# Minimum-Time Trajectory Optimization of Multiple Revolution Low-Thrust Earth-Orbit Transfers

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#### Abstract

The problem of determining high-accuracy minimum-time Earth-orbit transfers using low-thrust propulsion is considered. The optimal orbital transfer problem is posed as a constrained nonlinear optimal control problem and is solved using a variable-order Legendre-Gauss-Radau quadrature orthogonal collocation method. Initial guesses for the optimal control problem are obtained by solving a sequence of modified optimal control problems where the final true longitude is constrained and the mean square difference between the specified terminal boundary conditions and the computed terminal conditions is minimized. It is found that solutions to the minimum-time low-thrust optimal control problem are only locally optimal in that the solution has essentially the same number of orbital revolutions as that of the initial guess. A search method is then devised that enables computation of solutions with an even lower cost where the final true longitude is constrained to be different from that obtained in the original locally optimal solution. A numerical optimization study is then performed to determine optimal trajectories and control inputs for a range of initial thrust accelerations and constant specific impulses. The key features of the solutions are then determined, and relationships are obtained between the optimal transfer time and the optimal final true longitude as a function of the initial thrust acceleration and specific impulse. Finally, a detailed post-optimality analysis is performed to verify the close proximity of the numerical solutions to the true optimal solution.

### Nomenclature

a	=	Semi-major Axis, m
e	=	Eccentricity
f	=	Second Modified Equinoctial Element
g	=	Third Modified Equinoctial Element
$g_e$	=	Sea Level Acceleration Due to Earth Gravity, $m/s^2$
${\cal H}$	=	Optimal Control Augmented Hamiltonian
h	=	Second Modified Equinoctial Element
i	=	Inclination, deg or rad
$(\mathbf{i}_r,\mathbf{i}_ heta,\mathbf{i}_h)$	=	Rotating Radial Coordinate System
$J_2$	=	Second Zonal Harmonic
$J_3$	=	Third Zonal Harmonic
$J_4$	=	Fourth Zonal Harmonic
k	=	Fifth Modified Equinoctial Element

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L	=	Sixth Modified Equinoctial Element (True Longitude), rad or deg
m	=	Mass, kg
n	=	Mean Motion
$P_k$	=	Legendre Polynomial of Degree $k$
p	=	First Modified Equinoctial Element (Semi-Parameter), m
$R_e$	=	Radius of the Earth, m
T	=	Thrust, N
t	=	Time, s or d
u	=	Control Direction
$u_r$	=	Radial Component of Control
$u_{ heta}$	=	Tangential Component of Control
$u_h$	=	Normal Component of Control
$\Delta$	=	Spacecraft Specific Force, $m \cdot s^{-2}$
$\lambda$	=	Optimal Control Costate
$\mu$	=	Optimal Control Path Constraint Lagrange Multiplier
$\mu_e$	=	Earth Gravitational Parameter, $m^3 \cdot s^{-2}$
ν	=	True Anomaly, deg or rad
Ω	=	Longitude of Ascending Node, deg or rad
$\omega$	=	Argument of Periapsis, deg or rad

### 1 Introduction

Low-thrust propulsion systems are typically characterized by high specific impulses and small initial thrust accelerations (thrust-to-initial-mass) on the order of  $\mathcal{O}(10^{-4})$  m·s<sup>-2</sup>. The use of low-thrust propulsion has been studied extensively for orbital rendezvous, orbit maintenance, orbit transfer, and interplanetary space mission applications. While the efficiency of low-thrust propulsion is highly appealing, the resulting trajectory design problem is particularly challenging to solve. For example, the high specific impulse of a low-thrust engine combined with the small engine specific force leads to computational challenges due to the long duration of the orbital transfer. In addition, the trajectory design problem is particularly problematic when the initial and terminal orbits are widely spaced resulting in a trajectory that requires a large number of orbital revolutions in order to complete the transfer. Then, even if a solution is obtained, it is highly likely that the trajectory is not the global optimal solution.<sup>1</sup>

Low-thrust trajectory optimization has been the subject of much previous research. In Refs. 1–8 numerical optimization techniques were used for to optimize interplanetary space trajectories. In Ref. 9 a variation of parameters approach was employed to solve a minimum-fuel time-fixed rendezvous problem, while in Ref. 10 Pontryagin's minimum principle<sup>11,12</sup> was used to determine the optimal thrust acceleration for an orbit maintenance study. In Ref. 13 optimal control theory was used to solve minimum-time, circleto-circle, constant thrust orbit raising and simple graphical and analytical tools were used that related vehicle design parameters to orbit design parameters. In addition, a variety of approximation methods have been developed to overcome the computational challenge associated with the large number of orbital revolutions typical of a low-thrust orbital transfer. One of the most common approximation techniques is orbital averaging where simple approximations are derived to express incremental changes in the orbital elements for each orbital revolution. Using orbital averaging, in Ref. 14 the problem of minimum-fuel powerlimited transfers between coplanar elliptic orbits was studied, while in Ref. 15 near-optimal, minimum-time low-Earth orbit (LEO) to geostationary orbit (GEO) and geosynchronous transfer orbit (GTO) to GEO transfers were examined. In addition, in Ref. 16 a parameterized control law was employed together with orbital averaging in order to solve three common near-optimal, minimum-time Earth-orbit transfers. More recently, in Ref. 5 an orbital averaging approach was developed in conjunction with hybrid control formulations to solve LEO to GEO and GTO to GEO transfers. Next, in Ref. 17 a 100-revolution LEO to GEO coplanar transfer was solved using direct collocation paired with a Runge-Kutta parallel-shooting scheme, while in Ref. 18 a single shooting method was combined with a homotopic approach to solve a minimum-fuel transfer from a low, elliptic, and inclined orbit to GEO. In order to increase accuracy over orbital averaging techniques, in Ref. 19 sequential quadratic programming (SQP) was used with direct collocation to solve a minimum-fuel low-thrust near-polar Earth-orbit transfer with over 578 revolutions. In Ref. 20 an anti-aliasing method utilizing direction collocation was developed to obtain solutions to simple low-thrust trajectory optimization problems. Finally, in Ref. 21 a minimum-time LEO to high-Earth orbit (HEO) transfers was solved using direct collocation with a single specific impulse value.

While a great deal of progress has been made in low-thrust trajectory optimization, much of this work focuses on determining near-optimal solutions and very little work has been done to verify the optimality of the solutions obtained. The contribution of this research is on determining high-accuracy solutions to minimum-time low-thrust trajectory optimization problems for a wide-range of initial thrust accelerations and specific impulse values. Specifically, in this paper a variable-order Gaussian quadrature orthogonal collocation method<sup>22–29</sup> is used to determine minimum-time optimal trajectories of two common low-thrust Earth-orbit transfers. As a result, solutions to the optimal control problem are obtained without having to replace the equations of motion with averaged approximations over each orbital revolution such as in using an orbital averaging technique. Using the aforementioned collocation method, an initial guess generation method is used together with a simple search method to determine the solution that has the lowest cost amongst a range of locally optimal solutions. Numerical solutions are generated for a range of initial thrust acceleration values and specific impulse values that are typical of a low-thrust propulsion system. Then, in a manner similar to that of Ref. 21, regression analyses are performed to determine the transfer time as a function of the initial thrust acceleration and the specific impulse and to determine the final true longitude as a function of the transfer time. From these regressions it is possible to estimate the transfer time and final true longitude for different initial thrust accelerations and specific impulses without having to re-solve the optimal control problem. A post-optimality analysis is then performed to verify the optimality of the solutions obtained in this study.

