Chapter 2

The Geometry Of Spacetime

2.1 Introduction

The presence of a massive object (compact or otherwise) would deform the ambient Minkowski geometry of spacetime in accordance with the Einstein equation. The gravitational field is described by the metric tensor $g_{\mu\nu}$ of the manifold. On any region of the manifold we can place a coordinate system (t, x^1, x^2, x^3) such that t is a timelike coordinate, and x^i are spacelike for $i = 1, 2, 3$. The metric tensor can then be written in the form

$$g = g_{tt} dt \otimes dt + g_{ti} dt \otimes dx^i + g_{it} dx^i \otimes dt + g_{ij} dx^i \otimes dx^j. \qquad (2.1)$$

Here $g_{\mu\nu} = g_{\nu\mu}$, and since t is a timelike coordinate, $g_{tt} < 0$. At every point of our manifold, the tangent space is isomorphic to Minkowski space, in particular, the timelike vectors are contained within the two lightcones. We can assign any one of the lightcones as future directed. After having chosen the orientation of the future cone at one point, one must extend this to every point of the manifold in a continuous manner assigning light cones as future directed and past directed consistently. It may not be possible to do this in general if the manifold has a non-trivial topology. In the event that such a designation can be made, we say that the manifold is *time orientable*. By going into a local Minkowiski tangent frame at any point p of the manifold, we can easily see that any two distinct causal (timelike or lightlike) vectors at p belong to the same light cone if and only if their inner product is less than zero. In addition to time-orientability, if the metric satisfies the Einstein equation for an appropriate energy momentum tensor, the manifold is physically relevant and is a candidate *spacetime*.

The spacetime is determined by the gravitational interactions of all the matter and other causal fields in it. In turn, the geometry determines the motion of all matter and the time evolution of the fields it contains. Therefore, it is necessary to solve for the geometry and the fields (including matter) at the same time. This is nearly an impossible task save for a few ideal, but very important, examples. There is however a practical solution to this problem. Gravitation is a very weak force. Consequently, it takes a large amount of energy and momentum to deform the geometry of spacetime. Thus, it is possible to introduce the concept of a fixed ambient spacetime in which other particles and fields may evolve. For example, in the absence of massive stars and black holes, we may take the ambient geometry to be Minkowiski. This is the situation in special relativity. The Minkowiski metric descibes perfect vacuua. However, we may still talk about particles falling in it, and solve the *twin paradox* problem for example. Our situation will be very similar. We will fix the ambient geometry so that it describes the gravitational field of a rotating star or a black hole. In chapter 15, we will consider the nature of electromagnetic fields and currents in this fixed geometry.

2.2 Splitting Spacetime Into Space And Time

It will be convenient in our discussions to visualize spacetime as time-stacked slices of absolute space. These spacelike slices are 3-dimensional manifolds whose geometry, as we shall see, is described by the metric $\gamma_{ij} \equiv g_{ij}$. In general, the metric coefficients can be a function of time, in which case, the properties of the absolute space also become time-dependent. For the case of a stationary spacetime however, the metric coefficients are by definition time-independent. Stationary spacetimes will be of great significance for us since it will be sufficient in describing the ambient exterior geometry of stars and black holes. When this happens, the nature of absolute space will not evolve with time and can therefore be thought of as a curved space counterpart of the familiar Galilean notion of space. We will rewrite eq. (2.1) in a form that will make the foliation of the geometry into spacelike slices manifest, i.e., we shall give meaning to the various components of the metric tensor $g_{\mu\nu}$. The discussion here will be applicable in general, and shall not require the stationarity of spacetime.

Consider a spacelike hypersurface Σ_t obtained by fixing the value of the timelike coordinate t. Henceforth, such hypersurfaces will be referred to as absolute space. The geometric nature of the various slices of absolute space will in general evolve in time. Clearly,

$$T(\Sigma_t) = \text{span}\{\frac{\partial}{\partial x^i}\} \ . \tag{2.2}$$

From eq. (2.1), we see that the induced metric on our $3d$ absolute space is given by

$$\hat{\gamma} = g_{ij}\ dx^i \otimes dx^j \equiv \gamma_{ij}\ dx^i \otimes dx^j \tag{2.3}$$

since $dt = 0$ on Σ_t. Let $\tilde{t}$ denote the timelike vector field $\partial/\partial t$ and let n be the unit normal vector field on Σ_t that points in the direction of increasing t (as shown in Fig 1.1). We pick the time coordinate t such that $\tilde{t}$ is future pointing. Consequently, n is timelike and future pointing and so $g(n, \tilde{t}) < 0$. By definition,

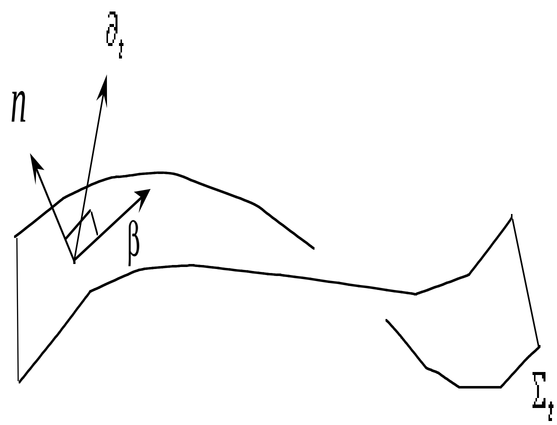

Figure 2.1: Σ_t is a sample spacelike surface which may in general be curved and evolve in time.

the component of $\tilde{t}$ in the direction of n is

$$\alpha \equiv -g(\tilde{t}, n). \tag{2.4}$$

To obtain the components of n, define a 3-dual vector in our absolute space by

$$\beta = \beta_i dx^i \equiv g_{ti} dx^i \ . \tag{2.5}$$

The corresponding tangent vector (which we also denote as β) is given by

$$\beta = \beta^i \frac{\partial}{\partial x^i} = \gamma^{ij} \beta_j \frac{\partial}{\partial x^i} \ . \tag{2.6}$$

We raise β by the induced metric $\hat{\gamma}$ since it belongs to $T(\Sigma_t)$. Vector fields like β will be given a new life in our absolute space. We shall refer to vectors belonging to $T(\Sigma_t)$ as spatial vectors. They are, however, to be distinguished from spacelike vectors. Spacelike vectors can have a component along $\tilde{t}$. Spatial vectors are 3-dimensional. Since n is normal to $T(\Sigma_t)$

$$g(n, \partial_i) = 0 \ . \tag{2.7}$$

In components, the above equation becomes

$$\beta_j n^t + \gamma_{ij} n^i = 0 \ . \tag{2.8}$$

From eq. (2.4) we get

$$g_{tt} n^t + \beta_i n^i = -\alpha \tag{2.9}$$

and since n is a unit timelike vector

$$g_{tt}(n^t)^2 + 2\beta_i n^i n^t + \gamma^{ij} n^i n^j = -1 \ . \tag{2.10}$$

Eq. (2.8) - eq. (2.10) can be solved immediately to obtain

$$n = \frac{1}{\alpha}(\partial_t - \beta^i \partial_i) \ . \tag{2.11}$$

Equivalently,

$$\partial_t = \alpha n + \beta^i \partial_i \ . \tag{2.12}$$

We now proceed to write the spacetime metric components in terms of α and β.

$$\begin{aligned} g_{tt} &= g(\tilde{t}, \tilde{t}) = g(\alpha n + \beta, \alpha n + \beta) \\ &= -\alpha^2 + 2\alpha g_(n, \beta) + g(\beta, \beta) \end{aligned}$$

