# Stability of a vacuum nonsingular black hole 

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#### Abstract

This is the first of series of papers in which we investigate stability of the spherically symmetric space-time with de Sitter center. Geometry, asymptotically Schwarzschild for large $r$ and asymptotically de Sitter as $r \rightarrow 0$, describes a vacuum nonsingular black hole for $m \geq m_{c r}$ and particle-like self-gravitating structure for $m<m_{c r}$ where a critical value $m_{c r}$ depends on the scale of the symmetry restoration to de Sitter group in the origin. In this paper we address the question of stability of a vacuum non-singular black hole with de Sitter center to external perturbations. We specify first two types of geometries with and without changes of topology. Then we derive the general equations for an arbitrary density profile and show that in the whole range of the mass parameter $m$ objects described by geometries with de Sitter center remain stable under axial perturbations. In the case of the polar perturbations we find criteria of stability and study in detail the case of the density profile $\rho(r)=\rho_{0} e^{-r^{3} / r_{0}^{2} r_{g}}$ where $\rho_{0}$ is the density of de Sitter vacuum at the center, $r_{0}=\sqrt{3 / \kappa \rho_{0}}$ is de Sitter radius and $r_{g}$ is the Schwarzschild radius.


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## I. INTRODUCTION

The idea of replacing of a Schwarzschild singularity with de Sitter vacuum goes back to 1965 papers of Sakharov [1] who considered $p=-\rho$ as the equation of state for superhigh density and of Gliner who interpreted $p=-\rho$ as corresponding to a vacuum and suggested that it could be a final state in a gravitational collapse [2].

In 1968 Bardeen presented the spherically symmetric metric of the same form as Schwarzschild and ReissnerNordström metric, describing a non-singular black hole ( BH ) without specifying the behavior at the center [3]. The very important point was noted in [3] for the first time: that the considered space-time exhibits the smooth changes of topology.

Direct matching of Schwarzschild metric to de Sitter metric within a short transitional space-like layer of the Planckian depth [4-8] results in metrics typically with a jump at the junction surface.

The situation with transition to de Sitter as $r \rightarrow 0$, has been analyzed in 1988 by Poisson and Israel who found necessary to introduce a transitional layer of "noninflationary material" of uncertain depth at the characteristic scale $\left(r_{0}^{2} r_{g}\right)^{1 / 3}\left(r_{0}\right.$ is de Sitter radius, and $r_{g}$ is the Schwarzschild radius), where geometry can be selfregulatory and describable semiclassically down a few Planckian radii by the Einstein equations with a source term representing vacuum polarization effects [9].

Generic properties of "noninflationary material" have been considered in 1990 in Ref. [10]. For a smooth de Sitter-Schwarzschild transition a source term satisfies [10]

$$
\begin{equation*}
T_{t}^{t}=T_{r}^{r} ; \quad T_{\theta}^{\theta}=T_{\phi}^{\phi} \tag{1.1}
\end{equation*}
$$

and the equation of state, following from $T_{\nu ; \mu}^{\mu}=0$, is

$$
\begin{equation*}
p_{r}=-\rho ; \quad p_{\perp}=-\rho-\frac{r}{2} \rho^{\prime} \tag{1.2}
\end{equation*}
$$

Here $\kappa=8 \pi G$ (we adopted $c=1$ for simplicity), $\rho(r)=T_{t}^{t}$ is the energy density, $p_{r}(r)=-T_{r}^{r}$ is the radial pressure, and $p_{\perp}(r)=-T_{\theta}^{\theta}=-T_{\phi}^{\phi}$ is the tangential pressure for anisotropic perfect fluid [11].

The stress-energy tensor with the algebraic structure (1.1) has an infinite set of comoving reference frames and is identified therefore as describing a spherically symmetric vacuum [10], invariant under boosts in the radial direction and defined by the symmetry of its stress-energy tensor (for review [12-16]).

The exact analytical solution was found in 1990 for the case of the density profile [10]

$$
\begin{equation*}
\rho(r)=\rho_{0} e^{-r^{3} / r_{0}^{2} r_{g}} ; \quad r_{0}^{2}=3 / \kappa \rho_{0} ; \quad r_{g}=2 G m \tag{1.3}
\end{equation*}
$$

which describes a vacuum asymptotically de Sitter as $r \rightarrow$ 0 in a simple semiclassical model for vacuum polarization in the spherically symmetric gravitational field [17].

In 1991 Morgan has considered a black hole in a simple model for quantum gravity with quantum effects represented by an upper cutoff on the curvature, and obtained de Sitter-like past and future cores replacing singularities [18]. In 1992 Strominger demonstrated the possibility of natural, not ad hoc, arising of de Sitter core inside a black hole in the model of two-dimensional dilaton gravity conformally coupled to $N$ scalar fields [19].

In 1996 it was shown that in the course of Hawking evaporation a vacuum nonsingular black hole evolves towards a self-gravitating particle-like vacuum structure
without horizons [17], kind of gravitational vacuum soliton called G-lump [20]. The form of temperature-mass diagram is generic for de Sitter-Schwarzschild black hole [17] and dictated by the Schwarzschild asymptotic and by the existence of two horizons - when decreasing during evaporation mass reaches a certain critical value $m_{c r}$ evaporation stops [17,21].

In 1997 in Ref. [22] it was shown that in a large class of space-times that satisfy the Weak Energy Condition (WEC), the existence of a regular black hole requires topology change. Bardeen metric and the metric generated by the density profile (1.3) belong to this class.

In 2000 studying the quantum gravitational effects by the effective average action with the running Newton constant, and improving Schwarzschild black hole with renormalization group, Bonnano and Reuter [23] have constructed nonsingular black hole metric and confirmed the results of [17] concerning the form of temperaturemass diagram and the fundamental fact that evaporation stops when the mass approaches the critical value $m_{c r}$.

Also in 2000 the regular BH solution with a charged de Sitter core has been considered by Kao [24] with using the density profile (1.3) for distribution of a charged material.

In the same year 2000 regular magnetic black hole and monopole solutions are found by Bronnikov [25] in Nonlinear Electrodynamics (NED) coupled to gravity with the stress-energy tensor of the structure (1.1).

Existence of regular electrically charged structures in nonlinear electrodynamics coupled to general relativity was proved recently in Ref. [26], where it was shown that in NED coupled to GR and satisfying WEC, regular charged structures must have de Sitter center.

In 2001 the non-singular quasi-black-hole model representing a compact object without horizons, was constructed by Mazur and Mottola [27] by extending the concept of Bose-Einstein condensation to gravitational systems. An interior de Sitter condensate phase is matched to an exterior Schwarzschild geometry of arbitrary mass through a phase boundary of a small but finite thickness with equation of state $p=\rho$.

In 2002 nonsingular BH solution was found by Nashed [28] as a general solution of Möller tetrad theory of gravitation by assuming the same specific form of the vacuum stress-energy tensor as in Ref. [10] with the density profile (1.3). Later it was extended to the case of teleparallel theory of gravitation [29]. Stability condition of geodesic motion in the field of vacuum nonsingular black hole described by the regular analytic solution [10] with the density profile (1.3), has been considered in [30].

Model-independent analysis of the Einstein spherically symmetric minimally coupled equations has shown [20,31] which geometry they can describe if certain general requirements are satisfied: (a) regularity of density; (b) finiteness of the ADM mass; (c) Dominant Energy Condition (DEC) for $T_{\mu \nu}$. These conditions lead to existence of regular structures with de Sitter center including
regular black holes without topological changes. The example of such a case is the exact analytic solution [26] describing in certain mass range a regular charged black hole with de Sitter center.

The condition (c) can be loosed to (c2): weak energy condition for $T_{\mu \nu}$ and regularity of pressures [31]. WEC which is contained in DEC, in both cases is needed for de Sitter asymptotic at the center.

The requirements (a)-(c) either (a)-(c2) define the family of asymptotically flat solutions with the regular center which includes the class of metrics asymptotically de Sitter as $r \rightarrow 0$. A source term connects de Sitter vacuum in the origin with the Minkowskli vacuum at infinity. Spacetime symmetry changes smoothly from de Sitter group at the center to the Poincare group at infinity through the radial boosts in between, and the standard formula for the ADM mass relates it to both de Sitter vacuum trapped inside an object and smooth breaking of spacetime symmetry [20].

Cases (c)-(c2) differ by behavior of the curvature scalar $R$. In the case (c) it is non-negative which evidences the existence of regular black holes without topological changes. So, the class of metrics with de Sitter center includes two subclasses with and without topological changes.

In this paper we specify conditions of existence of two types of geometries with the de Sitter center and investigate stability of configurations described by these geometries, by studying perturbations in geometry via Einstein equations linearized about the unperturbed space-time. Results are valid for geometries of both types.

This paper is organized as follows. In Sect. 2 we outline the conditions of existence and basic properties of spherically symmetric geometries with de Sitter center. In Sect. 3 we introduce the basic equations describing axially symmetric time-dependent perturbations of a spherically symmetric system with de Sitter center. In Sect. 4 we prove stability of such a system to axial perturbations. In Sect. 5 we analyze the case of polar perturbations and derive criteria of stability to these perturbations. In Sect. 6 we apply the results to the case of the vacuum nonsingular black hole with the density profile (1.3). Section 7 contains summary and discussion.

## II. SPHERICALLY SYMMETRIC SPACE-TIME WITH DE SITTER CENTER

A static spherically symmetric line element can be written in the form [11]

$$
\begin{equation*}
d s^{2}=e^{\mu(r)} d t^{2}-e^{\nu(r)} d r^{2}-r^{2} d \Omega^{2} \tag{2.1}
\end{equation*}
$$

where $d \Omega^{2}$ is the metric of a unit 2 -sphere. Integration of the Einstein equations gives

$$
\begin{equation*}
e^{-\nu(r)}=1-\frac{2 G M(r)}{r} ; \quad M(r)=4 \pi \int_{0}^{r} \rho(x) x^{2} d x \tag{2.2}
\end{equation*}
$$

whose asymptotic for large $r$ is $e^{-\nu}=1-2 G m / r$, with the mass parameter

$$
\begin{equation*}
m=4 \pi \int_{0}^{\infty} \rho(r) r^{2} d r \tag{2.3}
\end{equation*}
$$

Requirement of regularity of density, $\rho_{0}=\rho(r \rightarrow 0)<\infty$, leads to behavior of mass function $M(r) \sim r^{3}$ as $r \rightarrow 0$ and thus $\nu(0)=0$.

To outline the conditions of existence of spherically symmetric space-time with de Sitter center, we need the Oppenheimer equation [32]

$$
\begin{equation*}
T_{t}^{t}-T_{r}^{r}=p_{r}+\rho=\frac{1}{\kappa} \frac{e^{-\nu}}{r}\left(\nu^{\prime}+\mu^{\prime}\right) \tag{2.4}
\end{equation*}
$$

The dominant energy condition $T^{00} \geq\left|T^{a b}\right|$ for each $a, b=1,2,3$, holds if and only if [33] $\rho \geq 0 ; \rho+p_{k} \geq$ $0 ; \rho-p_{k} \geq 0 ; k=1,2,3$. It includes the weak energy condition which implies $\rho \geq 0 ; \rho+p_{k} \geq 0$. Together with the condition of regularity of density, DEC (via $p_{k} \leq \rho$ ) leads to $\mu^{\prime}+\nu^{\prime}=0$ as $r \rightarrow 0$ [20].

The same result can be achieved by requirement of regularity of pressure (the subclass satisfying (a-c2)).

In the limit $r \rightarrow \infty$ the condition of finiteness of the mass (2.3) requires density profile $\rho(r)$ to vanish at infinity quicker than $r^{-3}$. In the case (c) the dominant energy condition requires pressures to vanish as $r \rightarrow \infty$. Then $\mu^{\prime}=0$ and $\mu=$ const at infinity. Rescaling the time coordinate allows one to put the standard boundary condition $\mu \rightarrow 0$ as $r \rightarrow \infty$ which ensures asymptotic flatness needed to identify (2.3) as the ADM mass [34].

The same result can be achieved in the case (c2) by postulating regularity of pressures including vanishing of $p_{r}$ at infinity sufficient to get $\mu^{\prime}=0$ needed for asymptotic flatness.

The weak energy condition requires $\mu^{\prime}+\nu^{\prime} \geq 0$. The function $\mu+\nu$ is growing from $\mu=\mu(0)$ at $r=0$ to $\mu=0$ at $r \rightarrow \infty$, which gives $\mu(0) \leq 0$ [20].

The range of family parameter $\mu(0)$ includes $\mu(0)=0$. In this case the function $\nu(r)+\mu(r)$ is zero at $r=0$ and at $r \rightarrow \infty$, its derivative is non-negative (by WEC via $\rho+p_{k} \geq 0$ ), it follows that $\nu(r)=-\mu(r)$ everywhere.

A source term for this class of metrics corresponds to anisotropic perfect fluid which satisfies the $r$-dependent equation of state (1.2), and the weak energy condition $p_{\perp}+\rho \geq 0$ demands monotonic decreasing of a density profile, $\rho^{\prime} \leq 0$ [20].

