# Periodic Swing-By Orbits between Earth and Venus 

Walter M. Hollister ${ }^{*}$ and Michael D. Menning $\dagger$<br>Massachusetts Institute of Technology, Cambridge, Mass.


#### Abstract

The general problem of finding fly-by dates for multiple swing-by missions involves an iterative search in a space with dimension equal to the number of swingbys assuming the launch date and final arrival date are specified. A double swing-by trajectory that visits Mars and Venus requires a search in two dimensions. Periodic orbits connecting Earth and Venus require a search in a space of dimension between 10 and 15 . This paper reports the results of a study of the latter class of orbits. The computing procedures are generally applicable to any multiple swing-by problem. A summary is given of the computational experience gained. Trajectory data are presented for those periodic orbits which were computed.


## Introduction

T'HE use of a multiple swingby as part of an interplanetary mission was considered as early as 1925 by Hohmann ${ }^{1}$ and 1956 by Crocco. ${ }^{2}$ They each proposed interplanetary fly-by missions that would take a vehicle past both Mars and Venus before returning to Earth. Several investigators ${ }^{3-5}$ have subsequently studied this class of mission in more detail. It was Minovitch, ${ }^{6}$ however, who first recognized the fundamental role which the planetary flyby can play in trajectory design. He saw the planets as sources of free thrust which could be utilized to project a vehicle from one planet to another without the use of fuel. In Ref. 7 he described, for example, a round trip mission leaving and arriving Earth with six intermediate flybys at Venus, Mars, Earth, Mars, Earth, and Venus. He further proposed an interplanetary transportation network, ${ }^{8}$ using multiple fly-by trajectories that would continue indefinitely. In the interest of finding such trajectories it has been proposed that the natural periodicity of the solar system be used to develop periodic orbits. ${ }^{9}$

For the orbit to be periodic, the spacecraft must recurrently flyby a sequence of planets. The first and last flyby of the sequence must therefore occur at the same planet with identical spacecraft velocities and absolute planet orientations. For this reason the duration of one period of an acceptable orbit is restricted to integral multiples of the time required for the encountered planets to repeat their absolute orientation. Ideally a perfectly established periodic orbit would result in no subsequent thrust requirements. In practical application injection errors and small perturbations from the periodic orbit will inevitably be present. Hickman ${ }^{10}$ has shown, however, that the guidance requirements for nominal periodic orbits connecting Earth and Venus are quite reasonable.

Although periodic orbits are perhaps the most difficult of the multiple swing-by problems to analyze, their large number of swingbys make them particularly interesting. The computing procedures are generally applicable to any multiple swing-by problem. Rall, ${ }^{11}$ for example, has used the techniques reported here to find periodic orbits connecting Earth and Mars.

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## Multiple Swing-By Orbits

All the results in this report are based on patched conic analysis. ${ }^{12}$ The interplanetary trajectory is completely defined by the dates at each planet. Assume that only the initial and final dates of a multiple flyby are given. Then the number of dates to be selected is equal to the number of swingbys. For the case of a double swingby, all the possible combinations of dates are represented by the points on a plane. The locus of dates that produce equal magnitude inbound and outbound hyperbolic velocity at one planet is a line in the plane. The locus of dates that produce equal magnitude inbound and outbound hyperbolic velocity at the other planet is another line in the plane. The intersection of the two lines represents a pair of dates that satisfies the first necessary condition for a multiple swingby. Figure 1 shows these loci for a double swingby of Venus and Mars. This example has six intersections. The problem of finding the intersections is equivalent to finding the zeros of a function defined as the sum of the squares of the differences in relative velocity magnitudes at the two flybys. At each intersection the value of the function vanishes and thereby achieves a global minimum. The function represents a surface in three dimensions. The value of the function determines the height of the surface above the date-at-Mars-date-at-Venus plane. Contours are sketched around one of the better behaved intersections. A study of Fig. 1 shows how complex the surface can be even with only two swing-by dates to consider. Successful solution by iterative methods is contingent upon the initial approximation of the independent variables. A poor initial choice may cause the iterative procedure to find a local minimum rather than the desired solution. It is also possible for the search process to bog down in a ravine of the surface. Since the problem is compounded for higher-dimensional spaces, the need for a good initial guess is apparent.

The feasibility of a swing-by orbit also rests on each planet's capability to turn the inbound relative velocity into the outbound relative velocity on a hyperbolic path that does not pass below the planet surface. If this is possible, the dates of the planet encounters and corresponding planet locations are the solution to the problem. In this report all flybys occurring beyond 1.1 planet radii are considered acceptable. Not all the intersections in Fig. 1 satisfy the second condition. Each point needs to be found and then tested to ascertain that the hyperbolic path does not go below the surface.

## Direct Return Orbits

A direct return orbit is a sun-centered ellipse that returns a spacecraft to the same planet from which it was launched


Fig. 1 Flyby dates for double swingby.
without interruption by encounters with other planets. These orbits find useful application when the elliptical trajectory between the planets results in large excess hyperbolic velocities. Large excess hyperbolic velocities reduce the maximum allowable turn angle at a given planet and hence the chances for making acceptable flybys above the planet's surface. Large values of the excess hyperbolic velocity can often be reduced by delaying the interplanetary flight with direct returns until the relative planet positions permit interplanetary transfers with lower excess hyperbolic velocities. The long delay is unfortunate, however, the planet and spacecraft remain relatively close to one another during this period, which may be of practical use in the ultimate mission. Two commonly used direct return orbits are the "full-revolution return" and the "symmetric return."

