

## Curiosity's Fight Path to Mars

### A Project for Differential Equations (Math 256)



On November 25<sup>th</sup>, 2011, NASA launched a rocket that will carry a rover called Curiosity to Mars. The rover is scheduled to land at Mars's Gale crater on August 6<sup>th</sup>, 2012. In this project you will model the 352 million mile (567 million Km) flight path of this rover, and will send it into orbit around the Sun towards Mars.

You are part of the Initial Condition Group at NASA that will work on sending the rover to Mars. Your task is to model the orbital path, and find the correct initial velocity of the spaceship to send it into orbit around the Sun towards Mars. NASA's Curiosity Rover groups are:

The Launch Group (launches the satellite into space out of Earth's gravitational field)

**The Initial Condition Group (your group, models orbital path, finds initial velocity)**

The Course Correction Group (corrects the course to Mars every few weeks)

The Entry and Landing Group (works on entry into Mars's atmosphere, and landing)

## Introduction:

Johann Kepler in 1609 discovered that planets orbit the Sun in elliptic orbits, and that their orbital velocity is not constant but varies. The following summarizes Kepler's first two laws (See Figure 1):

- 1) **The planets orbit the Sun in Elliptic orbits with the Sun at one of the focal points.**
- 2) **The line joining the Sun to a planet sweeps out equal areas in equal time.**

His second law, simply said, means that planets slow down when they are farther from the Sun, and speed up when they are closer. Since the line joining the Sun to the planet is shorter when the planet is closer, the length of the orbit covered by the planet in a given interval of time would be larger to make the areas swept equal. The spaceship carrying Curiosity rover to Mars will be orbiting the Sun as well as Earth and Mars, and the above laws applies to it and any other celestial object orbiting the Sun, or any other star.

The closest point of orbit to the Sun is called **perihelion (periapsis)**, and the farthest point is called **aphelion (apoapsis)**.

Kepler did not have the physics and mathematical tools to prove his own discovery, and it was left for the genius of Sir Isaac Newton to do that, in 1665, using his second law of motion and his law of gravitation (  $F = m \cdot a$ , and  $F = -\frac{G \cdot M \cdot m}{d^2}$  ). The 23-year old was a student at the University of Cambridge when an outbreak of the Plague forced the university to close down for 2 years. Those two years were to be the most creative in Newton's life. He conceived the law of gravitation, the laws of motion, differential calculus, and the proof of Kepler's law.

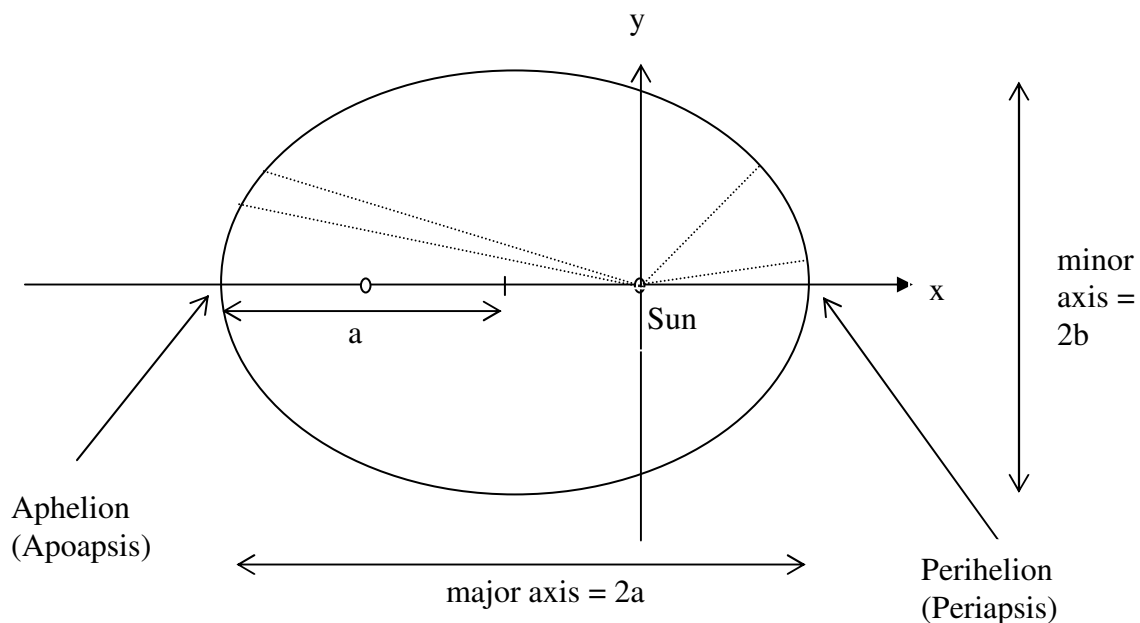


Figure 1.

