The river model of black holes

Andrew J. S. Hamilton[a] and Jason P. Lisle
JILA and Department of Astrophysical and Planetary Sciences, Box 440, University of Colorado, Boulder, Colorado 80309

(Received 11 October 2004; accepted 3 December 2007)

We present a lesser known way to conceptualize stationary black holes, which we call the river model. In this model, space flows like a river through a flat background, while objects move through the river according to the rules of special relativity. In a spherical black hole, the river of space falls into the black hole at the Newtonian escape velocity, hitting the speed of light at the horizon. Inside the horizon, the river flows inward faster than light, carrying everything with it. The river model also works for rotating (Kerr–Newman) black holes, though with a surprising twist. As in the spherical case, the river of space can be regarded as moving through a flat background. However, the river does not spiral inward, but falls inward with no azimuthal swirl. The river has at each point not only a velocity but also a rotation or twist. That is, the river has a Lorentz structure, characterized by six numbers (velocity and rotation). As an object moves through the river, it changes its velocity and rotation in response to tidal changes in the velocity and twist of the river along its path. An explicit expression is given for the river field, a six-component bivector field that encodes the velocity and twist of the river at each point and encapsulates all the properties of a stationary rotating black hole.
[DOI: 10.1119/1.2830526]

I. INTRODUCTION

As was first pointed out Gullstrand[1] and Painlevé,[2] the Schwarzschild[3,4] metric can be expressed in the form

$$ds^2 = -dt_{ff}^2 + (dr + \beta dt_{ff})^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{1}$$

where β is the Newtonian escape velocity in units of the speed of light at radius r from a spherical object of mass M

$$\beta = \left(\frac{2GM}{r}\right)^{1/2} \tag{2}$$

and t_{ff} (ff for free fall) is the proper time experienced by an object that free falls radially inward from zero velocity at infinity.

Although Gullstrand's paper was published in 1922 after Painlevé's paper, it appears that Gullstrand's work has priority. Gullstrand's paper was dated 25 May 1921, and Painlevé's paper is a write up of a presentation to the Académie des Sciences in Paris on 24 October 1921. Gullstrand seems to have had a better grasp of what he had discovered than Painlevé, for Gullstrand recognized that observables such as the redshift of light from the Sun are unaffected by the choice of coordinates in the Schwarzschild geometry. Painlevé noted that the spatial metric was flat at constant free-fall time, $dt_{ff}=0$, and concluded that as regards the redshift of light and such, "c'est pure imagination de prétendre tirer du ds^2 des conséquences de cette nature."

As shown in Sec. II, the Gullstrand–Painlevé metric provides a delightfully simple conceptual picture of the Schwarzschild geometry: it looks like ordinary flat space, with the distinctive feature that space itself is flowing radially inward at the Newtonian escape velocity. The place where the inward velocity reaches the speed of light, $\beta=1$, marks the horizon, the Schwarzschild radius. Inside the horizon, the inward velocity exceeds the speed of light, carrying everything with it (see Fig. 1).

Picture space as flowing like a river into the Schwarzschild black hole. Imagine light rays, photons, as fishes swimming fiercely in the current. Outside the horizon, photon fishes swimming upstream can make way against the flow. But inside the horizon, the space river is flowing inward so fast that it beats all fishes, carrying them inevitably toward their ultimate fate, the central singularity.

The river model of black holes offers a mental image of black holes that can be understood by nonexperts (at least in the spherical case) without the benefit of mathematics. It explains why light cannot escape from inside the horizon, and why no star can come to rest within the horizon. It explains how an extended object will be stretched radially by the inward acceleration of the river, and compressed transversely by the spherical convergence of the flow. It explains why an object that falls through the horizon appears to an outsider to be redshifted and frozen at the horizon: as the object approaches the horizon, light emitted by it takes an ever increasing time to forge against the inrushing torrent of

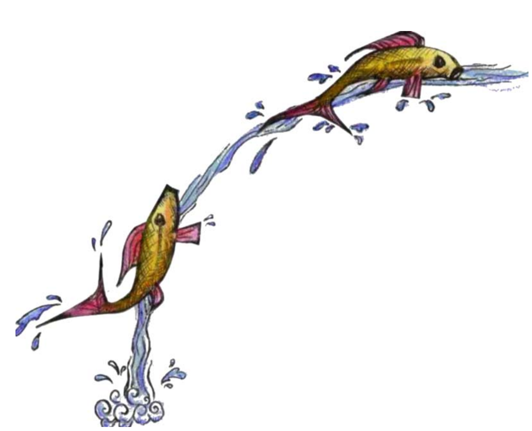


Fig. 1. (Color online) The fish upstream can make way against the current, but the fish downstream is swept to the bottom of the waterfall. Figure 1 of Ref. 5 presents a similar depiction.

space and to reach an outside observer. The river model paints a picture that is radically different from the Newtonian picture envisaged by Michell[6] and Laplace.[7]

The picture of space falling like a river into a black hole may seem discomfortingly concrete, but the aetherial overtones[12] are no more substantial than the familiar cosmological picture of expanding space (see for example Ref. 8).

As reviewed by Visser[9,10] and by Martel and Poisson,[11] the Gullstrand–Painlevé metric has been discovered and rediscovered repeatedly.[12–23] Surprisingly, the Gullstrand–Painlevé metric is widely neglected in texts on general relativity. An admirable exception is the text by Taylor and Wheeler,[24] which devotes an entire section, Project B, to the Gullstrand–Painlevé metric, calling it the "rain frame" (the metric appears on p. B-13). Taylor and Wheeler attribute (p. B-26) the idea for the rain frame to the book by Thorne, Price and MacDonald,[25] although the metric does not appear explicitly in that book.

It has been long recognized that some aspects of general relativity can be conceptualized in terms of flows. In the Arnowitt–Deser–Misner (ADM) formalism[26] (see, for example, Ref. 27 for a pedagogical review), we consider fiducial observers (FIDOs)[25] whose worldlines are orthogonal to hypersurfaces of constant time. The shift vector in the ADM formalism is just the velocity of these FIDOs through the spatial coordinates. Alcubierre[28,29] constructed his famous warp-drive metric by positing a superluminal (faster-than-light) shift vector.

In a seminal, albeit initially unremarked paper, Unruh[30,31] pointed out that the equations governing sound waves propagating in an inviscid, barotropic (pressure is a specified function of density), and irrotational fluid are the same as those for a massless scalar field propagating in a certain general relativistic metric. Unruh showed that this similarity implied that sound horizons would emit Hawking radiation in much the same way as event horizons in black holes, and he proposed that Hawking radiation might be detected from sonic black holes, or "dumb holes," in the laboratory.

As excellently reviewed in Ref. 31, Unruh's paper eventually led to a now thriving industry on "analog gravity," in which fluid flows with prescribed velocity fields simulate general relativistic spacetimes. The primary aim of the work on analog gravity is to try to understand, and maybe in the not-too-distant future to probe experimentally, quantum gravity through sonic analogs.

It is generally assumed that the fluid or river analogy applies to a limited class of general relativistic spacetimes, those in which the metric can be expressed up to an overall factor (a conformal factor) in terms of a shift vector (the velocity of the river) on an otherwise flat background space. The three-dimensional shift vector and the conformal factor provide four degrees of freedom, whereas at least six degrees of freedom are required to specify an arbitrary spacetime (the metric has ten degrees of freedom, of which four are removed by an arbitrary coordinate transformation). As a corollary, it has been thought that any general relativistic geometry admitting a fluid analog must necessarily be (up to a conformal factor) spatially flat at constant time,[32,33] as is the case in the Gullstrand–Painlevé metric. In particular, it has been thought that no river model for stationary rotating black holes exists,[33] because the Kerr–Newman geometry does not admit conformally flat slices.[34,35]

In the present paper we start from a somewhat different conceptual picture. We notice that fishes swimming in the Gullstrand–Painlevé river move according to the rules of special relativity, being boosted by tidal differences in the river velocity from place to place. We wonder, might there be an analogous behavior for rotating black holes? It comes as a magical surprise (see Sec. III) that the answer is yes. From this perspective there is a river model of the Kerr–Newman geometry. The rotating analog of the Gullstrand–Painlevé metric proves to be as expected[33] the Doran[36] form of the Kerr–Newman metric. The new feature that emerges from the mathematics is that the river of a rotating black hole is a fully six-dimensional Lorentz river, with a twist as well as a velocity. Just as a velocity is a generator of a space-time rotation (a Lorentz boost), so is a twist a generator of a space-space rotation (an ordinary spatial rotation). As a fish swims through the Doran river, it is not only boosted but also rotated by tidal differences in the river velocity and twist from place to place.

This novel point of view leads to a different notion of what is meant by the flat background space through which the river flows and twists. Mathematically, the essential feature of the river model is given by Eq. (72), which states that the connection coefficients, expressed in locally inertial frames co-moving with the infalling river of space, should equal the ordinary (noncovariant) gradient of the river field.

The property that the tetrad connection coefficients are equal to the ordinary gradient in Doran–Cartesian coordinates of the river field defines what we mean by the background space in the river model being flat. This feature appears to be a special property of stationary black holes. How this idea emerges from the mathematics is examined in Sec. III F and revisited in Sec. III I. We emphasize that the flat background does not mean that the metric is spatially flat, although the latter holds for spherical black holes. The notion that there is a sense in which stationary rotating black holes admit a flat background coordinate system might have application to numerical general relativity, for example in setting up initial conditions containing rotating black holes, for which traditional conformal imaging and puncture methods that assume a conformally flat 3-geometry are too restrictive to admit Kerr black holes.[37]

II. SPHERICAL BLACK HOLES

In this section we consider spherically symmetric black holes, and we justify the assertion that the Gullstrand–Painlevé metric, Eq. (1), can be interpreted as representing a river of space falling radially inward at velocity β. We demonstrate two features that are the essence of the river model for spherical black holes: first, that the river of space can be regarded as moving in Galilean fashion through a flat Galilean background space [Eqs. (14) and (15)]; and second, that as a freely-falling object moves through the flowing river of space, its 4-velocity, or more generally any 4-vector attached to the freely-falling object, can be regarded as evolving by a series of infinitesimal Lorentz boosts induced by the change in the velocity of the river from one place to the next [Eq. (18)]. Because the river moves in a Galilean fashion, it can, and inside the horizon does, move faster than light through the background. Objects moving in the river move according to the rules of special relativity, and so cannot move faster than light through the river.

In the following we will adopt the sign conventions and ordering of indices of Misner, Thorne and Wheeler.[38]

A. Mathematics of the river model

In general, a spherically symmetric metric of the form ($c = G = 1$)

$$\mathrm{d}s^2 = -[1 - 2M(r)/r]\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{[1 - 2M(r)/r]} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \tag{3}$$

can be expressed in the Gullstrand-Painlevé form Eq. (1) with infall velocity,

$$\beta(r) = \left[\frac{2M(r)}{r}\right]^{1/2}, \tag{4}$$

with the free-fall time t_{ff} given by

$$t_{\mathrm{ff}} = t - \int_r^\infty \frac{\beta}{1 - \beta^2}\mathrm{d}r. \tag{5}$$

The velocity β is commonly called the shift in the ADM formalism,[26,27] but we will refer to β as the river velocity. The river velocity β is positive for a black hole (infalling) and is negative for a white hole (outfalling). Horizons occur where the river velocity β equals the speed of light,

$$\beta = \pm 1, \tag{6}$$

with $\beta = 1$ for black hole horizons, and $\beta = -1$ for white hole horizons. The Reissner–Nordström metric for a spherically symmetric black hole of mass M and charge Q takes the form in Eq. (3) with the mass $M(r)$ interior to r, the Misner-Sharp[39,40] mass, given by

$$M(r) = M - \frac{Q^2}{2r}. \tag{7}$$

The river velocity β can also be considered to be a more general function of radius r. In Sec. II B we will return briefly to the Reissner–Nordström solution to see what its river looks like.

