

Black Hole Evaporation Rates without Spacetime

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Verlinde recently suggested that gravity, inertia, and even spacetime may be emergent properties of an underlying thermodynamic theory. This vision was motivated in part by Jacobson's 1995 surprise result that the Einstein equations of gravity follow from the thermodynamic properties of event horizons. Taking a first tentative step in such a program, we derive the evaporation rate (or radiation spectrum) from black hole event horizons in a spacetime-free manner. Our result relies on a Hilbert space description of black hole evaporation, symmetries therein which follow from the inherent high dimensionality of black holes, global conservation of the no-hair quantities, and the existence of Penrose processes. Our analysis is not wedded to standard general relativity and so should apply to extended gravity theories where we find that the black hole area must be replaced by some other property in any generalized area theorem.

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Jacobson's derivation of the Einstein field equations [1,2] indicates that the physics across event horizons determines the structure of gravity theory. To realize Verlinde's [3] vision, therefore, it seems reasonable to start by studying this physics, at the microscopic or quantum mechanical level, i.e., the particle production by the event horizon [1–3]. Although we focus primarily on black hole event horizons, much of the analysis should apply to general event horizons.

We require that the particle production mechanism be consistent with the complete unitary evaporation of a black hole. This strongly suggests tunneling as this mechanism [4–6]: Particles quantum mechanically tunnel out across the classically forbidden barrier associated with the event horizon to emerge as Hawking radiation [7,8]. Recent calculations of tunneling probabilities (the evaporation rate) from black holes have incorporated backreaction from an escaping particle on the classical black hole due to conservation laws of the no-hair quantities. Although only a limited number of black hole types, particle types, and WKB trajectories have been studied in this way (see, e.g., [7–9]), the tunneling probabilities appear to take the generic form

$$\Gamma \propto e^{\mathcal{S}_{\text{final}} - \mathcal{S}_{\text{initial}}}, \quad (1)$$

where $\mathcal{S}_{\text{initial(final)}}$ are the thermodynamic entropies of the black hole before (after) the tunneling process. In the simplest case of a spinless particle of energy ε evaporating from a Schwarzschild black hole of mass M along a radial trajectory, one has [8]

$$\Gamma \propto e^{4\pi(M-\varepsilon)^2 - 4\pi M^2}. \quad (2)$$

Tunneling moves subsystems from the black hole interior (int) to the exterior, appearing as radiation (R) [5]. Formally, the simplest Hilbert space description of such a process is given by

$$|i\rangle_{\text{int}} \rightarrow (U|i\rangle)_{BR}. \quad (3)$$

Here B denotes the reduced size subsystem corresponding to the remaining interior of the black hole, $|i\rangle$ is the initial state of the black hole interior (which we take here to be pure for convenience and without loss of generality [5]), and U denotes the unitary process “selecting” the subsystem to eject.

Note that spacetime and black hole geometry are not explicit in Eq. (3). Even the event horizon appears only as a generic Hilbert space tensor product structure separating what we call interior from exterior [6,10]. These observations provide support for the conjecture that Eq. (3) should apply to evaporation across arbitrary event horizons. For a more detailed motivation and history of this description, see Refs. [5,6].

We show that Eq. (3), symmetries therein, and global conservation laws imply Eq. (1) for evaporation across black hole event horizons. To apply the generic Eq. (3) specifically to such horizons we assume a correspondence between quantum and classical descriptions of black holes. In particular, we rely on their labeling by no-hair quantities and the existence of Penrose processes.

The first symmetry we investigate is a permutation symmetry in the order of “decay” products (evaporated particles). Consider a pair of distinct subsystems of the radiation. Interchanging them corresponds to a unitary operation which may be formally absorbed into the internal unitary U in Eq. (3). Because the Hilbert space dimensionalities needed to describe a black hole are so vast (at least $10^{10^{77}}$ for a stellar-mass black hole) random matrix theory [5,11] tells us that the statistical behavior of Eq. (3) is excellently approximated by treating U as a random unitary (i.e., by using a Haar average). Therefore, permuting the order in which particles appear as Hawking radiation will have no statistical effect.