A full-revolution return is a sun-centered elliptical orbit which returns the spacecraft to the launch planet in one launch planet period. Because the spacecraft and planet have equal periods, they must have equal velocity magnitudes relative to the sun. A double infinity of such orbits exists at each encountered planet. When the magnitude of the excess hyperbolic velocity is small with respect to the launch planet's orbital velocity, the two velocities will be nearly perpendicular. Furthermore, the orbital plane of the fullrevolution return will be only slightly inclined to the plane of the launch planet's orbit. A "half-revolution return" is a


Fig. 2 Symmetric return orbit.
special case of the full-revolution return in which the spacecraft has both the same period and eccentricity as the launch planet's orbit. In these cases the relative velocity vector is nearly perpendicular to the plane of the launch planet's orbit and the spacecraft returns to the launch planet after a half revolution of the sun.

A symmetric return is a sun-centered elliptical orbit which is coplanar with the launch planet's orbit and returns the spacecraft to the launch planet after a time greater than one launch planet period. ${ }^{4}$ In the construction of periodic orbits, the symmetric returns of most interest are those with times of flight greater than one, but less than two, launch planet periods. An example of a symmetric return is shown in Fig. 2. In this diagram the spacecraft and planet each pass through the arrival point (independently) and complete one circuit on their respective trajectories before the encounter is made. For a symmetric return orbit, the launch and arrival relative velocity magnitudes will be equal when the launch planet is in circular orbit.

## Iterative Solutions

Swing-by orbits are obtained by adjusting approximate dates of the planet flybys until differences in the relative velocity magnitudes at each flyby simultaneously vanish. Since the second flyby of a full-revolution return is constrained to occur one launch planet period after the first, only one fly-by date is an independent variable. Similarly, only one date can be considered independent for direct return orbits consisting of two or more consecutive full-revolution returns. In any event the number of independent dates $N$ equals the total number of interplanetary transfers and symmetric returns in a periodic orbit.
Successful solution by iterative methods is contingent upon the initial approximation of the independent variables. Approximate solutions to periodic orbits are not easily obtained. Hollister ${ }^{9}$ has discovered three periodic orbits that connect Earth and Venus. In the circular coplanar case each orbit includes a direct return orbit at Earth, an interplanetary transfer to Venus, two direct return orbits at Venus, and an interplanetary transfer back to Earth. The duration of each orbit is 3.2 yr ; they differ in the type of direct return orbits occurring at Earth and Venus. The first orbit has a full-revolution return at Earth and two consecutive fullrevolution returns at Venus. The second orbit consists of a

Table 1 Key to orbit descriptions ${ }^{a}$

| Periodic orbit number | Direct return orbits at Earth | Direct return orbits at Venus |
| :---: | :---: | :---: |
| 1 and 1H | 5 FR | 5 TFR |
| 2 and 2 H | 5 FR | 5 FRSY |
| 3 and 3H | 5SY | 5 TFR |
| 4 | 2FR, SY, 2FR | 5 TFR |
| 5 | FR, 2SY, 2FR | 5 TFR |
| 6 | 2FR, 2SY, FR | 5 TFR |
| 7 | FR, 3SY, FR | 5 TFR |
| 8 | FR, 4SY | 5 TFR |
| 9 | 5 FR | 2TFR, FRSY, 2TFR |
| 10 | 5 FR | 2TFR, 2FRSY, TFR |
| 11 | 5 FR | FRSY, TFR, 2FRSY, TFR |
| 12 | 5 FR | FRSY, TFR, 3FRSY |
| 13 | 2FR, SY, 2 FR | FRSY, 4TFR |
| 14 | $2 \mathrm{FR}, \mathrm{SY}, 2 \mathrm{FR}$ | FRSY, 2TFR, FRSY, TER |
| 15 | 2FR, SY, FR, SY | FRSY, 2TFR, FRSY, TFR |

${ }^{a}$ FR $=$ full-revolution return; $\mathrm{SY}=$ symmetric return; $\mathrm{FRSY}=$ fullrevolution return followed by a symmetric return; TFR = two consecutive full-revolution returns. Direct return orbits are listed in the order they occur.

Table 2 Flyby dates, (Julian date -2440000 ) orbits $1 \mathrm{H}-3 \mathrm{H}$

| Planet | Orbit 1H | Orbit 2 H | Orbit 3 H |
| :---: | :---: | :---: | :---: |
| E | 441 | 417 | 352 |
| E | 806 | 782 | 852 |
| V | 971 | 914 | 970 |
| V | 1196 | 1139 | 1195 |
| V | 1421 | 1470 | 14.20 |
| E | 1592 | 1612 | 1542 |
| E | 1957 | 1977 | 2042 |
| V | 2125 | 2086 | 2142 |
| V | 2350 | 2311 | 2367 |
| V | 2575 | 2642 | 2592 |
| E | 2797 | 2763 | 2697 |
| E | 3163 | 3128 | 3197 |
| V | 3316 | 3253 | 3297 |
| V | 3541 | 3478 | 3522 |
| V | 3765 | 3809 | 3747 |
| E | 3935 | 3927 | 3853 |
| E | 4300 | 4293 | 4353 |
| V | 4471 | 4427 | 4477 |
| V | 4696 | 4642 | 4702 |
| V | 4921 | 4953 | 4927 |
| E | 5077 | 5107 | 5038 |
| E | 5442 | 5472 | 5538 |
| V | 5664 | 5591 | 5644 |
| V | 5889 | 5816 | 5869 |
| V | 6114 | 6149 | 6094 |
| E | 6285 | 6261 | 6196 |
| Repeating after 16 years |  |  |  |

full-revolution return at Earth and a full-revolution and symmetric return at Venus. In the third orbit are a symmetric return at Earth and two full-revolution returns at Venus.

