FINITE-THRUST ORBIT TRANSFERS TO ANY CIRCULAR-EQUATORIAL ORBIT USING LYAPUNOV-BASED FEEDBACK CONTROL

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Abstract: A closed-loop solution is presented for finite-thrust orbit transfers to any circular-equatorial orbit, using Lyapunov stability theory. The model used is the Kustaanheimo-Stiefel transformation of the two-body problem, where the unperturbed equations of motion are equivalent to a simple harmonic oscillator. The guidance scheme is performed in two maneuver phases: first, a matching of the target's semimajor axis by allowing the spacecraft to continuously thrust, and second, a matching of inclination and eccentricity by means of thrust/coast maneuver phases. The control algorithm is robust, computationally fast, can be used for both low- and high-thrust problems.

Keywords: Orbit Transfers, Low-Thrust, Feedback Control, Lyapunov Stability

1. Introduction

The problem of computing finite thrust orbit transfers has been studied extensively in the past [1, 2]. Much work has concentrated on optimal transfers [3, 4, 5]; however, these solutions are computationally expensive and do not often result in closed-form solutions. Some attention has been given to heuristic models to achieve orbit transfers where feedback control laws are designed based on candidate Lyapunov functions, but these use mostly singular perturbation theory in conjunction with Lagrange's variational equations to establish suitable guidance laws [6, 7, 8]. In this paper, we take advantage of Lyapunov theory to design guidance laws for three-dimensional orbit transfers; however, the approach presented is quite novel due to the fact that not only are there are no additional time-scale related approximations involved, but the resulting new controllers require significantly lower computation time to permit easy onboard implementation.

To solve the orbit transfer problem, we work in a transformed model to design the Lyapunov-based control laws. The model we use is the Kustaanheimo-Stiefel (KS) regularization transformation of the two-body problem [9, 10]. The main idea behind regularization is to transform both the position and time coordinates to a new model. The position is transformed by expanding the 3D cartesian coordinates to a 4D formulation and the time is transformed by means of a Sundman transformation. This model has been mostly used in the past to deal with close-encounter-type problems [9, 11], but has only been slightly exploited for orbit transfer problems. For example, in Ref. 12 this model is used to solve the minimum-time transfer between coplanar circular orbits using Lagrange multipliers. We have recently used an equivalent planar regularized model to design guidance laws for both orbit-transfers and rendezvous problems [13, 14]. We expand on this planar work by allowing for out-of-plane change. One of the advantages of working in these transformed coordinates is that the solution

to the unperturbed equations of motion is that of a simple linear harmonic oscillator, where the frequency of oscillation is a function of *a*, the semi-major axis of the orbit.

The problem is to design a closed-loop guidance scheme to perform an orbit transfer from any three-dimensional closed orbit to any equatorial, circular orbit with a prescribed semi-major axis (this requires matching three desired parameters: semi-major axis a^* , eccentricity e^* , and inclination i^*). The spacecraft is allowed to thrust in threedimensions, where the thrust vector is given by a constant thrust magnitude T, and two time-varying spherical angles, β and δ . The idea is to design these control laws by using Lypaunov stability theory. This involves forming a quadratic artificial potential function that is always monotonic (i.e., non-increasing), and minimum at our desired final state. A detailed summary of Lyapunov-like algorithms previously developed is presented in [15], however, these methods typically all require computing the timederivatives associated with Lagrange's planetary equations and are restricted to lowthrust engines. Petropoulos [6] has developed an algorithm using Lyapunov stability to perform non-coplanar orbit transfers using a coast-thrust mechanism, but this algorithm is restricted to low-thrust solutions only and has singularities at e = 0 and i = 0.

In this paper, the convergence to the desired orbit is performed in two steps. The first step involves converging the spacecraft orbit to the desired semi-major axis a^* by continuously thrusting, by using a Lyapunov analysis that gives rise to an asymptotically stabilizing control law. Once a^* has been reached to within a specified tolerance, the second step of the algorithm involves matching the desired eccentricity and inclination (while always maintaining the specified target a^*). Note that while the guidance schemes are designed in the transformed regularized coordinates, the results can be explicitly characterized in the original coordinates; the transformation procedure merely acts as an enabler for the design. The algorithms designed are robust to any initial and final conditions, computationally fast, and can be used for both low- and high-thrust problems.

The paper is organized as follows. In Section 2 we outline how to obtain the equations of motion in the regularized coordinates (KS variables). Section 3 focuses on defining the different Lyapunov-like functions to converge to each of the three orbital elements targeted. In Section 4 we summarize the full control sequence and show several examples. Conclusions and future work in shown in Section 5.

2. Equations of Motion

The equations of motion of a spacecraft in a perturbed planar two-body model are

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} + \mathbf{f} \tag{1}$$

where μ is the gravitational parameter of the central body, the position vector

$$\mathbf{r} = \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix}^T$$

has magnitude $r = \sqrt{r_1^2 + r_2^2 + r_3^2}$, and $\mathbf{f} \in \mathbb{R}^3$ is the perturbing acceleration.

The main idea behind regularization is to transform both the position coordinates and time coordinates to a new model [9, 11]. The planar regularization problem, also known as Levi-Civita model, involves a mapping in the complex plane. However, this same procedure is not possible in 3D. Kustaanheimo and Stiefel worked around this

issue by creating a 4D formulation to arrive at the regularized model (known as the KS transformation).

