The Gravitational Energy of a Black Hole

Yuan K. Ha
Department of Physics, Temple University
Philadelphia, Pennsylvania 19122 U.S.A.
yuanha@temple.edu

General Relativity and Gravitation, Vol.35, No.11 (2003) p.2045-50 Abstract

An exact energy expression for a physical black hole is derived by considering

the escape of a photon from the black hole. The mass of the black hole within

its horizon is found to be twice its mass as observed at infinity. This result

is important in understanding gravitational waves in black hole collisions.

Keywords: Black hole; Gravitational energy

2

What is the energy of a black hole? This is a question which appears should have a simple answer. It is reasonable to conclude that the energy of a black hole is that which corresponds to its mass as determined by a distant observer by watching a satellite undergoing an orbiting motion around the black hole, using the equations of general relativity. This has been the empirical way of finding the mass of a planet or a star. The mass obtained in this way is the total mass of the system as seen by a distant observer. For a physical black hole, it is the net mass obtained from the difference between the constituent mass of the black hole and its gravitational energy. Since gravitational energy is known to be negative, therefore the constituent mass must be greater than the observed mass for the black hole.

To understand the nature of mass of a black hole, it is necessary to know the energy distribution of the black hole throughout all space. As the gravitational field of a black hole extends to infinity, its potential energy extends similarly and contributes also to its observed mass. The concept of a black hole comes from the Schwarzchild solution to Einstein's equation [1]. A Schwarzchild black hole has a mass M and a radius R_S according to a distant observer stationed at infinity. In this paper, the total energy expression for a nonrotating black hole including its gravitational energy is derived in a simple and physical way by considering the escape of a photon just outside the surface of a black hole in a gedanken experiment similar to the Hawking

process [2].

When a photon of a given energy is emitted just outside the horizon of a black hole it will have zero energy as it reaches infinity. This means the entire energy of the photon is used to escape the gravitational pull of the black hole. If the photon comes from the annihilation of a particle of mass m near the horizon, then it means the entire mass of the particle is used to make the photon escape from the black hole. This also means that the energy required to remove a mass m just outside the horizon to infinity is simply mc^2 . Now imagine that a mass m is removed from the horizon to infinity very slowly by an external agent so that no kinetic energy is generated in the process, the energy required to do this is still mc^2 . Eventually, the mass removed will reach infinity as a free mass. Consider next a particle of mass m being produced just outside the horizon and which has sufficient energy to escape to infinity on its own where it ends up as a free particle of mass m. The above consideration shows that the total energy required for this event is simply $2mc^2$. As a result, the black hole will lose energy by the same amount $2mc^2$ for each particle of mass m released at the horizon and observed at infinity. This energy is independent of the mass of the black hole. After a succession of processes in this manner, the entire black hole is transformed into asymptotic particles at infinity. If the total mass of the particles observed at infinity is M, then the original mass inside the black hole must

be equal to 2M, half of which is used to supply the gravitational energy of these particles, which is also the gravitational potential energy of the black hole itself. This is a remarkable result. Thus from the point of view of a distant observer, the constituent mass of the black hole is 2M, even though its observed mass is just M. This observed mass at infinity corresponds to the Arnowitt-Deser-Misner mass [3], which is a measure of the total energy of a gravitational system at spatial infinity in general relativity. A black hole thus has the maximum gravitational energy any system can have.

We therefore introduce the concept of the horizon mass and state the following theorem on the energy of a black hole:

If M is the mass of a black hole within its horizon, then its energy observed at infinity is $E = \frac{1}{2}Mc^2$.

Let us incorporate the above result into a mathematical formula. Far from the black hole, an observer should find a point mass M and the spacetime is the one described by the Schwarzschild metric. If a photon is emitted at coordinate r with energy ε_r and later observed at infinity, its energy there is given by

$$\varepsilon_{\infty} = \varepsilon_r \sqrt{1 - \frac{2GM}{rc^2}},\tag{1}$$

where G is the gravitational constant and c is the speed of light. The differ-

ence between the energies of the photon at the two locations is therefore

$$\varepsilon_r - \varepsilon_\infty = \varepsilon_r \left[1 - \sqrt{1 - \frac{2GM}{rc^2}} \right].$$
 (2)

The change in the photon's energy is a measure of the change of the gravitational potential energy of the black hole as a function of the coordinate r. Next, to describe the complete behavior of the energy of the black hole itself, we introduce a function f(r) interpolating between the surface of the black hole and infinity so that the energy of the black hole also becomes a function of the coordinate r. This energy expression gives the total energy of the black hole contained in a spherical volume from the origin up to the coordinate r and is given by

$$E(r) = f(r) \left[1 - \sqrt{1 - \frac{2GM}{rc^2}} \right]. \tag{3}$$

To determine the function f(r), we set the following conditions:

- 1. The total energy E(r) is always positive. Thus f(r) must be a positive function between R_S and ∞ .
- 2. The total energy E(r) decreases smoothly between R_S and ∞ . Thus its derivative dE/dr is always negative.
- 3. At large distances, the total energy E(r) approaches an asymptotic value. Thus $dE/dr \simeq 0$ at very large distances.

Taking the derivative dE/dr in Eq.(3) and subjecting it to the above conditions, we find at large distances an equation for f(r),

$$\frac{df(r)}{dr} = \frac{1}{r}f(r) \tag{4}$$

The solution is found to be $f(r) = \text{constant} \times r$.

