrak{d}\Gamma_g,\end{aligned}$$

we deduce that in order to obtain $\Im(V_1)=0$ we need to have

$$ {}^{(c)}\nabla_4\Im(\widetilde{C_1}) = -8\,{}^*\rho-8\mathrm{d}iv\,{}^*\underline{\eta}+r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b. \tag{D.4.19}$$

Therefore, if $\Im(\widetilde{C_1})=-4\,{}^{(a)}\mathrm{tr}\underline{\chi}$, relation (D.4.19) is satisfied, and V_1 is a real function.

We now explicitly compute V_1 for $\Re(C_1)=2\mathrm{tr}\,\underline{\chi}-2\frac{({}^{(a)}\mathrm{tr}\underline{\chi})^2}{\mathrm{tr}\,\underline{\chi}}$. We have

$$\Re(\widetilde{V}) = \frac{1}{2}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}+\frac{1}{2}\,{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi}-4\rho-4\eta\cdot\underline{\eta}-\Re(\hat{V}),$$

and

$$\Re(\hat{V}) = \Re(I_{33})+\Re(J_{33})+\Re(K_{33})+\Re(M_{33}).$$

Using (D.4.9), (D.4.10), (D.4.12), and (D.4.13), we compute

$$\begin{aligned}\Re(I_{33}) &= -2\rho-2\eta\cdot(\eta-2\underline{\eta})+{}^{(c)}\nabla_4\Re(C_1),\\ \Re(J_{33}) &= -5\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}+2\mathrm{d}iv\,\eta+4|\eta|^2,\\ \Re(K_{33}) &= -5\,{}^{(c)}\nabla_3\mathrm{tr}\,\chi=\frac{5}{2}\mathrm{tr}\,\underline{\chi}\mathrm{tr}\,\chi-\frac{5}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi}\,{}^{(a)}\mathrm{tr}\chi-10\mathrm{d}iv\,\eta-10|\eta|^2-10\rho,\\ \Re(M_{33}) &= \Re(2\eta\cdot\big(4(\eta+i\,{}^*\eta)+2\underline{\eta}\big))=8|\eta|^2+4\eta\cdot\underline{\eta},\end{aligned}$$

which gives

$$\Re(\hat{V}) = {}^{(c)}\nabla_4\Re(C_1)-\frac{5}{2}\mathrm{tr}\,\underline{\chi}\mathrm{tr}\,\chi-\frac{5}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi}\,{}^{(a)}\mathrm{tr}\chi-12\rho-8\mathrm{d}iv\,\eta+8\eta\cdot\underline{\eta},$$

and

$$\Re(\widetilde{V}) = 3\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}+3\,{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi}+8\rho+8\mathrm{d}iv\,\eta-12\eta\cdot\underline{\eta}-{}^{(c)}\nabla_4\Re(C_1).$$

Combining the above with Lemma D.4.6, we finally deduce

$$\begin{aligned} V_1 &= \Re(V_1) = \Re(f^{-1}\square_{\mathbf{g}}(f) + \widetilde{V}) \\ &= -2\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} + {}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} + 4\rho + 2\mathrm{d}iv\,(\eta - \underline{\eta}) + 3(|\eta|^2 + |\underline{\eta}|^2) + 2\eta\cdot\underline{\eta} \\ &\quad +8\mathrm{d}iv\,\eta - {}^{(c)}\nabla_4\Re(C_1). \end{aligned}$$

Using (3.4.2) and (3.4.2), we obtain

$$V_1 = -2\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} + {}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} + 4\rho + 8\mathrm{d}iv\,\underline{\eta} + 6|\underline{\eta}|^2 + 2\eta\cdot\underline{\eta} - {}^{(c)}\nabla_4\Re(C_1).$$

For $\Re(C_1) = 2\mathrm{tr}\,\underline{\chi} - 2\frac{({}^{(a)}\mathrm{tr}\underline{\chi})^2}{\mathrm{tr}\,\underline{\chi}}$, we have

$$\begin{aligned} {}^{(c)}\nabla_4\Re(C_1) &= 2\,{}^{(c)}\nabla_4\mathrm{tr}\,\underline{\chi} - 2\,{}^{(c)}\nabla_4\left(\frac{{}^{(a)}\mathrm{tr}\underline{\chi}^2}{\mathrm{tr}\,\underline{\chi}}\right) \\ &= 2\,{}^{(c)}\nabla_4\mathrm{tr}\,\underline{\chi} - 4\frac{{}^{(a)}\mathrm{tr}\underline{\chi}}{\mathrm{tr}\,\underline{\chi}}\,{}^{(c)}\nabla_4\,{}^{(a)}\mathrm{tr}\underline{\chi} + 2\frac{{}^{(a)}\mathrm{tr}\underline{\chi}^2}{\mathrm{tr}\,\underline{\chi}^2}\,{}^{(c)}\nabla_4\mathrm{tr}\,\underline{\chi} \\ &= -\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} + {}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} + 4\mathrm{d}iv\,\underline{\eta} + 4|\underline{\eta}|^2 + 4\rho - 4\frac{{}^{(a)}\mathrm{tr}\underline{\chi}}{\mathrm{tr}\,\underline{\chi}}\big(2\mathrm{c}url\,\underline{\eta} + 2\,{}^*\rho\big) \\ &\quad +2\frac{{}^{(a)}\mathrm{tr}\underline{\chi}^2}{\mathrm{tr}\,\underline{\chi}^2}\left(-\frac{1}{2}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} + \frac{1}{2}\,{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} + 2\mathrm{d}iv\,\underline{\eta} + 2|\underline{\eta}|^2 + 2\rho\right). \end{aligned}$$

We therefore obtain

$$\begin{aligned} V_1 &= -\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} + 4\mathrm{d}iv\,\underline{\eta} + 2|\underline{\eta}|^2 + 2\eta\cdot\underline{\eta} \\ &\quad +\frac{{}^{(a)}\mathrm{tr}\underline{\chi}}{\mathrm{tr}\,\underline{\chi}}\Big[{}^{(a)}\mathrm{tr}\underline{\chi}\mathrm{tr}\,\chi - \frac{{}^{(a)}\mathrm{tr}\underline{\chi}^2}{\mathrm{tr}\,\underline{\chi}}\,{}^{(a)}\mathrm{tr}\chi - \frac{{}^{(a)}\mathrm{tr}\underline{\chi}}{\mathrm{tr}\,\underline{\chi}}\big(4\mathrm{d}iv\,\underline{\eta} + 4|\underline{\eta}|^2 + 4\rho\big) \\ &\quad +8\mathrm{c}url\,\underline{\eta} + 8\,{}^*\rho\Big]. \end{aligned}$$

Using the values in Kerr given in Section 3.3, we have

$$\begin{aligned} V_1 &= \frac{4r^2\Delta}{|q|^6} + \frac{4a^4\sin^2\theta\cos^2\theta}{|q|^6} + 4\mathrm{d}iv\,\underline{\eta} \\ &\quad -\frac{a\cos\theta}{r}\left(\frac{4a\Delta\cos\theta}{r|q|^4} + \frac{a\cos\theta}{r}\big(4\mathrm{d}iv\,\underline{\eta} + 4|\underline{\eta}|^2 + 4\rho\big) + 8\mathrm{c}url\,\underline{\eta} + 8\,{}^*\rho\right) \\ &= \frac{4r^2\Delta}{|q|^6} + \frac{r^2 - a^2\cos^2\theta}{r^2}4\mathrm{d}iv\,\underline{\eta} - \frac{a\cos\theta}{r}8\mathrm{c}url\,\underline{\eta} \\ &\quad -\frac{4a^2\cos^2\theta}{r^2}\left(\frac{\Delta}{|q|^4} + \frac{a^4\sin^2\theta\cos^2\theta}{|q|^6}\right) - \frac{4a\cos\theta}{r}\left(\frac{a\cos\theta}{r}\rho + 2\,{}^*\rho\right). \end{aligned}$$

By writing

$$\begin{aligned} \mathrm{d}iv\, \underline{\eta} &= e_1(\underline{\eta}_1) + \Lambda \underline{\eta}_1 = \frac{1}{|q|^6}((-3\cos^2\theta + 1)a^2r^2 + a^4\cos^2\theta(-3+\cos^2\theta)),\\ curl\, \underline{\eta} &= e_1(\underline{\eta}_2) + \Lambda \underline{\eta}_2 = \frac{a\cos\theta}{|q|^6}(-2r^3 + 2a^2\cos^2\theta r - 4a^2 r), \end{aligned}$$

we obtain

$$\begin{aligned} \frac{1}{4}V_1 &= \frac{r^2(r^2-2mr+a^2)}{|q|^6} + \frac{1}{|q|^6}((-3\cos^2\theta+1)a^2r^2 + a^4\cos^2\theta(-3+\cos^2\theta))\\ &\quad -\frac{a^2\cos^2\theta}{r^2|q|^6}(-4r^4 + (\cos^2\theta - 7)a^2r^2 - 2a^4\cos^2\theta)\\ &\quad -\frac{a^2\cos^2\theta}{r^2|q|^4}\Delta - \frac{a^2\cos^2\theta}{r|q|^6}\big(10mr^2 + 2ma^2\cos^2\theta\big)\\ &= \frac{r^2(r^2-2mr+2a^2)}{|q|^6} - \frac{a^2\cos^2\theta}{r^2|q|^6}(8mr^3 - 3a^2r^2 + a^2\cos^2\theta r^2 - a^4\cos^2\theta). \end{aligned}$$

Writing

$$\begin{aligned} &\frac{1}{|q|^2}\frac{r^2-2mr+2a^2}{r^2}\\ &= \frac{(r^4 + 2a^2r^2\cos^2\theta + a^4\cos^4\theta)}{|q|^6}\frac{(r^2-2mr+2a^2)}{r^2}\\ &= \frac{r^2(r^2-2mr+2a^2)}{|q|^6}\\ &\quad + \frac{a^2\cos^2\theta}{r^2|q|^6}(2r^4 - 4mr^3 + 4a^2r^2 + a^2\cos^2\theta r^2 - 2mra^2\cos^2\theta + 2a^4\cos^2\theta), \end{aligned}$$

we obtain

$$\begin{aligned} \frac{1}{4}V_1 &= \frac{1}{|q|^2}\frac{r^2-2mr+2a^2}{r^2}\\ &\quad -\frac{a^2\cos^2\theta}{r^2|q|^6}(2r^4 + 4mr^3 + a^2r^2 + 2a^2\cos^2\theta r^2 - 2mra^2\cos^2\theta + a^4\cos^2\theta), \end{aligned}$$

as stated.

Recall from Proposition D.4.1, that

$$Z_{43} = {}^{(c)}\nabla_3\widetilde{C_1} + \mathrm{tr}\,\underline{\chi}\widetilde{C_1}.$$

In particular we have

$$\Im(Z_{43}) = {}^{(c)}\nabla_3 \Im(\widetilde{C_1}) + \mathrm{tr}\,\underline{\chi} \Im(\widetilde{C_1}) = r^{-1}\Gamma_b$$

if $\Im(\widetilde{C_1}) = n\,{}^{(a)}\mathrm{tr}\underline{\chi}$ for any n. We can also explicitly compute for $\Re(\widetilde{C_1}) = -2\frac{({}^{(a)}\mathrm{tr}\underline{\chi})^2}{\mathrm{tr}\,\underline{\chi}}$,

$$\begin{aligned}
Z_{43} &= -2\,{}^{(c)}\nabla_3 \left(\frac{({}^{(a)}\mathrm{tr}\underline{\chi})^2}{\mathrm{tr}\,\underline{\chi}} \right) + \mathrm{tr}\,\underline{\chi} \left(-2\frac{({}^{(a)}\mathrm{tr}\underline{\chi})^2}{\mathrm{tr}\,\underline{\chi}} \right) \\
&= -4\frac{{}^{(a)}\mathrm{tr}\underline{\chi}}{\mathrm{tr}\,\underline{\chi}}\,{}^{(c)}\nabla_3\,{}^{(a)}\mathrm{tr}\underline{\chi} + 2\frac{{}^{(a)}\mathrm{tr}\underline{\chi}^2}{\mathrm{tr}\,\underline{\chi}^2}\,{}^{(c)}\nabla_3 \mathrm{tr}\,\underline{\chi} - 2({}^{(a)}\mathrm{tr}\underline{\chi})^2 \\
&= -4\frac{{}^{(a)}\mathrm{tr}\underline{\chi}}{\mathrm{tr}\,\underline{\chi}}(-\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\underline{\chi}) - \frac{{}^{(a)}\mathrm{tr}\underline{\chi}^2}{\mathrm{tr}\,\underline{\chi}^2}(\mathrm{tr}\,\underline{\chi}^2 - {}^{(a)}\mathrm{tr}\underline{\chi}^2) - 2({}^{(a)}\mathrm{tr}\underline{\chi})^2 + r^{-2}\Gamma_b \\
&= {}^{(a)}\mathrm{tr}\underline{\chi}^2 + \frac{{}^{(a)}\mathrm{tr}\underline{\chi}^4}{\mathrm{tr}\,\underline{\chi}^2} + r^{-2}\Gamma_b.
\end{aligned}$$

We now consider Z_{a3}. Recall from Proposition D.4.1, that

$$Z_{a3} = I_{a3} + J_{a3} + L_{a3} + M_{a3} - 4C_1\underline{\eta},$$

with

$$\begin{aligned}
I_{a3} &= -2\,{}^{(c)}\nabla_3(\eta - \underline{\eta}) - (2C_1 - \mathrm{tr}\,\underline{\chi})(\eta - \underline{\eta}) - {}^{(a)}\mathrm{tr}\underline{\chi}\,{}^*(\eta - \underline{\eta}) \\
J_{a3} &= 2\,{}^{(c)}\nabla_3\eta - 2\,{}^{(c)}\nabla(C_1) + 2C_1\eta + \mathrm{tr}\,\underline{\chi}(8\underline{\eta} + 2\eta) - 2\,{}^{(a)}\mathrm{tr}\underline{\chi}\,{}^*\underline{\eta} \\
&\quad +i\left(4\mathrm{tr}\,\underline{\chi}\,{}^*\eta + 2\,{}^{(a)}\mathrm{tr}\underline{\chi}\eta + 2\,{}^{(a)}\mathrm{tr}\underline{\chi}\underline{\eta}\right) \\
L_{a3} &= -2(\mathrm{tr}\,\underline{\chi} - i\,{}^{(a)}\mathrm{tr}\underline{\chi})(\eta - \underline{\eta}) \\
M_{a3} &= 2\,{}^{(c)}\nabla_3\left(4H + \underline{H} + \overline{\underline{H}}\right) - \mathrm{tr}\,\underline{\chi}\left(4H + \underline{H} + \overline{\underline{H}}\right) + {}^{(a)}\mathrm{tr}\underline{\chi}\,{}^*\left(4H + \underline{H} + \overline{\underline{H}}\right).
\end{aligned}$$

We compute

$$\begin{aligned}
\Im(L_{a3}) &= 2\,{}^{(a)}\mathrm{tr}\underline{\chi}(\eta - \underline{\eta}), \\
\Im(M_{a3}) &= \Im(2\,{}^{(c)}\nabla_3\,(4\eta + 4i\,{}^*\eta) - \mathrm{tr}\,\underline{\chi}\,(4\eta + 4i\,{}^*\eta) + {}^{(a)}\mathrm{tr}\underline{\chi}\,{}^*\,(4\eta + 4i\,{}^*\eta)) \\
&= 8\,{}^{(c)}\nabla_3\,{}^*\eta - 4\mathrm{tr}\,\underline{\chi}\,{}^*\eta - 4\,{}^{(a)}\mathrm{tr}\underline{\chi}\eta.
\end{aligned}$$

and, recalling that $\Im(C_1) = \Im(\widetilde{C_1})$,

$$\begin{aligned}
\Im(I_{a3}) &= -2\Im(\widetilde{C_1})(\eta - \underline{\eta}), \\
\Im(J_{a3}) &= -2\,{}^{(c)}\nabla\Im(\widetilde{C_1}) + 2\Im(\widetilde{C_1})\eta + 4\mathrm{tr}\,\underline{\chi}\,{}^*\eta + 2\,{}^{(a)}\mathrm{tr}\underline{\chi}\eta + 2\,{}^{(a)}\mathrm{tr}\underline{\chi}\underline{\eta}.
\end{aligned}$$

We therefore obtain

$$\Im(Z_{a3}) = -2\,{}^{(c)}\nabla\Im(\widetilde{C_1}) - 2\Im(\widetilde{C_1})\underline{\eta} + 8\,{}^{(c)}\nabla_3\,{}^*\eta.$$

Using that, see (3.4.8),

$$\nabla_3\,{}^*\eta + \mathrm{tr}\,\underline{\chi}\,{}^*\eta - {}^{(a)}\mathrm{tr}\underline{\chi}\eta = \mathfrak{d}^{\leq 1}\Gamma_g,$$

we deduce that in order to have $\Im(Z_{a3}) = 0$, we need to have

$$ {}^{(c)}\nabla\Im(\widetilde{C_1}) + \Im(\widetilde{C_1})\underline{\eta} = -4(\mathrm{tr}\,\underline{\chi}\,{}^*\eta - {}^{(a)}\mathrm{tr}\underline{\chi}\eta) + \mathfrak{d}^{\leq 1}\Gamma_g. \tag{D.4.20}$$

Observe that, using (3.4.10), i.e.

$$\nabla({}^{(a)}\mathrm{tr}\underline{\chi}) = -\frac{3}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi}(\underline{\eta}+\eta) + \frac{1}{2}\mathrm{tr}\,\underline{\chi}({}^*\eta - {}^*\underline{\eta}) + r^{-1}\Gamma_g,$$

we obtain

$$\begin{aligned}
{}^{(c)}\nabla\,{}^{(a)}\mathrm{tr}\underline{\chi} + {}^{(a)}\mathrm{tr}\underline{\chi}\underline{\eta} &= \nabla\,{}^{(a)}\mathrm{tr}\underline{\chi} + {}^{(a)}\mathrm{tr}\underline{\chi}(\underline{\eta}-\zeta)\\
&= -\frac{3}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi}(\underline{\eta}+\eta) + \frac{1}{2}\mathrm{tr}\,\underline{\chi}({}^*\eta - {}^*\underline{\eta}) + {}^{(a)}\mathrm{tr}\underline{\chi}(\underline{\eta}-\zeta) + r^{-1}\Gamma_g\\
&= \frac{1}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi}\underline{\eta} - \frac{3}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi}\eta + \frac{1}{2}\mathrm{tr}\,\underline{\chi}({}^*\eta - {}^*\underline{\eta}) + r^{-1}\Gamma_g\\
&= \mathrm{tr}\,\underline{\chi}\,{}^*\eta - {}^{(a)}\mathrm{tr}\underline{\chi}\eta + r^{-1}\Gamma_g,
\end{aligned}$$

where we used, see (3.4.8), that $\mathrm{tr}\,\underline{\chi}({}^*\eta + {}^*\underline{\eta}) - {}^{(a)}\mathrm{tr}\underline{\chi}(\underline{\eta}-\eta) = r^{-1}\Gamma_g$. Therefore, if $\Im(\widetilde{C_1}) = -4\,{}^{(a)}\mathrm{tr}\underline{\chi}$, relation (D.4.20) is satisfied, and Z_{a3} is a real one-form.

We now consider $\Re(Z_{a3})$. We compute

$$\begin{aligned}
\Re(L_{a3}) &= -2\mathrm{tr}\,\underline{\chi}(\eta-\underline{\eta})\\
\Re(M_{a3}) &= 4\,{}^{(c)}\nabla_3\left(2\eta+\underline{\eta}\right) - \mathrm{tr}\,\underline{\chi}\left(4\eta+2\underline{\eta}\right) + {}^{(a)}\mathrm{tr}\underline{\chi}\,{}^*\left(4\eta+2\underline{\eta}\right),
\end{aligned}$$

and, recalling that $\Re(C_1) = 2\mathrm{tr}\,\underline{\chi} + \Re(\widetilde{C_1})$,

$$\begin{aligned}
\Re(I_{a3}) &= -2\,{}^{(c)}\nabla_3(\eta-\underline{\eta}) - (2\Re(C_1) - \mathrm{tr}\,\underline{\chi})(\eta-\underline{\eta}) - {}^{(a)}\mathrm{tr}\underline{\chi}\,{}^*(\eta-\underline{\eta})\\
&= -2\,{}^{(c)}\nabla_3(\eta-\underline{\eta}) - (2\Re(\widetilde{C_1}) + 3\mathrm{tr}\,\underline{\chi})(\eta-\underline{\eta}) - {}^{(a)}\mathrm{tr}\underline{\chi}\,{}^*(\eta-\underline{\eta})\\
\Re(J_{a3}) &= 2\,{}^{(c)}\nabla_3\eta - 2\,{}^{(c)}\nabla\Re(C_1) + 2\Re(C_1)\eta + \mathrm{tr}\,\underline{\chi}(8\underline{\eta}+2\eta) - 2\,{}^{(a)}\mathrm{tr}\underline{\chi}\,{}^*\underline{\eta}\\
&= 2\,{}^{(c)}\nabla_3\eta - 4\,{}^{(c)}\nabla\mathrm{tr}\,\underline{\chi} - 2\,{}^{(c)}\nabla\Re(\widetilde{C_1}) + 2(2\mathrm{tr}\,\underline{\chi} + \Re(\widetilde{C_1}))\eta\\
&\quad +\mathrm{tr}\,\underline{\chi}(8\underline{\eta}+2\eta) - 2\,{}^{(a)}\mathrm{tr}\underline{\chi}\,{}^*\underline{\eta}\\
&= 2\,{}^{(c)}\nabla_3\eta - 4\,{}^{(c)}\nabla\mathrm{tr}\,\underline{\chi} - 2\,{}^{(c)}\nabla\Re(\widetilde{C_1}) + 2\Re(\widetilde{C_1})\eta + \mathrm{tr}\,\underline{\chi}(8\underline{\eta}+6\eta) - 2\,{}^{(a)}\mathrm{tr}\underline{\chi}\,{}^*\underline{\eta}.
\end{aligned}$$

We therefore obtain

$$
\begin{aligned}
\Re(Z_{a3}) &= -2\,{}^{(c)}\nabla_3(\eta-\underline{\eta}) - (2\Re(\widetilde{C_1}) + 3\mathrm{tr}\,\underline{\chi})(\eta-\underline{\eta}) - {}^{(a)}\mathrm{tr}\underline{\chi}\,{}^*(\eta-\underline{\eta})\\
&\quad +2\,{}^{(c)}\nabla_3\eta - 4\,{}^{(c)}\nabla\mathrm{tr}\,\underline{\chi} - 2\,{}^{(c)}\nabla\Re(\widetilde{C_1}) + 2\Re(\widetilde{C_1})\eta + \mathrm{tr}\,\underline{\chi}(8\underline{\eta}+6\eta) - 2\,{}^{(a)}\mathrm{tr}\underline{\chi}\,{}^*\underline{\eta}\\
&\quad -2\mathrm{tr}\,\underline{\chi}(\eta-\underline{\eta}) + 4\,{}^{(c)}\nabla_3\left(2\eta+\underline{\eta}\right) - \mathrm{tr}\,\underline{\chi}\left(4\eta+2\underline{\eta}\right) + {}^{(a)}\mathrm{tr}\underline{\chi}\,{}^*\left(4\eta+2\underline{\eta}\right)\\
&\quad -4(2\mathrm{tr}\,\underline{\chi} + \Re(\widetilde{C_1}))\underline{\eta}\\
&= 6\,{}^{(c)}\nabla_3\underline{\eta} + 8\,{}^{(c)}\nabla_3\eta - 4\,{}^{(c)}\nabla\mathrm{tr}\,\underline{\chi} - 2\,{}^{(c)}\nabla\Re(\widetilde{C_1}) - 2\Re(\widetilde{C_1})\underline{\eta}\\
&\quad +\mathrm{tr}\,\underline{\chi}(3\underline{\eta}-3\eta) + {}^{(a)}\mathrm{tr}\underline{\chi}\,{}^*\left(3\eta+\underline{\eta}\right).
\end{aligned}
$$

Using the null structure equation for ${}^{(c)}\nabla_3\underline{\eta}$, (3.4.8) and (3.4.10), we obtain

$$
\begin{aligned}
\Re(Z_{a3}) &= 6\big(-\frac{1}{2}\mathrm{tr}\,\underline{\chi}(\underline{\eta}-\eta) + \frac{1}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi}({}^*\underline{\eta} - {}^*\eta)\big) + 8(-\mathrm{tr}\,\underline{\chi}\eta - {}^{(a)}\mathrm{tr}\underline{\chi}\,{}^*\eta)\\
&\quad -4\big(-\frac{3}{2}\mathrm{tr}\,\underline{\chi}\left(\underline{\eta}+\eta\right) - \frac{1}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi}\left({}^*\eta - {}^*\underline{\eta}\right) + \mathrm{tr}\,\underline{\chi\eta}\big) - 2\,{}^{(c)}\nabla\Re(\widetilde{C_1}) - 2\Re(\widetilde{C_1})\underline{\eta}\\
&\quad +\mathrm{tr}\,\underline{\chi}(3\underline{\eta}-3\eta) + {}^{(a)}\mathrm{tr}\underline{\chi}\,{}^*\left(3\eta+\underline{\eta}\right) + r^{-1}\Gamma_b\\
&= -2\,{}^{(c)}\nabla\Re(\widetilde{C_1}) - 2\Re(\widetilde{C_1})\underline{\eta} + 2\mathrm{tr}\,\underline{\chi}\left(\underline{\eta}-\eta\right) + 2\,{}^{(a)}\mathrm{tr}\underline{\chi}({}^*\underline{\eta} - 3\,{}^*\eta) + r^{-1}\Gamma_b.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
{}^{(c)}\nabla\left(\frac{{}^{(a)}\mathrm{tr}\underline{\chi}^2}{\mathrm{tr}\,\underline{\chi}}\right) &= \nabla\left(\frac{{}^{(a)}\mathrm{tr}\underline{\chi}^2}{\mathrm{tr}\,\underline{\chi}}\right) - \zeta\frac{{}^{(a)}\mathrm{tr}\underline{\chi}^2}{\mathrm{tr}\,\underline{\chi}}\\
&= \nabla\left(\frac{{}^{(a)}\mathrm{tr}\underline{\chi}^2}{\mathrm{tr}\,\underline{\chi}}\right) + \underline{\eta}\frac{{}^{(a)}\mathrm{tr}\underline{\chi}^2}{\mathrm{tr}\,\underline{\chi}}\\
&= -\frac{{}^{(a)}\mathrm{tr}\underline{\chi}^2}{\mathrm{tr}\,\underline{\chi}^2}\nabla\mathrm{tr}\,\underline{\chi} + \frac{2\,{}^{(a)}\mathrm{tr}\underline{\chi}}{\mathrm{tr}\,\underline{\chi}}\nabla\,{}^{(a)}\mathrm{tr}\underline{\chi} + \underline{\eta}\frac{{}^{(a)}\mathrm{tr}\underline{\chi}^2}{\mathrm{tr}\,\underline{\chi}}.
\end{aligned}
$$

Using (3.4.10), we obtain

$$
\begin{aligned}
&{}^{(c)}\nabla\left(\frac{{}^{(a)}\mathrm{tr}\underline{\chi}^2}{\mathrm{tr}\,\underline{\chi}}\right)\\
&= -\frac{{}^{(a)}\mathrm{tr}\underline{\chi}^2}{\mathrm{tr}\,\underline{\chi}^2}\left(-\frac{3}{2}\mathrm{tr}\,\underline{\chi}\left(\underline{\eta}+\eta\right) - \frac{1}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi}\left({}^*\eta - {}^*\underline{\eta}\right)\right)\\
&\quad +\frac{2\,{}^{(a)}\mathrm{tr}\underline{\chi}}{\mathrm{tr}\,\underline{\chi}}\left(-\frac{3}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi}(\underline{\eta}+\eta) + \frac{1}{2}\mathrm{tr}\,\underline{\chi}({}^*\eta - {}^*\underline{\eta})\right) + \underline{\eta}\frac{{}^{(a)}\mathrm{tr}\underline{\chi}^2}{\mathrm{tr}\,\underline{\chi}} + r^{-2}\Gamma_b\\
&= -\frac{{}^{(a)}\mathrm{tr}\underline{\chi}^2}{\mathrm{tr}\,\underline{\chi}}\left(\frac{1}{2}\underline{\eta} + \frac{3}{2}\eta\right) + \frac{1}{2}\frac{{}^{(a)}\mathrm{tr}\underline{\chi}^3}{\mathrm{tr}\,\underline{\chi}^2}\left({}^*\eta - {}^*\underline{\eta}\right) + {}^{(a)}\mathrm{tr}\underline{\chi}({}^*\eta - {}^*\underline{\eta}) + r^{-2}\Gamma_b,
\end{aligned}
$$

which implies

$$
\begin{aligned}
& {}^{(c)}\nabla\left(\frac{{}^{(a)}\mathrm{tr}\underline{\chi}^2}{\mathrm{tr}\,\underline{\chi}}\right)+\underline{\eta}\frac{{}^{(a)}\mathrm{tr}\underline{\chi}^2}{\mathrm{tr}\,\underline{\chi}}\\
= \ & \frac{{}^{(a)}\mathrm{tr}\underline{\chi}^2}{\mathrm{tr}\,\underline{\chi}}\left(\frac{1}{2}\underline{\eta}-\frac{3}{2}\eta\right)+{}^{(a)}\mathrm{tr}\underline{\chi}({}^*\eta-{}^*\underline{\eta})+\frac{1}{2}\frac{{}^{(a)}\mathrm{tr}\underline{\chi}^3}{\mathrm{tr}\,\underline{\chi}^2}\left({}^*\eta-{}^*\underline{\eta}\right)+r^{-2}\Gamma_b.
\end{aligned}
$$

Therefore, if $\Re(\widetilde{C_1})=-2\frac{({}^{(a)}\mathrm{tr}\underline{\chi})^2}{\mathrm{tr}\,\underline{\chi}}$, we have

$$
\begin{aligned}
& \Re(Z_{a3})\\
= \ & 4\Big({}^{(c)}\nabla\left(\frac{({}^{(a)}\mathrm{tr}\underline{\chi})^2}{\mathrm{tr}\,\underline{\chi}}\right)+\frac{({}^{(a)}\mathrm{tr}\underline{\chi})^2}{\mathrm{tr}\,\underline{\chi}}\underline{\eta}\Big)+2\mathrm{tr}\,\underline{\chi}\left(\underline{\eta}-\eta\right)+2\,{}^{(a)}\mathrm{tr}\underline{\chi}({}^*\underline{\eta}-3\,{}^*\eta)+r^{-1}\Gamma_b\\
= \ & 4\Big(\frac{{}^{(a)}\mathrm{tr}\underline{\chi}^2}{\mathrm{tr}\,\underline{\chi}}\left(\frac{1}{2}\underline{\eta}-\frac{3}{2}\eta\right)+{}^{(a)}\mathrm{tr}\underline{\chi}({}^*\eta-{}^*\underline{\eta})+\frac{1}{2}\frac{{}^{(a)}\mathrm{tr}\underline{\chi}^3}{\mathrm{tr}\,\underline{\chi}^2}\left({}^*\eta-{}^*\underline{\eta}\right)\Big)\\
& +2\mathrm{tr}\,\underline{\chi}\left(\underline{\eta}-\eta\right)+2\,{}^{(a)}\mathrm{tr}\underline{\chi}({}^*\underline{\eta}-3\,{}^*\eta)+r^{-1}\Gamma_b\\
= \ & 2\frac{{}^{(a)}\mathrm{tr}\underline{\chi}^2}{\mathrm{tr}\,\underline{\chi}}\left(\underline{\eta}-3\eta\right)+2\frac{{}^{(a)}\mathrm{tr}\underline{\chi}^3}{\mathrm{tr}\,\underline{\chi}^2}\left({}^*\eta-{}^*\underline{\eta}\right)+2\mathrm{tr}\,\underline{\chi}\left(\underline{\eta}-\eta\right)-2\,{}^{(a)}\mathrm{tr}\underline{\chi}({}^*\underline{\eta}+{}^*\eta)+r^{-1}\Gamma_b.
\end{aligned}
$$

Evaluating the above to $e_a=e_1$ and using that $\eta_1-\underline{\eta}_1=\Gamma_g$, ${}^*\eta_1+{}^*\underline{\eta}_1=\Gamma_g$, we obtain

$$
\begin{aligned}
\Re(Z_{13}) &= -4\frac{{}^{(a)}\mathrm{tr}\underline{\chi}^2}{\mathrm{tr}\,\underline{\chi}}\underline{\eta}_1-4\frac{{}^{(a)}\mathrm{tr}\underline{\chi}^3}{\mathrm{tr}\,\underline{\chi}^2}\,{}^*\underline{\eta}_1+r^{-1}\Gamma_b\\
&= -4\frac{{}^{(a)}\mathrm{tr}\underline{\chi}^2}{\mathrm{tr}\,\underline{\chi}}\Big(-\frac{a^2\sin\theta\cos\theta}{|q|^3}+\frac{a\cos\theta}{r}\frac{ar\sin\theta}{|q|^3}\Big)+r^{-1}\Gamma_b=r^{-1}\Gamma_b.
\end{aligned}
$$

Evaluating the above to $e_a=e_2$ and using that $\eta_2+\underline{\eta}_2=\Gamma_g$, ${}^*\eta_2-{}^*\underline{\eta}_2=\Gamma_g$

$$
\begin{aligned}
\Re(Z_{23}) &= 8\frac{{}^{(a)}\mathrm{tr}\underline{\chi}^2}{\mathrm{tr}\,\underline{\chi}}\underline{\eta}_2+4\mathrm{tr}\,\underline{\chi}\underline{\eta}_2-4\,{}^{(a)}\mathrm{tr}\underline{\chi}\,{}^*\underline{\eta}_2+r^{-1}\Gamma_b\\
&= 8\frac{a\cos\theta}{r}\frac{2a\Delta\cos\theta}{|q|^4}\frac{ar\sin\theta}{|q|^3}+4\frac{2r\Delta}{|q|^4}\frac{ar\sin\theta}{|q|^3}-4\frac{2a\Delta\cos\theta}{|q|^4}\frac{a^2\sin\theta\cos\theta}{|q|^3}+r^{-1}\Gamma_b\\
&= \frac{8a\sin\theta\Delta}{|q|^5}+r^{-1}\Gamma_b,
\end{aligned}
$$

as stated. □

We therefore deduce the following intermediate theorem for the gRW equation.

Theorem D.4.7. *For the following choices of complex scalar functions* C_1, C_2, *i.e.*

$$C_1 \;=\; 2tr\underline{\chi} - 2\frac{{}^{(a)}tr\underline{\chi}^2}{tr\underline{\chi}} - 4i\,{}^{(a)}tr\underline{\chi}, \qquad C_2 = \frac{1}{2}tr\underline{\chi}^2 + \widetilde{C_2}, \tag{D.4.21}$$

where $\widetilde{C_2}$ *is any complex function satisfying* $\widetilde{C_2} = O(|a|r^{-3})$, *the invariant symmetric traceless* 2*-tensor* $\mathfrak{q} \in \mathfrak{s}_2(\mathbb{C})$ *satisfies the equation*

$$\dot{\square}_2\mathfrak{q} - i\frac{4a\cos\theta}{|q|^2}\nabla_{\mathbf{T}}\mathfrak{q} - V_1\mathfrak{q} \;=\; \widetilde{L_{\mathfrak{q}}[A]} + Err[\dot{\square}_2\mathfrak{q}], \tag{D.4.22}$$

where

- *The potential* V_1 *is a* ***real*** *scalar function given by* $V_1 = -tr\,\chi tr\underline{\chi} + O(\frac{|a|}{r^4})$,
- $\widetilde{L_{\mathfrak{q}}[A]}$ *is a linear second order operator in* A *vanishing for zero angular momentum, given by*

$$\begin{aligned}-\widetilde{L_{\mathfrak{q}}[A]} \;&=\; q\overline{q}^3\Big[Z_{43}\ {}^{(c)}\nabla_4\,{}^{(c)}\nabla_3 A + Z_{23}\,{}^{(c)}\nabla_2\,{}^{(c)}\nabla_3 A\\ &\quad + Z_4\ {}^{(c)}\nabla_4 A + Z_3\,{}^{(c)}\nabla_3 A + Z_a\,{}^{(c)}\nabla_a A + Z_0 A\Big],\end{aligned}$$

  where in the outgoing frame

$$Z_{43} = \frac{4a^2\Delta^2\cos^2\theta}{r^2|q|^6}, \qquad Z_{23} = \frac{8a\sin\theta\Delta}{|q|^5}$$

  and Z_4, Z_3, Z_a, Z_0 *have the following fall-off in* r*:*

$$Z_4 = O(|a|r^{-4}), \qquad Z_3 = O(|a|r^{-4}), \qquad Z_a = O(|a|r^{-4}), \qquad Z_0 = O(|a|r^{-5}).$$

- $Err[\dot{\square}_2\mathfrak{q}]$ *is the nonlinear correction term, which is given schematically by the expression*

$$\begin{aligned}Err[\dot{\square}_2\mathfrak{q}] \;&=\; r^2\mathfrak{d}^{\leq 3}(\Gamma_g\cdot(A,B)) + \nabla_3(r^2\mathfrak{d}^{\leq 2}(\Gamma_b\cdot(A,B)))\\ &\quad + \mathfrak{d}^{\leq 1}(\Gamma_g\cdot\mathfrak{q}) + r^3\mathfrak{d}^{\leq 2}\big(\Gamma_b\cdot\Gamma_g\cdot\Gamma_g\big).\end{aligned}$$

Observe that, by expanding the conformal derivatives, the linear second order operator $\widetilde{L_{\mathfrak{q}}[A]}$ can be written as

$$-\widetilde{L_{\mathfrak{q}}[A]} \;=\; q\overline{q}^3\Big[Z_{43}\ \nabla_4\nabla_3 A + Z_{23}\nabla_2\nabla_3 A + W_4\ \nabla_4 A + W_3\nabla_3 A + W_a\nabla_a A + W_0 A\Big],$$

where W_4, W_3, W_a, W_0 have the same fall-off in r as Z_4, Z_3, Z_a, Z_0 respectively.

We now combine parts of the potential with the lower order terms $\widetilde{L_{\mathfrak{q}}[A]}$. In particular, we can write

$$\begin{aligned}
&-\frac{|q|^2}{\Delta}Z_{43}\ \mathfrak{q}+\widetilde{L_{\mathfrak{q}}[A]}\\
=\ & -\frac{|q|^2}{\Delta}Z_{43}q\overline{q}^3\big[\nabla_3\nabla_3 A+O(r^{-1})\nabla_3 A+O(r^{-2})A\big]\\
&-q\overline{q}^3\Big[Z_{43}\ \nabla_4\nabla_3 A+Z_{23}\nabla_2\nabla_3 A+W_4\ \nabla_4 A+W_3\nabla_3 A+W_a\nabla_a A+W_0 A\Big]\\
=\ & -q\overline{q}^3\big[Z_{43}\big(\frac{|q|^2}{\Delta}\nabla_3+\nabla_4\big)\nabla_3 A+Z_{23}\nabla_2\nabla_3 A\\
&+W_4\ \nabla_4 A+W_3\nabla_3 A+W_a\nabla_a A+W_0 A\Big].
\end{aligned}$$

Using that with the outgoinng normalization of (e_3, e_4),

$$\frac{1}{2}\left(\frac{\Delta}{r^2+a^2}e_4+\frac{|q|^2}{r^2+a^2}e_3\right)=\mathbf{T}+\frac{a}{r^2+a^2}\mathbf{Z},$$

and $e_2=\frac{a\sin\theta}{|q|}\nabla_{\mathbf{T}}+\frac{1}{|q|\sin\theta}\nabla_{\mathbf{Z}}$, we obtain

$$\begin{aligned}
-\frac{|q|^2}{\Delta}Z_{43}\ \mathfrak{q}+\widetilde{L_{\mathfrak{q}}[A]} &= -q\overline{q}^3\big[\big(2Z_{43}\frac{r^2+a^2}{\Delta}+Z_{23}\frac{a\sin\theta}{|q|}\big)\nabla_{\mathbf{T}}\nabla_3 A\\
&\quad+\big(2Z_{43}\frac{a}{\Delta}+Z_{23}\frac{1}{|q|\sin\theta}\big)\nabla_{\mathbf{Z}}\nabla_3 A\\
&\quad+W_4\ \nabla_4 A+W_3\nabla_3 A+W_a\nabla_a A+W_0 A\Big]\\
&= -q\overline{q}^3\big[\frac{8a^2\Delta}{r^2|q|^4}\nabla_{\mathbf{T}}\nabla_3 A+\frac{8a\Delta}{r^2|q|^4}\nabla_{\mathbf{Z}}\nabla_3 A\\
&\quad+W_4\ \nabla_4 A+W_3\nabla_3 A+W_a\nabla_a A+W_0 A\Big],
\end{aligned}$$

where we used the values of Z_{43} and Z_{23} obtained in Proposition D.4.5. Combining the above with equation (D.4.22), we finally obtain

$$\dot{\square}_2\mathfrak{q}-i\frac{4a\cos\theta}{|q|^2}\nabla_{\mathbf{T}}\mathfrak{q}-\big(V_1+\frac{|q|^2}{\Delta}Z_{43}\big)\ \mathfrak{q}\ \ =\ \ L_{\mathfrak{q}}[A]+\mathrm{Err}[\dot{\square}_2\mathfrak{q}],$$

where $L_{\mathfrak{q}}[A]:=-\frac{|q|^2}{\Delta}Z_{43}\ \mathfrak{q}+\widetilde{L_{\mathfrak{q}}[A]}$, is given by

$$\begin{aligned}
-L_{\mathfrak{q}}[A] &= q\overline{q}^3\big[\frac{8a^2\Delta}{r^2|q|^4}\nabla_{\mathbf{T}}\nabla_3 A+\frac{8a\Delta}{r^2|q|^4}\nabla_{\mathbf{Z}}\nabla_3 A\\
&\quad+W_4\ \nabla_4 A+W_3\nabla_3 A+W_a\nabla_a A+W_0 A\big].
\end{aligned}$$

By defining

$$\begin{aligned} V &:= V_1 + \frac{|q|^2}{\Delta} Z_{43} \\ &= \frac{4}{|q|^2} \frac{r^2 - 2mr + 2a^2}{r^2} - \frac{4a^2 \cos^2\theta}{|q|^6} (r^2 + 6mr + a^2 \cos^2\theta), \end{aligned}$$

we finally completed the proof of Theorem 5.2.9.

D.5 Proof of Proposition 5.3.1

D.5.1 Preliminaries

We make use of our gauge conditions $\Xi = 0$, $\underline{\widecheck{H}} = 0$ to derive the following linearized equations.

Lemma D.5.1. *We have*

$$\begin{aligned} {}^{(c)}\mathcal{D}\overline{trX} &= 2i\Im(trX)(H - \widecheck{H}) + r^{-1}\Gamma_g, \\ {}^{(c)}\mathcal{D}trX &= -2trX\,\underline{H} + r^{-1}\Gamma_g. \end{aligned} \tag{D.5.1}$$

Also,

$$\begin{aligned} {}^{(c)}\mathcal{D}\overline{tr\underline{X}} &= 2i\Im(tr\underline{X})\,\underline{H} + r^{-1}\Gamma_g, \\ {}^{(c)}\mathcal{D}tr\underline{X} &= -2tr\underline{X}(H - \widecheck{H}) + r^{-1}\Gamma_g. \end{aligned} \tag{D.5.2}$$

Proof. Since $\mathrm{tr}X = \frac{2\Delta\overline{q}}{|q|^4} + \Gamma_g$, and $Z = \frac{aq}{|q|^2}\mathfrak{J} + \Gamma_g$, we have

$$\begin{aligned} {}^{(c)}\mathcal{D}(\overline{\mathrm{tr}X}) &= \mathcal{D}\overline{\mathrm{tr}X} + \overline{\mathrm{tr}X}Z = \mathcal{D}\left(\frac{2\Delta q}{|q|^4} + \Gamma_g\right) + \left(\frac{2\Delta q}{|q|^4} + \Gamma_g\right)\left(\frac{aq}{|q|^2}\mathfrak{J} + \Gamma_g\right) \\ &= \mathcal{D}\left(\frac{2\Delta q}{|q|^4}\right) + \frac{2\Delta q}{|q|^4}\frac{aq}{|q|^2}\mathfrak{J} + r^{-1}\Gamma_g. \end{aligned}$$

Since $\mathcal{D}q = -a\mathfrak{J} + r\Gamma_g$, $\mathcal{D}\overline{q} = a\mathfrak{J} + r\Gamma_g$, $\mathcal{D}(\Delta) = r^2\Gamma_g$, we have

$$\begin{aligned} \mathcal{D}\left(\frac{2q\Delta}{|q|^4}\right) &= 2\Delta\mathcal{D}\left(\frac{1}{\overline{q}^2 q}\right) + r^{-1}\Gamma_g = 2\Delta\big(-\overline{q}^{-2}\,q^{-2}\mathcal{D}(q) - 2\overline{q}^{-3}\,q^{-1}\mathcal{D}(\overline{q})\big) + r^{-1}\Gamma_g \\ &= -\frac{2\Delta}{|q|^4}\Big(\mathcal{D}(q) + \frac{2q}{\overline{q}}\mathcal{D}(\overline{q})\Big) + r^{-1}\Gamma_g = \frac{2a\Delta}{|q|^4}\Big(\mathfrak{J} - \frac{2q}{\overline{q}}\mathfrak{J}\Big) + r^{-1}\Gamma_g \\ &= \frac{2aq\Delta}{|q|^4}\Big(\frac{\overline{q} - 2q}{|q|^2}\Big)\mathfrak{J} + r^{-1}\Gamma_g, \end{aligned}$$

which gives, using that $H = \check{H} + \frac{aq}{|q|^2}\mathfrak{J}$,

$$
\begin{aligned}
{}^{(c)}\mathcal{D}(\overline{\mathrm{tr}X}) &= \frac{2aq\Delta}{|q|^4}\Big(\frac{\overline{q}-2q}{|q|^2}\Big)\mathfrak{J} + \frac{2\Delta q}{|q|^4}\frac{aq}{|q|^2}\mathfrak{J} + r^{-1}\Gamma_g \\
&= \frac{2aq\Delta}{|q|^4}\Big(\frac{\overline{q}-q}{|q|^2}\Big)\mathfrak{J} + r^{-1}\Gamma_g = 2i\Im(\mathrm{tr}X)(H-\check{H}) + r^{-1}\Gamma_g.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
{}^{(c)}\mathcal{D}(\mathrm{tr}X) &= \mathcal{D}\mathrm{tr}X + \mathrm{tr}XZ = \mathcal{D}\left(\frac{2\Delta\overline{q}}{|q|^4} + \widecheck{\mathrm{tr}X}\right) + \left(\frac{2\Delta\overline{q}}{|q|^4} + \widecheck{\mathrm{tr}X}\right)\left(\frac{aq}{|q|^2}\mathfrak{J} + \check{Z}\right) \\
&= \mathcal{D}\left(\frac{2\Delta\overline{q}}{|q|^4}\right) + \frac{2\Delta\overline{q}}{|q|^4}\frac{aq}{|q|^2}\mathfrak{J} + \mathcal{D}\widecheck{\mathrm{tr}X} + \mathrm{tr}X\check{Z} + \widecheck{\mathrm{tr}X}Z,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{D}\left(\frac{2\overline{q}\Delta}{|q|^4}\right) &= 2\Delta\mathcal{D}\left(\frac{1}{q^2\overline{q}}\right) + \frac{2\overline{q}}{|q|^4}\mathcal{D}(\Delta) = 2\Delta\big(-q^{-2}\,\overline{q}^{-2}\mathcal{D}(\overline{q}) - 2q^{-3}\,\overline{q}^{-1}\mathcal{D}(q)\big) + r^{-1}\Gamma_g \\
&= -\frac{2\Delta}{|q|^4}\Big(\mathcal{D}(\overline{q}) + \frac{2\overline{q}}{q}\mathcal{D}(q)\Big) + r^{-1}\Gamma_g = -\frac{2a\Delta}{|q|^4}\left(\mathfrak{J} - \frac{2\overline{q}}{q}\mathfrak{J}\right) + r^{-1}\Gamma_g \\
&= -\frac{2a\overline{q}\Delta}{|q|^4}\frac{q-2\overline{q}}{|q|^2}\mathfrak{J} + r^{-1}\Gamma_g,
\end{aligned}
$$

which gives

$$
\begin{aligned}
{}^{(c)}\mathcal{D}(\mathrm{tr}X) &= \frac{4a\overline{q}^2\Delta}{|q|^6}\mathfrak{J} + r^{-1}\Gamma_g = -\frac{4\Delta\overline{q}}{|q|^4}\left(-\frac{a}{q}\mathfrak{J}\right) + r^{-1}\Gamma_g \\
&= -2\mathrm{tr}X\,\underline{H} + r^{-1}\Gamma_g,
\end{aligned}
$$

as stated.