This paper is organized as follows. Section 2 describes the minimum-time low-thrust orbit transfer trajectory optimization problem solved in this research. Section 3 describes the direct collocation method used to solve the optimal orbital transfer problem. Section 4 describes the numerical results obtained in this study and includes a post-optimality analysis to verify the optimality of the solutions obtained. Finally, Section 5 provides conclusions on this work.

# 2 Low-Thrust Earth-Orbit Transfer Optimal Control Problem

Consider the problem of transferring a spacecraft from an initial Earth-orbit to a final Earth-orbit using low-thrust propulsion. The objective is to determine the minimum-time trajectory and control that transfer the spacecraft from the specified initial orbit to the specified terminal orbit. The low-thrust optimal control problem for the orbit transfer is now described.

### 2.1 Equations of Motion

The dynamics of the spacecraft, modeled as a point mass, are described using modified equinoctial elements together with a fourth-order oblate gravity model and a continuous thrust propulsion system. The state of the spacecraft is comprised of the modified equinoctial elements  $(p, f, g, h, k, L)^{30}$  together with the mass, m, where p is the semi-parameter, f and g are modified equinoctial elements that describe the eccentricity of the orbit, h and k are modified equinoctial elements that describe the inclination of the orbit, and L is the true longitude. The control is the thrust direction,  $\mathbf{u}$ , where  $\mathbf{u}$  is expressed in rotating radial

coordinates as  $\mathbf{u} = (u_r, u_{\theta}, u_h)$ . The differential equations of motion of the spacecraft are given as

$$\frac{dp}{dt} = \sqrt{\frac{p}{\mu_e}} \frac{2p}{q} \Delta_{\theta} \equiv F_p,$$

$$\frac{df}{dt} = \sqrt{\frac{p}{\mu_e}} \left( \sin L \,\Delta_r + \frac{1}{q} \left( (q+1) \cos L + f \right) \,\Delta_{\theta} - \frac{q}{q} \left( h \sin L - k \cos L \right) \,\Delta_h \right) \equiv F_f,$$

$$\frac{dg}{dt} = \sqrt{\frac{p}{\mu_e}} \left( -\cos L \,\Delta_r + \frac{1}{q} \left( (q+1) \sin L + g \right) \,\Delta_\theta + \frac{f}{q} \left( h \sin L - k \cos L \right) \,\Delta_h \right) \equiv F_g,$$

$$\frac{dh}{dt} = \sqrt{\frac{p}{\mu_e}} \frac{s^2 \cos L}{2q} \,\Delta_h \equiv F_h,$$

$$\frac{dL}{dt} = \sqrt{\frac{p}{\mu_e}} \left( h \sin L - k \cos L \right) \,\Delta_h + \sqrt{\mu_e p} \left( \frac{q}{p} \right)^2 \equiv F_L,$$

$$\frac{dm}{dt} = -\frac{T}{g_e I_{sp}} \equiv F_m,$$
(1)

where

$$q = 1 + f \cos L + g \sin L, \qquad r = p/q, \alpha^2 = h^2 - k^2, \qquad s^2 = 1 + \sqrt{h^2 + k^2}.$$
(2)

In this research, time is replaced as the independent variable in favor of the true longitude, L, because L provides a more intuitive understanding of the transfer. Since the spacecraft moves from an orbit close to the Earth to GEO, more true longitude cycles will be completed in a given amount of time near the start of the transfer than will be completed near the terminus of the transfer. Using the true longitude as the independent variable, the differential equation for L is replaced with the differential equation

$$\frac{dt}{dL} = \frac{1}{F_L} = F_L^{-1} \equiv G_t,\tag{3}$$

while the remaining six differential equations for (p, f, g, h, k, m) that describe the dynamics of the spacecraft are given as

$$\frac{dp}{dL} = F_L^{-1}F_p \equiv G_p,$$

$$\frac{df}{dL} = F_L^{-1}F_f \equiv G_h,$$

$$\frac{dg}{dL} = F_L^{-1}F_g \equiv G_g,$$

$$\frac{dh}{dL} = F_L^{-1}F_h \equiv G_h,$$

$$\frac{dk}{dL} = F_L^{-1}F_k \equiv G_k,$$

$$\frac{dm}{dL} = F_L^{-1}F_m \equiv G_m.$$
(4)

Next, the spacecraft acceleration,  $\boldsymbol{\Delta} = (\Delta_r, \Delta_\theta, \Delta_h)$ , is modeled as

$$\boldsymbol{\Delta} = \boldsymbol{\Delta}_g + \boldsymbol{\Delta}_T,\tag{5}$$

where  $\Delta_g$  is the gravitational acceleration due to the oblateness of the Earth and  $\Delta_T$  is the thrust specific force. The acceleration due to Earth oblateness is expressed in rotating radial coordinates as

$$\mathbf{\Delta}_g = \mathbf{Q}_r^\mathsf{T} \delta \mathbf{g},\tag{6}$$

where  $\mathbf{Q}_r = \begin{bmatrix} \mathbf{i}_r & \mathbf{i}_\theta & \mathbf{i}_h \end{bmatrix}$  is the transformation from rotating radial coordinates to Earth-centered inertial coordinates and whose columns are defined as

$$\mathbf{i}_r = \frac{\mathbf{r}}{\|\mathbf{r}\|}, \quad \mathbf{i}_h = \frac{\mathbf{r} \times \mathbf{v}}{\|\mathbf{r} \times \mathbf{v}\|}, \quad \mathbf{i}_\theta = \mathbf{i}_h \times \mathbf{i}_r.$$
 (7)

Furthermore, the vector  $\delta \mathbf{g}$  is defined as

$$\delta \mathbf{g} = \delta g_n \mathbf{i}_n - \delta g_r \mathbf{i}_r \tag{8}$$

where  $\mathbf{i}_n$  is the local North direction and is defined as

$$\mathbf{i}_n = \frac{\mathbf{e}_n - (\mathbf{e}_n^\mathsf{T} \mathbf{i}_r) \mathbf{i}_r}{||\mathbf{e}_n - (\mathbf{e}_n^\mathsf{T} \mathbf{i}_r) \mathbf{i}_r||} \tag{9}$$

and  $\mathbf{e}_n = (0, 0, 1)$ . The oblate earth perturbations are then expressed as

$$\delta g_r = -\frac{\mu_e}{r^2} \sum_{k=2}^4 (k+1) \left(\frac{R_e}{r}\right)^k P_k(\sin\phi) J_k,$$
(10)

$$\delta g_n = -\frac{\mu_e \cos \phi}{r^2} \sum_{k=2}^4 \left(\frac{R_e}{r}\right)^k P'_k(\sin \phi) J_k, \tag{11}$$

where  $R_e$  is the equatorial radius of the earth,  $P_k$  is the  $k^{th}$ -degree Legendre polynomial,  $P'_k$  is the derivative of  $P_k$  with respect to  $\sin \phi$ , and  $J_k$  represents the zonal harmonic coefficients for k = (2, 3, 4). Next, the thrust specific force is given as

$$\mathbf{\Delta}_T = \frac{T}{m} \mathbf{u}.\tag{12}$$

Finally, the physical constants used in this study are given in Table 2.

Table 2: Physical Constants.