## Path of a Spaceship from Earth to Mars

It may seem that the best way to send a spaceship from Earth to Mars is to choose the shortest path straight out to Mars, and time it so that the spaceship and Mars will meet at the same time and place, as shown in the Figure 2. But since the spaceship is working against Sun's gravity, it will need an impractically large amount of fuel.

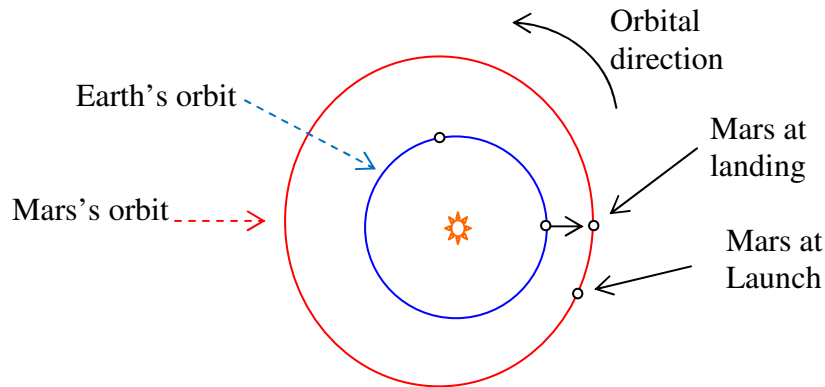


Figure 2. How not to send a spaceship to Mars

In 1925 German engineer Wolfgang Hohmann proposed that the most efficient way to send a spaceship to Mars (or to any outer planet) is to set the spaceship in orbit around the Sun with its perihelion at Earth's orbit, where it is launched, and its aphelion at Mars's orbit, where it will meet Mars. The spaceship needs almost no fuel to orbit the Sun, except for a small amount to give it the correct initial velocity to send it into orbit, and some course corrections. Figure 3 shows the flight path, with the standard  $xy$  coordinate system for the solar system. See **Appendix A** for a detailed explanation of the solar coordinate system.

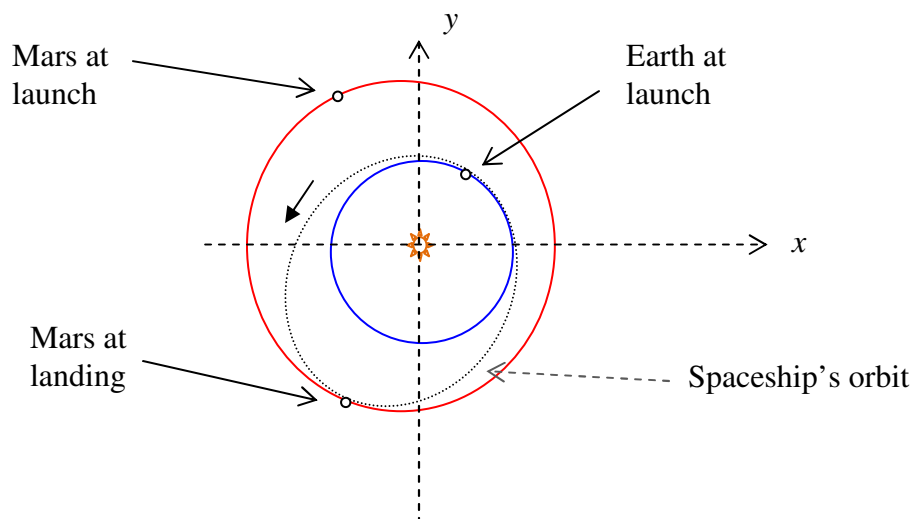


Figure 3. How to send a spaceship to Mars

## The Mathematics of the Project

Your task in this project is to model the orbit of the two planets, and the spaceship, and find the initial velocity of the spaceship to set it into orbit around the Sun so that it will meet Mars on August 6<sup>th</sup>, 2012.