Behavior at $r \rightarrow 0$ is dictated by the WEC [20]. The equation of state near the center becomes $p=-\rho$, which gives de Sitter asymptotic as $r \rightarrow 0$

$$
\begin{gather*}
d s^{2}=\left(1-\frac{r^{2}}{r_{0}^{2}}\right) d t^{2}-\frac{d r^{2}}{\left(1-\frac{r^{2}}{r_{0}^{2}}\right)}-r^{2} d \Omega^{2}  \tag{2.5}\\
T_{\mu \nu}=\rho_{0} g_{\mu \nu} ; \quad r_{0}^{2}=\frac{3}{\Lambda} ; \quad \Lambda=\kappa \rho_{0} \tag{2.6}
\end{gather*}
$$

where $\rho_{0}=\rho(r \rightarrow 0)$ and $\Lambda$ is the cosmological constant which appeared at the origin although was not present in the basic equations.

Requirements (a-c) either (a-c2) lead thus to the existence of the class of metrics

$$
\begin{gather*}
d s^{2}=g(r) d t^{2}-\frac{d r^{2}}{g(r)}-r^{2} d \Omega^{2}  \tag{2.7}\\
g(r)=1-\frac{R_{g}(r)}{r} ; \quad R_{g}(r)=2 G M(r)  \tag{2.8}\\
M(r)=4 \pi \int_{0}^{r} \rho(x) x^{2} d x \tag{2.9}
\end{gather*}
$$

which are asymptotically de Sitter as $r \rightarrow 0$, and asymptotically Schwarzschild at large $r$

$$
\begin{equation*}
d s^{2}=\left(1-\frac{r_{g}}{r}\right)-\frac{d r^{2}}{\left(1-\frac{r_{g}}{r}\right)}-r^{2} d \Omega^{2} ; \quad r_{g}=2 G m \tag{2.10}
\end{equation*}
$$

The weak energy condition defines the form of the metric function $g(r)$. In the region $r>0$ it has only minimum and the geometry can have not more than two horizons: a black hole horizon $r_{+}$and an internal horizon $R_{-}$[20].

The scalar curvature $R$, proportional to the trace of stress-energy tensor $T$, is proportional to $\rho-p_{\perp}$ for geometries satisfying (1.2), so that conditions (a-c) and (ac2) distinguish two types of geometries. In the case (ac) satisfying DEC requirement, scalar curvature remains non-negative, since DEC requires $\rho-p_{k} \geq 0$. The subclass satisfying (a-c) does not exhibit changes of topology by virtue of DEC and can be specified as DEC-subclass. Dominant energy condition requires that each principal pressure does not exceed the density which guarantees that speed of sound can not exceed speed of light. In nonlinear electrodynamics coupled to gravity, photons do not follow null geodesics of background geometry but propagate along null geodesics of an effective geometry [35], and propagation of photons resembles propagation inside a dielectric medium [36]. In the case of the regular NED structure satisfying DEC [26], it allows one to avoid problems with speed of sound exceeding speed of light.

In the case (a-c2) scalar curvature $R(r)$ changes sign somewhere and geometry experiences topological changes. This subclass satisfying only weak energy condition (needed in both cases for de Sitter behavior at the center) can be specified as WEC-subclass.

The case of the density profile (1.3) belongs to WECsubclass satisfying (c2). The metric function and the mass function are given by [10]
$g(r)=1-\frac{r_{g}}{r}\left(1-e^{-r^{3} / r_{0}^{2} r_{g}}\right) ; M(r)=m\left(1-e^{-r^{3} / r_{0}^{2} r_{g}}\right)$

Dominant Energy Condition is not satisfied so that a surface of zero scalar curvature exists at which $R(r)=0$. Zero curvature surface $r=r_{s}$ is shown in fig. 1 together with two horizons (a black hole event horizon $r_{+}$and an internal horizon $r_{-}$), and the characteristic surface of any geometry with de Sitter center: a zero-gravity surface $r=r_{c}$ beyond which the strong energy condition of singularities theorems [33], is violated (zero-gravity surface is defined by $\left.2 \rho+r \rho^{\prime}=0[17]\right)$.


FIG. 1. Characteristic surfaces of a spherically symmetric space-time of WEC type with de Sitter center.

Two horizons come together at the value of a mass parameter $m_{c r}$, which puts a lower limit on a black hole mass (see fig.2). For the case of a density profile (1.3) the critical mass is given by [17]

$$
\begin{equation*}
m_{c r} \simeq 0.3 m_{P l} \sqrt{\rho_{P l} / \rho_{0}} \tag{2.12}
\end{equation*}
$$



FIG. 2. Metric function $g(r)$ for de Sitter-Schwarzschild configurations. Mass $m$ is normalized to $m_{c r}$.

Temperature-mass diagram is shown in fig. 3 . Its form does not depend on particular choice of a density profile. Temperature drops to zero at $m=m_{c r}$, while the Schwarzschild asymptotic requires $T_{+} \rightarrow 0$ as $m \rightarrow \infty$. As a result the temperature-mass diagram should have a maximum between $m_{c r}$ and $m \rightarrow \infty$ [17]. In a maximum, at $m=m_{\text {cr } 2}$, a specific heat is broken and changes sign testifying to a second-order phase transition in the course of Hawking evaporation [21].


FIG. 3. Temperature-mass diagram for a vacuum nonsingular black hole with de Sitter center.

For $m \geq m_{c r}$ de Sitter-Schwarzschild geometry describes the vacuum nonsingular black hole, and global structure of space-time shown in fig. 4 [17], contains an infinite sequence of black and white holes whose future and past singularities are replaced with regular cores $\mathcal{R C}$ asymptotically de Sitter as $r \rightarrow 0$ [17].


FIG. 4. Global structure of space-time of a vacuum nonsingular black hole with de Sitter center [17].

## III. BASIC EQUATIONS FOR PERTURBATIONS

The perturbations of a spherically symmetric system are on essence time-dependent axially symmetric modes; the reason is the absence of a preferred axes in a spherically symmetric background [37].

They are described by the line element [37]

$$
\begin{gather*}
d s^{2}=e^{2 \nu} d t^{2}-e^{2 \psi}\left(d \phi-\omega d t-q_{2} d r-q_{3} d \theta\right)^{2} \\
-e^{2 \mu_{2}}(d r)^{2}-e^{2 \mu_{3}}(d \theta)^{2}, \tag{3.1}
\end{gather*}
$$

in which metric functions $\nu, \psi, \mu_{2}, \mu_{3}, \omega, q_{2}, q_{3}$ are functions only of $t, r, \theta$. They satisfy the Einstein equations

$$
\begin{equation*}
G_{i j}=R_{i j}-\frac{1}{2} g_{i j} R=-\kappa T_{i j} \tag{3.2}
\end{equation*}
$$

where the Ricci tensor is defined by $R_{i j}=g^{k l} R_{i k j l}$.
Non-zero components of stress-energy tensor read

$$
\begin{gather*}
T_{t t}=e^{2 \nu} \rho ; \quad T_{\phi \phi}=e^{2 \psi} p_{\phi} \\
T_{r r}=e^{2 \mu_{2}} p_{r} ; \quad T_{\theta \theta}=e^{2 \mu_{3}} p_{\theta} \tag{3.3}
\end{gather*}
$$

where $p_{r}, p_{\theta}, p_{\phi}$ are the principal pressures.
We obtain the relevant perturbation equations by linearizing the field equations around the spherically symmetric solution with de Sitter center. This solution considered as a special case of the line element (3.1) with

$$
\begin{equation*}
\mu_{2}=-\nu(r) ; \quad \psi=\ln (r \sin (\theta)) ; \quad \mu_{3}=\ln (r) \tag{3.4}
\end{equation*}
$$

has the form (2.7) with

$$
\begin{equation*}
g(r)=e^{2 \nu(r)}=1+\frac{C_{1}}{r}-\frac{\kappa}{r} \int \rho(r) r^{2} d r \tag{3.5}
\end{equation*}
$$

The particular solution (3.5) is specified by the choice of the constant $C_{1}$ which we choose in such a way to have unperturbed metric given by (2.8)-(2.9).

Our task is to investigate stability of the spherically symmetric system with de Sitter center to external perturbations in general case of a regular density profile $\rho(r)$.

The class of metrics with de Sitter center and a source term of structure (1.1), is extended to the case of nonzero cosmological constant $(\lambda<\Lambda)$ at infinity [38] corresponding to extension of the Einstein cosmological term $\Lambda g_{i k}$ to an $r$-dependent second rank symmetric tensor

$$
\begin{equation*}
\Lambda_{i k}=\kappa T_{i k} \tag{3.6}
\end{equation*}
$$

with the algebraic structure (1.1), connecting smoothly two de Sitter vacua with different values of a cosmological constant [39]. In this approach a constant scalar $\Lambda$ associated with a vacuum density $\Lambda=\kappa \rho_{v a c}$, becomes a tensor component $\Lambda_{t}^{t}$ associated explicitly with a density
component of a perfect fluid tensor whose vacuum properties follow from its symmetry (1.1) and whose variability follows from the Bianchi identities $[39,20]$.

Here we investigate stability for the particular case when $\lambda=0$ and spherically-symmetric space-time with de Sitter center is asymptotically flat.

Since an anisotropic fluid with the stress-energy tensor of type (1.1) admits identifying it as a vacuum-like medium associated with a time-evolving and spatially inhomogeneous cosmological term [10,39,20,40], we can write the Einstein equations in the form

$$
\begin{equation*}
G_{i k}+\Lambda_{i k}=0 \tag{3.7}
\end{equation*}
$$

(for discussion of where to put cosmological term see [41,20]). Then the quantities $\rho, p_{k}$ are treated as corresponding (in one-to-one way) components of the variable cosmological term $\Lambda_{t}^{t}=\kappa \rho, \Lambda_{k}^{k}=-\kappa p_{k}[39]$.

Since we apply the approach of studying direct perturbations of geometry via Einstein equations, we consider behavior of small perturbations for both the metric tensor and a stress-energy tensor associated with $\Lambda_{i k}$.

A general perturbation of a background geometry will result in $\omega, q_{2}, q_{3}$ becoming small quantities of the first order and the functions $\nu, \mu_{2}, \mu_{3}, \psi$ and $\rho, p_{k}$ experiencing small increments $\delta \nu, \delta \mu_{2}, \delta \mu_{3}, \delta \psi$ and $\delta \rho, \delta p_{k}$.

The perturbations leading to non-vanishing values of $\omega, q_{2}$ and $q_{3}$ induce a dragging of the inertial frame and impart a rotation, for this reason they are called axial perturbations [37].

Perturbations which do impart no rotation are called polar perturbations [37]. In the considered case they lead to increments in $\nu, \mu_{2}, \mu_{3}, \psi$ and $\rho, p_{k}$.

The equations governing the axial and the polar perturbations decouple.

Axial perturbations are governed by equations [37]

$$
\begin{equation*}
R_{r \phi}=R_{\theta \phi}=0 \tag{3.8}
\end{equation*}
$$

The equations governing the polar perturbations read

$$
\begin{gather*}
-R_{t r}=\left(\psi+\mu_{3}\right)_{, r t}+\psi_{, r}\left(\psi-\mu_{2}\right)_{, t} \\
+\mu_{3, r}\left(\mu_{3}-\mu_{2}\right)_{, t}-\nu_{, r}\left(\psi+\mu_{3}\right)_{, t}=0  \tag{3.9}\\
-R_{t \theta}=\left(\psi+\mu_{2}\right)_{, \theta t}+\psi_{, \theta}\left(\psi-\mu_{3}\right)_{, t} \\
+\mu_{2, \theta}\left(\mu_{2}-\mu_{3}\right)_{, t}-\nu_{, \theta}\left(\psi+\mu_{2}\right)_{, t}=0  \tag{3.10}\\
-R_{r \theta}=(\psi+\nu)_{, r \theta}+\psi_{, r}\left(\psi-\mu_{2}\right)_{, \theta} \\
+\nu_{, r}\left(\nu-\mu_{2}\right)_{, \theta}-\mu_{3, r}(\psi+\nu)_{, \theta}=0 \tag{3.11}
\end{gather*}
$$

