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Towards a general description of the interior structure of rotating black holes

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The purpose of this paper is to present a number of proposals about the interior structure of a rotating black hole that is accreting slowly, but in an arbitrary time- and space-dependent fashion. The proposals could potentially be tested with numerical simulations. Outgoing and ingoing particles free-falling in the parent Kerr geometry become highly focused along the principal outgoing and ingoing null directions as they approach the inner horizon, triggering the mass inflation instability. The original arguments of Barrabés, Israel & Poisson (1990) regarding inflation in rotating black holes are reviewed, and shown to be based on Raychauduri's equation applied along the outgoing and ingoing null directions. It is argued that gravitational waves should behave in the geometric optics limit, and consequently that the spacetime should be almost shear-free. A full set of shear-free equations is derived. A specific line-element is proposed, which is argued should provide a satisfactory approximation during early inflation. Finally, it is argued that super-Planckian collisions between outgoing and ingoing particles will lead to entropy production, bringing inflation to an end, and precipitating collapse.

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I. INTRODUCTION

What happens if you fall inside an astronomical black hole? Singularity theorems are quite unspecific [1], stating only that, subject to a trapped surface condition, an energy condition, and a causality condition, a geodesic is incomplete. Singularity theorems do not prescribe the nature of the incompleteness, nor do they assert that any matter actually moves along the incomplete geodesic, and they certainly do not imply that an infalling observer must fall to a singularity.

A major advance was made when Poisson & Israel [2, 3] discovered the mass inflation instability at the inner horizon of a charged, spherical (Reissner-Nordström) black hole. The inflationary instability is the nonlinear realization of the infinite blueshift at the inner horizon first pointed out by Penrose [4]. Many subsequent analytic and numerical investigations confirmed the instability (see [5] for a review and references). Chan, Chan & Mann [6] extended the Poisson-Israel argument to an azimuthally symmetrically accreting rotating black hole in 1+2 dimensions.

Barrabès, Israel & Poisson [7], hereafter BIP, soon generalized Poisson & Israel's argument to the case of a rotating black hole. The BIP argument is reviewed and critiqued below, §II. Further efforts to explore inflation in rotating black holes were made by Ori [8, 9].

In a recent series of papers [10–12], the author presented some explicit solutions for the interior structure of a rotating black hole that slowly accretes a collisionless fluid. The solutions followed the self-consistent nonlinear evolution of the spacetime through inflation to collapse down to an exponentially tiny scale, where rotation reasserted itself. However, the solutions were limited to the conformally separable special case, where the accretion flow incident on the inner horizon was uniform.

The purpose of the present paper is to set forward arguments about how inflation should develop in a rotating black hole that is accreting slowly, but in an arbitrary time- and space-dependent fashion. The arguments are in places qualitative rather than rigorous, so should be construed as tentative. Specifically, the argument that the spacetime is likely to be almost shearfree, §III, the possibility that the line-element (20) may provide an adequate approximation, and the argument that inflation will be terminated by super-Planckian collisions between outgoing and ingoing particles, §VII, are proposals to be tested. The proposals could be tested by numerical simulations. Whether true or false, I hope that simulators will find the ideas herein a useful guide as to what might be expected.

At first sight, it might seem that the problem of the interior structure of a rotating black hole accreting in an arbitrary time- and space-dependent fashion would be too complex to be analytically tractable. However, the problem simplifies if the black hole is accreting slowly, and therefore has a geometry well described by Kerr away from its inner horizon. Aside from rare (but interesting!) events of high accretion, such as when a black hole first forms, or when two black holes merge, real astronomical black holes accrete slowly for most of their lives.

A slowly accreting black hole has a number of simplifying features. First, as shown by [11], freelyfalling particles focus along the principal outgoing and ingoing null directions of the Kerr geometry as they approach the inner horizon, regardless of the initial orbital parameters of the particles. Moreover, for small accretion rates, outgoing and ingoing particles are

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already hyper-relativistic relative to each other when inflation ignites, so massive as well as massless, and charged as well as uncharged particles, all focus along the principal null directions as they approach the inner horizon. Thus the situation originally envisaged by BIP of outgoing and ingoing null streams crossing near the inner horizon is in fact realized in the real case.

The second simplifying feature is that during inflation, although the streaming energy-momentum and the Weyl curvature grow exponentially huge, the underlying geometry remains scarcely changed. In a sense, nothing is happening, despite the exponentially growing curvature. The reason for this is that the proper time experienced by an infaller during inflation is tiny, so that volume elements have little time to become distorted despite the enormous tidal forces, a fact first pointed out by [13].

Eventually however the spacetime does respond to the huge acceleration, by collapsing.

The metric signature in this paper is -+++.

II. THE BIP ARGUMENT

It is helpful to start by summarizing BIP's argument. For generality, I will recast BIP's [7] argument into continuum form. As will be seen, BIP's argument is essentially founded on the Raychaudhuri equation applied along two null directions (whereas Penrose's original singularity theorem [14] invoked the Raychaudhuri equation along a single null direction). In §IIB, I point out issues left open by BIP's argument.

A. Argument

BIP start by positing two null shells that cross each other near the inner horizon of a rotating black hole. As noted in the Introduction, freely-falling particles do in fact focus along the principal outgoing and ingoing null directions of the Kerr geometry as they approach the inner horizon, regardless of the initial orbital parameters or mass of the particles [11]. Thus the situation of two special null directions envisaged by BIP does in fact occur in the real case, at least for a slowly accreting black hole.

The principal null directions of the Kerr geometry are geodesic. More generally, as the geometry departs from Kerr thanks to the inflationary back-reaction, the two special null directions posited by BIP can be taken to be the geodesic continuation of the principal null directions. Choose a Newman-Penrose double-null tetrad frame $\{\gamma_v, \gamma_u, \gamma_+, \gamma_-\}$ such that the outgoing and ingoing tetrad axes γ_v and γ_u point along the two special outgoing and ingoing null geodesic directions, while the spinor axes γ_+ and γ_- (which are complex conjugates of each other) span the two-dimensional spatial plane orthogonal to the null directions.

In what follows, it is convenient to follow a convention in which early latin indices a, b, ... run over spinor indices +, -, late latin indices z, y, ... run over null indices v, u, and mid latin indices k, l, ... run over all four indices.

The condition that the special null directions be geodesic means that the tetrad-frame momenta p^a orthogonal to the geodesics satisfy $Dp^a/D\lambda = 0$, where D denotes the covariant derivative, and λ is an affine parameter. This in turn implies the vanishing of 4 of the 24 distinct tetrad-frame connections Γ_{klm} (recall that the tetrad-frame connections Γ_{klm} , being generators of Lorentz transformations, are antisymmetric in their first two indices, $\Gamma_{klm} = -\Gamma_{lkm}$), namely

$$\Gamma_{+vv} = \Gamma_{-vv} = \Gamma_{+uu} = \Gamma_{-uu} = 0 .$$
 (1)

These conditions use up four of the six gauge degrees of freedom of Lorentz transformations of the tetrad. The two remaining tetrad degrees of freedom correspond to a Lorentz boost in the vu plane, and a spatial rotation in the +- plane.

The extrinsic curvatures along the two null directions v and u are defined in the usual way to be the two 2 × 2 matrices Γ_{avb} and Γ_{aub} . The extrinsic curvatures are tetrad-frame tensors with respect to the two unfixed gauge degrees of freedom of the tetrad, a fact proven in Appendix A. Being tetrad-frame quantities, all tetrad-frame connections Γ_{klm} are automatically coordinate gauge-invariant. The standard definition of the tetrad-frame Riemann tensor R_{klmn} , coupled with the conditions (1), implies the generalized Raychaudhuri equations for the extrinsic curvatures along each of the null directions z = v, u (no implicit summation over the two lowered indices z),

$$D_z \Gamma_{azb} + \Gamma_{azc} \Gamma^c{}_{zb} + R_{zazb} = 0 .$$
 (2)

The four components of the extrinsic curvature Γ_{azb} along each null direction are commonly decomposed into a trace part, the expansion ϑ_z , an antisymmetric part, the twist ϖ_z , and a traceless symmetric part, the complex shear σ_z :

$$\Gamma_{+v-} = \Gamma_{-v+}^* \equiv \vartheta_v + i\varpi_v , \quad \Gamma_{+v+} = \Gamma_{-v-}^* \equiv \sigma_v ,$$
(3a)
$$\Gamma_{-u+} = \Gamma_{+u-}^* \equiv \vartheta_u + i\varpi_u , \quad \Gamma_{-u-} = \Gamma_{+u+}^* \equiv \sigma_u .$$
(3b)