There are two main steps to convert Eq. 1 into the regularized model. First, a change of position coordinates, and second, a time transformation. Following the notation in Ref. [9], we begin with the first step.

1. Change of position coordinates:

Let the position vector in the KS model be

$$\mathbf{u} = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix}^T \in \mathbb{R}^4$$
(2)

and define an expanded position vector

$$\mathbf{R} = \begin{bmatrix} \mathbf{r} \\ r_4 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} \in \mathbb{R}^4$$
(3)

where $r_4 = 0$. The transformation from **u** to **R** can be expressed in terms of the following linear operator

$$\mathcal{L}(\mathbf{u}) = \begin{bmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_2 \\ u_4 & -u_3 & u_2 & -u_1 \end{bmatrix}$$
(4)

such that

$$\mathbf{R} = \mathcal{L}(\mathbf{u})\mathbf{u} \tag{5}$$

The position coordinates of R are therefore

$$\begin{aligned} r_1 &= u_1^2 - u_2^2 - u_3^2 + u_4^2 \\ r_2 &= 2 \left(u_1 u_2 - u_3 u_4 \right) \\ r_3 &= 2 \left(u_1 u_3 + u_2 u_4 \right) \\ r_4 &= 0 \end{aligned}$$
 (6)

which verifies the constraint that $r_4 = 0$, since it is just a fictitious variable. The magnitude of the position vector is

$$R = \|\mathbf{R}\| = \|\mathbf{r}\| = \sqrt{r_1^2 + r_2^2 + r_3^2} = u_1^2 + u_2^2 + u_3^2 + u_4^2$$
(7)

The linear operator $\mathcal{L}(\mathbf{u})$ has the following properties [9]:

- (a) $\mathcal{L}^{T}(\mathbf{u})\mathcal{L}(\mathbf{u}) = r\mathbf{I}$, where \mathbf{I} is the identity matrix
- (b) $\mathcal{L}'(\mathbf{u}) = \mathcal{L}(\mathbf{u}')$, where ' denotes the time derivative
- (c) $\mathcal{L}(\mathbf{u})\mathbf{v} = \mathcal{L}(\mathbf{v})\mathbf{u}$
- (d) $(\mathbf{u} \cdot \mathbf{u})\mathcal{L}(\mathbf{v})\mathbf{v} 2(\mathbf{u} \cdot \mathbf{v})\mathcal{L}(\mathbf{u})\mathbf{v} + (\mathbf{v} \cdot \mathbf{v})\mathcal{L}(\mathbf{u})\mathbf{u} = \mathbf{0}$, which only holds in 4D if $u_4u'_1 u_3u'_2 + u_2u'_3 u_1u'_4 = 0$. This last constraint is necessary to ensure $r'_4 = 0$

2. Perform Sundman transformation to change the time variable

The second step in the derivation involves introducing a velocity transformation by performing a Sundman transformation,

$$\dot{\mathbf{R}} = \frac{1}{R}\mathbf{r}' \iff \frac{d\mathbf{R}}{dt} = \frac{1}{R}\frac{d\mathbf{R}}{ds}$$

Note, that $\dot{()} = \frac{d}{dt}()$, denotes the time derivative in **r** coordinates, whereas $()' = \frac{d}{ds}()$ is the time derivative in **u** coordinates. Using properties 2 and 3 of the linear operator $\mathcal{L}(\mathbf{u})$, the derivative of **R** in terms of **u** coordinates are

$$\mathbf{R}' = \frac{d}{ds}\mathbf{R} = \frac{d}{ds}\left(\mathcal{L}(\mathbf{u})\mathbf{u}\right) = 2\mathcal{L}(\mathbf{u})\mathbf{u}'$$
(8)

such that

$$\mathbf{u}' = \frac{1}{2} \mathcal{L}^T(\mathbf{u}) \dot{\mathbf{R}} \iff \dot{\mathbf{R}} = \frac{2}{R} \mathcal{L}(\mathbf{u}) \mathbf{u}'$$
 (9)

By following the same steps as in the derivation of the planar regularized model, which is outlined in Ref. [13, 14], the perturbed two-body equations of motion in Eq. 1 in the regularized form are

$$\mathbf{u}'' = \frac{E}{2}\mathbf{u} + \frac{1}{2}R\mathcal{L}^{T}(\mathbf{u})\mathbf{F}$$

$$E' = 2\mathbf{u}' \cdot \mathcal{L}^{T}(\mathbf{u})\mathbf{F}$$

$$t' = \mathbf{u} \cdot \mathbf{u}$$
(10)

where the augmented perturbing acceleration vector is

$$\mathbf{F} = \begin{bmatrix} \mathbf{f} \\ 0 \end{bmatrix} \in \mathbb{R}^4 \tag{11}$$

For implementation purposes it is convenient to introduce a new parameter

$$\alpha = \frac{1}{a} \tag{12}$$

where *a* is the semi-major axis of the orbit. The parameter α is related to the energy *E* of an orbit through the relation $E = -\mu/(2a) = -(\mu/2)\alpha$ and its derivative $\alpha' = -(2/\mu)E'$. Combining this into Eq. 10, results in the new equations of motion

$$\mathbf{u}'' = -\frac{\mu\alpha}{4}\mathbf{u} + \frac{1}{2}R\mathcal{L}^{T}(\mathbf{u})\mathbf{F}$$

$$\alpha' = -\frac{4}{\mu}\mathbf{u}' \cdot \mathcal{L}^{T}(\mathbf{u})\mathbf{F}$$

$$t' = \mathbf{u} \cdot \mathbf{u}$$
(13)

Note, that one of the components of \mathbf{u} is arbitrary, since the transformation involves a transformation from \mathbb{R}^3 to \mathbb{R}^4 . Therefore, there is some freedom in the choice of initial conditions, when transforming from $\mathbf{r}(t_0)$ to $\mathbf{u}(s_0)$ [9].