To make the argument simpler, we rewrite the Gullstrand–Painlevé metric in Eq. (1) in Cartesian coordinates $x^\mu = (x^0, x^1, x^2, x^3) = (t_{\mathrm{ff}}, x, y, z)$ instead of spherical coordinates:

$$\mathrm{d}s^2 = \eta_{\mu\nu}(\mathrm{d}x^\mu - \beta^\mu \mathrm{d}t_{\mathrm{ff}})(\mathrm{d}x^\nu - \beta^\nu \mathrm{d}t_{\mathrm{ff}}), \tag{8}$$

where $\eta_{\mu\nu}$ is the Minkowski metric, and

$$\beta^\mu = \beta\left(0, -\frac{x}{r}, -\frac{y}{r}, -\frac{z}{r}\right) \tag{9}$$

are the components of the radial river velocity.

Let $\mathbf{g}_\mu$ denote the basis of tangent vectors in the Gullstrand–Painlevé–Cartesian coordinate system x^μ. By definition, the scalar products of the tangent vectors constitute the metric $g_{\mu\nu}$

$$\mathbf{g}_\mu \cdot \mathbf{g}_\nu = g_{\mu\nu}. \tag{10}$$

Let $v^\mu \equiv \mathrm{d}x^\mu / \mathrm{d}\tau$ denote the 4-velocity of a particle falling freely (not necessarily radially) in the geometry, where τ is the proper time experienced by the particle. In particular, observers who free-fall radially from zero velocity at infinity have 4-velocity

$$v^\mu_{\mathrm{ff}} = (1, \beta^1, \beta^2, \beta^3). \tag{11}$$

Such observers are co-moving with the inflowing river of space. Let γ_m and the associated local coordinates $\xi^m = (\xi^0 \xi^1, \xi^2, \xi^3)$ denote a system of locally inertial orthonormal frames, tetrads, attached to observers who free-fall radially from zero velocity at infinity. We will use Latin indices to signify tetrad frames, and reserve Greek indices for curved space-time frames. Orthonormal means that the scalar products of the tetrad basis at each point of spacetime form the Minkowski metric

$$\gamma_m \cdot \gamma_n = \eta_{mn}. \tag{12}$$

That the tetrad frames move with the radially free-falling observers without precessing requires that the vectors γ_m be "parallel-transported" along the worldlines of these observers, that is,

$$v^\mu_{\mathrm{ff}} \frac{\partial \gamma_m}{\partial x^\mu} = 0. \tag{13}$$

Assume without loss of generality that the tetrad frames γ_m are aligned with the Gullstrand–Painlevé–Cartesian frame at infinity. Then the tetrad frames γ_m are related to the Gullstrand–Painlevé basis $\mathbf{g}_\mu$ at each point by

$$\gamma_0 = \mathbf{g}_0 + \beta^i \mathbf{g}_i \tag{14a}$$

$$\gamma_i = \mathbf{g}_i (i = 1, 2, 3) \tag{14b}$$

which is most easily deduced from Eqs. (10) and (12) and considerations of symmetry; the result is confirmed by checking that Eq. (13) holds. Remarkably, the relations given by Eq. (14) are those of a Galilean transformation, which shifts the time axis by the velocity β along the direction of motion, but leaves unchanged both the time component of the time axis and all of the spatial axes.

The 4-velocity u^m of a freely-falling particle with respect to the tetrad frame γ_m at the position of the particle follows from $u^m \gamma_m = v^\mu \mathbf{g}_\mu$, which implies that

$$u^0 = v^0 \tag{15a}$$

$$u^i = v^i - \beta^i v^0 (i = 1, 2, 3). \tag{15b}$$

Physically, the 4-velocity u^m is the 4-velocity of the particle relative to the inflowing river of space. For example, the spatial components u^i of the 4-velocity are zero if the particle is motionless in the river. Again, the relations given by Eq. (15) resemble those of a Galilean transformation, which shifts only the spatial components of the vector while leaving the time component unchanged. The only non-Galilean (relativistic) feature of Eq. (15) is that the 4-velocities u^m and v^μ are derivatives with respect to proper time. But proper time is a property of the objects moving in the river, not of the river itself. Objects moving in the river move through it according to the rules of special relativity. The river itself flows in Galilean fashion through a flat Galilean background.

Equations (14) and (15) demonstrate the first of the two features of the river model for spherical black holes: the river of space moves in Galilean fashion through a flat Galilean background.

We proceed to demonstrate the second feature of the river model, which is that objects moving in the river of space move according to the rules of special relativity, being Lorentz boosted by tidal differences in the river velocity from place to place.

The tetrad frames have been constructed so that observers who free fall from zero velocity at infinity find their own frame aligned at all times with the tetrad frame. In general, other observers who free fall along a different geodesic will find their locally inertial frame becoming misaligned with the tetrad frame. This misalignment is determined by the equations of motion of objects, 4-vectors, expressed with respect to the tetrad frame. Let $\mathbf{p}=p^m\gamma_m=p^\mu\mathbf{g}_\mu$ be a 4-vector. Its components p^m in the tetrad frame are related to those p^μ in the coordinate frame by $p^m=\delta^m_\mu p^\mu-\beta^m p^0$. For a 4-vector in free fall, the equations of motion for the components p^m in the tetrad frame are (see Sec. III E)

$$\frac{\mathrm{d}p^k}{\mathrm{d}\tau}+\Gamma^k_{mn}u^n p^m=0, \tag{16}$$

where Γ^k_{mn} are the tetrad frame connection coefficients, the tetrad frame analog of the coordinate frame Christoffel symbols. For spherically symmetric black holes the nonzero tetrad frame connection coefficients are given by the spatial gradient of the river velocity (see Sec. III H)

$$\Gamma^0_{ij}=\Gamma^i_{0j}=\frac{\partial\beta^i}{\partial x^j}\quad(i,j=1,2,3). \tag{17}$$

From Eqs. (16) and (17) it follows that

$$\frac{\mathrm{d}p^0}{\mathrm{d}\tau}=-\frac{\partial\beta_i}{\partial x^j}u^j p^i \tag{18a}$$

$$\frac{\mathrm{d}p^i}{\mathrm{d}\tau}=-\frac{\partial\beta^i}{\partial x^j}u^j p^0\quad(i=1,2,3). \tag{18b}$$

The summations over paired indices in Eq. (18) are formally over all four indices $0,1,2,3$, but in practice reduce to sums over only the three spatial indices 1, 2, and 3 because the infall velocity has zero time component, $\beta^0=0$, as we have defined it, and, the infall velocity has a zero time derivative, $\partial\beta^\mu/\partial x^0=0$.

In the context of the river model, the equations of motion, Eq. (18), have the following interpretation. In an interval $\delta\tau$ of proper time, a particle moves a distance $\delta x^i=v^i\delta\tau$ in the background Gullstrand–Painlevé–Cartesian coordinates, and a proper distance $\delta\xi^i=u^i\delta\tau=\delta x^i-\beta^i\delta t_{\mathrm{ff}}$ relative to the infalling river of space. The proper distance $\delta\xi^i$ equals the distance δx^i minus the distance $\beta^i\delta t_{\mathrm{ff}}$ moved by the river. In Gullstrand–Painlevé–Cartesian coordinates the velocity β^i of the infalling river at the new position differs from the velocity at the old position by $\delta x^j\,\partial\beta^i/\partial x^j$. In the river model, a particle moving in the river sees not the full change in river velocity relative to the background coordinates, but only the tidal change

$$\delta\beta^i=\frac{\partial\beta^i}{\partial x^j}\delta\xi^j \tag{19}$$

in the river velocity relative to the infalling locally inertial river frame. For example, if the particle is co-moving with the inflowing river, so that $\delta\xi^i=0$, then the particle sees no change at all in the river velocity as time goes by, $\delta\beta^i=0$. The infinitesimal tidal change $\delta\beta^i$ in the river velocity induces a Lorentz boost in the 4-vector p^m

$$p^0\to p^0-\delta\beta_i p^i \tag{20a}$$

$$p^i\to p^i-\delta\beta^i p^0\quad(i=1,2,3). \tag{20b}$$

Equations (19) and (20) reproduce the equations of motion, Eq. (18).

We have thus demonstrated the second of the claimed features of the river model for spherical black holes: as a particle moves through the river of space, its 4-velocity, or more generally any 4-vector attached to it, is Lorentz boosted by tidal changes in the river velocity along its path.

B. Reissner–Nordström metric

We conclude this section by commenting on the Reissner–Nordström metric for a spherical black hole of mass M and charge Q. In this case the mass $M(r)=M-Q^2/(2r)$ interior to r, Eq. (7), can be interpreted as the mass M at infinity, less the mass $\int_r^\infty(E^2/8\pi)4\pi r^2\mathrm{d}r=Q^2/(2r)$ contained in the electric field $E=Q/r^2$ outside r. The Reissner–Nordström geometry exhibits both outer and inner horizons r_+ and r_-

$$r_\pm=M\pm(M^2-Q^2)^{1/2}. \tag{21}$$

The inflow velocity β hits the speed of light at the outer horizon r_+, reaches a maximum velocity between the outer and inner horizons, slows back down to the speed of light at the inner horizon r_-, and slows all the way to zero velocity at the turnaround radius

$$r_0=\frac{Q^2}{2M}. \tag{22}$$

At this point the flow of space turns around, accelerates back outward through another inner horizon, the Cauchy horizon, into a white hole, and bursts through the outer horizon of the white hole into a new universe.

Sadly, the Reissner–Nordström solution is not realistic, and its promise of passages to other universes is moot. Penrose[41] first pointed out that an infaller passing through the inner horizon of a Reissner–Nordström black hole would see the outside universe infinitely blueshifted, and suggested that this shift would destabilize the Reissner–Nordström solution. The full nonlinear nature of the instability was eventually clarified in a seminal paper by Poisson and Israel,[42] who showed that relativistic counter streaming between ingoing fluids (going with the flow of the river, in the parlance of this paper) and outgoing fluids (going against the flow) just above the inner horizon of a Reissner–Nordström black hole produces an exponentially growing instability which they dubbed "mass inflation." See Ref. 43 for an entry to the literature on the intriguing subject of what happens inside charged black holes.

The infall velocity β is imaginary inside the turnaround radius r_0 of the Reissner–Nordström geometry, the interior mass $M(r)$ being negative inside this radius. The imaginary velocity might be considered a defect of the river model, but it might also be considered an asset, signaling the presence of unphysical negative mass. Whatever the case, the formalism remains valid even where the river velocity β is imaginary.