In the case of a Schwarzschild black hole of mass M undergoing a pair of consecutive evaporation events producing spinless particles in an s wave of energies ε_1 and ε_2 , this permutation symmetry implies an equality between tunneling probabilities: $\Gamma(\varepsilon_1, \varepsilon_2|M) = \Gamma(\varepsilon_2, \varepsilon_1|M)$, or in terms of conditional probabilities,

$$\Gamma(\varepsilon_1|M)\Gamma(\varepsilon_2|\varepsilon_1, M) = \Gamma(\varepsilon_2|M)\Gamma(\varepsilon_1|\varepsilon_2, M). \quad (4)$$

Now in field theory calculations, tunneling probabilities concern transitions between classical macroscopic spacetime geometries. Abstracting this into a spacetime-free language we would say that earlier decays should only affect subsequent decays through their backreaction on the black hole's identity via conservation laws. In the simple scenario above, particles only carry away black hole mass as energy, so $\Gamma(\varepsilon|\varepsilon', M) = \Gamma(\varepsilon|M - \varepsilon')$. This leaves the single-particle functional relation

$$\Gamma(\varepsilon_1|M)\Gamma(\varepsilon_2|M - \varepsilon_1) = \Gamma(\varepsilon_2|M)\Gamma(\varepsilon_1|M - \varepsilon_2). \quad (5)$$

Theorem 1: Suppose the function Γ is continuously differentiable in its domain of definition and satisfies Eq. (5). Then its general solution is

$$\Gamma(\varepsilon|M) = e^{f(M-\varepsilon)-f(M)+h(\varepsilon)}, \quad (6)$$

where f and h are arbitrary functions, continuously differentiable except possibly at some boundary points.

The general solution provided by this functional equation (see Ref. [6] for proofs of all our theorems) easily matches the known result of Eq. (2). Thus at least for this scenario, the permutation symmetry predicted by the Hilbert space description of Eq. (3) is supported by quantum field theoretic tunneling calculations on curved spacetime [8]. (Consistency with Hawking's original result of a thermal distribution for black hole radiation [12] when backreaction is negligible, i.e., when the energy ε carried away is infinitesimal, would immediately implicate f as the black hole's thermodynamic entropy.)

To generalize this result to more general scenarios we need only assume that any changes that occur in an event horizon's identity due to evaporation are characterized by linear conservation laws. We will now explicitly show that this approach is valid for evaporation of black holes.

Recall that the no-hair theorem [13] tells us that a black hole is characterized solely by its mass M , charge Q , and angular momentum J along some axis \hat{n} . The parameters $\vec{X} \equiv (M, Q, J)$ can be “readout” [14] (copied) by arbitrarily many observers throughout the spacetime geometry—they correspond to classical information about the quantum state of the black hole. It is therefore natural to associate a classical black hole with a quantum state which is the simultaneous eigenstate of M , Q , and $J_{\hat{n}}$. To ensure that angular momentum is described by a single quantum number (as required by the no-hair theorem), the angular momentum state for a black hole must correspond to a spin-coherent state $|J, J\rangle_{\hat{n}} = R(\theta, \phi)|j = J, m = J\rangle$,

where $|j, m\rangle$ are the usual simultaneous eigenstates of total angular momentum J^2 and J_z and $R(\theta, \phi)$ is a rotation operator which maps \hat{z} to \hat{n} . This correspondence has the added feature of making the quantum description of a black hole a minimum uncertainty angular momentum state—i.e., as classical as possible in its angular momentum degrees of freedom.

Now the ability of an infinite set of observers throughout spacetime to copy the classical information about the (black hole) geometry they are sitting in places a very strong constraint on any physical process. In particular, any process that yields a superposition of black hole states can only preserve this “copyability” if the superposition can be expressed as a sum over mutually orthogonal black hole states. This property and the presumed conservation during black hole evaporation of total energy, charge, and angular momentum yield the following.