If $r_{1_0} \ge 0$			If $r_{1_0} < 0$		
u_{4_0}	=	0	u_{3_0}	=	0
u_{1_0}	=	$\left[\frac{1}{2}\left(r_{1_{0}}+r_{0}\right)\right]^{1/2}$	u_{2_0}	=	$\left[\frac{1}{2}\left(r_{0}-r_{1_{0}}\right)\right]^{1/2}$
u_{2_0}	=	$\frac{1}{2}\left(\frac{r_{2_0}}{u_{1_0}}\right)$	u_{1_0}	=	$\frac{1}{2}\left(\frac{r_{2_0}}{u_{2_0}}\right)$
u_{3_0}	=	$\frac{1}{2}\left(\frac{r_{3_0}}{u_{1_0}}\right)$	u_{4_0}	=	$\frac{1}{2}\left(\frac{r_{3_0}}{u_{2_0}}\right)$

2.1. Unperturbed Equations of Motion

Note that in the case when there is no perturbing acceleration acting on the spacecraft, that is, $\mathbf{F} = \mathbf{0}$, the equations of motion in Eq. 13 become de-coupled linear harmonic oscillators and $\alpha' = 0$.

$$\mathbf{u}'' = -\frac{\mu\alpha}{4}\mathbf{u}$$
 and $\alpha' = 0$ (14)

The analytical solution is known, with a frequency of oscillation of $W = \sqrt{\mu \alpha/4}$. Therefore, during any coast phase, the analytical solution is always known and given by

$$\mathbf{u}(s) = \mathbf{u}_0 \cos\left(\mathcal{W}s\right) + \frac{\mathbf{u}_0'}{\mathcal{W}} \sin(\mathcal{W}s)$$
(15)

2.2. Defining the Thrust Vector F

Suppose that the perturbation f in Eq. 1 now comes from a finite thrust engine, with constant thrust magnitude T and specific impulse c. The acceleration magnitude $\|\mathbf{f}\| = T/m$ and m is the mass of the spacecraft at any point in time, which varies according to $\dot{m} = -T/c$. We choose the acceleration vector direction to be defined in a body-fixed frame of the spacecraft, with two spherical variable angles δ and β , such that

$$\mathbf{f} = \frac{T}{m} \left[\hat{\mathbf{r}}' \sin \beta + \hat{\mathbf{d}} \cos \delta \cos \beta + \hat{\mathbf{e}} \sin \delta \cos \beta \right] \in \mathbb{R}^3$$
(16)

where the unit vectors $\hat{\mathbf{r}}'$, $\hat{\mathbf{d}}$, and $\hat{\mathbf{e}}$ form a basis in the body-frame and are defined as

$$\begin{array}{lll} \hat{\mathbf{r}}' &=& \mathbf{r}'/\|\mathbf{r}'\| \\ \hat{\mathbf{d}} &=& \mathbf{d}/\|\mathbf{d}\|, \text{ where } \mathbf{d} = (\mathbf{r}/\|\mathbf{r}\|) \times \mathbf{r}' \\ \hat{\mathbf{e}} &=& \hat{\mathbf{r}}' \times \hat{\mathbf{d}} \end{array}$$

Note that the two varying angles δ and β are the design control inputs to reach the desired orbit.

The augmented acceleration vector \mathbf{F} implemented in Eq. 13 is then defined as

$$\mathbf{F} = \begin{bmatrix} \mathbf{f} \\ 0 \end{bmatrix} = \frac{T}{m} \begin{bmatrix} \mathbf{R}' \\ \|\mathbf{R}'\| \\ \sin\beta + \hat{\mathbf{D}}\cos\delta\cos\beta + \hat{\mathbf{E}}\sin\delta\cos\beta \end{bmatrix}$$
(17)

where

$$\hat{\mathbf{D}} = \begin{bmatrix} \hat{\mathbf{d}} \\ 0 \end{bmatrix}, \ \hat{\mathbf{E}} = \begin{bmatrix} \hat{\mathbf{e}} \\ 0 \end{bmatrix}, \ \text{and} \ \mathbf{R}' = \begin{bmatrix} \mathbf{r}' \\ 0 \end{bmatrix},$$

Note that \mathbf{R}' is the natural representation of the system since $r'_4 = 0$ from property 4 of the linear operator.

3. Lyapunov-Based Algorithms to Match Each Desired Orbital Element

The goal of this paper is to target a circular-equatorial orbit of any desired radius. In order to do so, three orbital elements need to be matched: the semi-major axis a^* , eccentricity $e^* = 0$, and inclination $i^* = 0$, where * denotes the desired value. We define the error between the current orbital elements and the desired ones as $e_a = a - a^*$, $e_e = e - e^*$, and $e_i = i - i^*$. The idea is to design the control algorithms by using Lyapunov stability theory. This involves forming a quadratic artificial potential function that is always monotonic (i.e., non-increasing), and minimum at our desired final state.