Quantity	Value
$g_e$	$9.80665 \text{ m} \cdot \text{s}^{-2}$
$\mu_e$	$3.9860047 \times 10^{14} \text{ m}^3 \cdot \text{s}^{-2}$
$R_e$	6378140 m
$J_2$	$1082.639 \times 10^{-6}$
$J_3$	$-2.565  imes 10^{-6}$
$J_4$	$-1.608 \times 10^{-6}$

### 2.2 Boundary Conditions and Path Constraints

The boundary conditions for the orbit transfer are described in terms of both classical orbital elements and modified equinoctial elements. The spacecraft starts in either a near circular inclined low-Earth orbit (LEO) or a geostationary transfer orbit (GTO) at time  $t_0 = 0$ . The initial orbit is specified in terms of classical orbital elements<sup>31</sup> as

$$\begin{aligned}
a(L_0) &= a_0, & \Omega(L_0) &= \Omega_0, \\
e(L_0) &= e_0, & \omega(L_0) &= \omega_0, \\
i(L_0) &= i_0, & \nu(L_0) &= \nu_0,
\end{aligned} \tag{13}$$

where a is the semi-major axis, e is the eccentricity, i is the inclination,  $\Omega$  is the longitude of the ascending node,  $\omega$  is the argument of periapsis, and  $\nu$  is the true anomaly. Equation (13) can be expressed equivalently in terms of the modified equinoctial elements as

$$p(L_0) = a_0(1 - e_0^2), \qquad h(L_0) = \tan(i_0/2)\sin\Omega_0,$$
  

$$f(L_0) = e_0\cos(\omega_0 + \Omega_0), \qquad k(L_0) = \tan(i_0/2)\cos\Omega_0, \qquad (14)$$
  

$$g(L_0) = e_0\sin(\omega_0 + \Omega_0), \qquad L_0 = \Omega_0 + \omega_0 + \nu_0.$$

For both cases considered, the spacecraft terminates in a geostationary orbit (GEO). The GEO terminal orbit is specified in classical orbital elements as

$$a(L_f) = a_f, \qquad \Omega(L_f) = \text{Free}, e(L_f) = e_f, \qquad \omega(L_f) = \text{Free}, i(L_f) = i_f, \qquad \nu(L_f) = \text{Free}.$$
(15)

Equation (15) can be expressed equivalently in terms of the modified equinoctial elements as

$$p(L_f) = a_f (1 - e_f^2),$$

$$\left(f^2(L_f) + g^2(L_f)\right)^{1/2} = e_f,$$

$$\left(h^2(L_f) + k^2(L_f)\right)^{1/2} = \tan(i_f/2).$$
(16)

Finally, during the transfer the thrust direction must be a vector of unit length. Thus, equality path constraint

$$\|\mathbf{u}\|_2 = u_r^2 + u_\theta^2 + u_h^2 = 1 \tag{17}$$

is enforced throughout the orbital transfer.

### 2.3 Optimal Control Problem

The goal of this study is to determine solutions to the following constrained nonlinear optimal control problem. Determine the trajectory (p(L), f(L), g(L), h(L), k(L), m(L), t(L)) and the control  $(u_r(L), u_{\theta}(L), u_h(L))$  that minimize the cost functional

$$J = \alpha t_f, \tag{18}$$

subject to the dynamic constraints of Eqs. (3) and (4), the initial conditions of Eq. (14), the terminal conditions of Eq. (16), and the path constraints of Eq. (17). Finally, it is noted that  $\alpha = 1/86400$  is the conversion factor from units of seconds to units of days.

### 3 Numerical Solution of Low-Thrust Optimal Control Problem

The minimum-time low-thrust optimal control problem described in this paper was solved using the optimal control software  $\mathbb{GPOPS} - \mathbb{II}$ .<sup>29</sup>  $\mathbb{GPOPS} - \mathbb{II}$  is a MATLAB software that transcribes the optimal control problem to a nonlinear programming problem (NLP) that implements the variable-order Legendre-Gauss-Radau quadrature collocation method described in Refs. 26,27, and 32 together with an hp adaptive mesh refinement method (see Refs. 28, 33, and 34). In this study the NLP arising from the LGR collocation method is solved using the open-source NLP solver  $IPOPT^{35}$  with analytical first and second derivatives obtained using the open-source algorithmic differentiation package ADiGator whose method is described in Ref. 36. The remainder of this section is organized as follows. First, an approach is described for generating initial guesses for solving the optimal control problem. Second, because the solutions obtained from the NLP are only locally optimal, a motivation is provided for developing a simple search method to obtain solutions that are closer to the global optimal. Finally, the developed simple search method is described and is applied to the low-thrust trajectory optimization problem.

### 3.1 Initial Guess Generation

In order to solve the low-thrust orbital transfer optimal control problem described in Section 2, it was necessary to provide initial guesses from which the NLP solver would converge to a solution. Because this research is focused on solving a problem whose solution will result in a large number of orbital revolutions, the initial guess must itself contain a number of orbital revolutions that is reasonably close to the actual number of orbital revolutions of the solution obtained by the NLP solver. In this paper, an initial guess procedure was devised where a sequence of optimal control sub-problems were solved. The goal of each sub-problem was to determine the state and control that transfer the spacecraft from the initial orbit to the terminal conditions that minimize the following mean square relative difference:

$$J = \left[\frac{p(L_f) - p_d}{1 + p_d}\right]^2 + \left[\frac{f^2(L_f) + g^2(L_f) - e_d^2}{1 + e_d^2}\right]^2 + \left[\frac{h^2(L_f) + k^2(L_f) - \tan^2(\frac{i_d}{2})}{1 + \tan^2(\frac{i_d}{2})}\right]^2.$$
 (19)

In other words, the objective of the optimal control sub-problem is to attain a solution that is as close in proximity to the desired terminal semi-parameter,  $p_d$ , eccentricity,  $e_d$ , and inclination,  $i_d$ . Each sub-problem is evaluated at most over one true longitude cycle using the terminal state of the previous sub-problem as the initial state of the current sub-problem. The continuous-time optimal control sub-problem is then stated as follows. Minimize the cost functional of Eq. (19) subject to the dynamic constraints of Eqs. (3) and (4), the path constraint of Eq. (17), and the boundary conditions

$$p^{(r)}(L_0^{(r)}) = p^{(r-1)}(L_f^{(r-1)}), \qquad p^{(r)}(L_f^{(r)}) = \text{Free},$$

$$f^{(r)}(L_0^{(r)}) = f^{(r-1)}(L_f^{(r-1)}), \qquad f^{(r)}(L_f^{(r)}) = \text{Free},$$

$$g^{(r)}(L_0^{(r)}) = g^{(r-1)}(L_f^{(r-1)}), \qquad g^{(r)}(L_f^{(r)}) = \text{Free},$$

$$h^{(r)}(L_0^{(r)}) = h^{(r-1)}(L_f^{(r-1)}), \qquad h^{(r)}(L_f^{(r)}) = \text{Free},$$

$$m^{(r)}(L_0^{(r)}) = m^{(r-1)}(L_f^{(r-1)}), \qquad h^{(r)}(L_f^{(r)}) = \text{Free},$$

$$m^{(r)}(L_0^{(r)}) = m^{(r-1)}(L_f^{(r-1)}), \qquad m^{(r)}(L_f^{(r)}) = \text{Free},$$

$$L_0^{(r)} = L_f^{(r-1)}, \qquad L_f^{(r)} \leq L_f^{(r-1)} + 2\pi,$$

$$(20)$$

for r = 1, ..., R where R represents the total number of true longitude cycles. The initial conditions for the first cycle, when r = 1, are simply the initial conditions stated in Eq. (14). Once the desired terminal conditions, as stated in Eq. (16), are obtained within a user specified tolerance, the sub-problem solutions are then combined to form the initial guess. The initial mesh is comprised of intervals based on the total number of true longitude cycles and an arbitrarily chosen number of collocation points assigned to each interval.