III. ROTATING BLACK HOLES

Does the river model work also for stationary rotating black holes? We will show that the answer is yes. There is a

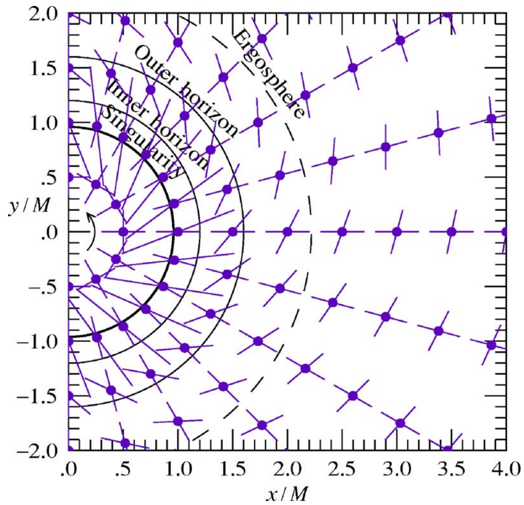


Fig. 2. (Color online) Sets of horizontal radial and azimuthal tetrad axes $\gamma_\rightarrow$ and $\gamma_\uparrow$, Eq. (35), in the equatorial x-y plane ($\theta=\pi/2$) of an uncharged (Kerr) black hole with angular momentum per unit mass $a=0.96$ plotted in Doran–Cartesian coordinates. The azimuthal axis at each point is tilted radially, reflecting the fact that the spatial metric is sheared.

river of space, and it moves through a flat background. Fishes move through the river special relativistically, as though they were being carried with it. But the river has a surprising twist. It might be anticipated that the river would spiral into the black hole like a whirlpool, but that is not the case. Rather, the river velocity has no azimuthal component. Instead of a spiral, the river possesses, besides a velocity at each point, a rotation or twist at each point. The river is characterized not by three numbers, a velocity vector, but by six numbers, a velocity vector and a twist vector. As a fish swims through the river, it is Lorentz boosted by gradients in the velocity of the river and rotated spatially by gradients in the twist of the river. A key result is the expression in Eq. (73) for the river field ω_{km}. This is a 6-component bivector[38,44] field antisymmetric in its indices km, whose electric part specifies the river velocity, and whose magnetic part specifies the river twist. The river field ω_{km} encapsulates all the properties of a stationary rotating black hole.

How can a river move and twist without spiralling? The answer to this conundrum is that, unlike the Gullstrand–Painlevé case, the spatial metric is not flat, but sheared (see Fig. 2). We can regard the twist in the river as inducing the shear in the spatial metric; or equally well we can regard the shear in the spatial metric as requiring a twist in the river. Whatever the case, the twist and the shear act together to ensure that locally inertial frames moving through the infalling river co-move with the geodesic motion of points at rest in a small neighborhood of the frame.

Recall from special relativity that Lorentz transformations are generated by a combination of changes in velocity, or Lorentz boosts, and spatial rotations. Lorentz boosts are rotations in a plane defined by a space axis and a time axis, and spatial rotations are rotations in a plane defined by two spatial axes. Gradients in the velocity of the river make the metric nonflat with respect to the time components, while leaving the spatial metric at constant time flat. Gradients in the rotation, or twist, of the river make the metric nonflat with respect to the spatial components, while leaving the time part of the metric flat; that is, the metric becomes $-\mathrm{d}t^2+g_{ij}\mathrm{d}x^i\mathrm{d}x^j$ where g_{ij} is a purely spatial metric. We see that the reason that the Gullstrand–Painlevé metric for spherical black holes is flat along hypersurfaces of constant free-fall time is that the river has no twist component. However, the Gullstrand–Painlevé river does have a velocity component, so the Gullstrand–Painlevé metric is not flat in the time direction. For rotating black holes the river has both velocity and twist components, and the metric is flat neither in time nor in space.

A. Doran metric

Doran[36] has pointed out that the Kerr–Newman metric for a rotating black hole of angular momentum a per unit mass (for positive a the black hole rotates right handedly about its axis), can be cast in oblate spheroidal coordinates $(t_{\mathrm{ff}}, r, \theta, \phi_{\mathrm{ff}})$ in the form

$$\mathrm{d}s^2 = -\,\mathrm{d}t_{\mathrm{ff}}^2 + \left[\frac{\rho\mathrm{d}r}{R} + \frac{\beta R}{\rho}(\mathrm{d}t_{\mathrm{ff}} - a\sin^2\theta\,\mathrm{d}\phi_{\mathrm{ff}})\right]^2 + \rho^2\mathrm{d}\theta^2 + R^2\sin^2\theta\,\mathrm{d}\phi_{\mathrm{ff}}^2, \tag{23}$$

where $\beta(r)$ is the river velocity,

$$R \equiv (r^2+a^2)^{1/2}, \quad \rho \equiv (r^2 + a^2\cos^2\theta)^{1/2}, \tag{24}$$

and the free-fall time t_{ff} and free-fall azimuthal angle ϕ_{ff} are related to the usual Boyer-Lindquist[38,45] time t and azimuthal angle ϕ by

$$t_{\mathrm{ff}} = t - \int_r^\infty \frac{\beta\mathrm{d}r}{1-\beta^2} \tag{25}$$

$$\phi_{\mathrm{ff}} = \phi - a\int_r^\infty \frac{\beta\mathrm{d}r}{R^2(1-\beta^2)}. \tag{26}$$

As before, we adopt the convention that the river velocity β is positive for a black hole (infalling) and is negative for a white hole (outfalling). Horizons occur (see Sec. III D) where the river velocity β equals the speed of light

$$\beta = \pm 1. \tag{27}$$

The boundaries of ergospheres[38] (where little children come from, because nothing can remain at rest there) occur where $\mathrm{d}s^2=0$ at $\mathrm{d}r=\mathrm{d}\theta=\mathrm{d}\phi_{\mathrm{ff}}=0$, which happens at

$$\beta = \pm\frac{\rho}{R}, \tag{28}$$

again with $\beta=\rho/R$ for black hole ergospheres, and $\beta=-\rho/R$ for white hole ergospheres. For a Kerr–Newman black hole with mass M and charge Q, the river velocity β is given by

$$\beta(r) = \frac{(2Mr - Q^2)^{1/2}}{R}. \tag{29}$$

For the present purpose the river velocity can be considered to be a more general function of the radial coordinate r. Note that the river velocity β as defined here differs from Doran's velocity[36] by a factor of ρ/R. Doran defines the velocity to equal the magnitude $(\beta_\mu\beta^\mu)^{1/2}=\beta R/\rho$ of the velocity vector β^μ given by Eq. (31), a seemingly natural choice. The point of the convention adopted here is that $\beta(r)$ is any and only a function of r, rather than depending also on θ through ρ

$\equiv(r^2+a^2\cos^2\theta)^{1/2}$. Moreover, with our convention the river velocity is ± 1 at the horizons, Eq. (27), as we will demonstrate in Sec. III D.

If the river velocity β is zero, then the metric, Eq. (23), reduces to the flat space metric in oblate spheroidal coordinates. However, unlike the spherical case, the metric is not flat along hypersurfaces of constant free-fall time, $dt_{\text{ff}}=0$.

B. Doran–Cartesian metric

The Doran coordinate system turns out (Secs. III F and III I) to provide the coordinates of the flat background through which the river of space flows into the black hole. We therefore express the Doran metric in Cartesian coordinates $x^\mu=(x^0,x^1,x^2,x^3)=(t_{\text{ff}},x,y,z)$ $=(t_{\text{ff}},R\sin\theta\cos\phi_{\text{ff}},R\sin\theta\sin\phi_{\text{ff}},r\cos\theta)$ with the rotation axis along the z direction:

$$ds^2=\eta_{\mu\nu}(dx^\mu-\beta^\mu\alpha_\kappa dx^\kappa)(dx^\nu-\beta^\nu\alpha_\lambda dx^\lambda). \tag{30}$$

The components of the river velocity β^μ are

$$\beta^\mu=\frac{\beta R}{\rho}\left(0,-\frac{xr}{R\rho},-\frac{yr}{R\rho},-\frac{zR}{r\rho}\right), \tag{31}$$

and $\alpha_\mu dx^\mu=dt_{\text{ff}}-a\sin^2\theta\, d\phi_{\text{ff}}$ has components

$$\alpha_\mu=\left(1,\frac{ay}{R^2},-\frac{ax}{R^2},0\right). \tag{32}$$

The vector α_μ is related to the 4-velocity of the horizon, Eq. (46), and we refer to it as the azimuthal vector, because its spatial components point in the (negative) azimuthal direction, in the direction opposite to the rotation of the black hole. The spheroidal radial coordinate r is given implicitly in terms of x,y,z by

$$r^4-r^2(x^2+y^2+z^2-a^2)-a^2z^2=0. \tag{33}$$

C. River tetrad

In modeling black holes as an inflowing river of space, it is natural to work in the orthonormal tetrad formalism. Let $\mathbf{g}_\mu$ denote the basis of tangent vectors in the Doran–Cartesian coordinate system x^μ, and let γ_m and the associated local coordinates $\xi^m=(\xi^0,\xi^1,\xi^2,\xi^3)$ denote a system of locally inertial frames, tetrads, attached to observers who free fall from zero velocity (with zero angular momentum) at infinity. Such freely falling observers are co-moving with the infalling river of space. They fall along trajectories of constant θ and ϕ_{ff}, and have 4-velocities $v^\mu_{\text{ff}}=(1,\beta^1,\beta^2,\beta^3)$ in the Doran–Cartesian coordinate system. The scalar products of the tangent vectors $\mathbf{g}_\mu$ at each point constitute the metric $g_{\mu\nu}$, Eq. (10), and the scalar products of the tetrad vectors γ_m at each point form the Minkowski metric, Eq. (12). If the tetrad frames γ_m are assumed without loss of generality to be aligned with the tangent vectors $\mathbf{g}_\mu$ at infinity, the relation between γ_m and $\mathbf{g}_\mu$ is[36]

$$\gamma_{t_{\text{ff}}}=\mathbf{g}_{t_{\text{ff}}}+\beta^i\mathbf{g}_i \tag{34a}$$

$$\gamma_x=\mathbf{g}_x+\alpha_x\beta^i\mathbf{g}_i \tag{34b}$$

$$\gamma_y=\mathbf{g}_y+\alpha_y\beta^i\mathbf{g}_i \tag{34c}$$

$$\gamma_z=\mathbf{g}_z, \tag{34d}$$

which may be confirmed by checking that the scalar products of γ_m form the Minkowski metric and that their derivatives vanish along the worldlines of observers who free fall from zero velocity at infinity, $v^\mu_{\text{ff}}\partial\gamma_m/\partial x^\mu=0$. If the horizontal radial and azimuthal axes are defined by $(\gamma_\rightarrow,\gamma_\uparrow)$ $\equiv(\cos\phi_{\text{ff}}\gamma_x+\sin\phi_{\text{ff}}\gamma_y,-\sin\phi_{\text{ff}}\gamma_x+\cos\phi_{\text{ff}}\gamma_y)$ and likewise $(\mathbf{g}_\rightarrow,\mathbf{g}_\uparrow)\equiv(\cos\phi_{\text{ff}}\mathbf{g}_x+\sin\phi_{\text{ff}}\mathbf{g}_y,-\sin\phi_{\text{ff}}\mathbf{g}_x+\cos\phi_{\text{ff}}\mathbf{g}_y)$, then

$$\gamma_\rightarrow=\mathbf{g}_\rightarrow \tag{35a}$$

$$\gamma_\uparrow=\mathbf{g}_\uparrow-\frac{a\sin\theta}{R}\beta^i\mathbf{g}_i. \tag{35b}$$

Equation (34a) shows that the time axis $\gamma_{t_{\text{ff}}}$ is shifted by velocity $\beta^i\mathbf{g}_i$, similar to the spherical case, Eq. (14a). Equation (35) shows that in addition the azimuthal axis $\gamma_\uparrow$ is shifted by $-(a\sin\theta/R)\beta^i\mathbf{g}_i$. Figure 2 illustrates the horizontal radial and azimuthal axes $\gamma_\rightarrow$ and $\gamma_\uparrow$ at several points in the equatorial plane of a Kerr black hole. The azimuthal axes $\gamma_\uparrow$ are tilted radially in accordance with Eq. (35), reflecting the fact that the spatial metric is sheared.