3.1. Matching Semi-Major Axis

Suppose that the semi-major axis of the target orbit is given by a^* , and therefore, its inverse is $\alpha^* = 1/a^*$. At any point in time (in *s* coordinates) the chaser spacecraft, whose motion is governed by Eq. 13, has a semi-major axis a(s), and correspondingly $\alpha(s) = 1/a(s)$. The difference between the current $\alpha(s)$ and the target α^* is denoted by $e_{\alpha} = \alpha - \alpha^*$. The quadratic candidate Lyapunov function

$$V_{\alpha} = \frac{1}{2}e_{\alpha}^{2} = \frac{1}{2}(\alpha - \alpha^{*})^{2}$$
(18)

is minimized when the semi-major axis of the chaser spacecraft matches the semimajor axis of the target. The time-derivative of V along solutions of Eq. 13 is given by

$$V'_{\alpha} = e_{\alpha}e'_{\alpha} = (\alpha - \alpha^*)\alpha'$$

By using the definition of the thrust vector \mathbf{F} in Eq. 17, the value α' becomes

$$\begin{aligned} \alpha' &= -\frac{4}{\mu} \mathbf{u}' \cdot \mathcal{L}^T(\mathbf{u}) \mathbf{F} = \frac{4}{\mu} \left(\mathcal{L}^T(\mathbf{u}) \mathbf{u}' \right)^T \mathbf{F} \\ &= -\frac{2}{\mu} \frac{T}{m} \mathbf{R'}^T \left[\frac{\mathbf{R'}}{\|\mathbf{R'}\|} \sin\beta + \hat{\mathbf{L}} \cos\beta \cos\delta + \hat{\mathbf{e}} \cos\beta \sin\delta \right] \\ &= -\frac{2}{\mu} \frac{T}{m} \|\mathbf{R'}\| \sin\beta \end{aligned}$$

Note that $\mathbf{R}' \cdot \hat{\mathbf{D}} = \mathbf{0}$ and $\mathbf{R}' \cdot \hat{\mathbf{E}} = \mathbf{0}$. Therefore, it is not necessary to have δ as a control input for the semi-major axis matching section, but will become important in the next sections. By choosing

$$\beta = \sin^{-1}(Ke_{\alpha})$$
 where $K = 1/\|e_{\alpha_0}\|$ (19)

results in

$$\alpha' = -\frac{2}{\mu} \frac{T}{m} \|\mathbf{R}'\| K e_{\alpha} \tag{20}$$

which ensures that V_{α} is non-increasing.

$$V'_{\alpha} = -\frac{2}{\mu} \frac{T}{m} K e_{\alpha}^2 \le 0 \tag{21}$$

Proposition 1. Let a spacecraft with a constant available maximum thrust $T = T_{max}$ and initial mass m_0 be on a closed orbit with initial semi-major axis $a(s_0) = a_0$. All solutions governed by the force model in Eq. 13 with thrust vector in Eq. 17 converge to any prescribed final semi-major axis a^* , with control angle β defined in Eq. 19, regardless of the choice of control angle δ .

3.2. Matching Eccentricity

The eccentricity e of an orbit expressed in the KS variables is [9]

$$e^{2} = (1 - r\alpha)^{2} + \frac{4}{\mu}\alpha(\mathbf{u} \cdot \mathbf{u}')^{2}$$
(22)

To target a circular orbit $e^* = 0$, a Lyapunov-candidate function (which is minimum when e = 0) is

$$V_e = \frac{1}{2}(e - e^*)^2 = \frac{1}{2}e^2 \ge 0$$
(23)

Note $r' = 2\mathbf{u} \cdot \mathbf{u}'$. The time derivative of V_e along solutions of Eq. 13 is

$$V'_{e} = \frac{1}{2} \frac{d}{ds} e^{2} = r\alpha^{2} (\mathbf{u} \cdot \mathbf{u}') + \left[-r(1 - r\alpha) + \frac{2}{\mu} (\mathbf{u} \cdot \mathbf{u}')^{2} \right] \alpha' + \frac{4}{\mu} \alpha (\mathbf{u} \cdot \mathbf{u}') [\mathbf{u} \cdot \mathbf{u}'']$$
$$= \frac{4}{\mu} \left[\frac{1}{2} \alpha (\mathbf{u} \cdot \mathbf{u}') r \left[\mathcal{L}(\mathbf{u}) \mathbf{u} \right]^{T} + \left(r(1 - r\alpha) - \frac{2}{\mu} (\mathbf{u} \cdot \mathbf{u}')^{2} \right) \left[\mathcal{L}(\mathbf{u}) \mathbf{u}' \right]^{T} \right] \mathbf{F}$$

We now use the relations in Eq. 5 and 9 and the control law F in Eq. 17. Also note that $\mathbf{R} \cdot \hat{\mathbf{D}} = \mathbf{0}$, $\mathbf{R}' \cdot \hat{\mathbf{D}} = \mathbf{0}$, and $\mathbf{R}' \cdot \hat{\mathbf{E}} = \mathbf{0}$. The time-derivative of V_e simplifies to

$$V'_{e} = \frac{2}{\mu} \frac{T}{m} \|\mathbf{R}'\| \left[\left(\frac{r^2 \alpha}{\|\mathbf{R}'\|^2} - \frac{1}{\mu} \right) (\mathbf{u} \cdot \mathbf{u}')^2 + r(1 - r\alpha) \right] \sin \beta \\ + \frac{2}{\mu} \frac{T}{m} r \alpha (\mathbf{u} \cdot \mathbf{u}') (\mathbf{R} \cdot \hat{\mathbf{E}}) \cos \beta \sin \delta$$