#### 3.2 Search Method to Assist in Obtaining Globally Optimal Solution

It is generally the case that gradient-based optimization methods converge to locally optimal solutions as opposed to globally optimal solutions. In the case of the low-thrust orbital transfer problems solved in this research, it was found that the optimal solution typically contained the same number of true longitude cycles as that of the initial guess due to the fact that the initial guess was very close to satisfying the terminal constraints. Thus, while the NLP solver converges with the initial guess provided, the solution is usually not the global optimal. In order to obtain a solution with a number of true longitude cycles different from that of the initial guess, the final true longitude was bounded within a specified cycle (that is, to lie within a specified interval of  $2\pi$ ). By bounding the final true longitude, the NLP solver is forced to deviate from the initial guess and potentially move closer to a global solution. Figure 1 shows the cost obtained when the final true longitude is bounded as described above and when the final true longitude is free for the GTO to GEO orbit transfer with  $(T/m_0, I_{sp}) = (4.00 \times 10^{-4} \text{ m} \cdot \text{s}^{-2}, 3000 \text{ s})$ . It is seen from

Fig. 1 that the solution obtained using the provided initial guess with a free terminal true longitude does not have the lowest cost solution. Instead, the lowest cost lies somewhere between 80 and 90 true longitude cycles.

Based on the structure shown in Fig. 1, the following simple search method is used to identify the approximate location of the globally optimal solution. First, an initial guess is generated as described in Section 3.1 and the final true longitude that is obtained from this initial guess, denoted  $L_f^{(0)}$ , is the starting point for the search method. An iteration on the final true longitude is then performed as follows, where K is the iteration number and K = 0 corresponds to  $L_f^{(0)}$ . The optimal control problem is solved for a final true longitude  $L_f \in [L_f^{(K)} - 2\pi, L_f^{(K)}] = \mathcal{I}_l^{(K)}$  and the cost obtained from this solution is denoted  $J_l^{(K)}$ . Next, in order to determine which direction to search for a lower cost solution, the optimal control problem is solved again for a final true longitude  $L_f \in [L_f^{(K)}, L_f^{(K)} + 2\pi] = \mathcal{I}_r^{(K)}$  and the cost associated with this solution is denoted  $J_r^{(K)}$ . The cycle that contains the lowest cost is then obtained using the following iterative process:

- Set  $K \to K + 1$ .
- Case 1 (minimum lies to the right of the initialization): If  $J_l^{(K-1)} > J_r^{(K-1)}$ , then set  $L_f^{(K)} = L_f^{(K-1)} + 2\pi$ ,  $\mathcal{I}_l^{(K)} = \mathcal{I}_r^{(K-1)}$ ,  $J_l^{(K)} = J_r^{(K-1)}$ , and  $\mathcal{I}_r^{(K)} = [L_f^{(K)}, L_f^{(K)} + 2\pi]$ . Then solve the optimal control problem again for  $L_f \in \mathcal{I}_r^{(K)}$  and the cost obtained is denoted  $J_r^{(K)}$ . Repeat until  $J_l^{(K)} < J_r^{(K)}$ . The lowest cost is  $J_l^{(K)}$  with  $L_f \in \mathcal{I}_l^{(K)}$ .
- Case 2 (minimum lies to the left of the initialization): If  $J_l^{(K-1)} < J_r^{(K-1)}$ , then set  $L_f^{(K)} = L_f^{(K-1)} 2\pi$ ,  $\mathcal{I}_r^{(K)} = \mathcal{I}_l^{(K-1)}$ ,  $J_r^{(K)} = J_l^{(K-1)}$ , and  $\mathcal{I}_l^{(K)} = [L_f^{(K)} 2\pi, L_f^{(K)}]$ . Then solve the optimal control problem again for  $L_f \in \mathcal{I}_l^{(K)}$  and the cost obtained is denoted  $J_l^{(K)}$ . Repeat until  $J_r^{(K)} < J_l^{(K)}$ . The lowest cost is  $J_r^{(K)}$  with  $L_f \in \mathcal{I}_r^{(K)}$ .

### 4 Results and Discussion

The GTO to GEO Earth-orbit transfer problem was solved using the following initial orbit:

$$\begin{aligned}
a(L_0) &= 24443 \text{ km}, & \Omega(L_0) &= 0 \text{ deg}, \\
e(L_0) &= 0.725, & \omega(L_0) &= 0 \text{ deg}, \\
i(L_0) &= 7 \text{ deg}, & \nu(L_0) &= 0 \text{ deg},
\end{aligned} \tag{21}$$

while the LEO to GEO Earth-orbit transfer problem was solved using the following initial orbit:

$$a(L_0) = 6656 \text{ km}, \qquad \Omega(L_0) = 0 \text{ deg},$$
  

$$e(L_0) = 0.001, \qquad \omega(L_0) = 0 \text{ deg},$$
  

$$i(L_0) = 28.5 \text{ deg}, \qquad \nu(L_0) = 0 \text{ deg}.$$
(22)

Both the GTO to GEO and LEO to GEO Earth-orbit transfer problems were solved using the following terminal orbit:

$$a(L_f) = 42164 \text{ km}, \qquad \Omega(L_f) = \text{Free},$$
  

$$e(L_f) = 0, \qquad \qquad \omega(L_f), = \text{Free},$$
  

$$i(L_f) = 0 \text{ deg}, \qquad \nu(L_f) = \text{Free}.$$
(23)

The minimum-time GTO to GEO and LEO to GEO transfers were solved with initial thrust acceleration values of  $T/m_0 = (2.000, 1.000, 0.667, 0.500, 0.400, 0.333, 0.286, 0.250, 0.222, 0.200) \times 10^{-3} \text{ m} \cdot \text{s}^{-2}$  and specific impulse values of  $I_{sp} = (500, 1000, 3000, 5000)$  s.



Figure 1: Locally optimal solutions obtained with free and bounded final true longitude vs.  $L_f/(2\pi)$  for GTO to GEO transfer with  $(T/m_0, I_{sp}) = (4.00 \times 10^{-4} \text{ m} \cdot \text{s}^{-2}, 3000 \text{ s}).$ 

The results of this study are divided into six sections. First, the key features of the GTO to GEO transfer are described and analyzed. Second, the key features of the LEO to GEO transfer are examined. Third, the relationship between the transfer time, initial thrust acceleration, and specific impulse is identified through regression techniques. Fourth, regression techniques are utilized further to identify the relationship between the final true longitude, transfer time, and specific impulse. Fifth, the coefficients of determination for the previously defined relationships are shown to validate the fit of the regressions. Finally, a post-optimality analysis of the solutions is provided to verify the optimality of the solutions obtained.

### 4.1 Key Features of Optimal GTO to GEO Transfers

Figure 2 shows a view in Earth-centered inertial Cartesian coordinates (x, y, z) of a typical optimal GTO to GEO trajectory for the case  $(T/m_0, I_{sp}) = (3.33 \times 10^{-4} \text{ m} \cdot \text{s}^{-2}, 3000 \text{ s})$ . The optimal solution has a 7.78% change in mass, a duration of 74 days, and nearly 103 orbital revolutions. It is seen from Fig. 3a that the semi-major axis increases nearly linearly throughout the entire transfer, and this rate of increase varies only slightly as a function of  $T/m_0$ . Furthermore, Fig. 3b shows that the eccentricity decreases slowly near the start of the transfer and decreases more rapidly starting from approximately one third of the way into the transfer and beyond. Also, Fig. 3c shows that the inclination decreases at an approximately linear rate throughout the entire transfer.



Figure 2: GTO to GEO transfer trajectory for  $(T/m_0, I_{sp}) = (3.33 \times 10^{-4} \text{ m} \cdot \text{s}^{-2}, 3000 \text{ s}).$ 



Figure 3: a, e, and i vs.  $\nu/(2\pi)$  for GTO to GEO transfer with various values of  $T/m_0$  and  $I_{sp} = 3000$  s.