We will let  $t = 0$  designate Nov. 25<sup>th</sup>, 2011. Counting the number of days to August 6<sup>th</sup>, and dividing it by 365.24 days per year, we can find that  $t = 0.70$  on August 6<sup>th</sup>, 2012.

$t = 0$  is Nov. 25<sup>th</sup>, 2011 (Launch)  
 $t = 0.70$  is August 6<sup>th</sup>, 2012 (Landing)

We will use differential equations to model the orbits and locations of Earth, Mars, and the spaceship using Newton's two laws mentioned above.

Newton's second law of motion in vector form is:

$$\vec{F} = m \vec{a} \quad (1)$$

where  $\vec{F}$  is the force vector in N (Newtons), and  $\vec{a}$  is the acceleration vector in  $m/s^2$ , and  $m$  is the mass in  $kg$ .

Newton's law of gravitation in vector form is:

$$\vec{F} = - \frac{GMm}{|\vec{r}|^2} \cdot \frac{\vec{r}}{|\vec{r}|} \quad (2)$$

where  $G = 6.67 \times 10^{-11} \frac{m^3}{s^2 \cdot kg}$  is the universal gravitational constant,  $M$  is the mass of the larger object (the Sun), and is  $2 \times 10^{30} \text{ kg}$ , and  $m$  is the mass the smaller one (the planets or the spaceship). The vector  $\vec{r}$  is the vector connecting the Sun to the orbiting objects. Note that in (1) and (2), the variables  $\vec{F}$ ,  $\vec{a}$  and  $\vec{r}$  are all functions of time, and  $\vec{a}(t) = \vec{r}''(t)$ .

The following will describe the steps you need to take to do this project.

1) Astronomers use Astronomical Units (AU) for distance, and years instead of seconds. An Astronomical Unit is the average distance between the Sun and the Earth. This is the same as the semi-major axis for the Earth ( $a$  in Figure 1 above). There are  $149.598 \times 10^9 \text{ m/AU}$ , there are  $3.15569 \times 10^7 \text{ s/y}$ . In order to use more conventional units that astronomers use, convert the units of universal gravitational constant from  $\frac{m^3}{s^2 \cdot kg}$  to  $\frac{AU^3}{y^2 \cdot kg}$  using the information above. Use at least three decimal places.

2) The motion force in Equation (1) and the gravitational force in Equation (2) are equal. Equate the right hand sides of equations (1) and (2), and cancel the common factor on the left and right sides.

3) Using  $\vec{a}(t) = \vec{r}''(t)$ , and  $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$ , convert the equation in step 2 above to an equation involving  $x(t)$ ,  $y(t)$  and their second derivatives. Note that these are the  $xy$ -coordinates of each orbiting object.

4) When two vectors are equal, their components are equal (this means that when  $a\vec{i} + b\vec{j} = c\vec{i} + d\vec{j}$ , then  $a = c$  and  $b = d$ ). Equate the  $x$  and the  $y$  components on each side of the equation you got in step 3. This should give you two second-order differential equations, one involving  $x''(t)$ ,  $x(t)$ , and  $y(t)$ , and the other involving  $y''(t)$ ,  $x(t)$ , and  $y(t)$ . These equations will not have  $\vec{i}$  and  $\vec{j}$  in them.

5) Let  $v_x(t) = x'(t)$ , and  $v_y(t) = y'(t)$ , and turn each equation into a system of two first-order differential equations. Collect these into a system of four first-order differential equations (the order should be  $x(t)$ ,  $v_x(t)$ ,  $y(t)$ ,  $v_y(t)$ ). The four unknowns are the  $x$  and the  $y$  coordinate, and the velocities in the  $x$  and the  $y$  directions of each celestial object orbiting the Sun. This system of four first-order differential equations will separately model the orbits of Earth, Mars and the spaceship, if we use the correct initial conditions for each orbiting object. In other words, if you use the four initial conditions for Mars in this system of equations, and solve them, you can get the  $x$ , and the  $y$  coordinates of Mars, and its velocity in the  $x$  and the  $y$  directions for a given time.