If we assume that the semi-major axis of the chaser has been previously matched to be that of the target (to within some specified tolerance) by using the control \mathbf{F} with β in Eq. 19, then $\sin \beta \approx 0$ and $\cos \beta \approx 1$. The time-derivative of the potential becomes

$$V'_{e} = \frac{2}{\mu} \frac{T}{m} r \alpha (\mathbf{u} \cdot \mathbf{u}') (\mathbf{R} \cdot \hat{\mathbf{e}}) \sin \delta$$
(24)

which can be made negative semi-definite by choosing

$$\delta = \begin{cases} \sin^{-1}(e) & \text{if } (\mathbf{u} \cdot \mathbf{u}')(\mathbf{R} \cdot \hat{\mathbf{E}}) < 0\\ \sin^{-1}(-e) & \text{otherwise} \end{cases}$$
(25)

Note we have defined this δ to only match the desired eccentricity and does not take into account any inclination matching.

3.3. Matching Inclination

We now continue to derive a guidance scheme that will converge the inclination of an orbit. The inclination is defined as

$$\cos i = \frac{h_k}{h} \tag{26}$$

where h_k is the k component of the angular momentum vector \mathbf{h} , and $h = ||\mathbf{h}||$. The angular momentum is expressed as $h = \sqrt{\mu a(1 - e^2)}$. Because we are already matching a^* and e^* by the thrusting protocol in the previous sections, the magnitude of the desired h^* will already be targeted. Also, because $0 \le i \le 180^\circ$, there is no ambiguity or quadrant check necessary. Therefore, in order to achieve a desired i^* , is equivalent to target a desired h_k^* .

The angular momentum $\mathbf{h} = \mathbf{r} \times \mathbf{v}$. Therefore, the $\hat{\mathbf{k}}$ component is

$$h_k = x\dot{y} - y\dot{x} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \mathbf{r}^T \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\mathbf{r}}$$
(27)

In order to target h_k^* , we need to express h_k in the KS variables.

$$h_{k} = \left[\mathcal{L}(\mathbf{u})\mathbf{u}\right]^{T}\mathbf{Y}\left[\frac{2}{r}\mathcal{L}(\mathbf{u})\mathbf{u}'\right]$$
$$= \frac{2}{r}\mathbf{u}^{T}\left(\mathcal{L}^{T}(\mathbf{u})\mathbf{Y}\mathcal{L}(\mathbf{u})\right)\mathbf{u}'$$
$$= \frac{2}{r}\mathbf{u}^{T}\mathbf{B}\mathbf{u}'$$
(28)

where

$$\mathbf{B} = \mathcal{L}^T(\mathbf{u})\mathbf{Y}\mathcal{L}(\mathbf{u}) \tag{29}$$

$$\mathbf{Y} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}_{4 \times 4} \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}_{2 \times 2} \text{ and } \mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2 \times 2}$$

The matrix **B** is skew-symmetric, $\mathbf{B} = -\mathbf{B}^T$

$$\mathbf{B} = \begin{bmatrix} 0 & a & -b & -c \\ -a & 0 & c & -b \\ b & -c & 0 & d \\ c & b & -d & 0 \end{bmatrix} \qquad \text{where} \qquad \begin{aligned} a &= u_1^2 + u_2^2 \\ b &= u_1 u_4 - u_2 u_3 \\ c &= u_1 u_3 + u_2 u_4 \\ d &= u_3^2 + u_4^2 \end{aligned} \tag{30}$$

The time-derivative of h_k is

$$h'_{k} = \frac{2}{r} \left(\mathbf{u}'^{T} \mathbf{B} \mathbf{u}' + \mathbf{u}^{T} \mathbf{B}' \mathbf{u}' + \mathbf{u}^{T} \mathbf{B} \mathbf{u}'' \right) + \frac{2}{r^{3}} (\mathbf{r} \cdot \mathbf{r}') \mathbf{u}^{T} \mathbf{B} \mathbf{u}'$$
$$= \frac{2}{r} \left[-\frac{\mu}{4} \alpha \mathbf{u}^{T} \mathbf{B} \mathbf{u} + \mathbf{u}'^{T} \mathbf{B} \mathbf{u}' + \mathbf{u}^{T} (\mathbf{B}' - 2(\mathbf{u} \cdot \mathbf{u}')B) \mathbf{u}' + \frac{r}{2} \mathbf{u}^{T} \mathbf{B} \mathcal{L}^{T} (\mathbf{u}) \mathbf{F} \right]$$

Note that because **B** is skew-symmetric, for any vector $\mathbf{x} \in \mathbb{R}^4$, $\mathbf{x}^T \mathbf{B} \mathbf{x} = \mathbf{0}$. Therefore, the first two terms in h'_k vanish. The third term in h'_k , $\mathbf{u}^T (\mathbf{B}' - 2(\mathbf{u} \cdot \mathbf{u}')B)\mathbf{u}'$, also vanishes. Therefore, the time-derivative of h_k is

$$h'_{k} = \mathbf{u}^{T} \mathbf{B} \mathcal{L}^{T}(\mathbf{u}) \mathbf{F} = -(\mathcal{L}(\mathbf{u}) \mathbf{B} \mathbf{u}) \cdot \mathbf{F}$$
 (31)