Therefore, we get that

$$g_{tt} = \beta^2 - \alpha^2.$$

The spacetime metric eq. (2.1) can now be written in the form:

$$g = (\beta^2 - \alpha^2) dt \otimes dt + \beta_i [dt \otimes dx^i + dx^i \otimes dt] + \gamma_{ij} dx^i \otimes dx^j . \tag{2.13}$$

α is referred to as the *lapse* function, and β is the *shift* vector. Lowering the index to obatin the one-form corresponding to n^μ we see that

$$n_\mu = \frac{1}{\alpha}(g_{t\mu} - g_{\mu i}\beta^i).$$

Therefore,

$$n_t = \frac{1}{\alpha}(g_{tt} - g_{ti}\beta^i) = \frac{1}{\alpha}(\beta^2 - \alpha^2 - \beta_i\beta^i) = -\alpha$$

and

$$n_j = \frac{1}{\alpha}(g_{tj} - g_{ji}\beta^i) = 0.$$

That is

$$n_\mu = (-\alpha, 0, 0, 0).$$

It is now easy to verify that in matrix form

$$g_{\mu\nu} = \begin{bmatrix} \beta^2 - \alpha^2 & \beta_1 \quad \beta_2 \quad \beta_3 \\ \begin{matrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{matrix} & \begin{bmatrix} & & \\ & \gamma_{ij} & \\ & & \end{bmatrix} \end{bmatrix} .$$

It is not too difficult to obtain a projection operator from $T(M) \to T(\Sigma_t)$. The interested reader is referred to the footnote below. [1] A quick calculation reveals that

$$\sqrt{-g} \equiv \sqrt{-\,det(g)} = \alpha\sqrt{det(\hat{\gamma})} \equiv \alpha\sqrt{\hat{\gamma}},$$

[1]Let X be any vector in $T(M)$. Then clearly, by an orthogonal decomposition

$$X = -g(X, n)\, n + \chi$$

where $det(g)$, and $det(\hat{\gamma})$ are the determinant of the matrix represetations of g, and $\hat{\gamma}$ (the induced spatial metric given in eq. (2.3)). It is important to distinguish the tensor g from the square root of its absolute value of the determinat $\sqrt{-g}$ (this rule applies to metric tensors of spacetime and the absolute spaces). It should be clear to the reader by now that the absolute spaces are hardly unique. They depend entirely on the time function t that was chosen. It is common to give the invariant interval ds^2 of a spacetime instead of the metric tensor g. They are of course, closely related. Let α be any curve in our spacetime parameterized by the variable τ as given in eq. (1.41). Then

$$ds^2 \equiv g(\dot{\alpha}, \dot{\alpha})\, d\tau^2\ . \tag{2.14}$$

Clearly, it is sufficient to given the expression for ds^2 instead of the metric tensor g. It will useful to briefly consider some simple examples from a $3+1$ space and time point of view. As remarked earlier, far away from regions of strong gravitation the Minkowski spacetime will be adequate in describing the background geometry. Here

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2, \tag{2.15}$$

where (t, x, y, z) are the spacetime coordinates. Here (as per the notation above) $\alpha = 1$, $\beta_i = 0$ and $\gamma_{ij} = \delta_{ij}$, where as usual $\delta_{ij} = 1$ if $i = j$, and 0 otherwise. The absolute spaces are the usual Cartesian space $\Re^3$ with coordinates (x, y, z) endowed with the metric

$$\hat{\gamma} = dx^2 + dy^2 + dz^2. \tag{2.16}$$

It will be instructive to write the above metric using the familiar spherical coordinate system. In this case

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{2.17}$$

The coordinates used are (t, r, θ, φ), where $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, and $z = r\cos\theta$. Once again $\alpha = 1$, $\beta_i = 0$, and the non-trivial components of $\hat{\gamma}$ are $\gamma_{rr} = 1$, $\gamma_{\theta\theta} = r^2$, and $\gamma_{\varphi\varphi} = r^2\sin^2\theta$. Since all we have done is a coordinate

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for some unique $\chi^i\partial_i \in T(\Sigma_t)$, where $g(n, \chi) = 0$ from eq. (2.7). But clearly,

$$X^\mu = -g(X, n)\, n^\mu + X^\mu + g(X, n)\, n^\mu = -g(X, n)\, n^\mu + (g^\mu_\nu + n^\mu n_\nu)X^\nu\ .$$

Therefore,

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$$

is the projection operator we need. In components

$$h_{\mu\nu} = \begin{bmatrix} \beta^2 & \beta_1 \quad \beta_2 \quad \beta_3 \\ \begin{matrix}\beta_1\\ \beta_2\\ \beta_3\end{matrix} & \begin{bmatrix} & & \\ & \gamma_{ij} & \\ & & \end{bmatrix} \end{bmatrix}\ .$$

As expected, the purely spatial components of the tensors g and h agree.

change, the absolute spaces continues to be flat, and in spherical coordinates (r, θ, φ), the spatial metric takes the form

$$\hat{\gamma} = dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{2.18}$$

For the case of the Schwarzschild metric

$$ds^2 = -(1 - \frac{2M}{r})\, dt^2 + (1 - \frac{2M}{r})^{-1}\, dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{2.19}$$

Since there aren't any mixed time and space components in the metric, here $\beta_i = 0$, and $\alpha^2 = (1 - 2M/r)$. The coordinate t is timelike when $r > 2M$, and in this region, the absolute space is described by the metric

$$\hat{\gamma} = (1 - \frac{2M}{r})^{-1}\, dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{2.20}$$

2.3 The Kerr Metric

Ever since its inception in 1963 [6], the Kerr solution is the only candidate metric to describe the exterior gravitational field of massive, stationary, compact objects. From a theoretical point of view, the Kerr solution has been supported by uniqueness theorems of varying sophistication. But the physical relevancy of the Kerr solution can only be discerned by careful astrophysical observations of rapidly rotating compact objects. The observational data has to be then matched up with theoretical predications and calculations. Therefore, it is crucial that we strive to do physics in a Kerr background. As a first step, in this section, we will describe the salient properties of the Kerr solution. The discussion here is merely functional. The penitent reader is referred to the exhaustive book on Kerr geometry by Barrett O'Neill [7]. However, the analysis that follows is sufficient and self-contained.

The Kerr metric describes the time-independent, axis-symmetric gravitational field of a collapsed object that has retained its angular momentum. All matter having collapsed, the Kerr metric (were defined) satisfies the vacuum Einstein equation:

$$R_{\mu\nu} = 0\ . \tag{2.21}$$

Here $R_{\mu\nu}$ is the Ricci tensor defined in eq. (4.69). In Boyer-Lindquist coodinate system (t, r, θ, φ), the Kerr metric takes the form:

$$ds^2 = g_{tt}dt^2 + 2\beta_\varphi dt d\varphi + \gamma_{rr}dr^2 + \gamma_{\theta\theta}d\theta^2 + \gamma_{\varphi\varphi}d\varphi^2. \tag{2.22}$$