For the inclined elliptic case, Earth and Venus repeat their absolute orientation to within several degrees accuracy every $16 \mathrm{yr} .{ }^{13}$ The error made by assuming exact periodicity of the solar system is of the same order of magnitude as the error made by the patched conic model. By using five circular coplanar orbits in succession (see Table 1) as an initial approximation, Hollister found solutions for each of the three periodic orbits in the inclined elliptic case. To simplify analysis, the duration of symmetric returns was assumed constant and the launch and arrival relative velocity ( $V_{r}$ ) magnitudes on the symmetric returns were assumed equal. The duration of the symmetric returns was chosen in accordance with expected values for the semimajor axes of the symmetric return orbits. The solutions for these three orbits are reproduced in Table 2 where they are denoted as orbits 1 H , 2 H , and 3 H , respectively. Since orbit 1 H contains no symmetric returns, the solution for this orbit is rigorous in the patched conic sense. Dates of planet flybys in orbits 2 H and 3 H are used herein as initial approximations for a solution that eliminates the assumptions of constant time of flight and equal relative velocity magnitudes on symmetric returns.

The function to be minimized can be considered an $N$ dimensional surface whose arguments are the dates of the $N$ planet flybys. The gradient of the function is an $N$ dimensional vector in the direction of the greatest rate of change of the function value. In steepest-descent iterations, dates of the $N$ flybys are incremented to correspond with movement along the gradient vector. Reduction of the function value is guaranteed for sufficiently small date increments. Following reduction of the function value, a new gradient vector is calculated and the iteration repeated.

In Newton-Raphson iterations the difference in the velocity magnitudes at each of the $N$ fybys is expanded in a Taylor series about the current values of the $N$ fly-by dates. Only first-order terms are retained in each Taylor series. Date increments are made to cause each of the linearized expressions for velocity difference to simultaneously vanish. Upon reduction of the function value, velocity differences are expanded in Taylor series about the new fly-by dates and the iteration repeated. Convergence is likely only when the initial date approximations are sufficiently accurate and higher derivatives of the velocity difference expressions are excessively large.

In both the steepest-descent and Newton-Raphson iterations, the first partial derivatives of the differences in velocity magnitude are required with respect to the fly-by dates. Approximate values of the partial derivatives are obtained by calculating the change in velocity difference at each flyby which results from making small changes in the $N$ fly by dates one at a time.

First attempts at obtaining rigorous solutions to the periodic orbit problem employed both steepest-descent and New-ton-Raphson procedures. Steepest-descent methods were first used to reduce the sum of the absolute differences in relative velocity ( $V_{r}$ ) magnitudes to 0.1 EMOS (Earth Mean Orbital Speed Unit). Newton-Raphson methods were then used to reduce the value from 0.1 EMOS to assumed convergence at 0.005 EMOS. Sometimes oscillations in the date increments indicated that a "ravine" problem had been encountered on the $N$ dimensional surface. The Davidon ${ }^{14}$ and conjugate gradient ${ }^{15}$ methods were employed when this situation developed.

Attempts to obtain a solution to Hollister's third periodic orbit resulted in convergence to a local minimum rather than a solution. Endeavors were made to sequentially modify orbit 1 H until an accurate approximation for orbit 3 H could be obtained.

Although orbit 1 H and 3 H have the same type of direct return orbits at Venus, orbit 1 H has five full-revolution returns at Earth and orbit 3 H has five symmetric returns at Earth. A solution was first attempted for orbit 1 H modified to include one symmetric return and four full-revolution returns at the Earth encounters. An approximate solution for this orbit was obtained by merely replacing one of the fullrevolution returns at Earth in orbit 1 H by a symmetric return of 1.37 yr duration. The symmetric return was inserted so as to equate the times of flight for the interplanetary transfers on either side of the Earth encounter. Rapid convergence to the orbit solution was achieved with the steepest-descent and Newton-Raphson procedures. Following this favorable outcome, the modified orbit was altered by replacing one of the remaining full-revolution returns by a second symmetric return. A solution was easily obtained in this case and also for orbits with three, four, and finally five symmetric returns at the Earth encounters. Solutions for each of these orbits are listed in Table 3 (orbits 3-8). For orbits 4, 7, and 8, convergence was rapid and predictable. In orbits 5 and 6 oscillation was encountered and convergence slowed at low levels of the function value. In these cases the conjugate gradient method was used to further reduce the function value.

Although a solution for each of the modified erbits was required to obtain a rigorous solution to orbit 3 H , the modified orbits are more than a means to an end. Each is a unique periodic orbit with characteristics far different from similar orbits. This fact is illustrated by a comparison of orbits 5 and 6. Although these orbits each have two symmetric and three full-revolution returns at Earth, the order in which the returns occur differs. In orbit 5 the minimum pass distance occurs during a flyby of Earth in which the spacecraft comes within 1.16 Earth radii of the planet surface. In orbit 6 the minimum pass distance also occurs during a flyby