Equation (34) may be abbreviated $\gamma_m=e_m{}^\mu\mathbf{g}_\mu$ where $e_m{}^\mu$ is the vierbein (German for four-leg) matrix

$$e_m{}^\mu=\delta^\mu_m+\alpha_m\beta^\mu, \tag{36}$$

with δ^μ_m the Kronecker delta. The inverse vierbein $e^m{}_\mu$ is

$$e^m{}_\mu=\delta^m_\mu-\alpha_\mu\beta^m. \tag{37}$$

The product of the vierbein and its inverse given by Eqs. (36) and (37) is the unit matrix, $e_m{}^\mu e^n{}_\mu=\delta^n_m$ and $e^m{}_\mu e_m{}^\nu$ $=\delta^\nu_\mu$, as follows from the orthogonality of the azimuthal and velocity vectors α_μ and β^μ, namely $\alpha_\mu\beta^\mu=0$. The vectors α_m with a Latin index in the vierbein, Eq. (36), and β^m with a Latin index in the inverse vierbein, Eq. (37), are defined by

$$\alpha_m\equiv\delta^\mu_m\alpha_\mu,\quad \beta^m\equiv\delta^m_\mu\beta^\mu, \tag{38}$$

and transform with the tetrad frame γ_m rather than the coordinate frame $\mathbf{g}_\mu$. The coordinates of α_m and β^m are the same as those of α_μ and β^μ in the particular tetrad frame and coordinate system we are using, but would be different in a different tetrad frame or a different coordinate system.

In general, the vierbein matrix $e_m{}^\mu$ and its inverse $e^m{}_\mu$ provide a means of transforming the components p_μ or p^μ of any arbitrary 4-vector between the coordinate frame and the tetrad frame

$$p_m=e_m{}^\mu p_\mu,\quad p^m=e^m{}_\mu p^\mu. \tag{39}$$

The indices on the vectors p_m and p^m in the tetrad frame are raised and lowered with the Minkowski metric η_{mn}, and the indices on vectors p_μ and p^μ in the coordinate frame are raised and lowered with the coordinate metric $g_{\mu\nu}$.

A special case of Eq. (39) is

$$\alpha_m=e_m{}^\mu\alpha_\mu,\quad \beta^m=e^m{}_\mu\beta^\mu \tag{40}$$

which reduces to the definitions in Eq. (38) thanks to the orthogonality of α_μ and β^μ. If the coordinate system or tetrad frame is changed, then the vierbein change accordingly, and α_m and β^m change in accordance with Eq. (40).

The components u^m of the 4-velocity of a particle relative to the tetrad frame are related to the components v^μ in the coordinate frame by $u^m=e^m{}_\mu v^\mu$, or explicitly

$$u^0 = v^0 \tag{41a}$$

$$u^i = v^i - \beta^i \alpha_\mu v^\mu \quad (i=1,2,3). \tag{41b}$$

Equation (41) implies that if in an interval of proper time $\delta\tau$ the particle moves a coordinate distance $\delta x^\mu = v^\mu \delta\tau$, then relative to the tetrad frame, that is, relative to the locally inertial frame of an observer who is comoving with the infalling river, the particle moves a proper distance

$$\delta\xi^m = e^m{}_\mu \delta x^\mu = \delta x^m - \beta^m \alpha_\mu \delta x^\mu. \tag{42}$$

We recognize the right-hand side of Eq. (42) as having the same form as a factor of the Doran–Cartesian metric, Eq. (30). The temporal displacement $\delta\xi^0$ of the particle in the tetrad frame is the Galilean time change δt_{ff}, as in the spherical case. The proper spatial displacement $\delta\xi^i$ of the particle in the tetrad frame differs from the displacement δx^i in the coordinate frame not by the Galilean distance $\beta^i \delta t_{\text{ff}}$ that the river moves in the time δt_{ff}, as in the spherical case, but by $\beta^i \alpha_\mu \delta x^\mu = \beta^i(\delta t_{\text{ff}} - a \sin^2\theta \delta\phi_{\text{ff}})$. The extra term $-\beta^i a \sin^2\theta\, \delta\phi_{\text{ff}}$ arises from the spatial shear in the metric, as illustrated in Fig. 2.

D. Horizons

It is now possible to understand how the position of the horizons is set by $\beta = \pm 1$, as was asserted in Eq. (27). It follows from the previous paragraph that the effective velocity of the river, from the point of view of an object in the river, depends on the state of motion of the object. The effective river velocity is $\beta^i \alpha_\mu \mathrm{d}x^\mu / \mathrm{d}t_{\text{ff}}$, which differs from β^i by the factor $\alpha_\mu v^\mu / v^0 = \alpha_\mu \mathrm{d}x^\mu / \mathrm{d}t_{\text{ff}} = 1 - a \sin^2\theta\, \mathrm{d}\phi_{\text{ff}} / \mathrm{d}t_{\text{ff}}$. Irrespective of this factor, the effective river velocity always points radially inward (along lines of constant θ and ϕ_{ff}) along the direction of β^i. If we restrict our consideration temporarily to objects with a fixed value of $\alpha_\mu v^\mu / v^0$, then such objects can escape outward only if their radial velocity,

$$v^r = \frac{\partial r}{\partial x^\mu} v^\mu, \tag{43}$$

exceeds zero. To determine the position of the horizon, we first solve the slightly more general problem of maximizing the radial velocity v^r subject to constraints on v^0 (which can be set to 1 without loss of generality), $\alpha_\mu v^\mu$, and $v_\mu v^\mu$. The last constraint comes from the fact that the 4-velocity must be time-like or light-like, requiring $v_\mu v^\mu \leq 0$. Equivalently, we can minimize $v_\mu v^\mu$ subject to constraints on v^0, $\alpha_\mu v^\mu$, and v^r, which gives

$$v_\mu = \lambda \delta^0_\mu + \mu \alpha_\mu + \nu \frac{\partial r}{\partial x^\mu}, \tag{44}$$

where λ, μ, and ν are Lagrange multipliers whose values are determined by fixing any three of the four quantities v^0, v^r, $\alpha_\mu v^\mu$, and $v_\mu v^\mu$. Not surprisingly, the largest value of v^r at fixed v^0 and $\alpha_\mu v^\mu$ occurs when the 4-velocity is light-like, $v_\mu v^\mu = 0$. We eliminate the Lagrange multipliers λ, μ, and ν in favor of $v^0 = 1$, $v^r = 0$, and $v_\mu v^\mu = 0$, and obtain

$$\frac{\alpha_\mu v^\mu}{v^0} = \frac{\rho^2[R \pm a \sin\theta (1-\beta^2)^{1/2}]}{R(\rho^2 + \beta^2 a^2 \sin^2\theta)}, \tag{45}$$

which has a real solution if $\beta^2 \leq 1$, with $\alpha_\mu v^\mu / v^0 = \rho^2 / R^2$ at $\beta^2 = 1$. The position of the horizon is thus set by $\beta^2 = 1$ as claimed. If $\beta^2 < 1$, there are geodesics on which a particle can escape, $v^r > 0$; if $\beta^2 > 1$, then all geodesics are trapped, and an object is compelled to fall inward (or outward for the case of a white hole).

The 4-velocity of a photon that just holds steady on the horizon, a member of the outgoing principal null congruence, satisfies $v_\mu = (\rho^2 / R^2)\, \partial r / \partial x^\mu$, and is

$$v^\mu = \left(1, -\frac{ay}{R^2}, \frac{ax}{R^2}, 0\right). \tag{46}$$

Interestingly, the contravariant components v^μ of this 4-velocity coincide, up to a minus sign, with the covariant components α_μ of the azimuthal vector, Eq. (32). Relative to the river frame, the horizon rotates right handedly with angular velocity

$$\frac{\mathrm{d}\phi_{\text{ff}}}{\mathrm{d}t_{\text{ff}}} = \frac{a}{R^2}, \tag{47}$$

which is also the angular velocity of the horizon perceived by an observer at rest at infinity.[38]

E. Equations of motion in the tetrad formalism

Our aim in this section is to derive the equations of motion Eq. (62) for objects moving relative to the inflowing river of space. For clarity we start from basic principles to derive the equations of motion of 4-vectors in the tetrad frame. We will then describe what these equations mean physically. In Sec. III F we will apply these equations to the special case of black holes, where the vierbein are given by Eq. (36).

Let $\mathbf{p}$ be an arbitrary 4-vector. The 4-vector $\mathbf{p} = p^m \gamma_m = p^\mu \mathbf{g}_\mu$ is an invariant object, independent of the choice of tetrad or coordinate system. According to the principle of equivalence, an unaccelerated 4-vector $\mathbf{p}$ remains at rest in its own free-fall frame, meaning that its derivative with respect to its own proper time τ is zero in its own frame

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\tau} = 0. \tag{48}$$

If the 4-vector $\mathbf{p}$ experiences an acceleration in its own frame (perhaps because of an electromagnetic field or rockets being fired), the zero on the right-hand side of Eq. (48) should be replaced by an appropriate invariant acceleration 4-vector. Here we set any such acceleration to zero, recognizing that an acceleration could be introduced if desired at the end of the calculation. Because $\mathbf{p}$ is invariant, Eq. (48) must be true in all frames. In the tetrad frame we have

$$\gamma_m \frac{\mathrm{d}p^m}{\mathrm{d}\tau} + \frac{\mathrm{d}\gamma_m}{\mathrm{d}\tau} p^m = 0. \tag{49}$$

The proper time derivative $\mathrm{d}/\mathrm{d}\tau$ can be written as

$$\frac{\mathrm{d}}{\mathrm{d}\tau} = v^\nu \frac{\partial}{\partial x^\nu} = u^n e_n{}^\nu \frac{\partial}{\partial x^\nu} = u^n \partial_n, \tag{50}$$

where the directed derivative ∂_n is defined to be the space-time derivative along the axis γ_n

$$\partial_n \equiv e_n{}^\nu \frac{\partial}{\partial x^\nu} = \gamma_n \cdot \partial, \tag{51}$$

where $\partial = \mathbf{g}^\mu \partial / \partial x^\mu = g^{\mu\nu} \mathbf{g}_\nu \partial / \partial x^\mu$ is the invariant space-time vector derivative. In other words, ∂_n constitute the tetrad frame components of the invariant 4-vector derivative ∂

$=\gamma^n\partial_n=\mathbf{g}^\nu\partial/\partial x^\nu$. The derivative ∂_n defined by Eq. (51) is independent of the choice of coordinates x^ν, as suggested by the absence of any Greek index. Unlike the partial derivatives $\partial/\partial x^\nu$, the directed derivatives ∂_n do not commute. In terms of the vierbein derivatives d_{kmn} defined by

$$\mathrm{d}_{kmn}\equiv\eta_{kl}e^l{}_\lambda e_n{}^\nu\frac{\partial e_m{}^\lambda}{\partial x^\nu}, \tag{52}$$

the commutator $[\partial_k,\partial_m]$ of two directed derivatives is

$$[\partial_k,\partial_m]=f_{km}{}^n\partial_n,\quad f_{kmn}\equiv\mathrm{d}_{nmk}-\mathrm{d}_{nkm}. \tag{53}$$

The f_{kmn} are the structure coefficients of the commutators of directed derivatives.