Here

$$g_{tt} = -1 + \frac{2Mr}{\rho^2}, \quad g_{t\varphi} \equiv \beta_\varphi = \frac{-2Mra\sin^2\theta}{\rho^2}, \quad \gamma_{rr} = \frac{\rho^2}{\Delta},$$

$$\gamma_{\theta\theta} = \rho^2, \quad \gamma_{\varphi\varphi} = \frac{\Sigma^2\sin^2\theta}{\rho^2}\ , \tag{2.23}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta,$$
$$\Delta = r^2 - 2Mr + a^2,$$

and

$$\Sigma^2 = (r^2 + a^2)^2 - \Delta\, a^2 \sin^2\theta.$$

Additionally

$$\alpha^2 = \frac{\rho^2\Delta}{\Sigma^2}, \qquad \beta^2 = \frac{\beta_\varphi^2}{\gamma_{\varphi\varphi}}$$

and

$$\sqrt{\hat{\gamma}} = \sqrt{\frac{\rho^2\Sigma^2}{\Delta}}\sin\theta, \quad \text{and} \quad \sqrt{-g} = \rho^2\sin\theta\ .$$

Here, M can be interpreted as the mass, and aM the angular momentum of the black hole. It is convenient to pick the time orientation of the Kerr metric consistently so that as $r \to \infty$, $\tilde{t}$ is future directed. The metric coefficient functions are independent of t and φ as expected from the assumed symmetry. When $a \to 0$, the Kerr metric reduces to the Schwarzschild metric given by eq. (1.46). The Schwarzschild metric is both static and spherically symmetric, and consequently describes the end product of a non-rotating spherically symmetric collapse. The contravariant form of the Kerr metric tensor is given by:

$$g = -\frac{\Sigma^2}{\rho^2\Delta}\partial_t\otimes\partial_t - \frac{2aMr}{\rho^2\Delta}\partial_t\otimes\partial_\varphi - \frac{2aMr}{\rho^2\Delta}\partial_\varphi\otimes\partial_t + \frac{\Delta}{\rho^2}\partial_r\otimes\partial_r$$
$$+\frac{1}{\rho^2}\partial_\theta\otimes\partial_\theta + \frac{(\Delta - a^2\sin^2\theta)}{\rho^2\Delta\sin^2\theta}\partial_\varphi\otimes\partial_\varphi\ . \tag{2.24}$$

We will find the following relationships obeyed by the components of the Kerr metric in Boyer-Lindquist coordinates useful. Since they can be easily verified by algebraic manipulation, we state them without proof:

$$a\sin^2\theta g_{tt} + g_{t\varphi} = -a\sin^2\theta, \tag{2.25}$$

$$(r^2 + a^2)g_{t\varphi} + a\gamma_{\varphi\varphi} = a\sin^2\theta\Delta, \tag{2.26}$$

$$(r^2 + a^2)g_{tt} + ag_{t\varphi} = -\Delta, \tag{2.27}$$

and

$$a\sin^2\theta g_{t\varphi} + \gamma_{\varphi\varphi} = (r^2 + a^2)\sin^2\theta. \tag{2.28}$$

It is clear that the Kerr metric is singular at $\rho^2 = 0$. This is a ture singularity of the geometry and cannot be removed away by a coordinate transformation as can be verified by computing scalar quantities at $\rho^2 = 0$ which do not change with coordinate systems. In particular the contraction of the Riemann tensor with itself is not well defined at $\rho^2 = 0$. Explicitly, $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \to \infty$ as $\rho^2 \to 0$. The singularity at $\rho^2 = r^2 + a^2\cos^2\theta = 0$, has an additional interesting feature in that it happens only when $\theta = \pi/2$. For an excellent description and the consequences of the Kerr singularity and related matters see Chandrashekar [8].

On the other hand, the singularity when $\Delta = 0$ in the contravariant form of the Kerr metric is unphysical. These apparent singularities are located at $r = r_\pm$, where

$$r_\pm = M \pm \sqrt{M^2 - a^2} \tag{2.29}$$

are the roots to the equation $\Delta = 0$. To understand the properties of these surfaces and the region bounded by them, we will have to rewrite the Kerr metric in a coordinate system that is well behaved at $r = r_\pm$. We will get to these matters shortly.

2.3.1 The Geodesic Equation and its Integrability in Kerr Geometry

The nature of the Kerr geodesics will play a vital role in understanding the process of energy extraction from black holes. Energy extraction from rotating black holes will be our chief concern in the chapters to follow. In a four dimensional spacetime we would need four constants along geodesics to successfully integrate the geodesic equation. It is clear that we should expect at least three such quantities:

1. The speed of a geodesic is a conserved quantity. In particular, the geodesic tangent vector u satisfies $u^2 = q^2$ where $q^2 = -1$ and $q^2 = 0$ for timelike and null geodesics respectively.
2. Since the Kerr metric is time-independent, we would expect a conserved quantity that is related to the energy of the particle.
3. Owing to the axial-symmetry of the geometry, or equivalently φ independence of the metric functions, the angular momentum of the particle corresponding to the geodesic would remain a constant.

Indeed, a fourth conserved quantity exists [9], and as such has enabled a complete geometric analysis of the Kerr spacetime. But, before we embark on deriving the fourth Carter's constant (as it is called), lets quantify the remaining two items above. Let u denote the tangent vector of a proper time parameterized geodesic, i.e.,

$$u(\tau) = \dot{t}\partial_t + \dot{r}\partial_r + \dot{\theta}\partial_\theta + \dot{\varphi}\partial_\varphi \ . \tag{2.30}$$

Here, (as always) the dot refers to derivative with respect to proper time τ. The symmetry properties of the Kerr geometry is reflected in its Killing vectors. A vector field ξ is Killing if it satisfies the following equation:

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \ . \tag{2.31}$$

It is not difficult to see that along any geodesic, $g(\xi, u) = \text{const}$, where u is the geodesic tangent:

$$ug(\xi, u) = g(u^\mu \nabla_\mu \xi, u) + g(\xi, \nabla_u u) = 0 \ .$$

In the above equation
$$u^\mu u^\nu \nabla_\mu \xi_\nu = 0$$
since the symmetric sum of an anti-symmetric object in trivial. Since the metric coefficients in the Kerr geometry are independent of t, $\tilde{t}$ is a Killing vector field. To check this, we see if $\tilde{t}$ satisfies the Killing equation, note that

$$\nabla_\mu \tilde{t}_\nu + \nabla_\nu \tilde{t}_\mu = g_{\nu\gamma} \nabla_\mu \tilde{t}^\gamma + g_{\mu\gamma} \nabla_\nu \tilde{t}^\gamma \; . \tag{2.32}$$

But,

$$\begin{aligned} g_{\nu\gamma} \nabla_\mu \tilde{t}^\gamma = g_{\nu\gamma} \Gamma^\gamma_{\mu\alpha} \tilde{t}^\alpha = g_{\nu\gamma} \Gamma^\gamma_{\mu t} &= \frac{1}{2} g_{\nu\gamma} g^{\gamma\alpha} (\partial_\mu g_{\alpha t} + \partial_t g_{\mu\alpha} - \partial_\alpha g_{\mu t}) \\ &= \frac{1}{2} (\partial_\mu g_{\nu t} - \partial_\nu g_{\mu t}) \; . \end{aligned}$$

Substituting the above into eq. (2.32) we find

$$\nabla_\mu \tilde{t}_\nu + \nabla_\nu \tilde{t}_\mu = \frac{1}{2} (\partial_\mu g_{\nu t} - \partial_\nu g_{\mu t} + \partial_\nu g_{\mu t} - \partial_\mu g_{\nu t}) = 0,$$

i.e., $\tilde{t}$ satisfies the Killing equation eq. (2.31).

Exercise 3.1 Show that $m = \partial_\varphi$ is a Killing vector field of the Kerr geometry.