In terms of the expansion, twist, and shear, the generalized Raychaudhuri equations (2) along the outgoing null direction v are

$$(D_v + \vartheta_v)\vartheta_v - \varpi_v^2 + \sigma_v\sigma_v^* + 4\pi T_{vv} = 0 , \qquad (4a)$$

$$(D_v + 2\vartheta_v)\varpi_v = 0 , \qquad (4b)$$

$$(D_v + 2\vartheta_v)\sigma_v + C_{v+v+} = 0 , \qquad (4c)$$

where C_{klmn} is the Weyl tensor, the traceless part of the Riemann tensor R_{klmn} . A similar set of equations holds for the ingoing direction u, with C_{v+v+} replaced by C_{u-u-} in equation (4c) for the shear. The covariant derivative D_v in equations (4) can be interpreted as acting either on rank-1 vectors ϑ_v , ϖ_v , and σ_v , or on their rank-3 tensor antecedents, equations (3a); the result is identical, given conditions (1). In the Raychaudhuri equation (4a) for the expansion, Einstein's equations have been invoked to replace the Ricci tensor R_{vv} by the energy-momentum $8\pi T_{vv}$. The energy-momentum T_{vv} that goes into the equation for the outgoing expansion is the ingoing energy flux: if the energy-momentum is characterized as a sum over streams of particles with tetrad-frame number currents n^k and momenta p^k , then the ingoing energy flux T_{vv} is

$$T_{vv} = T^{uu} = \sum_{\text{streams}} n^u p^u .$$
 (5)

In the BIP scenario of two null shells crossing, the Raychaudhuri equations (4a) can be integrated immediately over the crossing shells, leading to the result that the expansions ϑ_v and ϑ_u in the outgoing and ingoing directions are changed respectively by the integrals $4\pi \int T_{vv} dv$ and $4\pi \int T_{uu} du$ over the deltafunction of flux in the opposing direction.

BIP pointed out that the product $\vartheta_v \vartheta_u$ of the null expansions is a tetrad-frame scalar (with respect to the two unfixed tetrad degrees of freedom), as well as a coordinate scalar, and they proposed that this product would define an effective scalar mass function, as it does in spherically symmetric black holes. In fact for a (non-inflating) Kerr black hole of mass M_{\bullet} and specific angular momentum a, the product of the expansions is (in standard Boyer-Lindquist spheroidal coordinates)

$$\vartheta_v \vartheta_u = \frac{r^2 (r^2 + a^2)}{2(r^2 + a^2 \cos^2 \theta)^3} \left(\frac{2M_{\bullet}r}{r^2 + a^2} - 1\right) .$$
(6)

BIP argued that the closer to the inner horizon that the shells passed through each other, the more blueshifted the energy-momentum in the opposite shell would appear, and therefore the greater the increase of mass.

B. Critique

As BIP acknowledge, their analysis falls short of a complete description of inflation in rotating black holes.

At least initially, and for small accretion rates, the assumption that energy-momentum is focused along two special null directions appears robust, since outgoing and ingoing particles necessarily focus along the principal null directions of the Kerr geometry as they approach the inner horizon [11]. It seems highly likely that geodesics will remain focused, because the Raychaudhuri equation (4a) indicates that the effect of energy-momentum focused along the outgoing and ingoing directions is to cause further focusing (make the expansion more negative).

BIP's assumption of two thin shells allows the Raychaudhuri equations (4a) to be integrated immediately, but neglects the effects of twist and shear. Are twist and shear in fact unimportant? In the conformally separable solutions of [10–12], inflation was followed by collapse to an exponentially tiny scale, where rotation reasserted itself. The finite angular momentum of a null geodesic bundle is encoded in its twist. The Raychaudhuri equation (4b) shows that twist is amplified by collapse, and the Raychaudhuri equation (4a) shows that the effect of twist is to slow collapse. The Raychaudhuri equation (4c) shows similarly that shear is amplified by collapse. Notwithstanding the argument in §III that shear is probably small, shear will probably also become important following collapse. Whereas twist slows collapse, shear speeds it.

This suggests that twist and shear are probably unimportant prior to collapse, but are likely to become important following collapse. The subsequent evolution of the spacetime could be complicated, possibly with BKL-like behavior [15–19].

A potentially more serious defect is that the BIP analysis does not take into account the possible back-reaction of the inflating energy-momenta on the horizon. In the solutions of [10-12], inflation is followed by collapse, a fate inaccessible to BIP's analysis.

Ideally, it would be nice to have an explicit, general solution that self-consistently follows the evolution of the spacetime and energy-momentum through inflation to whatever happens then. The intent of this paper is to make some proposals about what that solution might look like.

III. THE ROLE OF GRAVITATIONAL WAVES

In this section it is argued that shear is likely to be small.

BIP, following [3], suggested that the outgoing energymomentum needed for inflation to occur would be provided by a Price [20] tail of gravitational radiation generated by the collapse of the black hole. On the other hand [11] argued that in real astronomical black holes, direct accretion of baryons and dark matter from outside the outer horizon provides a continual source of both outgoing and ingoing streams incident on the inner horizon, that probably soon overwhelm the Price tails generated by collapse. Direct accretion of matter can provide not only ingoing but also outgoing streams. Although a stream falling through the outer horizon is necessarily ingoing, it will become outgoing at the inner horizon if its angular momentum is large enough in the same direction as the angular momentum of the black hole.

This raises the question of how important gravitational waves are in shaping the structure of spacetime during inflation and thereafter?

One place where gravitational waves play an essential role is in radiating away the "hair" that a black hole acquires when it accretes anisotropically. The loss of hair by gravitational radiation is what makes it reasonable to treat a slowly accreting black hole as being approximated accurately by the Kerr geometry down to near its inner horizon.

However, from the perspective of what is happening in the inflationary zone near the inner horizon, gravitational waves should behave in the high-frequency, geometricoptics limit, as if they were any other kind of (massless) particle, as both BIP and [11] argued. Averaged over regions much larger than a wavelength, the gravitational wave contribution to the Einstein tensor can be taken over to the right hand side of the Einstein equations and reinterpreted as "real" energy-momentum.

Mathematically, the proposition that gravitational waves behave in the geometric optics limit is equivalent to the assertion that the high-frequency spin ± 2 components of the Weyl tensor vanish when averaged over the rapidly oscillating field,

$$\langle C_{z+z+} \rangle = \langle C_{z-z-} \rangle = 0 . \tag{7}$$

It then follows from integrating the Raychaudhuri equation (4c) that the average high-frequency shear also vanishes, given that the shear is initially zero, as it is in the Kerr geometry,

$$\langle \sigma_z \rangle = 0 \ . \tag{8}$$

On the other hand, the absolute value squared of the high-frequency shear does not vanish on averaging, and indeed could possibly be substantial,

$$\langle \sigma_z \sigma_z^* \rangle \neq 0 . \tag{9}$$

This averaged mean square shear (9) makes a contribution to the Raychaudhuri equation (4a) for the expansion, behaving effectively as if it were equivalent to energymomentum $4\pi T_{zz}$.

The analogous situation in electromagnetism is that electromagnetic waves behave as photons, in which the high-frequency components of the electric and magnetic fields vanish on average, but their squares do not vanish, and carry energy-momentum.

What about shear at long wavelengths, comparable to the size of the black hole? For slow accretion, the geometry away from the inner horizon will be close to Kerr, which is shear-free. Thus even long-wavelength shear is likely to be small.

Of course, just as photons can scatter, so also gravitons can scatter. However, I argue in §VII that scattering processes are likely to become important only when center-of-mass collision energies exceed the Planck scale.

IV. SHEAR-FREE EQUATIONS

If shear is negligible, as argued in the previous section §III, then a more complete set of equations can be derived, still without any explicit line-element.

A. Equations

As remarked in §II, the 8 tetrad-frame connections embodied by the extrinsic curvatures along the two null directions form tetrad-frame tensors with respect to the two unfixed tetrad gauge degrees of freedom. A further 8 tetrad-frame connections are tetrad-frame tensors, namely the extrinsic curvatures Γ_{yaz} along the angular tetrad directions a = +, -. The extrinsic curvature along each angular direction can be decomposed into a trace part, the expansion ϑ_a , and an antisymmetric part, the twist ϖ_a , analogously to equations (3),

$$\frac{1}{2}(\Gamma_{+vu} + \Gamma_{+uv}) \equiv \vartheta_{+} , \quad \frac{1}{2}(\Gamma_{+vu} - \Gamma_{+uv}) \equiv i\varpi_{+}$$
(10a)
$$\frac{1}{2}(\Gamma_{-uv} + \Gamma_{-vu}) \equiv \vartheta_{-} , \quad \frac{1}{2}(\Gamma_{-uv} - \Gamma_{-vu}) \equiv i\varpi_{-} .$$
(10b)

Note that $\vartheta_{-} = \vartheta_{+}^{*}$ and $\varpi_{-} = \varpi_{+}^{*}$. The angular shears σ_{a} vanish identically, equations (1).