Theorem 2: Consider a lone black hole $\vec{X} \equiv (M, Q, J)$ oriented along some direction \hat{n} that undergoes an evaporative process yielding a particle and a daughter black hole. If the particle's energy ε , charge q , and total (spatial plus spin) angular momentum j along the \hat{n} axis are measured, then the remaining state of the daughter black hole will be described by the no-hair triple $\vec{X}' - \vec{x}$ along \hat{n} , where $\vec{x} \equiv (\varepsilon, q, j)$. (Note, \hat{n} is arbitrary for $J = 0$.)

This theorem tells us that the transition from black hole mother to daughter by evaporation satisfies a simple set of linear conservation laws. Note, Theorem 2 should not be taken to imply that the particle is fully described by $\vec{x} \equiv (\varepsilon, q, j)$ and has no other “hair.” For example, the particle need not be in an overall spin-coherent state.

It immediately follows from Theorem 2 and the permutation symmetry already discussed that the probability $\Gamma(\vec{x}|\vec{X})$ for a particle with triple $\vec{x} \equiv (\varepsilon, q, j)$ to tunnel from a black hole with no-hair triple $\vec{X} = (M, Q, J)$ will satisfy $\Gamma(\vec{x}|\vec{X})\Gamma(\vec{x}'|\vec{X} - \vec{x}) = \Gamma(\vec{x}'|\vec{X})\Gamma(\vec{x}|\vec{X} - \vec{x}')$, again presuming that prior evaporative events only affect subsequent decays via their (linear) conservation laws, so $\Gamma(\vec{x}|\vec{x}', \vec{X}) = \Gamma(\vec{x}|\vec{X} - \vec{x}')$. It is therefore natural to extend Theorem 1 to the multivariate case.

Theorem 3: Let $\Gamma(\vec{x}|\vec{X})$ be a positive real function that is continuously differentiable in each of the variables, $\vec{x}, \vec{X} \in D \subset \mathbb{R}^n$, where the domain D of each of the (vector) arguments is assumed to be a closed subset of \mathbb{R}^n . Suppose that Γ satisfies the functional equation

$$\Gamma(\vec{x}|\vec{X})\Gamma(\vec{x}'|\vec{X} - \vec{x}) = \Gamma(\vec{x}'|\vec{X})\Gamma(\vec{x}|\vec{X} - \vec{x}'). \quad (7)$$

Then the function Γ is given by

$$\Gamma(\vec{x}|\vec{X}) = e^{f(\vec{X}-\vec{x})-f(\vec{X})+h(\vec{x})}, \quad (8)$$

where $f(\vec{X})$ and $h(\vec{x})$ are continuously differentiable functions of their arguments.

Moving beyond black holes, if a general event horizon were characterized by some vector of attributes \vec{X} and if

evaporation produced a linear backreaction to these attributes, then permutation symmetry and this theorem would imply the same functional form quite generally. Again, were we to assume consistency with a thermal spectrum [12] when backreaction is negligible, we would find that $f(\vec{X})$ is the thermodynamic entropy associated with the event horizon. Instead of this route, let us consider another symmetry specifically possessed by black holes that will allow us to determine both f and h and to uncover further structure. This will have the added benefit of allowing us to infer an almost certain breakdown of the area theorem [15] in extended gravity theories.

In quantum theory a reversible process should be represented by a unitary operation in Hilbert space. A reversible Penrose process [16] allows one to freely interconvert between two black holes with no-hair triples \vec{X}_1 and \vec{X}_2 , provided only that their irreducible masses I are equal. Consider a pair of such reversible processes applied to a black hole so as to bracket a single tunneling event. If the unitary describing that event can be well approximated by its Haar average, we must have

$$\Theta(\vec{X}_1, \vec{X}'_1) = \Theta(\vec{X}_2, \vec{X}'_2), \quad (9)$$

whenever $I(\vec{X}_1) = I(\vec{X}_2)$ and $I(\vec{X}'_1) = I(\vec{X}'_2)$. Where for convenience we introduce transition probabilities $\Theta(\vec{X}, \vec{X}') \equiv \Gamma(\vec{X} - \vec{X}' | \vec{X})$ for a mother black hole \vec{X} to yield a daughter \vec{X}' after a single tunneling event.