Since Killing vectors give rise to conserved quantities, we are now able to define the constants of motion arising from $\tilde{t}$ and m.

Definition 2.1 The energy E of the geodesic is given by

$$E = -g(\tilde{t}, u) = -(g_{tt} \dot{t} + g_{t\varphi} \dot{\varphi}) \; . \tag{2.33}$$

Definition 2.2 The angular momentum L of the geodesic with four-velocity u is given by

$$L = g(m, u) = (g_{t\varphi} \dot{t} + \gamma_{\varphi\varphi} \dot{\varphi}) \; . \tag{2.34}$$

Clearly, E and L are constant along each geodesic. There is a slight abuse of terminology here. For the case of particles with mass, since our timelike geodesics are of unit speed, the quantities E and L defined above corresponds to the energy and angular momentum per unit mass of the particle. It should be clear that the definitions above will determine the time evolution of the geodesic coordinates t and φ. The geodesic equations for t and φ in eq. (2.30) are

$$\dot{t} = \frac{\Sigma^2 E - 2aMrL}{\rho^2 \Delta}, \tag{2.35}$$

and

$$\dot{\varphi} = \frac{2aMrE\sin^2\theta + (\rho^2 - 2Mr)L}{\rho^2\Delta\sin^2\theta}. \tag{2.36}$$

This can be see by substituting expressions in eq. (2.23) for the metric coefficients in eq. (2.33). The energy of the geodesic is given by

$$-E = (-1 + \frac{2Mr}{\rho^2})\,\dot{t} - \frac{2aMr\sin^2\theta}{\rho^2}\dot{\varphi}\,. \tag{2.37}$$

Similarly, from eq. (2.23) and eq. (2.34) we see that

$$L = -\frac{2aMr\sin^2\theta}{\rho^2}\dot{t} + \frac{\Sigma^2\sin^2\theta}{\rho^2}\dot{\varphi}\,. \tag{2.38}$$

The above two equations can be inverted to obtain eq. (2.35) and eq. (2.36).

The promised fourth conserved quantity along Kerr geodesics is not so immediately obtained. The most efficient way of deriving the remaining constant of motion is by recalling that the geodesic equation is implied by an Euler-Lagrange set of equations. Having a Lagrangian in our possession lends itself to the sometimes very powerful set of equations of the Hamilton-Jacobi theory. Carter [9] was able to show that the action is indeed completely separable in this case and the conserved quatities can be accordingly obtained. With this in mind, we proceed along with our analysis of the geodesics.

The lagrangian for geodesic motion is given by eq. (1.36)

$$\mathbf{L}(x^\mu, \dot{x}^\mu, \tau) = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\,. \tag{2.39}$$

Here, x^μ are the coordinates, and can therefore in our case take on values t, r, θ and φ. In relativity theory, the proper time parameter τ takes on the usual role of time. To pass from a Lagrangian formalism to Hamilton's method, we must first obtain the conjugate momenta. By definition

$$P_\mu = \frac{\partial\mathbf{L}}{\partial\dot{x}^\mu} = g_{\mu\nu}\dot{x}^\nu\,, \tag{2.40}$$

where P_μ are the generalized momenta. Therefore, the Hamiltonian $H \equiv H(x^\mu, P_\mu, \tau)$ becomes

$$H = P_\mu\dot{x}^\mu - \mathbf{L} = \frac{1}{2}g^{\mu\nu}P_\mu P_\nu\,. \tag{2.41}$$

The Hamiltonian here is not an explicit function of τ and therefore is a conserved quantity. Clearly, $H = q^2/2$. To utilize the Hamilton-Jacobi method we must introduce a function S such that it is a function of τ, the old coordinates t, r, θ, φ

and the new set of conserved quantities of geodesic motion: q^2, E, L, K. Here, K will turn out to be Carter's constant. That is

$$S = S(\tau, t, r, \theta, \varphi, q^2, E, L, K) \ . \tag{2.42}$$

In addition, S is related to the conjugate momenta of the old coordinates as follows.

$$\partial_t S = P_t \ , \quad \partial_r S = P_r \ , \quad \partial_\theta S = P_\theta \ , \& \quad \partial_\varphi S = P_\varphi \ . \tag{2.43}$$

From eq. (2.41) and eq. (2.43) we get that

$$H = \frac{1}{2} g^{\mu\nu} \partial_\mu S \partial_\nu S \ . \tag{2.44}$$

The Hamilton-Jacobi equation $\partial_\tau S + H = 0$ can now be written in the form

$$\partial_\tau S + \frac{1}{2} g^{\mu\nu} \partial_\mu S \partial_\nu S = 0 \ . \tag{2.45}$$

From eq. (2.24) and eq. (2.45) we have

$$2\partial_\tau S = \frac{\Sigma^2}{\rho^2 \Delta}(\partial_t S)^2 + \frac{4aMr}{\rho^2\Delta}\partial_t S \partial_\varphi S - \frac{\Delta}{\rho^2}(\partial_r S)^2 - \frac{1}{\rho^2}(\partial_\theta S)^2$$
$$-\frac{(\Delta - a^2 \sin^2\theta)}{\rho^2 \Delta \sin^2\theta}(\partial_\varphi S)^2 \ . \tag{2.46}$$

Assuming separability, let us try to write S in the form

$$S = -\frac{1}{2} q^2 \tau - Et + L\varphi + S_r(r) + S_\theta(\theta) \ . \tag{2.47}$$

As suggested S_r is only dependent on r and S_θ only on θ. Here the τ dependence of S was chosen so that

$$\partial_\tau S = -\frac{1}{2} q^2 = -\frac{1}{2} g^{\mu\nu} P_\mu P_\nu = -H$$

as required by the Hamilton-Jacobi equation. From eq. (2.46) and eq. (2.47) we get

$$[\Delta(\frac{dS_r}{dr})^2 - \frac{C^2}{\Delta} - q^2 r^2] + [(\frac{dS_\theta}{d\theta})^2 + \frac{D^2}{\sin^2\theta} - q^2 a^2 \cos^2\theta] = 0 \ . \tag{2.48}$$

Here,

$$C = C(r) = (r^2 + a^2)E - aL \quad \text{and} \quad D = D(\theta) = L - aE\sin^2\theta \ . \tag{2.49}$$

The terms in the first square bracket above are functions of r alone, while in the second square bracket they are functions of only θ. This gives us the necessary separation constant K. Set

$$\Delta(\frac{dS_r}{dr})^2 - \frac{C^2}{\Delta} - q^2 r^2 = -K \ , \tag{2.50}$$

and

$$(\frac{dS_\theta}{d\theta})^2 + \frac{D^2}{\sin^2\theta} - q^2 a^2 \cos^2\theta = K \ . \tag{2.51}$$

Define functions $R(r)$ and $\Theta(\theta)$ by

$$R(r) \equiv C^2 + \Delta(q^2 r^2 - K), \tag{2.52}$$

and

$$\Theta(\theta) \equiv K + q^2 a^2 \cos^2\theta - \frac{D^2}{\sin^2\theta} \ . \tag{2.53}$$

Therefore, modulo an irrelevant additive constant we get the following expression for S.

$$S = -\frac{1}{2} q^2 \tau - Et + L\varphi + \int \frac{\sqrt{R}}{\Delta} dr + \int \sqrt{\Theta}\, d\theta \ . \tag{2.54}$$