$$
\begin{align*}
& G_{t t}=e^{-2 \mu_{2}}\left[\left(\psi+\mu_{3}\right)_{, r r}+\psi_{, r}\left(\psi-\mu_{2}+\mu_{3}\right)_{, r}+\mu_{3, r}\left(\mu_{3}-\mu_{2}\right)_{, r}\right] \\
& +e^{-2 \mu_{3}}\left[\left(\psi+\mu_{2}\right)_{, \theta \theta}+\psi_{, \theta}\left(\psi+\mu_{2}-\mu_{3}\right)_{, \theta}+\mu_{2, \theta}\left(\mu_{2}-\mu_{3}\right)_{, \theta}\right] \\
& \quad-e^{-2 \nu}\left[\psi_{, t}\left(\mu_{2}+\mu_{3}\right)_{, t}+\mu_{2, t} \mu_{3, t}\right]=-\kappa e^{2 \nu} \rho,  \tag{3.12}\\
& G_{\phi \phi}=e^{-2 \mu_{2}}\left[\left(\nu+\mu_{3}\right)_{, r r}+\nu_{, r}\left(\nu-\mu_{2}+\mu_{3}\right)_{,_{r}}+\mu_{3, r}\left(\mu_{3}-\mu_{2}\right)_{, r}\right] \\
& +e^{-2 \mu_{3}}\left[\left(\nu+\mu_{2}\right)_{, \theta \theta}+\nu_{, \theta}\left(\nu+\mu_{2}-\mu_{3}\right)_{, \theta}+\mu_{2, \theta}\left(\mu_{2}-\mu_{3}\right)_{, \theta}\right] \\
& -e^{-2 \nu}\left[\left(\mu_{2}+\mu_{3}\right)_{, t t}+\mu_{2, t}\left(\mu_{2}+\mu_{3}-\nu\right)_{, t}+\mu_{3, t}\left(\mu_{3}-\nu\right)_{, t}\right] \\
& \quad G_{r r}=e^{-2 \mu_{2}}\left[\psi_{, r}\left(\nu+\mu_{3}\right)_{, r}+\nu_{, r} \mu_{3, r}\right]  \tag{3.13}\\
& +e^{-2 \mu_{3}}\left[(\psi+\nu)_{, \theta \theta}+\psi_{, \theta}\left(\psi+\nu-\mu_{3}\right)_{, \theta}+\nu_{, \theta}\left(\nu-\mu_{3}\right)_{, \theta}\right] \\
& -e^{-2 \nu}\left[\left(\psi+\mu_{3}\right)_{, t t}+\psi_{, t}\left(\psi+\mu_{3}-\nu\right)_{, t}+\mu_{3, t}\left(\mu_{3}-\nu\right)_{, t}\right] \\
& +e_{\theta \theta}^{2 \mu_{2}} p_{r}, \\
& +e^{-2 \mu_{2}}\left[(\psi+\nu)_{, r r}+\psi_{, r}\left(\psi+\nu-\mu_{2}\right)_{, r}+\nu_{, r}\left(\nu-\mu_{2}\right)_{, r}\right]  \tag{3.14}\\
& +e^{-2 \mu_{3}}\left[\psi \psi_{, \theta}\left(\nu+\mu_{2}\right)_{, \theta}+\nu_{, \theta} \mu_{2, \theta}\right]-e^{-2 \nu}\left[\left(\psi+\mu_{2}\right)_{, t t}\right. \\
& \left.\left.+\mu_{2}-\nu\right)_{, t}+\mu_{2, t}\left(\mu_{2}-\nu\right)_{, t}\right]=\kappa e^{2 \mu_{3}} p_{\theta} . \quad(3.15)
\end{align*}
$$

We perturb equations (3.8)-(3.15) up to the first order, and in equations (3.12)-(3.15) we disturb both left and right hand sides. As a result we obtain the linear system of 7 partial differential equations for the polar perturbations, and the linear system of 2 equations for the axial perturbations.

## IV. AXIAL PERTURBATIONS

Axial perturbations corresponds to appearing of nonzero values $\omega, q_{2}, q_{3}$ which vanish for unperturbed system.

They are governed by the Einstein equations (3.8). This gives 2 equations for 3 functions which read

$$
\begin{align*}
& \frac{e^{2 \nu(r)}}{r^{2} \sin ^{3} \theta}\left[\sin ^{3} \theta\left(q_{2, \theta}-q_{3, r}\right)\right]_{, \theta}=-\left(\omega_{, r}-q_{2, t}\right)_{, t}  \tag{4.1}\\
& \frac{e^{2 \nu(r)}}{r^{2}}\left[r^{2} e^{2 \nu(r)}\left(q_{2, \theta}-q_{3, r}\right)\right]_{, r}=\left(\omega_{, \theta}-q_{3, t}\right)_{, t} \tag{4.2}
\end{align*}
$$

Now we take

$$
\omega(r, \theta, t)=\tilde{\omega}(r, \theta) e^{i \sigma t}
$$

and similarly for $q_{2}, q_{3}$; in what follows we retain the same symbols for the amplitudes of the perturbations which satisfy equations

$$
\begin{align*}
\frac{e^{2 \nu(r)}}{r^{2} \sin ^{3} \theta}\left[\sin ^{3} \theta\left(q_{2, \theta}-q_{3, r}\right)\right]_{, \theta} & =-\left(i \sigma \omega_{, r}+\sigma^{2} q_{2}\right)  \tag{4.3}\\
\frac{e^{2 \nu(r)}}{r^{2}}\left[r^{2} e^{2 \nu(r)}\left(q_{2, \theta}-q_{3, r}\right)\right]_{, r} & =i \sigma \omega_{, \theta}+\sigma^{2} q_{3} \tag{4.4}
\end{align*}
$$

Expressing $\omega_{, r}$ from (4.3) and $\omega_{, \theta}$ from (4.4), differentiating and equating $\omega_{, r \theta}=\omega_{, \theta r}$ we get one equation
$r^{4} \frac{\partial}{\partial r}\left(\frac{e^{2 \nu}}{r^{2}} \frac{\partial Q}{\partial r}\right)+\sin ^{3} \theta \frac{\partial}{\partial \theta}\left(\frac{1}{\sin ^{3} \theta} \frac{\partial Q}{\partial \theta}\right)+\sigma^{2} r^{2} e^{-2 \nu} Q=0$
Here

$$
\begin{equation*}
Q(r, \theta)=e^{2 \nu} r^{2} \sin ^{3} \theta\left(q_{2, \theta}-q_{3, r}\right) \tag{4.6}
\end{equation*}
$$

For the Schwarzschild metric, eq.(4.5) coincides with the analogous Chandrasekhar equation ( [37], Ch.4, eq.(18)). Separating variables by $Q(r, \theta)=R(r) \Theta(\theta)$, we get

$$
\begin{array}{r}
r^{2} e^{2 \nu} \frac{d}{d r}\left(\frac{e^{2 \nu}}{r^{2}} \frac{d R}{d r}\right)-\lambda \frac{e^{2 \nu}}{r^{2}} R+\sigma^{2} R=0 \\
\frac{d}{d \theta}\left(\frac{1}{\sin ^{3} \theta} \frac{d \Theta}{d \theta}\right)+\frac{\lambda}{\sin ^{3} \theta} \Theta=0 \tag{4.8}
\end{array}
$$

Solutions to (4.8) are Gegenbauer polynomials

$$
\begin{equation*}
Q_{l}(\theta)=C_{l+2}^{-\frac{3}{2}}(\theta)=\left(P_{l, \theta \theta}-P_{l, \theta} \operatorname{ctg} \theta\right) \sin ^{2} \theta \tag{4.9}
\end{equation*}
$$

which gives

$$
\begin{equation*}
\lambda_{l}=(l+2)(l-1) ; \quad l=2,3, \ldots \tag{4.10}
\end{equation*}
$$

General solution can be written in the form

$$
\begin{equation*}
Q(r, \theta)=\sum_{l=2}^{\infty} R_{l}(r) \Theta_{l}(\theta) \tag{4.11}
\end{equation*}
$$

Equation for $R_{l}(r)$ reads

$$
\begin{equation*}
r^{2} e^{2 \nu} \frac{d}{d r}\left(\frac{e^{2 \nu}}{r^{2}} \frac{d R_{l}}{d r}\right)-\frac{e^{2 \nu}}{r^{2}}(l+2)(l-1) R_{l}+\sigma_{l}^{2} R_{l}=0 \tag{4.12}
\end{equation*}
$$

In "tortoise" coordinate $r_{*}=\int d r / g(r)$, we get Schrödinger equation for $Z_{l}\left(r_{*}\right)=r^{-1} R_{l}\left(r_{*}\right)$

$$
\begin{equation*}
\left(\frac{d^{2}}{d r_{*}^{2}}+\sigma_{l}^{2}\right) Z_{l}=V_{l} Z_{l} \tag{4.13}
\end{equation*}
$$

where the potential is given by

$$
\begin{equation*}
V_{l}(r)=\frac{e^{2 \nu}}{r^{2}}\left(\mu^{2}+2 e^{2 \nu}-2 r \nu_{, r} e^{2 \nu}\right) \tag{4.14}
\end{equation*}
$$

with

$$
\begin{equation*}
\mu^{2}=(l+2)(l-1) \tag{4.15}
\end{equation*}
$$

For the Schwarzschild geometry (4.14) coincides with the Regge-Wheeler potential ([37], Ch.4, eq.(28)).

With the asymptotic behavior of (4.14)

$$
\begin{array}{r}
V_{l} \rightarrow \frac{l(l+1)}{r^{2}} \text { as } \quad r \rightarrow r_{*} \rightarrow \infty \\
V_{l} \rightarrow \text { const } e^{g^{\prime}\left(r_{+}\right) r_{*}} \text { as } \quad r_{*} \rightarrow-\infty \tag{4.16}
\end{array}
$$

solutions of (4.13) have asymptotic $e^{ \pm i \sigma r_{*}}$ as $r_{*} \rightarrow \pm \infty$ as in the case of the Schwarzschild geometry [37]. For real $\sigma$ they describe propagation of ingoing and outgoing waves through one-dimensional potential barrier, so that we have to look for solutions to (4.13), which satisfy the boundary conditions [37]

$$
\begin{align*}
& Z_{l} \rightarrow e^{+i \sigma_{l} r_{*}}+R_{l}\left(\sigma_{l}\right) e^{-i \sigma_{l} r_{*}} \quad\left(r_{*} \rightarrow+\infty\right) \\
& Z_{l} \rightarrow T_{l}\left(\sigma_{l}\right) e^{+i \sigma_{l} r_{*}} \quad\left(r_{*} \rightarrow-\infty\right) \tag{4.17}
\end{align*}
$$

These boundary conditions tell us that each $l$ component of an incident wave of the unit amplitude coming from infinity gives rise to a reflected wave of amplitude $R_{l}\left(\sigma_{l}\right)$ at infinity and a transmitted wave of amplitude $T_{l}\left(\sigma_{l}\right)$ at the horizon [37].

If such solutions exist only for real values of the time parameter $\sigma_{l}$ and form complete basic set, then any smooth initial perturbation defined at the finite interval of $r_{*}$ (with compact support), can be expanded on these functions, and since dependence of perturbations from time coordinate has the form $\exp \left(i \sigma_{l} t\right)$, this is followed by stability of geometry in question.

In terms of the metric function $g(r)$

$$
\begin{equation*}
V_{l}(r)=\frac{g(r)}{r^{2}}\left(\mu^{2}+2 g-r g^{\prime}\right) \tag{4.18}
\end{equation*}
$$

It is easily seen that $V_{l}(r)$ is constrained by the function

$$
V_{l}(r) \geq \frac{3 g}{r^{2}}\left(1+g+\frac{1}{3} \kappa \rho r^{2}\right)
$$

which is certainly positive and presents the value of the potential for the mode $l=2$.

We see that axial perturbations are governed by onedimensional wave equation (4.13) with a non-negative potential. In terms of a one-dimensional Schrödinger equation, an unstable mode exists if a potential has a bound
state, which corresponds to the negative eigenvalue $\sigma_{l}^{2}$. In the case of non-negative potential the system obeys the theorem [42] which guarantees the absence of negative eigenvalues in the standard one-dimensional Schrödinger equation with the non-negative potential. The absence of negative eigenvalues in the spectrum of (4.13) guarantees the absence of exponentially growing modes.

For the case when the potential is real, smooth and short-range, standard theorems of quantum mechanics guarantee that eigenfunctions of (4.13) form a complete set and any square-integrable state function can be expanded on them [37].

We have however to be careful about the behavior of the solution to (4.13) in the extremal regime near the double horizon $r_{ \pm}$which satisfies $g\left(r_{ \pm}\right)=0 ; g^{\prime}\left(r_{ \pm}\right)=0$.

To study the extreme case we introduce dimensionless variable $x$ by normalizing the variable $r$ to the characteristic scale $\left(r_{0}^{2} r_{g}\right)^{1 / 3}$.

Then the equation for the function $R_{l}$ reads

$$
\begin{equation*}
x^{2} g^{2} R_{l}^{\prime \prime}+2 x\left(\frac{x g^{\prime}}{2}-g\right) g R_{l}^{\prime}+\left(\sigma_{l}^{2} x^{2}-\mu^{2} g\right) R_{l}=0 \tag{4.19}
\end{equation*}
$$

where $\sigma_{l}=\sigma_{l}\left(r_{0}^{2} r_{g}\right)^{1 / 3}$. In what follows we retain the notation $\sigma_{l}$ keeping in mind that it is multiplied by $\left(r_{0}^{2} r_{g}\right)^{1 / 3}$. (A multiplier is not essential since in studying stability we are interested only in the sign of $\sigma_{l}^{2}$.)