A holographic view of an event horizon might be stated as saying that the Hilbert space beyond the event horizon is effectively encoded entirely at or near the event horizon itself. In such a case, the Haar symmetry for each individual tunneling event would be an ideal description of the random sampling of near event-horizon degrees of freedom for ejection as radiation. Specifically for black holes, Ref. [5] proves that, in order to preserve the equivalence principle during (unitary) evaporation over a black hole's lifetime, virtually the entire Hilbert space of the black hole must be encoded at the surface (in the form of trans-event-horizon entanglement), with at most a vanishingly small proportion located within the black hole interior. In this case Haar symmetry for each tunneling event would still be an excellent approximation. (In contrast, if the Hilbert space of the black hole interior were not encoded primarily near the surface, then its Haar invariance would require the dynamical assumption of a very short ‘‘global thermalization time’’ for the black hole—little more than the time for charge to spread across a black hole's surface.)

Theorem 4: Let $I : \Sigma \rightarrow \mathbb{R}$ be continuous on Σ (a closed subset of \mathbb{R}^n) and continuously differentiable on its (nonempty) interior Σ° . Assume further that the subset $K \subset \Sigma^\circ$ on which all partial derivatives $\partial I / \partial X_i$ vanish contains no open set. Furthermore, let $\Theta : \Sigma \times \Sigma \rightarrow \mathbb{R}$ be continuous. Suppose

$$\begin{aligned} I(\vec{X}_1) = I(\vec{X}_2) \quad \text{and} \quad I(\vec{X}'_1) = I(\vec{X}'_2) \\ \Rightarrow \Theta(\vec{X}_1, \vec{X}'_1) = \Theta(\vec{X}_2, \vec{X}'_2), \end{aligned} \quad (10)$$

then $\Theta(\vec{X}, \vec{X}') = \theta(I(\vec{X}), I(\vec{X}'))$ for some function θ .

Combining this with Theorem 3 implies that the universal function $f(\vec{X}) = u(I(\vec{X}))$ is some function solely of the irreducible mass I and that $h(\vec{x})$ must be a constant. In other words,

$$\Gamma(\vec{x} | \vec{X}) = \mathcal{N} e^{u(I(\vec{X}-\vec{x})) - u(I(\vec{X}))}, \quad (11)$$

where \mathcal{N} is a normalization constant.

We will now determine the function u from the fact that the Hilbert space description of evaporation in Eq. (3) is manifestly reversible. We assume that a black hole can evaporate away completely, since any stable black hole remnant would itself be tantamount to a failure of unitarity [17] and hence of quantum mechanics. Consider therefore the complete evaporation of our Hilbert space black hole with initial no-hair triple \vec{X} , leaving nothing but radiation. The probability for seeing a specific stream of radiation with triples $\{\vec{x}_1, \vec{x}_2, \dots\}$ may be precisely computed from Eq. (11) as

$$\Gamma(\vec{x}_1, \vec{x}_2, \dots | \vec{X}) \equiv \mathcal{N}' e^{u(I(\vec{0})) - u(I(\vec{X}))}, \quad (12)$$

where $u(I(\vec{0}))$ must be finite to ensure that complete evaporation is possible. Because Eq. (12) is independent of the specific radiation triples, it implies that the thermodynamic entropy of the radiation is exactly

$$S_{\text{rad}} = u(I(\vec{X})) - u(I(\vec{0})) - \ln \mathcal{N}' \quad (13)$$

(see Refs. [18,19] for related arguments). To ensure reversibility the entropy in the radiation must equal the thermodynamic entropy of the original black hole $\mathcal{S}(\vec{X})$. In other words, we must have $S_{\text{rad}} = \mathcal{S}(\vec{X})$, which in turn implies that $u(I(\vec{X}))$ is just the thermodynamic entropy associated with the black hole.