To determine the constant, we notice at large distances, the square root in Eq.(3) expands as $\sqrt{1-2GM/rc^2} \simeq 1-GM/rc^2$, the energy of the black hole should approach the asymptotic value Mc^2 as seen by the distant observer. Thus,

$$E(r) \simeq f(r) \left(\frac{GM}{rc^2}\right) \to Mc^2, \quad r \to \infty,$$
 (5)

giving finally

$$f(r) = \frac{rc^4}{G}. (6)$$

The overall energy expression for the black hole is now

$$E(r) = \frac{rc^4}{G} \left[1 - \sqrt{1 - \frac{2GM}{rc^2}} \right].$$
 (7)

With this result, we recover the energy of the black hole inside the Schwarzschild horizon as concluded earlier by the distant observer. Setting $r = R_S = 2GM/c^2$, we obtain from Eq.(7)

$$E(r = R_S) = \left(\frac{2GM}{c^2}\right)\frac{c^4}{G} = 2Mc^2.$$
 (8)

The expression given by Eq.(7) agrees with the analysis of the quasilocal energy of the Schwarzchild solution by Brown and York [4], and also agrees with the calculation of the energy in a black hole in the teleparallel equivalent formulation of general relativity by Maluf [5]. Those developments are however more mathematical and framework dependent than the present physical approach. The significance of the present result is that the total energy of a black hole can be found in general relativity without requiring the use of any illusive local gravitational energy density at all [6].

Figure 1 shows the variation of the mass of a black hole starting at $r = R_S$ to $r = 10R_S$, using the mass equivalence of Eq.(7). As can be seen, the mass decreases quickly from 2M at R_S and levels off to slightly above M at $10R_S$. At large distances, the mass is practically indistinguishable from its asymptotic value M. However, at close distances, the mass is quite different from M as seen by the distant observer. Here the mass function is defined by

$$M' = M'(r) = \frac{rc^2}{G} \left[1 - \sqrt{1 - \frac{2GM}{rc^2}} \right].$$
 (9)

An important consequence of the black hole energy formula is in understanding black hole collisions. Consider the following example. When a black hole of asymptotic mass 5M collides with a black hole of asymptotic mass 12M, the minimum result is a black hole of asymptotic mass 13M. This

follows from the area non-decrease theorem for black holes. The area of a black hole $A=4\pi R_S^2$ is proportional to the square of its asymptotic mass. Therefore, according to a distant observer watching the collision, the amount of mass radiated away during the collision process in the form of gravitational waves is (5M+12M)-13M=4M.

Without knowing the black hole energy formula in Eq. (7), a local observer close to the collision process believes that the above result is always correct. This local horizon observer firmly believes that the horizon mass is the same as the asymptotic mass because he cannot detect any measurable changes in particle motions outside a black hole even if he were told that the horizon mass is different from the asymptotic mass. Any particle motion is determined completely by the Schwarzschild metric based on the asymptotic mass. Thus the local horizon observer calculates his own orbit near the black hole based on the Schwarzchild metric and readily concludes that the mass of the black hole is the same as when he started out from infinity. He cannot justifiably accept any other result. But with the knowledge of the black hole energy formula, we can understand the collision better. The collision involves a black hole of horizon mass 10M with a black hole of horizon mass 24M, resulting in a black hole of horizon mass 26M, again following the area nondecrease theorem. Therefore the total mass radiated away in the collision process is (10M + 24M) - 26M = 8M. This is twice the amount as that

concluded by the distant observer, and also twice the amount concluded by the local horizon observer. Where has the extra mass 4M gone to?

If one believes that gravitational waves are responsible for the difference in mass of the black holes before and after the collision, then this means that an additional energy of the amount 4M is required to allow these waves to propagate from the final black hole to infinity for the distant observer. This is because when gravitational waves of mass 4M reach infinity they will have gained potential energy of the equal amount 4M. Energy is inertia. The total energy lost from the final black hole is hence 8M, consistent with our above observation. If the local horizon observer was correct, there would be no change in the potential energy of the gravitational waves at all. The gravitational waves in this case cannot propagate away from the black hole. Therefore in detecting any gravitational signal from a black hole collision such as that proposed in the LIGO project, any conclusion about the strength of the signals near its source should be based on the black hole energy formula. Understanding the collisions of black holes in galaxies is one of the outstanding problems in cosmology.

References

- K. Schwarzchild, Sitzber. Deut. Akad. Wiss. Berlin, KL. Math.-Phys. Tech., (1916) 189.
- [2] S. Hawking, Commun. Math. Phys. **43**, (1975) 199.
- [3] R. Arnowitt, S. Deser and C.W. Misner, in *Gravitation: An Introduction to Current Research*, (ed. L. Witten, Wiley, New York 1962).
- [4] J.D. Brown and J.W. York, Jr., Phys. Rev. D47, (1993) 1407.
- [5] J.W. Maluf, J. Math. Phys. **36**, (1995) 4242.
- [6] C.C. Chang, J.M. Nester and C.M. Chen, Phys. Rev. Letts. 83, (1999) 1897.

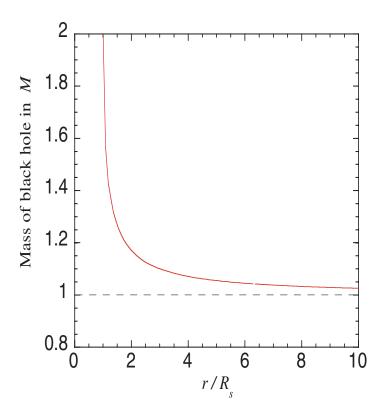


Figure 1. Mass of black hole as a function of radial coordinates