6) The four initial conditions for this system, for each orbiting object at  $t = 0$  (Nov. 25<sup>th</sup>, 2011) are given below. The letter e is for Earth, m is for Mars, and c is for the curiosity spaceship. Note that the initial positions for the spaceship are the same as Earth's. That is because when the spaceship is launched outside of Earth's strong gravitational field, it is about 200 miles above Earth, which is negligible compared to cosmic distances (one AU is about 93 million miles).

$$\begin{aligned} x_e(0) &= 0.44503 && \text{AU} \\ v_{xe}(0) &= -5.71113 && \text{AU / y} \\ y_e(0) &= 0.88106 && \text{AU} \\ v_{ye}(0) &= 2.80924 && \text{AU / y} \end{aligned}$$

$$\begin{aligned} x_m(0) &= -0.81449 && \text{AU} \\ v_{xm}(0) &= -4.23729 && \text{AU / y} \\ y_m(0) &= 1.41483 && \text{AU} \\ v_{ym}(0) &= -2.11473 && \text{AU / y} \end{aligned}$$

$$\begin{aligned} x_c(0) &= 0.44503 && \text{AU} \\ v_{xc}(0) &= \text{Yours to find out} && \text{AU / y}, \\ y_c(0) &= 0.88106 && \text{AU} \\ v_{yc}(0) &= \text{Yours to find out} && \text{AU / y} \end{aligned}$$

7) Use **MAPLE**, or your calculator, to solve this system numerically as described in a) or b) below. The initial conditions for Earth and Mars for  $t = 0$  are given above. You will have to choose the initial velocity for the spaceship ( $v_{xc}(0)$ , and  $v_{yc}(0)$ ), so that the spaceship is launched into orbit around the Sun, and is within 0.08 AU of Mars on August 6<sup>th</sup> ( $t = 0.7$ ).

**Hint:** First solve the system for Earth and Mars, to make sure that the orbits look reasonable, and your calculations are correct. Then input reasonable numbers for  $v_{xc}(0)$ , and  $v_{yc}(0)$  to get a reasonable orbit for the spaceship. Note that since the spaceship's orbit is a bit larger and more elliptic than Earth's, the spaceship's initial velocity should be a tad bit bigger than Earth's.

a) With **MAPLE** use the DEplot( ... ) command and the dsolve( ... type=numeric) command. The first one will give you the graph of the orbit, while the second one will give you the positions at different times after  $t = 0$  (Nov. 25<sup>th</sup>, 2011). Make stepsize equal to .005, and use scene=[x,y]. You will have to do this three times, one for Earth, one for Mars, and one for the Curiosity spaceship. The only difference between these MAPLE commands will be the initial conditions. Once you solve the Earth's orbit, you can do copy and paste and edit to get the other two orbits.

See **Appendix B** for an example of how you can use MAPLE to a system numerically and plot locations

b) If you use your **TI calculator**, you can, unfortunately, only get one solution and graph at a time. You need to set the following setting:

MODE→Graph→Diff Eq

y = → Let  $x = y1$ ,  $v_x = y2$ ,  $y = y3$ ,  $v_y = y4$ , and type your four equations and four initial conditions in:  $y1' =$ ,  $y1i =$ ,  $y2' =$ ,  $y2i =$ ,  $y3' =$ ,  $y3i =$ ,  $y4' =$ ,  $y4i =$

y = → F1 → Format → Coord → RECT

→ Solution → RK

→ Fields → FLDOFF

y = → F7 → Axes → CUSTOM, x Axis  $y1$ , y Axis  $y3$

WINDOW →  $t0 = 0$ ,  $tmax = 0.7$ ,  $tstep = 0.005$ ,  $xmin = -2*7/3$ ,  $xmax = 2*7/3$ ,  $ymin = -2$ ,  $ymax = 2$  (This will give you equal scaling on the axes, as in zoom square)

8) Present the initial velocity for the spaceship in a table, and find the  $xy$ -coordinates of each celestial object and present that in a second table as shown blow. Use four decimal places for the initial velocity table, and three decimal places for the  $xy$ -coordinates.

$v_{xc}(0)$	$v_{yc}(0)$

t (y)	xy-coord. of Earth	xy-coord. of Mars	xy-coord. of Curiosity
0			
0.2			
0.4			
0.6			
0.7			

9) Present the graph of orbits and their locations superimposed on it like the figure in Appendix B. See Appendix B for an example of how to attach these plots so the positions are superimposed on the orbits.

10) Calculate the distance between Mars and the Curiosity spaceship at  $t = 0.7$  y. This distance should be no more than 0.08 AU. Present it in a table as shown below.