We next calculate, using that $\mathrm{tr}\underline{X} = -\frac{2}{\overline{q}} + \Gamma_g$, and $H = \check{H} + \frac{aq}{|q|^2}\mathfrak{J}$

$$
\begin{aligned}
{}^{(c)}\mathcal{D}(\mathrm{tr}\underline{X}) &= \mathcal{D}\mathrm{tr}\underline{X} - \mathrm{tr}\underline{X}Z = \mathcal{D}\left(-\frac{2}{\overline{q}} + \Gamma_g\right) - \left(-\frac{2}{\overline{q}} + \Gamma_g\right)\left(\frac{aq}{|q|^2}\mathfrak{J} + \Gamma_g\right) \\
&= -\mathcal{D}\left(\frac{2}{\overline{q}}\right) + \frac{2}{\overline{q}}\frac{aq}{|q|^2}\mathfrak{J} + r^{-1}\Gamma_g = \frac{2}{\overline{q}^2}\mathcal{D}\overline{q} + \frac{2}{\overline{q}}\frac{aq}{|q|^2}\mathfrak{J} + r^{-1}\Gamma_g \\
&= \frac{2a}{\overline{q}^2}\mathfrak{J} + \frac{2a}{\overline{q}^2}\mathfrak{J} + r^{-1}\Gamma_g = -2\mathrm{tr}\underline{X}(H-\check{H}) + r^{-1}\Gamma_g.
\end{aligned}
$$

Similarly,

$$\begin{aligned}
{}^{(c)}\mathcal{D}(\overline{\mathrm{tr}\underline{X}}) &= \mathcal{D}\overline{\mathrm{tr}\underline{X}} - \overline{\mathrm{tr}\underline{X}}Z = \mathcal{D}\left(-\frac{2}{q}+\Gamma_g\right) - \left(-\frac{2}{q}+\Gamma_g\right)\left(\frac{aq}{|q|^2}\mathfrak{J}+\Gamma_g\right)\\
&= -\mathcal{D}\big(\frac{2}{q}\big) + \frac{2}{q}\frac{aq}{|q|^2}\mathfrak{J} + r^{-1}\Gamma_g = \frac{2}{q^2}\mathcal{D}q + \frac{2a}{|q|^2}\mathfrak{J} + r^{-1}\Gamma_g\\
&= -\frac{2a}{q^2}\mathfrak{J} + \frac{2a}{|q|^2}\mathfrak{J} + r^{-1}\Gamma_g = \big(\frac{2}{\overline{q}} - \frac{2}{q}\big)\frac{a}{q}\mathfrak{J} + r^{-1}\Gamma_g = 2i\Im(\mathrm{tr}\underline{X})\,\underline{H} + r^{-1}\Gamma_g,
\end{aligned}$$

as stated. □

Lemma D.5.2. *In the frame for which* $\underline{H} = -\frac{a}{q}\mathfrak{J}$ *we have*

$$\nabla_4\,\underline{H} + trX\,\underline{H} = r^{-2}\Gamma_g.$$

Proof. Recall that we have, see Definitions 4.1.3 and 4.1.5,

$$\nabla_4\mathfrak{J} = -\frac{\Delta\overline{q}}{|q|^4}\mathfrak{J} + r^{-1}\Gamma_g,\quad \nabla_4 q = \frac{\Delta}{|q|^2} + \Gamma_g,\quad \mathrm{tr}X = \frac{2\overline{q}\Delta}{|q|^4} + \Gamma_g,\quad {}^{(c)}\nabla_4 q = \frac{1}{2}\mathrm{tr}Xq + r\Gamma_g.$$

Hence

$$\begin{aligned}
\nabla_4\,\underline{H} &= \nabla_4\left(-\frac{a}{q}\mathfrak{J}\right) = \frac{a\nabla_4 q}{q^2}\mathfrak{J} - \frac{a}{q}\left(-\frac{\Delta\overline{q}}{|q|^4}\mathfrak{J} + r^{-1}\Gamma_g\right),\\
&= \Big(\frac{a\Delta}{q^2|q|^2} + \frac{a\overline{q}\Delta}{q|q|^4}\Big)\mathfrak{J} + r^{-2}\Gamma_g = \frac{a\Delta}{|q|^2}\Big(\frac{1}{q^2} + \frac{\overline{q}}{q|q|^2}\Big)\mathfrak{J} + r^{-2}\Gamma_g\\
&= 2\frac{a\Delta}{|q|^2}\frac{\overline{q}^2}{|q|^4}\mathfrak{J} + r^{-2}\Gamma_g\\
&= -\frac{2\Delta\overline{q}}{|q|^4}\,\underline{H} + r^{-2}\Gamma_g = -\mathrm{tr}X\,\underline{H} + r^{-2}\Gamma_g,
\end{aligned}$$

as stated. □

Proposition D.5.3. *The following equations hold true.*

$$\begin{aligned}
{}^{(c)}\nabla_4\check{H} + \frac{1}{2}\overline{trX}\check{H} &= -B - \frac{1}{2}\widehat{X}\cdot\overline{\check{H}} + r^{-2}\Gamma_g,\\
{}^{(c)}\nabla_4\underline{\Xi} &= -\frac{1}{2}\overline{trX}\check{\underline{H}} - \underline{B} - \frac{1}{2}\widehat{\underline{X}}\cdot\check{H} + r^{-2}\Gamma_b.
\end{aligned}\tag{D.5.3}$$

Also

$$\begin{aligned}
{}^{(c)}\mathcal{D}(\overline{\widecheck{trX}}) &= 2i\Im(tr\underline{X})\check{H} + 2B + \overline{{}^{(c)}\mathcal{D}}\cdot\widehat{X} + r^{-1}\Gamma_g,\\
{}^{(c)}\mathcal{D}\overline{\widecheck{tr\underline{X}}} &= -2\underline{B} + \overline{{}^{(c)}\mathcal{D}}\cdot\widehat{\underline{X}} + 2i\Im(trX)\underline{\Xi} + r^{-1}\Gamma_g.
\end{aligned}\tag{D.5.4}$$

Proof. Recall that H verifies the equation

$$ {}^{(c)}\nabla_4 H \quad = \quad -\frac{1}{2}\overline{\mathrm{tr} X}(H-\underline{H}) - \frac{1}{2}\widehat{X}\cdot(\overline{H}-\overline{\underline{H}}) - B. $$

Writing $H = \check{H} + \frac{aq}{|q|^2}\mathfrak{J}$ and $\underline{H} = -\frac{a\overline{q}}{|q|^2}\mathfrak{J}$ we deduce

$$ {}^{(c)}\nabla_4\check{H} + {}^{(c)}\nabla_4\left(\frac{aq}{|q|^2}\mathfrak{J}\right) = -\frac{1}{2}\overline{\mathrm{tr} X}\check{H} - \frac{1}{2}\overline{\mathrm{tr} X}\left(\frac{aq}{|q|^2}\mathfrak{J} + \frac{a\overline{q}}{|q|^2}\mathfrak{J}\right) - \frac{1}{2}\widehat{X}\cdot\overline{\check{H}} - B + r^{-2}\Gamma_g, $$

or,

$$ \begin{aligned} {}^{(c)}\nabla_4\check{H} + \frac{1}{2}\overline{\mathrm{tr} X}\check{H} &= -B - \frac{1}{2}\widehat{X}\cdot\overline{\check{H}} + r^{-2}\Gamma_g - E \\ E &= {}^{(c)}\nabla_4\left(\frac{aq}{|q|^2}\mathfrak{J}\right) + \frac{1}{2}\overline{\mathrm{tr} X}\left(\frac{aq}{|q|^2}\mathfrak{J} + \frac{a\overline{q}}{|q|^2}\mathfrak{J}\right). \end{aligned} $$

Using

$$ \nabla_4\mathfrak{J} \quad = \quad -\frac{\Delta\overline{q}}{|q|^4}\mathfrak{J} + r^{-1}\Gamma_g, \qquad \nabla_4 q = \frac{\Delta}{|q|^2} + \Gamma_g, \qquad \mathrm{tr} X = \frac{2\overline{q}\Delta}{|q|^2} + \Gamma_g, $$

we calculate

$$ \begin{aligned} E &= {}^{(c)}\nabla_4\Big(\frac{a}{\overline{q}}\Big)\mathfrak{J} + \frac{a}{\overline{q}}\Big(-\frac{\Delta\overline{q}}{|q|^4}\mathfrak{J} + r^{-1}\Gamma_g\Big) + \Big(\frac{q\Delta}{|q|^2} + \Gamma_g\Big)\Big(\frac{aq}{|q|^2} + \frac{a\overline{q}}{|q|^2}\Big)\mathfrak{J} \\ &= \Big(-\frac{a}{\overline{q}^2}\frac{\Delta}{|q|^2} - \frac{a\Delta}{|q|^4} + \frac{q\Delta}{|q|^2}\Big(\frac{aq}{|q|^2} + \frac{a\overline{q}}{|q|^2}\Big)\Big)\mathfrak{J} + r^{-2}\Gamma_g = r^{-2}\Gamma_g. \end{aligned} $$

Similarly the equation

$$ {}^{(c)}\nabla_3\,\underline{H} - {}^{(c)}\nabla_4\underline{\Xi} \quad = \quad -\frac{1}{2}\overline{\mathrm{tr}\underline{X}}(\,\underline{H} - H) - \frac{1}{2}\underline{\widehat{X}}\cdot(\,\overline{\underline{H}} - \overline{H}) + \underline{B}, $$

takes the form

$$ \begin{aligned} {}^{(c)}\nabla_4\underline{\Xi} &= -\frac{1}{2}\overline{\mathrm{tr} X}\check{H} - \underline{B} + r^{-2}\Gamma_b - E \\ E &= {}^{(c)}\nabla_3\left(\frac{a}{q}\mathfrak{J}\right) + \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\left(\frac{a}{q}\mathfrak{J} + \frac{aq}{|q|^2}\mathfrak{J}\right). \end{aligned} $$

Using the relations

$$ \mathrm{tr}\underline{X} = -\frac{2}{\overline{q}} + \Gamma_g, \;\; {}^{(c)}\nabla_3\mathfrak{J} = \frac{1}{\overline{q}}\mathfrak{J} + r^{-1}\Gamma_b, \;\; e_3(q) = -1 + r\Gamma_b, $$

we deduce

$$E = -\frac{a}{q^2}\big(-1+r\Gamma_b\big)\mathfrak{J}+\frac{a}{q}\left(\frac{1}{\overline{q}}\mathfrak{J}+r^{-1}\Gamma_b\right)+\frac{1}{2}\left(-\frac{2}{q}+\Gamma_g\right)\big(\frac{a}{q}+\frac{aq}{|q|^2}\big)\mathfrak{J}=r^{-2}\Gamma_b,$$

and thus

$$^{(c)}\nabla_4\underline{\Xi} = -\frac{1}{2}\overline{\mathrm{tr}X}\check{H}-\underline{B}-\frac{1}{2}\underline{\widehat{X}}\cdot\check{H}+r^{-2}\Gamma_b,$$

as stated. To derive the equations for $\widecheck{\mathrm{tr}\underline{X}}, \widecheck{\mathrm{tr}X}$ we start with the corresponding Codazzi equation written in the form[2]

$$\begin{aligned}\mathcal{D}\overline{\mathrm{tr}X}+\overline{\mathrm{tr}X}Z &= 2i\Im(\mathrm{tr}X)H+2B+\overline{^{(c)}\mathcal{D}}\cdot\widehat{X}+2i\Im(\mathrm{tr}X)\Xi+r^{-2}\Gamma_g,\\ \mathcal{D}\overline{\mathrm{tr}\underline{X}}-\overline{\mathrm{tr}\underline{X}}Z &= 2i\Im(\mathrm{tr}\underline{X})\,\underline{H}-2\underline{B}+\overline{^{(c)}\mathcal{D}}\cdot\underline{\widehat{X}}+2i\Im(\mathrm{tr}X)\underline{\Xi}+r^{-2}\Gamma_g.\end{aligned}\tag{D.5.5}$$

For the first equation, since $\mathrm{tr}X=\frac{2\Delta\overline{q}}{|q|^4}+\Gamma_g$, $2i\Im(\mathrm{tr}X)=\mathrm{tr}X-\overline{\mathrm{tr}X}=-\frac{2\Delta(q-\overline{q})}{|q|^4}+\Gamma_g$, $H=\check{H}+\frac{aq}{|q|^2}\mathfrak{J}$ and $Z=\frac{aq}{|q|^2}+\Gamma_g$ we write

$$\begin{aligned}&\mathcal{D}\overline{\mathrm{tr}X}+\overline{\mathrm{tr}X}Z-2i\Im(\mathrm{tr}X)H\\ =\;&\mathcal{D}(\widecheck{\overline{\mathrm{tr}X}}+\frac{2q\Delta}{|q|^4})+(\widecheck{\overline{\mathrm{tr}X}}+\frac{2q\Delta}{|q|^4})(\frac{aq}{|q|^2}\mathfrak{J}+\Gamma_g)-2i\Im(\mathrm{tr}X)(\check{H}+\frac{aq}{|q|^2}\mathfrak{J})\\ =\;&\mathcal{D}\widecheck{\overline{\mathrm{tr}X}}-2i\Im(\mathrm{tr}X)\check{H}+\mathcal{D}\big(\frac{2q\Delta}{|q|^4}\big)+\frac{2q\Delta}{|q|^4}\frac{aq}{|q|^2}\mathfrak{J}-2i\Im(\mathrm{tr}X)\check{H}+\frac{2\Delta(q-\overline{q})}{|q|^4}\frac{aq}{|q|^2}\mathfrak{J}\\ &+r^{-2}\Gamma_g\\ =\;&\mathcal{D}\widecheck{\overline{\mathrm{tr}X}}-2i\Im(\mathrm{tr}X)\check{H}+\mathcal{D}\big(\frac{2q\Delta}{|q|^4}\big)+\frac{2aq\Delta}{|q|^4}\mathfrak{J}\big(\frac{q}{|q|^2}+\frac{q-\overline{q}}{|q|^2}\big)+r^{-1}\Gamma_g.\end{aligned}$$

Since $\mathcal{D}q=-a\mathfrak{J}+r\Gamma_g$, $\mathcal{D}\overline{q}=a\mathfrak{J}+r\Gamma_g$,

$$\begin{aligned}\mathcal{D}\big(\frac{2q\Delta}{|q|^4}\big)&=2\Delta\mathcal{D}\big(\frac{1}{q\overline{q}^2}\big)=2\Delta\big(-q^{-2}\overline{q}^{-2}\mathcal{D}(q)-2q^{-1}\overline{q}^{-3}\mathcal{D}(\overline{q})\big)\\ &=-\frac{2\Delta}{|q|^4}\Big(\mathcal{D}(q)+\frac{2q}{\overline{q}}\mathcal{D}(\overline{q})\Big)=\frac{2a\Delta}{|q|^4}\Big(\mathfrak{J}-\frac{2q}{\overline{q}}\mathfrak{J}\Big)=\frac{2aq\Delta}{|q|^4}\Big(\frac{\overline{q}}{|q|^2}-\frac{2q}{|q|^2}\big)\mathfrak{J}.\end{aligned}$$

Hence

$$\begin{aligned}\mathcal{D}\overline{\mathrm{tr}X}+\overline{\mathrm{tr}X}Z-2i\Im(\mathrm{tr}X)H &= \mathcal{D}(\widecheck{\overline{\mathrm{tr}X}})-2i\Im(\mathrm{tr}X)\check{H}+r^{-1}\Gamma_g\\ &= {}^{(c)}\mathcal{D}(\widecheck{\overline{\mathrm{tr}X}})-2i\Im(\mathrm{tr}X)\check{H}+r^{-1}\Gamma_g,\end{aligned}$$

and therefore

$$^{(c)}\mathcal{D}(\widecheck{\overline{\mathrm{tr}X}}) = 2i\Im(\mathrm{tr}\underline{X})\check{H}+2B+\overline{^{(c)}\mathcal{D}}\cdot\widehat{X}+r^{-1}\Gamma_g.$$

The second equation in (D.5.4) is proved in the same manner. □

[2]Recall that $^{(c)}\mathcal{D}\overline{\mathrm{tr}\underline{X}}={}^{(c)}\mathcal{D}\overline{\mathrm{tr}\underline{X}}-Z\overline{\mathrm{tr}\underline{X}}$, $^{(c)}\mathcal{D}\overline{\mathrm{tr}X}=\mathcal{D}\overline{\mathrm{tr}X}+\overline{\mathrm{tr}X}Z$.

Lemma D.5.4. *The following identity holds true*

$$\begin{aligned} {}^{(c)}\mathcal{D}P &= -3P\,\underline{H} + {}^{(c)}\mathcal{D}\check{P} + r^{-3}\Gamma_g, \\ {}^{(c)}\mathcal{D}\overline{P} &= -3\overline{P}(H-\check{H}) + {}^{(c)}\mathcal{D}\overline{\check{P}} + r^{-3}\Gamma_g. \end{aligned} \tag{D.5.6}$$

Also,

$$ {}^{(c)}\nabla_4 P + \frac{3}{2}trXP \;=\; \nabla_4\check{P} + \frac{3}{2}trX\check{P} + r^{-3}\Gamma_g. $$

Proof. We write $P = -\frac{2m}{q^3} + \check{P}$. Thus, since $\mathcal{D}(q) = -a\mathfrak{J} + r\Gamma_g$ and $\underline{H} = -\frac{a}{q}\mathfrak{J}$,

$$ {}^{(c)}\mathcal{D}P = \mathcal{D}\big(-\frac{2m}{q^3} + \check{P}\big) = \frac{6m}{q^4}\mathcal{D}(q) + \mathcal{D}\check{P} = \frac{6m}{q^3}\,\underline{H} + \mathcal{D}\check{P} + r^{-3}\Gamma_g. $$

Similarly, since $\mathcal{D}\overline{q} = a\mathfrak{J} + r\Gamma_g$ and $H = \check{H} + \frac{aq}{|q|^2}\mathfrak{J}$,

$$ {}^{(c)}\mathcal{D}\overline{P} \;=\; \frac{6m}{\overline{q}^4}\mathcal{D}(\overline{q}) + \mathcal{D}\overline{\check{P}} = \frac{6m}{\overline{q}^4}(a\mathfrak{J} + r\Gamma_g) + \mathcal{D}\overline{\check{P}} = -3\overline{P}(H-\check{H}) + \mathcal{D}\overline{\check{P}} + r^3\Gamma_g. $$

Writing $P = \frac{2m}{q^3} + \check{P}$, $\nabla_4 q = \frac{\Delta}{|q|^2} + \Gamma_g$ and $\mathrm{tr}X = \frac{2\overline{q}\Delta}{|q|^2} + \Gamma_g$ we derive

$$\begin{aligned} {}^{(c)}\nabla_4 P + \frac{3}{2}\mathrm{tr}XP &= \nabla_4\big(\frac{2m}{q^3} + \check{P}\big) + \frac{3}{2}\mathrm{tr}X\big(\frac{2m}{q^3} + \check{P}\big) \\ &= \nabla_4\check{P} + \frac{3}{2}\mathrm{tr}X\check{P} - \frac{6m}{q^4}\,{}^{(c)}\nabla_4 q + \frac{3}{2}\big(\frac{2\overline{q}\Delta}{|q|^2} + \Gamma_g\big)\frac{2m}{q^3} \\ &= \nabla_4\check{P} + \frac{3}{2}\mathrm{tr}X\check{P} - \frac{6m}{q^4}\big(\frac{\Delta}{|q|^2} + \Gamma_g\big) + \frac{6m}{q^3}\big(\frac{\overline{q}\Delta}{|q|^2} + \Gamma_g\big) \\ &= \nabla_4\check{P} + \frac{3}{2}\mathrm{tr}X\check{P} + r^{-3}\Gamma_g, \end{aligned}$$

as stated. □

Proposition D.5.5. *The following linearized Bianchi equations hold true.*

$$\begin{aligned} {}^{(c)}\nabla_4\underline{B} + trX\underline{B} &= -\,{}^{(c)}\mathcal{D}\check{P} + \overline{B}\cdot\widehat{\underline{X}} + r^{-3}\Gamma_g \\ {}^{(c)}\nabla_4\check{P} + \frac{3}{2}trX\check{P} &= \frac{1}{2}\,{}^{(c)}\mathcal{D}\cdot\overline{B} + \underline{H}\cdot\overline{B} - \frac{1}{4}\widehat{\underline{X}}\cdot\overline{A} + r^{-3}\Gamma_g. \end{aligned} \tag{D.5.7}$$

Proof. In view of Lemma D.5.4, ${}^{(c)}\mathcal{D}P = -3P\,\underline{H} + {}^{(c)}\mathcal{D}\check{P} + r^{-3}\Gamma_g$, the equation

$$ {}^{(c)}\nabla_4\underline{B} + \mathrm{tr}X\underline{B} \;=\; -\,{}^{(c)}\mathcal{D}P + \overline{B}\cdot\widehat{\underline{X}} - 3P\,\underline{H} - \frac{1}{2}\underline{A}\cdot\Xi $$

becomes

$$
{}^{(c)}\nabla_4\underline{B} + \mathrm{tr}X\underline{B} \;=\; -{}^{(c)}\mathcal{D}\check{P} + \overline{B}\cdot\widehat{\underline{X}} + r^{-3}\Gamma_g.
$$

Using equation ${}^{(c)}\nabla_4 P + \frac{3}{2}\mathrm{tr}XP = \nabla_4\check{P} + \frac{3}{2}\mathrm{tr}X\check{P} + r^{-3}\Gamma_g$, the equation

$$
{}^{(c)}\nabla_4 P - \frac{1}{2}{}^{(c)}\mathcal{D}\cdot\overline{B} \;=\; -\frac{3}{2}\mathrm{tr}XP + \underline{H}\cdot\overline{B} - \Xi\cdot\underline{B} - \frac{1}{4}\widehat{\underline{X}}\cdot\overline{A}
$$

becomes

$$
{}^{(c)}\nabla_4\check{P} + \frac{3}{2}\mathrm{tr}X\check{P} \;=\; \frac{1}{2}{}^{(c)}\mathcal{D}\cdot\overline{B} + \underline{H}\cdot\overline{B} - \frac{1}{4}\widehat{\underline{X}}\cdot\overline{A} + r^{-3}\Gamma_g.
$$

as stated. □

D.5.2 Proof of Proposition 5.3.1

We start with the Bianchi identity

$$
\underline{A}_4 \;=\; -\frac{1}{2}{}^{(c)}\mathcal{D}\widehat{\otimes}\underline{B} - 2\,\underline{H}\widehat{\otimes}\underline{B} - 3P\widehat{\underline{X}}.
$$

To the above, we apply the operator ${}^{(c)}\nabla_3 + \big(2\overline{\mathrm{tr}\underline{X}} + \frac{1}{2}\mathrm{tr}\underline{X}\big)$ and we deduce

$$
\begin{aligned}
-{}^{(c)}\nabla_3\underline{A}_4 - \Big(2\overline{\mathrm{tr}\underline{X}} + \frac{1}{2}\mathrm{tr}\underline{X}\Big)\underline{A}_4 \;&=\; \frac{1}{2}{}^{(c)}\nabla_3{}^{(c)}\mathcal{D}\widehat{\otimes}\underline{B} + \frac{1}{2}\Big(2\overline{\mathrm{tr}\underline{X}} + \frac{1}{2}\mathrm{tr}\underline{X}\Big){}^{(c)}\mathcal{D}\widehat{\otimes}\underline{B}\\
&\quad +2\,{}^{(c)}\nabla_3(\underline{H}\widehat{\otimes}\underline{B}) + 2\Big(2\overline{\mathrm{tr}\underline{X}} + \frac{1}{2}\mathrm{tr}\underline{X}\Big)\underline{H}\widehat{\otimes}\underline{B}\\
&\quad +3\,{}^{(c)}\nabla_3(P\widehat{\underline{X}}) + 3\Big(2\overline{\mathrm{tr}\underline{X}} + \frac{1}{2}\mathrm{tr}\underline{X}\Big)P\widehat{\underline{X}}\\
&=\; I + J + K.
\end{aligned}
$$

Step 1: Calculation of I. Using the commutation formula (4.2.12) applied to $\underline{B}$ of signature -1, we have

$$
\begin{aligned}
[{}^{(c)}\nabla_3, {}^{(c)}\mathcal{D}\widehat{\otimes}]\underline{B} \;&=\; -\frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\underline{B} + H\widehat{\otimes}\,{}^{(c)}\nabla_3\underline{B} + \underline{\Xi}\widehat{\otimes}\,{}^{(c)}\nabla_4\underline{B}\\
&\quad -2\underline{B}\widehat{\otimes}\underline{B} - \mathrm{tr}X\underline{\Xi}\widehat{\otimes}\underline{B} - \frac{1}{2}\widehat{\underline{X}}\cdot\overline{{}^{(c)}\mathcal{D}}\underline{B}\\
&\quad +\frac{1}{2}\widehat{\underline{X}}(\overline{H}\cdot\underline{B}) + \frac{1}{2}(\widehat{\underline{X}}\cdot\overline{H})\widehat{\otimes}\underline{B} + \widehat{X}\cdot\underline{\Xi}\cdot\underline{B},
\end{aligned}
$$

which can be written as

$$
\begin{aligned}
[\,{}^{(c)}\nabla_3,\,{}^{(c)}\mathcal{D}\widehat{\otimes}]\underline{B} &= -\frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\underline{B}+H\widehat{\otimes}\,{}^{(c)}\nabla_3\underline{B}\\
&\quad+\underline{\Xi}\widehat{\otimes}\,{}^{(c)}\nabla_4\underline{B}-2\underline{B}\widehat{\otimes}\underline{B}-\mathrm{tr}X\underline{\Xi}\widehat{\otimes}\underline{B}-\frac{1}{2}\widehat{\underline{X}}\cdot\overline{{}^{(c)}\mathcal{D}}\underline{B}+r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b)\\
&\quad+(\widehat{\underline{X}}\cdot\overline{\underline{H}})\underline{B},
\end{aligned}
$$

where we can write $(\widehat{X}\cdot\underline{\Xi})\underline{B}=r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b)$. In view of the equation for ${}^{(c)}\nabla_4\underline{B}$, $\Xi=0$ and, see (D.5.6), ${}^{(c)}\mathcal{D}P=-3P\,\underline{H}+r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g$ we deduce

$$
\begin{aligned}
\underline{\Xi}\widehat{\otimes}\,{}^{(c)}\nabla_4\underline{B} &= \underline{\Xi}\widehat{\otimes}\big(-\,{}^{(c)}\mathcal{D}P-\mathrm{tr}X\underline{B}+\overline{B}\cdot\widehat{\underline{X}}-3P\,\underline{H}\big)\\
&= -\mathrm{tr}X\underline{\Xi}\widehat{\otimes}\underline{B}+r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b)+(\Gamma_b\cdot\Gamma_b)\cdot B.
\end{aligned}
$$

This gives

$$
\begin{aligned}
{}^{(c)}\nabla_3\,{}^{(c)}\mathcal{D}\widehat{\otimes}\underline{B} &= {}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\nabla_3\underline{B}-\frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\underline{B}+H\widehat{\otimes}\,{}^{(c)}\nabla_3\underline{B}\\
&\quad+\mathrm{Err}_1+r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b),
\end{aligned}
$$

where

$$
\mathrm{Err}_1 = -2\underline{B}\widehat{\otimes}\underline{B}-2\mathrm{tr}X\underline{\Xi}\widehat{\otimes}\underline{B}-\frac{1}{2}\widehat{\underline{X}}\cdot\overline{{}^{(c)}\mathcal{D}}\underline{B}+(\widehat{\underline{X}}\cdot\overline{\underline{H}})\underline{B}+(\Gamma_b\cdot\Gamma_b)\cdot B.
$$

Using the definition $\underline{B}_3={}^{(c)}\nabla_3\underline{B}+2\overline{\mathrm{tr}\underline{X}}\underline{B}$, we have

$$
\begin{aligned}
&{}^{(c)}\nabla_3\,{}^{(c)}\mathcal{D}\widehat{\otimes}\underline{B}\\
&= {}^{(c)}\mathcal{D}\widehat{\otimes}\big(\underline{B}_3-2\overline{\mathrm{tr}\underline{X}}\underline{B}\big)-\frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\underline{B}+H\widehat{\otimes}\big(\underline{B}_3-2\overline{\mathrm{tr}\underline{X}}\underline{B}\big)\\
&\quad+\mathrm{Err}_1+r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b)\\
&= {}^{(c)}\mathcal{D}\widehat{\otimes}\underline{B}_3+H\widehat{\otimes}\underline{B}_3-\big(2\overline{\mathrm{tr}\underline{X}}+\frac{1}{2}\mathrm{tr}\underline{X}\big)\,{}^{(c)}\mathcal{D}\widehat{\otimes}\underline{B}-\big(2\,{}^{(c)}\mathcal{D}\overline{\mathrm{tr}\underline{X}}+2\overline{\mathrm{tr}\underline{X}}H\big)\widehat{\otimes}\underline{B}\\
&\quad+\mathrm{Err}_1+r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b).
\end{aligned}
$$

This implies

$$
\begin{aligned}
I &= \frac{1}{2}\,{}^{(c)}\nabla_3\,{}^{(c)}\mathcal{D}\widehat{\otimes}\underline{B}+\frac{1}{2}\big(2\overline{\mathrm{tr}\underline{X}}+\frac{1}{2}\mathrm{tr}\underline{X}\big)\,{}^{(c)}\mathcal{D}\widehat{\otimes}\underline{B}\\
&= \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\underline{B}_3+\frac{1}{2}H\widehat{\otimes}\underline{B}_3-\big({}^{(c)}\mathcal{D}\overline{\mathrm{tr}\underline{X}}+\overline{\mathrm{tr}\underline{X}}H\big)\widehat{\otimes}\underline{B}+\mathrm{Err}_1+r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b).
\end{aligned}
$$

Making use of the equation

$$
\underline{B}_3 = -\frac{1}{2}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)-3P\,\underline{\Xi},
$$

we deduce

$$
\begin{aligned}
I &= -\frac{1}{4}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)-\frac{1}{4}H\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)\\
&\quad -\frac{3}{2}P\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,\underline{\Xi}-\frac{3}{2}({}^{(c)}\mathcal{D}P)\widehat{\otimes}\,\underline{\Xi}-\frac{3}{2}PH\widehat{\otimes}\,\underline{\Xi}-\big({}^{(c)}\mathcal{D}\overline{\mathrm{tr}\underline{X}}+\overline{\mathrm{tr}\underline{X}}H\big)\widehat{\otimes}\underline{B}\\
&\quad +\mathrm{Err}_1+r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b).
\end{aligned}
$$

Using again (D.5.6) to write $\mathcal{D}P=-3P\,\underline{H}+r^{-2}\Gamma_g$, we obtain

$$
\begin{aligned}
I &= -\frac{1}{4}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)-\frac{1}{4}H\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)\\
&\quad -\frac{3}{2}P\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,\underline{\Xi}+\big(\frac{9}{2}P\,\underline{H}-\frac{3}{2}PH\big)\widehat{\otimes}\,\underline{\Xi}-\big({}^{(c)}\mathcal{D}\overline{\mathrm{tr}\underline{X}}+\overline{\mathrm{tr}\underline{X}}H\big)\widehat{\otimes}\underline{B}\\
&\quad +\mathrm{Err}_1+r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b).
\end{aligned}
$$

Step 2: Calculation of J. We have

$$
\begin{aligned}
J &= 2\,{}^{(c)}\nabla_3(\,\underline{H}\widehat{\otimes}\underline{B})+2\big(2\overline{\mathrm{tr}\underline{X}}+\frac{1}{2}\mathrm{tr}\underline{X}\big)\,\underline{H}\widehat{\otimes}\underline{B}\\
&= 2\,{}^{(c)}\nabla_3\,\underline{H}\widehat{\otimes}\underline{B}+2\,\underline{H}\widehat{\otimes}\,{}^{(c)}\nabla_3\underline{B}+4\overline{\mathrm{tr}\underline{X}}(\,\underline{H}\widehat{\otimes}\underline{B})+\mathrm{tr}\underline{X}\,\underline{H}\widehat{\otimes}\underline{B}\\
&= 2\,{}^{(c)}\nabla_3\,\underline{H}\widehat{\otimes}\underline{B}+2\,\underline{H}\widehat{\otimes}\big(\underline{B}_3-2\overline{\mathrm{tr}\underline{X}}\underline{B}\big)+4\overline{\mathrm{tr}\underline{X}}(\,\underline{H}\widehat{\otimes}\underline{B})+\mathrm{tr}\underline{X}\,\underline{H}\widehat{\otimes}\underline{B}\\
&= 2\,{}^{(c)}\nabla_3\,\underline{H}\widehat{\otimes}\underline{B}+2\,\underline{H}\widehat{\otimes}\underline{B}_3+\mathrm{tr}\underline{X}\,\underline{H}\widehat{\otimes}\underline{B}.
\end{aligned}
$$

Making use of the equations

$$
{}^{(c)}\nabla_3\,\underline{H} = -\frac{1}{2}\overline{\mathrm{tr}\underline{X}}(\,\underline{H}-H)+\underline{B}+r^{-2}\Gamma_b,
$$

and

$$
\underline{B}_3 = -\frac{1}{2}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)-3P\,\underline{\Xi},
$$

we deduce

$$
\begin{aligned}
J &= -\overline{\mathrm{tr}\underline{X}}(\,\underline{H}-H)\widehat{\otimes}\underline{B}+2\,\underline{H}\widehat{\otimes}\underline{B}_3+\mathrm{tr}\underline{X}\,\underline{H}\widehat{\otimes}\underline{B}+2\underline{B}\widehat{\otimes}\underline{B}+r^{-3}(\Gamma_b\cdot\Gamma_b)\\
&= 2\,\underline{H}\widehat{\otimes}\big(-\frac{1}{2}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)-3P\,\underline{\Xi}\big)-\overline{\mathrm{tr}\underline{X}}(\,\underline{H}-H)\widehat{\otimes}\underline{B}+\mathrm{tr}\underline{X}\,\underline{H}\widehat{\otimes}\underline{B}\\
&\quad +2\underline{B}\widehat{\otimes}\underline{B}+r^{-3}(\Gamma_b\cdot\Gamma_b)\\
&= -\,\underline{H}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\underline{A}\cdot\overline{H}\big)+\big(-\overline{\mathrm{tr}\underline{X}}(\,\underline{H}-H)+\mathrm{tr}\underline{X}\,\underline{H}\big)\widehat{\otimes}\underline{B}-6P\,\underline{H}\widehat{\otimes}\,\underline{\Xi}\\
&\quad +2\underline{B}\widehat{\otimes}\underline{B}+r^{-3}(\Gamma_b\cdot\Gamma_b).
\end{aligned}
$$

Step 3. Calculation of K. We have

$$\begin{aligned} K &= 3\,{}^{(c)}\nabla_3(P\widehat{\underline{X}}) + 3\big(2\overline{\mathrm{tr}\underline{X}} + \frac{1}{2}\mathrm{tr}\underline{X}\big)P\widehat{\underline{X}} \\ &= 3\,{}^{(c)}\nabla_3 P\widehat{\underline{X}} + 3P\,{}^{(c)}\nabla_3\widehat{\underline{X}} + 6\overline{\mathrm{tr}\underline{X}}P\widehat{\underline{X}} + \frac{3}{2}\mathrm{tr}\underline{X}P\widehat{\underline{X}}. \end{aligned}$$

Making use of the equations,

$$\begin{aligned} {}^{(c)}\nabla_3 P &= -\frac{3}{2}\overline{\mathrm{tr}\underline{X}}P - \frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot\underline{B} - \overline{H}\cdot\underline{B} + \underline{\Xi}\cdot\overline{B} - \frac{1}{4}\widehat{\underline{X}}\cdot\overline{A}, \\ {}^{(c)}\nabla_3\widehat{\underline{X}} &= -\frac{1}{2}(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}})\widehat{\underline{X}} + \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\underline{\Xi} + \frac{1}{2}\underline{\Xi}\widehat{\otimes}(H + \underline{H}) - \underline{A}, \end{aligned}$$

we obtain

$$\begin{aligned} K &= 3\big(-\frac{3}{2}\overline{\mathrm{tr}\underline{X}}P - \frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot\underline{B} - \overline{H}\cdot\underline{B} + (A,B)\cdot\Gamma_b\big)\widehat{\underline{X}} \\ &\quad +3P\big(-\frac{1}{2}(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}})\widehat{\underline{X}} + \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\underline{\Xi} + \frac{1}{2}\underline{\Xi}\widehat{\otimes}(H + \underline{H}) - \underline{A}\big) \\ &\quad +6\overline{\mathrm{tr}\underline{X}}P\widehat{\underline{X}} + \frac{3}{2}\mathrm{tr}\underline{X}P\widehat{\underline{X}} \\ &= \frac{3}{2}P\,{}^{(c)}\mathcal{D}\widehat{\otimes}\underline{\Xi} + \frac{3}{2}P\underline{\Xi}\widehat{\otimes}(H + \underline{H}) - 3P\underline{A} \\ &\quad +\big(-\frac{3}{2}\overline{{}^{(c)}\mathcal{D}}\cdot\underline{B} - 3\overline{H}\cdot\underline{B}\big)\widehat{\underline{X}} + (\Gamma_b\cdot\Gamma_b)\cdot(A,B) \end{aligned}$$

Step 4. Final sum. By summing the above three terms we obtain the cancellation of the terms in $\underline{\Xi}$, and we deduce

$$\begin{aligned} -\,{}^{(c)}\nabla_3\underline{A}_4 - \big(2\overline{\mathrm{tr}\underline{X}} + \frac{1}{2}\mathrm{tr}\underline{X}\big)\underline{A}_4 &= I + J + K \\ &= -\frac{1}{4}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A} + \overline{H}\cdot\underline{A}\big) \\ &\quad -\frac{1}{4}(H + 4\,\underline{H})\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A} + \overline{H}\cdot\underline{A}\big) - 3P\underline{A} \\ &\quad -\big({}^{(c)}\mathcal{D}\overline{\mathrm{tr}\underline{X}} + (\overline{\mathrm{tr}\underline{X}} - \mathrm{tr}\underline{X})\,\underline{H}\big)\widehat{\otimes}\underline{B} \\ &\quad +\mathrm{Err}_1 + r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b) + 2\underline{B}\widehat{\otimes}\underline{B} + r^{-3}(\Gamma_b\cdot\Gamma_b) \\ &\quad +\big(-\frac{3}{2}\overline{{}^{(c)}\mathcal{D}}\cdot\underline{B} - 3\overline{H}\cdot\underline{B}\big)\widehat{\underline{X}} + (\Gamma_b\cdot\Gamma_b)\cdot(A,B). \end{aligned}$$

Using (D.5.2), i.e.

$${}^{(c)}\mathcal{D}\overline{\mathrm{tr}\underline{X}} = 2i\Im(\mathrm{tr}\underline{X})\,\underline{H} + r^{-1}\Gamma_g,$$

we finally obtain

$$
\begin{aligned}
-{}^{(c)}\nabla_3\underline{A}_4-\big(2\overline{\mathrm{tr}\underline{X}}+\frac{1}{2}\mathrm{tr}\underline{X}\big)\underline{A}_4 \quad = \quad & -\frac{1}{4}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)\\
& -\frac{1}{4}(H+4\,\underline{H})\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)-3P\underline{A}\\
& +\mathrm{Err}_1+r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b)+2\underline{B}\widehat{\otimes}\underline{B}+r^{-3}(\Gamma_b\cdot\Gamma_b)\\
& +\left(-\frac{3}{2}\overline{{}^{(c)}\mathcal{D}}\cdot\underline{B}-3\overline{H}\cdot\underline{B}\right)\widehat{\underline{X}}+(\Gamma_b\cdot\Gamma_b)\cdot(A,B).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
-{}^{(c)}\nabla_3\underline{A}_4-\big(2\overline{\mathrm{tr}\underline{X}}+\frac{1}{2}\mathrm{tr}\underline{X}\big)\underline{A}_4 = & -\frac{1}{4}\big({}^{(c)}\mathcal{D}+H+4\,\underline{H}\big)\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)\\
& -3P\underline{A}+\mathrm{Err}_{TE}[\underline{A}],
\end{aligned}
$$

with error term $\mathrm{Err}_{TE}[\underline{A}]$ given by

$$
\begin{aligned}
\mathrm{Err}_{TE} \quad = \quad & \mathrm{Err}_1+r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b)+2\underline{B}\widehat{\otimes}\underline{B}+r^{-3}(\Gamma_b\cdot\Gamma_b)\\
& +\left(-\frac{3}{2}\overline{{}^{(c)}\mathcal{D}}\cdot\underline{B}-3\overline{H}\cdot\underline{B}\right)\widehat{\underline{X}}+(\Gamma_b\cdot\Gamma_b)\cdot(A,B)\\
= \quad & -2\underline{B}\widehat{\otimes}\underline{B}-2\mathrm{tr}X\underline{\Xi}\widehat{\otimes}\underline{B}-\frac{1}{2}\widehat{\underline{X}}\cdot\overline{{}^{(c)}\mathcal{D}}\underline{B}+(\widehat{\underline{X}}\cdot\overline{\check{H}})\underline{B}+(\Gamma_b\cdot\Gamma_b)\cdot B\\
& +r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b)+2\underline{B}\widehat{\otimes}\underline{B}+r^{-3}(\Gamma_b\cdot\Gamma_b)\\
& +\left(-\frac{3}{2}\overline{{}^{(c)}\mathcal{D}}\cdot\underline{B}-3\overline{H}\cdot\underline{B}\right)\widehat{\underline{X}}+(\Gamma_b\cdot\Gamma_b)\cdot(A,B),
\end{aligned}
$$

which gives

$$
\begin{aligned}
\mathrm{Err}_{TE} \quad = \quad & -2\mathrm{tr}X\underline{\Xi}\widehat{\otimes}\underline{B}-\frac{1}{2}\widehat{\underline{X}}\cdot\overline{{}^{(c)}\mathcal{D}}\underline{B}-\frac{3}{2}(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{B})\widehat{\underline{X}}\\
& +(\widehat{\underline{X}}\cdot\overline{\check{H}})\underline{B}+(\Gamma_b\cdot\Gamma_b)\cdot(A,B)+r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b),
\end{aligned}
$$

as stated.

D.6 Proof of Theorem 5.3.6

We take a modified ${}^{(c)}\nabla_4$ derivative of the Teukolsky equation.

Proposition D.6.1. *We have*

$$\begin{aligned}&\Big({}^{(c)}\nabla_4+trX+\frac{1}{2}\overline{trX}\Big)\Big({}^{(c)}\nabla_3+2\overline{tr\underline{X}}+\frac{1}{2}tr\underline{X}\Big)\underline{A}_4\\&=\frac{1}{4}\big({}^{(c)}\mathcal{D}+H+5\,\underline{H}\big)\widehat{\otimes}\Big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\Big)+3P\underline{A}_4\\&+3\Big(\frac{1}{2}\overline{trX}-trX\Big)P\underline{A}+\mathcal{J}_{434}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)+Err_{434},\end{aligned}\tag{D.6.1}$$

where $\mathcal{J}_{434}$ *is a one-form given by*

$$\begin{aligned}\mathcal{J}_{434}&=\frac{1}{4}\Big(-\frac{1}{2}{}^{(c)}\mathcal{D}(trX+\overline{trX})+\frac{1}{2}(trX-\overline{trX})H-4trX\,\underline{H}\Big)-\frac{1}{4}B\\&\quad+r^{-2}\Gamma_g+\widehat{X}\cdot\check{H}\end{aligned}\tag{D.6.2}$$

$$=-\frac{3}{4}trX\,\underline{H}+r^{-1}\mathfrak{d}^{\leq 1}\Gamma_g,\tag{D.6.3}$$

and the error terms are schematically given by

$$\begin{aligned}Err_{434}=&\ {}^{(c)}\nabla_4Err_{TE}+\big(\frac{1}{2}trX+\overline{trX}\big)Err_{TE}\\&+{}^{(c)}\mathcal{D}\widehat{\otimes}(\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}\underline{A})+{}^{(c)}\mathcal{D}\widehat{\otimes}(({}^{(c)}\mathcal{D}\cdot\overline{\widehat{X}})\cdot\underline{A})\\&+r^{-1}\mathfrak{d}^{\leq 1}((A,B)\cdot\Gamma_b)+r^{-3}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b).\end{aligned}\tag{D.6.4}$$

Proof. We apply the operator ${}^{(c)}\nabla_4+\big(\mathrm{tr}X+\frac{1}{2}\overline{\mathrm{tr}X}\big)$ to the Teukolsky equation (5.3.1), and we deduce

$$\begin{aligned}&\Big({}^{(c)}\nabla_4+\mathrm{tr}X+\frac{1}{2}\overline{\mathrm{tr}X}\Big)\Big({}^{(c)}\nabla_3\underline{A}_4+\big(2\overline{\mathrm{tr}\underline{X}}+\frac{1}{2}\mathrm{tr}\underline{X}\big)\underline{A}_4\Big)\\=&\ I+J+K\\&+{}^{(c)}\nabla_4\mathrm{Err}_{TE}+\big(\mathrm{tr}X+\frac{1}{2}\overline{\mathrm{tr}X}\big)\mathrm{Err}_{TE}+r^{-3}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b)\end{aligned}$$

where

$$\begin{aligned}I&=\frac{1}{4}{}^{(c)}\nabla_4\,{}^{(c)}\mathcal{D}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)+\frac{1}{4}\big(\mathrm{tr}X+\frac{1}{2}\overline{\mathrm{tr}X}\big)\,{}^{(c)}\mathcal{D}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)\\J&=\frac{1}{4}{}^{(c)}\nabla_4\Big((H+4\,\underline{H})\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)\Big)\\&\quad+\frac{1}{4}\big(\mathrm{tr}X+\frac{1}{2}\overline{\mathrm{tr}X}\big)(H+4\,\underline{H})\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)\\K&=3\,{}^{(c)}\nabla_4(P\underline{A})+3\big(\mathrm{tr}X+\frac{1}{2}\overline{\mathrm{tr}X}\big)P\underline{A}.\end{aligned}$$

Step 1. Calculation of I. Here we use the commutation formula (4.2.12) applied to $F = \big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)$ of signature $s=-2$, and we obtain[3]

$$
\begin{aligned}
& {}^{(c)}\nabla_4\,{}^{(c)}\mathcal{D}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)\\
= \;& {}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\nabla_4\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)+\,\underline{H}\widehat{\otimes}\,{}^{(c)}\nabla_4\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)\\
& -\frac{1}{2}\mathrm{tr}X\,\Big({}^{(c)}\mathcal{D}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)+3\,\underline{H}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)\Big)\\
& +B\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}\big)-\frac{1}{2}\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}\big)+r^{-3}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b).
\end{aligned}
$$

We now compute ${}^{(c)}\nabla_4\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)$. We first apply (4.2.14) to $U=\underline{A}$ of signature $s=-2$ and we obtain

$$
\begin{aligned}
[\,{}^{(c)}\nabla_4,\overline{{}^{(c)}\mathcal{D}}\cdot]\underline{A} \;=\;& -\frac{1}{2}\overline{\mathrm{tr}X}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}\big)+\overline{\underline{H}}\cdot\,{}^{(c)}\nabla_4\underline{A}\\
& +4\overline{B}\cdot\underline{A}-\frac{1}{2}\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}\underline{A}+r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b).
\end{aligned}
$$

The above gives

$$
\begin{aligned}
{}^{(c)}\nabla_4\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big) \;=\;& \overline{{}^{(c)}\mathcal{D}}\cdot\big({}^{(c)}\nabla_4\underline{A}\big)-\frac{1}{2}\overline{\mathrm{tr}X}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}\big)+(\overline{H}+\overline{\underline{H}})\cdot\,{}^{(c)}\nabla_4\underline{A}\\
& +{}^{(c)}\nabla_4\overline{H}\cdot\underline{A}+4\overline{B}\cdot\underline{A}-\frac{1}{2}\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}\underline{A}+r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b)\\
=\;& \overline{{}^{(c)}\mathcal{D}}\cdot\big({}^{(c)}\nabla_4\underline{A}\big)-\frac{1}{2}\overline{\mathrm{tr}X}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}\big)+(\overline{H}+\overline{\underline{H}})\cdot\,{}^{(c)}\nabla_4\underline{A}\\
& -\frac{1}{2}\mathrm{tr}X(\overline{H}-\overline{\underline{H}})\cdot\underline{A}+3\overline{B}\cdot\underline{A}-\frac{1}{2}\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}\underline{A}+r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b),
\end{aligned}
$$

where we used the null structure equation

$$
{}^{(c)}\nabla_4 H \;=\; -\frac{1}{2}\overline{\mathrm{tr}X}(H-\underline{H})-B+r^{-2}\Gamma_g.
$$

[3]Because of the gauge conditions $\Xi=0$, $\widecheck{\underline{H}}=0$ there is no cubic term involving $\underline{A}$.