Near the double horizon $x_{ \pm}$, the metric function is $g(x)=g^{\prime \prime}\left(x_{ \pm}\right)\left(x-x_{ \pm}\right)^{2} / 2+\ldots$. Introducing the variable $z=x-x_{ \pm}$and the notation $\gamma=g^{\prime \prime}\left(x_{ \pm}\right) / 2$, we get the limiting equation

$$
\begin{equation*}
z^{4} R_{l, z z}+2 z^{3} R_{l, z}+\frac{\sigma_{l}^{2}}{\gamma^{2}} R_{l}=0 \tag{4.20}
\end{equation*}
$$

General solution to (4.20) (found by taking $R_{l}=$ $w_{l} / z$ ), reads [44]

$$
\begin{equation*}
R_{l}(z)=C_{1 l} \cos \left(\frac{\kappa_{l}}{z}\right)+C_{2 l} \sin \left(\frac{\kappa_{l}}{z}\right) \tag{4.21}
\end{equation*}
$$

(with $\kappa_{l}^{2}=\sigma_{l}^{2} / \gamma^{2}$ ), and tells us that axial perturbations are restricted near double horizon.

In case without horizons asymptotic behavior of the potential at infinity and near zero is $V_{l}=l(l+1) / r^{2}$. This is behavior of centrifugal part of radial Schrödinger operator in the spherically symmetric case.

We see that potential for axial perturbations is smooth, short-range and positive for all types of configurations described by spherically symmetric geometry with de Sitter center. We conclude that geometry with de Sitter center is stable to axial perturbations.

## V. POLAR PERTURBATIONS

## A. General equations

Linearizing the system (3.9)-(3.15) about the background geometry, we get the system describing polar perturbations

$$
\begin{gathered}
{\left[\left(\delta \psi+\delta \mu_{3}\right)_{, r}-\left(\nu_{, r}-\frac{1}{r}\right)\left(\delta \psi+\delta \mu_{3}\right)-\frac{2}{r} \delta \mu_{2}\right]_{, t}=0,} \\
{\left[\left(\delta \psi+\delta \mu_{2}\right)_{, \theta}+\operatorname{ctg}(\theta)\left(\delta \psi-\delta \mu_{3}\right)\right]_{, t}=0} \\
{\left[\delta \psi_{, \theta}+\operatorname{ctg}(\theta)\left(\delta \psi-\delta \mu_{3}\right)\right]_{, r}+\delta \nu_{, r \theta}} \\
+\left(\nu_{, r}-\frac{1}{r}\right) \delta \nu_{, \theta}-\left(\nu_{, r}+\frac{1}{r}\right) \delta \mu_{2, \theta}=0, \\
-e^{2 \nu}\left[\left(\delta \psi+\delta \mu_{3}\right)_{, r r}+\left(\nu_{, r}+\frac{3}{r}\right)\left(\delta \psi+\delta \mu_{3}\right)_{, r}\right. \\
\left.-\frac{2}{r} \delta \mu_{2, r}-2\left(\frac{2}{r} \nu_{, r}+\frac{1}{r^{2}}\right) \delta \mu_{2}\right]-\frac{1}{r^{2}}\left[\delta \psi \psi_{, \theta \theta}+c t g(\theta)\left(2 \delta \psi-\delta \mu_{3}\right)_{, \theta}\right.
\end{gathered}
$$

$$
\left.+2 \delta \mu_{3}+\delta \mu_{2, \theta \theta}+\operatorname{ctg}(\theta) \delta \mu_{2, \theta}\right]=\kappa \delta \rho,
$$

$$
e^{2 \nu}\left[\left(\delta \nu+\delta \mu_{3}\right)_{, r r}+\left(3 \nu_{, r}+\frac{1}{r}\right) \delta \nu_{, r}+2\left(\nu_{, r}+\frac{1}{r}\right) \delta \mu_{3, r}\right.
$$

$$
\left.-\left(\nu_{, r}+\frac{1}{r}\right) \delta \mu_{2, r}-2\left(\nu_{, r r}+2 \nu_{, r}^{2}+\frac{2}{r} \nu_{, r}\right) \delta \mu_{2}\right]
$$

$$
+\frac{1}{r^{2}}\left(\delta \nu+\delta \mu_{2}\right)_{, \theta \theta}-e^{-2 \nu}\left(\delta \mu_{2}+\delta \mu_{3}\right)_{, t t}=\kappa \delta p_{\phi}
$$

$$
e^{2 \nu}\left[\frac{2}{r} \delta \nu_{, r}+\left(\nu_{, r}+\frac{1}{r}\right)\left(\delta \psi+\delta \mu_{3}\right)_{, r}-2\left(\frac{2}{r} \nu_{, r}+\frac{1}{r^{2}}\right) \delta \mu_{2}\right]
$$

$$
+\frac{1}{r^{2}}\left[\delta \psi_{, \theta \theta}+\operatorname{ctg}(\theta)\left(2 \delta \psi-\delta \mu_{3}\right)_{, \theta}+2 \delta \mu_{3}+\delta \nu_{, \theta \theta}+\operatorname{ctg}(\theta) \delta \nu_{, \theta}\right]
$$

$$
-e^{-2 \nu}\left(\delta \psi+\delta \mu_{3}\right)_{, t t}=\kappa \delta p_{r},
$$

$$
e^{2 \nu}\left[(\delta \nu+\delta \psi)_{, r r}+\left(3 \nu_{, r}+\frac{1}{r}\right) \delta \nu_{, r}+2\left(\nu_{, r}+\frac{1}{r}\right) \delta \psi_{, r}\right.
$$

$$
\left.-\left(\nu_{, r}+\frac{1}{r}\right) \delta \mu_{2, r}-2\left(\nu_{, r r}+2 \nu_{, r}^{2}+\frac{2}{r} \nu_{, r}\right) \delta \mu_{2}\right]
$$

$$
\begin{equation*}
+\frac{1}{r^{2}} \operatorname{ctg}(\theta)\left(\delta \nu+\delta \mu_{2}\right)_{, \theta}-e^{-2 \nu}\left(\delta \mu_{2}+\delta \psi\right)_{, t t}=\kappa \delta p_{\theta} \tag{5.7}
\end{equation*}
$$

The system (5.1)-(5.7) is the system of 7 linear partial differential equations of the first order for 8 quantities: 4 small perturbations of metric tensor and 4 small perturbations of stress-energy tensor (which is in considered case can be associated with a variable cosmological term) whose unperturbed components are related by the equation of state, in our case (1.2). To investigate this system we should make an assumption concerning perturbation of $p_{r}$ valid for the case of small perturbations. Since for the background geometry we have $p_{r}=-\rho$, i.e. $p_{r}=p_{r}(\rho)$, we can assume (see, e.g., [43])

$$
\begin{equation*}
\delta p_{r}=\frac{d p_{r}}{d \rho} \delta \rho \tag{5.8}
\end{equation*}
$$

which results in

$$
\begin{equation*}
\delta p_{r}=-\delta \rho \tag{5.9}
\end{equation*}
$$

The possibility to connect perturbations $\delta p_{r}$ and $\delta \rho$ is implied by our system which contains 7 equations for 8 functions. The relation (5.9) is valid only for small perturbations, since only in this case the relation (5.8) is valid. So, if we prove that the system is stable, i.e. growing perturbation modes are absent, this will justify the validity of (5.9).

Taking into account (5.4) and (5.6), the equation (5.9) can be written in the form

$$
\begin{gather*}
e^{2 \nu}\left[-\left(\delta \psi+\delta \mu_{3}\right)_{, r r}-\frac{2}{r}\left(\delta \psi+\delta \mu_{3}\right)_{, r}+\frac{2}{r} \delta \nu_{, r}+\frac{2}{r} \delta \mu_{2, r}\right] \\
+\frac{1}{r^{2}}\left[\delta \nu_{, \theta \theta}+\operatorname{ctg}(\theta) \delta \nu_{, \theta}-\delta \mu_{2, \theta \theta}-\operatorname{ctg}(\theta) \delta \mu_{2, \theta}\right] \\
-e^{-2 \nu}\left(\delta \psi+\delta \mu_{3}\right)_{, t t}=0 \tag{5.10}
\end{gather*}
$$

In this way we obtain the system of 7 equations for 7 unknown functions which splits into uniform system of 4 linear partial differential equations (5.1), (5.2), (5.3), (5.10) for 4 small perturbations of the metric tensor, $\delta \nu(r, \theta, t), \delta \mu_{2}(r, \theta, t), \delta \mu_{3}(r, \theta, t), \delta \psi(r, \theta, t)$; and 3 linear algebraic equations (5.6), (5.5), (5.7), determining expressions for $\delta p_{r}, \delta p_{\phi}, \delta p_{\theta}$ through expressions for metric perturbations.

The problem ultimately reduces to investigation of the uniform linear system (5.1), (5.2), (5.3), (5.10).

Following Chandrasekhar [37] we assume the time dependence $e^{i \sigma t}$ which corresponds to the Fourier analysis of perturbations. The variables $r$ and $\theta$ are separated by the Friedman substitutions [37].

As a result we present perturbations as series

$$
\begin{equation*}
\delta \nu(r, \theta, t)=\sum_{l=2}^{+\infty} N_{l}(r) P_{l}(\cos \theta) e^{i \sigma_{l} t} \tag{5.11}
\end{equation*}
$$

$$
\begin{gather*}
\delta \mu_{2}(r, \theta, t)=\sum_{l=2}^{+\infty} L_{l}(r) P_{l}(\cos \theta) e^{i \sigma_{l} t},  \tag{5.12}\\
\delta \mu_{3}(r, \theta, t)=\sum_{l=2}^{+\infty}\left[T_{l}(r) P_{l}(\cos \theta)\right. \\
\left.+V_{l}(r) P_{l, \theta \theta}(\cos \theta)\right] e^{i \sigma_{l} t},  \tag{5.13}\\
\delta \psi(r, \theta, t)=\sum_{l=2}^{+\infty}\left[T_{l}(r) P_{l}(\cos \theta)\right. \\
\left.+V_{l}(r) P_{l, \theta}(\cos \theta) c t g \theta\right] e^{i \sigma_{l} t}  \tag{5.14}\\
\delta \rho(r, \theta, t)=\sum_{l=2}^{+\infty} C_{l}(r) P_{l}(\cos \theta) e^{i \sigma_{l} t}  \tag{5.15}\\
\delta p_{r}(r, \theta, t)=\sum_{l=2}^{+\infty} D_{l}(r) P_{l}(\cos \theta) e^{i \sigma_{l} t}  \tag{5.16}\\
\left.+H_{l}(r) P_{l, \theta \theta}(\cos \theta)\right] e^{i \sigma_{l} t} \\
\left.+H_{l}(r) P_{l, \theta}(\cos \theta) c t g \theta\right] e^{i \sigma_{l} t}  \tag{5.17}\\
\delta p_{\theta}(r, \theta, t)=\sum_{l=2}^{+\infty}\left[E_{l}(r) P_{l}(\cos \theta)\right. \\
\delta p_{\phi}(r, \theta, t)=\sum_{l=2}^{+\infty}\left[E_{l}(r) P_{l}(\cos \theta)\right. \tag{5.18}
\end{gather*}
$$

Let us introduce the function $X_{l}$ which will be useful in our further reductions

$$
\begin{equation*}
X_{l}(r)=n V_{l}(r) \tag{5.19}
\end{equation*}
$$

where

$$
n=\frac{l(l+1)}{2}-1 ; \quad l=2,3, \ldots ; \quad n=2,5,9, \ldots
$$

Using the properties of the Legendre polynomials

$$
\begin{gathered}
\left(\sin \theta P_{l, \theta}\right)_{, \theta}+l(l+1) \sin \theta P_{l}(\cos \theta)=0 \\
P_{l, \theta \theta}+P_{l, \theta} \operatorname{ctg} \theta=-l(l+1) P_{l}(\cos \theta)
\end{gathered}
$$

we get from Eqs (5.1)-(5.3), (5.10) after some algebra, the following relations between amplitudes

$$
T_{l}(r)=V_{l}(r)-L_{l}(r)
$$

$$
\begin{aligned}
& \left(X_{l}(r)+L_{l}(r)\right)_{, r}-\left(\nu_{, r}-\frac{1}{r}\right)\left(X_{l}(r)+L_{l}(r)\right)+\frac{1}{r} L_{l}(r)=0 \\
& \left(T_{l}(r)-V_{l}(r)\right)_{, r}-\left(\nu_{, r}+\frac{1}{r}\right) L_{l}(r)+N_{l, r}(r)+\left(\nu_{, r}-\frac{1}{r}\right) N_{l}(r)=0 \\
& e^{2 \nu}\left[\left(X_{l}(r)+L_{l}(r)\right)_{, r, r}+\frac{1}{r}\left(N_{l}(r)+2 X_{l}(r)+3 L_{l}(r)\right)_{, r}\right] \\
& -\frac{l(l+1)}{2 r^{2}}\left(N_{l}(r)-L_{l}(r)\right)-e^{-2 \nu} \sigma_{l}^{2}\left(X_{l}(r)+L_{l}(r)\right)=0
\end{aligned}
$$