The 16 tetrad-frame connections comprising the extrinsic curvatures along each of the 4 tetrad axes satisfy the 16 generalized Raychaudhuri equations

$$D_z \Gamma_{azk} + \Gamma_{azb} \Gamma^b{}_{zk} + R_{zazk} = 0 , \qquad (11)$$

where, as before, early latin indices a, b run over spinor indices +, -, the late latin index z runs over null indices v, u, and the mid latin index k runs over all four indices. Only the paired index b with one raised and one lowered is summed over; the pair of lowered z indices is not summed over. The 16 equations (11) include 4 whose terms all vanish identically, namely the equations for the angular shear Γ_{azz} . Equations (11) are valid without any assumption about the spacetime: they hold given any arbitrary pair of null geodesic directions, and a tetrad aligned with them.

Further progress can be made if, as argued in §III, the shear along the two null directions can be neglected, equation (8), or equivalently

$$\Gamma_{+v+} = \Gamma_{-v-} = \Gamma_{+u+} = \Gamma_{-u-} = 0 .$$
 (12)

Under the condition (12) of vanishing shear, the extrinsic curvatures along each of the 4 tetrad directions satisfy a further 16 generalized Raychaudhuri equations,

$$D_a \Gamma_{zak} + \Gamma_{zay} \Gamma^y{}_{ak} + R_{azak} = 0 .$$
 (13)

The two sets of Raychaudhuri equations (11) and (13) comprise two distinct sets of 8 non-vanishing equations for the expansions and twists along each of the tetrad directions, a total of 16 non-vanishing equations altogether.

Combining the Raychaudhuri equations (11) and (13), and invoking Einstein's equations to replace the Ricci tensor by energy-momenta, yields 8 Einstein equations,

$$D_z \Gamma^a{}_{za} + \Gamma^a{}_{zc} \Gamma^c{}_{za} + 8\pi T_{zz} = 0 , \quad (14a)$$

$$D_z \Gamma_{azy} + D_a \Gamma_{azb} + 2\Gamma_{azb} \Gamma_{azy} + 8\pi T_{za} = 0 , \quad (14b)$$

$$D_a \Gamma^{z}{}_{az} + \Gamma^{z}{}_{ay} \Gamma^{y}{}_{az} + 8\pi T_{aa} = 0 .$$
(14c)

In equation (14b), y is the null index opposite to z, and b is the spinor index opposite to a.

The Raychaudhuri equations (11) and (13) combine in a different way to yield an expression for the 4 spin ± 1 components $C_{zazy} = C_{azab}$ of the Weyl tensor,

$$D_z \Gamma_{azy} - D_a \Gamma_{azb} + 2C_{zazy} = 0 , \qquad (15)$$

where again y is the null index opposite to z, and b is the spinor index opposite to a. Altogether the 16 nonvanishing generalized Raychaudhuri equations comprise 8 Einstein equations, 4 Weyl equations, and 4 twist equations.

The above equations account for all but 8 of the equations relating the Riemann tensor to derivatives of the tetrad connections. The remaining 8 equations involve the scalar (with respect to the two unfixed degrees of freedom) components of the tetrad-frame Riemann tensor. Four of the equations involve only the expansion and twist connections. Of these, one constitutes an equation for a combination of the spin 0 polar (real) Weyl component C_{vuvu} and the Ricci scalar R, and another constitutes an equation for the spin 0 axial (imaginary) Weyl component C_{vu+-} :

$$R_{v+u-} + R_{v-u+} = -C_{vuvu} - \frac{1}{6}R$$
(16a)

$$= D^{z}\vartheta_{z} + D^{a}\vartheta_{a} + \vartheta^{z}\vartheta_{z} + \vartheta^{a}\vartheta_{a} + \varpi^{z}\varpi_{z} + \varpi^{a}\varpi_{a} ,$$

$$C_{vu+-} = R_{vu+-} = i\left(D^{z}\varpi_{z} + D^{a}\varpi_{a}\right) .$$
(16b)

In equations (16), radial and angular vectors (super/subscripted z and a respectively) are to be interpreted as lying in their respective 2-dimensional tangent spaces (thus for example $\vartheta_z = \{\vartheta_v, \vartheta_u, 0, 0\}$), and the covariant derivatives act on the corresponding 2-dimensional space. The two other equations involving only expansions and twists are:

$$D_{v}\vartheta_{u} - D_{u}\vartheta_{v} = i \left[(D_{+} + 2\vartheta_{+})\varpi_{-} - (D_{-} + 2\vartheta_{-})\varpi_{+} \right],$$
(17a)
$$D_{+}\vartheta_{-} - D_{-}\vartheta_{+} = i \left[(D_{v} + 2\vartheta_{v})\varpi_{u} - (D_{u} + 2\vartheta_{u})\varpi_{v} \right].$$
(17b)

The remaining 4 Riemann equations include 2 that determine the remaining scalar components R_{vuvu} and R_{+-+-} of the tetrad-frame Riemann tensor. These two components cannot be expressed solely in terms of expansions and twists and their covariant derivatives. Rather, they depend on two further gauge-invariant quantities that correspond physically to the Lorentz boost between the affinely-parameterized outgoing and ingoing null directions, and the spatial rotation angle between the affinely-parameterized angular directions. An explicit example will be seen in §V.

If the energy-momenta T_{kl} are arranged to satisfy covariant conservation of energy-momentum, as they should, and if the 8 Einstein equations (14) are satisfied, then the Einstein tensor will automatically satisfy covariant conservation, as it should,

$$D^k G_{kl} = 0 . (18)$$

The conservation equations (18) provide 2 evolution equations for G_{vu} (along each of the null directions), and 2 constraint equations for G_{+-} (along each of the spinor directions), that will be satisfied automatically.

B. Schematic solution of the equations

The equations of the previous subsection, §IV A, were derived without the benefit of any explicit line-element or choice of coordinates, the only assumption being that shear vanishes, as argued in §III. It is helpful to outline how these equations could in principle be solved.

Suppose that the energy-momenta T_{kl} are given. As argued in §VII, outgoing and ingoing streams should stream through each other collisionlessly at least until collision energies exceed the Planck scale. As long as the freely-falling particles remain focused along the null directions, the energy-momenta will satisfy the hierarchy of conditions

$$T_{zz} \gg T_{za} \gg T_{aa} \ . \tag{19}$$

The dominant energy-momenta are those T_{zz} along the null directions. Given these, the Raychaudhuri equations (4a) and (4b), integrated along the outgoing and ingoing null directions, yield the expansions and twists ϑ_z and ϖ_z along the null directions.

The next largest energy-momenta are the 4 radialangular components T_{za} . Given these, the Einstein equations (14b), integrated along the outgoing and ingoing null directions, yield the expansions and twists ϑ_a and ϖ_a along the angular directions. These 4 Einstein equations essentially express conservation of angular momentum.

The purely angular energy-momenta T_{aa} are subsub-dominant. The Einstein equations (14c) for T_{aa} depend on the angular expansions and twists ϑ_a and ϖ_a , but since these have already been determined by equations (14b) for T_{za} , it should be that, if all is consistent, the equations (14c) for T_{aa} should be satisfied identically. If equations (14c) are not satisfied to adequate accuracy, then it is a signal that the assumption of vanishing shear is breaking down.

The final two energy-momenta are T_{vu} and T_{+-} . As noted in §IV A, if the energy-momenta T_{kl} are arranged to satisfy covariant energy-momentum conservation, as they should, and if all is consistent, then the Einstein equations for T_{vu} and T_{+-} should be satisfied automatically.

V. MODEL LINE-ELEMENT

In this section I propose a particular line-element (20) that I argue should, during the early phases of inflation, provide an adequate description of the spacetime of a rotating black hole that accretes slowly, and undergoes inflation at (just above) its inner horizon.

A physical motivation for the line-element (20) is that the initial effect of inflationary energy-momentum, which is focused along the principal null directions, is to generate curvature (tidal accelerations) aligned with the principal frame. The amount of curvature at any point varies depending on the amplitude of the inflationary energy-momentum there, which in turn depends on the accretion flow into the two principal directions that pass through the point. The incipient tidal deformation is encoded in the radial and angular conformal factors ρ_x and ρ_y in the line-element (20).

The line-element (20) is overly simple, $\S V E$, but permits explicit expressions for the components G_{vu} and G_{+-} of the Einstein tensor, equations (35), that clarify how inflation affects the evolution of the conformal factors, $\S V D$, and provide the basis for some of the arguments in the next section, $\S VI$.