Further insight into the behavior of the optimal GTO to GEO transfers is obtained by examining the control  $\mathbf{u} = (u_r, u_\theta, u_h)$  along different segments of the optimal solution. The typical overall behavior of  $\mathbf{u}$  is shown in Fig. 4 for  $(T/m_0, I_{sp}) = (3.33 \times 10^{-4} \text{ m} \cdot \text{s}^{-2}, 3000)$  s. A closer examination of  $\mathbf{u}$  reveals that the following four segments identify the key features of the optimal control: (1) the first few orbital revolutions of the transfer, (2) the region where  $u_r \approx 0$ , (3) the region where  $u_h$  becomes -1, and (4) the final revolutions of the transfer. First, the effect of the control on the optimal trajectory in the first few orbital revolutions, can be explained via the following differential equations for the semi-major axis, eccentricity, and inclination:<sup>37</sup>

$$\frac{da}{dt} = \frac{2e\sin\nu}{nx}u_r + \frac{2ax}{nr}u_\theta, \qquad (24)$$

$$\frac{de}{dt} = \frac{x\sin\nu}{na}u_r + \frac{x}{na^2e}\left(\frac{a^2x^2}{r} - r\right)u_\theta,\tag{25}$$

$$\frac{di}{dt} = \frac{r\cos(\nu+\omega)}{na^2x}u_h,\tag{26}$$

where n is the mean motion and  $x = \sqrt{1 - e^2}$ . It is seen from Eq. (24) that the semi-major axis will increase when the control points either in the positive  $u_{\theta}$ -direction, radially outward near  $\nu = \pi/2$  (halfway between periapsis and apoapsis), or radially inward near  $\nu = 3\pi/2$  (halfway between apoapsis and periapsis). Furthermore, this cyclic behavior of  $u_r$  increases apoapsis and decreases periapsis when  $\nu \in [0, \pi]$  and decreases apoapsis and increases periapsis when  $\nu \in [\pi, 2\pi]$ . Equivalently, thrusting radially in this manner increases both the semi-major axis and the eccentricity. On the other hand, from Eq. (25) the eccentricity will decrease when the control points either in the positive  $u_{\theta}$ -direction, radially inward near  $\nu = \pi/2$ , or radially outward near  $\nu = 3\pi/2$ . It is seen from Fig. 5a that  $u_r$  is positive near  $\nu = \pi/2$  and negative near  $\nu = 3\pi/2$ , while  $u_{\theta} \approx 1$  in both cases. Even though  $u_r$  increases the semi-major axis, it simultaneously *increases* eccentricity. This small effect of  $u_r$  increasing eccentricity, however, is negated by the fact that **u** lies predominantly in the positive  $u_{\theta}$ -direction, thereby resulting in an overall increase in semi-major axis and decrease in eccentricity. Lastly, it is seen from Eq. (26) that di/dt is most negative when  $\cos(\nu + \omega) u_h$ is most negative. Examining Fig. 6a, it is seen that  $u_h$  is most positive and  $\cos(\nu + \omega) = -1$  when the spacecraft is at apoapsis, thereby resulting in the largest negative slope in di/dt as seen in Fig. 6b.

Next, Fig. 5b shows **u** in the segment of an optimal GTO to GEO transfer where  $u_r \approx 0$ . For every orbital revolution on the optimal solution beyond where  $u_r$  becomes zero (that is, all values beyond  $\nu/(2\pi) = 38.5$  as shown in Fig. 5b), **u** points radially inward near  $\nu = \pi/2$  such that apoapsis decreases and periapsis increases when  $\nu \in [0, \pi]$  and points radially outward near  $\nu = 3\pi/2$  such that apoapsis increases and periapsis decreases when  $\nu \in [\pi, 2\pi]$ . Thrusting radially in this manner decreases both the semi-major axis and eccentricity. Although  $u_r$  decreases the semi-major axis, the thrust direction lies predominantly in the positive  $u_{\theta}$ -direction, thereby increasing the semi-major axis and decreasing the eccentricity. Finally, because  $u_h$  is most positive near  $\nu = \pi$  and  $\cos(\nu + \omega) = -1$ , the inclination decreases most rapidly near apoapsis.

Next, Fig. 5c shows **u** in the segment of an optimal GTO to GEO transfer where  $u_h$  drops to -1. It is seen in this segment that  $u_r$  continues to point inward near  $\nu = \pi/2$  and outward near  $\nu = 3\pi/2$ , decreasing both the semi-major axis and the eccentricity. Moreover,  $u_{\theta}$  no longer dominates the thrust direction, reaching its most positive value at apoapsis and a gradually decreasing value at periapsis. Thrusting in this manner in the  $u_r$ - and  $u_{\theta}$ -directions raises periapsis when  $\nu \in [0, \pi]$  and lowers apoapsis when  $\nu \in [\pi, 2\pi]$ . Consequently, the semi-major axis increases while eccentricity decreases. Lastly, from Fig. 6c,  $u_h$  attains its most negative value near  $\nu = \pi$  and  $\cos(\nu + \omega) = -1$  and near  $\nu = 2\pi$  and  $\cos(\nu + \omega) = +1$ . While the inclination continues to decrease significantly near apoapsis of the transfer, Fig. 6d shows that di/dt is also negative near periapsis.

Finally, Fig. 5d shows **u** during the final few revolutions of an optimal GTO to GEO transfer. It is seen that  $u_r$  is negative near  $\nu = \pi/2$  and positive near  $\nu = 3\pi/2$ , while  $u_{\theta}$  is positive near  $\nu = 2\pi$  and is negative near  $\nu = \pi$ . Consequently, by thrusting in this manner periapsis increases and apoapsis decreases,

thereby resulting in a larger semi-major axis and a smaller eccentricity. Finally,  $u_h$  is negative near  $\nu = 2\pi$  and is positive near  $\nu = \pi$ . Because the orbit is nearly circular near the end of the transfer, the rate at which inclination decreases is essentially the same near periapsis and apoapsis.



Figure 4: **u** vs.  $\nu/(2\pi)$  for GTO to GEO transfer with  $(T/m_0, I_{sp}) = (3.33 \times 10^{-4} \text{ m} \cdot \text{s}^{-2}, 3000 \text{ s}).$ 



Figure 5: **u** vs.  $\nu/(2\pi)$  for GTO to GEO transfer with  $(T/m_0, I_{sp}) = (3.33 \times 10^{-4} \text{ m} \cdot \text{s}^{-2}, 3000 \text{ s}).$ 



Figure 6:  $u_h$ ,  $\cos(\nu + \omega)$ , and *i* vs.  $\nu/(2\pi)$ ) for GTO to GEO transfer with  $(T/m_0, I_{sp}) = (3.33 \times 10^{-4} \text{ m} \cdot \text{s}^{-2}, 3000 \text{ s}).$ 

### 4.2 Key Features of Optimal LEO to GEO Transfers

Figure 7 shows a three-dimensional view in Earth-centered inertial Cartesian coordinates (x, y, z) of a typical optimal LEO to GEO trajectory for the case  $(T/m_0, I_{sp}) = (3.33 \times 10^{-4} \text{ m} \cdot \text{s}^{-2}, 1000 \text{ s})$ . The optimal trajectory has a 44.73% change in mass, takes approximately 152 days, and contains nearly 1,023 revolutions. The semi-major axis and inclination for all values of  $T/m_0$  and  $I_{sp} = 1000$  s are shown in Figs. 8a and 8b, respectively. It is seen that the semi-major axis increases at a slower rate at the beginning of the transfer and increases more rapidly towards the end of the transfer. The inclination decreases at a slower rate at the beginning of the transfer and decreases more rapidly towards the end of the transfer. The inclination decreases at a slower rate at the beginning of the transfer and decreases more rapidly towards the end of the transfer. The inclination decreases at a slower rate at the beginning of the transfer and decreases more rapidly towards the end of the transfer. The inclination decreases at a slower rate at the beginning of the transfer and decreases more rapidly towards the end of the transfer. The inclination decreases at a slower rate at the beginning of the transfer and decreases more rapidly towards the end of the transfer.