The result we need is obtained from the requirement

$$\frac{\partial S}{\partial q^2} = 0 = -\frac{1}{2}\tau + \int \frac{1}{2\sqrt{R}\Delta} \frac{\partial R}{\partial q^2} dr + \int \frac{1}{2\sqrt{\Theta}} \frac{\partial \Theta}{\partial q^2}\, d\theta \ , \tag{2.55}$$

and

$$\frac{\partial S}{\partial K} = 0 = \int \frac{1}{2\sqrt{R}\Delta} \frac{\partial R}{\partial K} dr + \int \frac{1}{2\sqrt{\Theta}} \frac{\partial \Theta}{\partial K}\, d\theta \ . \tag{2.56}$$

Taking the derivative with respect to proper time after substituting the explicit form of R in eq. (2.55) we get

$$1 = \frac{r^2}{\sqrt{R}} \dot{r} + \frac{a^2 \cos^2\theta}{\sqrt{\Theta}} \dot{\theta} \ , \tag{2.57}$$

Similarly, from eq. (2.56) we see that

$$\frac{\dot{r}}{\sqrt{R}} = \frac{\dot{\theta}}{\sqrt{\Theta}} \ . \tag{2.58}$$

Therefore, the geodesic equations for r and θ in eq. (2.30) for the Kerr Geometry are given by

$$\rho^4 \dot{r}^2 = R(r) \ , \tag{2.59}$$

and

$$\rho^4 \dot{\theta}^2 = \Theta(\theta) \ . \tag{2.60}$$

Here, the Carter constant K brought about a separation of variables thus permitting the integrability of Kerr geodesic equations eq. (2.59) and eq. (2.60).

Null geodesics that stay on a constant value of θ will be important to our analysis. For null geodesics $q^2 = 0$. Here, we will also set $K = 0$, $E = 1$, and $L = a \sin^2\theta$. When this happens, from eq. (2.53) we see that $\dot{\theta} = 0$ as required, for only then is $L = a\sin^2\theta$ is a constant of motion. Consequently, from the above derived equations of motion, we see that the components of the geodesic tangent vectors takes the simple form

$$\dot{t} = \frac{r^2+a^2}{\Delta}\ , \quad \dot{r} = \pm 1\ , \quad \dot{\theta} = 0 \quad \& \quad \dot{\varphi} = \frac{a}{\Delta}.$$

Definition 2.3 It will be convenient to define the following two null geodesic tangent vectors

$$l_+ = \frac{1}{\Delta}((r^2+a^2)\partial_t + \Delta\partial_r + a\partial_\varphi), \tag{2.61}$$

and

$$l_- = \frac{1}{\Delta}((r^2+a^2)\partial_t - \Delta\partial_r + a\partial_\varphi). \tag{2.62}$$

Here l_+ is outgoing ($\dot{r} > 0$), and l_- is infalling ($\dot{r} < 0$).

As was mentioned before, the Boyer-Lindquist coordinates fail at $r = r_+$. Even the null geodesics defined above are not valid when $\Delta = 0$. In order for our analysis to be valid beyond this value of r, we must be able to transform all the relevent quantities to a coordinate system that is well defined across this region. This is the central purpose of the Kerr-Schild coordinate system that we will discuss in the following section.

2.3.2 The Kerr Metric in Kerr-Schild Coordinates

As expected, the coordinate transformation will be singular at $r = r_+$ if the new coordinates are to remove the existing unphysical singularity. Clearly, this is the case below. The Kerr-Schild coordinates are $\bar{t}, \bar{r}, \bar{\theta}$, and $\bar{\varphi}$. They are related to the Boyer-Lindquist coordinates by the following relations:

$$\bar{r} = r\ , \quad \bar{\theta} = \theta\ , \quad d\bar{t} = dt + \frac{2Mr}{\Delta}dr\ , \quad \& \quad d\bar{\varphi} = d\varphi + \frac{a}{\Delta}dr. \tag{2.63}$$

The "bar" is placed on r and θ so that no confusions arise while performing coordinate tranformations. We will have plenty of opportunities to compare various components of tensors in the Boyer-Lindquist and Kerr-Schild coordinates. Therefore, it will be crucial to estabilish the transformation properties as early as possible. Clearly,

$$\begin{bmatrix} d\bar{t} \\ d\bar{r} \\ d\bar{\theta} \\ d\bar{\varphi} \end{bmatrix} = \begin{bmatrix} 1 & G & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & H & 0 & 1 \end{bmatrix} \begin{bmatrix} dt \\ dr \\ d\theta \\ d\varphi \end{bmatrix}, \tag{2.64}$$

where

$$G = \frac{2Mr}{\Delta} \quad \& \quad H = \frac{a}{\Delta}. \tag{2.65}$$

We can write the above equation as

$$d\bar{x}^\mu = A^\mu_\nu dx^\nu. \tag{2.66}$$

Here, as will be the case always, "bar" quantities refer to the Kerr-Schild objects, and the "unbarred" objects are the equivalent Boyer-Lindquist objects. Also, A^μ_ν is the transformation matrix defined in eq. (2.64). For a 1-form X

$$\bar{X}_\mu d\bar{x}^\mu = \bar{X}_\mu A^\mu_\nu dx^\nu \equiv X_\nu dx^\nu \ , \tag{2.67}$$

i.e, in component form

$$X_\nu = A^\mu_\nu \bar{X}_\mu \ . \tag{2.68}$$

Taking the inverse of A^μ_ν we find

$$\bar{X}_\mu = (A^{-1})^\nu_\mu X_\nu \ , \tag{2.69}$$

where

$$(A^{-1})^\nu_\mu = \begin{bmatrix} 1 & -G & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -H & 0 & 1 \end{bmatrix} . \tag{2.70}$$

It is now a trivial matter to work out the transformation properties of vectors.

$$d\bar{x}^\mu \left(\frac{\partial}{\partial \bar{x}^\nu}\right) = \delta^\mu_\nu, \tag{2.71}$$

where δ^μ_ν is the Kronecker-delta. Let

$$\frac{\partial}{\partial \bar{x}^\nu} = B^\beta_\nu \frac{\partial}{\partial x^\beta}. \tag{2.72}$$

Then using eq. (2.66), eq. (2.71) becomes

$$A^\mu_\alpha dx^\alpha \left(B^\beta_\nu \frac{\partial}{\partial x^\beta}\right) = \delta^\mu_\nu. \tag{2.73}$$

Therefore

$$A^\mu_\alpha B^\alpha_\nu = \delta^\mu_\nu, \tag{2.74}$$

i.e.,

$$B^\alpha_\nu = (A^{-1})^\alpha_\nu. \tag{2.75}$$

Therefore from eq. (2.72) and the above equation we get the transformation properties for the components of a vector Y.

$$\bar{Y}^\alpha = A^\alpha_\beta Y^\beta. \tag{2.76}$$

Eq. (2.69) and eq. (2.76) can be used to transform general tensors. We are now in a position to compute the metric tensor in the Kerr-Schild coordinate system. Various metric identities listed in eqs. (2.25)-(2.28) will be required to simplify the expressions. To illustrate the nature of the simplifications, will carry out the calculation of $\bar{g}_{\bar{r}\bar{r}}$ explicitly, leaving the others to the reader to verify.

$$\bar{g}_{\bar{r}\bar{r}} = B_1^\alpha B_1^\beta g_{\alpha\beta} = [G^2 g_{tt} + GH g_{t\varphi}] + [GH g_{t\varphi} + H^2 \gamma_{\varphi\varphi}] + \gamma_{rr}. \tag{2.77}$$