I refer to the tetrad frame defined by the lineelement (20) as "principal" even though it is technically not so (the complex self-dual Weyl tensor is not diagonal in this frame). The frame is almost principal in the sense that its null directions are the geodesic continuation of the Kerr principal null directions, and the spin ± 2 (but not spin ± 1) components of the Weyl tensor vanish (so the Petrov type is general, not Type D).

A. Line-element

The proposed line-element is essentially the Kerr lineelement modulated by two conformal factors, a radial conformal factor ρ_x and an angular conformal factor ρ_y , each of which could in principle be an arbitrary function of all 4 coordinates x, t, y, ϕ :

$$ds^{2} = \rho_{x}^{2} \left[\frac{dx^{2}}{\Delta_{x}} - \frac{\Delta_{x}}{(1 - \omega_{x}\omega_{y})^{2}} \left(dt - \omega_{y} d\phi \right)^{2} \right] + \rho_{y}^{2} \left[\frac{dy^{2}}{\Delta_{y}} + \frac{\Delta_{y}}{(1 - \omega_{x}\omega_{y})^{2}} \left(d\phi - \omega_{x} dt \right)^{2} \right] .$$
(20)

The expressions in square brackets are those of the Kerr line-element. The coordinates t and ϕ are time and azimuthal coordinates, while the coordinates x and y are radial and angular coordinates related to the usual Boyer-Linquist spheroidal coordinates r and θ by

$$r \equiv a \cot(ax) , \quad \cos \theta \equiv -y .$$
 (21)

The quantities ω_x and ω_y are functions respectively only of the radial and angular coordinates,

$$\omega_x = \frac{a}{r^2 + a^2} , \quad \omega_y = a \sin^2 \theta , \qquad (22)$$

and likewise the radial and angular horizon functions Δ_x and Δ_y are functions respectively only of the radial and angular coordinates,

$$\Delta_x = \frac{1}{r^2 + a^2} \left(1 - \frac{2M_{\bullet}r}{r^2 + a^2} \right) , \quad \Delta_y = \sin^2\theta , \quad (23)$$

where M_{\bullet} is the black hole's mass.

The situation of interest is near the inner horizon, where the radial horizon function is negative and tending to zero, $\Delta_x \rightarrow -0$. The radial coordinate x is then timelike (and increasing inward, the direction of increasing proper time), while the time coordinate t is spacelike.

Since the black hole is slowly accreting, at any given time, call it t = 0, the line-element from just above the inner horizon upward should closely approximate the Kerr line-element, in which case the conformal factors ρ_x and ρ_y are given by their Kerr value

$$\rho_x = \rho_y = \rho_{\text{Kerr}} = \sqrt{r^2 + a^2 \cos^2\theta} .$$
 (24)

Because the conformal factors ρ_x and ρ_y are allowed to be functions of time t, the black hole can grow. From just above the inner horizon inward, inflation accelerates the conformal factors away from the Kerr value (24).

The line-element (20) generalizes the conformally separable line-element considered by [10–12]. Specifically, the transformations (there \rightarrow here)

$$\rho \to \rho_y , \quad dx \to \frac{\rho_x^2}{\rho_y^2} dx , \quad \Delta_x \to \frac{\rho_x^2}{\rho_y^2} \Delta_x ,$$
(25)

bring the conformally separable line-element of [10–12] to the form (20). The advantage of this transformation is that, if the conformal factors ρ_x and ρ_y are allowed to be arbitrary, then approximate solutions with arbitrarily time- and space-dependent accretion flow can be admitted. The solutions are no longer conformally separable, so null geodesics are no longer exactly solvable, but they prove do be solvable to adequate accuracy, §V B.

The line-element (20) defines not only a metric $g_{\kappa\lambda}$, which is an inner product of coordinate tangent vectors g_{κ} , but also, through

$$\boldsymbol{g}_{\kappa} \cdot \boldsymbol{g}_{\lambda} = g_{\kappa\lambda} = \eta_{kl} e^{k}{}_{\kappa} e^{l}{}_{\lambda} = e^{k}{}_{\kappa} \boldsymbol{\gamma}_{k} \cdot e^{l}{}_{\lambda} \boldsymbol{\gamma}_{l} , \qquad (26)$$

an inverse vierbein e^k_{κ} , and a corresponding orthonormal tetrad $\{\gamma_x, \gamma_t, \gamma_y, \gamma_\phi\}$, whose inner products form the Minkowski metric, $\gamma_k \cdot \gamma_l = \eta_{kl}$. The Newman-Penrose tetrad $\{\gamma_v, \gamma_u, \gamma_+, \gamma_-\}$ corresponding to the orthonormal tetrad is

$$\gamma_u^v \equiv \frac{1}{\sqrt{2}} (\gamma_x \pm \gamma_t) , \quad \gamma_{\pm} \equiv \frac{1}{\sqrt{2}} (\gamma_y \pm i \gamma_{\phi}) .$$
 (27)

B. Geodesics

The low-density, hyper-relativistically counterstreaming streams in the inflationary zone of a typically slowly accreting astronomical black hole should behave collisionlessly, at least until collision energies reach the Planck scale, §VII. Determining the collisionless energymomentum requires solution of the number density N and tetrad-frame momentum p_k along freely-falling trajectories. The energy-momentum tensor T_{kl} will then be a sum over collisionless streams,

$$T_{kl} = \sum_{\text{streams}} n_k p_l , \qquad (28)$$

where n^k is the number current

$$n^k = N p^k . (29)$$

Conformally separability is the proposition that the equations of motion of massless particles are Hamilton-Jacobi separable. The condition is weaker than strict separability (that is, Hamilton-Jacobi separability for massive as well as massless geodesics). Whereas strict separability leads to electrovac and cognate spacetimes [21], conformal separability, by allowing an arbitrary overall conformal factor, also admits inflationary solutions [10–12].

The line-element (20) is not conformally separable, because it has two arbitrary conformal factors instead of just one. However, under the conditions peculiar to inflation, namely that motions are hyper-relativistic, and the radial horizon function Δ_x is almost zero, the spacetime is close enough to being conformally separable that geodesics, massless or massive, are given adequately by a Hamilton-Jacobi approximation. Conformal separability and the accuracy of Hamilton-Jacobi separation are discussed at length in [11]. Here I sketch only the main points relevant to the present lineelement (20).

Write the covariant components p_k of the tetrad-frame momentum of a freely-falling particle of mass m in terms of a set of Hamilton-Jacobi parameters P_k ,

$$\{p_x, p_t, p_y, p_\phi\} = \left\{\frac{P_x}{\rho_x \sqrt{-\Delta_x}}, \frac{P_t}{\rho_x \sqrt{-\Delta_x}}, \frac{P_y}{\rho_y \sqrt{\Delta_y}}, \frac{P_\phi}{\rho_y \sqrt{\Delta_y}}\right\} , \quad (30)$$

given by

$$P_t = \pi_t + \pi_\phi \omega_x , \qquad (31a)$$

$$P_{\phi} = \pi_{\phi} + \pi_t \omega_y , \qquad (31b)$$

$$P_x = \sqrt{P_t^2 - (\rho_x^2/\rho_y^2)} \left[\mathcal{K} + m^2(\rho_y^2 - a^2\cos^2\theta)\right] \Delta_x ,$$
(31c)

$$P_y = \sqrt{-P_{\phi}^2 + \left(\mathcal{K} - m^2 a^2 \cos^2\theta\right) \Delta_y} .$$
 (31d)

The constants π_t , π_{ϕ} , and \mathcal{K} here are the conserved energy, azimuthal angular momentum, and Carter constant of the particle in the parent Kerr spacetime above the inner horizon. The Hamilton-Jacobi parameters P_k differ from the Kerr parameters only in that P_x involves the conformal factors ρ_x and ρ_y . The Hamilton-Jacobi parameter P_x itself is chosen so that mass conservation is respected, $p^k p_k = -m^2$. That equations (30) and (31) provide an adequate approximation is plausible from the fact that the horizon function Δ_x is tiny during inflation, so the factor proportional to Δ_x under the square root in the expression (31c) for P_x is small, so the behavior of the conformal factors ρ_x and ρ_y is irrelevant. Conversely, the Hamilton-Jacobi approximation (30), (31) can be expected to break down when $(\rho_x^2/\rho_y^2)\Delta_x$ ceases to be small. This occurs when the angular conformal factor ρ_y has collapsed to the point that rotation becomes significant, that is, angular motions p_y , p_{ϕ} become comparable to radial motions p_t .

In the Hamilton-Jacobi approximation, the number density N along any collisionless stream satisfies [11]

$$N \propto \frac{1 - \omega_x \omega_y}{\rho_y^2 P_x P_y} \,. \tag{32}$$

Appendix B confirms that the Hamilton-Jacobi approximations (30), (31), and (32) are adequately accurate, as long as $(\rho_x^2/\rho_y^2)\Delta_x$ remains small.