Table 3 Solutions for orbits 1-15


| Orbit 4 |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
| E | 432 | 0.133 | 76.8 | 2.45 |
| E | 798 | 0.133 | 76.8 | 2.45 |
| V | 989 | 0.180 | 43.4 | 4.11 |
| V | 1214 | 0.180 | 43.4 | 4.11 |
| V | 1439 | 0.180 | 43.4 | 4.11 |
| E | 1614 | 0.188 | 78.6 | 1.16 |
| E | 1980 | 0.188 | 78.6 | 1.16 |
| V | 2133 | 0.282 | 18.9 | 3.79 |
| V | 2358 | 0.282 | 13.8 | 5.41 |
| V | 2573 | 0.282 | 19.9 | 3.58 |
| E | 2710 | 0.203 | 57.4 | 1.86 |
| E | 3201 | 0.202 | 35.5 | 3.94 |
| V | 3313 | 0.208 | 30.9 | 3.77 |
| V | 3538 | 0.208 | 26.5 | 4.59 |
| V | 3763 | 0.208 | 26.5 | 4.59 |
| E | 3929 | 0.159 | 86.5 | 1.28 |
| E | 4294 | 0.159 | 86.5 | 1.28 |
| V | 4468 | 0.185 | 57.4 | 1.86 |
| V | 4693 | 0.185 | 57.4 | 1.86 |
| V | 4918 | 0.185 | 57.4 | 1.86 |
| E | 5074 | 0.163 | 46.9 | 4.03 |
| E | 5439 | 0.163 | 46.9 | 4.03 |
| V | 5659 | 0.204 | 34.1 | 3.41 |
| V | 5884 | 0.204 | 34.1 | 3.41 |
| V | 6109 | 0.204 | 34.1 | 3.41 |
| E | 6276 | 0.133 | 76.8 | 2.45 |


| Orbit 5 |  |  |  |  |
| :--- | ---: | :--- | ---: | ---: |
| E | 442 | 0.133 | 80.3 | 2.21 |
| $\mathbf{E}$ | 808 | 0.133 | 80.3 | 2.21 |
| V | 977 | 0.165 | 34.0 | 5.25 |
| V | 1242 | 0.165 | 34.0 | 5.25 |
| V | 1467 | 0.165 | 38.7 | 3.37 |
| E | 1542 | 0.173 | 43.9 | 3.96 |
| E | 2036 | 0.173 | 15.5 | 15.16 |
| V | 2147 | 0.136 | 52.0 | 4.06 |
| V | 2372 | 0.136 | 52.0 | 4.06 |
| V | 2597 | 0.136 | 52.9 | 3.95 |
|  |  |  |  |  |
| E | 2698 | 0.171 | 13.2 | 18.71 |
| $\mathbf{E}$ | 3192 | 0.171 | 37.7 | 5.03 |
| V | 3307 | 0.186 | 32.9 | 4.30 |
| V | 3532 | 0.186 | 26.0 | 5.86 |
| V | 3757 | 0.186 | 29.4 | 4.00 |
|  |  |  |  |  |
| E | 3922 | 0.135 | 80.0 | 2.16 |
| E | 4287 | 0.135 | 80.0 | 2.16 |
| V | 4474 | 0.148 | 64.4 | 2.37 |
| V | 4699 | 0.148 | 64.4 | 2.37 |
| V | 4924 | 0.148 | 64.4 | 2.37 |
|  |  |  |  |  |
| E | 5088 | 0.143 | 52.3 | 4.37 |
| E | 5453 | 0.143 | 52.3 | 4.37 |
| V | 5662 | 0.188 | 35.6 | 3.78 |
| V | 5877 | 0.188 | 35.6 | 3.78 |
| V | 6102 | 0.188 | 35.6 | 3.78 |
| E | 6286 | 0.133 | 80.3 | 2.21 |


| Orbit 6 |  |  |  |  |
| :--- | ---: | :---: | :---: | :---: |
| E | 444 | 0.125 | 85.0 | 2.18 |
| E | 809 | 0.125 | 85.0 | 2.18 |
| V | 995 | 0.187 | 41.7 | 3.06 |
| V | 1220 | 0.187 | 41.7 | 3.06 |
| V | 1445 | 0.187 | 41.7 | 3.06 |
|  |  |  |  |  |
| E | 1620 | 0.200 | 75.7 | 1.12 |
| E | 1986 | 0.200 | 75.7 | 1.12 |
| V | 2135 | 0.302 | 17.5 | 3.60 |
| V | 2585 | 0.302 | 11.4 | 5.85 |
| V | 2810 | 0.302 | 18.0 | 3.48 |
|  |  |  |  |  |
| E | 2716 | 0.209 | 64.1 | 1.43 |
| E | 3206 | 0.209 | 11.4 | 14.62 |
| V | 3299 | 0.150 | 62.8 | 2.43 |
| V | 3524 | 0.150 | 62.8 | 2.43 |
| V | 3749 | 0.150 | 62.8 | 2.43 |
|  |  |  |  |  |
| E | 3870 | 0.186 | 48.0 | 2.98 |
| E | 4363 | 0.186 | 32.3 | 5.18 |
| V | 4474 | 0.134 | 54.9 | 3.87 |
| V | 4699 | 0.134 | 45.4 | 5.26 |
| V | 4924 | 0.134 | 45.4 | $\mathbf{5 . 2 6}$ |
|  |  |  |  |  |
| E | 5093 | 0.133 | 56.6 | 4.41 |
| E | 5458 | 0.133 | 56.6 | 4.41 |
| V | 5660 | 0.173 | 38.3 | 4.02 |
| V | 5885 | 0.173 | 38.3 | 4.02 |
| V | 6110 | 0.173 | 38.3 | 4.02 |
| E | 6288 | 0.125 | 85.0 | 2.18 |