With using these relations we transform ultimately our starting system to the system of 3 differential equations in the normal form for the functions $N_{l}(r), L_{l}(r), X_{l}(r)$

$$
\begin{array}{r}
N_{l, r}=(n+1) a 1 N_{l}+\left(\nu_{, r}+b 1\right. \\
\left.-(n+1) a 1+\sigma_{l}^{2} c 1\right) L_{l}+\left(b 1+\sigma_{l}^{2} c 1\right) X_{l} \tag{5.20a}
\end{array}
$$

$$
L_{l, r}=\left(\nu_{, r}-\frac{1}{r}+(n+1) a 1\right) N_{l}+\left(-\frac{1}{r}+b 1\right.
$$

$$
\begin{equation*}
\left.\left.-(n+1) a 1+\sigma_{l}^{2} c 1\right) L_{l}+\left(b 1+\sigma_{l}^{2} c 1\right)\right) X_{l} \tag{5.20b}
\end{equation*}
$$

$$
X_{l, r}=\left(-\nu_{, r}+\frac{1}{r}-(n+1) a 1\right) N_{l}
$$

$$
+\left(\nu_{, r}-\frac{1}{r}-b 1+(n+1) a 1-\sigma_{l}^{2} c 1\right) L_{l}
$$

$$
\begin{equation*}
+\left(\nu_{, r}-\frac{1}{r}-b 1-\sigma_{l}^{2} c 1\right) X_{l} \tag{5.20c}
\end{equation*}
$$

where
$a 1(r)=\frac{e^{-2 \nu(r)}}{r} ; b 1(r)=-r \nu_{, r r}-r \nu_{, r}^{2} ; c 1(r)=r e^{-4 \nu(r)}$,
and 4 equations which define amplitudes $D_{l}(r), E_{l}(r)$, $H_{l}(r)$, and $C_{l}(r)$ through solutions of (5.20)

$$
\begin{gather*}
D_{l}(r)=\frac{2}{\kappa}\left[e ^ { 2 \nu } \left[\frac{1}{r} N_{l, r}-\left(\nu_{, r}+\frac{1}{r}\right)\left(X_{l}+L_{l}\right)_{, r}\right.\right. \\
\left.-\left(\frac{2}{r} \nu_{, r}+\frac{1}{r^{2}}\right) L_{l}\right]-\frac{1}{r^{2}}\left[X_{l}-n L_{l}+(n+1) N_{l}\right] \\
\left.\quad-\sigma_{l}^{2} e^{-2 \nu}\left(X_{l}+L_{l}\right)\right] \tag{5.22}
\end{gather*}
$$

$$
\begin{gather*}
E_{l}(r)=\frac{1}{\kappa}\left[e ^ { 2 \nu } \left[\left(N_{l}-L_{l}+\frac{1}{n} X_{l}\right)_{, r r}\right.\right. \\
+\left(3 \nu_{, r}+\frac{1}{r}\right) N_{l, r}+2\left(\nu_{, r}+\frac{1}{r}\right)\left(\frac{1}{n} X_{l}-L_{l}\right)_{, r} \\
\left.-\left(\nu_{, r}+\frac{1}{r}\right) L_{l, r}-2\left(\nu_{, r r}+2 \nu_{, r}^{2}+\frac{2}{r} \nu_{, r}\right) L_{l}\right] \\
\left.+\sigma_{l}^{2} e^{-2 \nu} \frac{1}{n} X_{l}\right]  \tag{5.23}\\
H_{l}(r)=\frac{1}{n \kappa}\left[e^{2 \nu}\left[X_{l, r r}+2\left(\nu_{, r}+\frac{1}{r}\right) X_{l, r}\right]\right. \\
\left.+\frac{n}{r^{2}}\left(N_{l}+L_{l}\right)+\sigma_{l}^{2} e^{-2 \nu} X_{l}\right],  \tag{5.24}\\
C_{l}(r)=-D_{l}(r) \tag{5.25}
\end{gather*}
$$

Let us now introduce the dimensionless variables

$$
\begin{equation*}
x=\frac{r}{r_{1}} ; \quad \rho \rightarrow \frac{\rho}{\rho_{0}} ; \quad \text { where } \quad r_{1}^{3}=r_{0}^{2} r_{g} \tag{5.26}
\end{equation*}
$$

and the characteristic parameter

$$
\begin{equation*}
\alpha=\frac{r_{g}}{r_{1}} \tag{5.27}
\end{equation*}
$$

In these notations the unperturbed solution (2.7) reads

$$
\begin{gather*}
d s^{2}=g(x) d t^{2}-\frac{d x^{2}}{g(x)}-x^{2} d \Omega^{2} \\
g(x)=1-\frac{\alpha M(x)}{x} ; \quad M(x)=3 \int_{0}^{x} \rho(q) q^{2} d q \tag{5.28}
\end{gather*}
$$

In terms of $g(x)$ our basic system (5.20) takes the form

$$
\begin{gather*}
N_{l, x}=\frac{(n+1)}{x g} N_{l}+\left[\frac{1}{2} \frac{g^{\prime}}{g}+\frac{x}{4}\left(\frac{g^{\prime}}{g}\right)^{2}-\frac{x}{2} \frac{g^{\prime \prime}}{g}\right. \\
\left.-\frac{(n+1)}{x g}+\sigma_{l}^{2} \frac{x}{g^{2}}\right] L_{l}+\left[\frac{x}{4}\left(\frac{g^{\prime}}{g}\right)^{2}-\frac{x}{2} \frac{g^{\prime \prime}}{g}+\sigma_{l}^{2} \frac{x}{g^{2}}\right] X_{l} \tag{5.29a}
\end{gather*}
$$

$L_{l, x}=\left[\frac{1}{2} \frac{g^{\prime}}{g}-\frac{1}{x}+\frac{(n+1)}{x g}\right] N_{l}+\left[-\frac{1}{x}+\frac{x}{4}\left(\frac{g^{\prime}}{g}\right)^{2}-\frac{x}{2} \frac{g^{\prime \prime}}{g}\right.$

$$
\begin{equation*}
\left.-\frac{(n+1)}{x g}+\sigma_{l}^{2} \frac{x}{g^{2}}\right] L_{l}+\left[\frac{x}{4}\left(\frac{g^{\prime}}{g}\right)^{2}-\frac{x}{2} \frac{g^{\prime \prime}}{g}+\sigma_{l}^{2} \frac{x}{g^{2}}\right] X_{l} \tag{5.29b}
\end{equation*}
$$

$$
\begin{align*}
& X_{l, x}=\left[-\frac{1}{2} \frac{g^{\prime}}{g}+\frac{1}{x}-\frac{(n+1)}{x g}\right] N_{l}+\left[\frac{1}{2} \frac{g^{\prime}}{g}-\frac{1}{x}-\frac{x}{4}\left(\frac{g^{\prime}}{g}\right)^{2}+\frac{x}{2} \frac{g^{\prime \prime}}{g}\right. \\
& \left.+\frac{(n+1)}{x g}-\sigma_{l}^{2} \frac{x}{g^{2}}\right] L_{l}+\left[\frac{1}{2} \frac{g^{\prime}}{g}-\frac{1}{x}-\frac{x}{4}\left(\frac{g^{\prime}}{g}\right)^{2}+\frac{x}{2} \frac{g^{\prime \prime}}{g}-\sigma_{l}^{2} \frac{x}{g^{2}}\right] X_{l} \tag{5.29c}
\end{align*}
$$

This system can be transformed to the equivalent form

$$
\begin{gather*}
x g^{2} N_{l, x}=(n+1) g N_{l}+g\left(\frac{x}{2} g^{\prime}-(n+1)\right) L_{l} \\
+x^{2}\left(\frac{1}{4}\left(g^{\prime}\right)^{2}-\frac{1}{2} g g^{\prime \prime}+\sigma_{l}^{2}\right) \tilde{X}_{l}  \tag{5.30a}\\
x g L_{l, x}+\left(\frac{x}{2} g^{\prime}+g\right) L_{l}=x g N_{l, x}+\left(\frac{x}{2} g^{\prime}-g\right) N_{l}  \tag{5.30b}\\
x g \tilde{X}_{l, x}=-g L_{l}+\left(\frac{x}{2} g^{\prime}-g\right) \tilde{X}_{l} \tag{5.30c}
\end{gather*}
$$

where

$$
\begin{equation*}
\tilde{X}_{l}=X_{l}+L_{l} \tag{5.31}
\end{equation*}
$$

which can be compared with the analogous Chandrasekhar system ( [37], Ch.4, eqs. (46-47), (50)). Our equations (5.30b)-(5.30c) coincide with Chandrasekhar equations (46)-(47), while our equation (5.30a) coincides with the Chandrasekhar equation (50) if and only if $\rho^{\prime}=0$ which is equivalent to $\left(x^{2} g^{\prime \prime} / 2-g+1\right)=0$.

The basic system (5.30) can be directly applied to study extreme black hole case. In the next subsection we investigate first the case of a simple horizon to make clear peculiarity of the case of the double horizon.

## B. Extreme black hole case

## Behavior near the simple horizon

In the neighborhood of a simple horizon $x_{+}$we have $g(x)=g^{\prime}\left(x_{+}\right)\left(x-x_{+}\right)+\frac{1}{2} g^{\prime \prime}\left(x_{+}\right)\left(x-x_{+}\right)^{2}+\ldots$. To study behavior in the limit $x \rightarrow x_{+}+0$ we introduce the variable $z=x-x_{+}$. In a small neighborhood of $z=0$ limiting system for (5.30) reads

$$
\begin{gather*}
x_{+}\left(g^{\prime}\left(x_{+}\right)\right)^{2} z^{2} N_{l, z}=(n+1) g^{\prime}\left(x_{+}\right) z N_{l} \\
+g^{\prime}\left(x_{+}\right) z\left[\frac{x_{+}}{2} g^{\prime}\left(x_{+}\right)-(n+1)\right] L_{l}+x_{+}^{2}\left(\frac{1}{4}\left(g^{\prime}\left(x_{+}\right)\right)^{2}+\sigma_{l}^{2}\right) \tilde{X}_{l}  \tag{5.32}\\
z\left(N_{l}-L_{l}\right)_{, z}+\frac{1}{2}\left(N_{l}-L_{l}\right)=0 \tag{5.33}
\end{gather*}
$$

$$
\begin{equation*}
x_{+} z \tilde{X}_{l, z}=-z L_{l}+\frac{1}{2} x_{+} \tilde{X}_{l} \tag{5.34}
\end{equation*}
$$

One immediately sees from (5.33) that the restricted near $z=0$ solutions should satisfy $N_{l}(z)=L_{l}(z)$. Then (5.32) and (5.34) form the system of two first-order equations for functions $N_{l}, L_{l}$

$$
\begin{equation*}
\left(g^{\prime}\left(x_{+}\right)\right)^{2} z^{2} N_{l, z}=\frac{1}{2}\left(g^{\prime}\left(x_{+}\right)^{2} z N_{l}+x_{+}\left(\frac{1}{4}\left(g^{\prime}\left(x_{+}\right)\right)^{2}+\sigma_{l}^{2}\right) \tilde{X}_{l}\right. \tag{5.35}
\end{equation*}
$$

$$
\begin{equation*}
x_{+} z \tilde{X}_{l, z}=-z N_{l}+\frac{1}{2} x_{+} \tilde{X}_{l} \tag{5.36}
\end{equation*}
$$

This system reduces to one second-order equation for $\tilde{X}_{l}$

$$
\begin{equation*}
z^{2} \tilde{X}_{l, z z}-z \tilde{X}_{l, z}+\left(1+\frac{\sigma_{l}^{2}}{\left(g^{\prime}\left(x_{+}\right)\right)^{2}}\right) \tilde{X}_{l}=0 \tag{5.37}
\end{equation*}
$$

This is the Euler equation, and solutions of (5.32)-(5.34) restricted near zero, have the form

$$
\begin{align*}
& \tilde{X}_{l}(z)=\left[B_{1 l} \cos \left(\frac{\sigma_{l}}{g^{\prime}\left(x_{+}\right)} \ln z\right)+B_{2 l} \sin \left(\frac{\sigma_{l}}{g^{\prime}\left(x_{+}\right)} \ln z\right)\right] z,  \tag{5.38}\\
& N_{l}=L_{l}=-x_{+}\left[\left(\frac{1}{2} B_{1 l}+\frac{\sigma_{l}}{g^{\prime}\left(x_{+}\right)} B_{2 l}\right) \cos \left(\frac{\sigma_{l}}{g^{\prime}\left(x_{+}\right)} \ln z\right)\right. \\
& \left.\quad+\left(\frac{1}{2} B_{2 l}-\frac{\sigma_{l}}{g^{\prime}\left(x_{+}\right)} B_{1 l}\right) \sin \left(\frac{\sigma_{l}}{g^{\prime}\left(x_{+}\right)} \ln z\right)\right], \tag{5.39}
\end{align*}
$$

where $B_{1 l}, B_{2 l}$ are arbitrary constants. As a result in the small neighborhood of a simple horizon restricted solutions exist for all real values of $\sigma_{l}$.