The quadratic candidate function

$$V_i = \frac{1}{2}e_h^2 = \frac{1}{2}(h_k - h_k^*)^2$$
(32)

is minimized when the inclination matches the desired inclination. For a desired equatorial orbit, $h_k^* = 1$ and the other two component of the angular momentum will be zero. The time-derivative of V_i ,

$$V'_{i} = (h_{k} - h_{k}^{*})h'_{k}$$

= $-(h_{k} - h_{k}^{*})[(\mathcal{L}(\mathbf{u})\mathbf{B}\mathbf{u}) \cdot \mathbf{F}]$
= $-\frac{T}{m}(h_{k} - h_{k}^{*})\left[(\mathcal{L}(\mathbf{u})\mathbf{B}\mathbf{u}) \cdot \left(\frac{\mathbf{R}'}{\|\mathbf{R}'\|}\sin\beta + \hat{\mathbf{D}}\cos\beta\cos\delta + \hat{\mathbf{E}}\cos\beta\sin\delta\right)\right]$ (33)

Note, that if the semi-major axis of the desired orbit has already been matched to a^* (to within a certain tolerance), the angle $\beta \approx 0$. The time derivative

$$V'_{i} = -\frac{T}{m}(h_{k} - h_{k}^{*})\left[\left(\mathcal{L}(\mathbf{u})\mathbf{B}\mathbf{u}\right) \cdot \left(\hat{\mathbf{D}}\cos\delta + \hat{\mathbf{E}}\sin\delta\right)\right]$$
(34)

At this point, the control angle δ in Eq. 25 has already been designed to match the desired eccentricity $e^* = 0$. In order to make $V'_i \leq 0$, the only option is to begin a set of thrust/coast arcs in the following manner

$$T = \begin{cases} T_{max} & \text{if } (h_k - h_k^*) \left[(\mathcal{L}(\mathbf{u}) \mathbf{B} \mathbf{u}) \cdot \left(\hat{\mathbf{D}} \cos \delta + \hat{\mathbf{E}} \sin \delta \right) \right] > 0 \\ 0 & \text{otherwise} \end{cases}$$
(35)

This ensures not only that $V'_i \leq 0$, but also $V'_e \leq 0$. Therefore, both eccentricity and inclination will converge to the desired $e^* = 0$ and $i^* = 0$.

Proposition 2. Let a spacecraft with a constant available maximum thrust $T = T_{max}$ and initial mass m_0 be on a closed orbit with an initial semi-major axis equal to the desired one $a_0 = a^*$. All solutions governed by the force model in Eq. 13 with thrust vector in Eq. 17 converge to a circular-equatorial orbit with $e^* = 0$ and $i^* = 0$, when the control angles are chosen as β in Eq. 19 and δ in Eq. 25, with the thrust/coast protocol defined in Eq. 35.

Another option is to re-define δ to make $(\hat{\mathbf{D}} \cos \delta + \hat{\mathbf{E}} \sin \delta)$ in Eq. 34 lie in the same direction as $\mathcal{L}(\mathbf{u})\mathbf{B}\mathbf{u}$, which will result in $V'_i \leq 0$. Because the fourth component of $\hat{\mathbf{D}}$ and $\hat{\mathbf{E}}$ are null, we are only interested in aligning to the first three components of $\mathcal{L}(\mathbf{u})\mathbf{B}\mathbf{u}$. We define the vector \mathbf{q} as the first three components of $(\mathcal{L}(\mathbf{u})\mathbf{B}\mathbf{u})_{1:3}$. The vector \mathbf{q} can be written in two separate components: $\mathbf{q}_{\hat{\mathbf{d}}-\hat{\mathbf{e}}}$, which lies in the plane formed by $\hat{\mathbf{d}} - \hat{\mathbf{e}}$, and $\mathbf{q}_{\hat{\mathbf{r}}'}$, which is perpendicular to this plane (i.e., lies in the direction of $\hat{\mathbf{r}'}$). The time-derivative of V_i in Eq. 34 becomes

$$V'_{i} = -\frac{T}{m}(h_{k} - h_{k}^{*}) \left[\mathbf{q} \cdot \left(\hat{\mathbf{d}} \cos \delta + \hat{\mathbf{e}} \sin \delta \right) \right]$$

$$= -\frac{T}{m}(h_{k} - h_{k}^{*}) \left[\left(\mathbf{q}_{\hat{\mathbf{d}} - \hat{\mathbf{e}}} + \mathbf{q}_{\hat{\mathbf{r}}'} \right) \cdot \left(\hat{\mathbf{d}} \cos \delta + \hat{\mathbf{e}} \sin \delta \right) \right]$$

$$= -\frac{T}{m}(h_{k} - h_{k}^{*}) \left[\left(q_{\hat{\mathbf{d}} - \hat{\mathbf{e}}} \hat{\mathbf{u}}_{\hat{\mathbf{d}} - \hat{\mathbf{e}}} \right) \cdot \left(\hat{\mathbf{d}} \cos \delta + \hat{\mathbf{e}} \sin \delta \right) \right]$$

where $q_{\hat{\mathbf{d}}-\hat{\mathbf{e}}} = \|\mathbf{q}_{\hat{\mathbf{d}}-\hat{\mathbf{e}}}\|$. By designing δ as

$$\delta = \tan^{-1}\left(\frac{u_2}{u_1}\right) + k_i \pi \quad \text{where } k_i = \begin{cases} 0 & \text{if } (h_k - h_k^*) \ge 0\\ 1 & \text{otherwise} \end{cases}$$
(36)

where

$$u_1 = \hat{\mathbf{u}}_{\hat{\mathbf{d}}-\hat{\mathbf{e}}} \cdot \mathbf{d}$$
$$u_2 = \hat{\mathbf{u}}_{\hat{\mathbf{d}}-\hat{\mathbf{e}}} \cdot \hat{\mathbf{e}}$$

results in $V'_i \leq 0$.