Reminder: The distance equation is  $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ .

t (y)	Distance between Mars and Curiosity spaceship (AU)
0.7	

## **Your Report**

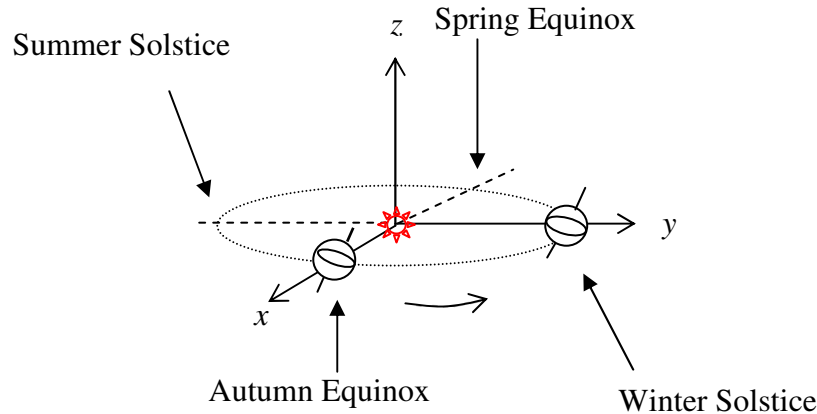
**Your report should be complete and easy to understand by a mathematician who has not seen this paper and has not been to our class.**

Your report should include:

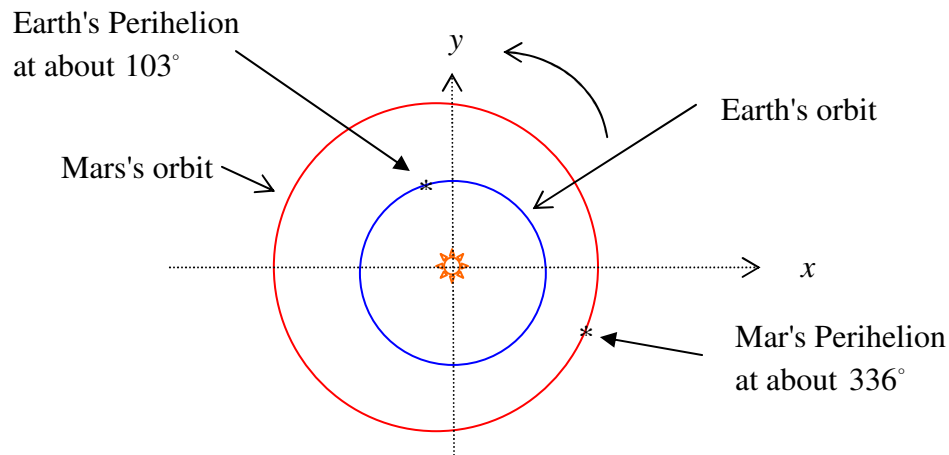
- I) A **cover sheet**.
- II) A **short and complete statement of the problem in your own words**.
- III) All your **math and calculations**.
- IV) All the **graphs and tables** (include the tables in steps 8, 9, and 10 above).
- V) All the MAPLE worksheets or calculator work (attached as an **Appendix** to the back of your report)
- VI) A short **conclusion** of what this project has contributed to your cosmic consciousness.
- VII) The report should be **all your own original words, graphs, equations, and tables** (with the possible exception of the figures in this handout you are reading now).

## Appendix A

### A short review of the heliocentric coordinate system



The three dimensional solar coordinate system, called the Heliocentric Coordinate System is shown above. The Sun is at the origin and the  $xy$ -plane is the plane of Earth's orbit. The Earth orbits the Sun, and rotates about its axis counterclockwise as seen from the positive  $z$ -axis. The Earth's rotation axis (north-south pole line) is in the  $z$  direction, but tilted towards the positive  $y$ -axis. The Earth is on the positive  $y$ -axis on Winter Solstice when the North Pole is tilted away from the Sun (approx. Dec. 21<sup>st</sup>), and on the negative  $y$ -axis on Summer Solstice when the North Pole is tilted into the Sun (approx. June 21<sup>st</sup>). The Earth is on the  $x$ -axis on the Equinoxes.



Top view of the orbits of Earth and Mars are shown above. The Earth's perihelion, measured clockwise from the positive  $x$ -axis, is at about 103° (approx. Jan. 3<sup>rd</sup>). Mars's perihelion is at about 336°.

The orbital period of Earth is 1 year, and that of Mars is 1.88 years. The Earth's semi-major axis is 1 AU, and that of Mars is 1.52 AU.