In general relativity both the thermodynamic entropy of a black hole and its irreducible mass are known to be functions of the black hole surface area, so their connection here is not too surprising. However, in higher curvature theories of gravity, Gauss-Bonnet gravity and other Lovelock extended gravities, the thermodynamic entropy (now the Noether charge entropy, up to quantum corrections) is not simply a function of area alone [20]. Although Penrose processes and the corresponding irreducible mass for black holes in these extended theories have not been analyzed, we have shown that the Hilbert space description of black hole evaporation implies that irreducible mass will be some function of the Noether charge entropy. This suggests that if it is possible to generalize Hawking's area theorem [15] to these theories, then some function of Noether charge entropy will instead play the role of black hole surface area.

Our above analysis shows that the black hole tunneling probabilities reduce to

$$\Gamma(\vec{x}|\vec{X}) = \mathcal{N} e^{S(\vec{X}-\vec{x})-S(\vec{X})}, \quad (14)$$

a result identical to the generic form of Eq. (1) [21]. In a sense then, black holes are not ideal but “real black bodies” that satisfy conservation laws, result in a non-thermal spectrum, and preserve thermodynamic entropy. In contrast, treated as ideal black bodies, black hole evaporation would lead to irreversible entropy production [23].

Our reasoning leading to Eq. (14) holds for all black hole and particle types (even in extended gravities) and is not limited to one-dimensional WKB analyses which underlie all previous quantum tunneling calculations. This supports our conjecture that Eq. (3) provides a spacetime-free description of evaporation across black hole horizons.

The physics deep inside the black hole is more elusive. Unfortunately, any analysis relying primarily on physics at or across the horizon cannot shed any light on the question of unitarity (which lies at the heart of the black hole information paradox). If unitarity holds globally, then Eq. (3) can be used to describe the entire time course of evaporation of a black hole and to learn how the information is retrieved (see, e.g., Ref. [5]). Specifically, in a unitarily evaporating black hole, there should exist some thermalization process, such that after what has been dubbed the black hole’s global thermalization (or scrambling) time, information that was encoded deep within the black hole can reach or approach its surface where it may be selected for evaporation as radiation. Alternatively, if the interior of the black hole is not unitary, some or all of this deeply encoded information may never reappear within the Hawking radiation.

At this stage we might take a step back and ask the obvious question: Does quantum information theory really bear any connection with the subtle physics associated with black holes and their spacetime geometry? After all, we do not yet have a proper theory of quantum gravity. However, whatever form such a theory may take, it should still be possible to argue, either due to the Hamiltonian constraint of describing an initially compact object with finite mass or by appealing to holographic bounds, that the dynamics of a black hole must be effectively limited to a finite-dimensional Hilbert space. Moreover, one can identify the most likely microscopic mechanism of black hole evaporation as tunneling [5,6]. Formally, these imply that evaporation should look very much like Eq. (3). Although finite, the dimensionalities of the Hilbert space are immense and from standard results in random unitary matrix theory and global conservation laws we obtain a number of invariances. These invariances completely determine the tunneling probabilities without needing to know the detailed dynamics (i.e., the underlying Hamiltonian). This result puts forth the Hilbert space description of black hole evaporation as a powerful tool. Put even more strongly, one

might interpret the analysis presented here as a quantum gravity calculation without any detailed knowledge of a theory of quantum gravity except the presumption of unitarity.

At a deeper level, the spacetime-free Hilbert space description and random matrix calculus should apply to arbitrary event horizons, not just those defining black holes (e.g., the Rindler horizon appears in the infinite mass limit of the Schwarzschild geometry [24]). In that case, Jacobson’s work [1,2] might suggest that the gravitational structure of spacetime and presumably spacetime itself along with related concepts could appear as emergent phenomena. If so, the approach presented here may provide a promising beginning towards achieving Verlinde’s vision [3]. However, to get even this far required a subtle but crucial change in that vision. Rather than emergence from a purely thermodynamic source, we should instead seek that source in quantum information.

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