The control protocol δ in Eq. 36 will ensure that the inclination converges to $i^* = 0$; however, we still need to match the eccentricity to e^* by ensuring that $V'_e \leq 0$ in Eq. 24. In order to do so, we begin a set of thrust/coast arcs such that

$$T = \begin{cases} T_{max} & \text{if } (\mathbf{u} \cdot \mathbf{u}')(\mathbf{R} \cdot \hat{\mathbf{e}}) \sin \delta < 0\\ 0 & \text{otherwise} \end{cases}$$
(37)

Proposition 3. Let a spacecraft with a constant available maximum thrust $T = T_{max}$ and initial mass m_0 be on a closed orbit with an initial semi-major axis equal to the desired one $a_0 = a^*$. All solutions governed by the force model in Eq. 13 with thrust vector in Eq. 17 converge to a circular-equatorial orbit with $e^* = 0$ and $i^* = 0$, when the control angles are chosen as β in Eq. 19 and δ in Eq. 36, with the thrust/coast protocol defined in Eq. 37.

A comparison example between the algorithms in Proposition 2 and Proposition 3 is shown in Fig. 1, with initial and final conditions given in Tab. 1¹ using a thrust $T_{max} = 20 \text{ N}$, $I_{sp} = 1,000 \text{ s}^2$, and initial spacecraft mass $m_0 = 1,000 \text{ kg}$. Both the initial and final orbits are identical in semi-major axis and eccentricity, but are offset in inclination. The position coordinates have been dimensionalized to be 1DU = 42,000 km, which is the value of a^* . Note, from Fig. 1(d-e), how neither the semi-major axis nor eccentricity vary during the process. A comparison in flight time and propellant used is shown in Tab. 2. The algorithm in Proposition 3 is faster in flight time (by 2.12 days), but uses more propellant (by 68.8 kg).



Figure 1: Orbit transfer from an initial orbit with $a_0 = 42,000$ km, $e_0 = 0.0$, and $i = 70.0^{\circ}$, to a final orbit with $a^* = 42,000$ km, $e^* = 0.0$, and $i^* = 0.0^{\circ}$ using $T_{max} = 20$ N, $I_{sp} = 1,000$ s, $m_0 = 1,000$ kg. Comparison between control algorithms defined in Proposition 2 and Proposition 3.

¹The orbital parameters, ω represents the argument of periapsis, and Ω is the right ascension of the ascending node. For a circular-equatorial orbit, these two parameters are not defined, and therefore, it is not necessary to target them.

²The specific impulse I_{sp} is an engine parameter used to describe its efficiency. The exhaust velocity c, which is used to calculate the fuel mass rate, is computed as $c = gI_{sp}$, where $g = 9.81 \text{ m/s}^2$.

Table 1: Initial and final orbital parameters for the example in Fig. 1

	a (km)	e	<i>i</i> (°)	ω (°)	Ω (°)
Initial Orbit	42,000	0.00	70.00	-	—
Final Orbit	42,000	0.00	0.00	-	-

Table 2: Propellant used and flight time for the example in Fig. 1

	Proposition 2	Proposition 3	
Propellant(kg)	439.2	508.0	
Flight Time (days)	5.46	3.34	

4. Transfer to any Circular-Equatorial Orbit

We now combine the results from the previous section, to discuss the algorithm to target a circular ($e^* = 0$), equatorial ($i^* = 0$) orbit with a prescribed semi-major axis a^* . The algorithm is a combination of Proposition 1 and Proposition 3¹. The algorithm is performed in two steps. First, a matching of the target semi-major axis a^* , by means of using β in Eq. 19. Recall, that the other spherical angle δ does not affect the semi-major axis convergence. For this reason, we choose δ as in Eq. 36, which does not necessarily guarantee the negative semi-definiteness of V'_i , but will approach it as $a \rightarrow a^*$. The second step, once the semi-major axis is converged to the desired one (to within a certain tolerance ϵ_a), involves converging the eccentricity and inclination to $e^* = 0$ and $i^* = 0$, by means of using the same δ , but now a set of thrust/coast solutions are implemented similarly to Eq. 37. The two steps are

1. $s_0 \leq s \leq s_i$: Semi-Major Axis Matching

$$T = T_{max}$$
 while $|a(s) - a^*| > \epsilon_a$ (38)

2. $s_i < s \le s_f$: Inclination and Eccentricity Matching

$$T = \begin{cases} T_{max} & \text{if } (\mathbf{u} \cdot \mathbf{u}')(\mathbf{R} \cdot \hat{\mathbf{e}}) \sin \delta < 0\\ 0 & \text{otherwise} \end{cases}$$
(39)

with control angles

$$\beta = \sin^{-1}(Ke_{\alpha}) \text{ where } K = 1/||e_{\alpha_0}||$$

$$\delta = \tan^{-1}\left(\frac{u_2}{u_1}\right) + k_i\pi \text{ where } k_i = \begin{cases} 0 \text{ if } (h_k - h_k^*) \ge 0\\ 1 \text{ otherwise} \end{cases}$$
(40)

Proposition 4. Let a spacecraft with a constant available maximum thrust $T = T_{max}$ and initial mass m_0 be on any closed orbit. All solutions governed by the force model in Eq. 13 with thrust vector in Eq. 17 converge to a circular ($e^* = 0$), equatorial ($i^* = 0$), orbit with a prescribed semi-major axis a^* , when the control angles β and δ are chosen as in Eq. 40, with the thrust/coast protocol defined in Eq. 38-39.