We now introduce the tetrad frame connection coefficients Γ^k_{mn}, also known as the Ricci rotation coefficients, defined by

$$\partial_n\gamma_m\equiv\Gamma^k_{mn}\gamma_k. \tag{54}$$

In terms of the vierbein $e_m{}^\mu$ and basis vectors $\mathbf{g}_\mu$, the tetrad frame connection coefficients with all indices lowered, $\Gamma_{kmn}\equiv\eta_{kl}\Gamma^l_{mn}$, are from Eq. (54),

$$\Gamma_{kmn}=\gamma_k\cdot\partial_n\gamma_m=e_k{}^\kappa\mathbf{g}_\kappa\cdot e_n{}^\nu\frac{\partial(e_m{}^\mu\mathbf{g}_\mu)}{\partial x^\nu}. \tag{55}$$

The usual coordinate frame connection coefficients, the Christoffel symbols $\Gamma_{\kappa\mu\nu}\equiv g_{\kappa\lambda}\Gamma^\lambda_{\mu\nu}$, are defined by

$$\frac{\partial\mathbf{g}_\mu}{\partial x^\nu}\equiv\Gamma^\kappa_{\mu\nu}\mathbf{g}_\kappa. \tag{56}$$

Equations (55) and (56) imply that the tetrad frame connection coefficients Γ_{kmn} are related to the Christoffel symbols $\Gamma_{\kappa\mu\nu}$ by

$$\Gamma_{kmn}=\mathrm{d}_{kmn}+e_k{}^\kappa e_m{}^\mu e_n{}^\nu\Gamma_{\kappa\mu\nu}. \tag{57}$$

Equation (54) and the fact that $\partial_n(\gamma_k\cdot\gamma_m)=\partial_n\eta_{km}=0$ implies that the tetrad frame connection coefficients Γ_{kmn} are antisymmetric in their first two indices,

$$\Gamma_{kmn}=-\Gamma_{mkn}. \tag{58}$$

The tangent vectors $\mathbf{g}_\mu$ can be regarded as coordinate derivatives of the invariant 4-vector interval $d\mathbf{x}\equiv\mathbf{g}_\mu dx^\mu$; that is, $\mathbf{g}_\mu=\partial\mathbf{x}/\partial x^\mu$. The commutativity of the partial derivatives, $\partial\mathbf{g}_\mu/\partial x^\nu=\partial^2\mathbf{x}/\partial x^\nu\partial x^\mu=\partial^2\mathbf{x}/\partial x^\mu\partial x^\nu=\partial\mathbf{g}_\nu/\partial x^\mu$, implies that the Christoffel symbols $\Gamma^\kappa_{\mu\nu}$ are symmetric in their last two indices,

$$\Gamma^\kappa_{\mu\nu}=\Gamma^\kappa_{\nu\mu}, \tag{59}$$

which is the usual no-torsion condition of general relativity. If we combine Eq. (57) with the antisymmetry relation Eq. (58) and the no-torsion condition Eq. (59), the result is an expression for the tetrad frame connection coefficients entirely in terms of the vierbein derivatives d_{kmn}

$$\Gamma_{kmn}=\frac{1}{2}(\mathrm{d}_{kmn}-\mathrm{d}_{mkn}+\mathrm{d}_{nmk}-\mathrm{d}_{nkm}+\mathrm{d}_{mnk}-\mathrm{d}_{knm}), \tag{60}$$

or equivalently, in terms of the structure coefficients f_{kmn}, Eq. (53),

$$\Gamma_{kmn}=\frac{1}{2}(f_{kmn}-f_{nkm}+f_{nmk}). \tag{61}$$

From Eqs. (49), (50), and (54) it follows that the equations of motion for the tetrad components p^k of an unaccelerated 4-vector $\mathbf{p}=p^k\gamma_k$ are

$$\frac{\mathrm{d}p^k}{\mathrm{d}\tau}+\Gamma^k_{mn}u^np^m=0. \tag{62}$$

The physical significance of Eq. (62) is as follows. The tetrad γ_m defines a set of locally inertial frames at each point of spacetime. In the present case these locally inertial frames have been constructed so that observers who free fall from zero velocity at infinity find their own frame aligned at all times with the tetrad frame. But in general other observers who free fall along a different geodesic will find their locally inertial frame becoming misaligned with the tetrad frame. Equation (62) expresses this misalignment of locally inertial frames. Because the misalignment is between locally inertial frames, it is a Lorentz transformation. This Lorentz transformation is encoded in the connections Γ^k_{mn}. In particular, if a 4-vector p^k is transported in free fall by an infinitesimal distance $\delta\xi^n=u^n\delta\tau$ relative to the tetrad frame γ_n, then the 4-vector experiences an infinitesimal Lorentz transformation $p^k\to p^k-\delta\xi^n\Gamma^k_{mn}p^m$. In other words, the connection coefficients Γ^k_{mn} for each final index n constitute the generator of a Lorentz transformation.

The antisymmetry of the tetrad frame connection coefficient with respect to its first two indices, Eq. (58), expresses the fact that Γ^k_{mn} for each n is the generator of a Lorentz transformation. The components of Γ^k_{mn} in which one of the first two indices k or m is 0 (time) generate Lorentz boosts. Components of Γ^k_{mn} in which both of the first two indices k and m are 1, 2, or 3 (space) generate spatial rotations.

F. The flat background

Section III E considered the equations of motion in the tetrad formalism in the general case. We now specialize to rotating black holes, where the vierbein are given by Eq. (36). We will see how the Doran-Cartesian coordinate system emerges as the coordinate system of a flat background. In Sec. III G, we will see how the connection coefficients are expressible as the flat space gradient of a river field. In Sec. III I we will revisit the notion of the flat background and what it means.

An explicit calculation of the structure coefficients f_{kmn}, Eq. (53), and hence of the connection coefficients Γ_{kmn}, Eq. (61), from the vierbein of Eq. (36) reveals that the sea of terms nonlinear in the vierbeins undergo a remarkable cancellation leaving only terms linear in the vierbeins $e_m{}^\lambda$. In other words, the structure coefficients and hence the connection coefficients reduce to the same expressions as Eqs. (53) and (61), but with the $e^l{}_\lambda e_n{}^\nu$ factors in Eq. (52) for d_{kmn} replaced by Kronecker deltas $\delta^l_\lambda\delta^\nu_n$

$$\mathrm{d}_{kmn}\to\eta_{kl}\delta^l_\lambda\delta^\nu_n\frac{\partial e_m{}^\lambda}{\partial x^\nu}=\delta^\nu_n\frac{\partial\alpha_m\beta_k}{\partial x^\nu}. \tag{63}$$

The fact that the derivative $e_n{}^\nu\partial/\partial x^\nu$ in Eq. (52) is replaced by $\delta^\nu_n\partial/\partial x^\nu$ in Eq. (63) motivates introducing a new set of flat space coordinates x^n (with Latin indices) with the defining property that in the particular coordinate and tetrad frame which we are using

$$\frac{\partial}{\partial x^n} \equiv \delta_n^\nu \frac{\partial}{\partial x^\nu}. \tag{64}$$

The invariant relation $\mathrm{d}x^n \partial/\partial x^n = \mathrm{d}x^\nu \partial/\partial x^\nu$ implies that the flat space differentials $\mathrm{d}x^n$ are related to the coordinate differentials $\mathrm{d}x^\nu$ by

$$\mathrm{d}x^n = \delta_\nu^n \mathrm{d}x^\nu. \tag{65}$$

The relations in Eqs. (64) and (65) should be interpreted as being valid only in the particular tetrad and coordinate frame that we are using. If the tetrad frame is subjected to a local gauge transformation[46] which rotates the locally inertial coordinates at each point by $\xi^n \to \xi'^n$, and if the coordinate system is subjected to a general coordinate transformation $x^\nu \to x'^\nu$, then the Kronecker deltas in Eqs. (64) and (65) should be replaced by

$$\delta_n^\nu \to \delta_m^\mu \frac{\partial \xi^m}{\partial \xi'^n} \frac{\partial x'^\nu}{\partial x^\mu}, \quad \delta_\nu^n \to \delta_\mu^m \frac{\partial \xi'^n}{\partial \xi^m} \frac{\partial x^\mu}{\partial x'^\nu}. \tag{66}$$

In the particular tetrad and coordinate frame that we are using, integrating the relation Eq. (65) arbitrarily through space yields (the constant of integration can be set to zero without loss of generality)

$$x^n = \delta_\nu^n x^\nu. \tag{67}$$

Notwithstanding the index notation, neither x^n nor x^ν is a 4-vector either under local gauge transformations of the tetrad or under general transformations of the coordinates (only the differentials $\mathrm{d}x^n$ and $\mathrm{d}x^\nu$ are 4-vectors). Hence Eq. (67) cannot be interpreted as a covariant equation relating the coordinates x^n and x^ν, even if the Kronecker delta is replaced according to Eq. (66). Rather, Eq. (67) should be interpreted as valid in the particular coordinate and tetrad frame that we are using. Equation (67) can be regarded as defining the flat space coordinates x^n: they are numerically the same as the curved space coordinates x^ν of the Doran–Cartesian metric, Eq. (30), but reincarnated as coordinates x^n of a flat space with a Minkowski metric. The Doran coordinate system[36] thus emerges as a special one, providing the coordinates of the flat background through which the river of space flows in rotating black holes.

The flat spacetime coordinates x^n are not the same as the locally inertial coordinates ξ^n attached to the tetrad γ_n at each point of spacetime. The locally inertial differentials $\mathrm{d}\xi^n$ are related to the coordinate differentials $\mathrm{d}x^\nu$ by

$$\mathrm{d}\xi^n = e^n{}_\nu \quad \mathrm{d}x^\nu, \tag{68}$$

which differs from the corresponding relation, Eq. (65), between $\mathrm{d}x^n$ and $\mathrm{d}x^\nu$.