Now writing ${}^{(c)}\nabla_4\underline{A} = \underline{A}_4 - \frac{1}{2}\mathrm{tr}X\underline{A}$, we obtain

$$\begin{aligned}
{}^{(c)}\nabla_4\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big) &= \overline{{}^{(c)}\mathcal{D}}\cdot(\underline{A}_4-\frac{1}{2}\mathrm{tr}X\underline{A})-\frac{1}{2}\overline{\mathrm{tr}X}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}\big)\\
&\quad+(\overline{H}+\overline{\underline{H}})\cdot(\underline{A}_4-\frac{1}{2}\mathrm{tr}X\underline{A})-\frac{1}{2}\mathrm{tr}X(\overline{H}-\overline{\underline{H}})\cdot\underline{A}\\
&\quad+3\overline{B}\cdot\underline{A}-\frac{1}{2}\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}\underline{A}+r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b)\\
&= \overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4-\frac{1}{2}(\mathrm{tr}X+\overline{\mathrm{tr}X})\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}\\
&\quad-\big(\frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\mathrm{tr}X+\mathrm{tr}X\overline{H}\big)\cdot\underline{A}\\
&\quad+3\overline{B}\cdot\underline{A}-\frac{1}{2}\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}\underline{A}+r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b).
\end{aligned}$$

Using the Codazzi equation

$$\overline{{}^{(c)}\mathcal{D}}\mathrm{tr}X = -(\mathrm{tr}X-\overline{\mathrm{tr}X})\overline{H}+2\overline{B}+{}^{(c)}\mathcal{D}\cdot\overline{\widehat{X}}+r^{-2}\Gamma_g,$$

we obtain

$$\begin{aligned}
&{}^{(c)}\nabla_4\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)\\
&= \overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4-\frac{1}{2}(\mathrm{tr}X+\overline{\mathrm{tr}X})\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)\\
&+2\overline{B}\cdot\underline{A}-\frac{1}{2}\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}\underline{A}-\frac{1}{2}\big({}^{(c)}\mathcal{D}\cdot\overline{\widehat{X}}\big)\cdot\underline{A}+r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b).
\end{aligned}\tag{D.6.5}$$

By applying the operator ${}^{(c)}\mathcal{D}\widehat{\otimes}$ to (D.6.5), we obtain

$$\begin{aligned}
&{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\nabla_4\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)\\
=\;&{}^{(c)}\mathcal{D}\widehat{\otimes}\Big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\Big)-\frac{1}{2}(\mathrm{tr}X+\overline{\mathrm{tr}X})\,{}^{(c)}\mathcal{D}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)\\
&-\frac{1}{2}\,{}^{(c)}\mathcal{D}(\mathrm{tr}X+\overline{\mathrm{tr}X})\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)\\
&+2\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{B}\cdot\underline{A})-\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}\underline{A})-\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(({}^{(c)}\mathcal{D}\cdot\overline{\widehat{X}})\cdot\underline{A})+r^{-3}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b).
\end{aligned}$$

Also, from (D.6.5) we obtain

$$\begin{aligned}
\underline{H}\widehat{\otimes}\,{}^{(c)}\nabla_4\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big) &= \underline{H}\widehat{\otimes}\Big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\Big)\\
&\quad-\frac{1}{2}(\mathrm{tr}X+\overline{\mathrm{tr}X})\,\underline{H}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)+r^{-3}\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\Gamma_b).
\end{aligned}$$

We therefore deduce

$$
\begin{aligned}
4I &= {}^{(c)}\mathcal{D}\widehat{\otimes}\Big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\Big)-\frac{1}{2}(\mathrm{tr}X+\overline{\mathrm{tr}X})\,{}^{(c)}\mathcal{D}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)\\
&\quad -\frac{1}{2}\,{}^{(c)}\mathcal{D}(\mathrm{tr}X+\overline{\mathrm{tr}X})\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)\\
&\quad +\underline{H}\widehat{\otimes}\Big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\Big)-\frac{1}{2}(\mathrm{tr}X+\overline{\mathrm{tr}X})\,\underline{H}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)\\
&\quad -\frac{1}{2}\mathrm{tr}X\,\Big({}^{(c)}\mathcal{D}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)+3\,\underline{H}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)\Big)\\
&\quad +\big(\mathrm{tr}X+\frac{1}{2}\overline{\mathrm{tr}X}\big)\,{}^{(c)}\mathcal{D}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)\\
&\quad +\frac{1}{4}B\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}\big)+\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{B}\cdot\underline{A})-\frac{1}{8}\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}\big)\\
&\quad -\frac{1}{8}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}\underline{A})-\frac{1}{8}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\big(({}^{(c)}\mathcal{D}\cdot\overline{\widehat{X}})\cdot\underline{A}\big)+r^{-3}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b),
\end{aligned}
$$

which gives

$$
\begin{aligned}
I &= \frac{1}{4}\big({}^{(c)}\mathcal{D}+\underline{H}\big)\widehat{\otimes}\Big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\Big)\\
&\quad -\frac{1}{8}\big({}^{(c)}\mathcal{D}(\mathrm{tr}X+\overline{\mathrm{tr}X})+(4\mathrm{tr}X+\overline{\mathrm{tr}X})\,\underline{H}\big)\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)\\
&\quad +\frac{1}{4}B\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}\big)+\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{B}\cdot\underline{A})-\frac{1}{8}\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}\big)\\
&\quad -\frac{1}{8}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}\underline{A})-\frac{1}{8}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\big(({}^{(c)}\mathcal{D}\cdot\overline{\widehat{X}})\cdot\underline{A}\big)+r^{-3}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b).
\end{aligned}
$$

Step 2. Calculation of J. We have, using (D.6.5),

$$
\begin{aligned}
J &= \frac{1}{4}(H+4\,\underline{H})\widehat{\otimes}\,{}^{(c)}\nabla_4\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)+\frac{1}{4}\,{}^{(c)}\nabla_4(H+4\,\underline{H})\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)\\
&\quad +\frac{1}{4}\big(\mathrm{tr}X+\frac{1}{2}\overline{\mathrm{tr}X}\big)(H+4\,\underline{H})\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)\\
&= \frac{1}{4}(H+4\,\underline{H})\widehat{\otimes}\Big[\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4-\frac{1}{2}(\mathrm{tr}X+\overline{\mathrm{tr}X})\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)\Big]\\
&\quad +\frac{1}{4}\,{}^{(c)}\nabla_4(H+4\,\underline{H})\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)\\
&\quad +\frac{1}{4}\big(\mathrm{tr}X+\frac{1}{2}\overline{\mathrm{tr}X}\big)(H+4\,\underline{H})\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)+r^{-3}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b),
\end{aligned}
$$

which gives

$$
\begin{aligned}
J &= \frac{1}{4}(H+4\,\underline{H})\widehat{\otimes}\Big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\Big)\\
&\quad +\Big(\frac{1}{4}\,{}^{(c)}\nabla_4(H+4\,\underline{H})+\frac{1}{8}\mathrm{tr}X(H+4\,\underline{H})\Big)\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)+r^{-3}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b).
\end{aligned}
$$

Step 3. Calculation of K. Writing ${}^{(c)}\nabla_4\underline{A} = \underline{A}_4 - \frac{1}{2}\mathrm{tr}X\underline{A}$, and making use of the equation

$$ {}^{(c)}\nabla_4 P - \frac{1}{2}\,{}^{(c)}\mathcal{D}\cdot\overline{B} \;=\; -\frac{3}{2}\mathrm{tr}XP + \underline{H}\cdot\overline{B} - \frac{1}{4}\underline{\widehat{X}}\cdot\overline{A}, $$

we have

$$ \begin{aligned} K &= 3P\,{}^{(c)}\nabla_4\underline{A} + 3\,{}^{(c)}\nabla_4(P)\underline{A} + 3\big(\mathrm{tr}X + \frac{1}{2}\overline{\mathrm{tr}X}\big)P\underline{A} \\ &= 3P\big(\underline{A}_4 - \frac{1}{2}\mathrm{tr}X\underline{A}\big) + 3\big(-\frac{3}{2}\mathrm{tr}XP\big)\underline{A} + 3\big(\mathrm{tr}X + \frac{1}{2}\overline{\mathrm{tr}X}\big)P\underline{A} \\ &\quad + \frac{3}{2}\big({}^{(c)}\mathcal{D}\cdot\overline{B}\big)\underline{A} + r^{-3}(\Gamma_g\cdot\Gamma_b) + \underline{\widehat{X}}\cdot\underline{A}\cdot A \\ &= 3P\underline{A}_4 + 3\big(\frac{1}{2}\overline{\mathrm{tr}X} - \mathrm{tr}X\big)P\underline{A} + \frac{3}{2}\big({}^{(c)}\mathcal{D}\cdot\overline{B}\big)\underline{A} + r^{-3}(\Gamma_g\cdot\Gamma_b) + r^{-1}(\Gamma_b\cdot A), \end{aligned} $$

where we wrote $\underline{\widehat{X}}\cdot\underline{A}\cdot A = r^{-1}(\Gamma_b\cdot A)$.

Step 4. Final sum. We deduce

$$ \begin{aligned} &\Big({}^{(c)}\nabla_4 + \mathrm{tr}X + \frac{1}{2}\overline{\mathrm{tr}X}\Big)\Big({}^{(c)}\nabla_3\underline{A}_4 + \big(2\overline{\mathrm{tr}\underline{X}} + \frac{1}{2}\mathrm{tr}\underline{X}\big)\underline{A}_4\Big) \\ =\; &\frac{1}{4}\big({}^{(c)}\mathcal{D} + H + 5\,\underline{H}\big)\widehat{\otimes}\Big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4 + (\overline{H} + \overline{\underline{H}})\cdot\underline{A}_4\Big) + 3P\underline{A}_4 \\ &+3\big(\frac{1}{2}\overline{\mathrm{tr}X} - \mathrm{tr}X\big)P\underline{A} + \mathcal{J}_{434}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A} + \overline{H}\cdot\underline{A}\big) + \mathrm{Err}_{434}, \end{aligned} $$

where $\mathcal{J}_{434}$ is the one form given by

$$ \mathcal{J}_{434} \;:=\; \frac{1}{4}\,{}^{(c)}\nabla_4(H + 4\,\underline{H}) + \frac{1}{8}\mathrm{tr}X(H + 4\,\underline{H}) - \frac{1}{8}\big({}^{(c)}\mathcal{D}(\mathrm{tr}X + \overline{\mathrm{tr}X}) + (4\mathrm{tr}X + \overline{\mathrm{tr}X})\,\underline{H}\big). $$

and Err_{434} are the error terms explicitly given by

$$ \begin{aligned} \mathrm{Err}_{434} \;=\; &{}^{(c)}\nabla_4\mathrm{Err}_{TE}[\underline{A}] + \big(\frac{1}{2}\mathrm{tr}X + \overline{\mathrm{tr}X}\big)\mathrm{Err}_{TE}[\underline{A}] \\ &+\frac{1}{4}B\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}\big) + \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{B}\cdot\underline{A}) + \frac{3}{2}\big({}^{(c)}\mathcal{D}\cdot\overline{B}\big)\underline{A} \\ &-\frac{1}{8}\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}\big) - \frac{1}{8}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}\underline{A}) - \frac{1}{8}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\big(({}^{(c)}\mathcal{D}\cdot\overline{\widehat{X}})\cdot\underline{A}\big) \\ &+r^{-3}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b) + r^{-1}(\Gamma_b\cdot A). \end{aligned} $$

Observe that the above error terms can schematically be written as

$$ \begin{aligned} \mathrm{Err}_{434} \;=\; &{}^{(c)}\nabla_4\mathrm{Err}_{TE} + \big(\frac{1}{2}\mathrm{tr}X + \overline{\mathrm{tr}X}\big)\mathrm{Err}_{TE} + {}^{(c)}\mathcal{D}\widehat{\otimes}(\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}\underline{A}) + {}^{(c)}\mathcal{D}\widehat{\otimes}\big(({}^{(c)}\mathcal{D}\cdot\overline{\widehat{X}})\cdot\underline{A}\big) \\ &+r^{-1}\mathfrak{d}^{\leq 1}((A,B)\cdot\Gamma_b) + r^{-3}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b). \end{aligned} $$

We now compute $\mathcal{J}_{434}$. Recall, see Lemma D.5.1,

$$\begin{aligned} {}^{(c)}\mathcal{D}\overline{\mathrm{tr}X} &= 2i\Im(\mathrm{tr}X)H + r^{-1}\Gamma_g \\ {}^{(c)}\mathcal{D}\mathrm{tr}X &= -2\mathrm{tr}X\,\underline{H} + r^{-1}\Gamma_g. \end{aligned}$$

Hence

$$ {}^{(c)}\mathcal{D}(\mathrm{tr}X + \overline{\mathrm{tr}X}) = (\mathrm{tr}X - \overline{\mathrm{tr}X})H - 2\mathrm{tr}X\,\underline{H} + r^{-1}\Gamma_g. $$

Recalling also the equations, see Lemma D.5.2,

$$\begin{aligned} \nabla_4\,\underline{H} + \mathrm{tr}X\,\underline{H} &= r^{-2}\Gamma_g \\ {}^{(c)}\nabla_4 H &= -\frac{1}{2}\overline{\mathrm{tr}X}(H - \,\underline{H}) - B + \widehat{X}\cdot\check{H} \end{aligned}$$

We deduce

$$\begin{aligned} \mathcal{J}_{434} &:= \frac{1}{4}\,{}^{(c)}\nabla_4(H + 4\,\underline{H}) + \frac{1}{8}\mathrm{tr}X(H + 4\,\underline{H}) - \frac{1}{8}\Big(\,{}^{(c)}\mathcal{D}(\mathrm{tr}X + \overline{\mathrm{tr}X}) + (4\mathrm{tr}X + \overline{\mathrm{tr}X})\,\underline{H}\Big) \\ &= \frac{1}{4}\Big(\Big(\,{}^{(c)}\nabla_4 H + \frac{1}{2}\mathrm{tr}X H\Big) + 4\Big(\,{}^{(c)}\nabla_4\,\underline{H} + \frac{1}{2}\mathrm{tr}X\,\underline{H}\Big)\Big) - \frac{1}{8}((\mathrm{tr}X - \overline{\mathrm{tr}X})H - 2\mathrm{tr}X\,\underline{H}) \\ &\quad -\frac{1}{8}(4\mathrm{tr}X + \overline{\mathrm{tr}X})\,\underline{H} + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_g \\ &= \frac{1}{4}\Big(\Big(- \frac{1}{2}\overline{\mathrm{tr}X}(H - \,\underline{H}) - B + \Gamma_g\cdot\Gamma_b + \frac{1}{2}\mathrm{tr}X H\Big) + 4\Big(- \mathrm{tr}X\,\underline{H} + \frac{1}{2}\mathrm{tr}X\,\underline{H}\Big)\Big) \\ &\quad -\frac{1}{8}((\mathrm{tr}X - \overline{\mathrm{tr}X})H - 2\mathrm{tr}X\,\underline{H}) - \frac{1}{8}(4\mathrm{tr}X + \overline{\mathrm{tr}X})\,\underline{H} + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_g \\ &= -\frac{3}{4}\mathrm{tr}X\,\underline{H} + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_g. \end{aligned}$$

Hence

$$ \mathcal{J}_{434} = -\frac{3}{4}\mathrm{tr}X\,\underline{H} + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_g. $$

We also have, by keeping only error terms that behave like $r^{-2}\Gamma_g$ or better,

$$\begin{aligned} \mathcal{J}_{434} &:= \frac{1}{4}\,{}^{(c)}\nabla_4(H + 4\,\underline{H}) + \frac{1}{8}\mathrm{tr}X(H + 4\,\underline{H}) - \frac{1}{8}\Big(\,{}^{(c)}\mathcal{D}(\mathrm{tr}X + \overline{\mathrm{tr}X}) + (4\mathrm{tr}X + \overline{\mathrm{tr}X})\,\underline{H}\Big) \\ &= \frac{1}{4}\Big(- \frac{1}{2}\overline{\mathrm{tr}X}(H - \,\underline{H}) - B + \widehat{X}\cdot\check{H} + \frac{1}{2}\mathrm{tr}X H + 4\Big(- \frac{1}{2}\mathrm{tr}X\,\underline{H} + r^{-2}\Gamma_g\Big)\Big) \\ &\quad -\frac{1}{8}\Big(\,{}^{(c)}\mathcal{D}(\mathrm{tr}X + \overline{\mathrm{tr}X}) + (4\mathrm{tr}X + \overline{\mathrm{tr}X})\,\underline{H}\Big) \\ &= \frac{1}{4}\Big(- \frac{1}{2}\,{}^{(c)}\mathcal{D}(\mathrm{tr}X + \overline{\mathrm{tr}X}) + \frac{1}{2}(\mathrm{tr}X - \overline{\mathrm{tr}X})H - 4\mathrm{tr}X\,\underline{H}\Big) \\ &\quad -\frac{1}{4}B + r^{-2}\Gamma_g + \widehat{X}\cdot\check{H}, \end{aligned}$$

as stated. This ends the proof of Proposition D.6.1. □

We now take another derivative of equation (D.6.1).

Proposition D.6.2. *We have*

$$\begin{aligned}
&\Big({}^{(c)}\nabla_4 + 3trX - \frac{1}{2}\overline{trX} - 2\frac{{}^{(a)}tr\chi^2}{tr\,\chi}\Big)\Big({}^{(c)}\nabla_4 + trX + \frac{1}{2}\overline{trX}\Big)\Big({}^{(c)}\nabla_3 + 2\overline{tr\underline{X}} + \frac{1}{2}tr\underline{X}\Big)\underline{A}_4 \\
&+ \big(2trX - \overline{trX}\big)3P\underline{A}_4 \\
&= \frac{1}{4}\big({}^{(c)}\mathcal{D} + H + 6\,\underline{H}\big)\widehat{\otimes}\Big(\overline{{}^{(c)}\mathcal{D}}\cdot\widetilde{\underline{Q}(\underline{A})} + (\overline{H} + 2\,\overline{\underline{H}})\cdot\widetilde{\underline{Q}(\underline{A})}\Big) + 3P\ \widetilde{\underline{Q}(\underline{A})} \\
&+ \mathscr{L}[\underline{A}, \underline{A}_4, \mathcal{D}\underline{A}_4, \mathcal{D}\underline{A}] + Err_{4434}
\end{aligned} \tag{D.6.6}$$

where $\mathscr{L}[\underline{A}, \underline{A}_4, \mathcal{D}\underline{A}_4, \mathcal{D}\underline{A}]$ *denote* $O(|a|)$ *linear order terms in* $\underline{A}, \underline{A}_4, \mathcal{D}\underline{A}, \mathcal{D}\underline{A}_4$ *explicitly given by* (D.6.11)*, schematically given by*

$$\begin{aligned}
\mathscr{L}[\underline{A}, \underline{A}_4, \mathcal{D}\underline{A}, \mathcal{D}\underline{A}_4] &= O(ar^{-3})\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4 + (\overline{H} + \overline{\underline{H}})\cdot\underline{A}_4\big) \\
&\quad + O(a^2r^{-4})\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A} + \overline{H}\cdot\underline{A}\big) + O(a^2r^{-7})\underline{A},
\end{aligned}$$

and the error terms are given by

$$\begin{aligned}
Err_{4434} &= {}^{(c)}\nabla_4 Err_{434} + \big(trX + \frac{3}{2}\overline{trX} + 2\frac{{}^{(a)}tr\chi^2}{tr\,\chi}\big)Err_{434} \\
&\quad + r^{-2}\mathfrak{d}^{\leq 1}((A,B)\cdot\Gamma_b) + r^{-4}\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\Gamma_b) \\
&\quad + r^{-1}\big({}^{(c)}\nabla_4(\widehat{X}\cdot\check{H}) + \frac{3}{2}trX(\widehat{X}\cdot\check{H})\big)\cdot\mathfrak{d}^{\leq 1}\underline{A}.
\end{aligned}$$

Proof. We apply the operator ${}^{(c)}\nabla_4 + 2\mathrm{tr}X + \frac{1}{2}\overline{\mathrm{tr}X}$ to equation (D.6.1), and we obtain

$$\begin{aligned}
&\Big({}^{(c)}\nabla_4 + 2\mathrm{tr}X + \frac{1}{2}\overline{\mathrm{tr}X}\Big)\Big({}^{(c)}\nabla_4 + \mathrm{tr}X + \frac{1}{2}\overline{\mathrm{tr}X}\Big)\Big({}^{(c)}\nabla_3 + 2\overline{\mathrm{tr}\underline{X}} + \frac{1}{2}\mathrm{tr}\underline{X}\Big)\underline{A}_4 \\
= \ & I + J + K + L + M \\
&+ {}^{(c)}\nabla_4\mathrm{Err}_{434} + \big(2\mathrm{tr}X + \frac{1}{2}\overline{\mathrm{tr}X}\big)\mathrm{Err}_{434} + r^{-4}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b),
\end{aligned}$$

where

$$
\begin{aligned}
I &= \frac{1}{4}\,{}^{(c)}\nabla_4\,{}^{(c)}\mathcal{D}\widehat{\otimes}\Big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\Big)\\
&\quad+\frac{1}{4}\big(2\mathrm{tr}X+\frac{1}{2}\overline{\mathrm{tr}X}\big)\,{}^{(c)}\mathcal{D}\widehat{\otimes}\Big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\Big),\\
J &= \frac{1}{4}\,{}^{(c)}\nabla_4\Big(\big(H+5\,\underline{H}\big)\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\big)\Big)\\
&\quad+\frac{1}{4}\big(2\mathrm{tr}X+\frac{1}{2}\overline{\mathrm{tr}X}\big)\big(H+5\,\underline{H}\big)\widehat{\otimes}\Big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\Big),\\
K &= 3\,{}^{(c)}\nabla_4(P\underline{A}_4)+3\big(2\mathrm{tr}X+\frac{1}{2}\overline{\mathrm{tr}X}\big)P\underline{A}_4,\\
L &= 3\,{}^{(c)}\nabla_4\Big(\big(\frac{1}{2}\overline{\mathrm{tr}X}-\mathrm{tr}X\big)P\underline{A}\Big)+3\big(2\mathrm{tr}X+\frac{1}{2}\overline{\mathrm{tr}X}\big)\big(\frac{1}{2}\overline{\mathrm{tr}X}-\mathrm{tr}X\big)P\underline{A},\\
M &= {}^{(c)}\nabla_4\Big(\mathcal{J}_{434}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)\Big)+\big(2\mathrm{tr}X+\frac{1}{2}\overline{\mathrm{tr}X}\big)\mathcal{J}_{434}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big).
\end{aligned}
$$

Step 1. Calculation of I. Here we use the commutation formula (4.2.13) applied to $F=\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\big)$ of signature $s=-1$, where observe that $F=r^{-3}\Gamma_b$. We therefore obtain

$$
\begin{aligned}
&{}^{(c)}\nabla_4\,{}^{(c)}\mathcal{D}\widehat{\otimes}\Big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\Big)\\
=\;& {}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\nabla_4\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\big)+\underline{H}\widehat{\otimes}\,{}^{(c)}\nabla_4\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\big)\\
&-\frac{1}{2}\mathrm{tr}X\,\Big({}^{(c)}\mathcal{D}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\big)+2\,\underline{H}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\big)\Big)\\
&+r^{-4}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b).
\end{aligned}
$$

We now compute ${}^{(c)}\nabla_4\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\big)$ by applying (4.2.15) to $U=\underline{A}_4$ of signature $s=-1$ and we obtain

$$
\begin{aligned}
{}^{(c)}\nabla_4\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\big) &= \overline{{}^{(c)}\mathcal{D}}\cdot(\,{}^{(c)}\nabla_4\underline{A}_4)-\frac{1}{2}\overline{\mathrm{tr}X}\,(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4-\overline{\underline{H}}\cdot\underline{A}_4)\\
&\quad+(\overline{H}+2\overline{\underline{H}})\cdot\,{}^{(c)}\nabla_4\underline{A}_4+\,{}^{(c)}\nabla_4(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\\
&\quad+r^{-3}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b).
\end{aligned}
$$

Making use of the equations

$$
\begin{aligned}
{}^{(c)}\nabla_4 H &= -\frac{1}{2}\overline{\mathrm{tr}X}(H-\underline{H})+r^{-1}\Gamma_g,\\
\nabla_4\,\underline{H} &= -\mathrm{tr}X\,\underline{H}+r^{-2}\Gamma_g,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
{}^{(c)}\nabla_4\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\big) &= \overline{{}^{(c)}\mathcal{D}}\cdot\big({}^{(c)}\nabla_4\underline{A}_4\big)-\frac{1}{2}\overline{\mathrm{tr}X}\,\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+\overline{\underline{H}}\cdot\underline{A}_4\big)\\
&+(\overline{H}+2\overline{\underline{H}})\cdot\,{}^{(c)}\nabla_4\underline{A}_4-\frac{1}{2}\mathrm{tr}X(\overline{H}-\overline{\underline{H}})\cdot\underline{A}_4\\
&+r^{-3}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b).
\end{aligned}
$$

Using (5.3.6) to write ${}^{(c)}\nabla_4\underline{A}_4=\widetilde{\underline{Q}(\underline{A})}-\big(\frac{5}{2}\mathrm{tr}X-\overline{\mathrm{tr}X}-2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}\big)\underline{A}_4$, we obtain

$$
\begin{aligned}
&{}^{(c)}\nabla_4\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\big)\\
=\;&\overline{{}^{(c)}\mathcal{D}}\cdot\Big(\widetilde{\underline{Q}(\underline{A})}-\big(\frac{5}{2}\mathrm{tr}X-\overline{\mathrm{tr}X}-2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}\big)\underline{A}_4\Big)\\
&-\frac{1}{2}\overline{\mathrm{tr}X}\,\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+\overline{\underline{H}}\cdot\underline{A}_4\big)-\frac{1}{2}\mathrm{tr}X(\overline{H}-\overline{\underline{H}})\cdot\underline{A}_4\\
&+(\overline{H}+2\overline{\underline{H}})\cdot\Big(\widetilde{\underline{Q}(\underline{A})}-\left(\frac{5}{2}\mathrm{tr}X-\overline{\mathrm{tr}X}-2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}\right)\underline{A}_4\Big)+r^{-3}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b),
\end{aligned}
$$

which gives

$$
\begin{aligned}
&{}^{(c)}\nabla_4\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\big)\\
&=\overline{{}^{(c)}\mathcal{D}}\cdot\widetilde{\underline{Q}(\underline{A})}+(\overline{H}+2\overline{\underline{H}})\cdot\widetilde{\underline{Q}(\underline{A})}\\
&-\left(\frac{5}{2}\mathrm{tr}X-\frac{1}{2}\overline{\mathrm{tr}X}-2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}\right)\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\big)\\
&+\overline{\mathcal{I}}\cdot\underline{A}_4+r^{-3}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b),
\end{aligned}
\tag{D.6.7}
$$

where $\overline{\mathcal{I}}$ is the one-form given by

$$
\begin{aligned}
\overline{\mathcal{I}} &= -\overline{{}^{(c)}\mathcal{D}}\left(\frac{5}{2}\mathrm{tr}X-\overline{\mathrm{tr}X}-2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}\right)+\frac{1}{2}\overline{\mathrm{tr}X}\,\overline{H}-\frac{1}{2}\mathrm{tr}X(\overline{H}-\overline{\underline{H}})\\
&-\left(\frac{5}{2}\mathrm{tr}X-\overline{\mathrm{tr}X}-2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}\right)\overline{\underline{H}}.
\end{aligned}
$$

Using Lemma (D.5.1), we obtain for $\mathcal{I}$:

$$
\begin{aligned}
\overline{\mathcal{I}} &= 4i\Im(trX)\overline{H}-\left(2\mathrm{tr}X+\overline{\mathrm{tr}X}-2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}\right)\overline{\underline{H}}+2\,\overline{{}^{(c)}\mathcal{D}}\left(\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}\right)+r^{-1}\Gamma_g\\
&= O(ar^{-3})+r^{-1}\Gamma_g.
\end{aligned}
\tag{D.6.8}
$$

By applying the operator ${}^{(c)}\mathcal{D}\widehat{\otimes}$ to (D.6.7), we obtain

$$
\begin{aligned}
& {}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\nabla_4\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\big)\\
= & \ {}^{(c)}\mathcal{D}\widehat{\otimes}\Big(\overline{{}^{(c)}\mathcal{D}}\cdot\widetilde{\underline{Q}(\underline{A})}+(\overline{H}+2\,\overline{\underline{H}})\cdot\widetilde{\underline{Q}(\underline{A})}\Big)\\
& -\big(\frac{5}{2}\mathrm{tr}X-\frac{1}{2}\overline{\mathrm{tr}X}-2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}\big)\,{}^{(c)}\mathcal{D}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\big)\\
& -{}^{(c)}\mathcal{D}\big(\frac{5}{2}\mathrm{tr}X-\frac{1}{2}\overline{\mathrm{tr}X}-2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}\big)\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\big)\\
& +2\overline{\mathcal{I}}\cdot{}^{(c)}\mathcal{D}\underline{A}_4+2({}^{(c)}\mathcal{D}\cdot\overline{\mathcal{I}})\underline{A}_4+r^{-4}\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\Gamma_b).
\end{aligned}
$$

Also, from (D.6.7) we obtain

$$
\begin{aligned}
& \underline{H}\widehat{\otimes}\,{}^{(c)}\nabla_4\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\big)\\
= & \ \underline{H}\widehat{\otimes}\Big(\overline{{}^{(c)}\mathcal{D}}\cdot\widetilde{\underline{Q}(\underline{A})}+(\overline{H}+2\,\overline{\underline{H}})\cdot\widetilde{\underline{Q}(\underline{A})}\Big)\\
& -\big(\frac{5}{2}\mathrm{tr}X-\frac{1}{2}\overline{\mathrm{tr}X}-2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}\big)\,\underline{H}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\big)+\underline{H}\widehat{\otimes}\big(\overline{\mathcal{I}}\cdot\underline{A}_4\big)\\
& +r^{-4}\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\Gamma_b).
\end{aligned}
$$

We therefore finally deduce

$$
\begin{aligned}
I = & \ \frac{1}{4}\big({}^{(c)}\mathcal{D}+\underline{H}\big)\widehat{\otimes}\Big(\overline{{}^{(c)}\mathcal{D}}\cdot\widetilde{\underline{Q}(\underline{A})}+(\overline{H}+2\,\overline{\underline{H}})\cdot\widetilde{\underline{Q}(\underline{A})}\Big)\\
& -\frac{1}{4}\big(\mathrm{tr}X-\overline{\mathrm{tr}X}-2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}\big)\,{}^{(c)}\mathcal{D}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\big)\\
& +\frac{1}{4}\mathcal{J}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\big)+\frac{1}{2}\overline{\mathcal{I}}\cdot{}^{(c)}\mathcal{D}\underline{A}_4+\frac{1}{2}({}^{(c)}\mathcal{D}\cdot\overline{\mathcal{I}})\underline{A}_4+\frac{1}{4}\,\underline{H}\widehat{\otimes}\big(\overline{\mathcal{I}}\cdot\underline{A}_4\big)\\
& +r^{-4}\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\Gamma_b),
\end{aligned}
$$

where $\mathcal{J}$ is the one-form given by

$$
\mathcal{J} = -{}^{(c)}\mathcal{D}\left(\frac{5}{2}\mathrm{tr}X-\frac{1}{2}\overline{\mathrm{tr}X}-2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}\right)-\left(\frac{5}{2}\mathrm{tr}X-\frac{1}{2}\overline{\mathrm{tr}X}-2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}\right)\underline{H}-\mathrm{tr}X\,\underline{H}.
$$

Using Lemma (D.5.1), we obtain for $\mathcal{J}$:

$$
\begin{aligned}
\mathcal{J} = & -\frac{5}{2}(-2\mathrm{tr}X\,\underline{H})+\frac{1}{2}(2i\Im(\mathrm{tr}X)(H-\check{H}))+2\,{}^{(c)}\mathcal{D}\left(\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}\right)\\
& -\left(\frac{5}{2}\mathrm{tr}X-\frac{1}{2}\overline{\mathrm{tr}X}-2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}\right)\underline{H}-\mathrm{tr}X\,\underline{H}+r^{-1}\Gamma_g \qquad \text{(D.6.9)}\\
= & \ i\Im(\mathrm{tr}X)(H-\check{H})+2\,{}^{(c)}\mathcal{D}\left(\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}\right)+\left(\frac{3}{2}\mathrm{tr}X+\frac{1}{2}\overline{\mathrm{tr}X}+2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}\right)\underline{H}+r^{-1}\Gamma_g.
\end{aligned}
$$

Step 2. Calculation of J. We have, using (D.6.7),

$$\begin{aligned}
J &= \frac{1}{4}\big(H+5\,\underline{H}\big)\widehat{\otimes}\,{}^{(c)}\nabla_4\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\big)\\
&+\frac{1}{4}\,{}^{(c)}\nabla_4\big(H+5\,\underline{H}\big)\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\big)\\
&+\frac{1}{4}\big(2\mathrm{tr}X+\frac{1}{2}\overline{\mathrm{tr}X}\big)\big(H+5\,\underline{H}\big)\widehat{\otimes}\Big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\Big)\\
&= \frac{1}{4}\big(H+5\,\underline{H}\big)\widehat{\otimes}\Big[\overline{{}^{(c)}\mathcal{D}}\cdot\widetilde{\underline{Q}(\underline{A})}+(\overline{H}+2\overline{\underline{H}})\cdot\widetilde{\underline{Q}(\underline{A})}\\
&-\big(\frac{5}{2}\mathrm{tr}X-\frac{1}{2}\overline{\mathrm{tr}X}-2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}\big)\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\big)+\overline{\mathcal{I}}\cdot\underline{A}_4\Big]\\
&+\frac{1}{4}\,{}^{(c)}\nabla_4\big(H+5\,\underline{H}\big)\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\big)\\
&+\frac{1}{4}\big(2\mathrm{tr}X+\frac{1}{2}\overline{\mathrm{tr}X}\big)\big(H+5\,\underline{H}\big)\widehat{\otimes}\Big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\Big)+r^{-4}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b),
\end{aligned}$$

which gives

$$\begin{aligned}
J &= \frac{1}{4}\big(H+5\,\underline{H}\big)\widehat{\otimes}\Big(\overline{{}^{(c)}\mathcal{D}}\cdot\widetilde{\underline{Q}(\underline{A})}+(\overline{H}+2\overline{\underline{H}})\cdot\widetilde{\underline{Q}(\underline{A})}\Big)\\
&\mathcal{K}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\big)+\frac{1}{4}\big(H+5\,\underline{H}\big)\widehat{\otimes}(\overline{\mathcal{I}}\cdot\underline{A}_4)+r^{-4}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b)
\end{aligned}$$

where $\mathcal{K}$ is given by

$$\begin{aligned}
\mathcal{K} &= -\frac{1}{4}\big(\frac{5}{2}\mathrm{tr}X-\frac{1}{2}\overline{\mathrm{tr}X}-2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}\big)\big(H+5\,\underline{H}\big)+\frac{1}{4}\,{}^{(c)}\nabla_4\big(H+5\,\underline{H}\big)\\
&+\frac{1}{4}\big(2\mathrm{tr}X+\frac{1}{2}\overline{\mathrm{tr}X}\big)\big(H+5\,\underline{H}\big),
\end{aligned}$$

which gives, making use of equations ${}^{(c)}\nabla_4 H=-\frac{1}{2}\overline{\mathrm{tr}X}(H-\underline{H})+r^{-1}\Gamma_g$ and $\nabla_4\underline{H}=-\mathrm{tr}X\,\underline{H}+r^{-2}\Gamma_g$,

$$\begin{aligned}
\mathcal{K} &= -\frac{1}{4}\big(\frac{5}{2}\mathrm{tr}X-\frac{1}{2}\overline{\mathrm{tr}X}-2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}\big)\big(H+5\,\underline{H}\big)+\frac{1}{4}(-\frac{1}{2}\overline{\mathrm{tr}X}(H-\underline{H}))+\frac{5}{4}(-\mathrm{tr}X\,\underline{H})\\
&+\frac{1}{4}\big(2\mathrm{tr}X+\frac{1}{2}\overline{\mathrm{tr}X}\big)\big(H+5\,\underline{H}\big)+r^{-1}\Gamma_g \qquad \text{(D.6.10)}\\
&= -\frac{1}{4}\Big[\frac{1}{2}\big(\mathrm{tr}X-\overline{\mathrm{tr}X}-4\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}\big)H+\big(\frac{15}{2}\mathrm{tr}X-\frac{11}{2}\overline{\mathrm{tr}X}-10\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}\big)\,\underline{H}\Big]+r^{-1}\Gamma_g.
\end{aligned}$$

Step 3. Calculation of K. Writing ${}^{(c)}\nabla_4\underline{A}_4=\widetilde{\underline{Q}(\underline{A})}-\big(\frac{5}{2}\mathrm{tr}X-\overline{\mathrm{tr}X}-2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}\big)\underline{A}_4$, we have

$$
\begin{aligned}
K &= 3P\,{}^{(c)}\nabla_4\underline{A}_4 + 3\,{}^{(c)}\nabla_4(P)\underline{A}_4 + 3\big(2\mathrm{tr}X + \frac{1}{2}\overline{\mathrm{tr}X}\big)P\underline{A}_4\\
&= 3P\Big(\widetilde{\underline{Q}(\underline{A})} - \big(\frac{5}{2}\mathrm{tr}X - \overline{\mathrm{tr}X} - 2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}\big)\underline{A}_4\Big) + 3\big(-\frac{3}{2}\mathrm{tr}XP\big)\underline{A}_4\\
&\quad +3\big(2\mathrm{tr}X + \frac{1}{2}\overline{\mathrm{tr}X}\big)P\underline{A}_4 + r^{-4}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b)\\
&= 3P\,\widetilde{\underline{Q}(\underline{A})} - 3\big(2\mathrm{tr}X - \frac{3}{2}\overline{\mathrm{tr}X} - 2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}\big)P\underline{A}_4 + r^{-4}(\Gamma_g\cdot\Gamma_b).
\end{aligned}
$$

Step 4. Calculation of L. Writing ${}^{(c)}\nabla_4\underline{A} = \underline{A}_4 - \frac{1}{2}\mathrm{tr}X\underline{A}$, and making use of the equations

$$
\begin{aligned}
{}^{(c)}\nabla_4 P - \frac{1}{2}\,{}^{(c)}\mathcal{D}\cdot\overline{B} &= -\frac{3}{2}\mathrm{tr}XP + \underline{H}\cdot\overline{B} - \frac{1}{4}\widehat{\underline{X}}\cdot\overline{A},\\
{}^{(c)}\nabla_4\mathrm{tr}X + \frac{1}{2}(\mathrm{tr}X)^2 &= -\frac{1}{2}\widehat{X}\cdot\overline{\widehat{X}},
\end{aligned}
$$

we obtain

$$
\begin{aligned}
L &= 3\big(\frac{1}{2}\overline{\mathrm{tr}X} - \mathrm{tr}X\big)P\,{}^{(c)}\nabla_4\underline{A} + 3\big(\frac{1}{2}\overline{\mathrm{tr}X} - \mathrm{tr}X\big)\,{}^{(c)}\nabla_4(P)\underline{A}\\
&\quad +3\big(\frac{1}{2}\,{}^{(c)}\nabla_4\overline{\mathrm{tr}X} - {}^{(c)}\nabla_4\mathrm{tr}X\big)P\underline{A} + 3\big(2\mathrm{tr}X + \frac{1}{2}\overline{\mathrm{tr}X}\big)\big(\frac{1}{2}\overline{\mathrm{tr}X} - \mathrm{tr}X\big)P\underline{A}\\
&= 3\big(\frac{1}{2}\overline{\mathrm{tr}X} - \mathrm{tr}X\big)P\big(\underline{A}_4 - \frac{1}{2}\mathrm{tr}X\underline{A}\big) + 3\big(\frac{1}{2}\overline{\mathrm{tr}X} - \mathrm{tr}X\big)\big(-\frac{3}{2}\mathrm{tr}XP\big)\underline{A}\\
&\quad +3\big(-\frac{1}{2}\frac{1}{2}(\overline{\mathrm{tr}X})^2 + \frac{1}{2}(\mathrm{tr}X)^2\big)P\underline{A} + 3\big(2\mathrm{tr}X + \frac{1}{2}\overline{\mathrm{tr}X}\big)\big(\frac{1}{2}\overline{\mathrm{tr}X} - \mathrm{tr}X\big)P\underline{A}\\
&\quad +r^{-2}\mathfrak{d}^{\leq 1}((A,B)\cdot\Gamma_b),
\end{aligned}
$$

which gives

$$
L = \big(\frac{1}{2}\overline{\mathrm{tr}X} - \mathrm{tr}X\big)3P\underline{A}_4 + \frac{1}{2}\mathrm{tr}X\big(\mathrm{tr}X - \overline{\mathrm{tr}X}\big)3P\underline{A} + r^{-2}\mathfrak{d}^{\leq 1}((A,B)\cdot\Gamma_b).
$$

Step 5. Calculation of M. Using (D.6.5), we have

$$
\begin{aligned}
M &= \mathcal{J}_{434}\widehat{\otimes}\,{}^{(c)}\nabla_4\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A} + \overline{H}\cdot\underline{A}\big) + {}^{(c)}\nabla_4\big(\mathcal{J}_{434}\big)\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A} + \overline{H}\cdot\underline{A}\big)\\
&\quad +\big(2\mathrm{tr}X + \frac{1}{2}\overline{\mathrm{tr}X}\big)\mathcal{J}_{434}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A} + \overline{H}\cdot\underline{A}\big)\\
&= \mathcal{J}_{434}\widehat{\otimes}\Big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4 + (\overline{H} + \overline{\underline{H}})\cdot\underline{A}_4\Big)\\
&\quad +\Big({}^{(c)}\nabla_4\big(\mathcal{J}_{434}\big) + \frac{3}{2}\mathrm{tr}X\mathcal{J}_{434}\Big)\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A} + \overline{H}\cdot\underline{A}\big)\\
&\quad +r^{-4}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b) + r^{-2}\mathfrak{d}^{\leq 1}(B\cdot\Gamma_b).
\end{aligned}
$$

Step 6. Final sum. We deduce

$$
\begin{aligned}
& \Big({}^{(c)}\nabla_4+2\mathrm{tr}X+\frac{1}{2}\overline{\mathrm{tr}X}\Big)\Big({}^{(c)}\nabla_4+\mathrm{tr}X+\frac{1}{2}\overline{\mathrm{tr}X}\Big)\Big({}^{(c)}\nabla_3+2\overline{\mathrm{tr}\underline{X}}+\frac{1}{2}\mathrm{tr}\underline{X}\Big)\underline{A}_4\\
= \ & \frac{1}{4}\big({}^{(c)}\mathcal{D}+H+6\,\underline{H}\big)\widehat{\otimes}\Big(\overline{{}^{(c)}\mathcal{D}}\cdot\widetilde{\underline{Q}(\underline{A})}+(\overline{H}+2\,\overline{\underline{H}})\cdot\widetilde{\underline{Q}(\underline{A})}\Big)+3P\,\widetilde{\underline{Q}(\underline{A})}\\
& -\frac{1}{4}\big(\mathrm{tr}X-\overline{\mathrm{tr}X}-2\frac{{}^{(a)}\mathrm{tr}\chi^2}{tr\,\chi}\big)\,{}^{(c)}\mathcal{D}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\big)\\
& +\big(\frac{1}{4}\mathcal{J}+\mathcal{K}+\mathcal{J}_{434}\big)\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\big)+\frac{1}{2}\overline{\mathcal{I}}\cdot{}^{(c)}\mathcal{D}\underline{A}_4+\frac{1}{2}\big({}^{(c)}\mathcal{D}\cdot\overline{\mathcal{I}}\big)\underline{A}_4\\
& +\frac{1}{4}\big(H+6\,\underline{H}\big)\widehat{\otimes}(\overline{\mathcal{I}}\cdot\underline{A}_4)+\big(-3\mathrm{tr}X+2\overline{\mathrm{tr}X}+2\frac{{}^{(a)}\mathrm{tr}\chi^2}{tr\,\chi}\big)3P\underline{A}_4\\
& +\frac{1}{2}\mathrm{tr}X\big(\mathrm{tr}X-\overline{\mathrm{tr}X}\big)3P\underline{A}+\Big({}^{(c)}\nabla_4\big(\mathcal{J}_{434}\big)+\frac{3}{2}\mathrm{tr}X\mathcal{J}_{434}\Big)\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)\\
& +\widetilde{\mathrm{Err}}_{4434},
\end{aligned}
$$

where the error terms are given by

$$
\begin{aligned}
\widetilde{\mathrm{Err}_{4434}} \ := \ & {}^{(c)}\nabla_4\mathrm{Err}_{434}+\big(2\mathrm{tr}X+\frac{1}{2}\overline{\mathrm{tr}X}\big)\mathrm{Err}_{434}\\
& +r^{-2}\mathfrak{d}^{\leq 1}((A,B)\cdot\Gamma_b)+r^{-4}\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\Gamma_b).
\end{aligned}
$$

Finally, we use (D.6.1) to substitute the term ${}^{(c)}\mathcal{D}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\big)$, given by

$$
\begin{aligned}
& \frac{1}{4}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\Big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\Big)\\
= \ & \Big({}^{(c)}\nabla_4+\mathrm{tr}X+\frac{1}{2}\overline{\mathrm{tr}X}\Big)\Big({}^{(c)}\nabla_3+2\overline{\mathrm{tr}\underline{X}}+\frac{1}{2}\mathrm{tr}\underline{X}\Big)\underline{A}_4\\
& -\frac{1}{4}\big(H+5\,\underline{H}\big)\widehat{\otimes}\Big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\Big)-3P\underline{A}_4\\
& -3\Big(\frac{1}{2}\overline{\mathrm{tr}X}-\mathrm{tr}X\Big)P\underline{A}-\mathcal{J}_{434}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)+\mathrm{Err}_{434},
\end{aligned}
$$

and we obtain

$$
\begin{aligned}
& \Big({}^{(c)}\nabla_4+2\mathrm{tr}X+\frac{1}{2}\overline{\mathrm{tr}X}\Big)\Big({}^{(c)}\nabla_4+\mathrm{tr}X+\frac{1}{2}\overline{\mathrm{tr}X}\Big)\Big({}^{(c)}\nabla_3+2\overline{\mathrm{tr}\underline{X}}+\frac{1}{2}\mathrm{tr}\underline{X}\Big)\underline{A}_4\\
& +\big(\mathrm{tr}X-\overline{\mathrm{tr}X}-2\frac{{}^{(a)}\mathrm{tr}\chi^2}{tr\,\chi}\big)\Big({}^{(c)}\nabla_4+\mathrm{tr}X+\frac{1}{2}\overline{\mathrm{tr}X}\Big)\Big({}^{(c)}\nabla_3+2\overline{\mathrm{tr}\underline{X}}+\frac{1}{2}\mathrm{tr}\underline{X}\Big)\underline{A}_4\\
& +\big(2\mathrm{tr}X-\overline{\mathrm{tr}X}\big)3P\underline{A}_4\\
= \ & \frac{1}{4}\big({}^{(c)}\mathcal{D}+H+6\,\underline{H}\big)\widehat{\otimes}\Big(\overline{{}^{(c)}\mathcal{D}}\cdot\widetilde{\underline{Q}(\underline{A})}+(\overline{H}+2\,\overline{\underline{H}})\cdot\widetilde{\underline{Q}(\underline{A})}\Big)+3P\,\widetilde{\underline{Q}(\underline{A})}\\
& +\mathscr{L}[\underline{A},\underline{A}_4,{}^{(c)}\mathcal{D}\underline{A}_4,\mathcal{D}\underline{A}]+\mathrm{Err}_{4434},
\end{aligned}
$$

where

$$\begin{aligned}
\mathrm{Err}_{4434} &= {}^{(c)}\nabla_4\mathrm{Err}_{434} + \big(\mathrm{tr}X + \frac{3}{2}\overline{\mathrm{tr}X} + 2\frac{{}^{(a)}\mathrm{tr}\chi^2}{tr\,\chi}\big)\mathrm{Err}_{434}\\
&\quad + r^{-2}\mathfrak{d}^{\leq 1}((A,B)\cdot\Gamma_b) + r^{-4}\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\Gamma_b),
\end{aligned}$$

and where $\mathscr{L}[\underline{A}, \underline{A}_4, {}^{(c)}\mathcal{D}\underline{A}_4, \mathcal{D}\underline{A}]$ are $O(|a|)$ linear order terms in $\underline{A}, \underline{A}_4, {}^{(c)}\mathcal{D}\underline{A}_4, \mathcal{D}\underline{A}$ explicitly given by

$$\begin{aligned}
&\mathscr{L}[\underline{A}, \underline{A}_4, {}^{(c)}\mathcal{D}\underline{A}_4, \mathcal{D}\underline{A}]\\
&= \big(\frac{1}{4}\mathcal{J} + \mathcal{K} + \mathcal{J}_{434} + \frac{1}{4}\big(\mathrm{tr}X - \overline{\mathrm{tr}X} - 2\frac{{}^{(a)}\mathrm{tr}\chi^2}{tr\,\chi}\big)\big(H + 5\,\underline{H}\big)\big)\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4 + (\overline{H} + \overline{\underline{H}})\cdot\underline{A}_4\big)\\
&+ \Big({}^{(c)}\nabla_4\big(\mathcal{J}_{434}\big) + \frac{3}{2}\mathrm{tr}X\mathcal{J}_{434} + \big(\mathrm{tr}X - \overline{\mathrm{tr}X} - 2\frac{{}^{(a)}\mathrm{tr}\chi^2}{tr\,\chi}\big)\mathcal{J}_{434}\Big)\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A} + \overline{H}\cdot\underline{A}\big) \qquad \text{(D.6.11)}\\
&+ \Big(\big(\mathrm{tr}X - \overline{\mathrm{tr}X} - 2\frac{{}^{(a)}\mathrm{tr}\chi^2}{tr\,\chi}\big)\big(\frac{1}{2}\overline{\mathrm{tr}X} - \mathrm{tr}X\big) + \frac{1}{2}\mathrm{tr}X\big(\mathrm{tr}X - \overline{\mathrm{tr}X}\big)\Big)3P\underline{A}\\
&+ \frac{1}{2}\overline{\mathcal{I}}\cdot {}^{(c)}\mathcal{D}\underline{A}_4 + \frac{1}{2}\big({}^{(c)}\mathcal{D}\cdot\overline{\mathcal{I}}\big)\underline{A}_4 + \frac{1}{4}\big(H + 6\,\underline{H}\big)\widehat{\otimes}\big(\overline{\mathcal{I}}\cdot\underline{A}_4\big).
\end{aligned}$$

Step 7. Analysis of lower order terms. We now analyze each term in $\mathscr{L}[\underline{A}, \underline{A}_4, {}^{(c)}\mathcal{D}\underline{A}_4, \mathcal{D}\underline{A}]$.