But,

$$[G^2 g_{tt} + GH g_{t\varphi}] = \frac{G}{\Delta}[2Mr g_{tt} + a g_{t\varphi}] = \frac{G}{\Delta}[(r^2 + a^2) g_{tt} + a g_{t\varphi} - \Delta g_{tt}].$$

Using eq. (2.27), the above equation gives

$$[G^2 g_{tt} + GH g_{t\varphi}] = -G[1 + g_{tt}] \tag{2.78}$$

Similary, using eq. (2.26) we find

$$[GH g_{t\varphi} + H^2 \gamma_{\varphi\varphi}] = H[a \sin^2\theta - g_{t\varphi}]. \tag{2.79}$$

Placing eq. (2.78) and eq. (2.79) in eq. (2.77) we get

$$\bar{g}_{\bar{r}\bar{r}} = 1 + \frac{2Mr}{\rho^2}.$$

In a similar manner, we find that in Kerr-Schild coordinates, the metric components in the basis $\{\bar{t}, \bar{r}, \bar{\theta}, \bar{\varphi}\}$ become

$$\bar{g}_{\mu\nu} = \begin{bmatrix} z-1 & z & 0 & -za\sin^2\theta \\ z & 1+z & 0 & -a\sin^2\theta(1+z) \\ 0 & 0 & \rho^2 & 0 \\ -za\sin^2\theta & -a\sin^2\theta(1+z) & 0 & \Sigma^2\sin^2\theta/\rho^2 \end{bmatrix}, \tag{2.80}$$

where $z = 2Mr/\rho^2$. As required, the metric above is not singular when $\Delta = 0$. In going to a 3+1 space and global time formalism, we must remember that, here we have different foliations of space. Spacelike slices in the two coordinate systems are not equivalent. On spacelike slices $d\bar{t} = 0$, the 3-metric in a basis $\{\bar{r}, \bar{\theta}, \bar{\varphi}\}$ become

$$\bar{\gamma}_{ij} = \begin{bmatrix} 1+z & 0 & -a\sin^2\theta(1+z) \\ 0 & \rho^2 & 0 \\ -a\sin^2\theta(1+z) & 0 & \Sigma^2\sin^2\theta/\rho^2 \end{bmatrix}. \tag{2.81}$$

Also

$$\bar{\alpha} = 1/\sqrt{1+z} \qquad \& \qquad \bar{\beta} = z d\bar{r} - za\sin^2\theta d\bar{\varphi}. \tag{2.82}$$

Lowering the above 1-form β using eq. (2.81), the shift vector becomes

$$\bar{\beta} = \frac{z}{1+z}\frac{\partial}{\partial\bar{r}}. \tag{2.83}$$

Having removed the coordinate singularity at $\Delta = 0$, we are able to meaningfully extend our analysis beyond the $r = r_+$ mark.

2.3.3 The Ergosphere

The causal character of the coordinate function t changes even outside the event horizon. To see this explicitly, lets locate the set of points such that $g_{tt} = 0$. This is given by the surface

$$r_{erg}(\theta) = M + \sqrt{M^2 - a^2 \cos^2\theta}. \tag{2.84}$$

$g_{tt} > 0$ in the region

$$r_+ < r < r_{erg}(\theta), \tag{2.85}$$

and consequently $\tilde{t} \equiv \partial_t$ becomes spacelike in the above region. We shall refer to the region defined by eq. (2.85) as the *ergosphere*. Outside the ergosphere $\tilde{t}$ is timelike and future directed (as per our choice of time orientation). Inside the ergosphere we can no longer use $\tilde{t}$ as our candidate future pointing timelike vector. Therefore, the nature of the futurecones inside the ergosphere becomes unclear. The same problem exists beyond $r = r_+$ (as we shall see below, this region brings about other interesting features). Both of these issues can now be handled in one stroke. To this end, let us write the null geodesic l_- in the Kerr-Schild coordinate system. We will denote the corresponding Kerr-Schild vector as $\bar{l}$. Using the transformations listed above we get from eq. (2.62) that

$$\bar{l}_- = \partial_{\bar{t}} - \partial_{\bar{r}}. \tag{2.86}$$

The above vector-field is well defined so long as the Kerr-Schild coordinate system is. As $r \to \infty$ we have that $g(\bar{l}_-, \tilde{t}) \to -1$. Therefore, $\bar{l}_-$ is asymptotically future pointing. But, since $\bar{l}_-$ is nowhere vanishing, for consistency reasons $\bar{l}_-$ is future pointing everywhere. *Therefore, we shall take the timecone containing $\bar{l}_-$ as the futurecone at every point in the region defined by $r_- < r$* since the Kerr-Schild coordinates is single valued and well defined in this region. In particular, $\bar{l}_-$ prescibes the futurecone in the ergosphere.

Even though $\tilde{t}$ is not timelike in the ergosphere, the coordinate function t does increase for observers in the ergosphere. To see this, note that eq. (2.24) implies that

$$g_{\mu\nu} \nabla^\mu t \nabla^\nu t = g^{\mu\nu} \nabla_\mu t \nabla_\nu t = g^{tt} < 0\ . \tag{2.87}$$

Therefore, $\nabla^\mu t$ is timelike in the ergosphere. Since $\bar{l}_-$ is future pointing,

$$g_{\mu\nu} \bar{l}^\mu \nabla^\nu t = \bar{l}^\mu \nabla_\mu t = l^0 = \frac{r^2 + a^2}{\Delta} > 0 \tag{2.88}$$

implies that $\nabla^\mu t$ is past-directed timelike in the ergosphere. Consequently, for an observer with future pointing four velocity u

$$0 < g_{\mu\nu} u^\mu \nabla^\nu t = u^\mu \nabla_\mu t = \dot{t}. \tag{2.89}$$

Therefore, for an observer in the ergosphere with four velocity $u(\tau) = (\dot{t}, \dot{r}, \dot{\theta}, \dot{\varphi})$

$$\dot{t} > 0\ . \tag{2.90}$$

As always, the dot corresponds to derivative with respect to propertime τ. In the ergosphere, since $\tilde{t}$ is spacelike, an observer cannot remain static. A *static* observer is one whose curve traced out in the spacetime has a fixed value of r, θ, and φ. The required rotation of an observer in the ergosphere can be thought of as an extreme case of frame dragging. At best, all we can have are stationary observers. Stationary observers move along constant values of r and θ. More precisely, as an object with mass falls into the ergosphere, it starts to rotate along with the black hole. This can be seen as follows:

Let $\alpha(\tau)$ be the curve traced out the by a stationary observer. The four-velocity of the observer then takes the form

$$\dot{\alpha} = u(\tau) = (\dot{t}, 0, 0, \dot{\varphi}).$$

We also require that

$$-1 = u^2 = \left[(\beta^2 - \alpha^2)\dot{t}^2 + \gamma_{\varphi\varphi}\dot{\varphi}^2 + 2\beta_\varphi \dot{t}\dot{\varphi}\right]. \tag{2.91}$$

If such an observer is to be static, $\dot{\varphi}$ must vanish. Inside the ergosphere, all but the last term on the right hand side of eq. (2.91) is positive. Therefore, for eq. (2.91) to hold true

$$\beta_\varphi \, \dot{\varphi} < 0$$

for timelike curves, since $\dot{t} > 0$ even in the ergosphere. From eq. (2.23) the above inequality remains true only when

$$a \, \dot{\varphi} > 0.$$

In particular, there are no static observers in the ergosphere for the observer is forced to rotate along with the black hole.