C. Expansions, twists, Einstein tensor

The line-element (20) satisfies the conditions (1), (12) of vanishing shear along all 4 tetrad directions. Consequently all the equations derived in §IV hold for this line-element. The expansions and twists in the 4 tetrad directions are

$$\vartheta_z = \partial_z \ln \rho_y \,\,, \tag{33a}$$

$$\vartheta_a = \partial_a \ln \rho_x , \qquad (33b)$$

$$\varpi_v = \varpi_u = -\frac{\rho_x \sqrt{-\Delta_x}}{2\sqrt{2}\rho_y^2 (1 - \omega_x \omega_y)} \frac{d\omega_y}{dy} , \qquad (33c)$$

$$\varpi_{+} = \varpi_{-} = \frac{\rho_y \sqrt{\Delta_y}}{2\sqrt{2}\rho_x^2 (1 - \omega_x \omega_y)} \frac{d\omega_x}{dx} , \qquad (33d)$$

where ∂_k denote tetrad-frame directed derivatives (not coordinate-frame derivatives).

Explicit expressions for the components of the Einstein tensor not already given in §IV A, namely G_{vu} and G_{+-} , can be expressed in terms of ν and μ defined by

$$\nu \equiv \ln\left(\frac{1-\omega_x\omega_y}{\rho_x\sqrt{-\Delta_x}}\right) \ , \quad \mu \equiv \ln\left(\frac{1-\omega_x\omega_y}{\rho_y\sqrt{\Delta_y}}\right) \ . \tag{34}$$

The quantity ν is physically the boost angle (logarithm of the blueshift, or Lorentz γ -factor) of affinelyparameterized principal null geodesics relative to the tetrad frame. Affinely-parameterized outgoing and ingoing principal null geodesics are blueshifted by a factor $e^{2\nu}$ relative to each other. Similarly μ is the spatial rotation angle of frames parallel-transported along the angular directions, relative to the tetrad frame. The components G_{vu} and G_{+-} of the Einstein tensor are

$$G_{vu} = D^a \partial_a \mu - D^z \vartheta_z - D^a \vartheta_a - 2 \vartheta^z \vartheta_z + 2 \varpi^z \varpi_z - \vartheta^a \vartheta_a - \varpi^a \varpi_a , \quad (35a)$$

In equations (35), as previously in equations (16), radial and angular vectors are to be interpreted as lying in their respective 2-dimensional tangent spaces. Expressions (35) for the Einstein components G_{vu} and G_{+-} generalize equations (10) of [10].

D. Evolution equations for the conformal factors

During inflation, infalling streams become highly focused along the outgoing and ingoing principal null directions, and they generate a large streaming energymomentum along those directions. The Raychaudhuri equations (4) then imply that the expansions (but not twists) along the null directions grow large. For the lineelement (20), the radial expansions are radial gradients of the angular conformal factor, equation (33a). Thus during inflation radial gradients grow large, and these dominate the expressions (35) for G_{vu} and G_{+-} . The Einstein equations for the energy-momenta T_{vu} and T_{+-} then reduce to

$$-D^z \vartheta_z - 2\vartheta^z \vartheta_z \approx 8\pi T_{vu} , \qquad (36a)$$

$$-D^{z}\partial_{z}\nu + D^{z}\vartheta_{z} + \vartheta^{z}\vartheta_{z} \approx 8\pi T_{+-} .$$
 (36b)

The trace $\vartheta^z \vartheta_z$ of the radial expansion is minus twice BIP's mass parameter.

As long as the outgoing and ingoing flows counterstream collisionlessly, the energy-momenta T_{vu} and T_{+-} should be small, so that the right hand sides of equations (36) can be set to zero. Equations (36) with vanishing right hand sides effectively generalize equations (13a) and (13b) of [10], which formed the principal evolutionary equations of the latter paper. Note that the latter paper, elaborated by [11], was far more careful than the present paper. By restricting to conformally separable solutions, [10] was able to solve the Einstein equations by separation of variables in a fashion that encompassed evolution from electrovac through inflation to collapse down to a tiny scale.

In the inflationary (radial-gradient-dominated) regime, the Raychaudhuri equation (3a) for T_{vv} and Einstein equation (36a) for T_{vu} can be written

$$D_v \partial_v \rho_y \approx -4\pi \rho_y T_{vv}$$
, (37a)

$$D_v \partial_u \rho_y \approx -\frac{\mathcal{M}}{2\rho_y} + 4\pi r T_{vu} ,$$
 (37b)

where \mathcal{M} is the mass function

$$\mathcal{M} \equiv -\rho_y^2 \vartheta^z \vartheta_z \;. \tag{38}$$

In the case of spherical symmetry, the corresponding exact Einstein equations are

$$D_v \partial_v r = -4\pi r T_{vv} , \qquad (39a)$$

$$D_v \partial_u r = -\frac{M}{r^2} + 4\pi r T_{vu} , \qquad (39b)$$

where $r \equiv \rho_y$ is the circumferential radius, and M is the interior mass,

$$M \equiv \frac{r^3}{2} \left(D^a \partial_a \mu - \vartheta^z \vartheta_z \right) \ . \tag{40}$$

The spherically symmetric Einstein equation (39b) differs from the radial-gradient-dominated Einstein equation (37b) by the inclusion of the sub-dominant term $D^a \partial_a \mu = 1/r^2$ in the interior mass M, equation (40). Equations (39) are equivalent to the spherically symmetric equations (8) of [5], which constituted the principal evolutionary equations of the latter reference.

Thus in the inflationary (radial-gradient-dominated) regime, the equations governing the evolution of the radial expansions, either (36) or (37), coincide (modulo sub-dominant terms) with the spherically symmetric equations.

With the expression (33a) for the expansion ϑ_z and (34) for the boost ν , the Einstein equations (36) combine to yield

$$D^{z}\partial_{z}\ln\left(\frac{\rho_{x}\rho_{y}^{1/2}}{\rho_{\text{Kerr}}^{3/2}}\right) \approx 4\pi\left(T_{vu}+2T_{+-}\right) .$$
(41)

In deriving (41), use has been made of the fact that G_{vu} and G_{+-} , equations (35), vanish identically for Kerr, and that the radial horizon function Δ_x in ν , equation (34), is the Kerr horizon function (23) (as long as Δ_x is a function only of x, it can be set equal to the Kerr value by a gauge transformation of the coordinate x and the conformal factor ρ_x). If the flow is collisionless, so that T_{vu} and T_{+-} remain small, then, given the boundary conditions that the conformal factors are initially equal to the Kerr value, equation (24), equation (41) integrates to

$$\frac{\rho_x \rho_y^{1/2}}{\rho_{\text{Kerr}}^{3/2}} \approx \text{constant} .$$
(42)

E. Limitations

An obvious defect of the line-element (20) is that the Kerr parameters are fixed (in comoving coordinates). The arbitrary conformal factors allow the black hole to grow conformally, so the black hole mass M_{\bullet} can grow and the specific angular momentum along with it, $a \propto M_{\bullet}$. In reality however the angular momentum of a black hole will evolve arbitrarily (but slowly) in both magnitude and direction as the black hole accretes, which conformal growth does not admit.

The slow variation of the Kerr parameters of a slowly accreting black hole is potentially relevant to inflation because, even though an outgoing or ingoing infaller experiences an extremely short proper time during inflation, they can see a considerable time go by on the highly blueshifted opposing stream [5]. An outgoing infaller sees ingoing particles accreted in the future, while an ingoing infaller sees outgoing particles accreted in the past, at times when the Kerr parameters of the black hole could differ appreciably.

A second limitation of the line-element (20) is that it imposes that the radial expansion ϑ_z be pure gradient, equation (33a). The condition means that the expansion must be integrable: the angular conformal factor ρ_{μ} can be obtained by integrating the expansions along either of the outgoing and ingoing null directions, and the same value of ρ_{y} must be recovered regardless of the integration path. Consider the angular hypersurface inside the black hole defined by the intersection of outgoing and ingoing null streams accreted when the Kerr conformal factor of the black hole was respectively ρ_{out} and ρ_{in} . As long as gravitational waves keep the slowly accreting black hole close to hairless above its inner horizon, the ratio $\rho_{\rm in}/\rho_{\rm out}$ is independent of angular position. This condition is not a problem during early inflation when the conformal factors are still near their initial Kerr value, equation (24), but it becomes a serious constraint as inflation develops. The integrability condition on the conformal factor ρ_y translates into a condition that the ingoing and outgoing fluxes T_{vv} and T_{uu} must balance in a fashion that is consistent over all angular positions.