## Behavior near the double horizon

The double horizon $x_{ \pm}$corresponds to the case $\alpha=\alpha_{c r}$ in (5.28). For the case of the density profile (1.4)

$$
\begin{equation*}
\alpha_{c r} \simeq 1.456 \tag{5.40}
\end{equation*}
$$

In the small neighborhood of the point $x=x_{ \pm}$, the metric function is written as

$$
g(x)=\gamma\left(x-x_{ \pm}\right)^{2}+\ldots, \gamma=\frac{1}{2} g^{\prime \prime}\left(x_{ \pm}\right)
$$

In the variable $z=x-x_{ \pm}$, in the small neighborhood of $z=0$ the limiting system for (5.30) reads

$$
\begin{gather*}
x_{ \pm} \gamma^{2} z^{4} N_{l, z}=(n+1) \gamma z^{2}\left(N_{l}-L_{l}\right)+x_{ \pm}^{2} \sigma_{l}^{2} \tilde{X}_{l}  \tag{5.41}\\
z\left(N_{l}-L_{l}\right)_{, z}+\left(N_{l}-L_{l}\right)=0  \tag{5.42}\\
x_{ \pm} z \tilde{X}_{l, z}=-z L_{l}+x_{ \pm} \tilde{X}_{l} \tag{5.43}
\end{gather*}
$$

As follows from (5.42), for a restricted solution it should be $N_{l}=L_{l}$. Then equations (5.41) and (5.43) form a system of two first-order equations for $N_{l}, L_{l}$ :

$$
\begin{gather*}
z^{4} N_{l, z}=x_{ \pm} \frac{\sigma_{l}^{2}}{\gamma^{2}} \tilde{X}_{l},  \tag{5.44}\\
z \tilde{X}_{l, z}=-\frac{z}{x_{ \pm}} N_{l}+\tilde{X}_{l} \tag{5.45}
\end{gather*}
$$

This system reduces to one second-order equation for $N_{l}$

$$
\begin{equation*}
z^{4} N_{l, z z}+3 z^{3} N_{l, z}+\frac{\sigma_{l}^{2}}{\gamma^{2}} N_{l}=0 \tag{5.46}
\end{equation*}
$$

which differs essentially from the analogous equation (5.37) for a simple horizon case. General solution to (5.46) is [44]

$$
\begin{equation*}
N_{l}(z)=\frac{1}{z}\left[C_{1 l} J_{1}\left(\frac{\sigma_{l}}{\gamma} \frac{1}{z}\right)+C_{2 l} Y_{1}\left(\frac{\sigma_{l}}{\gamma} \frac{1}{z}\right)\right] \tag{5.47}
\end{equation*}
$$

where $C_{1 l}, C_{2 l}$ are arbitrary constants, $J_{1}, Y_{1}$ are Bessel functions.

With taking into account asymptotic behavior of Bessel functions for big values of argument, we find the behavior of function $N_{l}(z)$ for $z \rightarrow 0$
$N_{l}(z)=\frac{1}{z^{\frac{1}{2}}}\left[C_{1 l} \cos \left(\frac{\sigma_{l}}{\gamma} \frac{1}{z}-\frac{3 \pi}{4}\right)+C_{2 l} \sin \left(\frac{\sigma_{l}}{\gamma} \frac{1}{z}-\frac{3 \pi}{4}\right)\right.$
We see that solutions to (5.46) are unbounded as $z \rightarrow 0$ for all real values of the parameter $\sigma_{l}$.

From (5.44) we get

$$
\begin{equation*}
X_{l}(z)=-\frac{\gamma z}{\left(x_{ \pm}\right) \sigma_{l}}\left[C_{1 l} J_{0}\left(\frac{\sigma_{l}}{\gamma} \frac{1}{z}\right)+C_{2 l} Y_{0}\left(\frac{\sigma_{l}}{\gamma} \frac{1}{z}\right)\right] \tag{5.49}
\end{equation*}
$$

which gives in the limit $z \rightarrow 0$

$$
\begin{gather*}
X_{l}(z)=-\frac{1}{\left(x_{ \pm}\right)}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(\frac{\gamma z}{\sigma_{l}}\right)^{\frac{3}{2}}\left[C_{1 l} \cos \left(\frac{\sigma_{l}}{\gamma} \frac{1}{z}-\frac{\pi}{4}\right)\right. \\
\left.+C_{2 l} \sin \left(\frac{\sigma_{l}}{\gamma} \frac{1}{z}-\frac{\pi}{4}\right)\right] \tag{5.50}
\end{gather*}
$$

Analysis of our basic system (5.30) in small neighborhood of double horizon $x=x_{ \pm}$shows that for all real values of the parameter $\sigma_{l}$, there exist unbounded solutions as $x \rightarrow x_{ \pm}$. Therefore the method of linear perturbations, as well as the assumption (5.8), are not suitable in this case, but the behavior of perturbations suggest instability of the extreme configuration. It should be investigated separately, and we are currently working on this [45].

## C. The reduction of the system to a one-dimensional wave equation

Now our goal is to reduce the system (5.30) to a single second-order equation. We introduce the new functions $z_{1 l}, z_{2 l}, z_{3 l}$ using the linear transformations

$$
\begin{gather*}
N_{l}=\left[\frac{1}{x} z_{1 l}+\left(\frac{g^{\prime}}{2}-\frac{x}{4 g}\left(g^{\prime}\right)^{2}-\sigma_{l}^{2} \frac{x}{g}\right) z_{2 l}+z_{3 l}\right] g^{\frac{1}{2}}  \tag{5.51a}\\
L_{l}=\left[\left(-\frac{x}{2} g^{\prime \prime}+\frac{g^{\prime}}{2}\right) z_{2 l}+z_{3 l}\right] g^{\frac{1}{2}}  \tag{5.51b}\\
X_{l}=\left[\left(\frac{(n+1)}{x}-\frac{g}{x}+\frac{x}{2} g^{\prime \prime}\right) z_{2 l}-z_{3 l}\right] g^{\frac{1}{2}} \tag{5.51c}
\end{gather*}
$$

The inverse transformation to (5.51) reads

$$
\begin{gather*}
z_{1 l}(x)=\frac{x}{\sqrt{g} b}\left[b(x) N_{l}(x)+\left((b(x)-n-1) \frac{x}{2} \frac{g^{\prime}}{g}-x b^{\prime}\right.\right. \\
\left.\left.-b(x)+\sigma_{l}^{2} \frac{x^{2}}{g(x)}\right) L_{l}+\left((b-n-1) \frac{x^{2}}{2} \frac{g^{\prime}}{g}-x b_{, x}+\sigma_{l}^{2} \frac{x^{2}}{g(x)}\right) X_{l}\right] \tag{5.52a}
\end{gather*}
$$

$$
\begin{equation*}
z_{2 l}(x)=\left[L_{l}+X_{l}\right] \frac{x}{\sqrt{g} b(x)} \tag{5.52b}
\end{equation*}
$$

$$
\begin{equation*}
z_{3 l}(x)=\left[\left(\frac{1}{x} b(x)+b_{, x}\right) L_{l}+b_{, x} X_{l}\right] \frac{x}{\sqrt{g} b(x)} \tag{5.52c}
\end{equation*}
$$

where

$$
\begin{equation*}
b(x)=n+1+\frac{x}{2} g^{\prime}(x)-g(x)=n+\frac{3 \alpha}{2 x}\left(M(x)-x^{3} \rho\right) \tag{5.53}
\end{equation*}
$$

The sum of (5.51b) and (5.51c) gives

$$
\begin{equation*}
X_{l}+L_{l}=\left[\frac{n+1}{x}+\frac{1}{2} g^{\prime}-\frac{1}{x} g\right] g^{1 / 2} z_{2 l} \tag{5.54}
\end{equation*}
$$

As a result we get the following system

$$
\begin{gathered}
z_{1 l, x}=\left(\frac{2}{x}-\frac{g^{\prime}}{g}\right) z_{1 l}-\left(\frac{1}{2} x^{2} g^{\prime \prime \prime}+x g^{\prime \prime}-g^{\prime}\right) z_{2 l} \\
+\left[2+\frac{x^{2}}{b(x)}\left(\frac{g^{\prime \prime}}{2}-\frac{\left(g^{\prime}\right)^{2}}{4 g}-\sigma_{l}^{2} \frac{1}{g}\right)\right] z_{3 l} \\
z_{2 l, x}=-\frac{1}{b(x)} z_{3 l} \\
z_{3 l, x}=\frac{b(x) g^{-1}}{x^{2}} z_{1 l}-\left[\frac{2}{x}+\frac{\left(x g^{\prime \prime}-g^{\prime}\right)}{2 b(x)}\right] z_{3 l}
\end{gathered}
$$

$$
\begin{equation*}
+\frac{1}{x}\left(\frac{x^{2}}{2} g^{\prime \prime \prime}+x g^{\prime \prime}-g^{\prime}\right) z_{2 l} \tag{5.55c}
\end{equation*}
$$

It is easily to prove that

$$
\begin{equation*}
\frac{1}{2} x^{2} g^{\prime \prime \prime}+x g^{\prime \prime}-g^{\prime}=-\frac{3 \alpha}{2}\left(x^{3} \rho^{\prime}\right)^{\prime}=3 \alpha x^{2} p_{\perp}^{\prime} \tag{5.56}
\end{equation*}
$$

so that in the case when the density profile satisfies the condition

$$
\begin{equation*}
\left(x^{3} \rho^{\prime}\right)^{\prime}=0 \tag{5.57a}
\end{equation*}
$$

the system (5.55) splits on the system of two equations ( 5.55 a ), ( 5.55 c ) for $z_{1 l}, z_{3 l}$, and the equation ( 5.55 b ).

Condition (5.57a) is in turn equivalent to

$$
\begin{equation*}
p_{\perp}^{\prime}=0 \tag{5.57b}
\end{equation*}
$$

In the particular case $\left(x^{3} \rho^{\prime}\right)=$ const $=0$ this is the necessary and sufficient condition for coinciding of our system (5.30) with the Chandrasekhar system ( [37], eqs.(46)-(47), (50) Ch.4).

Differentiating ( 5.55 c ), we come to the system which includes one first-order equation, (5.55b), and one second-order equation

$$
\begin{equation*}
z_{3 l, x x}+2\left(\frac{g^{\prime}}{g}+\frac{1}{x}\right) z_{3 l, x}+q_{l}(x) z_{3 l}=r_{l}(x) z_{2 l} \tag{5.58}
\end{equation*}
$$

where

$$
\begin{gather*}
q_{l}(x)=\sigma_{l}^{2} \frac{1}{g^{2}}-\frac{2(n+1)}{x^{2} g}-\frac{1}{2} \frac{g^{\prime \prime}}{g}+\frac{1}{4}\left(\frac{g^{\prime}}{g}\right)^{2}+\frac{3}{x} \frac{g^{\prime}}{g} \\
-\frac{\left(x g^{\prime \prime}-g^{\prime}\right)}{b(x)}\left[\frac{\left(x g^{\prime \prime}-g^{\prime}\right)}{2 b(x)}-\frac{g^{\prime}}{g}+\frac{1}{x}\right]+\frac{x g^{\prime \prime \prime}}{2 b(x)}+\frac{3 \alpha x}{b(x)} p_{\perp}^{\prime}  \tag{5.59}\\
r_{l}(x)=-3 \alpha p_{\perp}^{\prime}\left[\frac{(n+1)}{g}-\frac{3 x}{2} \frac{g^{\prime}}{g}\right. \\
\left.+\frac{x}{2 b(x)}\left(x g^{\prime \prime}-g^{\prime}\right)\right]+\frac{3 \alpha}{x}\left(x^{2} p_{\perp}^{\prime}\right)^{\prime} \tag{5.60}
\end{gather*}
$$

It is easily to see that in the case when the condition (5.57) is satisfied, two equations (5.55b) and (5.58) split. Introducing the new function $\omega_{3 l}(x)$ by

$$
\begin{equation*}
z_{3 l}(x)=\frac{1}{x g} \omega_{3 l}(x) \tag{5.61}
\end{equation*}
$$

we reduce the equation (5.58) to the form which does not contain the first derivative:

$$
\begin{equation*}
\omega_{3 l, x x}+\left[\sigma_{l}^{2} \frac{1}{g^{2}}-V_{1 l}(x)\right] \omega_{3 l}=x g r_{l} z_{2 l} \tag{5.62}
\end{equation*}
$$

where the potential $V_{1 l}(x)$ is given by
$V_{1 l}(x)=\frac{l(l+1)}{x^{2}} \frac{1}{g}+\frac{3}{2} \frac{g^{\prime \prime}}{g}-\frac{1}{x} \frac{g^{\prime}}{g}+\frac{\left(x g^{\prime \prime}-g^{\prime}\right)}{b(x)}\left[\frac{\left(x g^{\prime \prime}-g^{\prime}\right)}{2 b(x)}\right.$
$\left.-\frac{g^{\prime}}{g}+\frac{1}{x}\right]-\frac{1}{4}\left(\frac{g^{\prime}}{g}\right)^{2}-\frac{x g^{\prime \prime \prime}}{2 b(x)}-\frac{1}{x b(x)}\left[\frac{1}{2} x^{2} g^{\prime \prime \prime}+x g^{\prime \prime}-g^{\prime}\right]$
With taking into account (5.55b) and (5.61), equation (5.62) can be rewritten as integro-differential equation of the form

$$
\begin{align*}
& \omega_{3 l, x x}+\left[g^{-2}(x) \sigma_{l}^{2}-V_{1 l}(x)\right] \omega_{3 l}(x) \\
& \quad=-x g(x) r_{l}(x) \int \frac{\omega_{3 l}(x)}{x g(x) b(x)} d x \tag{5.64}
\end{align*}
$$

In this form $\left(g^{-2}(x)\right.$ scales the spectral parameter $\left.\sigma^{2}\right)$ the equation (5.64) corresponds to the generalized spectral problem with the non-local potential

$$
\begin{equation*}
-\omega_{3 l, x x}+V_{1 l}(x) \omega_{3 l}(x)-T_{l} \omega_{3 l}(x)=\sigma_{l}^{2} \frac{1}{g^{2}} \omega_{3 l}(x) \tag{5.65}
\end{equation*}
$$

where

$$
\begin{equation*}
T_{l} u(x)=x g(x) r_{l}(x) \int_{d}^{x} \frac{u(z) d z}{z b(z) g(z)} \tag{5.66}
\end{equation*}
$$

is the integral Vol'terra operator. The lower limit is $d=$ $x_{+}$for a black hole case.

