Periodic Orbits for Interplanetary Flight

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The possibility exists that a spacecraft can be placed on a free-fall trajectory that goes back and forth between Earth and Venus forever. Such trajectories have been found using patched-conic analysis. The trajectory is first established for the case when the two planets are in circular, coplanar orbits. The synodic period of Earth and Venus is used to establish the time of flight and the transfer angle. These values permit a solution of Lambert's problem for the orbit. The search procedure in the general case of inclined, elliptic orbits is more complex. A ten-dimensional iteration is required to find the boundary conditions for ten simultaneous solutions of the Lambert problem. Three periodic orbits connecting Earth and Venus have been found in the general case. There is a strong probability of the existence of other similar orbits.

1. Introduction

PERIODIC orbits for interplanetary flight are free-fall trajectories which shuttle back and forth between the planets, making a flyby at each terminal. The energy exchange associated with the planet flyby establishes the new trajectory toward the next destination. No propulsion is required once the orbit is established other than that needed to correct for guidance errors. The orbits are unstable in the sense that small perturbations from the orbit destroy the periodicity if left uncorrected. For this reason practical use of the orbits requires some form of station keeping. Assuming perfect guidance, the thrust requirement for station keeping would be extremely small.

The analysis of periodic orbits is performed by means of patched conics. The trajectory between the planets is assumed to be an ellipse relative to a set of nonrotating, sun-centered coordinates. The ellipse is a solution of the problem of two bodies, namely the sun and the spaceship. The trajectory near the planet is assumed to be a hyperbola relative to a set of nonrotating, planet-centered coordinates. The hyperbola is also a solution of the problem of two bodies, namely the planet and the spaceship. The two trajectories are patched by equating the terminal velocity of the spacecraft relative to the planet on the sun-centered ellipse to the hyperbolic velocity of the spacecraft relative to the planet far out on the planet-centered hyperbola. The time duration of the patched trajectory is assumed equal to the time determined from the ellipses alone. The accuracy of patched conic analysis for interplanetary orbits has been shown to be quite reasonable.[1] All of the analysis in this report is based on patched conics.

The analytic problem is to find a trajectory that continually makes a flyby of one planet after another. From the patched-conic point of view, the flyby at each terminal planet results in a rotation of the hyperbolic velocity vector. Therefore, it is necessary to find a set of repeating sun-centered ellipses that have equal magnitude inbound and outbound velocities relative to the terminal planet at each terminal planet. In addition, the turn angle between the inbound and outbound relative velocity vector must be small enough to allow the flyby to take place above the surface of the planet. Any single interplanetary trajectory is determined by the specification of the two terminal points and the time of flight. This is equivalent to the specification of the location of the terminal planets at the terminal times. The dates of the terminal events are fixed by the planet locations at the time of the terminal events. Consequently, the solution to the problem is the set of terminal planet locations that produces equal-magnitude relative velocity vectors for the inbound and outbound trajectory with the turn angle between the vectors small enough to permit the flyby to take place above the planet.

The periodicity of the solar system causes planet orientations to reoccur at regular intervals. A repeated flyby will continue forever if a set of terminal locations satisfying the foregoing constraints reoccurs periodically.

2. Orbital Geometry

The main planets of interest for manned, interplanetary flight are Earth, Venus, and Mars. To an accuracy of a few degrees[2] their motion can be described as follows. Earth makes 32 revolutions of the sun in 32 yr, giving it an orbital period of 1 yr. Venus makes 52 revolutions of the sun in 32 yr, giving it an orbital period of $\frac{8}{13}$ yr. Mars makes 17 revolutions of the sun in 32 yr, giving it an orbital period of $\frac{32}{17}$ yr. After 32 yr the cycle is complete and the absolute orientations of the three sun planet-vectors repeat in the same part of the sky as they did 32 yr earlier. For Earth and Venus the absolute orientation repeats after only 8 yr. In that time Earth has gone 8 revolutions while Venus has gone 13. The five alignments of the two planets during this period are equally spaced around the Earth's orbit.