2.3.4 The Event Horizon

In the region $r_- < r < r_+$, $\Delta < 0$, i.e., ∂_r is timelike. Since Δ changes sign in this region, $-\partial_r$ is necessarily timelike. We will now show that $-\partial_r$ is contained in the same timecone as l_-. Or equivalently we need that

$$g(-\partial_r, l_-) < 0. \tag{2.92}$$

Using the transformation matrix constructed above we see that

$$-\frac{\partial}{\partial r} = -G\frac{\partial}{\partial \bar{t}} - \frac{\partial}{\partial \bar{r}} - H\frac{\partial}{\partial \bar{\varphi}}. \tag{2.93}$$

Therefore,

$$g(-\partial_r, l_-) = -[G(\bar{g}_{\bar{t}\bar{t}} - \bar{g}_{\bar{t}\bar{r}})] + H[(\bar{g}_{\bar{t}\bar{\varphi}} - \bar{g}_{\bar{\varphi}\bar{r}})] - (\bar{g}_{\bar{t}\bar{r}} - \bar{\gamma}_{\bar{r}\bar{r}})) = \frac{\rho^2}{\Delta}. \tag{2.94}$$

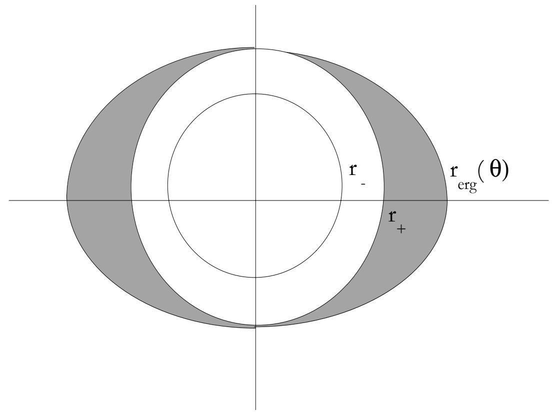

Figure 2.2: $r_{\pm}$ locate the horizons of the Kerr geometry. The shaded region is the ergosphere.

Since $\Delta < 0$ in the region of interest, we have the necessary result. Of course, the above inner product in not defined at $r = r_+$ since r is not a good coordinate function there. Consequently, $-\partial_r$ is future pointing and timelike when $r_- < r < r_+$.

This claim leads to a very important result in general relativity. If a particle (massless or otherwise) enters the region $r < r_+$, it will have necessarily have to move along decreasing values of r until it is thrown into the region $r < r_-$ where Δ is positive and $-\partial_r$ is no longer timelike. Therefore $r = r_+$ forms a one-way membrane. Particles entering it may never escape, thus $r = r_+$ is referred to as the *event horizon* and it forms the bounday of the *black hole* region. Consequently, what happens beyond $r = r_-$ will never effect physics (and indeed life) in our region of spacetime, since particles in the region $r < r_-$ may never enter the region $r > r_-$. For this reason, $r = r_-$ is called the *Cauchy horizon.* The region of spacetime beyond this is outside our domain of dependence.

2.4 The Penrose Process

As early as 1969, Roger Penrose [10] pointed out the possibility of extracting energy from rotating black holes. As we shall see, such an extraction of energy is possible only due to the existence of the Ergosphere.

We begin our analysis with a few preliminaries. Outside the ergosphere $\tilde{t}$ is future pointing and timelike. Therefore, for a particle (regardless of its mass) moving along a geodesic outside the ergosphere we have that

$$g(u, \tilde{t}) < 0. \tag{2.95}$$

From definition 2.1, we have that the energy E of such an object must be greater than zero. Positivity of energy is however not a requirement for geodesics in the ergosphere (since $\tilde{t}$ is noteven timelike here). Clearly, such an object will not be able to escape the ergosphere into the asymptotically flat region since the energy there must be greater than zero, for the 4-velocity of the particle to be future pointing, as stated above. For future pointing causal four velocity u we need $\dot{t} > 0$ and $\dot{\varphi} > 0$ in the ergosphere. From eq. (2.35) and eq. (2.36), we see that this is possible for geodesics so long as

$$\Sigma^2 E > 2aMrL \tag{2.96}$$

and

$$2aMrE\sin^2\theta > (\Delta - a^2\sin^2\theta)L. \tag{2.97}$$

As usual, here E and L are the energy and angular momentum of the geodesics as measured by an observer at infinity. Clearly, if we want orbits with $E < 0$, we must also have that $L < 0$. We shall see below that the above relations will place a restriction on the amount of energy that can be extracted from the black hole. Thus, we see that geodesics in the ergosphere are permitted to have "negative" values of energy E. Such *"negative-energy"* particles can be used for energy extraction from Kerr black holes. It is important to note that these particles are never observed in regions outside $r > r_{erg}(\theta)$.

Lets consider the simplest example put forth by Penrose in some detail. Our presentation of the Penrose process leans heavily on [8]. To extract energy from the black hole we would send an object with four momentum p_0 toward the hole via a timelike geodesic. Inside the ergosphere, this object is set to decay into two photons. One photon with negative energy will fall into the black hole, whereas the other photon would escape from the ergosphere into regions of large r. Conservation of energy would then imply that the photon emerging from the ergosphere will have a greater total energy than initial particle with mass. To see how this would happen let us set up the necessary notation:

1. The four-momentum of the initial infalling particle (along a timelike geodesic):
$$p_0 = m\,(\dot{t}_0, \dot{r}_0, \dot{\theta}_0, \dot{\varphi}_0) \tag{2.98}$$
2. The four-momentum of the decayed photon with negative energy (along a lightlike geodesic) that falls into the hole:
$$p_1 = (\dot{t}_1, \dot{r}_1, \dot{\theta}_1, \dot{\varphi}_1) \tag{2.99}$$
3. The four-momentum of the decayed photon that escape into regions of large r (along a lightlike geodesic):
$$p_2 = (\dot{t}_2, \dot{r}_2, \dot{\theta}_2, \dot{\varphi}_2) \tag{2.100}$$

At the point of decay $r = r_d$ (for $r_+ < r_d < r_{erg}(\theta)$), conservation of momentum would imply

$$p_0 = p_1 + p_2. \tag{2.101}$$

For simplicity, we would want all these geodesics to lie in the $\theta = \pi/2$ plane. Now, its just a matter of calculating the various geodesics constants for the three particle to make sure that all of what we want can be consistently done. To this end, lets recall the results of eq.(2.59) and eq.(2.60) and specialize it to the $\theta = \pi/2$ plane. The r and θ coordinates of all the geodesics above must satisfy

$$\rho^4 \dot{r}^2 = R(r) \equiv C^2 + \Delta(q^2 r^2 - K), \tag{2.102}$$

and

$$\rho^4 \dot{\theta}^2 = \Theta(\theta) \equiv K + q^2 a^2 \cos^2\theta - \frac{D^2}{\sin^2\theta}, \tag{2.103}$$

where

$$C = C(r) = (r^2 + a^2)E - aL\ , \& \qquad D = D(\theta) = L - aE\sin^2\theta.$$