The condition (5.57) (which leads to $r_{l}=0$ ) is necessary and sufficient condition to reduce (5.64) to the Schrödinger equation with the local potential.

Introducing "the tortoise coordinate" $x_{*}(x)=$ $\int d x / g(x)$ and the function $w\left(x_{*}\right)$ by

$$
\begin{equation*}
w_{3 l}\left(x_{*}\right)=x \sqrt{g(x)} z_{3 l}\left(x_{*}\right) \tag{5.67}
\end{equation*}
$$

we reduce the system (5.58), (5.55b) to the form

$$
\begin{gather*}
w_{3 l, x_{*} x_{*}}+\left[\sigma_{l}^{2}-W_{l}(x)\right] w_{3 l}\left(x_{*}\right)=x g^{\frac{5}{2}}(x) r_{l}(x) z_{2 l}\left(x_{*}\right)  \tag{5.68}\\
z_{2 l, x_{*}}=-\frac{g^{1 / 2}(x)}{x b(x)} w_{3 l}\left(x_{*}\right) \tag{5.69}
\end{gather*}
$$

where

$$
\begin{align*}
& W_{l}(x)=g\left[\frac{l(l+1)}{x^{2}}+g^{\prime \prime}-\frac{1}{x} g^{\prime}+\frac{g\left(x g^{\prime \prime}-g^{\prime}\right)}{b}\left(\frac{x g^{\prime \prime}-g^{\prime}}{2 b}\right.\right. \\
& \left.\left.-\frac{g^{\prime}}{g}+\frac{1}{x}\right)-\frac{x g g^{\prime \prime \prime}}{2 b}-\frac{g}{x b}\left(\frac{1}{2} x^{2} g^{\prime \prime \prime}+x g^{\prime \prime}-g^{\prime}\right)\right] \tag{5.70}
\end{align*}
$$

In the limit $x \rightarrow x_{+}$the integral term in (5.66) tends to zero, on essence due to $z_{2 l} \rightarrow 0$. Indeed, when $x \rightarrow x_{+}$, we get $z_{2 l} \sim \sqrt{x-x_{+}}$with using (5.52b) and taking into account asymptotic behavior of $\left(L_{l}+X_{l}\right) \sim\left(x-x_{+}\right)$ which follows from (5.38).

The potential (5.70) vanishes as $x_{*} \rightarrow+\infty$ as $x^{-2}$, while for $x_{*} \rightarrow-\infty$, it vanishes exponentially. Therefore solutions to (5.68) have asymptotic $e^{ \pm i \sigma_{l} x_{*}}$ as $x_{*} \rightarrow \pm \infty$, so that we have to look for solutions satisfying boundary conditions

$$
\begin{gather*}
w_{l} \rightarrow e^{i \sigma_{l} x_{*}}+R_{l}^{(w)} e^{-i \sigma_{l} x_{*}} \quad \text { as } \quad x_{*} \rightarrow \infty \\
w_{l} \rightarrow T_{l}^{(w)} e^{i \sigma_{l} x_{*}} \quad \text { as } \quad x_{*} \rightarrow-\infty \tag{5.71}
\end{gather*}
$$

In the particular case of validity of (5.57), $r_{l}(x)=0$, $p_{\perp}^{\prime}=0$, the system (5.68)-(5.69) splits and we get the Schrödinger equation

$$
-w_{3 l, x_{*} x_{*}}+W_{0 l}(x) w_{3 l}\left(x_{*}\right)=\sigma_{l}^{2} w_{3 l}\left(x_{*}\right)
$$

with the potential

$$
\begin{gathered}
W_{0 l}(x)=g\left[\frac{l(l+1)}{x^{2}}+g^{\prime \prime}-\frac{1}{x} g^{\prime}\right. \\
\left.+\frac{g\left(x g^{\prime \prime}-g^{\prime}\right)}{b(x)}\left(\frac{x g^{\prime \prime}-g^{\prime}}{2 b(x)}-\frac{g^{\prime}}{g}+\frac{2}{x}\right)\right]
\end{gathered}
$$

which for the Schwarzschild geometry coincides with the potential in the Zerilli equation ( [37], Ch.4, eq.(63)).

We have reduced the basic system of three first-order linear equations (5.29) to a single second-order equation for the particular combination of these function, $w_{3 l}\left(x_{*}\right)$. In our case this is the Schrödinger equation (5.72) with non-local potential. Its non-local part vanishes when the condition (5.57) is satisfied. As in Schwarzschild case, this empirically found reducibility (resulted from a sequence of mysterious cancellations), follows in fact from the existence of some particular solution to the system which we have to reduce [37].

In our case the condition (5.57) is the necessary and sufficient condition for vanishing non-local part in (5.72) and thus for reducing our system (5.29) to the standard Schrödinger equation. Actually, the condition (5.57) is also the necessary and sufficient condition for existence of particular solution which guarantees such a reduction (it is shown in detail in our paper [45] devoted to the case satisfying (5.57)). If the condition (5.57) is satisfied, then the particular solution reads

$$
\begin{gathered}
N_{l}^{p}=\sqrt{g}\left[\frac{g^{\prime \prime}}{2 g}-\left(\frac{g^{\prime}}{2 g}\right)^{2}-\sigma_{l}^{2} \frac{x}{g}-b^{\prime}(x)\right] \\
L_{l}^{p}=-\sqrt{g} b^{\prime}(x) ; \quad X_{l}^{p}=\sqrt{g}\left(\frac{b(x)}{x}+b^{\prime}(x)\right)
\end{gathered}
$$

The linear transformation (5.51) is needed to reduce the system of 3 linear equations to one equation, if the particular solution to (5.29) exists. Remarkable luck in our case is that this transformation works not only in the case when (5.57) is satisfied (i.e. obtained second order equation is the standard Schrödinger equation with the local potential), but also in general case. This fact allowed us to reduce our system (5.29) to the Schrödinger equation (5.72) with the non-local potential.

We see that the problem of stability of the spherically symmetric solution (2.7) to polar perturbations, reduces to investigation of the spectral problem (5.68)-(5.69) with the potential of the form (5.70) and with the boundary conditions (5.71).

Equation (5.68), with taking into account (5.69), can be written in the form corresponding to the spectral problem with a non-local potential

$$
\begin{equation*}
-w_{3 l, x_{*} x_{*}}+W_{l}(x) w_{3 l}\left(x_{*}\right)-\tilde{T}_{l} w_{3 l}\left(x_{*}\right)=\sigma_{l}^{2} w_{3 l}\left(x_{*}\right), \tag{5.72}
\end{equation*}
$$

where
$\tilde{T}_{l} u\left(x_{*}\right)=x\left(x_{*}\right) g^{\frac{5}{2}}\left(x\left(x_{*}\right)\right) r_{l}\left(x\left(x_{*}\right)\right) \int_{d_{*}}^{x_{*}} \frac{g^{\frac{1}{2}}\left(x\left(z_{*}\right)\right) u\left(z_{*}\right) d z_{*}}{x\left(z_{*}\right) b\left(x\left(z_{*}\right)\right)}$,
is the integral Vol'terra operator.
Ultimately the case reduces to investigation of the eigenvalue problem with integro-differential operator (5.72), which is the Schrödinger equation with the nonlocal potential.

The spectrum of eigenvalues contain all the values of the parameter $\sigma_{l}^{2}$, at which solutions exist which satisfy the imposed boundary conditions.

If such solutions exist only for real values of the time parameter $\sigma_{l}$ and if, in addition, they form the complete basic set of functions, then any smooth initial perturbation on the finite interval of variable $x_{*}$ (with compact measure) can be expanded on this basic set, and since dependence of each particular mode on time is given by $\exp \left(i \sigma_{l} t\right)$, it testifies for stability of geometry.

Indeed, a considered static configuration is stable if there are no integrable modes with negative $\sigma_{l}^{2}$. Appearance of negative eigenvalues $\sigma_{l}^{2}$ would lead to existence of exponentially growing modes of perturbations.

In the next subsection we study in detail the integrodifferential operator governing polar perturbations.

## D. Schrödinger equation with non-local potential

A system governed by Schrödinger equation with a non-local potential, obeys the following theorems:
i) If in the standard one-dimensional Schrödinger equation the potential is nonnegative, then negative eigenvalues are absent (see, e.g., [42]).
ii) The Weyl theorem [46] for self-conjugate operators:

The essential spectrum conserves under relatively compact perturbations [47].

The essential spectrum is defined as follows: If we remove from the spectrum of self-conjugate operator all isolated points which are eigenvalues of finite multiplicity, then the remaining of the spectrum is the essential spectrum.

Essential spectrum of non-perturbed (local) potential is continuous and represented by positive semi-axis $[0, \infty)$, isolated points are absent in case when negative values are excluded by non-negativity of the potential.

It follows that the essential spectrum of the problem with the non-local potential (5.72) is the same as the essential spectrum of the non-perturbed (local) potential. Essential spectrum of local potential is this total spectrum, since isolated points are absent, i.e., essential spectrum of the perturbed problem coincides with the total spectrum of the non-perturbed problem.

Non-local part of a potential represents the perturbation of the local potential. To not spoil an essential spectrum, this perturbation should be relatively compact.

So, our task now is to prove that non-local part represents a compact perturbation and to deduce criterion of non-negativity of a local potential.

In our case non-local part (perturbation of a local potential) is given by the integral Vol'terra operator (5.73). Such an operator is totally continuous, if it has smooth square integrable kernel.

Square integrability requires

$$
\begin{equation*}
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K^{2}\left(x_{*}, z_{*}\right) d x_{*} d z_{*}<\infty \tag{5.74}
\end{equation*}
$$

The kernel of our Vol'terra operator (5.73) is

$$
\begin{equation*}
K\left(x_{*}, z_{*}\right)=x\left(x_{*}\right) g^{\frac{5}{2}}\left(x\left(x_{*}\right)\right) r_{l}\left(x\left(x_{*}\right)\right) \frac{g^{\frac{1}{2}}\left(x\left(z_{*}\right)\right)}{x\left(z_{*}\right) b\left(x\left(z_{*}\right)\right)} \tag{5.75}
\end{equation*}
$$

Its smoothness is evident. The sufficient condition for square integrability of the kernel $K(x, y)$ is the condition on behavior of $K^{2}$ at infinity:

$$
K^{2}(x, y)<\frac{1}{x^{1+\delta_{1}}} \frac{1}{y^{1+\delta_{2}}}
$$

where $\delta_{1}, \delta_{2}$ are arbitrarily small.
For $K\left(x_{*}, z_{*}\right)$ given by (5.75), for $z_{*} \rightarrow-\infty, K^{2}$ vanishes as $g\left(x\left(z_{*}\right)\right)$. When $x_{*} \rightarrow-\infty$, then $K^{2}$ vanishes as $g^{3}\left(x\left(x_{*}\right)\right)$. The metric function $g\left(x\left(x_{*}\right)\right)$ near horizon behaves as $g\left(x_{*}\right) \simeq g^{\prime}\left(x_{*}\right) e^{g^{\prime}\left(x_{*}\right) x_{*}}, g^{\prime}$ is positive, so that metric as a function of $x_{*}$ vanishes exponentially at approaching the horizon.

When $z_{*} \rightarrow \infty$, then $K^{2}$ vanishes as $x^{-2}\left(z_{*}\right)$. From definition $z_{*}$ we see the main contribution at infinity is $z_{*} \sim x$. When $x_{*} \rightarrow \infty, K^{2}$ vanishes as $x^{2}\left(p_{\perp}^{\prime}\right)^{2}$. The tangential pressure $p_{\perp}$ for de Sitter-Schwarzschild
geometry vanishes at infinity quicker than $x^{-3}$, because it is related with density by the equation of state $p_{\perp}=$ $-\rho-x \rho^{\prime} / 2$, and density vanishes quicker than $x^{-\frac{1}{3}}$ to guarantee the finiteness of a mass. Hence in this limit $K^{2}$ vanishes quicker than $x^{-6}$.

So, for a BH case the kernel is square integrable.
As a result the totally continuous operator (5.73) gives a relatively compact perturbation to the local potential in the integro-differential equation (5.72).