The above defects both point to the fact that, even if shear is in fact small, gravitational waves cannot be neglected in a proper treatment of inflation inside an arbitrarily accreting black hole. It is gravitational waves that mediate the slow adjustment of the parent Kerr black hole from one angular momentum configuration to another. And it is gravitational waves that keep the black hole close to hairless above its inner horizon.

In practice these defects may after all be unimportant. If Planck scale collisions intervene, as argued in §VII, then a stream sees only a few hundred black hole crossing times elapse on the opposing stream before hitting the Planck wall. This is short enough that the slow evolution of the parent Kerr geometry should be unimportant.

VI. PHYSICAL PICTURE

The physical picture that emerges from the arguments and equations in §III–§V is consistent with that of BIP.

A. Picture

The geodesics of the line-element (20), §VB, support the idea that collisionless outgoing and ingoing streams remain focused along the geodesic continuation of the principal null directions, at least until the spacetime has collapsed to the point that the intrinsic angular momentum of a stream causes rotation to become important.

Inflationary energy-momentum focused along the principal null directions produces a tidal gravitational acceleration that tends to contract the angular conformal factor ρ_y and elongate the radial conformal factor ρ_x in a 2 to 1 ratio, equation (42),

$$\rho_x \propto \rho_y^{-1/2} . \tag{43}$$

For a typically slowly accreting black hole, the blueshift between outgoing and ingoing streams approaching the inner horizon grows so rapidly that the spacetime does not have time to react to the enormously growing tidal acceleration. During inflation, the conformal factors ρ_x and ρ_y are hugely accelerated (large second radial derivatives), and start to change rapidly (large first radial derivatives), but still the conformal factors themselves have hardly changed from their initial Kerr values. In effect, inflation produces an explosion of acceleration so rapid that it initially leaves the Kerr spacetime unchanged. The fact that volume elements are initially little distorted despite the enormous tidal acceleration was first pointed out by [13]. The enormous (in due course super-Planckian) tidal force vet little distortion is commonly characterized as a "weak null singularity."

If a black hole continues to accrete, as is always true in reality, and if the issue of super-Planckian curvature is set aside, then the end result of inflationary acceleration is collapse, not a null singularity [5]. Whether the acceleration causes inflation or collapse can be read off from the Einstein equations (37). These equations are essentially the same as those governing inflation in spherical black holes, equations (8) of [5]. As argued in that paper, collisionless counter-streaming tends to drive exponentially growing acceleration, while mass (the \mathcal{M} term in equation (37b)) tends to cause collapse. The reader is referred to [5] for a more detailed exposition.

B. Is inflation similar to that in spherical symmetry?

BIP advanced "tentative" arguments that inflation in rotating black holes would be similar to that in spherically symmetric black holes. Is it true?

From the perspective of an infaller, inflation takes place over an extremely short proper time. Consequently infallers belonging to the same (outgoing or ingoing) stream are causally connected over only an extremely short proper distance. During inflation, the blueshift of the opposing stream increases exponentially, and the proper time decreases inversely with the blueshift. An infaller sees approximately one black hole crossing time elapse on the opposing stream for each e-fold of blueshift [5]. Thus the evolution of the spacetime in any angular patch is determined only by the causal domain of that patch, which becomes increasingly narrow as inflation develops.

The results of §V support the idea that inflation in any angular patch behaves as in spherical symmetry. During early inflation, inflationary energy-momentum at any point produces an enormous tidal acceleration that depends only on the energy-momentum passing through that point. The tidal acceleration increases exponentially while the underlying volume element remains scarcely distorted. Once inflation gives way to collapse, it is plausible that volume elements will simply react as they have been accelerated: they will collapse in the angular direction and stretch in the radial direction. In this picture, the behavior in any angular patch is entirely local to that patch.

The effect of departures from uniform angular behavior can be assessed from the Raychaudhuri-Einstein equations (14b), which determine how the angular momentum of accreting matter affects the angular momentum of the black hole. Note that, as mentioned in §VE, the lineelement (20) is inadequate to follow the evolution of the angular momentum of the black hole. Applied to the lineelement (20), the Raychaudhuri-Einstein equations (14b) merely force the accreting matter to have an angular momentum consistent with that of the black hole.

Departures from uniform angular behavior arise from two causes: first, accretion with angular momentum per unit mass mismatched, in amplitude or direction, from that of the black hole; and second, accretion at different rates at different angular positions. The two causes provide two of the source terms, namely T_{za} and $D_a\Gamma_{azb}$, in the Raychaudhuri-Einstein equations (14b) governing the evolution $D_z\Gamma_{azy}$ of the angular expansions and twists.

Since angular energy-momenta T_{za} are sub-dominant to radial energy-momenta T_{zz} , order unity differences in the angular energy-momenta should have little effect on the dominant radial evolution of the spacetime (the radial tidal distortion). That the effect should be small is supported by the solutions of [10, 11], in which inflation was followed by collapse, and rotation became important only after the geometry had collapsed to an exponentially tiny scale. Since the specific angular momentum of accreted matter cannot be much larger than that of the black hole, otherwise the matter would not be accreted, it follows that the angular momentum T_{za} cannot be large enough to affect radial evolution substantially.

Accretion at different rates at different angular positions could potentially have a larger effect. Larger accretion rates cause more rapid collapse. Order unity angular gradients in the incident accretion flow translate into order unity differences across a causal angular patch. As with the accreted angular momentum T_{za} , order unity differences should not have much effect. But it is easy to think of situations, such as where accretion is confined to delta-functions of angular location, where angular gradients are large.

Gravitational waves are likely to be generated most strongly where accretion is most anisotropic. Gravitational waves should effectively spread out the accretion flow, by transforming some of the accretion energy into a collisionless fluid of gravitons more uniformly distributed over the inner horizon of the black hole. Thus gravitational waves should set an upper limit on anisotropy. It is not clear that the large angular gradients needed to produce significant departures from quasispherical collapse can be sustained in the presence of smoothing by gravitational waves.

While these arguments are not conclusive, they tend to support BIP's proosal that angular anisotropy plays a sub-dominant role in inflation in rotating black holes.

C. Ori

In one of the few studies of inflation in rotating black holes, Ori [8, 9] considered the situation of a rotating black hole that collapses and then remains isolated, in which case a Price tail of gravitational radiation provides the outgoing and ingoing streams that drive inflation. To the extent that gravitational waves can be treated as a collisionless fluid of gravitons, the situation considered by Ori can be compared to the results of the present paper.

Ori argued that perturbations of the geometry in the inflationary zone at the inner horizon would have some simplifying features. This is at least qualitatively consistent with two simplifying features of inflation noted in the present paper, namely that geodesics focus along the principal null directions of the parent Kerr geometry regardless of their orbital parameters, and that the effect of inflation is to generate large accelerations that leave the underlying Kerr spacetime at least initially unchanged.

Ori's principal conclusion was that the resulting null singularity in a rotating black hole would differ from that in a spherical black hole by being oscillatory. The oscillations would be dominated by quadrupole radiation, the lowest order of gravitational waves. The natural interpretation of this conclusion in the context of the present paper is that the quadrupole oscillations are the tail end of the Price tail of gravitational radiation witnessed by an infaller. Although the gravitational waves have a quadrupole angular pattern over the black hole, an infaller will see these waves hugely blueshifted, and concentrated into an intense, narrow beam focused along the opposing principal null direction.

VII. PLANCK WALL

At what point does the classical general relativistic description of inflation fail because of quantum gravitational effects? [5] pointed out that in a typical astronomical black hole, Planck-scale physics is first encountered in collisions between outgoing and ingoing particles, which reach Planck center-of-mass energies well before the curvature hits the Planck scale. The purpose of this section is to explore this idea to see where it leads. The units in this section are Planck units, $c = G = \hbar = 1$.

It is widely expected that collision cross-sections will become gravitational at super-Planckian energies and therefore increase rapidly with energy, increasing approximately as the horizon radius squared of a black hole whose mass equals the center-of-mass energy [22-25].By contrast, total cross-sections for nongravitational processes are dominated by small-angle scattering, and increase more slowly. Total cross-sections for electromagnetic inelastic scattering (bremsstrahlung) increase no faster than logarithmically with energy [26, 27]. Total cross-sections for short-range interactions, such as the nuclear force, are limited by unitarity to increasing no faster than the logarithm squared of the center-of-mass energy, the Froissart bound [28–32]. In the absence of collisions, inflation would easily accelerate outgoing and ingoing particles far above Planck center-Thus if collision cross-sections do of-mass energies. increase rapidly above the Planck scale, then inevitably inflation will reach a "Planck wall" where collisions become important, and presumably lead to a burst of entropy production.