G. The river field

The vectors α_m and β^m can be regarded as functions of the flat space coordinates x^n. The replacement of the vierbein derivatives d_{kmn}, Eq. (63), in the connection coefficients can be written as

$$\mathrm{d}_{kmn} \to \frac{\partial \alpha_m \beta_k}{\partial x^n}. \tag{69}$$

The connection coefficients, Eq. (60), are then given by flat space derivatives of α_m and β_m

$$\Gamma_{kmn} = \frac{1}{2}\left(\frac{\partial \alpha_m \beta_k}{\partial x^n} - \frac{\partial \alpha_k \beta_m}{\partial x^n} + \frac{\partial \alpha_m \beta_n}{\partial x^k} - \frac{\partial \alpha_n \beta_k}{\partial x^m} + \frac{\partial \alpha_n \beta_m}{\partial x^k} - \frac{\partial \alpha_k \beta_n}{\partial x^m} \right). \tag{70}$$

The connection coefficients with zero final index $n=0$ are all identically zero, $\Gamma_{km0}=0$. Taking the spatial curl of Γ_{kmn} on the n index yields another sea of terms which again undergo a remarkable cancellation

$$\varepsilon^{ijn} \frac{\partial \Gamma_{kmn}}{\partial x^j} = 0 \tag{71}$$

for all i,k,m. Equation (71) demonstrates that the connection coefficients Γ_{kmn} must be expressible as (minus) the flat space gradient $\partial/\partial x^n$ of an object ω_{km}. We call the latter the river field because it encapsulates all the properties of the river in the river model:

$$\Gamma_{kmn} = -\frac{\partial \omega_{km}}{\partial x^n}. \tag{72}$$

The river field ω_{km} is a bivector,[38,44] inheriting from Γ_{kmn} the property of being antisymmetric in km. That the connection coefficient Γ_{kmn} is the flat space gradient of the river field lies at the heart of the river model as a description of black holes. After some manipulation we find the desired bivector river field to be

$$\omega_{km} = \alpha_k \beta_m - \alpha_m \beta_k + \varepsilon_{0kmi} \zeta^i, \tag{73}$$

where the vector ζ^i is

$$\zeta^i = (0,0,0,\zeta), \quad \zeta = a \int_r^\infty \frac{\beta \mathrm{d}r}{R^2}, \tag{74}$$

which points vertically upward along the rotation axis of the black hole.

The river field ω_{km} given by Eq. (73) inherits from the connection coefficient Γ_{kmn} its Lorentz structure. The river field defines a velocity and a rotation or twist at each point of the black hole geometry. The components of ω_{km} in which one of the indices k or m is 0 (time) define a velocity; components in which both indices k and m are $1,2,3$ (space) define a spatial rotation or twist. The velocity is just the river velocity β_m,

$$\omega_{0m} = \beta_m, \tag{75}$$

and the angle and axis of the river twist are given by the rotation vector

$$\mu^i = \frac{1}{2} \varepsilon^{ikm} \omega_{km} = \varepsilon^{ikm} \alpha_k \beta_m + \zeta^i \quad (i,k,m=1,2,3). \tag{76}$$

Like the velocity vector β_i, the twist vector μ^i at each point lies in the plane of constant free-fall azimuthal angle ϕ_{ff}, because it is a sum of two vectors $\varepsilon^{ikm}\alpha_k \beta_m$ and ζ^i, both of which are orthogonal to the azimuthal vector α_k.

Figure 3 illustrates the velocity and twist fields β_i and μ_i for an uncharged black hole with angular momentum $a=0.96$.

Another familiar bivector is the electromagnetic field tensor F_{km}, and it can be useful to think of the river field bivector ω_{km} in these terms. The velocity vector β_i is the analog of the electric field vector E_i, and the twist vector μ_i is the

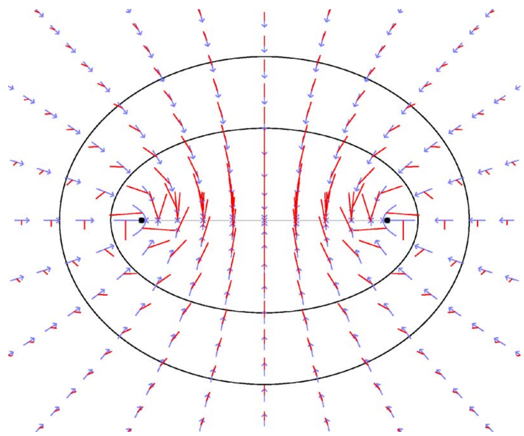


Fig. 3. (Color online) The velocity and twist fields for an uncharged (Kerr) black hole with $a=0.96$. The arrowed lines show the magnitude and direction of the river velocity, and the unarrowed lines emerging from the arrowed lines show the magnitude and axis of the river twist. The confocal ellipses show the outer and inner horizons, and the large dots at the foci of the ellipses indicate the ring singularity. In the vacuum Kerr solution the river velocity goes to zero at the horizontal disc bounded by the ring singularity, then turns around and rebounds through a white hole into a new universe.

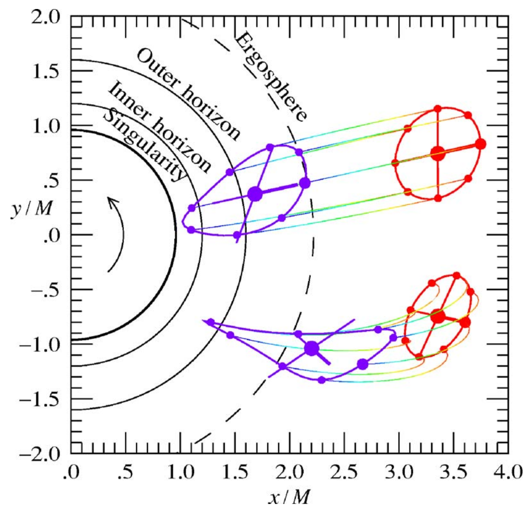


Fig. 4. (Color online) Two sample sets of geodesics in the equatorial x-y plane ($\theta=\pi/2$) of an uncharged (Kerr) black hole with $a=0.96$, plotted in Doran–Cartesian coordinates (see Sec. III B). Each set of geodesics shows a central point (large dot) and associated locally inertial axes (crossed thick lines), surrounded by a set of points that are initially uniformly spaced around and initially at rest relative to the central point in the locally inertial frame of the central point. The central point and its attendants follow geodesics (thin lines) into the black hole, and the ensemble becomes tidally distorted as it falls in. In the upper set of geodesics, the central point is co-moving with the infalling river of space; in the lower set of geodesics the central point is initially moving radially outward, but soon turns around and falls in. The lower ensemble illustrates how the locally inertial axes attached to the central point twist as the ensemble falls in; the twist acts to keep the frame co-moving with the geodesic motion of points in a small neighborhood of the central point.

analog of the magnetic field vector B_i. The analogy extends to the fact that, like a static electric field, the velocity vector β_i is the gradient of a potential ψ,

$$\beta_i = -\frac{\partial\psi}{\partial x^i}, \quad \psi \equiv -\int_r^\infty \beta \mathrm{d}r. \tag{77}$$

Unlike a magnetic field, the twist vector μ^i is not a pure curl, although curiously $\mu^i+\zeta^i$ is a pure curl, having zero divergence, $\partial(\mu^i+\zeta^i)/\partial x^i=0$.

H. Motion of objects in the river

We are now ready to demonstrate a fundamental feature of the river model for stationary rotating black holes: as an object moves through the river of space, it is Lorentz boosted and rotated by the tidal gradients in the velocity and twist fields of the river.

It follows from inserting the connection coefficients Γ_{kmn} from Eq. (72) into Eq. (62) that the equation of motion of an unaccelerated 4-vector p^k in the river frame is

$$\frac{\mathrm{d}p^k}{\mathrm{d}\tau} = \frac{\partial \omega^k{}_m}{\partial x^n} u^n p^m. \tag{78}$$

Equation (78) can be interpreted as follows. In an infinitesimal interval $\delta\tau$ of proper time a particle moves a distance $\delta\xi^n = u^n \delta\tau$ relative to the infalling river of space. As a result of its motion through the river, the particle experiences a tidal change

$$\delta\omega^k{}_m = \frac{\partial \omega^k{}_m}{\partial x^n} \delta\xi^n \tag{79}$$

in the river field, which generalizes Eq. (19) for spherical black holes. The tidal change $\delta\omega^k{}_m$ in the river field is an infinitesimal Lorentz transformation, and it induces a Lorentz boost and rotation in the 4-vector p^k

$$p^k \to p^k + \delta\omega^k{}_m p^m. \tag{80}$$

Equations (79) and (80) reproduce Eq. (78).

Figure 4 shows two ensembles of geodesics calculated using Eq. (78). Each ensemble consists of a central point and associated tetrad axes surrounded by a set of points that are initially uniformly spaced about and initially at rest relative to the central point in the locally inertial frame of the latter. In the upper ensemble the central point is co-moving with the infalling river of space; in the lower ensemble the central point is initially moving radially outward, but soon turns around and falls inward. The tetrad axes are skewed because the spatial metric is sheared (compare to Fig. 2). In the lower ensemble the tetrad axes are Lorentz contracted in the radial direction because of the initial outward motion of the ensemble relative to the infalling river. Each ensemble of points becomes tidally distorted as it falls into the black hole. If the locally inertial coordinates of a tetrad axis are denoted as $\delta\xi^k$, then the tetrad axis evolves according to Eq. (78) with $p^k \to \delta\xi^k$,

$$\frac{\mathrm{d}\,\delta\xi^k}{\mathrm{d}\tau} = \frac{\partial \omega^k{}_m}{\partial x^n} u^n \delta\xi^m. \tag{81}$$

Similarly, the tetrad 4-velocity u^k of each point in an ensemble evolves according to Eq. (78) with $p^k \to u^k$,

$$\frac{\mathrm{d}u^k}{\mathrm{d}\tau} = \frac{\partial \omega^k{}_m}{\partial x^n} u^n u^m. \tag{82}$$

Each point surrounding the central point is initially at rest relative to the latter in the central point's locally inertial frame. This condition requires that the covariant difference of tetrad 4-velocities between each point and the central point initially vanishes, which requires that the difference δu^k in the tetrad 4-velocity of a point separated from the central point by locally inertial separation $\delta\xi^k$ initially satisfies, to first order in the separation $\delta\xi^k$,

$$\delta u^k = \frac{\partial \omega^k{}_m}{\partial x^n} \delta\xi^n u^m . \tag{83}$$

The difference δu^k in Eq. (83) is to be understood as the tetrad 4-velocity of a point evaluated in the tetrad frame at that point, minus the tetrad 4-velocity of the central point evaluated in the tetrad frame at the central point. Note that the indices on $\delta\xi^n$ and u^m on the right-hand side of Eq. (83) are swapped compared to those on the right-hand side of Eq. (81). In Eq. (81) the axis $\delta\xi^k$ is transported along the 4-velocity u^n. In contrast, in Eq. (83) the 4-velocity u^k is transported along the axis $\delta\xi^n$.

The lower ensemble of points in Fig. 4 illustrates the twist in the locally inertial frame that develops as the ensemble moves through the river of space. The twist acts to keep the locally inertial frame co-moving with the geodesic motion of points in a small neighborhood of the frame.