As before, here, K is the Carter's constant. Since we want all geodesics to be in the $\theta = \pi/2$ plane, following eq. (2.103), we set

$$K_0 = D_0^2\ , \quad K_1 = D_1^2 \quad \& \quad K_2 = D_2^2. \tag{2.104}$$

The subscripts on all quantities refer to the particle labels as given in eq. (2.98 - 2.100). We will also pick the initial object to have unit mass, and we will drop it from rest at infinity, i.e.,

$$m = 1 \quad \& \qquad E_0 = 1. \tag{2.105}$$

At $r = r_d$, contracting the four-momentum conservation equation eq. (2.101) with the Killing vectors of the Kerr geometry, we get the conservation of energy and angular momentum relations, i.e.,

$$1 = E_1 + E_2 \quad \& \quad L_0 = L_1 + L_2. \tag{2.106}$$

The above equations will not uniquely specify the remaining constants. One way to insist on eq. (2.101), and a safe return of our energetic photon is to require that $r = r_d$ is the only turning point of all the three geodesics. That is we require that at $r = r_d$ we have that $\dot{r}_0 = \dot{r}_1 = \dot{r}_2 = 0$. Then clearly, eq. (2.106) will ensure that

$$(\dot{t}_0, 0, 0, \dot{\varphi}_0) = (\dot{t}_1, 0, 0, \dot{\varphi}_1) + (\dot{t}_2, 0, 0, \dot{\varphi}_2), \tag{2.107}$$

since $\dot{t}$ and $\dot{\varphi}$ are given by eq. (2.35) and eq. (2.36). Imposing the turning point condition, from eq. (2.102), setting $\dot{r}_0 = 0$ at $r = r_d$ for $E_0 = 1$ and $q^2 = -1$, we can solve for L_0. This gives

$$L_0 = \frac{1}{2M - r_d}[2aM - \sqrt{2Mr_d\,\Delta_{r_d}}]. \tag{2.108}$$

Here $r = 2M$ locates the outer boundary of the ergosphere in the $\theta = \pi/2$ plane, and $\Delta_{r_d} = \Delta(r = r_d)$. For r_d close to the event horizon, $\Delta \approx 0$ and so $L_0 > 0$

(we picked the appropriate root for the quadratic equation for L_0 in eq. (2.102) so that this happens). Similarly, from eq. (2.102), setting $\dot{r}_1 = 0$ at $r = r_d$ for as yet arbitrary but negative E_1 gives:

$$L_1 = \frac{1}{2M - r_d}[2aM + r_d\sqrt{\Delta_{r_d}}]E_1. \tag{2.109}$$

Here, we set $q^2 = 0$, since this geodesic describes a photon. Clearly, when E_1 is less than zero, so is L_1. Finally, for photon number 2 we get get in a similar manner that

$$L_2 = \frac{1}{2M - r_d}[2aM - r_d\sqrt{\Delta_{r_d}}]E_2. \tag{2.110}$$

With little difficulty we obtain from eq. (2.106), eq. (2.108), eq. (2.109) and eq. (2.110) the values for the photon energies:

$$E_1 = -\frac{1}{2}(\sqrt{\frac{2M}{r_d}} - 1) \quad \text{and} \quad E_2 = \frac{1}{2}(\sqrt{\frac{2M}{r_d}} + 1). \tag{2.111}$$

Indeed, the gain in energy in this process is given by

$$\Delta E = E_2 - E_0 = E_2 - 1 = \frac{1}{2}(\sqrt{\frac{2M}{r_d}} - 1) = -E_1 \tag{2.112}$$

as expected. This is the Penrose process.

As we have seen, negative energy particles in the ergosphere have negative angular momentum. As the black hole swallows such particles, the mass and the angular momentum of the black hole decreases. This will also lead to a decrease in the ergosphere region. Once the ergosphere vanishes, we cannot continue further with the extraction process. This places a natural limit on the amount of energy we can extract from the black hole. We now proceed to calculate this limit.

Consider the geodesic that describes the negative energy photon (or any other particle in general) that falls into the hole. From remarks made earlier, we know that for such a geodesic, in the ergosphere $\dot{t} \geq 0$ and $\dot{\varphi} \geq 0$. Therefore, at the event horizon, as the particle enters the black hole, from eq. (2.96) and eq. (2.97) we get the single condition

$$E \geq \Omega_H L. \tag{2.113}$$

Here, eq. (2.96) and eq. (2.97) were evaluated at $r = r_+$ (i.e., $\Delta = 0$), and

$$\Omega_H = \frac{a}{r_+^2 + a^2} = \frac{a}{2Mr_+}. \tag{2.114}$$

Ω_H is usually referred to as the angular velocity of the event horizon. After the negative energy particle falls into the black hole, it suffers a mass and angular momentum decrease subject to the condition

$$\delta M \geq \Omega_H \, \delta J. \tag{2.115}$$

Here, $J = aM$ is the angular momentum of the black hole. To consider the time evolution of the Kerr black hole which is subject to energy extraction, let us assume that this process is done in a very slow manner so that we may employ the adiabatic approximation. That is, the black hole metric is still given by the Kerr metric with its new value of M and J, i.e.,

$$M \to M + \delta M \quad \text{and} \quad J \to J + \delta J. \tag{2.116}$$

Of course $\delta J = \delta(aM) = M\ \delta a + a\ \delta M$. Christodoulou [11] defines the irreducible mass of the black hole to be

$$M_{irr}^2 = \frac{1}{2}[M^2 + \sqrt{M^4 - J^2}] = \frac{1}{2}M[M + \sqrt{M^2 - a^2}]. \tag{2.117}$$

To obtain the utility and the meaning of the above expression, lets compute the variation of the irreducible mass. We will do this in the usual manner by taking its variational derivative. Consider variations of the type

$$M \to M + \lambda\, \delta M \quad \text{and} \quad a \to a + \lambda\, \delta a. \tag{2.118}$$

Here λ is the variational parameter. By definition

$$\delta M_{irr}^2 = \frac{dM_{irr}^2}{d\lambda}\ |_{\lambda=0}$$

$$= \frac{1}{2}\delta M(M + \sqrt{M^2 - a^2}) + \frac{1}{2}M(\ \delta M + \frac{2M\ \delta M - 2a\ \delta a}{2\sqrt{M^2 - a^2}})$$

$$= \frac{1}{2\sqrt{M^2 - a^2}}[(2Mr_+ - a^2)\ \delta M - aM\ \delta a]. \tag{2.119}$$

But from eq. (2.115)

$$(2Mr_+ - a^2)\ \delta M > a\ \delta J - a^2\ \delta M = aM\ \delta a. \tag{2.120}$$

The above two equations imply that

$$\delta M_{irr}^2 > 0\ .$$

Therefore, the irreducible mass of a black hole cannot decrease as it expels energy and angular momentum. Eq. (2.117) can be inverted to give:

$$M^2 = M_{irr}^2 + \frac{J^2}{4M_{irr}^2}. \tag{2.121}$$

Now consider a Kerr black hole with some initial value of mass and angular momentum M_i and J_i. We can at best extract energy from the hole such that $\delta M_{irr}^2 = 0$ (the theoretical ideal). At the end of the energy extraction process, the mass of the black hole has now been reduced to the value $M = M_{irr}(M_i, J_i)$.

From the above equation we see that this happens when the new $J = 0$, i.e., the ergosphere has disappeared. Consequently, the term

$$\frac{J^2}{4M_{irr}^2} \tag{2.122}$$

can be thought of as the extractable rotational energy of the black hole. Since $J_i^2 = M_i^2$ for a maximally rotating Kerr black hole, the percent of energy that can be extracted from the hole is given by

$$(1 - \frac{M_{irr}}{M_i}) \times 100\% = (1 - \frac{1}{\sqrt{2}}) \times 100\% \approx 29\%. \tag{2.123}$$

Thus we see that the astrophysical black hole has little in common with its proverbial counterpart.