[^1]Table 3 Continued

| Planet | Date ${ }^{\text {a }}$ | $\begin{gathered} V_{r},{ }^{b} \\ \text { EMOS } \end{gathered}$ | $\begin{gathered} \theta_{b}^{b} \\ \operatorname{deg} \end{gathered}$ | $R_{\text {min }}{ }^{\text {b }}$ | Planet | Date ${ }^{\text {a }}$ | $\begin{gathered} V_{r}, b \\ \text { EMOS } \end{gathered}$ | $\begin{gathered} \theta, b^{b} \\ \operatorname{deg} \end{gathered}$ | $R_{\text {min }}{ }^{\text {b }}$ | Planet | Date ${ }^{\text {a }}$ | $\begin{gathered} V_{r}, b \\ \text { EMOS } \end{gathered}$ | $\begin{gathered} \theta^{b} \\ \text { deg } \end{gathered}$ | $R_{\text {min }}{ }^{\text {b }}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Orbit 7 |  |  |  |  | Orbit 8 |  |  |  |  | Orbit 9 |  |  |  |  |
| E | 452 | 0.135 | 79.2 | 2.21 | E | 454 | 0.129 | 81.1 | 2.30 | E | 430 | 0.133 | 61.2 | 3.86 |
| E | 817 | 0.135 | 79.2 | 2.21 | E | 819 | 0.129 | 81.1 | 2.30 | E | 795 | 0.133 | 61.2 | 3.86 |
| V | 1009 | 0.224 | 21.2 | 5.24 | V | 1008 | 0.218 | 21.8 | 5.34 | V | 973 | 0.148 | 60.6 | 2.65 |
| V | 1234 | 0.224 | 1.3 | 99.11 | V | 1233 | 0.218 | 2.0 | 71.47 | V | 1198 | 0.148 | 60.6 | 2.65 |
| V | 1459 | 0.224 | 23.6 | 4.59 | V | 1458 | 0.218 | 24.6 | 4.60 | V | 1423 | 0.148 | 60.6 | 2.65 |
| E | 1539 | 0.164 | 26.5 | 8.90 | E | 1539 | 0.163 | 26.1 | 9.16 | E | 1602 | 0.142 | 62.5 | 3.27 |
| E | 2034 | 0.163 | 18.6 | 13.82 | E | 2034 | 0.163 | 18.9 | 13.62 | E | 1967 | 0.142 | 62.5 | 3.27 |
| V | 2148 | 0.134 | 51.1 | 4.36 | V | 2149 | 0.136 | 49.8 | 4.42 | V | 2165 | 0.194 | 30.4 | 4.43 |
| V | 2373 | 0.134 | 51.1 | 4.36 | V | 2374 | 0.136 | 49.8 | 4.42 | V | 2390 | 0.194 | 30.4 | 4.43 |
| V | 2598 | 0.134 | 52.4 | 4.18 | V | 2599 | 0.136 | 54.4 | 3.81 | V | 2615 | 0.194 | 30.4 | 4.43 |
| E | 2700 | 0.164 | 14.3 | 18.53 | E | 2697 | 0.176 | 11.4 | 20.71 | E | 2;85 | 0.146 | 85.9 | 1.55 |
| E | 3195 | 0.164 | 22.2 | 11.05 | E | 3190 | 0.176 | 43.6 | 3.86 | E | 3150 | 0.146 | 85.9 | 1.55 |
| V | 3299 | 0.148 | 58.4 | 2.84 | V | 3308 | 0.205 | 62.6 | 1.30 | V | 3254 | 0.277 | 11.7 | 6.80 |
| V | 3524 | 0.148 | 58.4 | 2.84 | V | 3533 | 0.205 | 62.6 | 1.30 | V | 3479 | 0.277 | 10.9 | 7.31 |
| V | 3749 | 0.148 | 58.4 | 2.84 | V | 3758 | 0.205 | 62.6 | 1.30 | V | 3812 | 0.278 | 16.3 | 4.62 |
| E | 3870 | 0.185 | 47.2 | 3.09 | E | 3889 | 0.216 | 67.3 | 1.22 | E | 3918 | 0.148 | 62.5 | 2.99 |
| E | 4362 | 0.185 | 31.8 | 5.45 | E | 4374 | 0.216 | 26.4 | 5.11 | E | 4283 | 0.148 | 62.5 | 2.99 |
| V | 4473 | 0.131 | 56.3 | 3.83 | V | 4485 | 0.158 | 56.9 | 2.48 | V | 4467 | 0.153 | 57.1 | 2.75 |
| V | 4698 | 0.131 | 51.4 | 4.47 | V | 4710 | 0.158 | 56.9 | 2.48 | V | 4692 | 0.153 | 57.1 | 2.75 |
| V | 4923 | 0.131 | 51.4 | 4.47 | V | 4935 | 0.158 | 56.9 | 2.48 | V | 4917 | 0.153 | 57.1 | 2.75 |
| E | 5100 | 0.128 | 60.2 | 4.27 | E | 5039 | 0.164 | 27.9 | 8.30 | E | 5082 | 0.135 | 76.8 | 2.38 |
| E | 5465 | 0.128 | 60.2 | 4.27 | E | 5534 | 0.164 | 13.3 | 20.16 | E | 5447 | 0.135 | 76.8 | 2.38 |
| V | 5664 | 0.175 | 38.3 | 3.95 | V | 5645 | 0.158 | 46.3 | 3.66 | V | 5622 | 0.194 | 33.2 | 3.92 |
| V | 5889 | 0.175 | 38.3 | 3.95 | V | 4870 | 0.158 | 46.3 | 3.66 | V | 5847 | 0.194 | 33.2 | 3.92 |
| V | 6114 | 0.175 | 38.3 | 3.95 | V | 6095 | 0.158 | 46.3 | 3.66 | V | 6072 | 0.194 | 33.2 | 3.92 |
| E | 6296 | 0.135 | 79.2 | 2.21 | E | 6298 | 0.129 | 81.1 | 2.30 | E | 6274 | 0.133 | 61.2 | 3.86 |