Equation (83) holds initially when the ensemble of points is at rest relative to each other, satisfying $D\delta\xi^k/D\tau=0$. The more general form of Eq. (83), which is valid when the points are in relative motion, is to first order in the separation $\delta\xi^k$ (here we rewrite Eq. (72) in terms of the more familiar notation for the connections Γ^k_{mn}),

$$\delta u^k + \Gamma^k_{mn} \delta\xi^n u^m = \frac{D\delta\xi^k}{D\tau} , \tag{84}$$

or equivalently

$$\delta u^k = \frac{\mathrm{d}\delta\xi^k}{\mathrm{d}\tau} + (\Gamma^k_{nm} - \Gamma^k_{mn}) \delta\xi^n u^m . \tag{85}$$

Variation of the equation of motion (82) for u^k gives

$$\frac{\mathrm{d}\delta u^k}{\mathrm{d}\tau} + \partial_l \Gamma^k_{mn} \delta\xi^l u^n u^m + \Gamma^k_{mn}(u^n \delta u^m + u^m \delta u^n) = 0 . \tag{86}$$

If we substitute δu^k from Eq. (85) into Eq. (86), we obtain the familiar equation of geodesic deviation

$$\frac{D^2\delta\xi^k}{D\tau^2} = R_{lmn}{}^k u^m u^n \delta\xi^l , \tag{87}$$

where R_{klmn} is the Riemann curvature tensor

$$R_{klmn} = \partial_l \Gamma_{nmk} - \partial_k \Gamma_{nml} + \Gamma^j_{ml}\Gamma_{jnk} - \Gamma^j_{mk}\Gamma_{jnl} + (\Gamma^j_{lk} - \Gamma^j_{kl})\Gamma_{nmj} . \tag{88}$$

I. The flat background revisited

Now that we have completed the formalism of the river model, it is useful to revisit the question of the flat background, see Sec. III F, through which the river of space flows and twists into a rotating black hole. What exactly does flatness mean in this context?

The crucial relation is Eq. (72), which states that the connection coefficient is given by the flat space gradient of the river field. The fact that the gradient is an ordinary partial derivative with respect to Doran–Cartesian coordinates is what makes the background flat, and in a sense that is all there is to it. Equation (72) acquires physical significance because it propagates through to the equation of motion, Eq. (78), of objects swimming in the river. The equation of motion paints the physical picture of objects moving in the river being Lorentz boosted and rotated by the flat space tidal gradients in the velocity and twist components of the river field.

The statement that the background spacetime in the river model is flat is not a statement about the metric $g_{\mu\nu}$ being flat. Rulers and clocks swimming in the river of space measure not distances and times in the background space, but rather distances and times relative to the tidally twisting and stretching river. The presence of tides is the signature of curvature, so it makes sense that the metric measured by rulers and clocks is not flat.

We emphasize that the flat background has no physically observable meaning. It is a fictitious construct that emerges from the mathematics.

IV. SUMMARY

We have presented a way to conceptualize stationary black holes, which we call the river model. The river model offers a mental picture of black holes which is intuitively appealing, and whose basic elements can be grasped by nonexperts. In the river model, space itself flows like a river through a flat background, while objects move through the river according to the rules of special relativity. For a Schwarzschild (nonrotating, uncharged) black hole, the river falls radially inward at the Newtonian escape velocity, hitting the speed of light at the horizon. Inside the horizon, the river of space moves faster than light, carrying everything with it.

We have presented the details that place the river model on a sound mathematical basis. We have shown that the river model works for any stationary black hole, rotating as well as nonrotating, charged as well as uncharged. The Doran coordinate system[36] provides the coordinates of the flat background through which the river of space flows into the black hole.

The extension of the river model to rotating black holes proves to be surprising and pretty. Contrary to expectation, the river does not spiral into a rotating black hole: the azimuthal component of the river velocity is zero. Instead, the river has at each point not only a velocity, but also a rotation or twist. The river is thus a Lorentz river, characterized by all six generators of the Lorentz group. As an object moves through the river of space, it is Lorentz boosted by changes in the velocity of the river along its path and rotated by changes in the twist of the river. Equation (73) is an explicit expression for the river field, a six-component bivector field that specifies the velocity and twist of the river at each point of the black hole geometry.

The tidal boosts and twists experienced by an object in the river induce a curvature in the spacetime measured by the object, causing the metric to be nonflat. Changes in the river velocity rotate between space and time axes, and changes in the river twist rotate between two spatial axes. The river has zero twist for a spherical black hole, so objects experience no spatial rotation, with the consequence that the metric, the Gullstrand–Painlevé metric, is flat along spatial hypersurfaces at constant time, $\mathrm{d}t_{\mathrm{ff}}=0$. For a rotating black hole, the river has a finite twist, and the metric is not flat along spatial hypersurfaces.

ACKNOWLEDGMENTS

The authors thank Katharina Kohler for a translation of the final part of Ref. 1 and Wildrose Hamilton for drawing Fig. 1. We have benefited from conversations and correspondence with many colleagues, including but not limited to Peter Bender, James Bjorken, Robert Brandenberger, Jan Czerniawski, Nick Gnedin, and Scott Pollack. A.J.S.H. acknowledges support from NSF award No. ESI-0337286.

APPENDIX A: PROJECT: THE RIVER MODEL OF BLACK HOLES

The following project has been tested and refined over a period of several years in undergraduate classes on relativity and black holes at both lower-division nonscience-major and upper-division science-major levels. It was designed as a 45 min in-class group project in which students split into groups of 3 or 4, and by discussing with each other arrive at consensus answers to a series of concept questions. At the end of the project each group submits its answers for grade.

According to the river model of black holes, the behavior of objects near black holes is as if space were falling like a river into the black hole. For spherical black holes this model was discovered in 1921 by the Nobel prize winner Allvar Gullstrand[1] and independently by the mathematician Paul Painlevé.[2] In the model space falls inward at the Newtonian escape velocity $v=\sqrt{2GM/r}$. The infall velocity is less than the speed of light c outside the horizon, equals the speed of light c at the horizon, and exceeds the speed of light c inside the horizon.

What does the river model predict for the following questions? [For nonscience majors, use only the unstarred questions. For more advanced, science-major students, use all questions and drop or abbreviate the hints.]

Questions

*1. What radius does the river model predict for the horizon of a black hole?

2. Suppose that you are a light beam (therefore moving at the speed of light) exactly at the horizon. What would happen to you if were pointed directly outward? [Do you fall in? Do you move out? Do you move sideways?] What would happen to you if you were pointed mostly but not exactly outward?

3. In what way, if any, does this behavior differ from the predictions of the Newtonian corpuscular theory of light, which in the hands of John Michell in 1784[6] gave the "correct" result for the radius of the horizon? [In the corpuscular theory of light, a corpuscle of light is emitted at the speed of light, and behaves thereafter much like a massive particle: it moves outward and either goes to infinity or turns around and comes back depending on whether its initial velocity, the speed of light, is more or less than the escape velocity.]

4. Suppose that you are a light beam orbiting the black hole in a circular orbit. On this orbit, the "photon sphere," are you at the horizon, inside the horizon, or outside the horizon? Justify your answer.

5. Make a connection between the appearance of the sky if you hover just above the horizon of a black hole and special relativistic beaming. [How does a scene appear if you move through it at very close to the speed of light?]

6. Qualitatively, what would the river model predict for the tidal forces experienced by an infalling observer? [First, consider the tidal force in the vertical direction. Think about the fact that the river is accelerating inward. Next, consider the tidal force in the horizontal direction. Think about the fact that the river is converging (getting narrower) as it flows inward.]

*7. How does the river model account for redshifting and freezing at the horizon?

*8. Given that one of the fundamental propositions of special and general relativity is that spacetime has no absolute existence, what does it mean to say that space is falling into a black hole?

*9. In the river model the flow of space accelerates inward to the black hole. If the river were moving uniformly instead of accelerating, would there be any gravity?

Answers

1. The river velocity equals the speed of light when $(2GM/r_s)^{1/2}=c$, which rearranges to an expression for the radius of the event horizon, the Schwarzschild radius r_s,

$$r_s = \frac{2GM}{c^2}. \tag{A1}$$

2. If you were a light beam pointed directly outward at the horizon, then you would hang forever at the horizon. Your outward motion at the speed of light would be exactly canceled by the inward motion of the river of space at the speed of light. If you were a light beam not exactly pointed outward, then the outward component of your velocity would be a bit less than the speed of light, because part of your velocity would be sideways. The inflow of space would then carry you into the black hole.

3. Whereas in general relativity an outwardly pointed light beam at the horizon hangs there motionless forever, in the classical corpuscular theory the light never remains at rest. The light either keeps going outward forever (if its velocity exceeds the escape velocity), or it turns around and comes back. It is true that the light is motionless at the instant of turnaround, but otherwise the light is always moving. Another difference is that in general relativity the question of whether a light beam can escape from a point just above the horizon depends on the direction in which the light beam is pointed. If the light beam is pointed directly outward, then it will escape, but if it is pointed somewhat sideways, then it will fall into the black hole. In contrast, in the classical corpuscular theory, whether a corpuscle escapes from a given point depends only on whether its velocity exceeds the escape velocity, not on the direction in which it is pointed.

4. You cannot be at the horizon, because if you had any sideways motion, which you must because you are in circular orbit, then the inflow of space would drag you into the black hole. And you cannot be inside the horizon, because the inflow of space would again drag you inward. Therefore you must be in circular orbit somewhere outside the horizon. For a Schwarzschild black hole, the radius of the photon sphere turns out to be 1.5 Schwarzschild radii.

5. If you move through a scene at very close to the speed of light, then the scene ahead of you, in the direction you are moving, appears concentrated, brightened, blueshifted,

and speeded up. If you hover just above the horizon of a black hole, then according to the river model you must be moving very rapidly through the inflowing river of space. Consequently the view above you must appear concentrated, brightened, blueshifted, and speeded up. Hovering just above the horizon of a black hole is an unnatural and wasteful thing to do. In reality, you would surely "go with the flow" of space. If you free fall into a black hole, then you do not see the sky highly concentrated above you.

6. Because the river is accelerating inward, the velocity of the river is faster at your feet than at your head (presuming that you are upright, so that your feet are closer to the black hole than your head). The difference in river velocity means that you feel a tidal force in the vertical direction pulling your feet away from your head. In the horizontal direction the river is converging spherically toward the black hole, so you would feel tidally squashed in the horizontal direction.
7. Just above the horizon, a photon battling against the inrushing torrent of space takes a long time to reach an outside observer. As the emitter gets closer to the horizon, it takes longer and longer for the photon to get out, until at the horizon it takes an infinite time for a photon to lift off the horizon. Thus as an object approaches the horizon, it appears to an outside observer slower and slower, and thus more and more redshifted. Asymptotically, the object appears to freeze on the horizon, and the redshift goes to infinity.
8. The river model consists of a set of coordinates (the background) and a set of locally inertial frames that flow through those coordinates (the river that flows through the background). Attaching a set of coordinates and a set of locally inertial frames does not make the spacetime absolute.
9. According to the principle of equivalence, a gravitating frame is equivalent to an accelerating frame. So if there is no acceleration, then there is no gravity. If the river is falling at constant velocity in the vertical direction but still converging horizontally because of the spherical convergence of the flow, then you would feel a tidal squashing in the horizontal direction, so in that case there would be gravity.