Figure 7: LEO to GEO transfer trajectory with  $(T/m_0, I_{sp}) = (3.33 \times 10^{-4} \text{ m} \cdot \text{s}^{-2}, 1000 \text{ s}).$ 



(b) *i* vs.  $L/(2\pi)$ .

Figure 8: a and i vs.  $L/(2\pi)$  for LEO to GEO transfer using various values of  $T/m_0$  and  $I_{sp} = 3000$  s.

The structure of the optimal LEO to GEO transfers is examined in greater detail by studying the components of the control along the optimal solution. The typical overall behavior of the control  $\mathbf{u} = (u_r, u_\theta, u_h)$  is shown in Fig. 9 for  $(T/m_0, I_{sp}) = (3.33 \times 10^{-4} \text{m} \cdot \text{s}^{-2}, 1000 \text{ s})$ . As expected,  $u_r$  stays near zero, whereas  $u_\theta$  and  $u_h$  are non-zero throughout the entire transfer. Greater insight into the structure of the optimal control is now obtained by examining the control near the start and the terminus of the transfer. Figure 10b shows  $\mathbf{u}$  as a function of  $\nu$  near the start of the transfer (where  $\nu = L - \Omega - \omega \approx L$  because L,  $\omega$ , and  $\Omega$ , are approximately zero). It is seen from Fig. 10b that  $u_\theta$  points in the positive  $u_\theta$ -direction to increase the semi-major axis (see Eq. (24)), while  $u_h$  attains its most positive value near apoapsis and its most negative value near periapsis to decrease the inclination (see Eq. (26)). Near the terminus of the transfer,  $\nu = L - \Omega - \omega \approx L + 3\pi/2$  since  $\Omega$  approaches an approximate value of  $-3\pi/2$  (see Fig. 10a) and  $\omega$  is assumed to be zero. Fig. 10c shows  $\mathbf{u}$  as a function of  $\nu$  near the end of the transfer where it is seen that  $u_\theta$  points in the positive  $u_\theta$ -direction and  $u_h$  attains its most positive value near apoapsis and its most negative value near periapsis. Thus it is clear throughout the LEO to GEO transfer that the optimal thrust direction increases the semi-major axis and decreases the inclination while the eccentricity remains relatively unchanged near zero.



Figure 9: **u** vs.  $L/(2\pi)$  for LEO to GEO transfer with various values of  $T/m_0$  and  $I_{sp} = 1000$  s.



Figure 10:  $\Omega$  vs.  $L/2\pi$  and **u** vs.  $\nu/(2\pi)$  for LEO to GEO transfer with  $(T/m_0, I_{sp}) = (3.33 \times 10^{-4} \text{ m} \cdot \text{s}^{-2}, 1000 \text{ s}).$ 

### 4.3 Estimation of Minimum-Time Transfer Time

A key feature of the results is the ability to estimate the optimal transfer time as a function of initial thrust acceleration and specific impulse. Figure 11 shows the final time of the orbit transfer as a function of the specific impulse for each of the initial thrust acceleration values examined. For each value of  $T/m_0$ ,  $t_f$  increases slightly as  $I_{sp}$  increases in a manner similar to that of a power function

$$t_f = AI_{sp}^B + C, (27)$$

where the coefficients A, B, and C are functions of  $T/m_0$  because each value of  $T/m_0$  has an associated power function expression for  $t_f$  in terms of  $I_{sp}$ . The coefficients A, B, and C are determined as follows. Figures 12a and 12b show the coefficient A as a function of  $T/m_0$  for the GTO to GEO and LEO to GEO transfers, respectively. It is seen that the relationship between A and  $T/m_0$  has the form

$$A = a_1 (T/m_0)^{b_1}, (28)$$

where  $a_1$  and  $b_1$  are constant coefficients. Figures 12c and 12d show the coefficient B as a function of  $T/m_0$ for the GTO to GEO and LEO to GEO transfers, respectively. Because B has no significant change as a function of  $T/m_0$ , it is assumed that B is constant, and for any particular orbital transfer this constant is the average value of B over all values of  $T/m_0$  and  $I_{sp}$  for that transfer. Figures 12e and 12f show the coefficient C as a function of  $T/m_0$  for the GTO to GEO and LEO to GEO transfers, respectively. It is seen that the relationship between C and  $T/m_0$  is given as

$$C = a_2 (T/m_0)^{b_2}, (29)$$

where  $a_2$  and  $b_2$  are constants. Therefore, the estimated transfer time,  $\hat{t}_f$ , can be written as a function of both  $I_{sp}$  and  $T/m_0$  and is given as

$$\hat{t}_f = a_1 (T/m_0)^{b_1} (I_{sp})^B + a_2 (T/m_0)^{b_2}.$$
(30)

Values for the coefficients  $a_1$ ,  $b_1$ , B,  $a_2$ , and  $b_2$  are shown in Table 3. Equation 30 makes it possible to estimate the final transfer time,  $t_f$ , for values of  $I_{sp}$  that are different from those obtained in this study without having to re-solve the optimal control problem.



Figure 11:  $t_f$  vs.  $I_{sp}$ .

Table 3: Regression coefficients for  $\hat{t}_f$ .

Transfer	$a_1$	$b_1$	В	$a_2$	$b_2$
GTO to GEO	-1.3849	-0.9766	-0.8745	0.0253	-1.0018
LEO to GEO	-2.0361	-1.0005	-0.6757	0.0701	-1.0000

### 4.4 Estimation of Minimum-Time Final True Longitude

Another key feature of the results is the ability to approximate  $L(t_f)$  as a function of the final transfer time and specific impulse. Figure 13a shows  $L_f$  as a function of  $t_f$ . It is seen for each value of  $I_{sp}$  that  $L(t_f)$  increases linearly as a function of  $t_f$ , that is,

$$L_f = Dt_f + E,\tag{31}$$

where D and E are functions of  $I_{sp}$ . Expressions for coefficients D and E are determined as follows. Figures 14a and 14b show the coefficient D as a function of  $I_{sp}$  for the GTO to GEO and LEO to GEO transfers, respectively. It is seen that as  $I_{sp}$  increases, D decreases in a manner similar to that of an exponential function

$$D = a_3 e^{b_3 I_{sp}} + c_3, (32)$$

where  $a_3$ ,  $b_3$ , and  $c_3$  are constant coefficients. Figures 14c and 14d show the coefficient E as a function of  $I_{sp}$  for the GTO to GEO and LEO to GEO transfers, respectively. Is is seen that as  $I_{sp}$  increases, E is small. For example, E is no more than half a true longitude cycle for both the GTO to GEO and LEO to GEO transfers. Consequently, E is treated as a constant and for any orbital transfer, this constant is the average value of E over all values of  $T/m_0$  and  $I_{sp}$  for that transfer. Using the derived expressions for the coefficients D and E,  $L_f$  can be written as a function of both  $I_{sp}$  and  $t_f$  and is given by

$$\hat{L}_f = (a_3 e^{b_3 I_{sp}} + c_3) t_f + E \tag{33}$$

where  $a_3$ ,  $b_3$ ,  $c_3$ , and E are the regression constants and  $L_f$  denotes the estimate of  $L_f$ . Values for the regression constants are shown in Table 4. Together with Eq. 30, Eq. 33 makes it possible to quickly estimate  $L_f$  at points for which the results were not obtained.

Table 4: Regression coefficients for  $L_f$ .