According to the arguments of [5], if collisions convert sufficient streaming energy into center-of-mass energy, then exponential inflationary growth will cease, and the spacetime will collapse.

The blueshift of either of the outgoing or ingoing streams relative to the center-of-mass frame is e^{ν} . For brevity denote this blueshift by u,

$$u \equiv e^{\nu} . \tag{44}$$

During inflation, the blueshift u increases exponentially. By the estimate of [11], in the absence of collisions, and for an accretion rate of \dot{M}_{\bullet} , inflation gives way to collapse when the blueshift reaches the maximum value

$$u_{\max} \approx e^{-1/M_{\bullet}} . \tag{45}$$

For a typically slowly accreting black hole, the accretion rate is small, $\dot{M}_{\bullet} \ll 1$, and the maximum blueshift $u_{\rm max}$ is exponentially huge. For collision energies to reach the Planck scale before collapse, the accretion rate must be smaller than about 0.01,

$$M_{\bullet} \lesssim 0.01 . \tag{46}$$

If the accretion rate is larger, such as may happen when the black hole first collapses, or during a black hole merger, then the spacetime will collapse before the blueshift inflates to the Planck scale, and the situation envisaged in this section will not occur.

The proper time τ experienced in the center-of-mass frame as the blueshift increases by one *e*-fold is the black hole crossing time M_{\bullet} divided by the blueshift *u*,

$$\tau \approx M_{\bullet}/u$$
 . (47)

The proper time τ_{coll} in the center-of-mass frame for an outgoing or ingoing particle to collide with a particle in the opposing stream is

$$\tau_{\rm coll} \approx \frac{1}{n u \sigma} \approx \frac{m M_{\bullet}^2}{\dot{M}_{\bullet} u \sigma} , \qquad (48)$$

where n is the proper number density of particles in the opposing stream (in the latter's own frame), and σ is the collision cross-section. The proper number density n equals the accretion rate \dot{M}_{\bullet} , multiplied by the density of the black hole, which is M_{\bullet}^{-2} , divided by the rest mass m of the particle, $n \approx \dot{M}_{\bullet}/(M_{\bullet}^{2}m)$. The collision time (48) is less than an inflation time (47) if the cross-section σ exceeds the critical cross-section σ_{c} given by

$$\sigma_c \approx \frac{mM_{\bullet}}{\dot{M}_{\bullet}} \approx \frac{10 \,\mathrm{fb}}{\dot{M}_{\bullet}} \left(\frac{m}{1 \,\mathrm{TeV}}\right) \left(\frac{M_{\bullet}}{10^6 \,\mathrm{M}_{\odot}}\right) \,, \qquad (49)$$

where a femtobarn $1 \,\text{fb} = 10^{-43} \,\text{m}^2$ is about a weak interaction cross-section.

Nature will provide a broad range of accretion rates \dot{M}_{\bullet} . In a "typical" situation where the accretion rate is small but not too small, the collision time will be shorter than an inflation time for electromagnetic and strong interactions, but longer than an inflation time for weak interactions. In this situation, collisions will keep baryons, electrons, and photons tightly coupled, forcing them into a common outgoing or ingoing stream before inflation ignites. Magnetohydrodynamic processes will contribute to keeping the baryonic plasma tightly coupled [33, 34]. On the other hand particles that interact only by weak interactions or gravity, such as dark matter particles or gravitons, can occupy the opposing stream, and stream relativistically through the baryonic stream without collisions, driving inflation.

If the accretion rate \dot{M}_{\bullet} is large enough, and if dark matter particles are weakly interacting, then electroweak-scale collisions could lead to significant entropy production. Currently only upper limits to crosssections between dark matter particles and baryons are known [41–43].

Since wavelengths decrease inversely with momentum, unknown processes with thresholds at higher (sub-Planckian) energies will probably have correspondingly smaller cross-sections, and will probably not contribute significantly to collisions. Given the slow increase of non-gravitational cross-sections with energy, inflation will typically blueshift up to the Planck energy without meeting a major collisional barrier.

Above a Planck energy, cross-sections increase gravitationally. The Planck wall will be reached when the gravitational cross-section exceeds the critical crosssection (49). Notwithstanding the lack of a robust theory of quantum gravity, the common expectation is that super-Planckian collisions will lead to the production of mini black holes [22–25, 44–48]. Hereafter I will refer to the product of a super-Planckian collision as a proto mini black hole, or pmbh. If the Planck energy is at the standard value 10¹⁶ TeV, then the mass of a pmbh at the Planck wall is $m_{\bullet} \approx \sqrt{\sigma_c}$, or

$$m_{\bullet} \approx \sqrt{\frac{mM_{\bullet}}{\dot{M}_{\bullet}}} \approx \frac{10^{14}}{\sqrt{\dot{M}_{\bullet}}} \left(\frac{m}{1\,\mathrm{TeV}}\right)^{1/2} \left(\frac{M_{\bullet}}{10^6\,\mathrm{M}_{\odot}}\right)^{1/2} \,. \tag{50}$$

Modulo a factor of $\dot{M}_{\bullet}^{-1/2}$, the mass m_{\bullet} of the pmbh is the geometric mean of the masses m and M_{\bullet} of the colliding particle and the parent black hole. The collision energy of $m_{\bullet} \gtrsim 10^{14}$ Planck masses is large compared to any collision process one could imagine happening elsewhere in the Universe.

At the Planck wall, the blueshift u_{wall} is

$$u_{\text{wall}} \approx \frac{m_{\bullet}}{m} \approx \frac{10^{30}}{\sqrt{\dot{M}_{\bullet}}} \left(\frac{m}{1 \,\text{TeV}}\right)^{-1/2} \left(\frac{M_{\bullet}}{10^6 \,\text{M}_{\odot}}\right)^{1/2}, (51)$$

and the proper time $\tau_{\rm wall}$ for the blueshift to increase one $e\text{-}{\rm fold}$ is

$$\tau_{\text{wall}} \approx \frac{M_{\bullet}}{u_{\text{wall}}} \approx m_{\bullet} \dot{M}_{\bullet}$$
$$\approx 10^{14} \sqrt{\dot{M}_{\bullet}} \left(\frac{m}{1 \text{ TeV}}\right)^{1/2} \left(\frac{M_{\bullet}}{10^6 \text{ M}_{\odot}}\right)^{1/2} , \quad (52)$$

which is larger than a Planck time if the accretion rate is not too tiny, $\dot{M}_{\bullet} \gtrsim 10^{-28}$. Regardless of whether inflation continues or gives way to collapse, the time $\tau_{\rm wall}$ sets the characteristic proper time remaining before the background spacetime comes to an end (or at least, the spacetime reaches Planck density). The proper time $\tau_{\rm wall}$, equation (52), is less than of the order of the crossing time m_{\bullet} of the pmbh. Thus the pmbh does not have time to express itself as a black hole, which takes several crossing times. The pmbh certainly does not have time to evaporate by Hawking radiation, which takes a time m_{\bullet}^3 . Meanwhile, the mass density of the background spacetime is

$$mnu_{\text{wall}}^2 \approx \frac{\dot{M}_{\bullet}}{M_{\bullet}m} \approx \frac{1}{m_{\bullet}^2} ,$$
 (53)

which is already the density of the putative pmbh.

Does a collision actually occur, if the pmbh does not have time to express itself as a black hole? Yes. In the center-of-mass frame, the colliding outgoing and ingoing particles are Lorentz-contracted to size 1/(mu), and it takes that proper time for the particles to pass by each other,

$$\tau_{\rm pass} \approx \frac{1}{m_{\bullet}} , \qquad (54)$$

which is short compared to the time τ_{wall} available.

The fact that super-Planckian collisions do occur indicates that entropy production takes place, but the fact that there is not enough time for the collision products to express themselves as mini black holes indicates that the situation is messy. One possibility is that entropy production might lead to a stringy Hagedorn phase [49, 50].

String theory predicts the existence of higher dimensions. It has been speculated that if some of the extra dimensions are large, then the fundamental Planck energy could be as low as 1 TeV [51], and that super-Planckian collisions could create higher-dimensional mini black holes with radii between the fundamental Planck scale and the scale of the large extra dimensions [22– 25]. If this scenario is correct, then the critical crosssection (49) still sets the radius of pmbhs at the Planck wall, but their mass m_{\bullet} will be smaller (though still necessarily super-Planckian). Their dimensionality will be the number of dimensions larger than the radius of the critical cross-section. The possibility of extra dimensions does not change the conclusion that inflation will meet a Planck wall where collisions become important.

Before hitting the Planck wall, the inflationary zone of the black hole behaves like a particle accelerator, accelerating twin beams of particles through each other, and conducting numerous collision experiments at energies up to m_{\bullet} . Even highly improbable collision events might occur.