The trajectories between the planets depend on the relative orientation, which repeats with the synodic period. Venus passes Earth 20 times in 32 yr, giving an average synodic period between alignments of $\frac{8}{5}$ yr. Venus passes Mars 35 times in 32 yr giving an average synodic period of $\frac{32}{35}$ yr. Earth passes Mars 15 times in 32 yr, giving an average synodic period of $\frac{32}{15}$ yr. If all three planets were in circular, coplanar orbits the relative orientations of the three planets would reoccur identically every $\frac{32}{5}$ yr although in a different part of the sky. The alignment dates would be spaced in time as shown in Fig. 1, and the pattern would repeat every 6.4 yr. The eccentricities of the three orbits cause slight variations from this pattern with the most noticeable being due to the eccentricty of Mars ($e = 0.093$). For Earth and Venus the relative orientation repeats every 1.6 yr in a different part of the sky. The eccentricities of Earth (0.017) and Venus (0.007) are small enough to make the timing of the alignments quite regular. The relatively large orbital inclination of 3°24′ causes differences in the terminal velocities for similar trajectories in adjacent synodic periods. The assumption of

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circular, coplanar orbits, however, makes each 1.6-yr synodic period identical. That is, the planets and the spaceship move in the same way relative to one another, but they traverse a different part of the sky relative to the stars. The assumption greatly simplifies the problem of finding a repeating orbit since everything repeats after only 1.6 yr. This case is analyzed in Sec. 4. The general case of eccentric and inclined orbits for Earth and Venus is treated in Sec. 5.

3. Direct-Return Orbits

A direct-return orbit is a free-fall ellipse about the sun which returns the spaceship some time after launch to the same planet from whence it was launched. There exists a large class of such orbits, but for the purposes of this study only two types are considered. The first type, called the "full-revolution return," is an ellipse which has the same period as the launch planet. After one revolution of the sun both the spaceship and the planet return simultaneously to the launch point. There are a double infinity of ellipses passing through the launch point with period equal to that of the launch planet. The magnitude of the velocity of the spaceship relative to the sun is equal to that of the launch planet relative to the sun. Therefore, the locus of the tip of the velocity vector of the spaceship relative to the sun lies on a sphere in velocity space and the hyperbolic velocity of the spaceship relative to the launch planet is approximately perpendicular to the velocity vector of the launch planet relative to the sun. The "half-revolution return" is a special case when the ellipse has both the same period and the same eccentricity as the launch planet's orbit, but is inclined to it. The spaceship returns after only a half-revolution of the sun. There are a single infinity of these orbits and the hyperbolic velocity vector is approximately perpendicular to the plane of the launch planet's orbit.

The second type of direct return is shown in Fig. 2. The ellipse is coplanar with the launch planet orbit. The spaceship returns to the launch planet after about 1.4 solar revolutions. The exact transfer angle is found by an iterative solution of Lambert's equation. The condition for a return to a planet that is in circular orbit is that the time of flight in launch planet periods is equal to the transfer angle in revolutions. There are a single infinity of solutions for this case near the 1.41 point, as can be seen in Fig. 3. This type of transfer will be called a "symmetric return" after Ross.[3] The locus of the hyperbolic velocity of the spaceship relative to the launch planet at launch is also shown in Fig. 3.

All these direct-return orbits have the useful property that they can be achieved with an arbitrary initial velocity magnitude. Also, when the launch planet is in circular orbit, the arrival velocity magnitude is equal to the launch velocity magnitude. This makes it easy to pair the direct-return orbits with interplanetary transfers in order to synthesize a periodic orbit.

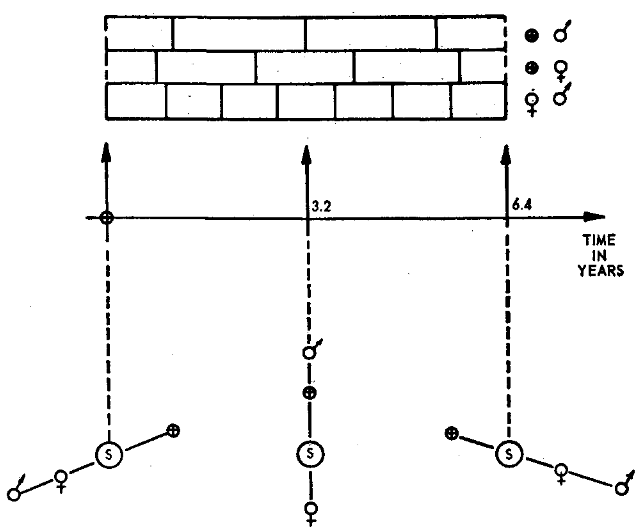

Fig. 1 Spacing of the planetary alignments.