[a]Author to whom correspondence should be addressed. Electronic mail: andrew.hamilton@colorado.edu

[1]A. Gullstrand, "Allgemeine Lösung des statischen Einkörperproblems in der Einsteinschen Gravitationstheorie" ("General solution of the static body problem in Einstein's gravitation theory")," Ark. Mat., Astron. Fys. **16**(8), 1–15 (1922).

[2]P. Painlevé, "La mécanique classique et la théorie de la relativité," C.R. Acad. Sci. (Paris) **173**, 677–680 (1921).

[3]K. Schwarzschild, "Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie," Sitzungsberichte der Deutschen Akademie der Wissencshaften zu Berlin, Klasse für Mathematik, Physik, und Technik **1**, 189–196 (1916). For an English translation see "On the gravitational field of a mass point according to Einstein's theory," Gen. Relativ. Gravit. **35**, 951–959 (2003).

[4]T. Rothman, "Editor's note: The field of a single centre in Einstein's theory of gravitation, and the motion of a particle in that field," Gen. Relativ. Gravit. **34**, 1541–1543 (2002) comments that the Schwarzschild metric was discovered independently by Johannes Droste in 1916.

[5]Tanmay Vachaspati, "Cosmic problems for condensed matter experiment," J. Low Temp. Phys. **136**, 361–377 (2004).

[6]J. Michell, "On the means of discovering the distance, magnitude, etc., of the fixed stars, in consequence of the diminution of their light, in case such a diminution should be found to take place in any of the them, and such other data should be procured from observations, as would be futher necessary for that purpose," Philos. Trans. R. Soc. London **74**, 35–57 (1784).

[7]P.-S. Laplace, "Proof of the theorem, that the attractive force of a heavenly body could be so large, that light could not flow out of it," in *Allgemeine Geographische Ephemeriden, verfasset von Einer Gesellschaft Gelehrten, 8vo Weimer, IV, I. Bd St*, edited by F. X. von Zach (1799); An English translation is in Stephen W. Hawking and George F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, 1973), Appendix A.

[8]B. Greene, *The Fabric of the Cosmos: Space, Time, and the Texture of Reality* (A. A. Knopf, New York, 2004), p. 237.

[9]M. Visser, "Acoustic black holes: Horizons, ergospheres, and Hawking radiation," Class. Quantum Grav. **15**, 1767–1791 (1998).

[10]M. Visser, "Heuristic approach to the Schwarzschild geometry," Int. J. Mod. Phys. D **14**, 2051–2068 (2005).

[11]K. Martel and E. Poisson, "Regular coordinate systems for Schwarzschild and other spherical spacetimes," Am. J. Phys. **69**, 476–480 (2001).

[12]W. Laschkarew, "Zure Theorie der Gravitation," Z. Phys. **35**, 473–476 (1926). A translation by babelfish.altavista.com/tr of the abstract is "The Einstein equivalence principle can be satisfied from the point of view of an aether theory, if one permits accelerated aether currents. All consequences of the Einstein theory of gravitation are derived in an elementary way by connection of the theory of special relativity with the above hypothesis."

[13]G. Lemaître, "L'univers en expansion," Ann. Soc. Sci. (Bruxelles) A **53**, 51–85 (1933).

[14]A. Trautman, "Comparison of Newtonian and relativistic theories of space-time," in *Perspectives in Geometry and Relativity; Essays in Honor of Václav Hlavatý*, edited by B. Hoffmann (Indiana University Press, Bloomington, 1966), pp. 413–425.

[15]H. P. Robertson and Thomas W. Noonan, *Relativity and Cosmology* (Saunders, Philadelphia, 1968).

[16]F. Estabrook, H. Wahlquist, S. Christensen, B. DeWitt, L. Smarr, and E. Tsiang, "Maximally slicing a black hole," Phys. Rev. D **7**, 2814–2817 (1973).

[17]R. Gautreau and B. Hoffmann, "The Schwarzschild radial coordinate as a measure of proper distance," Phys. Rev. D **17**, 2552–2555 (1978).

[18]R. Gautreau, "Light cones inside the Schwarzschild radius," Am. J. Phys. **63**, 431–439 (1995).

[19]P. Kraus and F. Wilczek, "A simple stationary line element for the Schwarzschild geometry, and some applications," Mod. Phys. Lett. A **9**, 3713–3719 (1994).

[20]K. Lake, "A Class of quasistationary regular line elements for the Schwarzschild geometry," gr-qc/9407005.

[21]A. Lasenby, C. Doran, and S. Gull, "Gravity, gauge theories and geometric algebra," Philos. Trans. R. Soc. London, Ser. A **356**, 487–582 (1998).

[22]P. Nurowski, E. Schucking, and A. Trautman, "Relativistic gravitational fields with close Newtonian analogs," in *On Einstein's Path: Essays in Honor of Engelbert Schucking*, edited by Alex Harvey (Springer, New York, 1999), pp. 329–337.

[23]J. Czerniawski, "What is wrong with Schwarzschild's coordinates?," Concepts Phys. **3**, 307–318 (2006).

[24]E. F. Taylor and J. A. Wheeler, *Exploring Black Holes: Introduction to General Relativity* (Addison–Wesley Longman, San Francisco, 2000).

[25]K. S. Thorne, R. H. Price, and D. A. MacDonald, *Black Holes: The Membrane Paradigm* (Yale University Press, New Haven, 1986).

[26]R. Arnowitt, S. Deser, and C. W. Misner, "The dynamics of general relativity," in *Gravitation, An Introduction to Current Research*, edited by Louis Witten (Wiley, New York, 1962), pp. 227–265.

[27]L. Lehner, "Numerical relativity: A review," Class. Quantum Grav. **18**, R25–86 (2001).

[28]M. Alcubierre, "The warp drive: Hyper-fast travel within general relativity," Class. Quantum Grav. **11**, L73–L79 (1994).

[29]F. S. N. Lobo and M. Visser, "Fundamental limitations on 'warp drive' spacetimes," Class. Quantum Grav. **21**, 5871–5892 (2004).

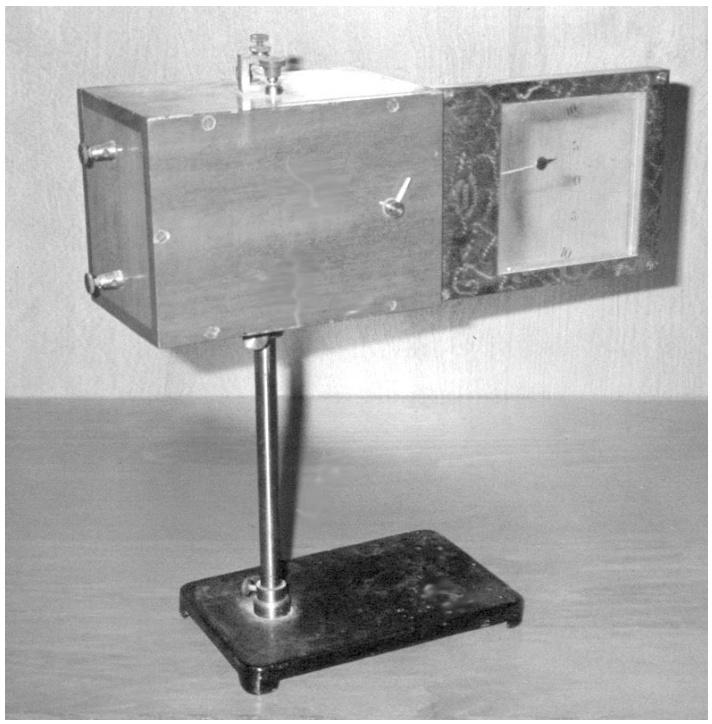
[30]W. G. Unruh, "Experimental black hole evaporation," Phys. Rev. D **14**, 1351–1353 (1981).

[31]C. Barceló, S. Liberati, and M. Visser, "Analogue gravity," Living Rev. Relativ. **8**, 12 (2005),⟨www.livingreviews.org/Irr-2005-12⟩.

[32]J. Natário, "Newtonian limits of warp drive spacetimes," Gen. Rel. Grav. **38**, 475–484 (2006).

[33]M. Visser and S. E. Ch. Weinfurtner, "Vortex analogue for the equatorial geometry of the Kerr black hole," Class. Quantum Grav. **22**, 2493–2510

(2005).
[34] A. Garat and R. H. Price, "Nonexistence of conformally flat slices of the Kerr spacetime," Phys. Rev. D **61**, 124011-1–4 (2000).
[35] J. A. Valiente Kroon, "Nonexistence of conformally flat slices in Kerr and other stationary spacetimes," Phys. Rev. Lett. **92**, 041101-1–4 (2004).
[36] C. Doran, "A new form of the Kerr solution," Phys. Rev. D **61**, 067503-1–4 (2000).
[37] G. B. Cook, "Initial data for numerical relativity," Living Rev. Relativ. **3**, 2000-5 (2000).
[38] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, New York, 1973).
[39] C. W. Misner and D. H. Sharp, "Relativistic equations for adiabatic, spherically symmetric gravitational collapse," Phys. Rev. **136**, B571–B576 (1964).
[40] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (Wiley, New York, 1972), p. 300.
[41] R. Penrose, "Structure of space-time," in *Battelle Rencontres: 1967 Lectures in Mathematics and Physics*, edited by C. M. de Witt and J. A. Wheeler (W. A. Benjamin, New York, 1968), pp. 121–235.
[42] E. Poisson and W. Israel, "Internal structure of black holes," Phys. Rev. D **41**, 1796–1809 (1990).
[43] A. J. S. Hamilton and S. E. Pollack, "Inside charged black holes I. Baryons," Phys. Rev. D **71**, 084031-1–29 (2005); A. J. S. Hamilton and S. E. Pollack, "Inside charged black holes II. Baryons plus dark matter," *ibid.* 084032-1–18 (2005).
[44] S. Gull, A. Lasenby, and C. Doran, "Imaginary numbers are not real – the geometric algebra of spacetime," Found. Phys. **23**, 1175–1201 (1993),⟨www.ucl.ac.uk/~ucesjph/reality/ga/intro.html⟩; C. Doran and A. Lasenby, *Geometric Algebra for Physicists* (Cambridge University Press, Cambridge, 2003).
[45] R. H. Boyer and R. W. Lindquist, "Maximal analytic extension of the Kerr metric," J. Math. Phys. **8**, 265–281 (1967).
[46] A local gauge transformation is a Lorentz transformation that varies from place to place.



Projection Galvanometer. Lecture table meters can only be made so large. The alternative is to go small, and display the readings by projection. This model, at Franklin and Marshall College in Lancaster, Pennsylvania, was made by James W. Queen of Philadelphia. It is probably designed to slide into a large slide projector, but the careful design of the mechanism holding it up suggests that it might also have been used for shadow projection, using a carbon arc as a light source. It is listed at $22 in the 1916 catalogue of the L.E. Knott Apparatus Co. of Boston. (Photograph and Notes by Thomas B. Greenslade, Jr., Kenyon College)