| Orbit 10 |  |  |  |  | Orbit 11 |  |  |  |  | Orbit 12 |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| E | 430 | 0.129 | 61.2 | 4.11 | E | 430 | 0.127 | 80.9 | 2.36 | E | 417 | 0.138 | 69.5 | 2.81 |
| E | 795 | 0.129 | 61.2 | 4.11 | E | 795 | 0.127 | 80.9 | 2.36 | E | 782 | 0.138 | 69.5 | 2.81 |
| V | 973 | 0.148 | 60.4 | 2.66 | V | 915 | 0.248 | 17.6 | 5.31 | V | 914 | 0.237 | 19.9 | 5.03 |
| V | 1198 | 0.148 | 60.4 | 2.66 | V | 1140 | 0.248 | 17.6 | 5.31 | V | 1139 | 0.237 | 17.6 | 5.82 |
| V | 1423 | 0.148 | 60.4 | 2.66 | V | 1471 | 0.247 | 23.1 | 3.85 | V | 1469 | 0.237 | 27.2 | 3.41 |
| E | 1602 | 0.142 | 62.7 | 3.23 | E | 1603 | 0.140 | 86.0 | 1.69 | E | 1605 | 0.138 | 85.0 | 1.79 |
| E | 1967 | 0.142 | 62.7 | 3.23 | E | 1968 | 0.140 | 86.0 | 1.69 | E | 1970 | 0.138 | 85.0 | 1.79 |
| V | 2164 | 0.192 | 30.5 | 4.48 | V | 2165 | 0.191 | 30.6 | 4.50 | V | 2165 | 0.188 | 31.1 | 4.56 |
| V | 2389 | 0.192 | 30.5 | 4.48 | V | 2390 | 0.191 | 30.6 | 4.50 | V | 2389 | 0.188 | 31.1 | 4.56 |
| V | 2614 | 0.192 | 30.5 | 4.48 | V | 2615 | 0.191 | 30.6 | 4.50 | V | 2614 | 0.188 | 31.1 | 4.56 |
| E | 2784 | 0.144 | 85.6 | 1.62 | E | 2784 | 0.144 | 85.8 | 1.61 | E | 2785 | 0.143 | 86.6 | 1.58 |
| E | 3149 | 0.144 | 85.6 | 1.62 | E | 3149 | 0.144 | 85.8 | 1.61 | E | 3150 | 0.143 | 86.6 | 1.58 |
| V | 3255 | 0.270 | 12.5 | 6.67 | V | 3255 | 0.270 | 12.4 | 6.68 | V | 3255 | 0.273 | 12.1 | 6.74 |
| V | 3480 | 0.270 | 11.5 | 7.30 | V | 3480 | 0.270 | 11.6 | 7.18 | V | 3480 | 0.273 | 12.0 | 6.80 |
| V | 3812 | 0.270 | 18.5 | 4.24 | V | 3813 | 0.270 | 18.3 | 4.30 | V | 3813 | 0.272 | 17.1 | 4.56 |
| E | 3926 | 0.151 | 84.8 | 1.49 | E | 3925 | 0.151 | 84.8 | 1.49 | E | 3923 | 0.151 | 86.0 | 1.45 |
| E | 4291 | 0.151 | 84.8 | 1.49 | E | 4291 | 0.151 | 84.8 | 1.49 | E | 4289 | 0.151 | 86.0 | 1.45 |
| V | 4422 | 0.259 | 18.5 | 4.60 | V | 4422 | 0.259 | 18.6 | 4.55 | V | 4426 | 0.244 | 22.1 | 4.18 |
| V | 4647 | 0.259 | 18.5 | 4.60 | V | 4647 | 0.259 | 18.6 | 4.55 | V | 4651 | 0.244 | 22.1 | 4.18 |
| V | 4980 | 0.259 | 13.6 | 6.55 | V | 4979 | 0.258 | 13.6 | 6.55 | V | 4982 | 0.244 | 12.5 | 8.10 |
| E | 5086 | 0.127 | 83.2 | 2.21 | E | 5086 | 0.127 | 83.3 | 2.21 | E | 5108 | 0.143 | 66.8 | 2.82 |
| E | 5451 | 0.127 | 83.2 | 2.21 | E | 5451 | 0.127 | 83.3 | 2.21 | E | 5473 | 0.143 | 66.8 | 2.82 |
| V | 5624 | 0.188 | 34.0 | 4.05 | V | 5624 | 0.188 | 34.3 | 4.01 | V | 5591 | 0.255 | 16.5 | 5.42 |
| V | 5849 | 0.188 | 34.0 | 4.05 | V | 5849 | 0.188 | 34.3 | 4.01 | V | 5816 | 0.255 | 10.4 | 9.05 |
| V | 6074 | 0.188 | 34.0 | 4.05 | V | 6074 | 0.188 | 34.3 | 4.01 | V | 6147 | 0.255 | 13.8 | 6.64 |
| E | 6274 | 0.129 | 61.2 | 4.11 | E | 6274 | 0.127 | 80.9 | 2.36 | E | 6261 | 0.138 | 69.5 | 2.81 |