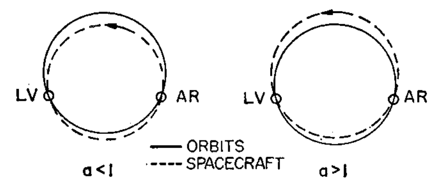

Fig. 2 Symmetric return orbits.

4. Circular, Coplanar Case for Earth and Venus

Consider Earth and Venus to be in circular, coplanar orbits so that the relative orientation of the two planets repeats exactly every 1.6 yr. A connecting orbit is periodic if it repeats after a multiple of 1.6 yr. Several combinations of orbits making a total of 4.2 solar revolutions in 3.2 yr have been found to satisfy this special condition.

For periodic orbit I, let the spaceship make one full-revolution return at Earth, transfer to Venus and make two full-revolution returns at Venus, then transfer back to Earth and repeat. By subtraction, the time for each transfer between Earth and Venus is 0.485 yr and the transfer angle is 0.6 revolution. These values allow a solution of Lambert's equation for the orbital elements of the transfer ellipse and the launch and arrival hyperbolic velocity vectors. The same transfer is used going to and from Venus, so the magnitudes of the hyperbolic velocity at each terminal planet are equal. The result of the Lambert computation shows the hyperbolic relative velocity at each Venus flyby to be 0.126 Earth Mean Orbital Speed unit (3.8 km/sec), and at each Earth flyby to be 0.107 Earth Mean Orbital Speed unit (3.2 km/sec). The gravities of Earth and Venus are strong enough to rotate these hyperbolic velocity vectors through the angles required to go from the interplanetary transfer to a planet-return transfer during flybys that take place well above the surface of each planet. The choice of full-revolution return at Earth and Venus is not unique. Any return which keeps the spacecraft at a distance greater than 1.1 planet radii during the flyby is considered acceptable. This amounts to a constraint on the magnitude of the turn angle. The maximum value of the turn angle is a function of the hyperbolic velocity magnitude V_H, as shown in Fig. 4. An example of periodic orbit I as it would be seen from Earth is shown in Fig. 5.

For periodic orbit II let the spaceship make one full-revolution return at Earth, transfer to Venus and make one symmetric return and one full-revolution return, then transfer back to Earth and repeat. For periodic orbit III let the spaceship make one symmetric return at Earth, transfer to Venus and make two full-revolution returns at Venus, then transfer back to Earth and repeat. Periodic orbits II and III

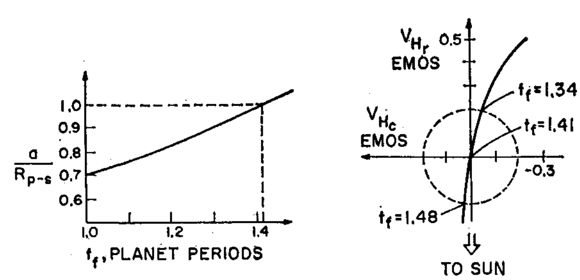

Fig. 3 Semimajor axis vs time of flight, locus of hyperbolic velocity vector (radial vs circumferential component), for symmetric return to a planet in circular orbit.

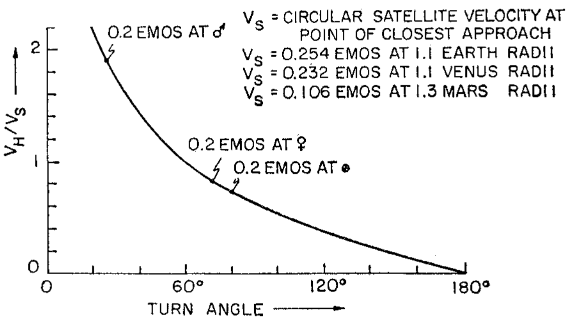

Fig. 4 Limiting turn angles.