The term in $\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4 + (\overline{H} + \overline{\underline{H}})\cdot\underline{A}_4\big)$ is given by, using (D.6.9), (D.6.10) and (D.6.3),

$$\begin{aligned}
&\frac{1}{4}\mathcal{J} + \mathcal{K} + \mathcal{J}_{434} + \frac{1}{4}\big(\mathrm{tr}X - \overline{\mathrm{tr}X} - 2\frac{{}^{(a)}\mathrm{tr}\chi^2}{tr\,\chi}\big)\big(H + 5\,\underline{H}\big)\\
&= \frac{1}{4}\Big(i\Im(\mathrm{tr}X)(H - \check{H}) + 2\,{}^{(c)}\mathcal{D}\big(\frac{{}^{(a)}\mathrm{tr}\chi^2}{tr\,\chi}\big) + \big(\frac{3}{2}\mathrm{tr}X + \frac{1}{2}\overline{\mathrm{tr}X} + 2\frac{{}^{(a)}\mathrm{tr}\chi^2}{tr\,\chi}\big)\,\underline{H}\Big)\\
&\quad -\frac{1}{4}\Big[\frac{1}{2}\big(\mathrm{tr}X - \overline{\mathrm{tr}X} - 4\frac{{}^{(a)}\mathrm{tr}\chi^2}{tr\,\chi}\big)H + \big(\frac{15}{2}\mathrm{tr}X - \frac{11}{2}\overline{\mathrm{tr}X} - 10\frac{{}^{(a)}\mathrm{tr}\chi^2}{tr\,\chi}\big)\,\underline{H}\Big]\\
&\quad -\frac{3}{4}\mathrm{tr}X\,\underline{H} + \frac{1}{4}\big(\mathrm{tr}X - \overline{\mathrm{tr}X} - 2\frac{{}^{(a)}\mathrm{tr}\chi^2}{tr\,\chi}\big)\big(H + 5\,\underline{H}\big) + r^{-1}\Gamma_g\\
&= \frac{1}{4}\Big[2\,{}^{(c)}\mathcal{D}\big(\frac{{}^{(a)}\mathrm{tr}\chi^2}{tr\,\chi}\big) - 3\mathrm{tr}X\,\underline{H} - \big(\mathrm{tr}X - \overline{\mathrm{tr}X} - 2\frac{{}^{(a)}\mathrm{tr}\chi^2}{tr\,\chi}\big)\,\underline{H} + \big(\mathrm{tr}X - \overline{\mathrm{tr}X}\big)H\Big]\\
&\quad + \Im(\mathrm{tr}X)\check{H} + r^{-1}\Gamma_g.
\end{aligned}$$

In particular, this is given by

$$\begin{aligned}
&\big(\frac{1}{4}\mathcal{J} + \mathcal{K} + \mathcal{J}_{434} + \frac{1}{4}\big(\mathrm{tr}X - \overline{\mathrm{tr}X} - 2\frac{{}^{(a)}\mathrm{tr}\chi^2}{tr\,\chi}\big)\big(H + 5\,\underline{H}\big)\big)\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4 + (\overline{H} + \overline{\underline{H}})\cdot\underline{A}_4\big)\\
&= O(ar^{-3})\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4 + (\overline{H} + \overline{\underline{H}})\cdot\underline{A}_4\big) + r^{-4}(\Gamma_g\cdot\Gamma_b).
\end{aligned}$$

To compute the term in $\left(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\right)$, we first compute using (D.6.2),

$$\begin{aligned}
{}^{(c)}\nabla_4\big(\mathcal{J}_{434}\big) &= \frac{1}{4}\Big[-\frac{1}{2}{}^{(c)}\nabla_4{}^{(c)}\mathcal{D}(\mathrm{tr}X+\overline{\mathrm{tr}X})+\frac{1}{2}(\mathrm{tr}X-\overline{\mathrm{tr}X}){}^{(c)}\nabla_4 H\\
&+\frac{1}{2}{}^{(c)}\nabla_4(\mathrm{tr}X-\overline{\mathrm{tr}X})H-4\mathrm{tr}X\,{}^{(c)}\nabla_4\underline{H}-4\,{}^{(c)}\nabla_4\mathrm{tr}X\,\underline{H}\Big]\\
&-\frac{1}{4}{}^{(c)}\nabla_4 B+r^{-3}\Gamma_g+{}^{(c)}\nabla_4(\widehat{X}\cdot\check{H})\\
&= \frac{1}{4}\Big[-\frac{1}{2}{}^{(c)}\nabla_4{}^{(c)}\mathcal{D}(\mathrm{tr}X+\overline{\mathrm{tr}X})-\frac{1}{4}\overline{\mathrm{tr}X}(\mathrm{tr}X-\overline{\mathrm{tr}X})(H-\underline{H})\\
&+\frac{1}{2}{}^{(c)}\nabla_4(\mathrm{tr}X-\overline{\mathrm{tr}X})H+4(\mathrm{tr}X)^2\,\underline{H}-4\,{}^{(c)}\nabla_4\mathrm{tr}X\,\underline{H}\Big]\\
&-\frac{1}{4}{}^{(c)}\nabla_4 B+r^{-3}\Gamma_g+{}^{(c)}\nabla_4(\widehat{X}\cdot\check{H}).
\end{aligned}$$

Using the commutator for a scalar function h of signature s

$$\begin{aligned}
[{}^{(c)}\nabla_4,{}^{(c)}\mathcal{D}]h &= -\frac{1}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}h+\underline{H}\,{}^{(c)}\nabla_4 h-\frac{1}{2}\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}h\\
&+s\left(\frac{1}{2}\mathrm{tr}X\,\underline{H}+\frac{1}{2}\widehat{X}\cdot\overline{\underline{H}}-B\right)h+(\Gamma_b\cdot\Gamma_g)h,
\end{aligned}$$

and making use of the equation

$$ {}^{(c)}\nabla_4\mathrm{tr}X = -\frac{1}{2}(\mathrm{tr}X)^2+\Gamma_g\cdot\Gamma_g, $$

we have

$$\begin{aligned}
{}^{(c)}\nabla_4\big(\mathcal{J}_{434}\big) &= \frac{1}{4}\Big[-\frac{1}{2}{}^{(c)}\mathcal{D}({}^{(c)}\nabla_4\mathrm{tr}X+{}^{(c)}\nabla_4\overline{\mathrm{tr}X})\\
&-\frac{1}{2}\big[-\frac{1}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}(\mathrm{tr}X+\overline{\mathrm{tr}X})+\underline{H}\,{}^{(c)}\nabla_4(\mathrm{tr}X+\overline{\mathrm{tr}X})\\
&-\frac{1}{2}\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}(\mathrm{tr}X+\overline{\mathrm{tr}X})+(\mathrm{tr}X+\overline{\mathrm{tr}X})\big(\frac{1}{2}\mathrm{tr}X\,\underline{H}-B\big)\big]\\
&-\frac{1}{4}\overline{\mathrm{tr}X}(\mathrm{tr}X-\overline{\mathrm{tr}X})(H-\underline{H})\\
&+\frac{1}{2}(-\frac{1}{2}(\mathrm{tr}X)^2+\frac{1}{2}(\overline{\mathrm{tr}X})^2)H+4(\mathrm{tr}X)^2\,\underline{H}-4(-\frac{1}{2}(\mathrm{tr}X)^2)\,\underline{H}\Big]\\
&-\frac{1}{4}{}^{(c)}\nabla_4 B+r^{-3}\Gamma_g+{}^{(c)}\nabla_4(\widehat{X}\cdot\check{H}).
\end{aligned}$$

which gives

$$\begin{aligned}{}^{(c)}\nabla_4\big(\mathcal{J}_{434}\big) &= \frac{1}{4}\Big[\frac{1}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}(\mathrm{tr}X)+\frac{1}{2}\overline{\mathrm{tr}X}\,{}^{(c)}\mathcal{D}(\overline{\mathrm{tr}X})+\frac{1}{4}\mathrm{tr}X\,{}^{(c)}\mathcal{D}(\mathrm{tr}X+\overline{\mathrm{tr}X})\\ &+\big(-\frac{1}{4}(\mathrm{tr}X)^2-\frac{1}{4}\overline{\mathrm{tr}X}\mathrm{tr}X+\frac{1}{2}(\overline{\mathrm{tr}X})^2\big)H+6(\mathrm{tr}X)^2\,\underline{H}\Big]\\ &-\frac{1}{4}\,{}^{(c)}\nabla_4 B+\frac{1}{8}(\mathrm{tr}X+\overline{\mathrm{tr}X})B+\frac{1}{16}\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}(\mathrm{tr}X+\overline{\mathrm{tr}X})\\ &+r^{-3}\Gamma_g+\,{}^{(c)}\nabla_4(\widehat{X}\cdot\check{H}).\end{aligned}$$

We therefore obtain for the term in $\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)$,

$$\begin{aligned}&{}^{(c)}\nabla_4\big(\mathcal{J}_{434}\big)+\frac{3}{2}\mathrm{tr}X\mathcal{J}_{434}+\big(\mathrm{tr}X-\overline{\mathrm{tr}X}-2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}\big)\mathcal{J}_{434}\\ =\;&\frac{1}{4}\Big[\frac{1}{2}(\overline{\mathrm{tr}X}-\mathrm{tr}X)\,{}^{(c)}\mathcal{D}(\overline{\mathrm{tr}X})+\big(\frac{1}{2}(\mathrm{tr}X)^2-\overline{\mathrm{tr}X}\mathrm{tr}X+\frac{1}{2}(\overline{\mathrm{tr}X})^2\big)H\Big]\\ &+\big(\mathrm{tr}X-\overline{\mathrm{tr}X}-2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}\big)\mathcal{J}_{434}+r^{-1}\mathfrak{d}^{\leq 1}B+r^{-3}\Gamma_g+\,{}^{(c)}\nabla_4(\widehat{X}\cdot\check{H})+\frac{3}{2}\mathrm{tr}X(\widehat{X}\cdot\check{H}).\end{aligned}$$

Observe that $\frac{1}{2}(\mathrm{tr}X)^2-\overline{\mathrm{tr}X}\mathrm{tr}X+\frac{1}{2}(\overline{\mathrm{tr}X})^2=\frac{1}{2}(\mathrm{tr}X-\overline{\mathrm{tr}X})^2=-2i(\,{}^{(a)}\mathrm{tr}\chi)^2=O(r^{-4})$, and therefore

$$\begin{aligned}&{}^{(c)}\nabla_4\big(\mathcal{J}_{434}\big)+\frac{3}{2}\mathrm{tr}X\mathcal{J}_{434}+\big(\mathrm{tr}X-\overline{\mathrm{tr}X}-2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}\big)\mathcal{J}_{434}\\ =\;&O(a^2r^{-5})+r^{-1}\mathfrak{d}^{\leq 1}B+r^{-3}\Gamma_g+\,{}^{(c)}\nabla_4(\widehat{X}\cdot\check{H})+\frac{3}{2}\mathrm{tr}X(\widehat{X}\cdot\check{H}).\end{aligned}$$

The term in $\underline{A}$ is given by

$$\begin{aligned}&\Big(\big(\mathrm{tr}X-\overline{\mathrm{tr}X}-2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}\big)\big(\frac{1}{2}\overline{\mathrm{tr}X}-\mathrm{tr}X\big)+\frac{1}{2}\mathrm{tr}X\big(\mathrm{tr}X-\overline{\mathrm{tr}X}\big)\Big)3P\underline{A}\\ =\;&\Big(-\frac{1}{2}\big(\mathrm{tr}X-\overline{\mathrm{tr}X}\big)^2-2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}\big(\frac{1}{2}\overline{\mathrm{tr}X}-\mathrm{tr}X\big)\Big)3P\underline{A}\\ =\;&\Big(2\,{}^{(a)}\mathrm{tr}\chi^2-2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}\big(-\frac{1}{2}\mathrm{tr}\,\chi+\frac{3}{2}i\,{}^{(a)}\mathrm{tr}\chi\big)\Big)3P\underline{A}\\ =\;&3\,{}^{(a)}\mathrm{tr}\chi^2\Big(1-\frac{i\,{}^{(a)}\mathrm{tr}\chi}{\mathrm{tr}\,\chi}\Big)3P\underline{A}=O(a^2r^{-7})\underline{A}+r^{-4}(\Gamma_g\cdot\Gamma_b).\end{aligned}$$

Finally, using (D.6.8) and putting together all the above we obtain from (D.6.11),

$$\begin{aligned}&\mathscr{L}[\underline{A},\underline{A}_4,\,{}^{(c)}\mathcal{D}\underline{A}_4,\mathcal{D}\underline{A}]\\ =\;&O(ar^{-3})\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}_4+(\overline{H}+\overline{\underline{H}})\cdot\underline{A}_4\big)+O(a^2r^{-4})\big(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A}+\overline{H}\cdot\underline{A}\big)+O(a^2r^{-7})\underline{A}\\ &+\big(r^{-2}\mathfrak{d}^{\leq 1}B,r^{-1}(\,{}^{(c)}\nabla_4(\widehat{X}\cdot\check{H})+\frac{3}{2}\mathrm{tr}X(\widehat{X}\cdot\check{H}))\big)\mathfrak{d}^{\leq 1}\underline{A}+r^{-4}(\Gamma_g\cdot\Gamma_b).\end{aligned}$$

This completes the proof of Proposition D.6.2. □

We finally show that the coefficient $\underline{W}_3 = O(a^2r^{-5})$. The coefficient comes from the symmetric of W_4 as in Theorem 5.2.9, whose lowest decaying term is given by Z_4 in Proposition D.4.1, i.e.

$$\underline{Z}_3 = {}^{(c)}\nabla_4\widetilde{\underline{C}_2} + 2\mathrm{tr}\,\chi\widetilde{\underline{C}_2} - \frac{1}{4}(\mathrm{tr}\,\chi^2 + {}^{(a)}\mathrm{tr}\chi^2)\widetilde{\underline{C}_1}.$$

Using the definition of $\underline{C}_1$ and $\underline{C}_2$ given by (5.3.3), we have

$$\begin{aligned}
\underline{Z}_3 &= {}^{(c)}\nabla_4(-2i\mathrm{tr}\,\chi\,{}^{(a)}\mathrm{tr}\chi) + 2\mathrm{tr}\,\chi(-2i\mathrm{tr}\,\chi\,{}^{(a)}\mathrm{tr}\chi) - \frac{1}{4}(\mathrm{tr}\,\chi^2 + {}^{(a)}\mathrm{tr}\chi^2)(-4i\,{}^{(a)}\mathrm{tr}\chi)\\
&\quad +O(a^2r^{-5})\\
&= -2i\left(-\frac{1}{2}\mathrm{tr}\,\chi^2\,{}^{(a)}\mathrm{tr}\chi + \mathrm{tr}\,\chi(-\mathrm{tr}\,\chi\,{}^{(a)}\mathrm{tr}\chi)\right) + 2\mathrm{tr}\,\chi(-2i\mathrm{tr}\,\chi\,{}^{(a)}\mathrm{tr}\chi)\\
&\quad +i\mathrm{tr}\,\chi^2\,{}^{(a)}\mathrm{tr}\chi + O(a^2r^{-5})\\
&= O(a^2r^{-5}),
\end{aligned}$$

as stated.

D.6.1 Treatment of the nonlinear terms

The nonlinear terms we have encountered so far are as follows.

1. Nonlinear terms in the Teukolsky equation, see Proposition 5.3.1, schematically given by

$$\begin{aligned}
\mathrm{Err}_{TE} &= \mathrm{tr}X\underline{\Xi}\widehat{\otimes}\underline{B} + (\overline{{}^{(c)}\mathcal{D}}\cdot\underline{B})\widehat{\underline{X}} + (\widehat{\underline{X}}\cdot\overline{\widecheck{H}})\underline{B}\\
&\quad +(\Gamma_b\cdot\Gamma_b)\cdot(A,B) + r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b).
\end{aligned}$$

2. Nonlinear terms after the first commutation, see Proposition D.6.1, written schematically in the form

$$\begin{aligned}
\mathrm{Err}_{434} &= {}^{(c)}\nabla_4\mathrm{Err}_{TE} + \Big(\frac{1}{2}\mathrm{tr}X + \overline{\mathrm{tr}X}\Big)\mathrm{Err}_{TE}\\
&\quad + {}^{(c)}\mathcal{D}\widehat{\otimes}(\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}\underline{A}) + {}^{(c)}\mathcal{D}\widehat{\otimes}((\,{}^{(c)}\mathcal{D}\cdot\overline{\widehat{X}})\cdot\underline{A})\\
&\quad +r^{-1}\mathfrak{d}^{\leq 1}((A,B)\cdot\Gamma_b) + r^{-3}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b).
\end{aligned}$$

3. Nonlinear terms after the second commutation, see Proposition D.6.2,

$$\begin{aligned}
\mathrm{Err}_{4434} &= {}^{(c)}\nabla_4 \mathrm{Err}_{434} + \big(\mathrm{tr}X + \frac{3}{2}\overline{\mathrm{tr}X} + 2\frac{{}^{(a)}\mathrm{tr}\chi^2}{\mathrm{tr}\,\chi}\big)\mathrm{Err}_{434}\\
&\quad + r^{-2}\mathfrak{d}^{\leq 1}((A,B)\cdot\Gamma_b) + r^{-4}\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\Gamma_b)\\
&\quad + r^{-1}\big({}^{(c)}\nabla_4(\widehat{X}\cdot\check{H}) + \frac{3}{2}\mathrm{tr}X(\widehat{X}\cdot\check{H})\big)\cdot\mathfrak{d}^{\leq 1}\underline{A}.
\end{aligned}$$

We can therefore separate the final error terms, Err_{4434}, as follows.

$$\begin{aligned}
\mathrm{Err}_{4434} &= \mathrm{Err}_1 + \mathrm{Err}_2 + r^{-2}\mathfrak{d}^{\leq 2}((A,B)\cdot\Gamma_b) + r^{-4}\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\Gamma_b)\\
&\quad + r^{-1}\left({}^{(c)}\nabla_4(\widehat{X}\cdot\check{H}) + \frac{3}{2}\mathrm{tr}X(\widehat{X}\cdot\check{H})\right)\cdot\mathfrak{d}^{\leq 1}\underline{A}.
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{Err}_1 &= {}^{(c)}\nabla_4\big({}^{(c)}\nabla_4\mathrm{Err}_{TE} + \big(\frac{1}{2}\mathrm{tr}X + \overline{\mathrm{tr}X}\big)\mathrm{Err}_{TE}\big)\\
&\quad + \big(\mathrm{tr}X + \frac{3}{2}\overline{\mathrm{tr}X}\big)\big({}^{(c)}\nabla_4\mathrm{Err}_{TE} + \big(\frac{1}{2}\mathrm{tr}X + \overline{\mathrm{tr}X}\big)\mathrm{Err}_{TE}\big),\\
\mathrm{Err}_2 &= {}^{(c)}\nabla_4\big({}^{(c)}\mathcal{D}\widehat{\otimes}(\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}\underline{A}}) + {}^{(c)}\mathcal{D}\widehat{\otimes}(({}^{(c)}\mathcal{D}\cdot\overline{\widehat{X}})\cdot\underline{A})\big)\\
&\quad + \big(\mathrm{tr}X + \frac{3}{2}\overline{\mathrm{tr}X}\big)\big({}^{(c)}\mathcal{D}\widehat{\otimes}(\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}\underline{A}}) + {}^{(c)}\mathcal{D}\widehat{\otimes}(({}^{(c)}\mathcal{D}\cdot\overline{\widehat{X}})\cdot\underline{A})\big).
\end{aligned}$$

We have the following for Err_{4434}.

Proposition D.6.3. *The following holds true:*

$$Err_{4434} = r^{-2}\mathfrak{d}^{\leq 2}((A,B)\cdot\Gamma_b) + r^{-4}\mathfrak{d}^{\leq 3}(\Gamma_g\cdot\Gamma_b).$$

Proof. We start with Err_1, which contains the terms coming from Err_{TE}. We consider each term in Err_{TE} separately.

Step 1. We consider the term $\mathrm{tr}X\underline{\Xi}\widehat{\otimes}\underline{B}$. We first compute,

$$\begin{aligned}
I_1 &:= {}^{(c)}\nabla_4(\mathrm{tr}X\underline{\Xi}\widehat{\otimes}\underline{B}) + \big(\frac{1}{2}\mathrm{tr}X + \overline{\mathrm{tr}X}\big)(\mathrm{tr}X\underline{\Xi}\widehat{\otimes}\underline{B})\\
&= \big({}^{(c)}\nabla_4(\mathrm{tr}X) + \frac{1}{2}(\mathrm{tr}X)^2\big)\underline{\Xi}\widehat{\otimes}\underline{B} + \mathrm{tr}X\,{}^{(c)}\nabla_4\underline{\Xi}\widehat{\otimes}\underline{B} + \mathrm{tr}X\underline{\Xi}\widehat{\otimes}\big({}^{(c)}\nabla_4\underline{B} + \mathrm{tr}X\underline{B}\big)\\
&\quad + \big(\overline{\mathrm{tr}X} - \mathrm{tr}X\big)(\mathrm{tr}X\underline{\Xi}\widehat{\otimes}\underline{B})\\
&= \mathrm{tr}X\big(-\frac{1}{2}\overline{\mathrm{tr}X}\check{H} - \underline{B} - \frac{1}{2}\widehat{\underline{X}}\cdot\check{H}\big)\widehat{\otimes}\underline{B} + r^{-3}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\Gamma_b),
\end{aligned}$$

where we used Proposition D.5.3 and Proposition D.5.5 to write

$$\begin{aligned}{}^{(c)}\nabla_4\underline{\Xi} &= -\frac{1}{2}\overline{\mathrm{tr}X}\check{H} - \underline{B} - \frac{1}{2}\widehat{\underline{X}}\cdot\check{H} + r^{-2}\Gamma_b\\ {}^{(c)}\nabla_4\underline{B} + \mathrm{tr}X\underline{B} &= r^{-2}\Gamma_g.\end{aligned}$$

We write schematically in the form

$$I_1 = (\mathrm{tr}X)^2\check{H}\widehat{\otimes}\underline{B} + \mathrm{tr}X(\underline{B}\widehat{\otimes}\underline{B}) + \mathrm{tr}X(\widehat{\underline{X}}\cdot\check{H})\widehat{\otimes}\underline{B} + r^{-3}\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\Gamma_b).$$

We then obtain that

$$I_2 := {}^{(c)}\nabla_4 I_1 + \big(\mathrm{tr}X + \frac{3}{2}\overline{\mathrm{tr}X}\big)I_1 + r^{-4}\mathfrak{d}^{\leq 3}(\Gamma_g\cdot\Gamma_b),$$

is given as the sum of the following terms

$$\begin{aligned}S_1 &= {}^{(c)}\nabla_4\big((\mathrm{tr}X)^2\check{H}\widehat{\otimes}\underline{B}\big) + \big(\mathrm{tr}X + \frac{3}{2}\overline{\mathrm{tr}X}\big)\big((\mathrm{tr}X)^2\check{H}\widehat{\otimes}\underline{B}\big)\\ S_2 &= {}^{(c)}\nabla_4\big(\mathrm{tr}X(\underline{B}\widehat{\otimes}\underline{B})\big) + \big(\mathrm{tr}X + \frac{3}{2}\overline{\mathrm{tr}X}\big)\big(\mathrm{tr}X(\underline{B}\widehat{\otimes}\underline{B})\big)\\ S_3 &= {}^{(c)}\nabla_4\big(\mathrm{tr}X(\widehat{\underline{X}}\cdot\check{H})\widehat{\otimes}\underline{B}\big) + \big(\mathrm{tr}X + \frac{3}{2}\overline{\mathrm{tr}X}\big)\big(\mathrm{tr}X(\widehat{\underline{X}}\cdot\check{H})\widehat{\otimes}\underline{B}\big).\end{aligned}$$

We now analyze each of the above.

Step 1a. We have

$$\begin{aligned}S_1 &= \big({}^{(c)}\nabla_4(\mathrm{tr}X)^2 + (\mathrm{tr}X)^3\big)\check{H}\widehat{\otimes}\underline{B} + (\mathrm{tr}X)^2\big({}^{(c)}\nabla_4\check{H} + \frac{1}{2}\overline{\mathrm{tr}X}\check{H}\big)\widehat{\otimes}\underline{B}\\ &+(\mathrm{tr}X)^2\check{H}\widehat{\otimes}\big({}^{(c)}\nabla_4\underline{B} + \mathrm{tr}X\underline{B}\big) + r^{-4}\mathfrak{d}^{\leq 3}(\Gamma_g\cdot\Gamma_b).\end{aligned}$$

Recall that H verifies the equation, see (D.5.3),

$${}^{(c)}\nabla_4\check{H} + \frac{1}{2}\overline{\mathrm{tr}X}\check{H} = -\frac{1}{2}\widehat{X}\cdot\overline{\check{H}} - B + r^{-2}\Gamma_g,$$

and using again Proposition D.5.5 and the null structure equation for ${}^{(c)}\nabla_4\mathrm{tr}X$ we obtain

$$S_1 = r^{-4}\mathfrak{d}^{\leq 3}(\Gamma_g\cdot\Gamma_b).$$

Step 1b. We have

$$\begin{aligned}S_2 &= {}^{(c)}\nabla_4\big(\mathrm{tr}X(\underline{B}\widehat{\otimes}\underline{B})\big) + \big(\mathrm{tr}X + \frac{3}{2}\overline{\mathrm{tr}X}\big)\big(\mathrm{tr}X(\underline{B}\widehat{\otimes}\underline{B})\big)\\ &= \big({}^{(c)}\nabla_4(\mathrm{tr}X) + \frac{1}{2}(\mathrm{tr}X)^2\big)(\underline{B}\widehat{\otimes}\underline{B}) + 2\mathrm{tr}X\big({}^{(c)}\nabla_4\underline{B} + \mathrm{tr}X\underline{B}\big)\widehat{\otimes}\underline{B} + r^{-4}\mathfrak{d}^{\leq 3}(\Gamma_g\cdot\Gamma_b)\\ &= r^{-4}\mathfrak{d}^{\leq 3}(\Gamma_g\cdot\Gamma_b),\end{aligned}$$

as above.

Step 1c. We have

$$
\begin{aligned}
S_3 &= {}^{(c)}\nabla_4\big(\mathrm{tr}X(\widehat{\underline{X}}\cdot\check{H})\widehat{\otimes}\underline{B}\big)+\big(\mathrm{tr}X+\frac{3}{2}\overline{\mathrm{tr}X}\big)\big(\mathrm{tr}X(\widehat{\underline{X}}\cdot\check{H})\widehat{\otimes}\underline{B}\big)\\
&= \big({}^{(c)}\nabla_4(\mathrm{tr}X)+\frac{1}{2}(\mathrm{tr}X)^2\big)(\widehat{\underline{X}}\cdot\check{H})\widehat{\otimes}\underline{B}+\big(\mathrm{tr}X(({}^{(c)}\nabla_4\widehat{\underline{X}}+\frac{1}{2}\mathrm{tr}X\widehat{\underline{X}})\cdot\check{H})\widehat{\otimes}\underline{B}\big)\\
&\quad+\big(\mathrm{tr}X(\widehat{\underline{X}}\cdot\big({}^{(c)}\nabla_4\check{H}+\frac{1}{2}\overline{\mathrm{tr}X}\check{H}\big))\widehat{\otimes}\underline{B}\big)+\big(\mathrm{tr}X(\widehat{\underline{X}}\cdot\check{H})\widehat{\otimes}({}^{(c)}\nabla_4\underline{B}+2\mathrm{tr}X\underline{B})\big)\\
&\quad+r^{-3}(\Gamma_b\cdot\Gamma_b)\cdot\underline{B}\\
&= r^{-4}\mathfrak{d}^{\leq 3}(\Gamma_g\cdot\Gamma_b),
\end{aligned}
$$

as above. This completes the term involving $\mathrm{tr}X\underline{\Xi}\widehat{\otimes}\underline{B}$, by showing that

$$
I_2=r^{-4}\mathfrak{d}^{\leq 3}(\Gamma_g\cdot\Gamma_b).
$$

Step 2. We consider the term $(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{B})\widehat{\underline{X}}$. We first compute

$$
\begin{aligned}
J_1 &:= {}^{(c)}\nabla_4((\overline{{}^{(c)}\mathcal{D}}\cdot\underline{B})\widehat{\underline{X}})+\big(\frac{1}{2}\mathrm{tr}X+\overline{\mathrm{tr}X}\big)((\overline{{}^{(c)}\mathcal{D}}\cdot\underline{B})\widehat{\underline{X}})\\
&= {}^{(c)}\nabla_4(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{B})\widehat{\underline{X}}+(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{B})\big({}^{(c)}\nabla_4\widehat{\underline{X}}+\frac{1}{2}\mathrm{tr}X\widehat{\underline{X}}\big)+\overline{\mathrm{tr}X}(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{B})\widehat{\underline{X}}\\
&= \overline{{}^{(c)}\mathcal{D}}\cdot({}^{(c)}\nabla_4\underline{B})\widehat{\underline{X}}+([{}^{(c)}\nabla_4,\overline{{}^{(c)}\mathcal{D}}\cdot]\underline{B})\widehat{\underline{X}}+\overline{\mathrm{tr}X}(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{B})\widehat{\underline{X}}\\
&\quad+(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{B})\big({}^{(c)}\nabla_4\widehat{\underline{X}}+\frac{1}{2}\mathrm{tr}X\widehat{\underline{X}}\big).
\end{aligned}
$$

Using the equation

$$
{}^{(c)}\nabla_4\widehat{\underline{X}}+\frac{1}{2}\mathrm{tr}X\,\widehat{\underline{X}} = r^{-1}\Gamma_g,
$$

and the commutator formula (4.2.15), written schematically

$$
[{}^{(c)}\nabla_4,\overline{{}^{(c)}\mathcal{D}}\cdot]\underline{B} = -\frac{1}{2}\overline{\mathrm{tr}X}\,(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{B})+r^{-3}\underline{B}+r^{-1}(\mathfrak{d}^{\leq 1}\Gamma_g)\cdot\underline{B},
$$

we have, using also from Lemma D.5.1, ${}^{(c)}\mathcal{D}\overline{\mathrm{tr}X}=a^2r^{-3}+r^{-2}\Gamma_b+r^{-1}\mathfrak{d}^{\leq 1}\Gamma_g$,

$$
\begin{aligned}
J_1 &= \overline{{}^{(c)}\mathcal{D}}\cdot({}^{(c)}\nabla_4\underline{B})\widehat{\underline{X}}+\frac{1}{2}\overline{\mathrm{tr}X}(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{B})\widehat{\underline{X}}+r^{-3}(\Gamma_g\cdot\Gamma_b)\\
&= \overline{{}^{(c)}\mathcal{D}}\cdot({}^{(c)}\nabla_4\underline{B}+\mathrm{tr}X\underline{B})\widehat{\underline{X}}-\frac{1}{2}\overline{\mathrm{tr}X}(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{B})\widehat{\underline{X}}+r^{-3}(\Gamma_g\cdot\Gamma_b)\\
&= -\frac{1}{2}\overline{\mathrm{tr}X}(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{B})\widehat{\underline{X}}+r^{-3}\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\Gamma_b).
\end{aligned}
$$

We write schematically in the form

$$J_1 = \overline{\mathrm{tr}X}(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{B})\widehat{\underline{X}} + r^{-3}\mathfrak{d}^{\leq 2}(\Gamma_g\cdot\Gamma_b),$$

and we then obtain that

$$J_2 = {}^{(c)}\nabla_4 J_1 + \big(\mathrm{tr}X + \frac{3}{2}\overline{\mathrm{tr}X}\big)J_1 + r^{-4}\mathfrak{d}^{\leq 3}(\Gamma_g\cdot\Gamma_b).$$

Using the equations stated above, we obtain

$$\begin{aligned}
J_2 &= {}^{(c)}\nabla_4(\overline{\mathrm{tr}X}(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{B})\widehat{\underline{X}}) + \big(\mathrm{tr}X + \frac{3}{2}\overline{\mathrm{tr}X}\big)\overline{\mathrm{tr}X}(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{B})\widehat{\underline{X}} + r^{-4}\mathfrak{d}^{\leq 3}(\Gamma_g\cdot\Gamma_b)\\
&= \big({}^{(c)}\nabla_4(\overline{\mathrm{tr}X}) + \frac{1}{2}(\overline{\mathrm{tr}X})^2\big)(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{B})\widehat{\underline{X}} + \overline{\mathrm{tr}X}\big({}^{(c)}\nabla_4(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{B}) + \frac{3}{2}\overline{\mathrm{tr}X}\,\overline{{}^{(c)}\mathcal{D}}\cdot\underline{B}\big)\widehat{\underline{X}}\\
&\quad + \overline{\mathrm{tr}X}(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{B})\big({}^{(c)}\nabla_4(\widehat{\underline{X}}) + \frac{1}{2}\mathrm{tr}X\widehat{\underline{X}}\big) + r^{-4}\mathfrak{d}^{\leq 3}(\Gamma_g\cdot\Gamma_b)\\
&= r^{-4}\mathfrak{d}^{\leq 3}(\Gamma_g\cdot\Gamma_b),
\end{aligned}$$

which completes the term involving $(\overline{{}^{(c)}\mathcal{D}}\cdot\underline{B})\widehat{\underline{X}}$.

Step 3. We consider the term $(\widehat{\underline{X}}\cdot\overline{\check{H}})\underline{B}$. Following a similar pattern as above we obtain

$$\begin{aligned}
K_1 &:= {}^{(c)}\nabla_4((\widehat{\underline{X}}\cdot\overline{\check{H}})\underline{B}) + \big(\frac{1}{2}\mathrm{tr}X + \overline{\mathrm{tr}X}\big)((\widehat{\underline{X}}\cdot\overline{\check{H}})\underline{B})\\
&= (({}^{(c)}\nabla_4\widehat{\underline{X}} + \frac{1}{2}\mathrm{tr}X\widehat{\underline{X}})\cdot\overline{\check{H}})\underline{B} + (\widehat{\underline{X}}\cdot({}^{(c)}\nabla_4\overline{\check{H}} + \frac{1}{2}\mathrm{tr}X\overline{\check{H}}))\underline{B}\\
&\quad + (\widehat{\underline{X}}\cdot\overline{\check{H}})({}^{(c)}\nabla_4\underline{B} + \frac{1}{2}\mathrm{tr}X\underline{B}) - \frac{1}{2}\overline{\mathrm{tr}X}(\widehat{\underline{X}}\cdot\overline{\check{H}})\underline{B} + r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g)\cdot\underline{B}\\
&= -\frac{1}{2}\overline{\mathrm{tr}X}(\widehat{\underline{X}}\cdot\overline{\check{H}})\underline{B} + r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_g)\cdot\underline{B}.
\end{aligned}$$

By taking the second derivative, we finally get

$$\begin{aligned}
J_2 &= {}^{(c)}\nabla_4 K_1 + \big(\mathrm{tr}X + \frac{3}{2}\overline{\mathrm{tr}X}\big)K_1 + r^{-4}\mathfrak{d}^{\leq 3}(\Gamma_g\cdot\Gamma_b)\\
&= {}^{(c)}\nabla_4(\overline{\mathrm{tr}X}(\widehat{\underline{X}}\cdot\overline{\check{H}})\underline{B}) + \big(\mathrm{tr}X + \frac{3}{2}\overline{\mathrm{tr}X}\big)\overline{\mathrm{tr}X}(\widehat{\underline{X}}\cdot\overline{\check{H}})\underline{B} + r^{-4}\mathfrak{d}^{\leq 3}(\Gamma_g\cdot\Gamma_b)\\
&= \big({}^{(c)}\nabla_4(\overline{\mathrm{tr}X}) + \frac{1}{2}(\overline{\mathrm{tr}X})^2\big)(\widehat{\underline{X}}\cdot\overline{\check{H}})\underline{B} + \overline{\mathrm{tr}X}(({}^{(c)}\nabla_4\widehat{\underline{X}} + \frac{1}{2}\mathrm{tr}X\widehat{\underline{X}})\cdot\overline{\check{H}})\underline{B}\\
&\quad + \overline{\mathrm{tr}X}(\widehat{\underline{X}}\cdot({}^{(c)}\nabla_4\overline{\check{H}} + \frac{1}{2}\mathrm{tr}X\overline{\check{H}}))\underline{B} + \overline{\mathrm{tr}X}(\widehat{\underline{X}}\cdot\overline{\check{H}})({}^{(c)}\nabla_4\underline{B} + \mathrm{tr}X\underline{B}) + r^{-4}\mathfrak{d}^{\leq 3}(\Gamma_g\cdot\Gamma_b)\\
&= r^{-4}\mathfrak{d}^{\leq 3}(\Gamma_g\cdot\Gamma_b).
\end{aligned}$$

Step 4. Combining the three steps above with the remaining terms in Err_{TE}, we finally obtain

$$\mathrm{Err}_1 = r^{-4}\mathfrak{d}^{\leq 3}(\Gamma_g\cdot\Gamma_b)+r^{-3}\mathfrak{d}^{\leq 2}\Gamma_b\cdot(A,B).$$

Step 5. We now prove the bound for Err_2. We first treat the term ${}^{(c)}\mathcal{D}\widehat{\otimes}(\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}\underline{A})$, while the other term can be treated in a similar manner. We have

$$\begin{aligned}
L &:= {}^{(c)}\nabla_4\big({}^{(c)}\mathcal{D}\widehat{\otimes}(\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}\underline{A})\big)+\big(\mathrm{tr}X+\frac{3}{2}\overline{\mathrm{tr}X}\big)\big({}^{(c)}\mathcal{D}\widehat{\otimes}(\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}\underline{A})\big)\\
&= {}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\nabla_4(\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}\underline{A})+[\,{}^{(c)}\nabla_4,\,{}^{(c)}\mathcal{D}\widehat{\otimes}](\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}\underline{A})\\
&\quad+\big(\mathrm{tr}X+\frac{3}{2}\overline{\mathrm{tr}X}\big)\big({}^{(c)}\mathcal{D}\widehat{\otimes}(\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}\underline{A})\big)\\
&= {}^{(c)}\mathcal{D}\widehat{\otimes}({}^{(c)}\nabla_4\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}\underline{A})+{}^{(c)}\mathcal{D}\widehat{\otimes}(\widehat{X}\cdot{}^{(c)}\nabla_4\overline{{}^{(c)}\mathcal{D}}\underline{A})-\frac{1}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}\underline{A})\\
&\quad+\big(\mathrm{tr}X+\frac{3}{2}\overline{\mathrm{tr}X}\big)\big({}^{(c)}\mathcal{D}\widehat{\otimes}(\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}\underline{A})\big)+r^{-4}(\Gamma_g\cdot\Gamma_b).
\end{aligned}$$

Using the null structure equation

$${}^{(c)}\nabla_4\widehat{X}+\Re(\mathrm{tr}X)\widehat{X} = -A,$$

we then obtain

$$\begin{aligned}
L &= {}^{(c)}\mathcal{D}\widehat{\otimes}\big(({}^{(c)}\nabla_4\widehat{X}+\Re(\mathrm{tr}X)\widehat{X})\cdot\overline{{}^{(c)}\mathcal{D}}\underline{A}\big)+{}^{(c)}\mathcal{D}\widehat{\otimes}(\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}\,{}^{(c)}\nabla_4\underline{A})\\
&\quad+{}^{(c)}\mathcal{D}\widehat{\otimes}(\widehat{X}\cdot[\,{}^{(c)}\nabla_4,\overline{{}^{(c)}\mathcal{D}}]\underline{A})+\big(\overline{\mathrm{tr}X}\big)\big({}^{(c)}\mathcal{D}\widehat{\otimes}(\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}\underline{A})\big)+r^{-4}(\Gamma_g\cdot\Gamma_b)\\
&= -\,{}^{(c)}\mathcal{D}\widehat{\otimes}(A\cdot\overline{{}^{(c)}\mathcal{D}}\underline{A})+{}^{(c)}\mathcal{D}\widehat{\otimes}(\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}\,{}^{(c)}\nabla_4\underline{A})\\
&\quad-\frac{1}{2}\overline{\mathrm{tr}X}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\widehat{X}\cdot{}^{(c)}\mathcal{D}\underline{A})+\big(\overline{\mathrm{tr}X}\big)\big({}^{(c)}\mathcal{D}\widehat{\otimes}(\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}\underline{A})\big)+r^{-4}(\Gamma_g\cdot\Gamma_b)\\
&= r^{-2}\mathfrak{d}^{\leq 2}(A\cdot\Gamma_b)+{}^{(c)}\mathcal{D}\widehat{\otimes}(\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}\underline{A}_4)+r^{-3}(\Gamma_g\cdot\Gamma_b)\\
&= r^{-4}\mathfrak{d}^{\leq 3}(\Gamma_g\cdot\Gamma_b)+r^{-2}\mathfrak{d}^{\leq 2}(A\cdot\Gamma_b).
\end{aligned}$$

This shows

$$\mathrm{Err}_2 = r^{-4}\mathfrak{d}^{\leq 3}(\Gamma_g\cdot\Gamma_b)+r^{-2}\mathfrak{d}^{\leq 2}(A\cdot\Gamma_b).$$

Step 6. We are now left to treat the following term in Err_{4434}:

$$\begin{aligned}
M &:= r^{-1}\big({}^{(c)}\nabla_4(\widehat{X}\cdot\check{H})+\frac{3}{2}\mathrm{tr}X(\widehat{X}\cdot\check{H})\big)\cdot\mathfrak{d}^{\leq 1}\underline{A}\\
&= r^{-1}\big(\big({}^{(c)}\nabla_4(\widehat{X})+\Re(\mathrm{tr}X)\widehat{X}\big)\cdot\check{H}+\widehat{X}\cdot\big({}^{(c)}\nabla_4(\check{H})+\frac{1}{2}\overline{\mathrm{tr}X}\check{H}\big)\big)\cdot\mathfrak{d}^{\leq 1}\underline{A}\\
&\quad+r^{-4}\mathfrak{d}^{\leq 3}(\Gamma_g\cdot\Gamma_b)\\
&= r^{-1}\big(A\cdot\check{H}+\widehat{X}\cdot\big(\widehat{X}\cdot\check{H}+B\big)\big)\cdot\mathfrak{d}^{\leq 1}\underline{A}+r^{-4}\mathfrak{d}^{\leq 3}(\Gamma_g\cdot\Gamma_b),
\end{aligned}$$

which gives

$$M = r^{-4}\mathfrak{d}^{\leq 3}(\Gamma_g \cdot \Gamma_b) + r^{-2}\mathfrak{d}^{\leq 2}((A, B) \cdot \Gamma_b).$$

Step 7. Finally, we combine

$$\begin{aligned} \mathrm{Err}_{4434} &= \mathrm{Err}_1 + \mathrm{Err}_2 + r^{-2}\mathfrak{d}^{\leq 2}((A, B) \cdot \Gamma_b) + r^{-4}\mathfrak{d}^{\leq 2}(\Gamma_g \cdot \Gamma_b) \\ &\quad + r^{-1}\left({}^{(c)}\nabla_4(\widehat{X} \cdot \check{H}) + \frac{3}{2}\mathrm{tr}X(\widehat{X} \cdot \check{H})\right) \cdot \mathfrak{d}^{\leq 1}\underline{A} \\ &= r^{-2}\mathfrak{d}^{\leq 2}((A, B) \cdot \Gamma_b) + r^{-4}\mathfrak{d}^{\leq 3}(\Gamma_g \cdot \Gamma_b), \end{aligned}$$

which concludes the proof of the Proposition. □

D.7 Proof of Proposition 5.4.1

D.7.1 Preliminaries

We use (3.4.1) and Lemma 3.4.5 to define the following $O(\epsilon)$ quantities in perturbations of Kerr.

Definition D.7.1. *We define the following $O(\epsilon)$ quantities in $O(\epsilon)$ perturbations of Kerr:*

$$\begin{aligned} \mathcal{A}_1 &= {}^{(c)}\mathcal{D}P + 3P\,\underline{H} \in \mathfrak{s}_1(\mathbb{C}) \\ \mathcal{A}_2 &= {}^{(c)}\mathcal{D}\widehat{\otimes}\,\underline{H} + \underline{H}\widehat{\otimes}\,\underline{H} \in \mathfrak{s}_2(\mathbb{C}) \\ \mathcal{A}_3 &= 2\,{}^{(c)}\nabla_4(\underline{H}) - {}^{(c)}\mathcal{D}\,(trX) \in \mathfrak{s}_1(\mathbb{C}) \\ \mathcal{A}_4 &= {}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\,(trX) + 3\,\underline{H}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\,(trX) \in \mathfrak{s}_2(\mathbb{C}). \end{aligned}$$

We also define

$$\mathcal{B} = {}^{(c)}\mathcal{D} \cdot \overline{B} + 2\,\underline{H} \cdot \overline{B} \in \mathfrak{s}_0.$$

We collect here some useful derivatives of the above.

Lemma D.7.2. *We have, modulo quadratic terms,*

$${}^{(c)}\nabla_4\mathcal{A}_1 = -2trX\mathcal{A}_1 + \frac{1}{2}\,{}^{(c)}\mathcal{D}\mathcal{B} + 2\,\underline{H}\mathcal{B} + \frac{3}{2}P\left(\overline{H} \cdot \widehat{X} - \overline{tr\underline{X}}\Xi + \mathcal{A}_3\right),$$

and

$$
\begin{aligned}
{}^{(c)}\nabla_4 {}^{(c)}\mathcal{D}\widehat{\otimes}\mathcal{A}_1 &= -\frac{5}{2}trX\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathcal{A}_1+\frac{3}{2}trX\,\underline{H}\widehat{\otimes}\mathcal{A}_1+\frac{1}{2}{}^{(c)}\mathcal{D}\widehat{\otimes}{}^{(c)}\mathcal{D}\mathcal{B}+\frac{5}{2}\underline{H}\widehat{\otimes}{}^{(c)}\mathcal{D}\mathcal{B}\\
&+{}^{(c)}\mathcal{D}\widehat{\otimes}\Big(\frac{3}{2}P\left(\overline{H}\cdot\widehat{X}-\overline{tr\underline{X}}\Xi+\mathcal{A}_3\right)\Big)\\
&+\underline{H}\widehat{\otimes}\Big(\frac{3}{2}P\left(\overline{H}\cdot\widehat{X}-\overline{tr\underline{X}}\Xi+\mathcal{A}_3\right)\Big).
\end{aligned}
$$

Proof. Applying Lemma 4.2.2, (4.2.9), to $h=P$ and $s=0$, we obtain

$$
\begin{aligned}
[{}^{(c)}\nabla_4, {}^{(c)}\mathcal{D}]P &= -\frac{1}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}P+\underline{H}\,{}^{(c)}\nabla_4P-\frac{1}{2}\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}P+\Xi\,{}^{(c)}\nabla_3P\\
&= -\frac{1}{2}\mathrm{tr}X\Big({}^{(c)}\mathcal{D}P+3P\,\underline{H}\Big)+\underline{H}\left(\frac{1}{2}{}^{(c)}\mathcal{D}\cdot\overline{B}+\underline{H}\cdot\overline{B}\right)\\
&\quad -\frac{1}{2}\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}P-\frac{3}{2}\overline{\mathrm{tr}\underline{X}}P\Xi\\
&= -\frac{1}{2}\mathrm{tr}X\mathcal{A}_1+\underline{H}\left(\frac{1}{2}{}^{(c)}\mathcal{D}\cdot\overline{B}+\underline{H}\cdot\overline{B}\right)+\frac{3}{2}P\overline{H}\cdot\widehat{X}-\frac{3}{2}\overline{\mathrm{tr}\underline{X}}P\Xi
\end{aligned}
$$

where we used $\overline{{}^{(c)}\mathcal{D}}P=-3P\overline{H}+O(\epsilon)$. This gives

$$
\begin{aligned}
{}^{(c)}\nabla_4\mathcal{A}_1 &= {}^{(c)}\nabla_4\Big({}^{(c)}\mathcal{D}P+3P\,\underline{H}\Big)\\
&= {}^{(c)}\mathcal{D}\,{}^{(c)}\nabla_4P+[{}^{(c)}\nabla_4, {}^{(c)}\mathcal{D}]P+3\,{}^{(c)}\nabla_4(P)\,\underline{H}+3P\,{}^{(c)}\nabla_4(\underline{H})\\
&= {}^{(c)}\mathcal{D}\left(-\frac{3}{2}\mathrm{tr}XP+\frac{1}{2}{}^{(c)}\mathcal{D}\cdot\overline{B}+\underline{H}\cdot\overline{B}\right)\\
&\quad +3\left(-\frac{3}{2}\mathrm{tr}XP+\frac{1}{2}{}^{(c)}\mathcal{D}\cdot\overline{B}+\underline{H}\cdot\overline{B}\right)\underline{H}+3P\,{}^{(c)}\nabla_4(\underline{H})\\
&\quad -\frac{1}{2}\mathrm{tr}X\mathcal{A}_1+\underline{H}\left(\frac{1}{2}{}^{(c)}\mathcal{D}\cdot\overline{B}+\underline{H}\cdot\overline{B}\right)+\frac{3}{2}P\overline{H}\cdot\widehat{X}-\frac{3}{2}\overline{\mathrm{tr}\underline{X}}P\Xi\\
&= -\frac{3}{2}\mathrm{tr}X\Big({}^{(c)}\mathcal{D}P+3P\,\underline{H}\Big)+{}^{(c)}\mathcal{D}\left(\frac{1}{2}{}^{(c)}\mathcal{D}\cdot\overline{B}+\underline{H}\cdot\overline{B}\right)\\
&\quad +3\,\underline{H}\left(\frac{1}{2}{}^{(c)}\mathcal{D}\cdot\overline{B}+\underline{H}\cdot\overline{B}\right)+3\left({}^{(c)}\nabla_4(\underline{H})-\frac{1}{2}{}^{(c)}\mathcal{D}\,(\mathrm{tr}X)\right)P\\
&\quad -\frac{1}{2}\mathrm{tr}X\mathcal{A}_1+\underline{H}\left(\frac{1}{2}{}^{(c)}\mathcal{D}\cdot\overline{B}+\underline{H}\cdot\overline{B}\right)+\frac{3}{2}P\overline{H}\cdot\widehat{X}-\frac{3}{2}\overline{\mathrm{tr}\underline{X}}P\Xi,
\end{aligned}
$$

which gives

$$
\begin{aligned}
{}^{(c)}\nabla_4\mathcal{A}_1 &= -2\mathrm{tr}X\mathcal{A}_1 + {}^{(c)}\mathcal{D}\left(\frac{1}{2}\,{}^{(c)}\mathcal{D}\cdot\overline{B} + \underline{H}\cdot\overline{B}\right) + 4\,\underline{H}\left(\frac{1}{2}\,{}^{(c)}\mathcal{D}\cdot\overline{B} + \underline{H}\cdot\overline{B}\right)\\
&\quad +3\left({}^{(c)}\nabla_4(\underline{H}) - \frac{1}{2}\,{}^{(c)}\mathcal{D}\,(\mathrm{tr}X)\right)P + \frac{3}{2}P\overline{H}\cdot\widehat{X} - \frac{3}{2}\overline{\mathrm{tr}\underline{X}}P\Xi.
\end{aligned}
$$

Recalling the definition of $\mathcal{B}$ and $\mathcal{A}_3$ we obtain the first identity.