[^2]Table 3 Concluded

| Planet | Date ${ }^{\text {a }}$ | $\begin{gathered} V_{r},{ }^{b} \\ \text { EMOS } \end{gathered}$ | $\begin{gathered} \theta, b \\ \mathrm{deg} \end{gathered}$ | $R_{\text {min }}{ }^{\text {b }}$ | Planet | Date ${ }^{a}$ | $\begin{gathered} V,{ }^{b} \\ \text { EMOS } \end{gathered}$ | $\begin{aligned} & \theta,{ }^{b} \\ & \operatorname{deg} \end{aligned}$ | $R_{\text {min }}{ }^{\text {b }}$ | Planet | Date ${ }^{\text {a }}$ | $\begin{gathered} V_{r},{ }^{b} \\ \text { EMOS } \end{gathered}$ | $\begin{gathered} \theta,{ }^{b} \\ \operatorname{deg} \end{gathered}$ | $R_{\text {min }}{ }^{\text {b }}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Orbit 13 |  |  |  |  | Orbit 14 |  |  |  |  | Orbit 15 |  |  |  |  |
| E | 437 | 0.129 | 74.6 | 2.99 | E | 443 | 0.122 | 78.5 | 2.76 | E | 448 | 0.121 | 86.5 | 2.23 |
| E | 802 | 0.124 | 74.6 | 2.99 | E | 808 | 0.122 | 78.5 | 2.76 | E | 813 | 0.121 | 86.5 | 2.23 |
| V | 916 | 0.250 | 17.0 | 5.47 | V | 916 | 0.255 | 16.2 | 5.53 | V | 917 | 0.261 | 15.4 | 5.62 |
| V | 1141 | 0.250 | 17.0 | 5.47 | V | 1141 | 0.255 | 16.2 | 5.53 | V | 1142 | 0.261 | 15.4 | 5.62 |
| V | 1473 | 0.250 | 22.9 | 3.80 | V | 1473 | 0.255 | 21.3 | 4.01 | V | 1474 | 0.261 | 20.2 | 5.07 |
| E | 1607 | 0.146 | 59.4 | 3.40 | E | 1605 | 0.146 | 60.8 | 4.25 | E | 1603 | 0.146 | 62.9 | 3.12 |
| E | 1972 | 0.146 | 59.4 | 3.40 | E | 1970 | 0.146 | 60.8 | 4.25 | E | 1968 | 0.146 | 62.0 | 3.12 |
| V | 2132 | 0.217 | 25.1 | 4.54 | V | 2130 | 0.213 | 25.6 | 4.56 | V | 2130 | 0.211 | 26.0 | 4.57 |
| V | 2357 | 0.217 | 22.5 | 5.19 | V | 2355 | 0.213 | 24.2 | 4.89 | V | 2355 | 0.211 | 26.0 | 4.57 |
| V | 2582 | 0.217 | 27.3 | 4.07 | V | 2580 | 0.213 | 28.6 | 3.96 | V | 2580 | 0.211 | 29.5 | 3.87 |
| E | 2701 | 0.180 | 40.1 | 4.19 | E | 2699 | 0.183 | 36.7 | 4.59 | E | 2696 | 0.189 | 33.6 | 4.88 |
| E | 3194 | 0.178 | 36.4 | 4.89 | E | 3192 | 0.183 | 44.0 | 3.53 | E | 3188 | 0.189 | 50.7 | 2.65 |
| V | 3309 | 0.190 | 32.7 | 4.16 | V | 3310 | 0.212 | 28.3 | 4.07 | V | 3310 | 0.233 | 24.9 | 3.96 |
| V | 3534 | 0.190 | 26.6 | 5.44 | V | 3535 | 0.212 | 21.5 | 5.73 | V | 3535 | 0.233 | 17.7 | 5.97 |
| V | 3759 | 0.190 | 28.7 | 4.94 | V | 3760 | 0.212 | 25.5 | 4.66 | V | 3760 | 0.233 | 23.0 | 4.35 |
| E | 3924 | 0.140 | 82.2 | 1.87 | E | 3921 | 0.154 | 51.8 | 3.86 | E | 3918 | 0.167 | 53.4 | 3.10 |
| E | 4289 | 0.140 | 82.2 | 1.87 | E | 4287 | 0.154 | 51.8 | 3.86 | E | 4284 | 0.167 | 53.4 | 3.10 |
| V | 4471 | 0.155 | 61.0 | 2.40 | V | 4423 | 0.253 | 20.3 | 4.32 | V | 4417 | 0.281 | 16.2 | 4.58 |
| V | 4696 | 0.155 | 61.0 | 2.40 | V | 4648 | 0.253 | 20.3 | 4.32 | V | 4642 | 0.281 | 16.2 | 4.58 |
| V | 4921 | 0.155 | 61.0 | 2.40 | V | 4980 | 0.252 | 13.9 | 4.74 | V | 4975 | 0.281 | 17.0 | 4.21 |
| E | 5084 | 0.144 | 52.1 | 4.39 | E | 5092 | 0.130 | 74.6 | 2.72 | E | 5042 | 0.173 | 28.0 | 7.39 |
| E | 5449 | 0.144 | 52.1 | 4.39 | E | 5457 | 0.130 | 74.6 | 2.72 | E | 5535 | 0.173 | 9.5 | 26.13 |
| V | 5659 | 0.184 | 36.6 | 3.82 | V | 5659 | 0.171 | 38.9 | 4.05 | V | 5644 | 0.154 | 48.7 | 3.53 |
| V | 5884 | 0.184 | 36.6 | 3.82 | V | 5884 | 0.171 | 38.9 | 4.05 | V | 5869 | 0.154 | 48.7 | 3.53 |
| V | 6109 | 0.184 | 36.6 | 3.82 | V | 6109 | 0.171 | 38.9 | 4.05 | V | 6094 | 0.154 | 48.7 | 3.53 |
| E | 6281 | 0.124 | 74.6 | 2.99 | E | 6287 | 0.122 | 78.5 | 2.76 | E | 6291 | 0.121 | 86.5 | 2.23 |

a Julian - 2440000 ; dates repeat after 16 yr (add 5844 days).
${ }^{b} V_{r}=$ relative velocity; $\theta=$ turn angle; $R_{\text {min }}=$ minimum distance to planet center, planet radii.
of the Earth. In this case the spacecraft comes within 0.12 Earth radii of the planet surface. Thus despite similarity of the orbits, each combination of direct return orbits requires individual evaluation. Details of the turn angle ( $\theta$ ) calculation are contained in Ref. 16.