## Appendix B

You will solve the system of four equations in your project numerically using Maple. The following is an example of how you can solve a system of three Diff. Eqs. in three unknowns  $x(t)$ ,  $y(t)$ , and  $z(t)$  numerically. You can use the same commands as in this example, but will have to modify it to reflect your own project.

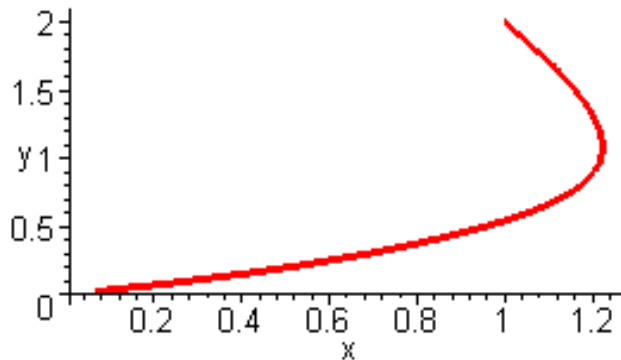
$$\begin{aligned}x'(t) &= -2x(t) + 3y(t) + z(t) \\ y'(t) &= 2x(t) - 6y(t) - 2z(t) \\ z'(t) &= x(t) - 2y(t) - 3z(t)\end{aligned}$$

DEplot gives the graph of the numeric solution.

dsolve gives the numeric solution that can be evaluated at any  $t$ .

You can then save the DEplot as p1, and the pointplot of locations as p2, and then display the graph and the locations of each solution on top of it. When saving plots, make sure to end the command with a colon (:), and not a semicolon (;). The colon suppresses the output until you are ready to display the plots.

```
> with(DEtools): with(plots):  
  
> DEplot({diff(x(t),t)=-2*x(t)+3*y(t)+z(t),  
diff(y(t),t)=2*x(t)-6*y(t)-2*z(t),diff(z(t),t)=x(t)-2*y(t)-  
3*z(t)},[x(t),y(t),z(t)],t=0..4,[[x(0)=1,y(0)=2,z(0)=-1]]  
,scene=[x,y],arrows=thin,linecolor=red,stepsize=.05);
```



```
> sol:=dsolve({diff(x(t),t)=-2*x(t)+3*y(t)+z(t),  
diff(y(t),t)=2*x(t)-6*y(t)-2*z(t),diff(z(t),t)=x(t)-2*y(t)-  
3*z(t),x(0)=1,y(0)=2,z(0)=-1},  
[x(t),y(t),z(t)],type=numeric);
```

```
sol := proc(rkf45_x) ... end proc
```

```
> sol(0);
```

```
[t = 0, x(t) = 1., y(t) = 2., z(t) = -1.]
```

```

> sol(0.2);
      [t = .2, x(t) = 1.22464334242087536 , y(t) = 1.08667890529896894 ,
      z(t) = -.797300639411056334 ]

> sol(.5);
      [t = .5, x(t) = 1.09121351452654158 , y(t) = .649786284216056998 ,
      z(t) = -.410865740179701122 ]

> sol(1);
      [t = 1, x(t) = .774048507820265241 , y(t) = .354922732807339714 ,
      z(t) = -.0932091994835609061 ]

> sol(2);
      [t = 2, x(t) = .349881052241647838 , y(t) = .127842021374964642 ,
      z(t) = .0272897365510226253 ]

> sol(3);
      [t = 3, x(t) = .151077207063233798 , y(t) = .0510881785512952366 ,
      z(t) = .0205865881299654302 ]

> sol(4);
      [t = 4, x(t) = .0643413765672093269 , y(t) = .0212120472193912287 ,
      z(t) = .00993510209622652276 ]

> p1:=DEplot({diff(x(t),t)=-2*x(t)+3*y(t)+z(t),
diff(y(t),t)=2*x(t)-6*y(t)-2*z(t), diff(z(t),t)=x(t)-
2*y(t)-3*z(t)}, [x(t),y(t),z(t)], t=0..4,
[[x(0)=1,y(0)=2,z(0)=-1]], scene=[x,y], arrows=thin,
linecolor=red,stepsize=.05):

> p2:=pointplot([1,2],[1.22,1.09],[1.09,0.65],[.77,0.35],
[.35,0.13],[.15,.05]),symbol=circle,color=black):

> display(p1,p2);

```