To obtain the second identity we apply Lemma 4.2.2, (4.2.13), to $F = \mathcal{A}_1$ and $s = 0$, and we have

$$
[\,{}^{(c)}\nabla_4,\,{}^{(c)}\mathcal{D}\widehat{\otimes}]\mathcal{A}_1 = -\frac{1}{2}\mathrm{tr}X\left({}^{(c)}\mathcal{D}\widehat{\otimes}\mathcal{A}_1 + \underline{H}\widehat{\otimes}\mathcal{A}_1\right) + \underline{H}\widehat{\otimes}\,{}^{(c)}\nabla_4\mathcal{A}_1.
$$

This gives

$$
\begin{aligned}
&{}^{(c)}\nabla_4\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathcal{A}_1\\
=\;& {}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\nabla_4\mathcal{A}_1 + [\,{}^{(c)}\nabla_4,\,{}^{(c)}\mathcal{D}\widehat{\otimes}]\mathcal{A}_1\\
=\;& {}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\nabla_4\mathcal{A}_1 - \frac{1}{2}\mathrm{tr}X\left({}^{(c)}\mathcal{D}\widehat{\otimes}\mathcal{A}_1 + \underline{H}\widehat{\otimes}\mathcal{A}_1\right) + \underline{H}\widehat{\otimes}\,{}^{(c)}\nabla_4\mathcal{A}_1\\
=\;& {}^{(c)}\mathcal{D}\widehat{\otimes}\Big(- 2\mathrm{tr}X\mathcal{A}_1 + \frac{1}{2}\,{}^{(c)}\mathcal{D}\mathcal{B} + 2\,\underline{H}\mathcal{B} + \frac{3}{2}P\left(\overline{H}\cdot\widehat{X} - \overline{\mathrm{tr}\underline{X}}\Xi + \mathcal{A}_3\right)\Big)\\
& -\frac{1}{2}\mathrm{tr}X\left({}^{(c)}\mathcal{D}\widehat{\otimes}\mathcal{A}_1 + \underline{H}\widehat{\otimes}\mathcal{A}_1\right)\\
& +\underline{H}\widehat{\otimes}\Big(- 2\mathrm{tr}X\mathcal{A}_1 + \frac{1}{2}\,{}^{(c)}\mathcal{D}\mathcal{B}_1 + 2\,\underline{H}\mathcal{B}_1 + \frac{3}{2}P\left(\overline{H}\cdot\widehat{X} - \overline{\mathrm{tr}\underline{X}}\Xi + \mathcal{A}_3\right)\Big)\\
=\;& -\frac{5}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathcal{A}_1 - \big(2\,{}^{(c)}\mathcal{D}\mathrm{tr}X + \frac{5}{2}\mathrm{tr}X\,\underline{H}\big)\widehat{\otimes}\mathcal{A}_1\\
& + {}^{(c)}\mathcal{D}\widehat{\otimes}\Big(\frac{1}{2}\,{}^{(c)}\mathcal{D}\mathcal{B} + 2\,\underline{H}\mathcal{B}\Big) + \underline{H}\widehat{\otimes}\Big(\frac{1}{2}\,{}^{(c)}\mathcal{D}\mathcal{B} + 2\,\underline{H}\mathcal{B}\Big)\\
& + {}^{(c)}\mathcal{D}\widehat{\otimes}\Big(\frac{3}{2}P\left(\overline{H}\cdot\widehat{X} - \overline{\mathrm{tr}\underline{X}}\Xi + \mathcal{A}_3\right)\Big) + \underline{H}\widehat{\otimes}\Big(\frac{3}{2}P\left(\overline{H}\cdot\widehat{X} - \overline{\mathrm{tr}\underline{X}}\Xi + \mathcal{A}_3\right)\Big)\\
=\;& -\frac{5}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathcal{A}_1 - \big(2\,{}^{(c)}\mathcal{D}\mathrm{tr}X + \frac{5}{2}\mathrm{tr}X\,\underline{H}\big)\widehat{\otimes}\mathcal{A}_1\\
& +\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathcal{B} + \frac{5}{2}\,\underline{H}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathcal{B} + 2\big({}^{(c)}\mathcal{D}\widehat{\otimes}\,\underline{H} + \underline{H}\widehat{\otimes}\,\underline{H}\big)\mathcal{B}\\
& + {}^{(c)}\mathcal{D}\widehat{\otimes}\Big(\frac{3}{2}P\left(\overline{H}\cdot\widehat{X} - \overline{\mathrm{tr}\underline{X}}\Xi + \mathcal{A}_3\right)\Big) + \underline{H}\widehat{\otimes}\Big(\frac{3}{2}P\left(\overline{H}\cdot\widehat{X} - \overline{\mathrm{tr}\underline{X}}\Xi + \mathcal{A}_3\right)\Big).
\end{aligned}
$$

In view of ${}^{(c)}\mathcal{D}\widehat{\otimes}\,\underline{H} + \underline{H}\widehat{\otimes}\,\underline{H} = O(\epsilon)$ and ${}^{(c)}\mathcal{D}(\mathrm{tr}X) = -2\mathrm{tr}X\,\underline{H} + O(\epsilon)$ we obtain the stated. □

Lemma D.7.3. *We have, modulo quadratic terms,*

$$\begin{aligned} {}^{(c)}\mathcal{D}\widehat{\otimes}\mathcal{A}_3 - 2\,{}^{(c)}\nabla_4\mathcal{A}_2 &= trX\mathcal{A}_2 - 3\,\underline{H}\widehat{\otimes}\mathcal{A}_3 - \mathcal{A}_4 + 2B\widehat{\otimes}\,\underline{H} \\ &\quad +(tr\underline{X} + \overline{tr\underline{X}})\,\underline{H}\widehat{\otimes}\Xi - \overline{tr\underline{X}}H\widehat{\otimes}\Xi + \widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}\,\underline{H} - \widehat{X}(\,\overline{\underline{H}}\cdot\,\underline{H}). \end{aligned}$$

Proof. Recalling the definitions of $\mathcal{A}_2$ and $\mathcal{A}_3$ we obtain

$$\begin{aligned} {}^{(c)}\mathcal{D}\widehat{\otimes}\mathcal{A}_3 - 2\,{}^{(c)}\nabla_4\mathcal{A}_2 &= {}^{(c)}\mathcal{D}\widehat{\otimes}\big(2\,{}^{(c)}\nabla_4(\,\underline{H}) - {}^{(c)}\mathcal{D}\,(\mathrm{tr}X)\,\big) - 2\,{}^{(c)}\nabla_4\big(\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,\underline{H} + \,\underline{H}\widehat{\otimes}\,\underline{H}\big) \\ &= 2[\,{}^{(c)}\mathcal{D}\widehat{\otimes},\,{}^{(c)}\nabla_4]\,\underline{H} - {}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\,(\mathrm{tr}X) - 4\,{}^{(c)}\nabla_4\,\underline{H}\widehat{\otimes}\,\underline{H}. \end{aligned}$$

Applying Lemma 4.2.2, (4.2.12), to $F = \,\underline{H}$ and $s = 0$, we obtain, up to quadratic terms,

$$\begin{aligned} [\,{}^{(c)}\nabla_4,\,{}^{(c)}\mathcal{D}\widehat{\otimes}]\,\underline{H} &= -\frac{1}{2}\mathrm{tr}X\,\big(\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,\underline{H} + \,\underline{H}\widehat{\otimes}\,\underline{H}\big) + \,\underline{H}\widehat{\otimes}\,{}^{(c)}\nabla_4\,\underline{H} + \Xi\widehat{\otimes}\,{}^{(c)}\nabla_3\,\underline{H} \\ &\quad -B\widehat{\otimes}\,\underline{H} - \frac{1}{2}\mathrm{tr}\underline{X}\Xi\widehat{\otimes}\,\underline{H} - \frac{1}{2}\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}\,\underline{H} + \frac{1}{2}\widehat{X}(\,\overline{\underline{H}}\cdot\,\underline{H}) \\ &= -\frac{1}{2}\mathrm{tr}X\mathcal{A}_2 + \,\underline{H}\widehat{\otimes}\,{}^{(c)}\nabla_4\,\underline{H} - \frac{1}{2}(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}})\,\underline{H}\widehat{\otimes}\Xi + \frac{1}{2}\overline{\mathrm{tr}\underline{X}}H\widehat{\otimes}\Xi \\ &\quad -B\widehat{\otimes}\,\underline{H} - \frac{1}{2}\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}\,\underline{H} + \frac{1}{2}\widehat{X}(\,\overline{\underline{H}}\cdot\,\underline{H}) \end{aligned}$$

where we used that $\nabla_3\,\underline{H} = -\frac{1}{2}\overline{\mathrm{tr}\underline{X}}(\,\underline{H} - H) + O(\epsilon)$. This gives

$$\begin{aligned} {}^{(c)}\mathcal{D}\widehat{\otimes}\mathcal{A}_3 - 2\,{}^{(c)}\nabla_4\mathcal{A}_2 &= \mathrm{tr}X\mathcal{A}_2 - 6\,\underline{H}\widehat{\otimes}\,{}^{(c)}\nabla_4\,\underline{H} + 2B\widehat{\otimes}\,\underline{H} - {}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\,(\mathrm{tr}X) \\ &\quad +(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}})\,\underline{H}\widehat{\otimes}\Xi - \overline{\mathrm{tr}\underline{X}}H\widehat{\otimes}\Xi + \widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}\,\underline{H} - \widehat{X}(\,\overline{\underline{H}}\cdot\,\underline{H}). \end{aligned}$$

Writing $2\,{}^{(c)}\nabla_4(\,\underline{H}) = \mathcal{A}_3 + {}^{(c)}\mathcal{D}\,(\mathrm{tr}X)$, we obtain

$$\begin{aligned} &{}^{(c)}\mathcal{D}\widehat{\otimes}\mathcal{A}_3 - 2\,{}^{(c)}\nabla_4\mathcal{A}_2 \\ = \;& \mathrm{tr}X\mathcal{A}_2 - 3\,\underline{H}\widehat{\otimes}\mathcal{A}_3 + 2B\widehat{\otimes}\,\underline{H} - {}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\,(\mathrm{tr}X) - 3\,\underline{H}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\,(\mathrm{tr}X) \\ & +(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}})\,\underline{H}\widehat{\otimes}\Xi - \overline{\mathrm{tr}\underline{X}}H\widehat{\otimes}\Xi + \widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}\,\underline{H} - \widehat{X}(\,\overline{\underline{H}}\cdot\,\underline{H}). \end{aligned}$$

Recalling the definition of $\mathcal{A}_4$, this concludes the lemma. □

Lemma D.7.4. *We have, modulo quadratic terms,*

$$\begin{aligned} &{}^{(c)}\nabla_4\mathcal{A}_4 \\ = \;& -2(trX)\mathcal{A}_4 - 3trX\,\underline{H}\widehat{\otimes}\mathcal{A}_3 - trX\,{}^{(c)}\mathcal{D}\widehat{\otimes}B + 2trX\,\underline{H}\widehat{\otimes}B + \mathcal{M}[\Xi] \\ & -\frac{1}{2}(\overline{trX} - trX)\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\widehat{X}\cdot\overline{H}) + \frac{1}{2}trX\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\widehat{X}\cdot\overline{\underline{H}}) \\ & -\frac{1}{2}(trX - \overline{trX})(H\cdot\overline{H})\widehat{X} + trX(\,\underline{H}\cdot\overline{\underline{H}})\widehat{X} + trX\underline{H}\widehat{\otimes}(\widehat{X}\cdot\overline{H}) + \widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}(trX\,\underline{H}) \end{aligned}$$

where $\mathcal{M}[\Xi]$ only depends on Ξ, and therefore vanishes if $\Xi = 0$.

Proof. Recalling the definition of $\mathcal{A}_4$ we obtain

$$
\begin{aligned}
& {}^{(c)}\nabla_4\mathcal{A}_4\\
= \;& {}^{(c)}\nabla_4\big({}^{(c)}\mathcal{D}\widehat{\otimes}\, {}^{(c)}\mathcal{D}\,(\mathrm{tr}X) + 3\,\underline{H}\widehat{\otimes}\, {}^{(c)}\mathcal{D}\,(\mathrm{tr}X)\,\big)\\
= \;& {}^{(c)}\mathcal{D}\widehat{\otimes}\, {}^{(c)}\nabla_4\, {}^{(c)}\mathcal{D}\,(\mathrm{tr}X) + [\, {}^{(c)}\nabla_4,\, {}^{(c)}\mathcal{D}\widehat{\otimes}]\, {}^{(c)}\mathcal{D}\,(\mathrm{tr}X)\\
& +3\, {}^{(c)}\nabla_4\,\underline{H}\widehat{\otimes}\, {}^{(c)}\mathcal{D}\,(\mathrm{tr}X) + 3\,\underline{H}\widehat{\otimes}\, {}^{(c)}\nabla_4\, {}^{(c)}\mathcal{D}\,(\mathrm{tr}X)\\
= \;& {}^{(c)}\mathcal{D}\widehat{\otimes}\, {}^{(c)}\mathcal{D}\,\big({}^{(c)}\nabla_4\mathrm{tr}X\big) + {}^{(c)}\mathcal{D}\widehat{\otimes}\big([\, {}^{(c)}\nabla_4,\, {}^{(c)}\mathcal{D}]\mathrm{tr}X\big) + [\, {}^{(c)}\nabla_4,\, {}^{(c)}\mathcal{D}\widehat{\otimes}]\, {}^{(c)}\mathcal{D}\,(\mathrm{tr}X)\\
& +3\, {}^{(c)}\nabla_4\,\underline{H}\widehat{\otimes}\, {}^{(c)}\mathcal{D}\,(\mathrm{tr}X) + 3\,\underline{H}\widehat{\otimes}\, {}^{(c)}\mathcal{D}\,\big({}^{(c)}\nabla_4\mathrm{tr}X\big) + 3\,\underline{H}\widehat{\otimes}[\, {}^{(c)}\nabla_4,\, {}^{(c)}\mathcal{D}]\,(\mathrm{tr}X)
\end{aligned}
$$

Applying Lemma 4.2.2 to $h = \mathrm{tr}X$ and $s = 1$, we obtain

$$
\begin{aligned}
& [\, {}^{(c)}\nabla_4,\, {}^{(c)}\mathcal{D}]\mathrm{tr}X\\
= \;& -\frac{1}{2}\mathrm{tr}X\, {}^{(c)}\mathcal{D}\mathrm{tr}X + \,\underline{H}\, {}^{(c)}\nabla_4\mathrm{tr}X - \frac{1}{2}\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}\mathrm{tr}X + \Xi\, {}^{(c)}\nabla_3\mathrm{tr}X\\
& + \left(\frac{1}{2}\mathrm{tr}X\,\underline{H} + \frac{1}{2}\widehat{X}\cdot\overline{\underline{H}} - \frac{1}{2}\mathrm{tr}\underline{X}\Xi - B\right)\mathrm{tr}X\\
= \;& -\frac{1}{2}\mathrm{tr}X\, {}^{(c)}\mathcal{D}\mathrm{tr}X + \,\underline{H}\big(-\frac{1}{2}(\mathrm{tr}X)^2 + {}^{(c)}\mathcal{D}\cdot\overline{\Xi} + \Xi\cdot\overline{H} + \overline{\Xi}\cdot H\big)\\
& -\frac{1}{2}(\overline{\mathrm{tr}X} - \mathrm{tr}X)\widehat{X}\cdot\overline{H} + \Xi\big(-\frac{1}{2}\mathrm{tr}\underline{X}\mathrm{tr}X + {}^{(c)}\mathcal{D}\cdot\overline{H} + H\cdot\overline{H} + 2P\big)\\
& + \left(\frac{1}{2}\mathrm{tr}X\,\underline{H} + \frac{1}{2}\widehat{X}\cdot\overline{\underline{H}} - \frac{1}{2}\mathrm{tr}\underline{X}\Xi - B\right)\mathrm{tr}X\\
= \;& -\frac{1}{2}\mathrm{tr}X\, {}^{(c)}\mathcal{D}\mathrm{tr}X - \mathrm{tr}XB\\
& +\underline{H}\big({}^{(c)}\mathcal{D}\cdot\overline{\Xi} + \Xi\cdot\overline{H} + \overline{\Xi}\cdot H\big) + \big(-\mathrm{tr}\underline{X}\mathrm{tr}X + {}^{(c)}\mathcal{D}\cdot\overline{H} + H\cdot\overline{H} + 2P\big)\Xi\\
& -\frac{1}{2}(\overline{\mathrm{tr}X} - \mathrm{tr}X)\widehat{X}\cdot\overline{H} + \frac{1}{2}\mathrm{tr}X\widehat{X}\cdot\overline{\underline{H}}.
\end{aligned}
$$

where we used that ${}^{(c)}\nabla_3\mathrm{tr}X + \frac{1}{2}\mathrm{tr}\underline{X}\mathrm{tr}X = {}^{(c)}\mathcal{D}\cdot\overline{H} + H\cdot\overline{H} + 2P + O(\epsilon^2)$, ${}^{(c)}\nabla_4\mathrm{tr}X + \frac{1}{2}(\mathrm{tr}X)^2 = {}^{(c)}\mathcal{D}\cdot\overline{\Xi} + \Xi\cdot\overline{H} + \overline{\Xi}\cdot H$.

Using that $\overline{\mathrm{tr}X}\,\underline{H} + \mathrm{tr}XH = O(\epsilon)$, we also simplify

$$
\begin{aligned}
-\frac{1}{2}(\overline{\mathrm{tr}X} - \mathrm{tr}X)\widehat{X}\cdot\overline{H} + \frac{1}{2}\mathrm{tr}X\widehat{X}\cdot\overline{\underline{H}} &= -\frac{1}{2}\overline{\mathrm{tr}X}\widehat{X}\cdot\overline{H} + \frac{1}{2}\mathrm{tr}X\widehat{X}\cdot\overline{H} + \frac{1}{2}\mathrm{tr}X\widehat{X}\cdot\overline{\underline{H}}\\
&= \mathrm{tr}X\widehat{X}\cdot\overline{\underline{H}} + \frac{1}{2}\mathrm{tr}X\widehat{X}\cdot\overline{H},
\end{aligned}
$$

which gives

$$
\begin{aligned}
[\,{}^{(c)}\nabla_4,\, {}^{(c)}\mathcal{D}]\mathrm{tr}X &= -\frac{1}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}\mathrm{tr}X - \mathrm{tr}XB\\
&\quad + \underline{H}\big({}^{(c)}\mathcal{D}\cdot\overline{\Xi} + \Xi\cdot\overline{H} + \overline{\Xi}\cdot H\big)\\
&\quad + \big(-\mathrm{tr}\underline{X}\mathrm{tr}X + {}^{(c)}\mathcal{D}\cdot\overline{H} + H\cdot\overline{H} + 2P\big)\Xi\\
&\quad + \mathrm{tr}X\widehat{X}\cdot\overline{\underline{H}} + \frac{1}{2}\mathrm{tr}X\widehat{X}\cdot\overline{H}.
\end{aligned}
$$

Applying Lemma 4.2.2, (4.2.12), to $F = {}^{(c)}\mathcal{D}\mathrm{tr}X$ and $s=1$ we obtain

$$
\begin{aligned}
&[\,{}^{(c)}\nabla_4,\, {}^{(c)}\mathcal{D}\widehat{\otimes}]\,{}^{(c)}\mathcal{D}\mathrm{tr}X\\
=\ & -\frac{1}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathrm{tr}X + \underline{H}\widehat{\otimes}\,{}^{(c)}\nabla_4\,{}^{(c)}\mathcal{D}\mathrm{tr}X + \Xi\widehat{\otimes}\,{}^{(c)}\nabla_3\,{}^{(c)}\mathcal{D}\mathrm{tr}X\\
& -2B\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathrm{tr}X - \mathrm{tr}\underline{X}\Xi\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathrm{tr}X - \frac{1}{2}\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}\,{}^{(c)}\mathcal{D}\mathrm{tr}X\\
& +\frac{1}{2}\widehat{X}(\overline{\underline{H}}\cdot\,{}^{(c)}\mathcal{D}\mathrm{tr}X) + \frac{1}{2}(\widehat{X}\cdot\overline{\underline{H}})\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathrm{tr}X\\
=\ & -\frac{1}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathrm{tr}X + \underline{H}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\,{}^{(c)}\nabla_4\mathrm{tr}X + \underline{H}\widehat{\otimes}[\,{}^{(c)}\nabla_4,\, {}^{(c)}\mathcal{D}]\mathrm{tr}X\\
& +\Xi\widehat{\otimes}\,{}^{(c)}\nabla_3(-2\mathrm{tr}X\,\underline{H}) - 2B\widehat{\otimes}(-2\mathrm{tr}X\,\underline{H}) - \mathrm{tr}\underline{X}\Xi\widehat{\otimes}(-2\mathrm{tr}X\,\underline{H})\\
& -\frac{1}{2}\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}(-2\mathrm{tr}X\,\underline{H})\\
& +\frac{1}{2}\widehat{X}(\overline{\underline{H}}\cdot(-2\mathrm{tr}X\,\underline{H})) + \frac{1}{2}(\widehat{X}\cdot\overline{\underline{H}})\widehat{\otimes}(-2\mathrm{tr}X\,\underline{H})\\
=\ & -\frac{1}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathrm{tr}X + \underline{H}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\,{}^{(c)}\nabla_4\mathrm{tr}X + \underline{H}\widehat{\otimes}[\,{}^{(c)}\nabla_4,\, {}^{(c)}\mathcal{D}]\mathrm{tr}X\\
& +4\mathrm{tr}XB\widehat{\otimes}\,\underline{H} - 2\Xi\widehat{\otimes}\big({}^{(c)}\nabla_3(\mathrm{tr}X\,\underline{H}) - \mathrm{tr}X\mathrm{tr}\underline{X}\,\underline{H}\big)\\
& +\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}(\mathrm{tr}X\,\underline{H}) - 2\mathrm{tr}X(\,\underline{H}\cdot\overline{\underline{H}})\widehat{X}
\end{aligned}
$$

where we used that ${}^{(c)}\mathcal{D}(\mathrm{tr}X) = -2\mathrm{tr}X\,\underline{H} + O(\epsilon)$.

We therefore obtain

$$
\begin{aligned}
{}^{(c)}\nabla_4\mathcal{A}_4 &= {}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\left({}^{(c)}\nabla_4\mathrm{tr}X\right) + {}^{(c)}\mathcal{D}\widehat{\otimes}\big([\,{}^{(c)}\nabla_4,\, {}^{(c)}\mathcal{D}]\mathrm{tr}X\big) + 4\underline{H}\widehat{\otimes}[\,{}^{(c)}\nabla_4,\, {}^{(c)}\mathcal{D}]\mathrm{tr}X\\
&\quad -\frac{1}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathrm{tr}X + 4\underline{H}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\,{}^{(c)}\nabla_4\mathrm{tr}X + 3\,{}^{(c)}\nabla_4\,\underline{H}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\,(\mathrm{tr}X)\\
&\quad +4\mathrm{tr}XB\widehat{\otimes}\,\underline{H} - 2\Xi\widehat{\otimes}\big({}^{(c)}\nabla_3(\mathrm{tr}X\,\underline{H}) - \mathrm{tr}X\mathrm{tr}\underline{X}\,\underline{H}\big)\\
&\quad +\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}(\mathrm{tr}X\,\underline{H}) - 2\mathrm{tr}X(\,\underline{H}\cdot\overline{\underline{H}})\widehat{X}.
\end{aligned}
$$

Using the Ricci identity ${}^{(c)}\nabla_4 \mathrm{tr} X = -\frac{1}{2}(\mathrm{tr} X)^2 + {}^{(c)}\mathcal{D}\cdot\overline{\Xi} + \Xi\cdot\overline{H} + \overline{\Xi}\cdot H + O(\epsilon^2)$ we compute

$$
\begin{aligned}
{}^{(c)}\mathcal{D}\left({}^{(c)}\nabla_4\mathrm{tr}X\right) &= {}^{(c)}\mathcal{D}\left(-\frac{1}{2}(\mathrm{tr}X)^2 + {}^{(c)}\mathcal{D}\cdot\overline{\Xi} + \Xi\cdot\overline{H} + \overline{\Xi}\cdot H\right)\\
&= -(\mathrm{tr}X)\,{}^{(c)}\mathcal{D}\mathrm{tr}X + {}^{(c)}\mathcal{D}\,{}^{(c)}\mathcal{D}\cdot\overline{\Xi} + {}^{(c)}\mathcal{D}\Xi\cdot\overline{H} + {}^{(c)}\mathcal{D}\overline{\Xi}\cdot H\\
&\quad +\Xi\,{}^{(c)}\mathcal{D}\cdot\overline{H} + \overline{\Xi}\,{}^{(c)}\mathcal{D}\cdot H
\end{aligned}
$$

and

$$
\begin{aligned}
& {}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\left({}^{(c)}\nabla_4\mathrm{tr}X\right)\\
={}& {}^{(c)}\mathcal{D}\widehat{\otimes}\Big(-(\mathrm{tr}X)\,{}^{(c)}\mathcal{D}\mathrm{tr}X + {}^{(c)}\mathcal{D}\,{}^{(c)}\mathcal{D}\cdot\overline{\Xi} + {}^{(c)}\mathcal{D}\Xi\cdot\overline{H} + {}^{(c)}\mathcal{D}\overline{\Xi}\cdot H\\
& +\Xi\,{}^{(c)}\mathcal{D}\cdot\overline{H} + \overline{\Xi}\,{}^{(c)}\mathcal{D}\cdot H\Big)\\
={}& -(\mathrm{tr}X)\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathrm{tr}X - {}^{(c)}\mathcal{D}(\mathrm{tr}X)\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathrm{tr}X + {}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\,{}^{(c)}\mathcal{D}\cdot\overline{\Xi}\\
& + {}^{(c)}\mathcal{D}\widehat{\otimes}\Big({}^{(c)}\mathcal{D}\Xi\cdot\overline{H} + {}^{(c)}\mathcal{D}\overline{\Xi}\cdot H + \Xi\,{}^{(c)}\mathcal{D}\cdot\overline{H} + \overline{\Xi}\,{}^{(c)}\mathcal{D}\cdot H\Big).
\end{aligned}
$$

Also, using the above we compute

$$
\begin{aligned}
{}^{(c)}\mathcal{D}\widehat{\otimes}\big([\,{}^{(c)}\nabla_4, {}^{(c)}\mathcal{D}]\mathrm{tr}X\big) &= -\frac{1}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathrm{tr}X - \frac{1}{2}\,{}^{(c)}\mathcal{D}\mathrm{tr}X\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathrm{tr}X\\
&\quad -\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}B - {}^{(c)}\mathcal{D}\mathrm{tr}X\widehat{\otimes}B\\
&\quad -\frac{1}{2}(\overline{\mathrm{tr}X} - \mathrm{tr}X)\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\widehat{X}\cdot\overline{H}) + \frac{1}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\widehat{X}\cdot\overline{\underline{H}})\\
&\quad -\frac{1}{2}({}^{(c)}\mathcal{D}\overline{\mathrm{tr}X} - {}^{(c)}\mathcal{D}\mathrm{tr}X)\widehat{\otimes}(\widehat{X}\cdot\overline{H}) + \frac{1}{2}\,{}^{(c)}\mathcal{D}\mathrm{tr}X\widehat{\otimes}(\widehat{X}\cdot\overline{\underline{H}}).
\end{aligned}
$$

Using that ${}^{(c)}\mathcal{D}(\mathrm{tr}X) = -2\mathrm{tr}X\,\underline{H} + O(\epsilon)$, ${}^{(c)}\mathcal{D}(\overline{\mathrm{tr}X}) = (\mathrm{tr}X - \overline{\mathrm{tr}X})H + O(\epsilon)$ and ${}^{(c)}\mathcal{D}\widehat{\otimes}\,\underline{H} = -\underline{H}\widehat{\otimes}\,\underline{H} + O(\epsilon)$, we obtain

$$
\begin{aligned}
{}^{(c)}\mathcal{D}\widehat{\otimes}\big([\,{}^{(c)}\nabla_4, {}^{(c)}\mathcal{D}]\mathrm{tr}X\big) &= -\frac{1}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathrm{tr}X - \frac{1}{2}\,{}^{(c)}\mathcal{D}\mathrm{tr}X\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathrm{tr}X\\
&\quad -\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}B + 2\mathrm{tr}X\,\underline{H}\widehat{\otimes}B\\
&\quad -\frac{1}{2}(\overline{\mathrm{tr}X} - \mathrm{tr}X)\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\widehat{X}\cdot\overline{H}) + \frac{1}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\widehat{X}\cdot\overline{\underline{H}})\\
&\quad -\frac{1}{2}((\mathrm{tr}X - \overline{\mathrm{tr}X})H + 2\mathrm{tr}X\,\underline{H})\widehat{\otimes}(\widehat{X}\cdot\overline{H}) - \mathrm{tr}X\,\underline{H}\widehat{\otimes}(\widehat{X}\cdot\overline{\underline{H}}).
\end{aligned}
$$

We finally put all together to obtain

$$\begin{aligned}
& {}^{(c)}\nabla_4\mathcal{A}_4\\
= \; & -2(\mathrm{tr}X)\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathrm{tr}X-\frac{3}{2}\big({}^{(c)}\mathcal{D}(\mathrm{tr}X)-2\,{}^{(c)}\nabla_4\,\underline{H}\big)\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathrm{tr}X-6\mathrm{tr}X\underline{H}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathrm{tr}X\\
& -\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}B+2\mathrm{tr}X\,\underline{H}\widehat{\otimes}B+\mathcal{M}[\Xi]\\
& -\frac{1}{2}(\overline{\mathrm{tr}X}-\mathrm{tr}X)\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\widehat{X}\cdot\overline{H})+\frac{1}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\widehat{X}\cdot\overline{\underline{H}})\\
& -\frac{1}{2}((\mathrm{tr}X-\overline{\mathrm{tr}X})H+2\mathrm{tr}X\,\underline{H})\widehat{\otimes}(\widehat{X}\cdot\overline{H})-\mathrm{tr}X\,\underline{H}\widehat{\otimes}(\widehat{X}\cdot\overline{\underline{H}})\\
& +4\underline{H}\widehat{\otimes}(\mathrm{tr}X\widehat{X}\cdot\overline{\underline{H}}+\frac{1}{2}\mathrm{tr}X\widehat{X}\cdot\overline{H})+\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}(\mathrm{tr}X\,\underline{H})-2\mathrm{tr}X(\,\underline{H}\cdot\overline{\underline{H}})\widehat{X}\\
= \; & -2(\mathrm{tr}X)\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathrm{tr}X-\frac{3}{2}\big({}^{(c)}\mathcal{D}(\mathrm{tr}X)-2\,{}^{(c)}\nabla_4\,\underline{H}\big)\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathrm{tr}X-6\mathrm{tr}X\underline{H}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathrm{tr}X\\
& -\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}B+2\mathrm{tr}X\,\underline{H}\widehat{\otimes}B+\mathcal{M}[\Xi]\\
& -\frac{1}{2}(\overline{\mathrm{tr}X}-\mathrm{tr}X)\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\widehat{X}\cdot\overline{H})+\frac{1}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\widehat{X}\cdot\overline{\underline{H}})\\
& -\frac{1}{2}(\mathrm{tr}X-\overline{\mathrm{tr}X})(H\cdot\overline{H})\widehat{X}+\mathrm{tr}X(\,\underline{H}\cdot\overline{\underline{H}})\widehat{X}+\mathrm{tr}X\underline{H}\widehat{\otimes}(\widehat{X}\cdot\overline{H})+\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}(\mathrm{tr}X\,\underline{H}).
\end{aligned}$$

Finally writing $\mathcal{A}_3=2\,{}^{(c)}\nabla_4\,\underline{H}-\,{}^{(c)}\mathcal{D}\mathrm{tr}X$ and $\mathcal{A}_4=\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\,(\mathrm{tr}X)+3\,\underline{H}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\,(\mathrm{tr}X)$, we obtain

$$\begin{aligned}
& {}^{(c)}\nabla_4\mathcal{A}_4\\
= \; & -2(\mathrm{tr}X)\mathcal{A}_4-3\mathrm{tr}X\,\underline{H}\widehat{\otimes}\mathcal{A}_3-\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}B+2\mathrm{tr}X\,\underline{H}\widehat{\otimes}B+\mathcal{M}[\Xi]\\
& -\frac{1}{2}(\overline{\mathrm{tr}X}-\mathrm{tr}X)\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\widehat{X}\cdot\overline{H})+\frac{1}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\widehat{X}\cdot\overline{\underline{H}})\\
& -\frac{1}{2}(\mathrm{tr}X-\overline{\mathrm{tr}X})(H\cdot\overline{H})\widehat{X}+\mathrm{tr}X(\,\underline{H}\cdot\overline{\underline{H}})\widehat{X}+\mathrm{tr}X\underline{H}\widehat{\otimes}(\widehat{X}\cdot\overline{H})+\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}(\mathrm{tr}X\,\underline{H}),
\end{aligned}$$

as stated. □

Lemma D.7.5. *We have, modulo quadratic terms,*

$$\begin{aligned}
{}^{(c)}\nabla_4\mathcal{B} &= -\frac{5}{2}trX\mathcal{B}+2trX\,\underline{H}\cdot\overline{B}+3P\,{}^{(c)}\mathcal{D}\cdot\overline{\Xi}\\
&\quad+\frac{1}{2}\big({}^{(c)}\mathcal{D}+3\,\underline{H}\big)\cdot\big({}^{(c)}\mathcal{D}\cdot\overline{A}+\overline{A}\cdot\,\underline{H}\big),\\
{}^{(c)}\nabla_4\,{}^{(c)}\mathcal{D}\mathcal{B} &= -3trX\,{}^{(c)}\mathcal{D}\mathcal{B}+5trX\,\underline{H}\mathcal{B}-8trX\,\underline{H}(\,\underline{H}\cdot\overline{B})\\
&\quad+3P\,{}^{(c)}\mathcal{D}\,{}^{(c)}\mathcal{D}\cdot\overline{\Xi}-6\,\underline{H}P\,{}^{(c)}\mathcal{D}\cdot\overline{\Xi}\\
&\quad+\frac{1}{2}(\,{}^{(c)}\mathcal{D}+\,\underline{H})\Big(\big({}^{(c)}\mathcal{D}+3\,\underline{H}\big)\cdot\big({}^{(c)}\mathcal{D}\cdot\overline{A}+\overline{A}\cdot\,\underline{H}\big)\Big),
\end{aligned}$$

and

$$
\begin{aligned}
& {}^{(c)}\nabla_4\, {}^{(c)}\mathcal{D}\widehat{\otimes}\, {}^{(c)}\mathcal{D}\mathcal{B} \\
= & -\frac{7}{2}trX\, {}^{(c)}\mathcal{D}\widehat{\otimes}\, {}^{(c)}\mathcal{D}\mathcal{B} + 8trX\, \underline{H}\widehat{\otimes}\, {}^{(c)}\mathcal{D}\mathcal{B} - 10trX\, \underline{H}\widehat{\otimes}\, \underline{H}\mathcal{B} \\
& ({}^{(c)}\mathcal{D} + \underline{H})\widehat{\otimes}\Big[-8trX\, \underline{H}(\underline{H}\cdot\overline{B}) + 3P\, {}^{(c)}\mathcal{D}\, {}^{(c)}\mathcal{D}\cdot\overline{\Xi} - 6\,\underline{H}P\, {}^{(c)}\mathcal{D}\cdot\overline{\Xi}\Big] \\
& +\frac{1}{2}({}^{(c)}\mathcal{D} + \underline{H})\widehat{\otimes}\Big(({}^{(c)}\mathcal{D} + \underline{H})\big(({}^{(c)}\mathcal{D} + 3\,\underline{H})\cdot\big({}^{(c)}\mathcal{D}\cdot\overline{A} + \overline{A}\cdot\underline{H}\big)\big)\Big).
\end{aligned}
$$

Proof. Recalling the definition of $\mathcal{B}$ we obtain

$$
\begin{aligned}
{}^{(c)}\nabla_4\mathcal{B} &= {}^{(c)}\nabla_4({}^{(c)}\mathcal{D}\cdot\overline{B} + 2\,\underline{H}\cdot\overline{B}) \\
&= {}^{(c)}\mathcal{D}\cdot({}^{(c)}\nabla_4\overline{B}) + [{}^{(c)}\nabla_4, {}^{(c)}\mathcal{D}\cdot]\overline{B} + 2\,\underline{H}\cdot{}^{(c)}\nabla_4\overline{B} + 2\,{}^{(c)}\nabla_4\,\underline{H}\cdot\overline{B}.
\end{aligned}
$$

From Lemma 4.2.2, (4.2.16), applied to $F = B$ and $s = 1$, we have

$$
[{}^{(c)}\nabla_4, {}^{(c)}\mathcal{D}\cdot]\overline{B} = -\frac{1}{2}\mathrm{tr}X\,({}^{(c)}\mathcal{D}\cdot\overline{B} - 2\,\underline{H}\cdot\overline{F}) + \underline{H}\cdot{}^{(c)}\nabla_4\overline{B},
$$

and therefore

$$
{}^{(c)}\nabla_4\mathcal{B} = {}^{(c)}\mathcal{D}\cdot({}^{(c)}\nabla_4\overline{B}) - \frac{1}{2}\mathrm{tr}X\,({}^{(c)}\mathcal{D}\cdot\overline{B} - 2\,\underline{H}\cdot\overline{B}) + 3\,\underline{H}\cdot{}^{(c)}\nabla_4\overline{B} + 2\,{}^{(c)}\nabla_4\,\underline{H}\cdot\overline{B}.
$$

Using the Bianchi identity

$$
{}^{(c)}\nabla_4\overline{B} = -2\mathrm{tr}X\overline{B} + 3P\,\overline{\Xi} + \frac{1}{2}\big({}^{(c)}\mathcal{D}\cdot\overline{A} + \overline{A}\cdot\underline{H}\big).
$$

we obtain

$$
\begin{aligned}
& {}^{(c)}\nabla_4\mathcal{B} \\
= & {}^{(c)}\mathcal{D}\cdot\Big(-2\mathrm{tr}X\overline{B} + 3P\,\overline{\Xi} + \frac{1}{2}\big({}^{(c)}\mathcal{D}\cdot\overline{A} + \overline{A}\cdot\underline{H}\big)\Big) \\
& -\frac{1}{2}\mathrm{tr}X\,({}^{(c)}\mathcal{D}\cdot\overline{B} - 2\,\underline{H}\cdot\overline{B}) + 3\,\underline{H}\cdot\Big(-2\mathrm{tr}X\overline{B} + 3P\,\overline{\Xi} + \frac{1}{2}\big({}^{(c)}\mathcal{D}\cdot\overline{A} + \overline{A}\cdot\underline{H}\big)\Big) \\
& +2\,{}^{(c)}\nabla_4\,\underline{H}\cdot\overline{B} \\
= & -2\mathrm{tr}X\,{}^{(c)}\mathcal{D}\cdot\overline{B} - 2\,{}^{(c)}\mathcal{D}\mathrm{tr}X\cdot\overline{B} + 3P\,{}^{(c)}\mathcal{D}\cdot\overline{\Xi} + 3\,{}^{(c)}\mathcal{D}P\cdot\overline{\Xi} + \frac{1}{2}\,{}^{(c)}\mathcal{D}\cdot\big({}^{(c)}\mathcal{D}\cdot\overline{A} + \overline{A}\cdot\underline{H}\big) \\
& -\frac{1}{2}\mathrm{tr}X\,({}^{(c)}\mathcal{D}\cdot\overline{B} - 2\,\underline{H}\cdot\overline{B}) + 3\,\underline{H}\cdot\Big(-2\mathrm{tr}X\overline{B} + 3P\,\overline{\Xi} + \frac{1}{2}\big({}^{(c)}\mathcal{D}\cdot\overline{A} + \overline{A}\cdot\underline{H}\big)\Big) \\
& +2\,{}^{(c)}\nabla_4\,\underline{H}\cdot\overline{B}.
\end{aligned}
$$

Using that ${}^{(c)}\mathcal{D}(\mathrm{tr}X)=-2\mathrm{tr}X\,\underline{H}+O(\epsilon)$, ${}^{(c)}\nabla_4(\,\underline{H})=-\mathrm{tr}X\,\underline{H}+O(\epsilon)$ and ${}^{(c)}\mathcal{D}P=-3\,\underline{H}P+O(\epsilon)$, we obtain

$$
\begin{aligned}
{}^{(c)}\nabla_4\mathcal{B} &= -\frac{5}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}\cdot\overline{B}-3\mathrm{tr}X\,\underline{H}\cdot\overline{B}+3P\,{}^{(c)}\mathcal{D}\cdot\Xi\\
&+\frac{1}{2}\big({}^{(c)}\mathcal{D}+3\,\underline{H}\big)\cdot\big({}^{(c)}\mathcal{D}\cdot\overline{A}+\overline{A}\cdot\,\underline{H}\big).
\end{aligned}
$$

Writing ${}^{(c)}\mathcal{D}\cdot\overline{B}=\mathcal{B}_1-2\,\underline{H}\cdot\overline{B}$ we obtain the first identity.

We now compute ${}^{(c)}\nabla_4\,{}^{(c)}\mathcal{D}\mathcal{B}$. Applying Lemma 4.2.2, (4.2.10), to $h=\mathcal{B}$ and $s=1$ we have, up to quadratic terms

$$
[\,{}^{(c)}\nabla_4,\,{}^{(c)}\mathcal{D}]\mathcal{B} = -\frac{1}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}\mathcal{B}+\,\underline{H}\,{}^{(c)}\nabla_4\mathcal{B}+\frac{1}{2}\mathrm{tr}X\,\underline{H}\mathcal{B},
$$

and therefore we obtain, using the previous obtained identity,

$$
\begin{aligned}
{}^{(c)}\nabla_4\,{}^{(c)}\mathcal{D}\mathcal{B} &= {}^{(c)}\mathcal{D}(\,{}^{(c)}\nabla_4\mathcal{B})-\frac{1}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}\mathcal{B}+\,\underline{H}\,{}^{(c)}\nabla_4\mathcal{B}+\frac{1}{2}\mathrm{tr}X\,\underline{H}\mathcal{B}\\
&= {}^{(c)}\mathcal{D}\left(-\frac{5}{2}\mathrm{tr}X\mathcal{B}+2\mathrm{tr}X\,\underline{H}\cdot\overline{B}+3P\,{}^{(c)}\mathcal{D}\cdot\Xi\right)-\frac{1}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}\mathcal{B}+\frac{1}{2}\mathrm{tr}X\,\underline{H}\mathcal{B}\\
&+\,\underline{H}\left(-\frac{5}{2}\mathrm{tr}X\mathcal{B}+2\mathrm{tr}X\,\underline{H}\cdot\overline{B}+3P\,{}^{(c)}\mathcal{D}\cdot\Xi\right)\\
&+\frac{1}{2}(\,{}^{(c)}\mathcal{D}+\,\underline{H})\Big(\big({}^{(c)}\mathcal{D}+3\,\underline{H}\big)\cdot\big({}^{(c)}\mathcal{D}\cdot\overline{A}+\overline{A}\cdot\,\underline{H}\big)\Big).
\end{aligned}
$$

Using Lemma 2.4.6 we simplify the above to

$$
\begin{aligned}
{}^{(c)}\nabla_4\,{}^{(c)}\mathcal{D}\mathcal{B} &= -\frac{5}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}\mathcal{B}-\frac{5}{2}\,{}^{(c)}\mathcal{D}\mathrm{tr}X\mathcal{B}+2\,{}^{(c)}\mathcal{D}\mathrm{tr}X\,\underline{H}\cdot\overline{B}+2\mathrm{tr}X(\,{}^{(c)}\mathcal{D}\,\underline{H})\cdot\overline{B}\\
&+2\mathrm{tr}X\,\underline{H}(\,{}^{(c)}\mathcal{D}\cdot\overline{B})+3\,{}^{(c)}\mathcal{D}P\,{}^{(c)}\mathcal{D}\cdot\Xi+3P\,{}^{(c)}\mathcal{D}\,{}^{(c)}\mathcal{D}\cdot\Xi\\
&-\frac{1}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}\mathcal{B}+\frac{1}{2}\mathrm{tr}X\,\underline{H}\mathcal{B}+\,\underline{H}(-\frac{5}{2}\mathrm{tr}X\mathcal{B}+2\mathrm{tr}X\,\underline{H}\cdot\overline{B}+3P\,{}^{(c)}\mathcal{D}\cdot\Xi)\\
&+\frac{1}{2}(\,{}^{(c)}\mathcal{D}+\,\underline{H})\Big(\big({}^{(c)}\mathcal{D}+3\,\underline{H}\big)\cdot\big({}^{(c)}\mathcal{D}\cdot\overline{A}+\overline{A}\cdot\,\underline{H}\big)\Big)\\
&= -3\mathrm{tr}X\,{}^{(c)}\mathcal{D}\mathcal{B}+3\mathrm{tr}X\,\underline{H}\mathcal{B}-2\mathrm{tr}X\,\underline{H}(\,\underline{H}\cdot\overline{B})\\
&+2\mathrm{tr}X(\,{}^{(c)}\mathcal{D}\,\underline{H})\cdot\overline{B}+2\mathrm{tr}X\,\underline{H}(\,{}^{(c)}\mathcal{D}\cdot\overline{B})+3P\,{}^{(c)}\mathcal{D}\,{}^{(c)}\mathcal{D}\cdot\Xi-6\,\underline{H}P\,{}^{(c)}\mathcal{D}\cdot\Xi\\
&+\frac{1}{2}(\,{}^{(c)}\mathcal{D}+\,\underline{H})\Big(\big({}^{(c)}\mathcal{D}+3\,\underline{H}\big)\cdot\big({}^{(c)}\mathcal{D}\cdot\overline{A}+\overline{A}\cdot\,\underline{H}\big)\Big),
\end{aligned}
$$

where we used that $(\,{}^{(c)}\mathcal{D}\,\underline{H})\cdot\overline{B}=-\,\underline{H}(\,\underline{H}\cdot\overline{B})+O(\epsilon)$. By writing ${}^{(c)}\mathcal{D}\cdot\overline{B}=\mathcal{B}-2\,\underline{H}\cdot\overline{B}$, we obtain the second identity.