## Criterion of non-negativity of local potential

Introducing the function $p(x)=x g^{\prime \prime}(x)-g^{\prime}(x)$, we write the potential (5.72) in the form
$W_{l}(x)=g\left[\frac{1}{2} g\left(\frac{p}{b}\right)^{2}+\frac{1}{2 b(x)}\left(g^{\prime}\right)^{2}+\frac{2(n+1)}{x^{2}}-\frac{1}{b x} I_{l}(x)\right]$,
where

$$
\begin{equation*}
I_{l}(x)=x^{2}\left(\frac{1}{2} g^{\prime} g^{\prime \prime}+g g^{\prime \prime \prime}\right)-(n+1-g) p(x) \tag{5.77}
\end{equation*}
$$

In (5.76) we should investigate the term $I_{l}$. The rest is positive, since $b(x) \geq n$ in a BH case.

Expressing $g(x)$, its derivatives, $p(x)$ and $b(x)$ in terms of mass function $M(x)$, density $\rho(x)$ and its derivatives, we transform $I_{l}(x)$ to the form

$$
\begin{gather*}
I_{l}(x)=\alpha\left[-4 \alpha \frac{M^{2}}{x^{3}}+\frac{9}{2} \alpha x^{4} \rho \rho^{\prime}+3(n-1) x^{2} \rho^{\prime}\right. \\
\left.+\frac{9}{2} \alpha x \rho^{\prime} M-3 n x \rho-3 g x\left(x^{2} \rho^{\prime \prime}+2 \rho\right)+\frac{3(n+2)}{x^{2}} M\right] \tag{5.77a}
\end{gather*}
$$

For a BH case $g(x)>0$ while the weak energy condition gives $\rho^{\prime}<0$. Then the sufficient condition for $W_{l} \geq 0$ is the condition on the equation of state

$$
\begin{equation*}
x^{2} \rho^{\prime \prime}(x)+2 \rho(x) \geq 0 \tag{5.78a}
\end{equation*}
$$

which constraints the growth of the derivative of $p_{\perp}+\rho$

$$
\begin{equation*}
x\left(p_{\perp}+\rho\right)^{\prime} \leq \rho+\left(p_{\perp}+\rho\right) \tag{5.78b}
\end{equation*}
$$

This condition is actually satisfied also for the case without horizons (then $g(x)>0$ for all $x$ ).

When (5.78) is satisfied, then proof of non-negativity of (5.76) reduces to proof of non-negativity of the function

$$
\begin{equation*}
\phi(x)=\frac{2(n+1)}{x^{2}}-\frac{3 \alpha(n+2) M}{x^{3} b(x)} \tag{5.79}
\end{equation*}
$$

It is bounded from below as follows

$$
\begin{gathered}
\phi(x)=\frac{2}{x^{2}}\left[n+1-\frac{3 \alpha M(n+2)}{2 x b(x)}\right] \\
\geq \frac{2}{x^{2}}\left[n+1-\frac{3}{2} \frac{(n+2)}{n} \frac{\alpha M}{x}\right]=\frac{2}{x^{2}}\left[n+1-\frac{3}{2} \frac{(n+2)}{n}(1-g)\right]
\end{gathered}
$$

$$
\begin{equation*}
>\frac{2}{x^{2}}\left[n-\frac{1}{2}-\frac{3}{n}\right] \geq \frac{2}{x^{2}}(n-2) \geq 0 \tag{5.80}
\end{equation*}
$$

As a result, we find the sufficient condition (5.78) for non-negativity of the potential (5.70) in all range of argument for which $g(x)>0$.

For the density profile (1.3) this condition is satisfied.
We can conclude that the essential spectrum of the integro-differential operator (5.72) is the same as the essential spectrum of its local potential. Now the key point is to find the condition on a perturbation of a local potential which guarantees the absence of isolated points in the total spectrum (negative values of $\sigma_{l}^{2}$ ) of the integrodifferential operator (5.72).

## Non-local contribution

Multiplying (5.68) by $w_{3 l}^{* *}$ and integrating by parts with taking into account asymptotic behavior of (5.60) at infinity, we obtain the following relation

$$
\begin{gather*}
\sigma_{l}^{2} \int_{-\infty}^{+\infty}\left|w_{3 l}\left(x_{*}\right)\right|^{2} d x_{*}+\left.w_{3 l, x_{*}} w_{3 l}^{*}\right|_{-\infty} ^{\infty}= \\
\int_{-\infty}^{+\infty}\left[\left|w_{3 l, x_{*}}\right|^{2}+W_{l}(x)\left|w_{3 l}\left(x_{*}\right)\right|^{2}+\psi_{l, x} g(x)\left|z_{2 l}\left(x_{*}\right)\right|^{2}\right] d x_{*} \tag{5.81}
\end{gather*}
$$

where

$$
\begin{equation*}
\psi_{l}(x)=\frac{x^{2}}{2} g^{2}(x) b(x) r_{l}(x) \tag{5.82}
\end{equation*}
$$

The Wronskian $\left.w_{3 l, x_{*}} w_{3 l}^{*}\right|_{-\infty} ^{+\infty}$ of two independent solutions $w_{3 l}$ and $w_{3 l}^{*}$ is constant (see [37], Par.27 Ch.4).

The contribution to the spectrum from the non-local part of the potential is given by

$$
\begin{equation*}
N=\int_{x_{+}}^{+\infty} \psi_{l, x}(x)\left|z_{2 l}(x)\right|^{2} d x \tag{5.83}
\end{equation*}
$$

If the condition of non-negativity of a local potential $W_{l}(x)$ is satisfied, then the requirement

$$
\begin{equation*}
N=\int_{x_{+}}^{+\infty} \psi_{l, x}(x)\left|z_{2 l}(x)\right|^{2} d x \geq 0 \tag{5.84}
\end{equation*}
$$

gives the sufficient condition for the absence of negative eigenvalues $\sigma_{l}^{2}$ of the considered spectral problem.

[^0]Fortunately, non-local contribution given by (5.83), does not grow with the mode number $n$, since $\left|z_{2 l}\right|^{2}$ is constrained from above by the function proportional to $n^{-2}$. This constraint is valid for any density profile and follows from (5.69) with taking into account that in a BH case $b(x) \geq n$.

In the case when the metric function $g(x)$ satisfies the condition (5.57), the sufficient condition (5.84) is trivially satisfied $(N=0)$, and negative eigenvalues do not appear in the spectrum. As a result spherically symmetric metrics satisfying (5.57) and (5.78) are stable to polar perturbations.

## VI. THE CASE OF DENSITY PROFILE (1.3)

In the case of the density profile $\rho(x)=e^{-x^{3}}$ the metric function in (5.28) reads

$$
\begin{equation*}
g(x)=e^{2 \nu(x)}=1-\frac{\alpha}{x}\left(1-e^{-x^{3}}\right) \tag{6.1}
\end{equation*}
$$

Potential (4.14) governing the axial perturbations is depicted in Figs.5-6 for the density profile (1.3) and two values of the characteristic parameter $\alpha$ (denoted in figures as $a$ ).


FIG. 5. Axial potential (4.14) for $m \simeq 2.8 m_{c r}$.


FIG. 6. Axial potential (4.14) for $m \simeq 11.2 m_{c r}$.

Potentials are smooth, short-range and positive for all values of the characteristic parameter $\alpha$. Therefore all types of vacuum configurations with de Sitter center including a vacuum nonsingular black hole, are stable to axial perturbations.

The local potential governing polar perturbations given by (5.70) is shown in figs.7-8 for the density profile (1.3) and two values of parameter $\alpha$ (denoted in figures as $a)$.


FIG. 7. Polar potential (5.70) for $m \simeq 2.8 m_{\text {crit }}$.


FIG. 8. Polar potential (5.70) for $m \simeq 11.2 m_{\text {crit }}$.
Both axial and polar potentials for bigger values of $\alpha$ become similar to those for the Schwarzschild case [37].

Polar local potentials are smooth short-range potentials, so that integrals of them are finite over all the region of variable $x$.

The potential (5.70) for the density profile (1.3) satisfies the criterion of non-negativity (5.78), but the condition (5.57) is not satisfied, so that appearance of negative eigenvalues $\sigma_{l}^{2}$ is in principle possible.

The question of existence of isolated points with negative values $\sigma_{l}^{2}$ in the total spectrum of the integrodifferential operator(5.72) requires the complicated numerical analysis which is in progress. Preliminary results
suggest that non-local contribution (5.83) does not lead to negative values $\sigma_{l}^{2}$ for the masses $m>m_{c r 2}$.

This result looks natural. The second critical mass value $m_{c r 2}$ is distinguished for the unperturbed geometry. The value $m_{c r 2}$ marks the point in the temperature-mass diagram at which specific heat is broken and changes its sign, so that a second order phase transition starts when in the course of Hawking evaporation the mass approaches the value $m_{c r 2}$. For the density profile (1.3) it is given by

$$
\begin{equation*}
m_{c r 2} \simeq 0.38 m_{P l} \sqrt{\rho_{P l} / \rho_{0}} \tag{6.2}
\end{equation*}
$$

The extreme state of non-singular black hole $\left(m=m_{c r}\right)$, can be unstable since some perturbations modes grow unlimited at the double horizon for any density profile. If the considered configuration would develop instability before achieving the extreme state, the most appropriate range gets beyond $m_{c r 2}$ where a phase transition starts.

The critical value $m_{c r 2}$ corresponds to the maximum at the temperature-mass curve (see fig.3). It is calculated from the condition $d T / d m=0$. In the units normalized to de Sitter radius $r_{0}$ (which is the characteristic scale related to de Sitter vacuum trapped in the origin)

$$
\begin{equation*}
y_{+}=\frac{r_{+}}{r_{0}} ; \quad s=\frac{r_{g}}{r_{0}} \tag{6.3}
\end{equation*}
$$

the temperature on a BH event horizon is given by [17]

$$
\begin{equation*}
T=\frac{1}{y_{+}}-\frac{3}{y_{+}}\left(1-\frac{y_{+}}{s}\right) \tag{6.4}
\end{equation*}
$$

The density profile and metric in these units read

$$
\begin{equation*}
\rho(y)=e^{-y^{3} / s} ; \quad g(y)=1-\frac{s}{y}\left(1-e^{-y^{3} / s}\right) \tag{6.5}
\end{equation*}
$$

From $d T / d s=0$ and $g\left(y_{+}\right)=0$, we get the critical value $s_{2}$ and the value $y_{+}$corresponding to $m=m_{\text {cr } 2}$.

$$
\begin{equation*}
s_{2}=\simeq 2.226 ; \quad y_{+}=\simeq 2.166 \tag{6.6}
\end{equation*}
$$

For comparison, the critical values for the extreme case $m_{c r}$ of the double horizon $r_{ \pm}$are [17]

$$
\begin{equation*}
s_{c r} \simeq 1.7576 ; \quad y_{ \pm} \simeq 1.4957 \tag{6.7}
\end{equation*}
$$

## VII. DISCUSSION

We present the conditions specifying two types of configurations with de Sitter center, including black holes with and without changes of topology.

We found that any configuration described by spherically symmetric geometry with de Sitter center is stable to axial perturbations.

The problem of stability to polar perturbation reduces to a one-dimensional Schrödinger equation with a nonlocal potential given by the Vol'terra integral operator
with square integrable smooth kernel representing a compact perturbation to the local potential.

We derived the criterion of non-negativity of the local potential which defines the essential spectrum of integrodifferential operator governing polar perturbations in general case.

We derived the criterion of vanishing of non-local part of the potential which distinguishes the class of geometries for which the problem of stability reduces to a standard one-dimensional Schrödinger equation. This class of metrics has been studied in [45].

For the case when perturbations are described by the Schrödinger equation with the non-local potential, we found the sufficient condition for the absence of the negative eigenvalues in the spectrum which guarantees the stability of investigated geometry.

For an extreme black hole, the method of small perturbations is not applicable due to existence of unlimited perturbation modes at approaching the double horizon.

Asymptotic behavior of the basic system near double horizon suggests instability of the extreme configuration. The behavior in this regime is very special, unrestricted solutions for perturbations exist for positive values of the spectral parameter $\sigma_{l}^{2}$. The limiting equation for perturbations near double horizon is essentially different from that for one-horizon case in which unrestricted solutions do not appear for positive values $\sigma_{l}^{2}$ and which cannot be smoothly continued to two-horizon case. The question arises what is the place of the metric with $m=m_{c r}$, at the set of metrics whose stability we investigate as one-parametric set of solutions to Einstein unperturbed equations with a given density profile.

The critical value of the mass parameter $m_{c r}$ is calculated from two transcendental equations: $g\left(r_{ \pm}\right)=$ $0 ; g^{\prime}\left(r_{ \pm}\right)=0$. This is the unique point for considered metric function $g(r)$ because it is the minimum of $g(r)$ and $g(r)$ has the only one minimum [20]. Therefore the transcendental system for $\left(r_{ \pm}, m_{c r}\right)$ has the unique solution for each particular one-parametric set with a given density profile.

The metric with double horizon represents an isolated singular point at the set of metrics $g(r)$ for each given density profile. It resembles a fixed point attractor behavior which is currently the key point of our efforts [45].

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[^0]:    *We denote the complex conjugate by * for convenience of comparison with the classical results presented in [37].