VIII. CONCLUSIONS

In this paper I have presented a number of proposals about how inflation should develop inside a rotating black hole accreting slowly but in an arbitrary time- and spacedependent fashion.

As shown by [11], particles freely-falling in the parent Kerr geometry of a rotating black hole become highly focused along the principal outgoing and ingoing null directions as they approach the inner horizon, regardless of their initial orbital parameters. If the Kerr geometry remained unchanged, then the streaming energy of the outgoing and ingoing streams would diverge at the inner In practice, the energy-momentum focused horizon. along the principal null directions produces a tidal acceleration that tends to collapse angular directions and stretch radial directions. For a slowly accreting black hole, the proper timescale over which the streaming energy grows is so short that the tidal acceleration grows exponentially huge before volume elements actually begin to distort, as first pointed out by [13]. In due course the enormous acceleration leads to collapse of the angular directions. Because of the extremely short proper time, angular patches are causally connected over extremely short proper distances, so the acceleration and collapse of an angular patch depends only on the streaming energy local to that patch.

The arguments of the present paper generally support BIP's [7] proposal that inflation in rotating black holes would evolve similarly to that in spherical black holes. Yet it is surprisingly difficult to demonstrate this proposal conclusively. In §III I argue that during inflation gravitational waves should behave like a collisionless fluid of gravitons, and consequently that the spacetime should be almost shear-free, as it is in the parent Kerr geometry. But even if the spacetime is shear-free, gravitational waves still play a crucial role both in keeping the parent Kerr black hole almost hairless as it evolves slowly from one angular momentum configuration to another, and in smoothing the angular distribution of accretion energy when the accretion flow is highly anisotropic.

It would be desirable to test the proposals of the present paper with full 4-dimensional numerical simulations. Such simulations will not be easy. One challenge is that inflation generates enormous spatial and temporal gradients. A second challenge is the need to model outgoing and ingoing streams that counter-A third challenge is how to stream relativistically. treat gravitational waves both in the familiar regime where they radiate away a black hole's hair, and in the inflationary regime where they behave effectively like a collisionless fluid. A final difficulty in comparing theory to simulation is that theoretical arguments are simplest when accretion rates are small, but numerical simulations should be easiest in the opposite regime, when accretion rates are high. It may help to work in a double-null 2+2 formalism, since the physics in any angular patch is that of two effectively null streams streaming through each other. Since angular patches become causally disconnected from each other as inflation progresses, it may help to confine the simulations to one causal patch.

As pointed out by [5], center-of-mass collision energies between outgoing and ingoing particles will hit the Planck energy well before the curvature reaches the Planck scale. It is thought that collision cross-sections will become gravitational, and therefore increase rapidly, at super-Planckian collision energies [22–25]. If so, then inflation will reach a "Planck wall" where the collision time is less than the inflation time. In a typical supermassive black hole, the collision energy at the Planck wall is $\gtrsim 10^{14}$ Planck masses, equation (50). It is thought that super-Planckian collisions will lead to

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the formation of mini black holes [22–25], but that does not happen here because the timescale for the spacetime to collapse is less than the size of a nascent mini black hole. Collisions do actually occur, since the collapse timescale is easily long enough to allow two colliding particles to pass by each other. Presumably collisions lead to entropy production, but the situation appears complicated. Qualitatively, entropy production should bring inflation to an end and precipitate collapse [5]. It is possible that collisions might lead to a stringy Hagedorn phase [50].

Prior to hitting the Planck wall, inflation acts like a particle accelerator of extraordinary power. It appears inescapable that Nature is conducting vast numbers of collision experiments over a broad range of peri- and super-Planckian energies in large numbers of black holes throughout our Universe. Does Nature do anything interesting with this extravagance – such as create baby universes – or is it merely a final hurrah en route to nothingness?

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Appendix A: Proof that extrinsic curvatures are tetrad-frame tensors

The tetrad frame is fixed so that the outgoing and ingoing null axes γ_v and γ_u point along the special outgoing and ingoing null geodesic directions. The remaining unfixed tetrad gauge transformations are a Lorentz boost in the vu plane, which transforms

$$\gamma_v \to e^{\nu} \gamma_v , \quad \gamma_u \to e^{-\nu} \gamma_u , \qquad (A1)$$

and a spatial rotation in the +- plane, which transforms

$$\gamma_+ \to e^{i\mu} \gamma_+ , \quad \gamma_- \to e^{-i\mu} \gamma_- .$$
 (A2)

Tetrad-frame connection coefficients are defined by $\Gamma_{klm} \equiv \gamma_k \cdot \partial_m \gamma_l$. In particular, the extrinsic curvature

component Γ_{+v-} (for example) is defined by

$$\Gamma_{+v-} \equiv \gamma_+ \cdot \partial_- \gamma_v \ . \tag{A3}$$

Upon a tetrad transformation (A1) and (A2), the only potentially non-tensorial contribution to the transformation of the extrinsic curvature component (A3) is

$$\gamma_+ \cdot \gamma_v \partial_- e^{\nu}$$
, (A4)

but this vanishes because the angular and null axes γ_+ and γ_v are orthogonal. A similar argument applies to each of the 16 components of the extrinsic curvatures in each of the 4 tetrad directions. Thus the extrinsic curvatures form tetrad-frame tensors as claimed.

Appendix B: Accuracy of the Hamilton-Jacobi approximation for geodesics

This Appendix confirms the accuracy of the the Hamilton-Jacobi approximation (30), (31) for particles moving in the spacetime defined by the line-element (20). For simplicity and brevity, the demonstration is restricted to massless particles. It is clear physically that, under the hyper-relativistic conditions characteristic of inflation, the motions of massive particles should be well-approximated by those of massless particles, as is demonstrated explicitly by [11] for the case of conformally separable spacetimes.

For a massless particle, m = 0, the Hamilton-Jacobi approximation (30), (31) to the tetrad-frame momentum p_k of a freely-falling particle yields the following expressions for the covariant derivatives $Dp_k/D\lambda$ with respect to affine parameter λ along the trajectory:

$$p^k \frac{Dp_k}{D\lambda} = 0 , \qquad (B1a)$$

$$\frac{Dp_k}{D\lambda} = \frac{\mathcal{K}}{\rho_y^2} \partial_k \ln\left(\frac{\rho_x}{\rho_y}\right) \quad (k = t, y, \phi) .$$
(B1b)

The covariant derivatives would vanish if the path were exactly geodesic; they do not vanish because the Hamilton-Jacobi approximation is not exact. The covariant derivatives do vanish along the principal null directions, which have vanishing Carter constant, $\mathcal{K} = 0$. The covariant derivatives would also vanish if the two conformal factors were equal, $\rho_x = \rho_y$, which is the conformally separable case.

The Hamilton-Jacobi approximation (30), (31) can be considered adequate if, integrated along the path of a particle, the difference Δp_k between the approximate Hamilton-Jacobi momentum p_k , which satisfies equations (B1), and the true momentum, which satisfies $Dp_k/D\lambda = 0$, is small, $\Delta p_k \ll p_k$. Appendix D of [11] gives general criteria by which such integrals may be judged small. The integrals of equations (B1b) easily satisfy those smallness criteria (specifically, the parameters α and β in Appendix D of [11] are $\alpha = 0$, $\beta = -1$ for p_y and p_{ϕ} , and $\alpha = 1/2$, $\beta = -5/2$ for p_t).

The Hamilton-Jacobi approximation to the number density N along a collisionless stream is given by equation (32). The corresponding number current n_k , equation (29), satisfies

$$\frac{1}{N}D^k n_k = -\left[\frac{P_x^2 - P_t^2}{P_x^2}p^t\partial_t + \frac{P_x^2 + P_t^2}{P_x^2}\left(p^y\partial_y + p^\phi\partial_\phi\right)\right]\ln\left(\frac{\rho_x}{\rho_y}\right) .$$
(B2)

The true number current should be covariantly conserved, $D^k n_k = 0$; the Hamilton-Jacobi number current is not conserved, equation (B2), because the Hamilton-Jacobi approximation is not exact. Since

$$\frac{d\ln N}{d\lambda} = \frac{1}{N} D_k n^k - D_k p^k , \qquad (B3)$$

and the momentum p^k has already been checked to be given accurately by the Hamilton-Jacobi approximation (30), (31), the Hamilton-Jacobi approximation (32) to the number density N can be deemed adequate if the right hand side of equation (B2), integrated over the path of a particle, is small. Again, the integral easily satisfies the smallness criteria given in Appendix D of [11] (the p^t term on the right hand side has $\alpha = 1$, $\beta = 0$, while the p^y and p^{ϕ} terms have $\alpha = 1/2$, $\beta = -3/2$).