We now compute ${}^{(c)}\nabla_4\, {}^{(c)}\mathcal{D}\widehat{\otimes}\, {}^{(c)}\mathcal{D}\mathcal{B}_1$. Applying Lemma 4.2.2, (4.2.13), to $F = {}^{(c)}\mathcal{D}\mathcal{B}$ and $s = 1$ we have, up to quadratic terms,

$$[\, {}^{(c)}\nabla_4,\, {}^{(c)}\mathcal{D}\widehat{\otimes}]\, {}^{(c)}\mathcal{D}\mathcal{B} \;=\; -\frac{1}{2}\mathrm{tr}X\, {}^{(c)}\mathcal{D}\widehat{\otimes}\, {}^{(c)}\mathcal{D}\mathcal{B} + \underline{H}\widehat{\otimes}\, {}^{(c)}\nabla_4\, {}^{(c)}\mathcal{D}\mathcal{B},$$

and therefore we obtain, using the previous obtained identity,

$$\begin{aligned}
& {}^{(c)}\nabla_4\, {}^{(c)}\mathcal{D}\widehat{\otimes}\, {}^{(c)}\mathcal{D}\mathcal{B} \\
= \;& {}^{(c)}\mathcal{D}\widehat{\otimes}(\, {}^{(c)}\nabla_4\, {}^{(c)}\mathcal{D}\mathcal{B}) - \frac{1}{2}\mathrm{tr}X\, {}^{(c)}\mathcal{D}\widehat{\otimes}\, {}^{(c)}\mathcal{D}\mathcal{B} + \underline{H}\widehat{\otimes}\, {}^{(c)}\nabla_4\, {}^{(c)}\mathcal{D}\mathcal{B} \\
= \;& {}^{(c)}\mathcal{D}\widehat{\otimes}\Big[- 3\mathrm{tr}X\, {}^{(c)}\mathcal{D}\mathcal{B} + 5\mathrm{tr}X\, \underline{H}\mathcal{B} - 8\mathrm{tr}X\, \underline{H}(\, \underline{H}\cdot\overline{B}) \\
& +3P\, {}^{(c)}\mathcal{D}\, {}^{(c)}\mathcal{D}\cdot\overline{\Xi} - 6\,\underline{H}P\, {}^{(c)}\mathcal{D}\cdot\overline{\Xi} \\
& +\frac{1}{2}(\, {}^{(c)}\mathcal{D} + \,\underline{H})\big(\big(\, {}^{(c)}\mathcal{D} + 3\,\underline{H}\big)\cdot\big(\, {}^{(c)}\mathcal{D}\cdot\overline{A} + \overline{A}\cdot\,\underline{H}\big)\big)\Big] - \frac{1}{2}\mathrm{tr}X\, {}^{(c)}\mathcal{D}\widehat{\otimes}\, {}^{(c)}\mathcal{D}\mathcal{B}_1 \\
& +\underline{H}\widehat{\otimes}\Big[- 3\mathrm{tr}X\, {}^{(c)}\mathcal{D}\mathcal{B} + 5\mathrm{tr}X\, \underline{H}\mathcal{B} - 8\mathrm{tr}X\, \underline{H}(\, \underline{H}\cdot\overline{B}) \\
& +3P\, {}^{(c)}\mathcal{D}\, {}^{(c)}\mathcal{D}\cdot\overline{\Xi} - 6\,\underline{H}P\, {}^{(c)}\mathcal{D}\cdot\overline{\Xi} \\
& +\frac{1}{2}(\, {}^{(c)}\mathcal{D} + \,\underline{H})\big(\big(\, {}^{(c)}\mathcal{D} + 3\,\underline{H}\big)\cdot\big(\, {}^{(c)}\mathcal{D}\cdot\overline{A} + \overline{A}\cdot\,\underline{H}\big)\big)\Big].
\end{aligned}$$

By denoting

$$\mathrm{Expr}_1(A) \;:=\; \frac{1}{2}(\, {}^{(c)}\mathcal{D} + \,\underline{H})\widehat{\otimes}\Big((\, {}^{(c)}\mathcal{D} + \,\underline{H})\big(\big(\, {}^{(c)}\mathcal{D} + 3\,\underline{H}\big)\cdot\big(\, {}^{(c)}\mathcal{D}\cdot\overline{A} + \overline{A}\cdot\,\underline{H}\big)\big)\Big)$$

we obtain

$$\begin{aligned}
& {}^{(c)}\nabla_4\, {}^{(c)}\mathcal{D}\widehat{\otimes}\, {}^{(c)}\mathcal{D}\mathcal{B}_1 \\
= \;& -3\mathrm{tr}X\, {}^{(c)}\mathcal{D}\widehat{\otimes}\, {}^{(c)}\mathcal{D}\mathcal{B} - 3\, {}^{(c)}\mathcal{D}\mathrm{tr}X\widehat{\otimes}\, {}^{(c)}\mathcal{D}\mathcal{B} \\
& +5\, {}^{(c)}\mathcal{D}\mathrm{tr}X\widehat{\otimes}\, \underline{H}\mathcal{B} + 5\mathrm{tr}X\, {}^{(c)}\mathcal{D}\widehat{\otimes}\, \underline{H}\mathcal{B}_1 + 5\mathrm{tr}X\, \underline{H}\widehat{\otimes}\, {}^{(c)}\mathcal{D}\mathcal{B} \\
& {}^{(c)}\mathcal{D}\widehat{\otimes}\Big[- 8\mathrm{tr}X\, \underline{H}(\, \underline{H}\cdot\overline{B}) + 3P\, {}^{(c)}\mathcal{D}\, {}^{(c)}\mathcal{D}\cdot\overline{\Xi} - 6\,\underline{H}P\, {}^{(c)}\mathcal{D}\cdot\overline{\Xi}\Big] \\
& -\frac{1}{2}\mathrm{tr}X\, {}^{(c)}\mathcal{D}\widehat{\otimes}\, {}^{(c)}\mathcal{D}\mathcal{B} + \underline{H}\widehat{\otimes}\Big[- 3\mathrm{tr}X\, {}^{(c)}\mathcal{D}\mathcal{B} + 5\mathrm{tr}X\, \underline{H}\mathcal{B} - 8\mathrm{tr}X\, \underline{H}(\, \underline{H}\cdot\overline{B}) \\
& +3P\, {}^{(c)}\mathcal{D}\, {}^{(c)}\mathcal{D}\cdot\overline{\Xi} - 6\,\underline{H}P\, {}^{(c)}\mathcal{D}\cdot\overline{\Xi}\Big] + \mathrm{Expr}_1(A) \\
= \;& -\frac{7}{2}\mathrm{tr}X\, {}^{(c)}\mathcal{D}\widehat{\otimes}\, {}^{(c)}\mathcal{D}\mathcal{B} + 8\mathrm{tr}X\, \underline{H}\widehat{\otimes}\, {}^{(c)}\mathcal{D}\mathcal{B} - 10\mathrm{tr}X\, \underline{H}\widehat{\otimes}\, \underline{H}\mathcal{B} \\
& (\, {}^{(c)}\mathcal{D} + \,\underline{H})\widehat{\otimes}\Big[- 8\mathrm{tr}X\, \underline{H}(\, \underline{H}\cdot\overline{B}) + 3P\, {}^{(c)}\mathcal{D}\, {}^{(c)}\mathcal{D}\cdot\overline{\Xi} - 6\,\underline{H}P\, {}^{(c)}\mathcal{D}\cdot\overline{\Xi}\Big] \\
& +\mathrm{Expr}_1(A)
\end{aligned}$$

where we used ${}^{(c)}\mathcal{D}\widehat{\otimes}\,\underline{H} = -\,\underline{H}\widehat{\otimes}\,\underline{H} + O(\epsilon)$. This proves the lemma. □

D.7.2 The derivation of the Teukolsky-Starobinski identity

The Teukolsky-Starobinski identity is obtained by taking three ${}^{(c)}\nabla_4$ derivatives of appropriately rescalings of the Bianchi identity for $\underline{A}$, i.e.

$$ {}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A} = -{}^{(c)}\mathcal{D}\widehat{\otimes}\underline{B} - 4\,\underline{H}\widehat{\otimes}\underline{B} - 3P\widehat{\underline{X}}. \tag{D.7.1} $$

The second e_4 derivative of $\underline{A}$

Lemma D.7.6. *The quantity $\mathfrak{F} := -{}^{(c)}\nabla_4\underline{A} - \frac{1}{2}trX\underline{A} \in \mathfrak{s}_2(\mathbb{C})$ satisfies, modulo quadratic terms,*

$$ \begin{aligned} {}^{(c)}\nabla_4\mathfrak{F} + \frac{3}{2}trX\mathfrak{F} &= -{}^{(c)}\mathcal{D}\widehat{\otimes}\mathcal{A}_1 - 5\underline{H}\widehat{\otimes}\mathcal{A}_1 - 2trX\,\underline{H}\widehat{\otimes}\underline{B} \\ &\quad -\frac{3}{2}P\Big(trX\,\widehat{\underline{X}} + \overline{tr\underline{X}}\widehat{X} - 2\mathcal{A}_2\Big). \end{aligned} $$

Proof. We infer from (D.7.1) that

$$ \mathfrak{F} = {}^{(c)}\mathcal{D}\widehat{\otimes}\underline{B} + 4\,\underline{H}\widehat{\otimes}\underline{B} + 3P\widehat{\underline{X}}. $$

We compute

$$ {}^{(c)}\nabla_4\mathfrak{F} = {}^{(c)}\nabla_4\,{}^{(c)}\mathcal{D}\widehat{\otimes}\underline{B} + 4\,\underline{H}\widehat{\otimes}\,{}^{(c)}\nabla_4\underline{B} + 4\,{}^{(c)}\nabla_4\,\underline{H}\widehat{\otimes}\underline{B} + 3P\,{}^{(c)}\nabla_4\widehat{\underline{X}} + 3\,{}^{(c)}\nabla_4 P\widehat{\underline{X}}. $$

Applying Lemma 4.2.2, (4.2.13), to $F = \underline{B}$ and $s = -1$, we have, modulo quadratic terms,

$$ \begin{aligned} {}^{(c)}\nabla_4\mathfrak{F} &= {}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\nabla_4\underline{B} - \frac{1}{2}\mathrm{tr}X\left({}^{(c)}\mathcal{D}\widehat{\otimes}\underline{B} + 2\,\underline{H}\widehat{\otimes}\underline{B}\right) + \underline{H}\widehat{\otimes}\,{}^{(c)}\nabla_4\underline{B} \\ &\quad +4\,\underline{H}\widehat{\otimes}\,{}^{(c)}\nabla_4\underline{B} + 4\,{}^{(c)}\nabla_4\,\underline{H}\widehat{\otimes}\underline{B} + 3P\,{}^{(c)}\nabla_4\widehat{\underline{X}} + 3\,{}^{(c)}\nabla_4 P\widehat{\underline{X}} \\ &= {}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\nabla_4\underline{B} + 5\underline{H}\widehat{\otimes}\,{}^{(c)}\nabla_4\underline{B} - \frac{1}{2}\mathrm{tr}X\left({}^{(c)}\mathcal{D}\widehat{\otimes}\underline{B} + 2\,\underline{H}\widehat{\otimes}\underline{B}\right) + 4\,{}^{(c)}\nabla_4\,\underline{H}\widehat{\otimes}\underline{B} \\ &\quad +3P\,{}^{(c)}\nabla_4\widehat{\underline{X}} + 3\,{}^{(c)}\nabla_4 P\widehat{\underline{X}}. \end{aligned} $$

Using the Bianchi identity

$$ {}^{(c)}\nabla_4\underline{B} = -\mathrm{tr}X\underline{B} - \left({}^{(c)}\mathcal{D}P + 3P\,\underline{H}\right) = -\mathrm{tr}X\underline{B} - \mathcal{A}_1 \tag{D.7.2} $$

we obtain

$$
\begin{aligned}
{}^{(c)}\nabla_4\mathfrak{F} &= {}^{(c)}\mathcal{D}\widehat{\otimes}\big(-\mathrm{tr}X\underline{B}-\mathcal{A}_1\big)+5\underline{H}\widehat{\otimes}\big(-\mathrm{tr}X\underline{B}-\mathcal{A}_1\big)\\
&\quad -\frac{1}{2}\mathrm{tr}X\left({}^{(c)}\mathcal{D}\widehat{\otimes}\underline{B}+2\,\underline{H}\widehat{\otimes}\underline{B}\right)+4\,{}^{(c)}\nabla_4\,\underline{H}\widehat{\otimes}\underline{B}+3P\,{}^{(c)}\nabla_4\widehat{\underline{X}}+3\,{}^{(c)}\nabla_4P\widehat{\underline{X}}\\
&= -\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}\underline{B}-{}^{(c)}\mathcal{D}\mathrm{tr}X\widehat{\otimes}\underline{B}-{}^{(c)}\mathcal{D}\widehat{\otimes}\mathcal{A}_1+5\underline{H}\widehat{\otimes}\big(-\mathrm{tr}X\underline{B}-\mathcal{A}_1\big)\\
&\quad -\frac{1}{2}\mathrm{tr}X\left({}^{(c)}\mathcal{D}\widehat{\otimes}\underline{B}+2\,\underline{H}\widehat{\otimes}\underline{B}\right)+4\,{}^{(c)}\nabla_4\,\underline{H}\widehat{\otimes}\underline{B}+3P\,{}^{(c)}\nabla_4\widehat{\underline{X}}+3\,{}^{(c)}\nabla_4P\widehat{\underline{X}}\\
&= -\frac{3}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}\underline{B}-6\mathrm{tr}X\,\underline{H}\widehat{\otimes}\underline{B}-{}^{(c)}\mathcal{D}\widehat{\otimes}\mathcal{A}_1-5\underline{H}\widehat{\otimes}\mathcal{A}_1\\
&\quad +\big(4\,{}^{(c)}\nabla_4\,\underline{H}-{}^{(c)}\mathcal{D}\mathrm{tr}X\big)\widehat{\otimes}\underline{B}+3P\,{}^{(c)}\nabla_4\widehat{\underline{X}}+3\,{}^{(c)}\nabla_4P\widehat{\underline{X}}.
\end{aligned}
$$

By writing ${}^{(c)}\mathcal{D}\widehat{\otimes}\underline{B}+4\,\underline{H}\widehat{\otimes}\underline{B}=\mathfrak{F}-3P\widehat{\underline{X}}$ we obtain

$$
\begin{aligned}
{}^{(c)}\nabla_4\mathfrak{F} &= -\frac{3}{2}\mathrm{tr}X\big(\mathfrak{F}-3P\widehat{\underline{X}}\big)-{}^{(c)}\mathcal{D}\widehat{\otimes}\mathcal{A}_1-5\underline{H}\widehat{\otimes}\mathcal{A}_1\\
&\quad +\big(4\,{}^{(c)}\nabla_4\,\underline{H}-{}^{(c)}\mathcal{D}\mathrm{tr}X\big)\widehat{\otimes}\underline{B}+3P\,{}^{(c)}\nabla_4\widehat{\underline{X}}+3\big(-\frac{3}{2}\mathrm{tr}XP\big)\widehat{\underline{X}},
\end{aligned}
$$

where we used ${}^{(c)}\nabla_4P=-\frac{3}{2}\mathrm{tr}XP+O(\epsilon)$, and therefore

$$
\begin{aligned}
{}^{(c)}\nabla_4\mathfrak{F}+\frac{3}{2}\mathrm{tr}X\mathfrak{F} &= -{}^{(c)}\mathcal{D}\widehat{\otimes}\mathcal{A}_1-5\underline{H}\widehat{\otimes}\mathcal{A}_1\\
&\quad +\big(4\,{}^{(c)}\nabla_4\,\underline{H}-{}^{(c)}\mathcal{D}\mathrm{tr}X\big)\widehat{\otimes}\underline{B}+3P\,{}^{(c)}\nabla_4\widehat{\underline{X}}.
\end{aligned}
$$

Using the Ricci identity

$$
{}^{(c)}\nabla_4\widehat{\underline{X}}+\frac{1}{2}\mathrm{tr}X\,\widehat{\underline{X}} = {}^{(c)}\mathcal{D}\widehat{\otimes}\,\underline{H}+\,\underline{H}\widehat{\otimes}\,\underline{H}-\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\widehat{X}=\mathcal{A}_2-\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\widehat{X},
$$

we finally have

$$
\begin{aligned}
{}^{(c)}\nabla_4\mathfrak{F}+\frac{3}{2}\mathrm{tr}X\mathfrak{F} &= -{}^{(c)}\mathcal{D}\widehat{\otimes}\mathcal{A}_1-5\underline{H}\widehat{\otimes}\mathcal{A}_1+3P\mathcal{A}_2\\
&\quad +\big(4\,{}^{(c)}\nabla_4\,\underline{H}-{}^{(c)}\mathcal{D}\mathrm{tr}X\big)\widehat{\otimes}\underline{B}-\frac{3}{2}P\big(\mathrm{tr}X\,\widehat{\underline{X}}+\overline{\mathrm{tr}\underline{X}}\widehat{X}\big).
\end{aligned}
$$

Using that

$$
4\,{}^{(c)}\nabla_4\,\underline{H}-{}^{(c)}\mathcal{D}\mathrm{tr}X = 4(-\mathrm{tr}X\,\underline{H})-(-2\mathrm{tr}X\,\underline{H})+O(\epsilon)=-2\mathrm{tr}X\,\underline{H}+O(\epsilon),
$$

we obtain the stated. □

The third e_4 derivative of $\underline{A}$

Lemma D.7.7. *The quantity* $\mathfrak{G} := {}^{(c)}\nabla_4\mathfrak{F} + \frac{3}{2}trX\mathfrak{F} \in \mathfrak{s}_2(\mathbb{C})$ *satisfies, modulo quadratic terms,*

$$\begin{aligned}
{}^{(c)}\nabla_4\mathfrak{G} + \frac{5}{2}trX\mathfrak{G} &= -\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathcal{B} - 5\,\underline{H}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathcal{B} - 10(\underline{H}\widehat{\otimes}\,\underline{H})\mathcal{B} + 3trX\,\underline{H}\widehat{\otimes}\mathcal{A}_1\\
&+\frac{3}{2}P\overline{tr\underline{X}}A + \frac{3}{2}P\Big[\mathcal{A}_4 - 2B\widehat{\otimes}\,\underline{H} - {}^{(c)}\mathcal{D}\widehat{\otimes}\big(\overline{H}\cdot\widehat{X}\big)\\
&+\big(\overline{trX\,tr\underline{X}} - 2\,\overline{{}^{(c)}\mathcal{D}}\cdot\,\underline{H} - 2P\big)\widehat{X} - 3\,\underline{H}\widehat{\otimes}\big(\overline{H}\cdot\widehat{X}\big)\Big].
\end{aligned}$$

Proof. We infer from Lemma D.7.6 that

$$\mathfrak{G} = -\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathcal{A}_1 - 5\underline{H}\widehat{\otimes}\mathcal{A}_1 - 2\mathrm{tr}X\,\underline{H}\widehat{\otimes}\underline{B} - \frac{3}{2}P\big(\mathrm{tr}X\,\widehat{\underline{X}} + \overline{\mathrm{tr}\underline{X}}\widehat{X} - 2\mathcal{A}_2\big).$$

We compute

$$\begin{aligned}
{}^{(c)}\nabla_4\mathfrak{G} &= -\,{}^{(c)}\nabla_4\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathcal{A}_1 - 5\underline{H}\widehat{\otimes}\,{}^{(c)}\nabla_4\mathcal{A}_1 - 5\,{}^{(c)}\nabla_4\underline{H}\widehat{\otimes}\mathcal{A}_1\\
&-2\mathrm{tr}X\,\underline{H}\widehat{\otimes}\,{}^{(c)}\nabla_4\underline{B} - 2\mathrm{tr}X\,{}^{(c)}\nabla_4\,\underline{H}\widehat{\otimes}\underline{B} - 2\,{}^{(c)}\nabla_4\mathrm{tr}X\,\underline{H}\widehat{\otimes}\underline{B}\\
&-\frac{3}{2}P\,{}^{(c)}\nabla_4\big(\mathrm{tr}X\,\widehat{\underline{X}} + \overline{\mathrm{tr}\underline{X}}\widehat{X} - 2\mathcal{A}_2\big) - \frac{3}{2}\,{}^{(c)}\nabla_4P\big(\mathrm{tr}X\,\widehat{\underline{X}} + \overline{\mathrm{tr}\underline{X}}\widehat{X} - 2\mathcal{A}_2\big).
\end{aligned}$$

Using that ${}^{(c)}\nabla_4P = -\frac{3}{2}\mathrm{tr}XP + O(\epsilon)$, ${}^{(c)}\nabla_4\mathrm{tr}X = -\frac{1}{2}(\mathrm{tr}X)^2 + O(\epsilon)$, ${}^{(c)}\nabla_4\,\underline{H} = -\mathrm{tr}X\,\underline{H} + O(\epsilon)$ and ${}^{(c)}\nabla_4\underline{B} = -\mathrm{tr}X\underline{B} - \mathcal{A}_1$, we obtain

$$\begin{aligned}
{}^{(c)}\nabla_4\mathfrak{G} &= -\,{}^{(c)}\nabla_4\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathcal{A}_1 - 5\underline{H}\widehat{\otimes}\,{}^{(c)}\nabla_4\mathcal{A}_1 + 7\mathrm{tr}X\,\underline{H}\widehat{\otimes}\mathcal{A}_1 + 5(\mathrm{tr}X)^2\,\underline{H}\widehat{\otimes}\underline{B}\\
&-\frac{3}{2}P\,{}^{(c)}\nabla_4\big(\mathrm{tr}X\,\widehat{\underline{X}} + \overline{\mathrm{tr}\underline{X}}\widehat{X} - 2\mathcal{A}_2\big) + \frac{9}{4}\mathrm{tr}XP\big(\mathrm{tr}X\,\widehat{\underline{X}} + \overline{\mathrm{tr}\underline{X}}\widehat{X} - 2\mathcal{A}_2\big).
\end{aligned}$$

Using Lemma D.7.2 for the derivatives of $\mathcal{A}_1$, we obtain

$$
\begin{aligned}
& {}^{(c)}\nabla_4\mathfrak{G}\\
= & -\Big[-\frac{5}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathcal{A}_1+\frac{3}{2}\mathrm{tr}X\,\underline{H}\widehat{\otimes}\mathcal{A}_1+\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathcal{B}+\frac{5}{2}\,\underline{H}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathcal{B}\\
& +{}^{(c)}\mathcal{D}\widehat{\otimes}\Big(\frac{3}{2}P\left(\overline{H}\cdot\widehat{X}-\overline{\mathrm{tr}\underline{X}}\Xi+\mathcal{A}_3\right)\Big)+\underline{H}\widehat{\otimes}\Big(\frac{3}{2}P\left(\overline{H}\cdot\widehat{X}-\overline{\mathrm{tr}\underline{X}}\Xi+\mathcal{A}_3\right)\Big)\Big]\\
& -5\underline{H}\widehat{\otimes}\Big[-2\mathrm{tr}X\mathcal{A}_1+\frac{1}{2}\,{}^{(c)}\mathcal{D}\mathcal{B}+2\,\underline{H}\mathcal{B}+\frac{3}{2}P\left(\overline{H}\cdot\widehat{X}-\overline{\mathrm{tr}\underline{X}}\Xi+\mathcal{A}_3\right)\Big]\\
& +7\mathrm{tr}X\,\underline{H}\widehat{\otimes}\mathcal{A}_1+5(\mathrm{tr}X)^2\,\underline{H}\widehat{\otimes}\underline{B}-\frac{3}{2}P\,{}^{(c)}\nabla_4\big(\mathrm{tr}X\,\widehat{\underline{X}}+\overline{\mathrm{tr}\underline{X}}\widehat{X}-2\mathcal{A}_2\big)\\
& +\frac{9}{4}\mathrm{tr}XP\big(\mathrm{tr}X\,\widehat{\underline{X}}+\overline{\mathrm{tr}\underline{X}}\widehat{X}-2\mathcal{A}_2\big)\\
= & \frac{5}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathcal{A}_1+\frac{31}{2}\mathrm{tr}X\,\underline{H}\widehat{\otimes}\mathcal{A}_1+5(\mathrm{tr}X)^2\,\underline{H}\widehat{\otimes}\underline{B}\\
& -\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathcal{B}-5\,\underline{H}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathcal{B}-10(\underline{H}\widehat{\otimes}\,\underline{H})\mathcal{B}\\
& -{}^{(c)}\mathcal{D}\widehat{\otimes}\Big(\frac{3}{2}P\left(\overline{H}\cdot\widehat{X}-\overline{\mathrm{tr}\underline{X}}\Xi+\mathcal{A}_3\right)\Big)-6\underline{H}\widehat{\otimes}\Big(\frac{3}{2}P\left(\overline{H}\cdot\widehat{X}-\overline{\mathrm{tr}\underline{X}}\Xi+\mathcal{A}_3\right)\Big)\\
& -\frac{3}{2}P\Big[{}^{(c)}\nabla_4\big(\mathrm{tr}X\,\widehat{\underline{X}}+\overline{\mathrm{tr}\underline{X}}\widehat{X}\big)-2\,{}^{(c)}\nabla_4\mathcal{A}_2-\frac{3}{2}\mathrm{tr}X\big(\mathrm{tr}X\,\widehat{\underline{X}}+\overline{\mathrm{tr}\underline{X}}\widehat{X}-2\mathcal{A}_2\big)\Big].
\end{aligned}
$$

Writing

$$
{}^{(c)}\mathcal{D}\widehat{\otimes}\mathcal{A}_1+2\mathrm{tr}X\,\underline{H}\widehat{\otimes}\underline{B}=-\mathfrak{G}-5\underline{H}\widehat{\otimes}\mathcal{A}_1-\frac{3}{2}P\big(\mathrm{tr}X\,\widehat{\underline{X}}+\overline{\mathrm{tr}\underline{X}}\widehat{X}-2\mathcal{A}_2\big),
$$

we obtain

$$
\begin{aligned}
& {}^{(c)}\nabla_4\mathfrak{G}+\frac{5}{2}\mathrm{tr}X\mathfrak{G}\\
= & -\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathcal{B}-5\,\underline{H}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathcal{B}-10(\underline{H}\widehat{\otimes}\,\underline{H})\mathcal{B}+3\mathrm{tr}X\,\underline{H}\widehat{\otimes}\mathcal{A}_1\\
& -{}^{(c)}\mathcal{D}\widehat{\otimes}\Big(\frac{3}{2}P\left(\overline{H}\cdot\widehat{X}-\overline{\mathrm{tr}\underline{X}}\Xi+\mathcal{A}_3\right)\Big)-6\underline{H}\widehat{\otimes}\Big(\frac{3}{2}P\left(\overline{H}\cdot\widehat{X}-\overline{\mathrm{tr}\underline{X}}\Xi+\mathcal{A}_3\right)\Big)\\
& -\frac{3}{2}P\Big[{}^{(c)}\nabla_4\big(\mathrm{tr}X\,\widehat{\underline{X}}+\overline{\mathrm{tr}\underline{X}}\widehat{X}\big)-2\,{}^{(c)}\nabla_4\mathcal{A}_2+\mathrm{tr}X\big(\mathrm{tr}X\,\widehat{\underline{X}}+\overline{\mathrm{tr}\underline{X}}\widehat{X}-2\mathcal{A}_2\big)\Big].
\end{aligned}
$$

We now write for the second line, using that ${}^{(c)}\mathcal{D}P=-3P\,\underline{H}+O(\epsilon)$,

$$
\begin{aligned}
&-{}^{(c)}\mathcal{D}\widehat{\otimes}\Big(\frac{3}{2}P\left(\overline{H}\cdot\widehat{X}-\overline{\mathrm{tr}\underline{X}}\Xi+\mathcal{A}_3\right)\Big)-6\underline{H}\widehat{\otimes}\Big(\frac{3}{2}P\left(\overline{H}\cdot\widehat{X}-\overline{\mathrm{tr}\underline{X}}\Xi+\mathcal{A}_3\right)\Big)\\
=\;&-\frac{3}{2}P\,{}^{(c)}\mathcal{D}\widehat{\otimes}\big(\overline{H}\cdot\widehat{X}-\overline{\mathrm{tr}\underline{X}}\Xi+\mathcal{A}_3\big)-\frac{3}{2}(-3P\,\underline{H})\widehat{\otimes}\big(\overline{H}\cdot\widehat{X}-\overline{\mathrm{tr}\underline{X}}\Xi+\mathcal{A}_3\big)\\
&-6\underline{H}\widehat{\otimes}\Big(\frac{3}{2}P\left(\overline{H}\cdot\widehat{X}-\overline{\mathrm{tr}\underline{X}}\Xi+\mathcal{A}_3\right)\Big)\\
=\;&-\frac{3}{2}P\Big[{}^{(c)}\mathcal{D}\widehat{\otimes}\big(\overline{H}\cdot\widehat{X}-\overline{\mathrm{tr}\underline{X}}\Xi+\mathcal{A}_3\big)+3\,\underline{H}\widehat{\otimes}\big(\overline{H}\cdot\widehat{X}-\overline{\mathrm{tr}\underline{X}}\Xi+\mathcal{A}_3\big)\Big]\\
=\;&-\frac{3}{2}P\Big[{}^{(c)}\mathcal{D}\widehat{\otimes}\mathcal{A}_3+3\,\underline{H}\widehat{\otimes}\mathcal{A}_3-\overline{\mathrm{tr}\underline{X}}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\Xi-{}^{(c)}\mathcal{D}\overline{\mathrm{tr}\underline{X}}\widehat{\otimes}\Xi-3\overline{\mathrm{tr}\underline{X}}\,\underline{H}\widehat{\otimes}\Xi\\
&+{}^{(c)}\mathcal{D}\widehat{\otimes}\big(\overline{H}\cdot\widehat{X}\big)+3\,\underline{H}\widehat{\otimes}\big(\overline{H}\cdot\widehat{X}\big)\Big]\\
=\;&-\frac{3}{2}P\Big[{}^{(c)}\mathcal{D}\widehat{\otimes}\mathcal{A}_3+3\,\underline{H}\widehat{\otimes}\mathcal{A}_3-\overline{\mathrm{tr}\underline{X}}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\Xi-(\mathrm{tr}\underline{X}+2\overline{\mathrm{tr}\underline{X}})\,\underline{H}\widehat{\otimes}\Xi\\
&+{}^{(c)}\mathcal{D}\widehat{\otimes}\big(\overline{H}\cdot\widehat{X}\big)+3\,\underline{H}\widehat{\otimes}\big(\overline{H}\cdot\widehat{X}\big)\Big]
\end{aligned}
$$

where we used ${}^{(c)}\mathcal{D}\overline{\mathrm{tr}\underline{X}}=(\mathrm{tr}\underline{X}-\overline{\mathrm{tr}\underline{X}})\,\underline{H}+O(\epsilon)$.

Also, we have

$$
\begin{aligned}
{}^{(c)}\nabla_4\left(\mathrm{tr}X\,\widehat{\underline{X}}+\overline{\mathrm{tr}\underline{X}}\widehat{X}\right)\;=\;&-\frac{1}{2}(\mathrm{tr}X)^2\widehat{\underline{X}}+\overline{{}^{(c)}\nabla_4\mathrm{tr}\underline{X}}\widehat{X}\\
&+\mathrm{tr}X\left(-\frac{1}{2}\mathrm{tr}X\,\widehat{\underline{X}}+\mathcal{A}_2-\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\widehat{X}\right)\\
&+\overline{\mathrm{tr}\underline{X}}\left(-\frac{1}{2}(\mathrm{tr}X+\overline{\mathrm{tr}X})\widehat{X}+{}^{(c)}\mathcal{D}\widehat{\otimes}\Xi+\Xi\widehat{\otimes}(\,\underline{H}+H)-A\right)\\
=\;&-(\mathrm{tr}X)^2\widehat{\underline{X}}+\left(\overline{{}^{(c)}\nabla_4\mathrm{tr}\underline{X}}-\left(\mathrm{tr}X+\frac{1}{2}\overline{\mathrm{tr}X}\right)\overline{\mathrm{tr}\underline{X}}\right)\widehat{X}\\
&-\overline{\mathrm{tr}\underline{X}}A+\mathrm{tr}X\mathcal{A}_2+\overline{\mathrm{tr}\underline{X}}\left({}^{(c)}\mathcal{D}\widehat{\otimes}\Xi+\Xi\widehat{\otimes}(\,\underline{H}+H)\right).
\end{aligned}
$$

By putting all together we obtain

$$
\begin{aligned}
&{}^{(c)}\nabla_4\mathfrak{G}+\frac{5}{2}\mathrm{tr}X\mathfrak{G}\\
=\;&-\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathcal{B}_1-5\,\underline{H}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathcal{B}_1-10(\underline{H}\widehat{\otimes}\,\underline{H})\mathcal{B}_1+3\mathrm{tr}X\,\underline{H}\widehat{\otimes}\mathcal{A}_1+\frac{3}{2}P\overline{\mathrm{tr}\underline{X}}A\\
&-\frac{3}{2}P\Big[{}^{(c)}\mathcal{D}\widehat{\otimes}\mathcal{A}_3+3\,\underline{H}\widehat{\otimes}\mathcal{A}_3-2\,{}^{(c)}\nabla_4\mathcal{A}_2-\mathrm{tr}X\mathcal{A}_2\\
&-(\mathrm{tr}\underline{X}+\overline{\mathrm{tr}\underline{X}})\,\underline{H}\widehat{\otimes}\Xi+\overline{\mathrm{tr}\underline{X}}H\widehat{\otimes}\Xi\\
&+{}^{(c)}\mathcal{D}\widehat{\otimes}\big(\overline{H}\cdot\widehat{X}\big)+\big(\overline{{}^{(c)}\nabla_4\mathrm{tr}\underline{X}}-\frac{1}{2}\overline{\mathrm{tr}X\mathrm{tr}\underline{X}}\big)\widehat{X}+3\,\underline{H}\widehat{\otimes}\big(\overline{H}\cdot\widehat{X}\big)\Big].
\end{aligned}
$$

Finally, using Lemma D.7.3, we obtain

$$
\begin{aligned}
& {}^{(c)}\nabla_4\mathfrak{G} + \frac{5}{2}\mathrm{tr}X\mathfrak{G} \\
= \quad & -\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathcal{B}_1 - 5\,\underline{H}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathcal{B}_1 - 10(\underline{H}\widehat{\otimes}\,\underline{H})\mathcal{B}_1 + 3\mathrm{tr}X\,\underline{H}\widehat{\otimes}\mathcal{A}_1 + \frac{3}{2}P\overline{\mathrm{tr}\underline{X}}A \\
& -\frac{3}{2}P\Big[- \mathcal{A}_4 + 2B\widehat{\otimes}\,\underline{H} + \widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}\,\underline{H} + {}^{(c)}\mathcal{D}\widehat{\otimes}\big(\overline{H}\cdot\widehat{X}\big) \\
& +\big(\overline{{}^{(c)}\nabla_4\mathrm{tr}\underline{X}} - \frac{1}{2}\overline{\mathrm{tr}X\mathrm{tr}\underline{X}} - \overline{\underline{H}}\cdot\,\underline{H}\big)\widehat{X} + 3\,\underline{H}\widehat{\otimes}\big(\overline{H}\cdot\widehat{X}\big)\Big].
\end{aligned}
$$

Finally using (2.4.4) to write $\widehat{X}\cdot\overline{\mathcal{D}}\,\underline{H} = \widehat{X}(\overline{\mathcal{D}}\cdot\,\underline{H})$, and that

$$
{}^{(c)}\nabla_4\overline{\mathrm{tr}\underline{X}} \quad = \quad -\frac{1}{2}\overline{\mathrm{tr}X\mathrm{tr}\underline{X}} + \overline{{}^{(c)}\mathcal{D}}\cdot\,\underline{H} + \,\underline{H}\cdot\,\overline{\underline{H}} + 2P + O(\epsilon^2)
$$

we conclude the lemma. □

The fourth e_4 derivative of $\underline{A}$

Define

$$
\mathfrak{H} \quad := \quad {}^{(c)}\nabla_4\mathfrak{G} + \frac{5}{2}\mathrm{tr}X\mathfrak{G}.
$$

We infer from Lemma D.7.7,

$$
\mathfrak{H} \quad = \quad \mathfrak{H}_1 + \frac{3}{2}P\mathfrak{H}_2 + \frac{3}{2}P\overline{\mathrm{tr}\underline{X}}A.
$$

with

$$
\begin{aligned}
\mathfrak{H}_1 \quad &= \quad -\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathcal{B} - 5\,\underline{H}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathcal{B} - 10(\underline{H}\widehat{\otimes}\,\underline{H})\mathcal{B} + 3\mathrm{tr}X\,\underline{H}\widehat{\otimes}\mathcal{A}_1 \\
\mathfrak{H}_2 \quad &= \quad \mathcal{A}_4 - 2B\widehat{\otimes}\,\underline{H} - {}^{(c)}\mathcal{D}\widehat{\otimes}\big(\overline{H}\cdot\widehat{X}\big) + \big(\overline{\mathrm{tr}X\mathrm{tr}\underline{X}} - 2\,\overline{{}^{(c)}\mathcal{D}}\cdot\,\underline{H} - 2P\big)\widehat{X} - 3\,\underline{H}\widehat{\otimes}\big(\overline{H}\cdot\widehat{X}\big).
\end{aligned}
$$

This gives, using that ${}^{(c)}\nabla_4 P = -\frac{3}{2}\mathrm{tr}XP$,

$$
{}^{(c)}\nabla_4\mathfrak{H} \quad = \quad {}^{(c)}\nabla_4\mathfrak{H}_1 + \frac{3}{2}P\big[\,{}^{(c)}\nabla_4\mathfrak{H}_2 - \frac{3}{2}\mathrm{tr}X\mathfrak{H}_2\big] + {}^{(c)}\nabla_4\Big(\frac{3}{2}P\overline{\mathrm{tr}\underline{X}}A\Big). \qquad \text{(D.7.3)}
$$

Here we compute ${}^{(c)}\nabla_4\mathfrak{H}_1$.

Lemma D.7.8. *We have, modulo quadratic terms,*

$$\begin{aligned}
{}^{(c)}\nabla_4\mathfrak{H}_1 + \frac{7}{2}trX\mathfrak{H}_1 &= \frac{3}{2}P\Big[- {}^{(c)}\mathcal{D}\widehat{\otimes}({}^{(c)}\mathcal{D}\,{}^{(c)}\mathcal{D}\cdot\Xi) - 6\,\underline{H}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\,{}^{(c)}\mathcal{D}\cdot\Xi\\
&\quad -6\,\underline{H}\widehat{\otimes}\,\underline{H}\;{}^{(c)}\mathcal{D}\cdot\Xi - 3trX\overline{tr\underline{X}}\,\underline{H}\widehat{\otimes}\Xi\\
&\quad +3trX\,\underline{H}\widehat{\otimes}(\overline{H}\cdot\widehat{X}) + 3trX\,\underline{H}\widehat{\otimes}\mathcal{A}_3\Big] + Expr_2(A)
\end{aligned}$$

where

$$\begin{aligned}
Expr_2(A) &:= -\frac{1}{4}({}^{(c)}\mathcal{D} + 11\,\underline{H})\widehat{\otimes}\Big(({}^{(c)}\mathcal{D} + \,\underline{H})\big(({}^{(c)}\mathcal{D} + 3\,\underline{H})\cdot\big({}^{(c)}\mathcal{D}\cdot\overline{A} + \overline{A}\cdot\,\underline{H}\big)\big)\Big)\\
&\quad -5(\underline{H}\widehat{\otimes}\,\underline{H})\big({}^{(c)}\mathcal{D} + 3\,\underline{H}\big)\cdot\big({}^{(c)}\mathcal{D}\cdot\overline{A} + \overline{A}\cdot\,\underline{H}\big).
\end{aligned}$$

Proof. From the definition of $\mathfrak{H}_1$ we obtain

$$\begin{aligned}
{}^{(c)}\nabla_4\mathfrak{H}_1 &= -\frac{1}{2}\,{}^{(c)}\nabla_4\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathcal{B} - 5\,\underline{H}\widehat{\otimes}\,{}^{(c)}\nabla_4\,{}^{(c)}\mathcal{D}\mathcal{B} - 5\,{}^{(c)}\nabla_4\,\underline{H}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathcal{B}\\
&\quad -10(\underline{H}\widehat{\otimes}\,\underline{H})\,{}^{(c)}\nabla_4\mathcal{B} - 20({}^{(c)}\nabla_4\underline{H}\widehat{\otimes}\,\underline{H})\mathcal{B}\\
&\quad +3\mathrm{tr}X\,\underline{H}\widehat{\otimes}\,{}^{(c)}\nabla_4\mathcal{A}_1 + 3\mathrm{tr}X\,{}^{(c)}\nabla_4\,\underline{H}\widehat{\otimes}\mathcal{A}_1 + 3\,{}^{(c)}\nabla_4\mathrm{tr}X\,\underline{H}\widehat{\otimes}\mathcal{A}_1\\
&= -\frac{1}{2}\,{}^{(c)}\nabla_4\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathcal{B} - 5\,\underline{H}\widehat{\otimes}\,{}^{(c)}\nabla_4\,{}^{(c)}\mathcal{D}\mathcal{B} + 5\mathrm{tr}X\,\underline{H}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathcal{B}\\
&\quad -10(\underline{H}\widehat{\otimes}\,\underline{H})\,{}^{(c)}\nabla_4\mathcal{B} + 20\mathrm{tr}X(\,\underline{H}\widehat{\otimes}\,\underline{H})\mathcal{B} + 3\mathrm{tr}X\,\underline{H}\widehat{\otimes}\,{}^{(c)}\nabla_4\mathcal{A}_1 - \frac{9}{2}(\mathrm{tr}X)^2\,\underline{H}\widehat{\otimes}\mathcal{A}_1
\end{aligned}$$

where we used ${}^{(c)}\nabla_4\,\underline{H} = -\mathrm{tr}X\,\underline{H} + O(\epsilon)$ and ${}^{(c)}\nabla_4\mathrm{tr}X = -\frac{1}{2}(\mathrm{tr}X)^2 + O(\epsilon)$.

Using Lemma D.7.5 and Lemma D.7.2, we obtain

$$\begin{aligned}
{}^{(c)}\nabla_4\mathfrak{H}_1 &= \frac{7}{4}\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathcal{B} - 4\mathrm{tr}X\,\underline{H}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathcal{B} + 5\mathrm{tr}X\,\underline{H}\widehat{\otimes}\,\underline{H}\mathcal{B}\\
&\quad -\frac{1}{2}({}^{(c)}\mathcal{D} + \,\underline{H})\widehat{\otimes}\Big[- 8\mathrm{tr}X\,\underline{H}(\,\underline{H}\cdot\overline{B}) + 3P\;{}^{(c)}\mathcal{D}\,{}^{(c)}\mathcal{D}\cdot\Xi - 6\,\underline{H}P\;{}^{(c)}\mathcal{D}\cdot\Xi\Big]\\
&\quad -5\,\underline{H}\widehat{\otimes}\Big[- 3\mathrm{tr}X\,{}^{(c)}\mathcal{D}\mathcal{B} + 5\mathrm{tr}X\,\underline{H}\mathcal{B} - 8\mathrm{tr}X\,\underline{H}(\,\underline{H}\cdot\overline{B})\\
&\quad +3P\;{}^{(c)}\mathcal{D}\,{}^{(c)}\mathcal{D}\cdot\Xi - 6\,\underline{H}P\;{}^{(c)}\mathcal{D}\cdot\Xi\Big] + 5\mathrm{tr}X\,\underline{H}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathcal{B}\\
&\quad -10(\underline{H}\widehat{\otimes}\,\underline{H})\Big[- \frac{5}{2}\mathrm{tr}X\mathcal{B} + 2\mathrm{tr}X\,\underline{H}\cdot\overline{B} + 3P\;{}^{(c)}\mathcal{D}\cdot\Xi\Big] + 20\mathrm{tr}X(\,\underline{H}\widehat{\otimes}\,\underline{H})\mathcal{B}\\
&\quad +3\mathrm{tr}X\,\underline{H}\widehat{\otimes}\Big[- 2\mathrm{tr}X\mathcal{A}_1 + \frac{1}{2}\,{}^{(c)}\mathcal{D}\mathcal{B} + 2\,\underline{H}\mathcal{B} + \frac{3}{2}P\left(\overline{H}\cdot\widehat{X} - \overline{\mathrm{tr}\underline{X}}\Xi + \mathcal{A}_3\right)\Big]\\
&\quad -\frac{9}{2}(\mathrm{tr}X)^2\,\underline{H}\widehat{\otimes}\mathcal{A}_1 + \mathrm{Expr}_2(A)
\end{aligned}$$

where

$$\begin{aligned}\mathrm{Expr}_2(A) \;:=\;& -\frac{1}{4}\big({}^{(c)}\mathcal{D}+11\,\underline{H}\big)\widehat{\otimes}\Big(\big({}^{(c)}\mathcal{D}+\underline{H}\big)\Big(\big({}^{(c)}\mathcal{D}+3\,\underline{H}\big)\cdot\big({}^{(c)}\mathcal{D}\cdot\overline{A}+\overline{A}\cdot\underline{H}\big)\Big)\Big)\\ & -5(\underline{H}\widehat{\otimes}\underline{H})\big({}^{(c)}\mathcal{D}+3\,\underline{H}\big)\cdot\big({}^{(c)}\mathcal{D}\cdot\overline{A}+\overline{A}\cdot\underline{H}\big).\end{aligned}$$

We simplify the above to

$$\begin{aligned}{}^{(c)}\nabla_4\mathfrak{H}_1 \;=\;& \frac{7}{4}\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathcal{B}+\frac{35}{2}\mathrm{tr}X\,\underline{H}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathcal{B}+31\mathrm{tr}X\,\underline{H}\widehat{\otimes}\,\underline{H}\mathcal{B}-\frac{21}{2}(\mathrm{tr}X)^2\,\underline{H}\widehat{\otimes}\mathcal{A}_1\\ & -\frac{1}{2}\big({}^{(c)}\mathcal{D}+11\,\underline{H}\big)\widehat{\otimes}\Big[-8\mathrm{tr}X\,\underline{H}(\underline{H}\cdot\overline{B})+3P\,{}^{(c)}\mathcal{D}\,{}^{(c)}\mathcal{D}\cdot\Xi-6\,\underline{H}P\,{}^{(c)}\mathcal{D}\cdot\Xi\Big]\\ & -10(\underline{H}\widehat{\otimes}\underline{H})\Big[2\mathrm{tr}X\,\underline{H}\cdot\overline{B}+3P\,{}^{(c)}\mathcal{D}\cdot\Xi\Big]\\ & +\frac{3}{2}P3\mathrm{tr}X\,\underline{H}\widehat{\otimes}\Big[\overline{H}\cdot\widehat{X}-\overline{\mathrm{tr}\underline{X}}\Xi+\mathcal{A}_3\Big]+\mathrm{Expr}_2(A).\end{aligned}$$

By writing $\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathcal{B}=-\mathfrak{H}_1-5\,\underline{H}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\mathcal{B}-10(\underline{H}\widehat{\otimes}\underline{H})\mathcal{B}+3\mathrm{tr}X\,\underline{H}\widehat{\otimes}\mathcal{A}_1$, and reorganizing the above we obtain

$$\begin{aligned}{}^{(c)}\nabla_4\mathfrak{H}_1+\frac{7}{2}\mathrm{tr}X\mathfrak{H}_1 \;=\;& -4\mathrm{tr}X\,\underline{H}\widehat{\otimes}\,\underline{H}\mathcal{B}\\ & +4\big({}^{(c)}\mathcal{D}+11\,\underline{H}\big)\widehat{\otimes}\Big[\mathrm{tr}X\,\underline{H}(\underline{H}\cdot\overline{B})\Big]-20\mathrm{tr}X(\underline{H}\widehat{\otimes}\underline{H})(\underline{H}\cdot\overline{B})\\ & -\frac{3}{2}\big({}^{(c)}\mathcal{D}+11\,\underline{H}\big)\widehat{\otimes}\Big[P\,{}^{(c)}\mathcal{D}\,{}^{(c)}\mathcal{D}\cdot\Xi-2\,\underline{H}P\,{}^{(c)}\mathcal{D}\cdot\Xi\Big]\\ & -30P(\underline{H}\widehat{\otimes}\underline{H})\,{}^{(c)}\mathcal{D}\cdot\Xi\\ & +\frac{3}{2}P3\mathrm{tr}X\,\underline{H}\widehat{\otimes}\Big[\overline{H}\cdot\widehat{X}-\overline{\mathrm{tr}\underline{X}}\Xi+\mathcal{A}_3\Big]+\mathrm{Expr}_2(A).\end{aligned}$$

We now simplify the second line. We obtain

$$\begin{aligned}& 4\big({}^{(c)}\mathcal{D}+11\,\underline{H}\big)\widehat{\otimes}\Big[\mathrm{tr}X\,\underline{H}(\underline{H}\cdot\overline{B})\Big]-20\mathrm{tr}X(\underline{H}\widehat{\otimes}\underline{H})(\underline{H}\cdot\overline{B})\\ =\;& 4\,{}^{(c)}\mathcal{D}\widehat{\otimes}\Big[\mathrm{tr}X\,\underline{H}(\underline{H}\cdot\overline{B})\Big]+24\mathrm{tr}X(\underline{H}\widehat{\otimes}\underline{H})(\underline{H}\cdot\overline{B})\\ =\;& 4\Big[{}^{(c)}\mathcal{D}\mathrm{tr}X\widehat{\otimes}\,\underline{H}(\underline{H}\cdot\overline{B})+\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,\underline{H}(\underline{H}\cdot\overline{B})+\mathrm{tr}X\,\underline{H}\widehat{\otimes}\,{}^{(c)}\mathcal{D}(\underline{H}\cdot\overline{B})\Big]\\ & +24\mathrm{tr}X(\underline{H}\widehat{\otimes}\underline{H})(\underline{H}\cdot\overline{B})\\ =\;& 4\Big[-2\mathrm{tr}X\,\underline{H}\widehat{\otimes}\,\underline{H}(\underline{H}\cdot\overline{B})-\mathrm{tr}X\,\underline{H}\widehat{\otimes}\,\underline{H}(\underline{H}\cdot\overline{B})+\mathrm{tr}X\,\underline{H}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\,\underline{H}\cdot\overline{B}\\ & +\mathrm{tr}X\,\underline{H}\widehat{\otimes}\,\underline{H}({}^{(c)}\mathcal{D}\cdot\overline{B})\Big]+24\mathrm{tr}X(\underline{H}\widehat{\otimes}\underline{H})(\underline{H}\cdot\overline{B})\\ =\;& 4\mathrm{tr}X\,\underline{H}\widehat{\otimes}\,\underline{H}({}^{(c)}\mathcal{D}\cdot\overline{B})+8\mathrm{tr}X(\underline{H}\widehat{\otimes}\underline{H})(\underline{H}\cdot\overline{B})\\ =\;& 4\mathrm{tr}X\,\underline{H}\widehat{\otimes}\,\underline{H}\mathcal{B},\end{aligned}$$

where we used ${}^{(c)}\mathcal{D}(\mathrm{tr}X) = -2\mathrm{tr}X\,\underline{H} + O(\epsilon)$, ${}^{(c)}\mathcal{D}\widehat{\otimes}\,\underline{H} = -\,\underline{H}\widehat{\otimes}\,\underline{H} + O(\epsilon)$.