Transfer	$a_3$	$b_3$	$c_3$	E
GTO to GEO	0.0949	-0.0020	1.4006	0.3110
LEO to GEO	3.0633	-0.0021	6.3653	-0.1633

#### 4.5 Coefficient of Determination to Assess Quality of Regressions

To demonstrate the quality with which the regression models for  $\hat{t}_f$  and  $\hat{L}_f$  fit the observed data, the coefficients of determination,  $R^2$ , are calculated.<sup>38</sup> The coefficient of determination is a value between 0 and 1, where a value of unity indicates a perfect fit. The coefficient of determination for  $\hat{t}_f$  is calculated as follows. Let  $t_{fi}$  and  $\hat{t}_{fi}$  be the observed and predicted values, respectively, of the  $i^{th}$  value of  $t_f$ . Then, the sum of the squares of the error, denoted S, is calculated as

$$S = \sum_{i=1}^{n} \left( t_{fi} - \hat{t}_{fi} \right)^2.$$
(34)

Next, let  $S_{yy}$  denote the total sum of squares

$$S_{yy} = \sum_{i=1}^{n} \left( t_{fi} - \bar{t}_f \right)^2,$$
(35)

where  $\bar{t}_f$  is the mean of the observed values of  $t_f$ . Finally, the coefficient of determination is calculated using

$$R^2 = 1 - \frac{S}{S_{yy}}.$$
 (36)

The coefficient of determination for  $\hat{L}_f$  is calculated in a similar manner to  $\hat{t}_f$ . All coefficients of determination are shown in Table 5, where it is seen that  $R^2$  is close to unity in all cases.

Table 5: Coefficient of determination  $R^2$  for  $\hat{t}_f$  and  $\hat{L}_f$ .

Transfer	$\hat{t}_f$	$\hat{L}_f$
GTO to GEO	0.999950	0.999898
LEO to GEO	0.999984	0.999981



Figure 12: Regression coefficients A, B, and C vs.  $T/m_0$ .



Figure 13:  $L_f/(2\pi)$  vs.  $t_f$ .



Figure 14: Regression coefficients D and E vs.  $I_{sp}.$ 

### 4.6 Post-Optimality Analysis

It is known from previous research (see Refs. 26 and 27) that the first-order optimality conditions of the nonlinear programming problem arising from discretization of a continuous optimal control problem via the LGR collocation method are a discrete approximation of the first-order calculus of variations optimality conditions of the optimal control problem. Moreover, the costate of the optimal control problem can be obtained via a simple linear transformation of the Lagrange multipliers of the NLP arising from the LGR collocation. In addition to the equivalence between the NLP and calculus of variations optimality conditions, it has also been proven that the solution obtained using the variable-order (hp) LGR collocation method converges exponentially (that is, the state, control, and costate associated with the LGR collocation method all converge) at the convergence rate given in Ref. 39. Consequently, by solving the NLP arising from the LGR collocation method on an appropriate mesh, an accurate approximation to the solution of the optimal control problem is obtained to both the primal variables (that is, the state and control) and the dual variable (that is, the costate). Therefore by solving the variable-order LGR NLP on a sufficiently accurate mesh, it is possible to verify the extremality of the solutions without having to solve the Hamiltonian boundary-value problem that arises from the calculus of variations. In other words, by obtaining the solution to the variable-order LGR NLP on an appropriate mesh the optimality of the solution can be verified without having to resort to solving the optimal control problem using an indirect method.

In this study the proximity of the numerical solutions to the true optimal solutions is investigated by examining various aspects of the first-order calculus of variations conditions. In this analysis the firstorder variational conditions are presented in terms of the classical orbital elements (as opposed to the modified equinoctial elements which were used to solve the optimal control problem), where the firstorder optimality conditions are obtained in terms of the classical orbital elements as follows. First, the discrete approximation of the costate in terms of the modified equinoctial elements are obtained using the transformation of the NLP Lagrange multipliers as described in Refs. 26 and 27 (where it is noted that  $\mathbb{GPOPS}$  – II performs this costate computation after the NLP is solved). Next, the costate approximation in terms of the modified equinoctial elements obtained from the LGR collocation method is transformed to the costate in terms of classical orbital elements using the relationship between the modified equinoctial element costate and the classical orbital element costate as derived in the Appendix. Then, using the fact that the costate is the sensitivity of the cost with respect to the state along the optimal solution, the costate in terms of classical orbital elements at the initial time is also approximated by solving the optimal control problem at a perturbed initial orbital element and taking the ratio of the change in cost to the change in the orbital element of interest (for example, if it is interested in computing the costate associated with the eccentricity, then the ratio of the change in the cost to a perturbation in the eccentricity at the initial point is computed).

The costates associated with the classical orbital elements of interest were verified by resolving the problem with a small perturbation in the initial semi-major axis, initial eccentricity, and initial inclination. For a perturbation in the initial semi-major axis, the change in cost from the optimal cost is approximated as  $\begin{bmatrix} x & y & y \\ y & y \end{bmatrix}$ 

$$J^{\delta} \approx J^* + \left[\frac{\partial J}{\partial a(L_0)}\right]_* \left(a^{\delta}(L_0) - a^*(L_0)\right)$$
(37)

where  $J^{\delta}$  and  $J^*$  denote the cost on the perturbed and optimal solutions, respectively. Therefore, the estimated semi-major axis costate at  $L = L_0$  is approximated by

$$\left[\frac{\partial J}{\partial a(L_0)}\right]_* \approx \frac{J^{\delta} - J^*}{a^{\delta}(L_0) - a^*(L_0)} = \frac{\Delta J}{\Delta a}$$
(38)

which is then compared to the derived costate value,  $\lambda_a^*(L_0)$ . For a perturbation in the initial eccentricity or initial inclination, the estimated costate value is calculated in a similar manner (that is, replace the semi-major axis, *a*, with either the eccentricity, *e*, or the inclination, *i*, in Eqs. (37) and (38)). In this analysis, the perturbations in the initial semi-major axis, initial eccentricity, and initial inclination were  $\Delta a = 1000 \text{ m}, \Delta e = 0.0001 \text{ and } \Delta i = 0.00017453 \text{ rad} (= 0.01 \text{ deg}), \text{ respectively.}$  Tables 6a and 6b show the costate approximations  $(\lambda_a^*(L_0), \lambda_e^*(L_0), \lambda_i^*(L_0))$  alongside the ratios of the cost to the perturbations,  $(\Delta J/\Delta a, \Delta J/\Delta e, \Delta J/\Delta i)$  in the orbital elements for the GTO to GEO case with  $T/m_0 = 2.22 \times 10^{-4} \text{ m} \cdot s^{-2}$  and for the LEO to GEO case with  $T/m_0 = 4.00 \times 10^{-4} \text{ m} \cdot s^{-2}$ . For both cases the LGR costate approximations closely match the estimated change in cost due to a perturbation in the classical orbital element of interest, and the costate approximations are consistent with the expected behavior (for example, increasing the initial semi-major axis for either orbit transfer decreases the cost, while increasing the initial eccentricity increases the cost). Moreover, it is seen that perturbing the initial eccentricity significantly increases the cost for the GTO to GEO case but increases the cost. Finally, in all cases the magnitude of cost sensitivity increases as the specific impulse increases. This last result is consistent with the fact that the efficiency of the engine increases as the specific impulse increases.