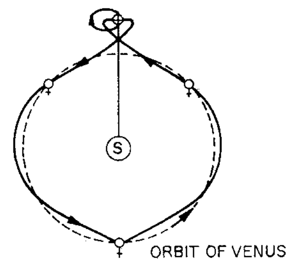

Fig. 5 Periodic orbit I as seen from Earth.

also give hyperbolic relative velocity magnitudes small enough to accomplish flybys above the surface of each planet.

5. General Case for Earth and Venus

The general case where Earth and Venus are considered to move in elliptic, inclined orbits is more complex. The periodic orbits of the last section can be used as starting points, but they now take 16 yr before repeating exactly. There are 10 transfers between Earth and Venus in the 16 yr, spaced in between direct-return orbits at each planet. Consider first the general case of periodic orbit I. Since the direct-return orbits start and end at the same point, there are only 10 terminal locations required to specify completely the interplanetary portions of the periodic orbit. Let t_i identify the 10 terminal locations of Earth and Venus and V_i the difference between inbound and outbound hyperbolic velocity for the 10 terminal locations. Ten solutions of Lambert's problem are required to find the V_i given the t_i. The problem is to adjust the $10t_i$ to make the $10V_i$ vanish simultaneously. A solution has been obtained using steepest-descent procedures. A step is taken in t_i space in the direction which causes the largest change in the sum of V_i^2. Convergence is slow but satisfactory. The procedure is as follows: 1) Use the circular, coplanar analysis to establish the $10t_i$ for the case of zero eccentricity and inclination. 2) Increase the eccentricity and inclination to their actual values. 3) Find the $10V_i$ corresponding to the $10t_i$ by 10 solutions of Lambert's problem. 4) Test

$$\sum_{i=1}^{10} V_i^2 < \epsilon$$

5) Form the matrix $\|\partial V_j/\partial t_i\|$ by making small changes in the t_i one at a time. 6) Change the t_i to t_i' by $t_i' = t_i - K\|\partial V_j/\partial t_i\|V_i$ (K is a constant used to control the step size and convergence rate. If the procedure fails to converge the magnitude of K is adjusted.) 7) Repeat 3–6 until test 4 is satisfied.

A Newton iteration[4] to solve for the $10t_i$ was also used successfully, but gave less satisfactory convergence than the steepest-descent procedure. Any future work would profit from modifications to provide more efficient convergence. The results of the computation are shown in Table 1. The hyperbolic velocity magnitudes are significantly higher than the coplanar case because of the high out-of-plane components. The full-revolution returns must be chosen so as to take the arrival velocity vector into the departure velocity vector after two flybys at Earth or after three flybys at Venus. As before, the choice of direct-return orbits at Earth and Venus is not unique. It was verified that a set existed that did not strike the planet by plotting the angle and elevation from Table 1 on a sphere, judiciously selecting return orbits, measuring the turn angles geometrically, and then verifying that the turn angles could be accomplished above the surface through the use of Fig. 4.

Periodic orbits II and III were also studied in the general case of elliptic, inclined orbits for Earth and Venus. The use of the symmetric return creates greater than 10 terminal locations of Earth and Venus. In order to keep the analysis relatively simple and also use the same computer program, the duration of the symmetric return was assumed to be a constant value selected from Fig. 3 for the approximate value of semimajor axis expected. The launch and arrival velocity magnitudes on the symmetric return were assumed equal. These approximations allow the periodic orbit again to be specified by only 10 values of t_i. Solutions were found for both periodic orbit II and periodic orbit III in the general case. Again the turn angles permitted the flybys to take place above the surface of each planet. The average magnitudes of the hyperbolic velocities in the three periodic orbits are shown in Table 2 for comparison.