We finally obtain

$$\begin{aligned}
& {}^{(c)}\nabla_4\mathfrak{H}_1 + \frac{7}{2}\mathrm{tr}X\mathfrak{H}_1 \\
= \;& -\frac{3}{2}P\Big[\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\,{}^{(c)}\mathcal{D}\,{}^{(c)}\mathcal{D}\cdot\overline{\Xi} - 2\,\underline{H}\,\,{}^{(c)}\mathcal{D}\cdot\overline{\Xi}) \\
& -3\,\underline{H}\widehat{\otimes}\,(\,{}^{(c)}\mathcal{D}\,{}^{(c)}\mathcal{D}\cdot\overline{\Xi} - 2\,\underline{H}\,\,{}^{(c)}\mathcal{D}\cdot\overline{\Xi})\Big] \\
& -\frac{3}{2}P(11\,\underline{H})\widehat{\otimes}\Big[\,{}^{(c)}\mathcal{D}\,{}^{(c)}\mathcal{D}\cdot\overline{\Xi} - 2\,\underline{H}\,\,{}^{(c)}\mathcal{D}\cdot\overline{\Xi}\Big] - 3P10(\underline{H}\widehat{\otimes}\,\underline{H})\,\,{}^{(c)}\mathcal{D}\cdot\overline{\Xi} \\
& +\frac{3}{2}P3\mathrm{tr}X\,\underline{H}\widehat{\otimes}\Big[\overline{H}\cdot\widehat{X} - \overline{\mathrm{tr}\underline{X}}\Xi + \mathcal{A}_3\Big] + \mathrm{Expr}_2(A) \\
= \;& \frac{3}{2}P\Big[-\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\,{}^{(c)}\mathcal{D}\,{}^{(c)}\mathcal{D}\cdot\overline{\Xi}) - 6\,\underline{H}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\,{}^{(c)}\mathcal{D}\cdot\overline{\Xi} - 6\,\underline{H}\widehat{\otimes}\,\underline{H}\,\,{}^{(c)}\mathcal{D}\cdot\overline{\Xi} \\
& -3\mathrm{tr}X\overline{\mathrm{tr}\underline{X}}\,\underline{H}\widehat{\otimes}\Xi + 3\mathrm{tr}X\,\underline{H}\widehat{\otimes}(\overline{H}\cdot\widehat{X}) + 3\mathrm{tr}X\,\underline{H}\widehat{\otimes}\mathcal{A}_3\Big] + \mathrm{Expr}_2(A)
\end{aligned}$$

as stated. ◻

Here we compute ${}^{(c)}\nabla_4\mathfrak{H}_2$.

Lemma D.7.9. *We have, modulo quadratic terms,*

$$\begin{aligned}
{}^{(c)}\nabla_4\mathfrak{H}_2 \;=\;& -2(trX)\mathfrak{H}_2 - 3trX\,\underline{H}\widehat{\otimes}\mathcal{A}_3 + \widetilde{\mathcal{M}}[\Xi] \\
& +\Big[\frac{1}{2}(3trX - 3\overline{trX})\overline{trX\,tr\underline{X}} + (3\overline{trX})\overline{{}^{(c)}\mathcal{D}}\cdot\,\underline{H} - 3trX(\,\underline{H}\cdot\overline{\underline{H}}) \\
& -3trX\overline{H}\cdot\,\underline{H} + 3(\overline{trX}P - trX\overline{P})\Big]\widehat{X} + Expr_4(A),
\end{aligned}$$

with

$$\begin{aligned}
Expr_4(A) \;=\;& -\,\underline{H}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot A + A\cdot\overline{\underline{H}}\big) + {}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{H}\cdot A) + 4\,\underline{H}\widehat{\otimes}(\overline{H}\cdot A) \\
& -\big(\overline{trX\,tr\underline{X}} - 2\,\overline{{}^{(c)}\mathcal{D}}\cdot\,\underline{H} - 2P\big)A + trX\big(-\,{}^{(c)}\nabla_3 A - \frac{1}{2}tr\underline{X}A\big),
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{\mathcal{M}}[\Xi] \;:=\;& \mathcal{M}[\Xi] - 6\overline{P}\,\Xi - \overline{H}\cdot\,{}^{(c)}\mathcal{D}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\Xi - (\,{}^{(c)}\mathcal{D}\cdot\overline{H})\,{}^{(c)}\mathcal{D}\widehat{\otimes}\Xi \\
& -\overline{H}\cdot\,{}^{(c)}\mathcal{D}(\Xi\widehat{\otimes}(\,\underline{H} + H)) - \,\underline{H}\widehat{\otimes}\big(\overline{H}\cdot\,{}^{(c)}\mathcal{D}\widehat{\otimes}\Xi\big) \\
& -(\,{}^{(c)}\mathcal{D}\cdot\overline{H})(\Xi\widehat{\otimes}(\,\underline{H} + H)) - \,\underline{H}\widehat{\otimes}\big(\overline{H}\cdot(\Xi\widehat{\otimes}(\,\underline{H} + H))\big) \\
& \big(\overline{trX\,tr\underline{X}} - \overline{{}^{(c)}\mathcal{D}}\cdot\,\underline{H} - 2P\big)\big(\,{}^{(c)}\mathcal{D}\widehat{\otimes}\Xi + \Xi\widehat{\otimes}(\,\underline{H} + H)\big) \\
& -3\,\underline{H}\widehat{\otimes}\big(\overline{H}\cdot\big(\,{}^{(c)}\mathcal{D}\widehat{\otimes}\Xi + \Xi\widehat{\otimes}(\,\underline{H} + H)\big)\big).
\end{aligned}$$

Proof. Recall that

$$\mathfrak{H}_2 = \mathcal{A}_4 - 2B\widehat{\otimes}\,\underline{H} - {}^{(c)}\mathcal{D}\widehat{\otimes}\big(\overline{H}\cdot\widehat{X}\big) + \big(\overline{\mathrm{tr}X\mathrm{tr}\underline{X}} - 2\,\overline{{}^{(c)}\mathcal{D}}\cdot\,\underline{H} - 2P\big)\widehat{X} - 3\,\underline{H}\widehat{\otimes}\big(\overline{H}\cdot\widehat{X}\big).$$

We therefore obtain

$$ {}^{(c)}\nabla_4\mathfrak{H}_2 = I_1 + I_2 + I_3 + I_4 + I_5. $$

with

$$\begin{aligned}
I_1 &= {}^{(c)}\nabla_4\mathcal{A}_4,\\
I_2 &= -2\,{}^{(c)}\nabla_4(B\widehat{\otimes}\,\underline{H}),\\
I_3 &= -\nabla_4\,{}^{(c)}\mathcal{D}\widehat{\otimes}\big(\overline{H}\cdot\widehat{X}\big)\\
I_4 &= {}^{(c)}\nabla_4\Big(\big(\overline{\mathrm{tr}X\mathrm{tr}\underline{X}} - 2\,\overline{{}^{(c)}\mathcal{D}}\cdot\,\underline{H} - 2P\big)\widehat{X}\Big)\\
I_5 &= -3\,{}^{(c)}\nabla_4\Big(\,\underline{H}\widehat{\otimes}\big(\overline{H}\cdot\widehat{X}\big)\Big).
\end{aligned}$$

We now compute each term above.

Using Lemma D.7.4, we deduce

$$\begin{aligned}
I_1 &= -2(\mathrm{tr}X)\mathcal{A}_4 - 3\mathrm{tr}X\,\underline{H}\widehat{\otimes}\mathcal{A}_3 - \mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}B + 2\mathrm{tr}X\,\underline{H}\widehat{\otimes}B + \mathcal{M}[\Xi]\\
&\quad -\frac{1}{2}(\overline{\mathrm{tr}X} - \mathrm{tr}X)\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\widehat{X}\cdot\overline{H}) + \frac{1}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\widehat{X}\cdot\overline{\underline{H}})\\
&\quad -\frac{1}{2}(\mathrm{tr}X - \overline{\mathrm{tr}X})(H\cdot\overline{H})\widehat{X} + \mathrm{tr}X(\,\underline{H}\cdot\overline{\underline{H}})\widehat{X} + \mathrm{tr}X\underline{H}\widehat{\otimes}(\widehat{X}\cdot\overline{H}) + \widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}(\mathrm{tr}X\,\underline{H}).
\end{aligned}$$

Using (2.4.4) to write $\widehat{X}\cdot\overline{{}^{(c)}\mathcal{D}}(\mathrm{tr}X\,\underline{H}) = \widehat{X}(\overline{{}^{(c)}\mathcal{D}}\cdot(\mathrm{tr}X\,\underline{H}))$ and using that $\overline{\mathrm{tr}XH} = -\mathrm{tr}X\,\overline{\underline{H}} + O(\epsilon)$,

$$\begin{aligned}
\overline{{}^{(c)}\mathcal{D}}\cdot(\mathrm{tr}X\,\underline{H}) &= \overline{{}^{(c)}\mathcal{D}}(\mathrm{tr}X)\cdot\,\underline{H} + \mathrm{tr}X\,\overline{{}^{(c)}\mathcal{D}}\cdot\,\underline{H}\\
&= (\overline{\mathrm{tr}X} - \mathrm{tr}X)\overline{H}\cdot\,\underline{H} + \mathrm{tr}X\,\overline{{}^{(c)}\mathcal{D}}\cdot\,\underline{H}\\
&= \overline{\mathrm{tr}XH}\cdot\,\underline{H} - \mathrm{tr}X\overline{H}\cdot\,\underline{H} + \mathrm{tr}X\,\overline{{}^{(c)}\mathcal{D}}\cdot\,\underline{H}\\
&= -\mathrm{tr}X\,\overline{\underline{H}}\cdot\,\underline{H} - \mathrm{tr}X\overline{H}\cdot\,\underline{H} + \mathrm{tr}X\,\overline{{}^{(c)}\mathcal{D}}\cdot\,\underline{H},
\end{aligned}$$

we finally obtain

$$\begin{aligned}
I_1 &= -2(\mathrm{tr}X)\mathcal{A}_4 - 3\mathrm{tr}X\,\underline{H}\widehat{\otimes}\mathcal{A}_3 - \mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}B + 2\mathrm{tr}X\,\underline{H}\widehat{\otimes}B + \mathcal{M}[\Xi]\\
&\quad -\frac{1}{2}(\overline{\mathrm{tr}X} - \mathrm{tr}X)\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\widehat{X}\cdot\overline{H}) + \frac{1}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\widehat{X}\cdot\overline{\underline{H}})\\
&\quad -\frac{1}{2}(\mathrm{tr}X - \overline{\mathrm{tr}X})(H\cdot\overline{H})\widehat{X} + \mathrm{tr}X\underline{H}\widehat{\otimes}(\widehat{X}\cdot\overline{H})\\
&\quad +\big(-\mathrm{tr}X\overline{H}\cdot\,\underline{H} + \mathrm{tr}X\,\overline{{}^{(c)}\mathcal{D}}\cdot\,\underline{H}\big)\widehat{X}.
\end{aligned}$$

We have

$$
\begin{aligned}
I_2 &= -2\,{}^{(c)}\nabla_4 B\widehat{\otimes}\,\underline{H} - 2B\widehat{\otimes}\,{}^{(c)}\nabla_4\,\underline{H}\\
&= -2\Big(-2\overline{\mathrm{tr}X}B + 3\overline{P}\,\Xi + \frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot A + \frac{1}{2}A\cdot\overline{\underline{H}}\Big)\widehat{\otimes}\,\underline{H} + 2\mathrm{tr}X\,B\widehat{\otimes}\,\underline{H}\\
&= (2\mathrm{tr}X + 4\overline{\mathrm{tr}X})\,\underline{H}\widehat{\otimes}B - 6\overline{P}\,\Xi - \,\underline{H}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot A + A\cdot\overline{\underline{H}}\big),
\end{aligned}
$$

where we used ${}^{(c)}\nabla_4\,\underline{H} = -\mathrm{tr}X\,\underline{H} + O(\epsilon)$ and the Bianchi identity ${}^{(c)}\nabla_4 B = -2\overline{\mathrm{tr}X}B + 3\overline{P}\,\Xi + \frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot A + \frac{1}{2}A\cdot\overline{\underline{H}}$.

Using Lemma 4.2.2, (4.2.13), applied to $F = \overline{H}\cdot\widehat{X}$ and $s=1$, we have, up to quadratic terms,

$$
[\,{}^{(c)}\nabla_4,\,{}^{(c)}\mathcal{D}\widehat{\otimes}]\overline{H}\cdot\widehat{X} = -\frac{1}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{H}\cdot\widehat{X}) + \,\underline{H}\widehat{\otimes}\,{}^{(c)}\nabla_4(\overline{H}\cdot\widehat{X}),
$$

and therefore

$$
\begin{aligned}
I_3 &= -\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\nabla_4\big(\overline{H}\cdot\widehat{X}\big) - [\,{}^{(c)}\nabla_4,\,{}^{(c)}\mathcal{D}\widehat{\otimes}]\big(\overline{H}\cdot\widehat{X}\big)\\
&= -\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\nabla_4\big(\overline{H}\cdot\widehat{X}\big) + \frac{1}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{H}\cdot\widehat{X}) - \,\underline{H}\widehat{\otimes}\,{}^{(c)}\nabla_4(\overline{H}\cdot\widehat{X})\\
&= -\,{}^{(c)}\mathcal{D}\widehat{\otimes}\big(\overline{H}\cdot\,{}^{(c)}\nabla_4\widehat{X} + \,{}^{(c)}\nabla_4\overline{H}\cdot\widehat{X}\big) + \frac{1}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{H}\cdot\widehat{X})\\
&\quad -\underline{H}\widehat{\otimes}\big(\overline{H}\cdot\,{}^{(c)}\nabla_4\widehat{X} + \,{}^{(c)}\nabla_4\overline{H}\cdot\widehat{X}\big).
\end{aligned}
$$

Using ${}^{(c)}\nabla_4\overline{H} = -\frac{1}{2}\mathrm{tr}X(\overline{H} - \overline{\underline{H}}) + O(\epsilon)$ and the Ricci identity ${}^{(c)}\nabla_4\widehat{X} = -\frac{1}{2}(\mathrm{tr}X + \overline{\mathrm{tr}X})\widehat{X} + \,{}^{(c)}\mathcal{D}\widehat{\otimes}\Xi + \Xi\widehat{\otimes}(\,\underline{H} + H) - A$, we obtain

$$
\begin{aligned}
\overline{H}\cdot\,{}^{(c)}\nabla_4\widehat{X} + \,{}^{(c)}\nabla_4\overline{H}\cdot\widehat{X} &= \overline{H}\cdot\Big(-\frac{1}{2}(\mathrm{tr}X + \overline{\mathrm{tr}X})\widehat{X} + \,{}^{(c)}\mathcal{D}\widehat{\otimes}\Xi + \Xi\widehat{\otimes}(\,\underline{H} + H) - A\Big)\\
&\quad -\frac{1}{2}\mathrm{tr}X(\overline{H} - \overline{\underline{H}})\cdot\widehat{X}\\
&= -\frac{1}{2}(2\mathrm{tr}X + \overline{\mathrm{tr}X})\overline{H}\cdot\widehat{X} + \frac{1}{2}\mathrm{tr}X\,\overline{\underline{H}}\cdot\widehat{X}\\
&\quad +\overline{H}\cdot\,{}^{(c)}\mathcal{D}\widehat{\otimes}\Xi + \overline{H}\cdot(\Xi\widehat{\otimes}(\,\underline{H} + H)) - \overline{H}\cdot A.
\end{aligned}
$$

Writing that $\overline{\mathrm{tr}X H} = -\mathrm{tr}X\,\overline{\underline{H}} + O(\epsilon)$, we obtain

$$
\begin{aligned}
\overline{H}\cdot\,{}^{(c)}\nabla_4\widehat{X} + \,{}^{(c)}\nabla_4\overline{H}\cdot\widehat{X} &= -\mathrm{tr}X\overline{H}\cdot\widehat{X} + \mathrm{tr}X\,\overline{\underline{H}}\cdot\widehat{X}\\
&\quad +\overline{H}\cdot\,{}^{(c)}\mathcal{D}\widehat{\otimes}\Xi + \overline{H}\cdot(\Xi\widehat{\otimes}(\,\underline{H} + H)) - \overline{H}\cdot A.
\end{aligned}
$$

We therefore obtain

$$
\begin{aligned}
I_3 &= -{}^{(c)}\mathcal{D}\widehat{\otimes}\Big(-\frac{1}{2}(2\mathrm{tr}X+\overline{\mathrm{tr}X})\overline{H}\cdot\widehat{X}+\frac{1}{2}\mathrm{tr}X\,\overline{\underline{H}}\cdot\widehat{X}-\overline{H}\cdot A\Big)+\frac{1}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{H}\cdot\widehat{X})\\
&\quad -\underline{H}\widehat{\otimes}\Big(-\mathrm{tr}X\overline{H}\cdot\widehat{X}+\mathrm{tr}X\,\overline{\underline{H}}\cdot\widehat{X}-\overline{H}\cdot A\Big)\\
&= \frac{1}{2}(3\mathrm{tr}X+\overline{\mathrm{tr}X})\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{H}\cdot\widehat{X})-\frac{1}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{\underline{H}}\cdot\widehat{X})\\
&\quad +\frac{1}{2}(2\,{}^{(c)}\mathcal{D}\mathrm{tr}X+{}^{(c)}\mathcal{D}\overline{\mathrm{tr}X})\widehat{\otimes}(\overline{H}\cdot\widehat{X})-\frac{1}{2}\,{}^{(c)}\mathcal{D}\mathrm{tr}X\widehat{\otimes}(\overline{\underline{H}}\cdot\widehat{X})\\
&\quad +\underline{H}\widehat{\otimes}\big(\mathrm{tr}X\overline{H}\cdot\widehat{X}-\mathrm{tr}X\,\overline{\underline{H}}\cdot\widehat{X}\big)+{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{H}\cdot A)+\underline{H}\widehat{\otimes}\overline{H}\cdot A.
\end{aligned}
$$

Using that ${}^{(c)}\mathcal{D}\mathrm{tr}X=-2\mathrm{tr}X\,\underline{H}+O(\epsilon)$ and ${}^{(c)}\mathcal{D}(\overline{\mathrm{tr}X})=(\mathrm{tr}X-\overline{\mathrm{tr}X})H+O(\epsilon)$ we obtain

$$
\begin{aligned}
I_3 &= \frac{1}{2}(3\mathrm{tr}X+\overline{\mathrm{tr}X})\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{H}\cdot\widehat{X})-\frac{1}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{\underline{H}}\cdot\widehat{X})\\
&\quad -\mathrm{tr}X\,\underline{H}\widehat{\otimes}(\overline{H}\cdot\widehat{X})+\frac{1}{2}(\mathrm{tr}X-\overline{\mathrm{tr}X})(H\cdot\overline{H})\widehat{X}\\
&\quad +{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{H}\cdot A)+\underline{H}\widehat{\otimes}\overline{H}\cdot A.
\end{aligned}
$$

We now compute

$$
\begin{aligned}
I_4 &= \big(\overline{\mathrm{tr}X\mathrm{tr}\underline{X}}-2\,\overline{{}^{(c)}\mathcal{D}}\cdot\underline{H}-2P\big)\,{}^{(c)}\nabla_4\widehat{X}+{}^{(c)}\nabla_4\big(\overline{\mathrm{tr}X\mathrm{tr}\underline{X}}-2\,\overline{{}^{(c)}\mathcal{D}}\cdot\underline{H}-2P\big)\widehat{X}\\
&= \big(\overline{\mathrm{tr}X\mathrm{tr}\underline{X}}-2\,\overline{{}^{(c)}\mathcal{D}}\cdot\underline{H}-2P\big)\Big(-\frac{1}{2}(\mathrm{tr}X+\overline{\mathrm{tr}X})\widehat{X}+{}^{(c)}\mathcal{D}\widehat{\otimes}\Xi+\Xi\widehat{\otimes}(\underline{H}+H)-A\Big)\\
&\quad +\Big({}^{(c)}\nabla_4(\overline{\mathrm{tr}X})\overline{\mathrm{tr}\underline{X}}+\overline{\mathrm{tr}X}\,{}^{(c)}\nabla_4(\overline{\mathrm{tr}\underline{X}})-2\,{}^{(c)}\nabla_4\overline{{}^{(c)}\mathcal{D}}\cdot\underline{H}-2\,{}^{(c)}\nabla_4P\Big)\widehat{X}\\
&= \Big[-\frac{1}{2}(\mathrm{tr}X+\overline{\mathrm{tr}X})\overline{\mathrm{tr}X\mathrm{tr}\underline{X}}+(\mathrm{tr}X+\overline{\mathrm{tr}X})\,\overline{{}^{(c)}\mathcal{D}}\cdot\underline{H}+(\mathrm{tr}X+\overline{\mathrm{tr}X})P\Big]\widehat{X}\\
&\quad +\big(\overline{\mathrm{tr}X\mathrm{tr}\underline{X}}-2\,\overline{{}^{(c)}\mathcal{D}}\cdot\underline{H}-2P\big)\big({}^{(c)}\mathcal{D}\widehat{\otimes}\Xi+\Xi\widehat{\otimes}(\underline{H}+H)\big)\\
&\quad -\big(\overline{\mathrm{tr}X\mathrm{tr}\underline{X}}-2\,\overline{{}^{(c)}\mathcal{D}}\cdot\underline{H}-2P\big)A+\Big[-\frac{1}{2}(\overline{\mathrm{tr}X})^2\overline{\mathrm{tr}\underline{X}}\\
&\quad +\overline{\mathrm{tr}X}\Big(-\frac{1}{2}\overline{\mathrm{tr}X\mathrm{tr}\underline{X}}+\overline{{}^{(c)}\mathcal{D}}\cdot\underline{H}+\underline{H}\cdot\overline{\underline{H}}+2P\Big)-2\,{}^{(c)}\nabla_4\overline{{}^{(c)}\mathcal{D}}\cdot\underline{H}+3\mathrm{tr}XP\Big]\widehat{X},
\end{aligned}
$$

where we used ${}^{(c)}\nabla_4\overline{\mathrm{tr}X}=-\frac{1}{2}(\overline{\mathrm{tr}X})^2+O(\epsilon)$, ${}^{(c)}\nabla_4\overline{\mathrm{tr}\underline{X}}=-\frac{1}{2}\overline{\mathrm{tr}X\mathrm{tr}\underline{X}}+\overline{{}^{(c)}\mathcal{D}}\cdot\underline{H}+\underline{H}\cdot\overline{\underline{H}}+2P+O(\epsilon^2)$, and ${}^{(c)}\nabla_4P=-\frac{3}{2}\mathrm{tr}XP+O(\epsilon)$. This gives

$$
\begin{aligned}
I_4 &= \Big[-\frac{1}{2}(\mathrm{tr}X+3\overline{\mathrm{tr}X})\overline{\mathrm{tr}X\mathrm{tr}\underline{X}}+(\mathrm{tr}X+2\overline{\mathrm{tr}X})\,\overline{{}^{(c)}\mathcal{D}}\cdot\underline{H}+\overline{\mathrm{tr}X}(\underline{H}\cdot\overline{\underline{H}})\\
&\quad -2\,{}^{(c)}\nabla_4\overline{{}^{(c)}\mathcal{D}}\cdot\underline{H}+(4\mathrm{tr}X+3\overline{\mathrm{tr}X})P\Big]\widehat{X}\\
&\quad +\big(\overline{\mathrm{tr}X\mathrm{tr}\underline{X}}-2\,\overline{{}^{(c)}\mathcal{D}}\cdot\underline{H}-2P\big)\big({}^{(c)}\mathcal{D}\widehat{\otimes}\Xi+\Xi\widehat{\otimes}(\underline{H}+H)\big)\\
&\quad -\big(\overline{\mathrm{tr}X\mathrm{tr}\underline{X}}-2\,\overline{{}^{(c)}\mathcal{D}}\cdot\underline{H}-2P\big)A.
\end{aligned}
$$

Finally, using Lemma 4.2.2, (4.2.16), applied to $F = \underline{H}$ and $s = 0$,

$$\begin{aligned}
[\,{}^{(c)}\nabla_4, \overline{{}^{(c)}\mathcal{D}\cdot}]\,\underline{H} &= -\frac{1}{2}\overline{\mathrm{tr}X}\,\Big(\overline{{}^{(c)}\mathcal{D}}\cdot\,\underline{H} - \overline{\underline{H}}\cdot\,\underline{H}\Big) + \overline{\underline{H}}\cdot\,{}^{(c)}\nabla_4\,\underline{H} + O(\epsilon)\\
&= -\frac{1}{2}\overline{\mathrm{tr}X}\,\Big(\overline{{}^{(c)}\mathcal{D}}\cdot\,\underline{H} - \overline{\underline{H}}\cdot\,\underline{H}\Big) - \mathrm{tr}X\,\overline{\underline{H}}\cdot\,\underline{H} + O(\epsilon),
\end{aligned}$$

we write

$$\begin{aligned}
&-2\,{}^{(c)}\nabla_4\,\overline{{}^{(c)}\mathcal{D}}\cdot\,\underline{H} = -2\,\overline{{}^{(c)}\mathcal{D}}\cdot(\,{}^{(c)}\nabla_4\,\underline{H}) - 2[\,{}^{(c)}\nabla_4, \overline{{}^{(c)}\mathcal{D}\cdot}]\,\underline{H}\\
=\;& 2\,\overline{{}^{(c)}\mathcal{D}}\cdot(\mathrm{tr}X\,\underline{H}) + \overline{\mathrm{tr}X}\,\Big(\overline{{}^{(c)}\mathcal{D}}\cdot\,\underline{H} - \overline{\underline{H}}\cdot\,\underline{H}\Big) + 2\mathrm{tr}X\,\overline{\underline{H}}\cdot\,\underline{H}\\
=\;& 2\,\overline{{}^{(c)}\mathcal{D}}\cdot(\mathrm{tr}X\,\underline{H}) + \overline{\mathrm{tr}X}\,\Big(\overline{{}^{(c)}\mathcal{D}}\cdot\,\underline{H} - \overline{\underline{H}}\cdot\,\underline{H}\Big) + 2\mathrm{tr}X\,\overline{\underline{H}}\cdot\,\underline{H}\\
=\;& 2\,\overline{{}^{(c)}\mathcal{D}}(\mathrm{tr}X)\cdot\,\underline{H} + 2\mathrm{tr}X\,\overline{{}^{(c)}\mathcal{D}}\cdot\,\underline{H} + \overline{\mathrm{tr}X}\,\Big(\overline{{}^{(c)}\mathcal{D}}\cdot\,\underline{H} - \overline{\underline{H}}\cdot\,\underline{H}\Big) + 2\mathrm{tr}X\,\overline{\underline{H}}\cdot\,\underline{H}\\
=\;& 2(\overline{\mathrm{tr}X} - \mathrm{tr}X)\overline{H}\cdot\,\underline{H} + (2\mathrm{tr}X + \overline{\mathrm{tr}X})\,\overline{{}^{(c)}\mathcal{D}}\cdot\,\underline{H} + (2\mathrm{tr}X - \overline{\mathrm{tr}X})\,\overline{\underline{H}}\cdot\,\underline{H}\\
=\;& -2\mathrm{tr}X\,\overline{\underline{H}}\cdot\,\underline{H} - 2\mathrm{tr}X\overline{H}\cdot\,\underline{H} + (2\mathrm{tr}X + \overline{\mathrm{tr}X})\,\overline{{}^{(c)}\mathcal{D}}\cdot\,\underline{H} + (2\mathrm{tr}X - \overline{\mathrm{tr}X})\,\overline{\underline{H}}\cdot\,\underline{H}\\
=\;& -2\mathrm{tr}X\overline{H}\cdot\,\underline{H} + (2\mathrm{tr}X + \overline{\mathrm{tr}X})\,\overline{{}^{(c)}\mathcal{D}}\cdot\,\underline{H} - \overline{\mathrm{tr}X}\,\overline{\underline{H}}\cdot\,\underline{H},
\end{aligned}$$

where we used that $\overline{\mathrm{tr}X H} = -\mathrm{tr}X\,\overline{\underline{H}} + O(\epsilon)$. We finally obtain

$$\begin{aligned}
I_4 \;=\;& \Big[-\frac{1}{2}(\mathrm{tr}X + 3\overline{\mathrm{tr}X})\overline{\mathrm{tr}X\mathrm{tr}\underline{X}} + (3\mathrm{tr}X + 3\overline{\mathrm{tr}X})\,\overline{{}^{(c)}\mathcal{D}}\cdot\,\underline{H}\\
&-2\mathrm{tr}X\overline{H}\cdot\,\underline{H} + (4\mathrm{tr}X + 3\overline{\mathrm{tr}X})P\Big]\widehat{X}\\
&+\Big(\overline{\mathrm{tr}X\mathrm{tr}\underline{X}} - 2\,\overline{{}^{(c)}\mathcal{D}}\cdot\,\underline{H} - 2P\Big)\big(\,{}^{(c)}\mathcal{D}\widehat{\otimes}\Xi + \Xi\widehat{\otimes}(\,\underline{H} + H)\big)\\
&-\Big(\overline{\mathrm{tr}X\mathrm{tr}\underline{X}} - 2\,\overline{{}^{(c)}\mathcal{D}}\cdot\,\underline{H} - 2P\Big)A.
\end{aligned}$$

We compute

$$\begin{aligned}
I_5 \;=\;& -3\,{}^{(c)}\nabla_4\,\underline{H}\widehat{\otimes}\big(\overline{H}\cdot\widehat{X}\big) - 3\,\underline{H}\widehat{\otimes}\big(\,{}^{(c)}\nabla_4\overline{H}\cdot\widehat{X}\big) - 3\,\underline{H}\widehat{\otimes}\big(\overline{H}\cdot\,{}^{(c)}\nabla_4\widehat{X}\big)\\
=\;& 3\mathrm{tr}X\,\underline{H}\widehat{\otimes}\big(\overline{H}\cdot\widehat{X}\big) + \frac{3}{2}\mathrm{tr}X\,\underline{H}\widehat{\otimes}\big((\overline{H} - \overline{\underline{H}})\cdot\widehat{X}\big)\\
&-3\,\underline{H}\widehat{\otimes}\Big(\overline{H}\cdot\Big(-\frac{1}{2}(\mathrm{tr}X + \overline{\mathrm{tr}X})\widehat{X} + \,{}^{(c)}\mathcal{D}\widehat{\otimes}\Xi + \Xi\widehat{\otimes}(\,\underline{H} + H) - A\Big)\Big)\\
=\;& 6\mathrm{tr}X\,\underline{H}\widehat{\otimes}\big(\overline{H}\cdot\widehat{X}\big) - \frac{3}{2}\mathrm{tr}X(\,\underline{H}\cdot\overline{\underline{H}})\widehat{X} + \frac{3}{2}\overline{\mathrm{tr}X}\,\underline{H}\widehat{\otimes}\big(\overline{H}\cdot\widehat{X}\big)\\
&-3\,\underline{H}\widehat{\otimes}\Big(\overline{H}\cdot\big(\,{}^{(c)}\mathcal{D}\widehat{\otimes}\Xi + \Xi\widehat{\otimes}(\,\underline{H} + H)\big)\Big) + 3\,\underline{H}\widehat{\otimes}\big(\overline{H}\cdot A\big)\\
=\;& 6\mathrm{tr}X\,\underline{H}\widehat{\otimes}\big(\overline{H}\cdot\widehat{X}\big) - 3\mathrm{tr}X(\,\underline{H}\cdot\overline{\underline{H}})\widehat{X}\\
&-3\,\underline{H}\widehat{\otimes}\Big(\overline{H}\cdot\big(\,{}^{(c)}\mathcal{D}\widehat{\otimes}\Xi + \Xi\widehat{\otimes}(\,\underline{H} + H)\big)\Big) + 3\,\underline{H}\widehat{\otimes}\big(\overline{H}\cdot A\big),
\end{aligned}$$

where we used ${}^{(c)}\nabla_4 \underline{H} = -\mathrm{tr}X\, \underline{H} + O(\epsilon)$, ${}^{(c)}\nabla_4 \overline{H} = -\frac{1}{2}\mathrm{tr}X(\overline{H} - \overline{\underline{H}}) + O(\epsilon)$ and the Ricci identity ${}^{(c)}\nabla_4 \widehat{X} = -\frac{1}{2}(\mathrm{tr}X + \overline{\mathrm{tr}X})\widehat{X} + {}^{(c)}\mathcal{D}\widehat{\otimes}\Xi + \Xi\widehat{\otimes}(\underline{H} + H) - A$. We also used that $\overline{\mathrm{tr}XH} = -\mathrm{tr}X\,\overline{\underline{H}} + O(\epsilon)$.

We obtain for the sum

$$\begin{aligned}
& {}^{(c)}\nabla_4\mathfrak{H}_2 \\
= \;& -2(\mathrm{tr}X)\mathcal{A}_4 - 3\mathrm{tr}X\, \underline{H}\widehat{\otimes}\mathcal{A}_3 - \mathrm{tr}X\, {}^{(c)}\mathcal{D}\widehat{\otimes}B + (4\mathrm{tr}X + 4\overline{\mathrm{tr}X})\, \underline{H}\widehat{\otimes}B \\
& +2\mathrm{tr}X\, {}^{(c)}\mathcal{D}\widehat{\otimes}(\widehat{X}\cdot\overline{H}) + + \Big[-\frac{1}{2}(\mathrm{tr}X + 3\overline{\mathrm{tr}X})\overline{\mathrm{tr}X\mathrm{tr}\underline{X}} + (4\mathrm{tr}X + 3\overline{\mathrm{tr}X})\,\overline{{}^{(c)}\mathcal{D}}\cdot\underline{H} \\
& -3\mathrm{tr}X(\underline{H}\cdot\overline{\underline{H}}) - 3\mathrm{tr}X\overline{H}\cdot\underline{H} + (4\mathrm{tr}X + 3\overline{\mathrm{tr}X})P\Big]\widehat{X} + 6\mathrm{tr}X\, \underline{H}\widehat{\otimes}\big(\overline{H}\cdot\widehat{X}\big) \\
& +\mathrm{Expr}_3(A) + \widetilde{\mathcal{M}}[\Xi],
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{Expr}_3(A) \;=\; & -\underline{H}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot A + A\cdot\overline{\underline{H}}\big) + {}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{H}\cdot A) + 4\,\underline{H}\widehat{\otimes}(\overline{H}\cdot A) \\
& -\big(\overline{\mathrm{tr}X\mathrm{tr}\underline{X}} - \overline{{}^{(c)}\mathcal{D}}\cdot\underline{H} - 2P\big)A.
\end{aligned}$$

Writing

$$\mathfrak{H}_2 \;=\; \mathcal{A}_4 - 2B\widehat{\otimes}\,\underline{H} - {}^{(c)}\mathcal{D}\widehat{\otimes}\big(\overline{H}\cdot\widehat{X}\big) + \big(\overline{\mathrm{tr}X\mathrm{tr}\underline{X}} - 2\,\overline{{}^{(c)}\mathcal{D}}\cdot\underline{H} - 2P\big)\widehat{X} - 3\,\underline{H}\widehat{\otimes}\big(\overline{H}\cdot\widehat{X}\big),$$

we obtain

$$\begin{aligned}
{}^{(c)}\nabla_4\mathfrak{H}_2 \;=\; & -2(\mathrm{tr}X)\mathfrak{H}_2 - 3\mathrm{tr}X\,\underline{H}\widehat{\otimes}\mathcal{A}_3 - \mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}B + 4\overline{\mathrm{tr}X}\,\underline{H}\widehat{\otimes}B \\
& +\Big[\frac{1}{2}(3\mathrm{tr}X - 3\overline{\mathrm{tr}X})\overline{\mathrm{tr}X\mathrm{tr}\underline{X}} + (3\overline{\mathrm{tr}X})\,\overline{{}^{(c)}\mathcal{D}}\cdot\underline{H} \\
& -3\mathrm{tr}X(\underline{H}\cdot\overline{\underline{H}}) - 3\mathrm{tr}X\overline{H}\cdot\underline{H} + 3\overline{\mathrm{tr}X}P\Big]\widehat{X} + \mathrm{Expr}_3(A) + \widetilde{\mathcal{M}}[\Xi].
\end{aligned}$$

Observe that $\overline{\mathrm{tr}X}\,\underline{H} = -\mathrm{tr}XH + O(\epsilon)$ and therefore on the right hand side the terms depending on B can be written in terms of A and $\widehat{X}$ using the Bianchi identity:

$$\begin{aligned}
-\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}B + 4\overline{\mathrm{tr}X}B\widehat{\otimes}\,\underline{H} \;&=\; -\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}B - 4\mathrm{tr}XB\widehat{\otimes}H \\
&=\; \mathrm{tr}X\big(-\,{}^{(c)}\nabla_3 A - \frac{1}{2}\mathrm{tr}\underline{X}A - 3\overline{P}\widehat{X}\big).
\end{aligned}$$

This gives

$$\begin{aligned}
{}^{(c)}\nabla_4\mathfrak{H}_2 \;=\; & -2(\mathrm{tr}X)\mathfrak{H}_2 - 3\mathrm{tr}X\,\underline{H}\widehat{\otimes}\mathcal{A}_3 \\
& +\Big[\frac{1}{2}(3\mathrm{tr}X - 3\overline{\mathrm{tr}X})\overline{\mathrm{tr}X\mathrm{tr}\underline{X}} + (3\overline{\mathrm{tr}X})\,\overline{{}^{(c)}\mathcal{D}}\cdot\underline{H} - 3\mathrm{tr}X(\underline{H}\cdot\overline{\underline{H}}) \\
& -3\mathrm{tr}X\overline{H}\cdot\underline{H} + 3(\overline{\mathrm{tr}X}P - \mathrm{tr}X\overline{P})\Big]\widehat{X} + \mathrm{Expr}_4(A) + \widetilde{\mathcal{M}}[\Xi],
\end{aligned}$$

with

$$\mathrm{Expr}_4(A) \;=\; \mathrm{Expr}_3(A) + \mathrm{tr}X\big(- {}^{(c)}\nabla_3 A - \frac{1}{2}\mathrm{tr}\underline{X}A\big).$$

This concludes the proof. □

We can finally conclude the derivation. For simplicity, we consider the above computations in a gauge where $\Xi = 0$. Then from (D.7.3), i.e.

$$ {}^{(c)}\nabla_4\mathfrak{H} \;=\; {}^{(c)}\nabla_4\mathfrak{H}_1 + \frac{3}{2}P\big[{}^{(c)}\nabla_4\mathfrak{H}_2 - \frac{3}{2}\mathrm{tr}X\mathfrak{H}_2\big] + {}^{(c)}\nabla_4\Big(\frac{3}{2}P\overline{\mathrm{tr}\underline{X}}A\Big),$$

and using Lemma D.7.8 and D.7.9, we obtain

$$\begin{aligned}
{}^{(c)}\nabla_4\mathfrak{H} \;=\;& -\frac{7}{2}\mathrm{tr}X\mathfrak{H}_1 + \frac{3}{2}P\Big[3\mathrm{tr}X\, \underline{H}\widehat{\otimes}(\overline{H}\cdot\widehat{X}) + 3\mathrm{tr}X\, \underline{H}\widehat{\otimes}\mathcal{A}_3\Big] + \mathrm{Expr}_2(A)\\
&+\frac{3}{2}P\Big[- \frac{7}{2}(\mathrm{tr}X)\mathfrak{H}_2 - 3\mathrm{tr}X\, \underline{H}\widehat{\otimes}\mathcal{A}_3\\
&+\Big(\frac{1}{2}(3\mathrm{tr}X - 3\overline{\mathrm{tr}X})\overline{\mathrm{tr}X\mathrm{tr}\underline{X}} + (3\overline{\mathrm{tr}X})\,\overline{{}^{(c)}\mathcal{D}}\cdot\underline{H} - 3\mathrm{tr}X(\underline{H}\cdot\overline{\underline{H}})\\
&-3\mathrm{tr}X\overline{H}\cdot\underline{H} + 3(\overline{\mathrm{tr}X}P - \mathrm{tr}X\overline{P})\Big)\widehat{X} + \mathrm{Expr}_4(A)\Big] + {}^{(c)}\nabla_4\Big(\frac{3}{2}P\overline{\mathrm{tr}\underline{X}}A\Big)\\
=\;& -\frac{7}{2}\mathrm{tr}X\big(\mathfrak{H}_1 + \frac{3}{2}P\mathfrak{H}_2\big)\\
&+\frac{3}{2}P\Big[3\mathrm{tr}X\, \underline{H}\widehat{\otimes}(\overline{H}\cdot\widehat{X}) + \Big(\frac{1}{2}(3\mathrm{tr}X - 3\overline{\mathrm{tr}X})\overline{\mathrm{tr}X\mathrm{tr}\underline{X}} + (3\overline{\mathrm{tr}X})\,\overline{{}^{(c)}\mathcal{D}}\cdot\underline{H}\\
&-3\mathrm{tr}X(\underline{H}\cdot\overline{\underline{H}}) - 3\mathrm{tr}X\overline{H}\cdot\underline{H} + 3(\overline{\mathrm{tr}X}P - \mathrm{tr}X\overline{P})\Big)\widehat{X}\Big]\\
&+\mathrm{Expr}_2(A) + \frac{3}{2}P\mathrm{Expr}_4(A) + {}^{(c)}\nabla_4\Big(\frac{3}{2}P\overline{\mathrm{tr}\underline{X}}A\Big).
\end{aligned}$$

Recalling that $\mathfrak{H} = \mathfrak{H}_1 + \frac{3}{2}P\mathfrak{H}_2 + \frac{3}{2}P\overline{\mathrm{tr}\underline{X}}A$, we write

$$\begin{aligned}
&{}^{(c)}\nabla_4\mathfrak{H} + \frac{7}{2}\mathrm{tr}X\mathfrak{H}\\
=\;& \frac{3}{2}P\big[3\mathrm{tr}X\, \underline{H}\widehat{\otimes}(\overline{H}\cdot\widehat{X}) + \Big(\frac{1}{2}(3\mathrm{tr}X - 3\overline{\mathrm{tr}X})\overline{\mathrm{tr}X\mathrm{tr}\underline{X}} + (3\overline{\mathrm{tr}X})\,\overline{{}^{(c)}\mathcal{D}}\cdot\underline{H}\\
&-3\mathrm{tr}X(\underline{H}\cdot\overline{\underline{H}}) - 3\mathrm{tr}X\overline{H}\cdot\underline{H} + 3(\overline{\mathrm{tr}X}P - \mathrm{tr}X\overline{P})\Big)\widehat{X}\big]\\
&+\mathrm{Expr}_2(A) + \frac{3}{2}P\mathrm{Expr}_4(A) + {}^{(c)}\nabla_4\Big(\frac{3}{2}P\overline{\mathrm{tr}\underline{X}}A\Big) + \frac{21}{4}P\mathrm{tr}X\overline{\mathrm{tr}\underline{X}}A.
\end{aligned}$$

Using (2.4.2), i.e. $\underline{H}\widehat{\otimes}(\overline{H}\cdot\widehat{X})+H\widehat{\otimes}(\overline{\underline{H}}\cdot\widehat{X})=(\underline{H}\cdot\overline{H}+\overline{\underline{H}}\cdot H)\,\widehat{X}$ and that $\overline{\mathrm{tr}X}\,\underline{H}=-\mathrm{tr}XH+O(\epsilon)$, we have

$$\begin{aligned}3\mathrm{tr}X\,\underline{H}\widehat{\otimes}(\overline{H}\cdot\widehat{X}) &= 3\mathrm{tr}X\Big(-H\widehat{\otimes}(\overline{\underline{H}}\cdot\widehat{X})+(\underline{H}\cdot\overline{H}+\overline{\underline{H}}\cdot H)\,\widehat{X}\Big)\\ &= 3\overline{\mathrm{tr}X}\,\underline{H}\widehat{\otimes}(\overline{\underline{H}}\cdot\widehat{X})+(3\mathrm{tr}X\,\underline{H}\cdot\overline{H}-3\overline{\mathrm{tr}X}\,\underline{H}\cdot\overline{\underline{H}})\,\widehat{X}\\ &= 3\mathrm{tr}X\,\underline{H}\cdot\overline{H}\,\widehat{X}\end{aligned}$$

and therefore obtain

$$\begin{aligned}&{}^{(c)}\nabla_4\mathfrak{H}+\frac{7}{2}\mathrm{tr}X\mathfrak{H}\\ =\;&\frac{3}{2}P\Big[\Big(\frac{1}{2}(3\mathrm{tr}X-3\overline{\mathrm{tr}X})\overline{\mathrm{tr}X\mathrm{tr}\underline{X}}+(3\overline{\mathrm{tr}X})\,\overline{{}^{(c)}\mathcal{D}}\cdot\underline{H}\\ &-3\mathrm{tr}X(\underline{H}\cdot\overline{\underline{H}})+3(\overline{\mathrm{tr}X}P-\mathrm{tr}X\overline{P})\Big)\widehat{X}\Big]\\ &+\mathrm{Expr}_2(A)+\frac{3}{2}P\mathrm{Expr}_4(A)+{}^{(c)}\nabla_4\Big(\frac{3}{2}P\overline{\mathrm{tr}\underline{X}}A\Big)+\frac{21}{4}P\mathrm{tr}X\overline{\mathrm{tr}\underline{X}}A.\end{aligned}$$

We now show that the terms in the parenthesis vanish in Kerr, i.e.

$$\frac{1}{2}(3\mathrm{tr}X-3\overline{\mathrm{tr}X})\overline{\mathrm{tr}X\mathrm{tr}\underline{X}}+3(\overline{\mathrm{tr}X}P-\mathrm{tr}X\overline{P})+3\overline{\mathrm{tr}X}\,\overline{{}^{(c)}\mathcal{D}}\cdot\underline{H}-3\mathrm{tr}X(\underline{H}\cdot\overline{\underline{H}})=O(\epsilon).$$

Recall that in Kerr

$$\begin{aligned}&\mathrm{tr}\,\chi=\frac{2r}{|q|^2},\quad {}^{(a)}\mathrm{tr}\chi=\frac{2a\cos\theta}{|q|^2},\quad \mathrm{tr}X=\frac{2}{q},\quad \mathrm{tr}\underline{X}=-\frac{2\Delta q}{|q|^4},\quad P=-\frac{2m}{q^3}\\ &H_1=\frac{ai\sin\theta\,q}{|q|^3},\quad H_2=\frac{a\sin\theta\,q}{|q|^3},\quad \underline{H}_1=-\frac{ai\sin\theta\,\overline{q}}{|q|^3},\quad \underline{H}_2=-\frac{a\sin\theta\,\overline{q}}{|q|^3}.\end{aligned}$$

We compute

$$\begin{aligned}\overline{\mathrm{tr}X\mathrm{tr}\underline{X}} &= -\frac{2}{\overline{q}}\frac{\overline{2\Delta q}}{|q|^4}=-\frac{4\Delta}{|q|^4}\\ \frac{1}{2}(3\mathrm{tr}X-3\overline{\mathrm{tr}X})\overline{\mathrm{tr}X\mathrm{tr}\underline{X}} &= -\frac{6\Delta}{|q|^4}\left(\frac{2}{q}-\frac{2}{\overline{q}}\right)=i\frac{24a\cos\theta\Delta}{|q|^6}\\ \overline{\mathrm{tr}X}P-\mathrm{tr}X\overline{P} &= \frac{2}{\overline{q}}\left(-\frac{2m}{q^3}\right)-\frac{2}{q}\left(-\frac{2m}{\overline{q}^3}\right)=-\frac{4m}{\overline{q}q^3}+\frac{4m}{q\overline{q}^3}=\frac{4m(q^2-\overline{q}^2)}{|q|^6}\\ &= i\frac{16a\cos\theta mr}{|q|^6}.\end{aligned}$$

Therefore

$$\begin{aligned}
& \frac{1}{2}(3\text{tr}X - 3\overline{\text{tr}X})\overline{\text{tr}X\text{tr}\underline{X}} + 3(\overline{\text{tr}X}P - \text{tr}X\overline{P}) \\
= \; & i\frac{24a\cos\theta(r^2-2mr+a^2)}{|q|^6} + i\frac{48a\cos\theta mr}{|q|^6} = i\frac{24a\cos\theta(r^2+a^2)}{|q|^6} \\
= \; & i\frac{4a\cos\theta}{|q|^8}(6r^4 + 6a^2\cos^2\theta r^2 + 6a^2r^2 + 6a^2a^2\cos^2\theta).
\end{aligned}$$

Also,

$$\begin{aligned}
\overline{^{(c)}\mathcal{D}}\cdot \underline{H} &= 2\text{d}iv\,\underline{\eta} + 2icurl\,\underline{\eta} \\
&= 2\frac{a^2}{|q|^6}(-2\cos^2\theta r^2 + \sin^2\theta r^2 - a^2\sin^2\theta\cos^2\theta - 2a^2\cos^2\theta) \\
&\quad + i2\frac{a\cos\theta}{|q|^6}(-2r^3 + 2a^2\cos^2\theta r - 4a^2 r).
\end{aligned}$$

Also,

$$\underline{H}\cdot\overline{\underline{H}} = 2\frac{a^2}{|q|^6}(\sin^2\theta r^2 + a^2\sin^2\theta\cos^2\theta).$$

We therefore obtain in Kerr

$$\begin{aligned}
& \frac{1}{2}(3\text{tr}X - 3\overline{\text{tr}X})\overline{\text{tr}X\text{tr}\underline{X}} + 3(\overline{\text{tr}X}P - \text{tr}X\overline{P}) + 3\overline{\text{tr}X}\,\overline{^{(c)}\mathcal{D}}\cdot\underline{H} - 3\text{tr}X(\underline{H}\cdot\overline{\underline{H}}) \\
= \; & i\frac{4a\cos\theta}{|q|^8}(6r^4 + 6a^2\cos^2\theta r^2 + 6a^2r^2 + 6a^2a^2\cos^2\theta) \\
& + \frac{4a\cos\theta}{|q|^8}\Big(-6r^4 - 12a^2r^2 + 6a^2\sin^2\theta r^2 + 3a^2(-2a^2\cos^2\theta)\Big) = 0.
\end{aligned}$$

We therefore have proved, that modulo quadratic terms,

$$^{(c)}\nabla_4\mathfrak{H} + \frac{7}{2}\text{tr}X\mathfrak{H} = \text{Expr}_2(A) + \frac{3}{2}P\text{Expr}_4(A) + {}^{(c)}\nabla_4\Big(\frac{3}{2}P\overline{\text{tr}\underline{X}}A\Big) + \frac{21}{4}P\text{tr}X\overline{\text{tr}\underline{X}}A.$$

By denoting

$$\begin{aligned}
\mathcal{P}(A) &:= \text{Expr}_2(A) + \frac{3}{2}P\text{Expr}_4(A) + {}^{(c)}\nabla_4\Big(\frac{3}{2}P\overline{\text{tr}\underline{X}}A\Big) + \frac{21}{4}P\text{tr}X\overline{\text{tr}\underline{X}}A\\
&= -\frac{1}{4}({}^{(c)}\mathcal{D} + 11\,\underline{H})\widehat{\otimes}\Big(({}^{(c)}\mathcal{D} + \underline{H})\big(({}^{(c)}\mathcal{D} + 3\,\underline{H})\cdot\big({}^{(c)}\mathcal{D}\cdot\overline{A} + \overline{A}\cdot\underline{H}\big)\big)\Big)\\
&\quad -5(\underline{H}\widehat{\otimes}\,\underline{H})\big({}^{(c)}\mathcal{D} + 3\,\underline{H}\big)\cdot\big({}^{(c)}\mathcal{D}\cdot\overline{A} + \overline{A}\cdot\underline{H}\big)\\
&\quad +\frac{3}{2}P\Big(-\underline{H}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot A + A\cdot\overline{\underline{H}}\big) + {}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{H}\cdot A) + 4\,\underline{H}\widehat{\otimes}(\overline{H}\cdot A)\\
&\quad -\big(\overline{\text{tr}X\text{tr}\underline{X}} - 2\,\overline{{}^{(c)}\mathcal{D}}\cdot\underline{H} - 2P\big)A\\
&\quad +\text{tr}X\big(-{}^{(c)}\nabla_3 A - \frac{1}{2}\text{tr}\underline{X}A).\Big) + {}^{(c)}\nabla_4\Big(\frac{3}{2}P\overline{\text{tr}\underline{X}}A\Big) + \frac{21}{4}P\text{tr}X\overline{\text{tr}\underline{X}}A,
\end{aligned}$$

we have, for $r \leq r_0$

$$ {}^{(c)}\nabla_4\mathfrak{H} + \frac{7}{2}\text{tr}X\mathfrak{H} = \mathcal{P}(A) + \mathfrak{d}^{\leq 3}(\Gamma_b\cdot\Gamma_g). $$

Observe that restricting the validity of the above expression to the region $r \leq r_0$, we can denote the error terms by $\mathfrak{d}^{\leq 3}(\Gamma_b\cdot\Gamma_g)$.