¹Note that combining Propositions 1 and 2 is also a viable option. The basis for our choice is that most of the simulations run (albeit, not exhaustive), appear to be more optimal both in time and fuel efficiency by using Propositions 1 and 3.

An example simulation is shown in Fig. 2 with initial and final parameters given in Tab. 3, $T_{max} = 1$ N, $I_{sp} = 3100$ s, and an initial spacecraft mass $m_0 = 300$ kg. The transfer sequence is performed in two steps: Seq.1 (shown in red) corresponds to the thrust protocol in Eq. 38 and Seq. 2 (shown in black) corresponds to Eq. 39. The final results of flight time and propellant are shown in Tab. 4 and compared to two other closed-loop solutions, which are more optimal in both time and propellant than the algorithm presented here¹. One of these solutions is Edelbaum's optimal analytical solution for circle to circle transfers [16], which is derived assuming a constant acceleration magnitude and a low eccentricity (e << 0.1) during the transfer. The other solution is Petropolous Q-Law [6], which is another closed-loop solution that uses Lyapunov stability theory based on Lagrange's planetary equation. The Q-Law, however, is singular at i = 0 and e = 0, and therefore, cannot fully target these values.



Figure 2: Orbit transfer from an initial orbit with $a_0 = 6,700$ km, $e_0 = 0.05$, and $i_0 = 28.4^{\circ}$, to a final orbit with $a^* = 42,100$ km, $e^* = 0.0$, and $i^* = 0.0^{\circ}$ using $T_{max} = 1 N$, $I_{sp} = 3100 s$, $m_0 = 300$ kg.

¹Results of these two methods are taken from an example in Ref. [15]

Table 3: Initial and final orbital parameters for the example in Fig. 2

	a (km)	e	<i>i</i> (°)	ω (°)	Ω (°)
Initial Orbit	6,700	0.05	28.40	0.00	0.00
Final Orbit	42,100	0.00	0.00	-	_

Table 4: Propellant used and flight time for the example in Fig. 2

Control Law	Proposed	Edelbaum [16]	Petropoulos [6]
Propellant (kg)	60.10	49.69	56.62
Flight Time (days)	21.43	17.48	19.92

Another example simulation is shown in Fig. 3 with initial and final parameters given in Tab. 5, $T_{max} = 10$ N, $I_{sp} = 3,100$ s, and an initial spacecraft mass $m_0 = 300$ kg. The position has been non-dimensionalized by $1DU = a^* = 40,000$ km. The transfer takes 2.35 days and the final mass is $m_f = 254.3$ kg.

Table 5: Initial and final orbital parameters for the example in Fig. 3

	a (km)	e	<i>i</i> (°)	ω (°)	Ω (°)
Initial Orbit	20,000	0.7	40.00	20.00	0.00
Final Orbit	40,000	0.0	0.00	—	—

5. Conclusion

In this paper we take on a new approach to the problem of orbit transfers by working in a transformed model to design Lyapunov-based control laws. The model we use is the KS regularization transformation of the two-body problem. One of the advantages of working in these transformed coordinates is that the solution to the unperturbed equations of motion is that of a simple linear harmonic oscillator, where the frequency of oscillation is a function of *a*, the semi-major axis of the orbit.

We design a closed-loop guidance scheme for orbit transfer from any initial elliptical orbit to any final circular-equatorial orbit utilizing a spacecraft with thrust-coast capabilities. The convergence to the desired formation is performed in two steps. The first step involves converging the spacecraft orbit to the desired semi-major axis a^* , even though no other parameters are the desired ones at this point, by using a Lyapunov analysis that gives rise to an asymptotically stabilizing control law. Once a^* has been reached to within a specified tolerance, the second step of the algorithm involves matching the other two desired orbital parameters (eccentricity $e^* = 0$ and inclination $i^* = 0$), using the same control law as before, but with an added on/off switching mechanism for coasting during certain intervals.

The algorithms designed is robust, computationally fast, and can be used for both low- and high-thrust problems, though fuel or time-optimality is not guaranteed. Several examples are given for various initial and final parameters as well as different engine capabilities.



Figure 3: Orbit transfer from an initial orbit with $a_0 = 20,000$ km, $e_0 = 0.7$, and $i_0 = 40.0^o$, to a final orbit with $a^* = 40,000$ km, $e^* = 0.0$, and $i^* = 0.0^o$ using $T_{max} = 10$ N, $I_{sp} = 3100$ s, $m_0 = 300$ kg.

The next major step is to expand this guidance schemes to do any general threedimensional orbit transfer. This is done by matching five orbital parameters: semimajor axis, eccentricity, inclination, argument of periapsis, and right ascension of the ascending node.

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