The method of sequential modification was also used to obtain solutions to a series of orbits which differ only in the direct return orbits at the Venus encounters. These orbits are numbered $9-12$ in Table 3. Solutions to orbits 13-15 demonstrate the effectiveness of sequential modification for periodic orbits with symmetric returns at both Earth and Venus. In all orbits numbered $9-15$ no additional convergence problems were encountered.

## Summary

Four types of iterative solutions were attempted in solving the periodic orbit problem. The Davidon method was found to be an inappropriate choice due to the excessive amount of time required for each iteration. The steepest-descent method proved to be extremely useful in obtaining initial reduction of the function value resulting from the approximate fly-by dates. Since only calculation of the function value and gradient vector are required, each steepest-descent interaction requires little time. When simplicity of the steepest-descent method slowed convergence at a "ravine," the conjugate gradient method was used successfully to obtain additional reduction of the function value. For periodic orbit problems, the conjugate gradient method is characterized by an appropriate tradeoff between the time required for each iteration and the sophistication required for satisfactory reduction of the function value. The Newton-Raph-
son method was used with success only after a sufficiently accurate approximation to the fly-by dates had been reached. Although Newton-Raphson iterations suffered from the time required to invert an $N$-order matrix, large reductions in the function value were obtained from each iteration.
An average of 1.2 min of computer time was required to reach a solution for each periodic orbit in which no local minimums were encountered between the initial approximation and the global minimum. Of the 1.2 min , approximately 0.3 min were required for compilation whereas approximately 0.9 min were required for execution. The longest time required to obtain a solution was approximately 2.5 min ( 70 iterations) while the shortest time required was approximately 0.4 min (3iterations).

The technique described herein is in no way restricted to the solution of periodic orbit problems. A periodic orbit is merely a multiple-fly-by problem in which the first and last flyby are constrained to occur at the same planet with equal relative velocity magnitudes on dates separated by an integral multiple of a specified time period. In a general multiple-fly-by problem any one or all of these constraints may be relaxed.
As specifically demonstrated by orbits 5 and 6, the order in which direct return orbits occur drastically affects the characteristics of each orbit. It is important to consider approximating the total number of acceptable periodic orbits which connect Earth and Venus. Each orbit contains five encounters at Earth and five encounters at Venus. At each encounter either one of two direct return orbits is available for selection. For this reason a minimum of 1024 acceptable orbits may exist. Although acceptable orbits containing as many as five symmetric returns have been shown to exist, the
times of flight for interplanetary transfers on these orbits are quite small. The inclusion of additional symmetric returns would tend to further reduce the time available for interplanetary transfers. It is reasonable to assume that periodic orbits with six or more symmetric returns would require flybys passing below the surface of a planet. This assumption reduces the total number of periodic orbits with acceptable flybys to 648 .

A larger number of periodic orbits are possible if additional variations in the direct return orbits are considered. Those Venus encounters with two consecutive full-revolution returns could be replaced by encounters consisting of a halfrevolution return followed by a full-revolution return and a second half-revolution return. This alternative would cause rotation of the relative velocity vector at four rather than three flybys. The additional flyby might reduce the largest turn angle enough to allow more desirable flybys. Still more orbits could be obtained by reversing the order of the full-revolution and symmetric returns which occur in succession at many of the Venus encounters. This variation would change the direction of the inbound and outbound relative velocities at the full-revolution and symmetric returns so that a reduction in the largest turn angle might result. No variations of this type were required for the orbits listed in the Appendix since the flybys at all such Venus encounters occur well above the planet surface.

The periodic orbits require 16 yr to complete each cycle. Earth and Venus repeat their absolute orientation every 8 yr . Therefore, two spacecraft are required to take advantage of all the opportunities for each periodic orbit. When one spacecraft is leaving Earth for Venus, the other is approaching Earth from Venus. The alternate sets of fly-by dates can be obtained by adding 8 yr to each set of fly-by dates listed in the Appendix. The 15 periodic orbits presented here would allow 30 spacecraft to simultaneously make periodic flights between Earth and Venus. The large number of trajectory choices provides considerable flexibility in establishing a particular mission.

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    * Associate Professor, Department of Aeronautics and Astronautics. Associate Fellow AIAA.
    $\dagger$ NASA Fellow; now Associate Engineer, Lockheed Missiles \& Space Company, Sunnyvale, Calif.

[^1]:    a Julian - 2440000; dates repeating after 16 yr (add 5844 days).
    ${ }^{b} V_{F}=$ relative velocity $; \theta=$ turn angle; $R_{\min }=$ minimum distance to planet center, planet radii.

[^2]:    a Julian - 2440000; dates repeating after 16 yr (add 5844 days).
    ${ }^{b} V_{r}=$ relative velocity; $\theta=$ turn angle; $R_{m i n}=$ minimum distance to planet center, planet radii.