Finally we show that ${}^{(c)}\nabla_4\mathfrak{H} + \frac{7}{2}\text{tr}X\mathfrak{H} = \Big({}^{(c)}\nabla_4 + 2\text{tr}X\Big)^4\underline{A}$. Recall that

$$\begin{aligned}
\mathfrak{F} &= -{}^{(c)}\nabla_4\underline{A} - \frac{1}{2}\text{tr}X\underline{A},\\
\mathfrak{G} &= {}^{(c)}\nabla_4\mathfrak{F} + \frac{3}{2}\text{tr}X\mathfrak{F},\\
\mathfrak{H} &= {}^{(c)}\nabla_4\mathfrak{G} + \frac{5}{2}\text{tr}X\mathfrak{G},
\end{aligned}$$

and

$$ \nabla_4\mathfrak{H} + \frac{7}{2}\text{tr}X\mathfrak{H} = \mathcal{P}(A). $$

Choosing a normalization such that $\omega = O(\epsilon)$, and hence $\text{tr}X = \frac{2}{q} + O(\epsilon)$, we obtain

$$ \mathfrak{F} = -\frac{1}{q}\,{}^{(c)}\nabla_4(q\underline{A}), \qquad \mathfrak{G} = \frac{1}{q^3}\nabla_4(q^3\mathfrak{F}), \qquad \mathfrak{H} = \frac{1}{q^5}\nabla_4(q^5\mathfrak{G}), \qquad \frac{1}{q^7}\nabla_4(q^7\mathfrak{H}) = \mathcal{P}(A). $$

We infer

$$ \frac{1}{q^7}\nabla_4\left(q^2\nabla_4\left(q^2\nabla_4\left(q^2\,{}^{(c)}\nabla_4(q\underline{A})\right)\right)\right) = \mathcal{P}(A). $$

The above can be written as

$$\begin{aligned}
\frac{1}{q^3}\nabla_4\left(q^2\nabla_4\right)^3(q\underline{A}) &= \frac{1}{q^2}\nabla_4\left(q^2\nabla_4\right)^2\Big(q^3\nabla_4\underline{A}+q^2\underline{A}\Big)\\
&= \frac{1}{q^3}\nabla_4\left(q^2\nabla_4\right)\Big(q^5\nabla_4^2\underline{A}+4q^4\nabla_4\underline{A}+2q^3\underline{A}\Big)\\
&= \frac{1}{q^3}\nabla_4\Big(q^7\nabla_4^3\underline{A}+9q^6\nabla_4^2\underline{A}+18q^5\nabla_4\underline{A}+6q^4\underline{A}\Big)\\
&= q^4\nabla_4^4\underline{A}+16q^3\nabla_4^3\underline{A}+72q^2\nabla_4^2\underline{A}+96q\nabla_4\underline{A}+24\underline{A},
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
\frac{1}{q^4}\,{}^{(c)}\nabla_4^4(q^4\underline{A}) &= q^4\nabla_4^4\underline{A}+4\nabla_4(q^4)\nabla_4^3\underline{A}+6\nabla_4^2(q^4)\nabla_4^2\underline{A}+4\nabla_4^3(q^4)\nabla_4\underline{A}+\nabla_4^4(q^4)\underline{A}\\
&= q^4\nabla_4^4\underline{A}+16q^3\nabla_4^3\underline{A}+72q^2\nabla_4^2\underline{A}+96q\nabla_4\underline{A}+24\underline{A}.
\end{aligned}$$

which can be written as

$$\Big(\nabla_4+2\mathrm{tr}X\Big)^4\underline{A}=\mathcal{P}(A).$$

This concludes the proof of Proposition 5.4.1.

D.8 Proof of Lemma 5.5.1

We make use of the Bianchi identity

$${}^{(c)}\nabla_4 P-\frac{1}{2}\,{}^{(c)}\mathcal{D}\cdot\overline{B} = -\frac{3}{2}\mathrm{tr}XP+\underline{H}\cdot\overline{B}-\Xi\cdot\underline{B}-\frac{1}{4}\widehat{\underline{X}}\cdot\overline{A}.$$

We differentiate w.r.t. ${}^{(c)}\nabla_3$ and obtain

$$\begin{aligned}
&{}^{(c)}\nabla_3\,{}^{(c)}\nabla_4 P-\frac{1}{2}\,{}^{(c)}\mathcal{D}\cdot\,{}^{(c)}\nabla_3\overline{B}-\frac{1}{2}[\,{}^{(c)}\nabla_3,\,{}^{(c)}\mathcal{D}\cdot]\overline{B}\\
=\;& -\frac{3}{2}P\,{}^{(c)}\nabla_3\mathrm{tr}X-\frac{3}{2}\mathrm{tr}X\,{}^{(c)}\nabla_3 P+\,{}^{(c)}\nabla_3\,\underline{H}\cdot\overline{B}+\underline{H}\cdot\,{}^{(c)}\nabla_3\overline{B}\\
&-\,{}^{(c)}\nabla_3(\Xi\cdot\underline{B})-\frac{1}{4}\,{}^{(c)}\nabla_3(\widehat{\underline{X}}\cdot\overline{A}).
\end{aligned}$$

Next we make use of

$${}^{(c)}\nabla_3 B-\,{}^{(c)}\mathcal{D}\overline{P} = -\mathrm{tr}\underline{X}B+3\overline{P}H+\overline{\underline{B}}\cdot\widehat{X}+\frac{1}{2}A\cdot\underline{\Xi},$$

and taking the complex conjugate, we infer

$$
{}^{(c)}\nabla_3\overline{B} - \overline{{}^{(c)}\mathcal{D}}P \;=\; -\overline{\mathrm{tr}\underline{X}B} + 3P\overline{H} + \underline{B}\cdot\overline{\widehat{X}} + \frac{1}{2}\overline{A}\cdot\underline{\Xi}.
$$

We deduce

$$
\begin{aligned}
& {}^{(c)}\nabla_3\,{}^{(c)}\nabla_4 P - \frac{1}{2}\,{}^{(c)}\mathcal{D}\cdot\Big(\overline{{}^{(c)}\mathcal{D}}P - \overline{\mathrm{tr}\underline{X}B} + 3P\overline{H}\Big) - \frac{1}{2}[\,{}^{(c)}\nabla_3,\,{}^{(c)}\mathcal{D}\cdot]\overline{B}\\
= \;& -\frac{3}{2}P\,{}^{(c)}\nabla_3\mathrm{tr}X - \frac{3}{2}\mathrm{tr}X\,{}^{(c)}\nabla_3 P + {}^{(c)}\nabla_3\,\underline{H}\cdot\overline{B} + \underline{H}\cdot\Big(\overline{{}^{(c)}\mathcal{D}}P - \overline{\mathrm{tr}\underline{X}B} + 3P\overline{H}\Big)\\
& - {}^{(c)}\nabla_3(\overline{\underline{\Xi}}\cdot\underline{B}) - \frac{1}{4}\,{}^{(c)}\nabla_3(\underline{\widehat{X}}\cdot\overline{A}) + \frac{1}{2}\,{}^{(c)}\mathcal{D}\cdot\Big(\underline{B}\cdot\overline{\widehat{X}} + \frac{1}{2}\overline{A}\cdot\underline{\Xi}\Big)\\
& + \underline{H}\cdot\Big(\underline{B}\cdot\overline{\widehat{X}} + \frac{1}{2}\overline{A}\cdot\underline{\Xi}\Big),
\end{aligned}
$$

and hence

$$
\begin{aligned}
& {}^{(c)}\nabla_3\,{}^{(c)}\nabla_4 P + \frac{3}{2}P\,{}^{(c)}\nabla_3\mathrm{tr}X + \frac{3}{2}\mathrm{tr}X\,{}^{(c)}\nabla_3 P\\
& -\frac{1}{2}\,{}^{(c)}\mathcal{D}\cdot(\overline{{}^{(c)}\mathcal{D}}P + 3P\overline{H}) - \underline{H}\cdot(\overline{{}^{(c)}\mathcal{D}}P + 3P\overline{H})\\
= \;& -\frac{1}{2}\,{}^{(c)}\mathcal{D}\cdot\Big(\overline{\mathrm{tr}\underline{X}B}\Big) + \frac{1}{2}[\,{}^{(c)}\nabla_3,\,{}^{(c)}\mathcal{D}\cdot]\overline{B} + {}^{(c)}\nabla_3\,\underline{H}\cdot\overline{B} - \overline{\mathrm{tr}\underline{X}}\,\underline{H}\cdot\overline{B}\\
& - {}^{(c)}\nabla_3(\overline{\underline{\Xi}}\cdot\underline{B}) - \frac{1}{4}\,{}^{(c)}\nabla_3(\underline{\widehat{X}}\cdot\overline{A}) + \frac{1}{2}\,{}^{(c)}\mathcal{D}\cdot\Big(\underline{B}\cdot\overline{\widehat{X}} + \frac{1}{2}\overline{A}\cdot\underline{\Xi}\Big)\\
& + \underline{H}\cdot\Big(\underline{B}\cdot\overline{\widehat{X}} + \frac{1}{2}\overline{A}\cdot\underline{\Xi}\Big).
\end{aligned}
$$

Next we make use of the commutator, see Lemma 4.2.2,

$$
\begin{aligned}
[\,{}^{(c)}\nabla_3,\,\overline{{}^{(c)}\mathcal{D}}\cdot]F \;=\; & -\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\,(\overline{{}^{(c)}\mathcal{D}}\cdot F + (s-1)\overline{H}\cdot F) + \overline{H}\cdot\,{}^{(c)}\nabla_3 F\\
& -\underline{B}\cdot\overline{F} + \overline{\underline{\Xi}}\cdot\nabla_4 F - \underline{\widehat{X}}\cdot\overline{{}^{(c)}\mathcal{D}}F - H\cdot\underline{\widehat{X}}\cdot\overline{F},
\end{aligned}
$$

which for B and $s=1$, we deduce

$$
\begin{aligned}
[\,{}^{(c)}\nabla_3,\,{}^{(c)}\mathcal{D}\cdot]\overline{B} \;=\; & -\frac{1}{2}\mathrm{tr}\underline{X}\;{}^{(c)}\mathcal{D}\cdot\overline{B} + H\cdot\,{}^{(c)}\nabla_3\overline{B}\\
& -\overline{\underline{B}}\cdot B + \underline{\Xi}\cdot\nabla_4\overline{B} - \overline{\underline{\widehat{X}}}\cdot\,{}^{(c)}\mathcal{D}\overline{B} - \overline{H}\cdot\overline{\underline{\widehat{X}}}\cdot B.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& {}^{(c)}\nabla_3\, {}^{(c)}\nabla_4 P + \frac{3}{2} P\, {}^{(c)}\nabla_3 \mathrm{tr} X + \frac{3}{2}\mathrm{tr} X\, {}^{(c)}\nabla_3 P \\
& -\frac{1}{2}\, {}^{(c)}\mathcal{D}\cdot\big(\overline{{}^{(c)}\mathcal{D}} P + 3P\overline{H}\big) - \underline{H}\cdot\big(\overline{{}^{(c)}\mathcal{D}} P + 3P\overline{H}\big) \\
= & -\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\, {}^{(c)}\mathcal{D}\cdot\overline{B} - \frac{1}{2}\, {}^{(c)}\mathcal{D}\overline{\mathrm{tr}\underline{X}}\cdot\overline{B} + \frac{1}{2}\Big(-\frac{1}{2}\mathrm{tr}\underline{X}\, {}^{(c)}\mathcal{D}\cdot\overline{B} + H\cdot\, {}^{(c)}\nabla_3\overline{B}\Big) \\
& + {}^{(c)}\nabla_3\, \underline{H}\cdot\overline{B} - \overline{\mathrm{tr}\underline{X}}\, \underline{H}\cdot\overline{B} \\
& - {}^{(c)}\nabla_3(\overline{\Xi}\cdot\underline{B}) - \frac{1}{4}\, {}^{(c)}\nabla_3(\widehat{\underline{X}}\cdot\overline{A}) + \frac{1}{2}\, {}^{(c)}\mathcal{D}\cdot\big(\underline{B}\cdot\overline{\widehat{X}} + \frac{1}{2}\overline{A}\cdot\underline{\Xi}\big) \\
& + \underline{H}\cdot\big(\underline{B}\cdot\overline{\widehat{X}} + \frac{1}{2}\overline{A}\cdot\underline{\Xi}\big) \\
& + \frac{1}{2}\Big(-\overline{\underline{B}}\cdot B + \underline{\Xi}\cdot\nabla_4\overline{B} - \overline{\widehat{\underline{X}}}\cdot\, {}^{(c)}\mathcal{D}\overline{B} - \overline{H}\cdot\overline{\widehat{\underline{X}}}\cdot B\Big).
\end{aligned}
$$

Observe that the LHS of the above becomes

$$
\begin{aligned}
LHS = & {}^{(c)}\nabla_3\, {}^{(c)}\nabla_4 P + \frac{3}{2}\mathrm{tr} X\, {}^{(c)}\nabla_3 P - \frac{1}{2}\, {}^{(c)}\mathcal{D}\cdot\overline{{}^{(c)}\mathcal{D}} P - \frac{3}{2}\overline{H}\cdot\, {}^{(c)}\mathcal{D} P - \underline{H}\cdot\overline{{}^{(c)}\mathcal{D}} P \\
& + \frac{3}{2}\Big[-\, {}^{(c)}\mathcal{D}\cdot\overline{H} + {}^{(c)}\nabla_3 \mathrm{tr} X - 2\,\underline{H}\cdot\overline{H}\Big] P,
\end{aligned}
$$

while the RHS, using

$$
\begin{aligned}
\frac{1}{2}\, {}^{(c)}\mathcal{D}\cdot\overline{B} &= {}^{(c)}\nabla_4 P + \frac{3}{2}\mathrm{tr} X P - \underline{H}\cdot\overline{B} + \overline{\Xi}\cdot\underline{B} + \frac{1}{4}\widehat{\underline{X}}\cdot\overline{A}, \\
{}^{(c)}\nabla_3\overline{B} &= \overline{{}^{(c)}\mathcal{D}} P + 3P\overline{H} - \overline{\mathrm{tr}\underline{X}B} + \underline{B}\cdot\overline{\widehat{X}} + \frac{1}{2}\overline{A}\cdot\underline{\Xi} \\
-\frac{1}{2}\, {}^{(c)}\mathcal{D}\overline{\mathrm{tr}\underline{X}} &= -i\Im(\mathrm{tr}\underline{X})\,\underline{H} - \frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot\widehat{\underline{X}} - i\Im(\mathrm{tr}X)\underline{\Xi} + \underline{B} \\
{}^{(c)}\nabla_3\,\underline{H} &= -\frac{1}{2}\overline{\mathrm{tr}\underline{X}}(\,\underline{H} - H) + {}^{(c)}\nabla_4\underline{\Xi} - \frac{1}{2}\widehat{\underline{X}}\cdot(\,\overline{\underline{H}} - \overline{H}) + \underline{B},
\end{aligned}
$$

gives the following

$$
\begin{aligned}
RHS &= -\frac{1}{4}\big(2\overline{\mathrm{tr}\underline{X}}+\mathrm{tr}\underline{X}\big)\,{}^{(c)}\mathcal{D}\cdot\overline{B}+\frac{1}{2}H\cdot{}^{(c)}\nabla_3\overline{B}\\
&\quad+\Big(-\frac{1}{2}{}^{(c)}\mathcal{D}\overline{\mathrm{tr}\underline{X}}+{}^{(c)}\nabla_3\,\underline{H}-\overline{\mathrm{tr}\underline{X}}\,\underline{H}\Big)\cdot\overline{B}-{}^{(c)}\nabla_3(\overline{\Xi}\cdot\underline{B})\\
&\quad-\frac{1}{4}{}^{(c)}\nabla_3(\widehat{\underline{X}}\cdot\overline{A})+\frac{1}{2}{}^{(c)}\mathcal{D}\cdot\Big(\underline{B}\cdot\overline{\widehat{X}}+\frac{1}{2}\overline{A}\cdot\underline{\Xi}\Big)+\underline{H}\cdot\Big(\underline{B}\cdot\overline{\widehat{X}}+\frac{1}{2}\overline{A}\cdot\underline{\Xi}\Big)\\
&\quad+\frac{1}{2}\Big(-\overline{\underline{B}}\cdot B+\underline{\Xi}\cdot\nabla_4\overline{B}-\overline{\widehat{\underline{X}}}\cdot{}^{(c)}\mathcal{D}\overline{B}-\overline{H}\cdot\overline{\widehat{\underline{X}}}\cdot B\Big)\\
&= -\frac{1}{2}\big(2\overline{\mathrm{tr}\underline{X}}+\mathrm{tr}\underline{X}\big)\Big({}^{(c)}\nabla_4 P+\frac{3}{2}\mathrm{tr}XP-\underline{H}\cdot\overline{B}\Big)\\
&\quad+\frac{1}{2}H\cdot\Big(\overline{{}^{(c)}\mathcal{D}}P+3P\overline{H}-\overline{\mathrm{tr}\underline{X}B}\Big)\\
&\quad+\Big(-i\Im(\mathrm{tr}\underline{X})\,\underline{H}-\frac{1}{2}\overline{\mathrm{tr}\underline{X}}(\,\underline{H}-H)-\overline{\mathrm{tr}\underline{X}}\,\underline{H}\Big)\cdot\overline{B}+\mathrm{Err}\\
&= -\frac{1}{2}\big(2\overline{\mathrm{tr}\underline{X}}+\mathrm{tr}\underline{X}\big)\Big({}^{(c)}\nabla_4 P+\frac{3}{2}\mathrm{tr}XP\Big)+\frac{1}{2}H\cdot\Big(\overline{{}^{(c)}\mathcal{D}}P+3P\overline{H}\Big)+\mathrm{Err}
\end{aligned}
$$

where

$$
\begin{aligned}
\mathrm{Err} &= -{}^{(c)}\nabla_3(\overline{\Xi}\cdot\underline{B})-\frac{1}{4}{}^{(c)}\nabla_3(\widehat{\underline{X}}\cdot\overline{A})+\frac{1}{2}{}^{(c)}\mathcal{D}\cdot\Big(\underline{B}\cdot\overline{\widehat{X}}+\frac{1}{2}\overline{A}\cdot\underline{\Xi}\Big)\\
&\quad+\underline{H}\cdot\Big(\underline{B}\cdot\overline{\widehat{X}}+\frac{1}{2}\overline{A}\cdot\underline{\Xi}\Big)\\
&\quad+\frac{1}{2}\Big(-\overline{\underline{B}}\cdot B+\underline{\Xi}\cdot\nabla_4\overline{B}-\overline{\widehat{\underline{X}}}\cdot{}^{(c)}\mathcal{D}\overline{B}-\overline{H}\cdot\overline{\widehat{\underline{X}}}\cdot B\Big)\\
&\quad-\frac{1}{2}\big(2\overline{\mathrm{tr}\underline{X}}+\mathrm{tr}\underline{X}\big)\Big(\overline{\Xi}\cdot\underline{B}+\frac{1}{4}\widehat{\underline{X}}\cdot\overline{A}\Big)+\frac{1}{2}H\cdot\Big(\underline{B}\cdot\overline{\widehat{X}}+\frac{1}{2}\overline{A}\cdot\underline{\Xi}\Big)\\
&\quad+\Big(-\frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot\widehat{\underline{X}}-i\Im(\mathrm{tr}X)\underline{\Xi}+\underline{B}+{}^{(c)}\nabla_4\underline{\Xi}-\frac{1}{2}\widehat{\underline{X}}\cdot(\overline{\underline{H}}-\overline{H})+\underline{B}\Big)\cdot\overline{B}.
\end{aligned}
$$

By putting the two together we obtain

$$
\begin{aligned}
&{}^{(c)}\nabla_3\,{}^{(c)}\nabla_4 P+\frac{3}{2}\mathrm{tr}X\,{}^{(c)}\nabla_3 P+\frac{1}{2}\big(2\overline{\mathrm{tr}\underline{X}}+\mathrm{tr}\underline{X}\big)\,{}^{(c)}\nabla_4 P-\frac{1}{2}{}^{(c)}\mathcal{D}\cdot\overline{{}^{(c)}\mathcal{D}}P\\
&-\frac{3}{2}\overline{H}\cdot{}^{(c)}\mathcal{D}P-\Big(\underline{H}+\frac{1}{2}H\Big)\cdot\overline{{}^{(c)}\mathcal{D}}P+\frac{3}{2}\Big[\overline{\mathrm{tr}\underline{X}}\mathrm{tr}X+2P-2\,\underline{H}\cdot\overline{H}\Big]P\\
=\ &\mathrm{Err}-\frac{3}{2}\Big(\underline{\Xi}\cdot\overline{\Xi}-\frac{1}{2}\widehat{\underline{X}}\cdot\overline{\widehat{X}}\Big)P,
\end{aligned}
$$

where we also used

$$
{}^{(c)}\nabla_3\mathrm{tr}X+\frac{1}{2}\mathrm{tr}\underline{X}\mathrm{tr}X = {}^{(c)}\mathcal{D}\cdot\overline{H}+H\cdot\overline{H}+2P+\underline{\Xi}\cdot\overline{\Xi}-\frac{1}{2}\widehat{\underline{X}}\cdot\overline{\widehat{X}}.
$$

From Lemma 4.7.4, we deduce

$$\square_{\mathbf{g}} P = -e_3(e_4(P)) + \left(2\underline{\omega} - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\right)\nabla_4 P - \frac{1}{2}tr\,\chi\nabla_3 P + 2\eta\cdot\nabla P + \Delta P.$$

Also, we have

$$\begin{aligned}
\frac{1}{2}\mathcal{D}\cdot\overline{\mathcal{D}}P &= \frac{1}{2}(\nabla + i\,{}^*\nabla)\cdot(\nabla - i\,{}^*\nabla)P = \Delta P + \frac{i}{2}({}^*\nabla\cdot\nabla - \nabla\cdot{}^*\nabla)P\\
&= \Delta P + i(\nabla_2\nabla_1 - \nabla_1\nabla_2)P\\
&= \Delta P + i\Big((\mathbf{D}_2\mathbf{D}_1 - \mathbf{D}_1\mathbf{D}_2)P + \mathbf{D}_{\mathbf{D}_2 e1 - \nabla_2 e_1 - \mathbf{D}_1 e_2 + \nabla_1 e_2}P\Big)\\
&= \Delta P + \frac{i}{2}\Big((\chi_{21} - \chi_{12})e_3 P + (\underline{\chi}_{21} - \underline{\chi}_{12})e_4 P\Big)\\
&= \Delta P - \frac{i}{2}\Big({}^{(a)}\mathrm{tr}\chi e_3 P + {}^{(a)}\mathrm{tr}\underline{\chi} e_4 P\Big),
\end{aligned}$$

and

$$\overline{H}\cdot\mathcal{D}P + H\cdot\overline{\mathcal{D}}P = (\eta - i\,{}^*\eta)\cdot(\nabla + i\,{}^*\nabla)P + (\eta + i\,{}^*\eta)\cdot(\nabla - i\,{}^*\nabla)P = 4\eta\cdot\nabla P.$$

We therefore deduce for P of conformal type 0:

$$\begin{aligned}
\square_{\mathbf{g}} P &= -{}^{(c)}\nabla_3\,{}^{(c)}\nabla_4 P + \frac{1}{2}\,{}^{(c)}\mathcal{D}\cdot\overline{{}^{(c)}\mathcal{D}}P - \frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(c)}\nabla_4 P - \frac{1}{2}\mathrm{tr}X\,{}^{(c)}\nabla_3 P\\
&\quad + \frac{1}{2}\overline{H}\cdot{}^{(c)}\mathcal{D}P + \frac{1}{2}H\cdot\overline{{}^{(c)}\mathcal{D}}P.
\end{aligned}$$

We then obtain from the above computations

$$\begin{aligned}
\square_{\mathbf{g}} P &= \mathrm{tr}X\,{}^{(c)}\nabla_3 P + \overline{\mathrm{tr}\underline{X}}\,{}^{(c)}\nabla_4 P - \overline{H}\cdot{}^{(c)}\mathcal{D}P - \underline{H}\cdot\overline{{}^{(c)}\mathcal{D}}P\\
&\quad + \frac{3}{2}\Big[\overline{\mathrm{tr}\underline{X}}\mathrm{tr}X + 2P - 2\,\underline{H}\cdot\overline{H}\Big]P + \mathrm{Err}[\square_{\mathbf{g}} P],
\end{aligned}$$

where

$$\mathrm{Err}[\square_{\mathbf{g}} P] = \mathrm{Err} - \frac{3}{2}\big(\underline{\Xi}\cdot\overline{\Xi} - \frac{1}{2}\widehat{\underline{X}}\cdot\overline{\widehat{X}}\big)P$$

as stated.

D.9 Proof of Proposition 5.6.1

From the Bianchi identity

$${}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A} = -\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\underline{B} - 2\,\underline{H}\widehat{\otimes}\underline{B} - 3P\widehat{\underline{X}},$$

we infer

$$
\begin{aligned}
{}^{(c)}\nabla_4\big({}^{(c)}\nabla_4\underline{A}+\frac{1}{2}\mathrm{tr}X\underline{A}\big) &= -\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\nabla_4\underline{B}-\frac{1}{2}[\,{}^{(c)}\nabla_4,\,{}^{(c)}\mathcal{D}\widehat{\otimes}]\underline{B}\\
&\quad -2\,\underline{H}\widehat{\otimes}\,{}^{(c)}\nabla_4\underline{B}-2\,{}^{(c)}\nabla_4\,\underline{H}\widehat{\otimes}\underline{B}-3P\,{}^{(c)}\nabla_4\widehat{\underline{X}}-3\,{}^{(c)}\nabla_4P\widehat{\underline{X}}.
\end{aligned}
$$

Using the commutator (4.2.13), i.e.

$$
\begin{aligned}
[\,{}^{(c)}\nabla_4,\,{}^{(c)}\mathcal{D}\widehat{\otimes}]\underline{B} &= -\frac{1}{2}\mathrm{tr}X\,\big({}^{(c)}\mathcal{D}\widehat{\otimes}\underline{B}+2\,\underline{H}\widehat{\otimes}\underline{B}\big)+\underline{H}\widehat{\otimes}\,{}^{(c)}\nabla_4\underline{B}\\
&\quad +\Xi\widehat{\otimes}\,{}^{(c)}\nabla_3\underline{B}+r^{-1}\Gamma_g\cdot\mathfrak{d}^{\leq1}\underline{B},
\end{aligned}
$$

we have

$$
\begin{aligned}
{}^{(c)}\nabla_4\big({}^{(c)}\nabla_4\underline{A}+\frac{1}{2}\mathrm{tr}X\underline{A}\big) &= -\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\nabla_4\underline{B}-\frac{5}{2}\,\underline{H}\widehat{\otimes}\,{}^{(c)}\nabla_4\underline{B}\\
&\quad +\frac{1}{4}\mathrm{tr}X\,\big({}^{(c)}\mathcal{D}\widehat{\otimes}\underline{B}+2\,\underline{H}\widehat{\otimes}\underline{B}\big)\\
&\quad -2\,{}^{(c)}\nabla_4\,\underline{H}\widehat{\otimes}\underline{B}-3P\,{}^{(c)}\nabla_4\widehat{\underline{X}}-3\,{}^{(c)}\nabla_4P\widehat{\underline{X}}\\
&\quad +\Xi\widehat{\otimes}\,{}^{(c)}\nabla_3\underline{B}+r^{-1}\Gamma_g\cdot\mathfrak{d}^{\leq1}\underline{B}.
\end{aligned}
$$

Next, using

$$
\begin{aligned}
{}^{(c)}\nabla_4\underline{B}+{}^{(c)}\mathcal{D}P &= -\mathrm{tr}X\underline{B}-3P\,\underline{H}+\Gamma_b\cdot B+\Xi\cdot\underline{A},\\
{}^{(c)}\nabla_4P-\frac{1}{2}\,{}^{(c)}\mathcal{D}\cdot\overline{\underline{B}} &= -\frac{3}{2}\mathrm{tr}XP+\,\underline{H}\cdot\overline{B}+\Xi\cdot\underline{B}+\Gamma_b\cdot A,\\
{}^{(c)}\nabla_4\widehat{\underline{X}}+\frac{1}{2}\mathrm{tr}X\,\widehat{\underline{X}} &= \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,\underline{H}+\frac{1}{2}\,\underline{H}\widehat{\otimes}\,\underline{H}-\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\widehat{X}+\frac{1}{4}\Xi\widehat{\otimes}\underline{\Xi}
\end{aligned}
$$

we deduce

$$
\begin{aligned}
&{}^{(c)}\nabla_4\big({}^{(c)}\nabla_4\underline{A}+\frac{1}{2}\mathrm{tr}X\underline{A}\big)\\
=\;& \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\big({}^{(c)}\mathcal{D}P+\mathrm{tr}X\underline{B}+3P\,\underline{H}\big)+\frac{5}{2}\,\underline{H}\widehat{\otimes}\big({}^{(c)}\mathcal{D}P+\mathrm{tr}X\underline{B}+3P\,\underline{H}\big)\\
&+\frac{1}{4}\mathrm{tr}X\,\big({}^{(c)}\mathcal{D}\widehat{\otimes}\underline{B}+2\,\underline{H}\widehat{\otimes}\underline{B}\big)-2\,{}^{(c)}\nabla_4\,\underline{H}\widehat{\otimes}\underline{B}\\
&-3P\big(-\frac{1}{2}\mathrm{tr}X\,\widehat{\underline{X}}+\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,\underline{H}+\frac{1}{2}\,\underline{H}\widehat{\otimes}\,\underline{H}-\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\widehat{X}\big)-3\big(-\frac{3}{2}\mathrm{tr}XP\big)\widehat{\underline{X}}\\
&+\Xi\widehat{\otimes}\,{}^{(c)}\nabla_3\underline{B}+r^{-1}\mathfrak{d}^{\leq1}(\Xi\cdot\underline{A})+r^{-1}\Gamma_g\cdot\mathfrak{d}^{\leq1}\underline{B}+r^{-1}\mathfrak{d}^{\leq1}(\Gamma_b\cdot B)+r^{-3}\Xi\cdot\underline{\Xi},
\end{aligned}
$$

which gives

$$
\begin{aligned}
& {}^{(c)}\nabla_4\big({}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A}\big) \\
= \quad & \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}P - \frac{3}{2}\mathrm{tr}X\big(-\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\underline{B} - 2\,\underline{H}\widehat{\otimes}\underline{B} - 3P\widehat{\underline{X}}\big) + 3P\big(\frac{1}{2}\mathrm{tr}X\,\widehat{\underline{X}} + \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\widehat{X}\big) \\
& +\big(\frac{1}{2}\,{}^{(c)}\mathcal{D}\mathrm{tr}X - 2\,{}^{(c)}\nabla_4\,\underline{H}\big)\widehat{\otimes}\underline{B} + \big(4\,{}^{(c)}\mathcal{D}P + 6P\,\underline{H}\big)\widehat{\otimes}\,\underline{H} \\
& +\Xi\widehat{\otimes}\,{}^{(c)}\nabla_3\underline{B} + r^{-1}\mathfrak{d}^{\leq 1}(\Xi\cdot\underline{A}) + r^{-1}\Gamma_g\cdot\mathfrak{d}^{\leq 1}\underline{B} + r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b\cdot B) + r^{-3}\Xi\cdot\underline{\Xi}.
\end{aligned}
$$

Using that $-\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\underline{B} - 2\,\underline{H}\widehat{\otimes}\underline{B} - 3P\widehat{\underline{X}} = {}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A}$, we have

$$
\begin{aligned}
& {}^{(c)}\nabla_4\big({}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A}\big) + \frac{3}{2}\mathrm{tr}X\big({}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A}\big) \\
= \quad & \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}P + \frac{3}{2}P\big(\mathrm{tr}X\,\widehat{\underline{X}} + \overline{\mathrm{tr}\underline{X}}\widehat{X}\big) \\
& +\big(\frac{1}{2}\,{}^{(c)}\mathcal{D}\mathrm{tr}X - 2\,{}^{(c)}\nabla_4\,\underline{H}\big)\widehat{\otimes}\underline{B} + \big(4\,{}^{(c)}\mathcal{D}P + 6P\,\underline{H}\big)\widehat{\otimes}\,\underline{H} \\
& +\Xi\widehat{\otimes}\,{}^{(c)}\nabla_3\underline{B} + r^{-1}\mathfrak{d}^{\leq 1}(\Xi\cdot\underline{A}) + r^{-1}\Gamma_g\cdot\mathfrak{d}^{\leq 1}\underline{B} + r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b\cdot B) + r^{-3}\Xi\cdot\underline{\Xi}.
\end{aligned}
$$

Using (4.1.13), we deduce that $\frac{1}{2}\,{}^{(c)}\mathcal{D}\mathrm{tr}X - 2\,{}^{(c)}\nabla_4\,\underline{H} = \mathrm{tr}X\,\underline{H} + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_g$, which gives

$$
\begin{aligned}
& {}^{(c)}\nabla_4\big({}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A}\big) + \frac{3}{2}\mathrm{tr}X\big({}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A}\big) \\
& = \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}P + \big(4\,{}^{(c)}\mathcal{D}P + 6P\,\underline{H} + \mathrm{tr}X\underline{B}\big)\widehat{\otimes}\,\underline{H} + \frac{3}{2}P\big(\mathrm{tr}X\,\widehat{\underline{X}} + \overline{\mathrm{tr}\underline{X}}\widehat{X}\big) \qquad (D.9.1) \\
& + \Xi\widehat{\otimes}\,{}^{(c)}\nabla_3\underline{B} + r^{-1}\mathfrak{d}^{\leq 1}(\Xi\cdot\underline{A}) + r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\underline{B}) + r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b\cdot B).
\end{aligned}
$$

Consider the LHS of (D.9.1). Using that ${}^{(c)}\nabla_4\mathrm{tr}X + \frac{1}{2}(\mathrm{tr}X)^2 = r^{-1}\mathfrak{d}^{\leq 1}\Xi + \Gamma_g\cdot\Gamma_g$, we obtain

$$
\begin{aligned}
LHS \quad = \quad & {}^{(c)}\nabla_4\big({}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A}\big) + \frac{3}{2}\mathrm{tr}X\big({}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A}\big) \\
= \quad & {}^{(c)}\nabla_4\,{}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\,{}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\,{}^{(c)}\nabla_4\mathrm{tr}X\underline{A} + \frac{3}{2}\mathrm{tr}X\big({}^{(c)}\nabla_4\underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A}\big) \\
= \quad & {}^{(c)}\nabla_4\,{}^{(c)}\nabla_4\underline{A} + 2\mathrm{tr}X\,{}^{(c)}\nabla_4\underline{A} + \frac{1}{2}(\mathrm{tr}X)^2\underline{A} + r^{-1}\mathfrak{d}^{\leq 1}\Xi\cdot\underline{A}.
\end{aligned}
$$

Using the definition (5.3.2), we deduce

$$
\begin{aligned}
LHS \quad = \quad & \underline{Q}(\underline{A}) + \left(2\frac{{}^{(a)}\mathrm{tr}\chi^2}{tr\,\chi} + 2i\,{}^{(a)}\mathrm{tr}\chi\right)\,{}^{(c)}\nabla_4\underline{A} \\
& + \left(-\frac{3}{2}\frac{{}^{(a)}\mathrm{tr}\chi^4}{tr\,\chi^2} + \frac{7}{2}\,{}^{(a)}\mathrm{tr}\chi^2 + i tr\,\chi\,{}^{(a)}\mathrm{tr}\chi - 4i\frac{{}^{(a)}\mathrm{tr}\chi^3}{tr\,\chi}\right)\underline{A} + r^{-1}\mathfrak{d}^{\leq 1}\Xi\cdot\underline{A} \\
= \quad & \underline{Q}(\underline{A}) + O(ar^{-3})\mathfrak{d}^{\leq 1}\underline{A} + r^{-2}\Gamma_g\cdot\mathfrak{d}^{\leq 1}\underline{A} + r^{-1}\mathfrak{d}^{\leq 1}\Xi\cdot\underline{A}.
\end{aligned}
$$

Writing ${}^{(c)}\nabla_4\underline{A}=\underline{A}_4-\frac{1}{2}\mathrm{tr}X\underline{A}$, we have

$$\begin{aligned}
LHS &= \underline{Q}(\underline{A})+\left(2\frac{{}^{(a)}\mathrm{tr}\chi^2}{tr\,\chi}+2i\,{}^{(a)}\mathrm{tr}\chi\right)\left(\underline{A}_4-\frac{1}{2}(tr\,\chi-i\,{}^{(a)}\mathrm{tr}\chi)\underline{A}\right)\\
&\quad+\left(-\frac{3}{2}\frac{{}^{(a)}\mathrm{tr}\chi^4}{tr\,\chi^2}+\frac{7}{2}\,{}^{(a)}\mathrm{tr}\chi^2+itr\,\chi\,{}^{(a)}\mathrm{tr}\chi-4i\frac{{}^{(a)}\mathrm{tr}\chi^3}{tr\,\chi}\right)\underline{A}+r^{-1}\mathfrak{d}^{\leq 1}\Xi\cdot\underline{A}\\
&= \underline{Q}(\underline{A})+\left(2\frac{{}^{(a)}\mathrm{tr}\chi^2}{tr\,\chi}+2i\,{}^{(a)}\mathrm{tr}\chi\right)\underline{A}_4\\
&\quad+\left(-\frac{3}{2}\frac{{}^{(a)}\mathrm{tr}\chi^4}{tr\,\chi^2}+\frac{3}{2}\,{}^{(a)}\mathrm{tr}\chi^2-3i\frac{{}^{(a)}\mathrm{tr}\chi^3}{tr\,\chi}\right)\underline{A}+r^{-1}\mathfrak{d}^{\leq 1}\Xi\cdot\underline{A}\\
&= \underline{Q}(\underline{A})+O(ar^{-2})\underline{A}_4+O(a^2r^{-4})\underline{A}+r^{-1}\mathfrak{d}^{\leq 1}\Xi\cdot\underline{A}.
\end{aligned}$$

Using again the Bianchi identity for $\underline{A}_4$ we also have

$$\begin{aligned}
LHS &= \underline{Q}(\underline{A})+O(ar^{-5})\Gamma_b+O(ar^{-3})\mathfrak{d}^{\leq 1}\underline{B}+O(ar^{-4})\underline{A}\\
&\quad+r^{-2}\Gamma_b\cdot\check{P}+r^{-1}\mathfrak{d}^{\leq 1}\Xi\cdot\underline{A}+r^{-2}\Gamma_b\cdot\underline{B}.
\end{aligned}$$

Now consider the RHS of (D.9.1). Using (D.5.6), i.e. ${}^{(c)}\mathcal{D}P=-3P\,\underline{H}+{}^{(c)}\mathcal{D}\check{P}+r^{-3}\Gamma_g$, we obtain

$$\begin{aligned}
RHS &= \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(-3P\,\underline{H}+{}^{(c)}\mathcal{D}\check{P})+\big(4(-3P\,\underline{H}+{}^{(c)}\mathcal{D}\check{P})+6P\,\underline{H}+\mathrm{tr}X\underline{B}\big)\widehat{\otimes}\,\underline{H}\\
&\quad+\frac{3}{2}P\big(\mathrm{tr}X\,\widehat{\underline{X}}+\overline{\mathrm{tr}\underline{X}}\widehat{X}\big)\\
&\quad+r^{-4}\,\not{\mathfrak{d}}^{\leq 1}\Gamma_g+\Xi\widehat{\otimes}\,{}^{(c)}\nabla_3\underline{B}+r^{-1}\mathfrak{d}^{\leq 1}(\Xi\cdot\underline{A})+r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\underline{B})+r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b\cdot B)\\
&= \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\check{P}-\frac{3}{2}\,{}^{(c)}\mathcal{D}P\widehat{\otimes}\,\underline{H}-\frac{3}{2}P\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,\underline{H}\\
&\quad+\big(4({}^{(c)}\mathcal{D}\check{P})-6P\,\underline{H}+\mathrm{tr}X\underline{B}\big)\widehat{\otimes}\,\underline{H}+\frac{3}{2}P\big(\mathrm{tr}X\,\widehat{\underline{X}}+\overline{\mathrm{tr}\underline{X}}\widehat{X}\big)\\
&\quad+r^{-4}\,\not{\mathfrak{d}}^{\leq 1}\Gamma_g+\Xi\widehat{\otimes}\,{}^{(c)}\nabla_3\underline{B}+r^{-1}\mathfrak{d}^{\leq 1}(\Xi\cdot\underline{A})+r^{-1}\,\not{\mathfrak{d}}^{\leq 1}(\Gamma_g\cdot\underline{B})+r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b\cdot B).
\end{aligned}$$

Using, see (4.1.13), that ${}^{(c)}\mathcal{D}\widehat{\otimes}\,\underline{H}=-\underline{H}\widehat{\otimes}\,\underline{H}+r^{-1}\mathfrak{d}^{\leq 1}\Gamma_g$, we obtain

$$\begin{aligned}
&RHS\\
&= \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\check{P}-\frac{3}{2}(-3P\,\underline{H}+{}^{(c)}\mathcal{D}\check{P})\widehat{\otimes}\,\underline{H}-\frac{3}{2}P(-\underline{H}\widehat{\otimes}\,\underline{H})\\
&\quad+\big(4({}^{(c)}\mathcal{D}\check{P})-6P\,\underline{H}+\mathrm{tr}X\underline{B}\big)\widehat{\otimes}\,\underline{H}+\frac{3}{2}P\big(\mathrm{tr}X\,\widehat{\underline{X}}+\overline{\mathrm{tr}\underline{X}}\widehat{X}\big)+r^{-1}\mathfrak{d}^{\leq 1}\Gamma_g\cdot\check{P}\\
&\quad+r^{-4}\,\not{\mathfrak{d}}^{\leq 1}\Gamma_g+\Xi\widehat{\otimes}\,{}^{(c)}\nabla_3\underline{B}+r^{-1}\mathfrak{d}^{\leq 1}(\Xi\cdot\underline{A})+r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\underline{B})+r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b\cdot B)\\
&= \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\check{P}+\frac{5}{2}({}^{(c)}\mathcal{D}\check{P})\widehat{\otimes}\,\underline{H}+\mathrm{tr}X\underline{B}\widehat{\otimes}\,\underline{H}+\frac{3}{2}P\big(\mathrm{tr}X\,\widehat{\underline{X}}+\overline{\mathrm{tr}\underline{X}}\widehat{X}\big)+r^{-1}\mathfrak{d}^{\leq 1}\Gamma_g\cdot\check{P}\\
&\quad+r^{-4}\,\not{\mathfrak{d}}^{\leq 1}\Gamma_g+\Xi\widehat{\otimes}\,{}^{(c)}\nabla_3\underline{B}+r^{-1}\mathfrak{d}^{\leq 1}(\Xi\cdot\underline{A})+r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_g\cdot\underline{B})+r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b\cdot B).
\end{aligned}$$

The above can be further simplified, and we obtain

$$\begin{aligned} RHS &= \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\check{P} + r^{-4}\,\mathfrak{d}\!\!/^{\leq 1}(\Gamma_b, r\Gamma_g) + O(ar^{-3})\underline{B} + O(ar^{-3})\mathfrak{d}^{\leq 1}\check{P} \\ &\quad + r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b \cdot (\check{P}, B)) + r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_g \cdot \underline{B}) + \Xi\widehat{\otimes}\,{}^{(c)}\nabla_3\underline{B} + r^{-1}\mathfrak{d}^{\leq 1}(\Xi \cdot \underline{A}). \end{aligned}$$

By combining the above with the LHS, we obtain

$$\begin{aligned} \underline{Q}(\underline{A}) &= \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\check{P} + r^{-4}\,\mathfrak{d}\!\!/^{\leq 1}(\Gamma_b, r\Gamma_g) + O(ar^{-2})\underline{A}_4 + O(ar^{-4})\underline{A} + O(ar^{-3})\mathfrak{d}^{\leq 1}\check{P} \\ &\quad + r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b \cdot (\check{P}, B)) + r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_g \cdot (r\underline{B}, \underline{A})) + \Xi\widehat{\otimes}\,{}^{(c)}\nabla_3\underline{B} + r^{-1}\mathfrak{d}^{\leq 1}(\Xi \cdot \underline{A}), \end{aligned}$$

or

$$\begin{aligned} \underline{Q}(\underline{A}) &= \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\mathcal{D}\check{P} + r^{-4}\,\mathfrak{d}\!\!/^{\leq 1}(\Gamma_b, r\Gamma_g) + O(ar^{-3})\mathfrak{d}^{\leq 1}\underline{B} + O(ar^{-4})\underline{A} + O(ar^{-3})\mathfrak{d}^{\leq 1}\check{P} \\ &\quad + r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b \cdot (\check{P}, B)) + r^{-2}\mathfrak{d}^{\leq 1}(\Gamma_g \cdot (r\underline{B}, \underline{A})) + \Xi\widehat{\otimes}\,{}^{(c)}\nabla_3\underline{B} + r^{-1}\mathfrak{d}^{\leq 1}(\Xi \cdot \underline{A}), \end{aligned}$$

as stated.

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