These three cases are the only combinations that have been investigated so far. They all lead to a satisfactory periodic orbit connecting Earth and Venus. However, it is not necessary that the five direct-return orbits used at each terminal be of the same type. It would be reasonable to use any combination that satisfied the conditions for a periodic orbit. There are a minimum of twenty combinations to investigate since each of the 10 direct-return orbits can be chosen in at

Table 1 Periodic orbit I[a,b]

	JD 244–	V, EMOS	Ang, deg	Elev, deg	JD 244–	V, EMOS	Ang, deg	Elev, deg
LV ⊕	0806	0.154	163	−52	4300	0.156	154	50
AR ♀	0971	0.178	38	55	4471	0.192	17	−64
LV ♀	1421	0.178	335	−60	4921	0.192	319	58
AR ⊕	1592	0.154	201	50	5077	0.174	183	−56
LV ⊕	1957	0.154	143	−44	5442	0.174	123	−12
AR ♀	2125	0.203	8	67	5664	0.223	69	−15
LV ♀	2575	0.203	294	14	6114	0.223	8	−69
AR ⊕	2797	0.191	240	12	6285	0.154	215	47
LV ⊕	3163	0.191	186	58	6650	0.154	163	−52
AR ♀	3316	0.194	36	−62	Repeating after 16 yr			
LV ♀	3765	0.194	343	65				
AR ⊕	3935	0.156	215	46				

[a] The angle is in the orbital plane clockwise from the circumferential direction.
[b] The elevation is positive when above the orbital plane.

Table 2 Velocities (EMOS × 10^3) for periodic orbits I, II, and III, lowest/average/highest

	I	II	III
V_H at Earth	154/165/191	136/145/159	173/196/214
V_H at Venus	178/198/223	232/243/249	144/174/232

least two ways. In general one could search for the particular combination that gives the optimum set of characteristics for a particular mission. The conclusion is that three periodic orbits connecting Earth and Venus have been shown to exist, but there is a strong probability of the existence of additional orbits similar in nature.

6. Applications

The Earth-Venus periodic orbits presented here take two synodic periods to complete one cycle. Therefore, two spaceships are required in order to take advantage of every Earth-Venus opportunity. With both spaceships operating, there is a flyby of Venus at intervals that average one every 6.4 months. One spaceship is going to Venus and one returning from Venus at the time of planetary alignment. When one is on a planet return near Earth, the other is on a planet return near Venus, and vice-versa. During the time spent near Venus the spaceship goes behind the sun relative to the Earth. During the course of the trip the two spaceships have the opportunity to see each planet from all sides. One application for the spaceship is as a reusable interplanetary vehicle, i.e., a long-life shelter for astronauts and their equipment. It could also serve as the location of a communications link, a rescue station, or an interplanetary navigation beacon or solar probe.[7]

The computing technique described here is useful for the analysis of any multiple-planet-flyby trajectory. Examples of such trajectories are missions to the outer planets via Jupiter[5] and Earth-Venus-Mars-Earth round trips.[6] The latter suggests the possibility that unending orbits exist which periodically visit Mars as well as Earth and Venus. The analysis, however, is even more complicated when Mars is included. The assumption of a circular orbit for Mars is very unrealistic ($e = 0.093$). There is also the problem that the surface gravity of Mars is small by comparison with Earth and Venus. From Fig. 4, the limiting turn angle at Mars for the same hyperbolic velocity is half as large as it is at Earth or Venus. The solution of the Lambert iteration might call for the spaceship to go beneath the surface during a Mars flyby. The flyby at Venus rarely exhibits this result for attractive transfers to Mars via Venus. It may be necessary first to establish the trajectories from Earth and Venus which fly by Mars above its surface, and then attempt to pair these trajectories at Earth and Venus where the limiting turn angles are not so small. It might also be reasonable to take the known Earth-Venus-Mars-Earth round-trip trajectories and attempt to pair them with other trajectories at Earth. The large number of possible combinations makes it a difficult search. Another approach to the problem is to work with continuous thrust trajectories instead of free-fall orbits. First establish a periodic, continuous-thrust trajectory which visits the three planets, and then search for the neighboring optimum trajectory satisfying constraints on the point of closest approach to the planets plus the periodicity condition. The absolute optimum would be a free-fall periodic orbit if such a trajectory existed in the neighborhood of the initial guess.

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