As a further verification of the close proximity of the numerical solutions to the true optimal solution, the final column of Tables 6a and 6b show the maximum absolute value of  $\mathcal{H}_{\mathbf{u}} \equiv \partial \mathcal{H} / \partial \mathbf{u} = (\mathcal{H}_{u_r}, \mathcal{H}_{u_{\theta}}, \mathcal{H}_{u_h})$ on  $L \in [L_0, L_f]$ , that is, Tables 6a and 6b show

$$\max_{L \in [L_0, L_f]} (|\mathcal{H}_{u_r}|, |\mathcal{H}_{u_\theta}|, |\mathcal{H}_{u_h}|),$$

where  $\mathcal{H}$  is computed as given in the Appendix using the costate approximation obtained from the LGR collocation method as described in Ref. 27. Because the control lies on the interior of the allowable control set for the problem studied in this paper, it is known theoretically that  $\mathcal{H}_{\mathbf{u}}$  is zero along the optimal solution. Commensurate with this known value of  $\mathcal{H}_{\mathbf{u}}$ , Tables 6a and 6b show that  $\mathcal{H}_{\mathbf{u}}$  is extremely small, further substantiating the close proximity of the numerical solution to the true optimal solution.

Table 6: Post-optimality results for GTO to GEO and LEO to GEO transfers.

(a) GTO to GEO p	post-optimality	results for	$T/m_0 = 2.50$	$0 \times 10^{-4}$	$m \cdot s^{-2}$ .
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Isp	$\lambda_a^*(t_0)$	$\Delta J/\Delta a$	$\lambda_e^*(t_0)$	$\Delta J/\Delta e$	$\lambda_i^*(t_0)$	$\Delta J/\Delta i$	$\max_{L \in [L_0, L_f]} ( \mathcal{H}_{u_r} ,  \mathcal{H}_{u_{\theta}} ,  \mathcal{H}_{u_h} )$
500	$-1.84 \times 10^{-6}$	$-1.84 \times 10^{-6}$	77.78	77.81	17.45	17.47	$2.65 \times 10^{-9}$
1000	$-2.31 \times 10^{-6}$	$-2.30 \times 10^{-6}$	97.58	97.70	21.92	21.98	$3.33 \times 10^{-10}$
3000	$-2.39 \times 10^{-6}$	$-2.39 \times 10^{-6}$	108.74	108.69	24.45	24.47	$2.20 \times 10^{-10}$
5000	$-2.75 \times 10^{-6}$	$-2.75 \times 10^{-6}$	116.72	116.78	26.24	26.27	$2.22 \times 10^{-10}$

(b)	LEO to	GEO	post-optimality	results for	$T/m_0 = 4$	$1.00 \times 10^{-4}$	$m \cdot s^{-2}$ .
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Isp	$\lambda_a^*(t_0)$	$\Delta J/\Delta a$	$\lambda_e^*(t_0)$	$\Delta J/\Delta e$	$\lambda_i^*(t_0)$	$\Delta J/\Delta i$	$\max_{L \in [L_0, L_f]} ( \mathcal{H}_{u_r} ,  \mathcal{H}_{u_\theta} ,  \mathcal{H}_{u_h} )$
500	$-4.73 \times 10^{-6}$	$-4.72 \times 10^{-6}$	0.14	0.14	36.75	36.75	$1.53 \times 10^{-10}$
1000	$-8.56 \times 10^{-6}$	$-8.53 \times 10^{-6}$	0.40	0.40	66.57	66.58	$1.67 \times 10^{-10}$
3000	$-1.30 \times 10^{-5}$	$-1.29 \times 10^{-5}$	0.59	0.59	99.20	99.22	$1.42 \times 10^{-10}$
5000	$-1.38 \times 10^{-5}$	$-1.37 \times 10^{-5}$	0.63	0.57	107.06	107.04	$2.08 \times 10^{-10}$

# 5 Conclusions

The problem of high-accuracy low-thrust minimum-time Earth-orbit transfers has been studied. The optimal orbital transfer problem is posed as a constrained nonlinear optimal control problem. It is solved using a variable-order Legendre-Gauss-Radau (LGR) quadrature orthogonal collocation method paired with a search method that helps the NLP solver determine the best locally optimal solution. A numerical optimization study has been conducted to determine optimal trajectories and controls for a range of initial thrust accelerations and constant specific impulses. The key features of the solutions have been identified and relationships have been obtained that relate the optimal transfer time to the optimal number of true longitude cycles as a function of the initial thrust acceleration and specific impulse. Finally, a post-optimality analysis has been performed that verifies the optimality of the solutions that were obtained in this study.

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### References

- Dachwald, B., "Optimization of Very-Low-Thrust Trajectories Using Evolutionary Neurocontrol," Acta Astronautica, Vol. 57, No. 2-8, 2005, pp. 175–185.
- [2] Tang, S. and Conway, B. A., "Optimization of Low-Thrust Interplanetary Trajectories Using Collocation and Nonlinear Programming," *Journal of Guidance, Control, and Dynamics*, Vol. 18, No. 3, May-June 1995, pp. 599–604.
- [3] Rauwolf, G. A. and Coverstone-Carroll, V. L., "Near-Optimal Low-Thrust Orbit Transfers Generated by a Genetic Algorithm," *Journal of Spacecraft and Rockets*, Vol. 33, No. 6, November-December 1996, pp. 859–862.
- [4] Coverstone-Carroll, V., Hartmann, J. W., and Mason, W. J., "Optimal Multi-Objective Low-Thrust Spacecraft Trajectories," *Computer Methods in Applied Mechanics and Engineering*, Vol. 186, No. 2-4, June 2000, pp. 387–402.
- [5] Falck, R. D. and Dankanich, J. D., "Optimization of Low-Thrust Spiral Trajectories by Collocation," AIAA/AAS Astrodynamics Specialist Conference, No. 2012-4423, Minneapolis, MN, August 2012.
- [6] Petropoulos, A. and Longuski, J. M., "Shaped-Based Algorithm for the Automated Design of Low-Thrust, Gravity Assist Trajectories," *Journal of Spacecraft and Rockets*, Vol. 41, No. 5, September-October 2004, pp. 787–796.
- [7] Starek, J. A. and Kolmanovsky, I. V., "Nonlinear model predictive control strategy for low thrust spacecraft missions," *Optimal Control Applications and Methods*, Vol. 35, No. 1, January-February 2014, pp. 1–20.
- [8] Bartholomew-Biggs, M. C., Dixon, L. C. W., Hersom, S. E., and Maany, Z. A., "The solution of some difficult problems in low-thrust interplanetary trajectory optimization," *Optimal Control Applications* and Methods, Vol. 9, No. 3, July-September 1988, pp. 229–251.
- Kechichian, J. A., "Optimal Low-Thrust Transfers Using Variable Bounded Thrust," Acta Astronautica, Vol. 36, No. 7, 1995, pp. 357–365.

- [10] da Silva Fernandes, S., "Optimum Low-Thrust Limited Power Transfers Between Neighbouring Elliptic Non-Equatorial Orbits in a Non-Central Gravity Field," Acta Astronautica, Vol. 35, No. 12, 1995, pp. 763–770.
- [11] Pontryagin, L. S., Boltyanskii, V. G., Gamkrelidze, R., and Mishchenko, E. F., The Mathematical Theory of Optimal Processes (Russian), English Translation: Interscience, 1962.
- [12] Athans, M. A. and Falb, P. L., Optimal Control: An Introduction to the Theory and Its Applications, Dover Publications, Mineola, New York, 2006.
- [13] Alfano, S. and Thorne, J. D., "Circle-to-Circle, Constant-Thrust Orbit Raising," The Journal of Astronautical Sciences, Vol. 42, No. 1, 1994, pp. 35–45.
- [14] Haissig, C. M., Mease, K. D., and Vinh, N. X., "Minimum-Fuel, Power-Limited Transfers Between Coplanar Elliptic Orbits," Acta Astronautica, Vol. 29, No. 1, January 1993, pp. 1–15.
- [15] Kluever, C. A. and Oleson, S. R., "Direct Approach for Computing Near-Optimal Low-Thrust Earth-Orbit Transfers," *Journal of Spacecraft and Rockets*, Vol. 35, No. 4, July-August 1998, pp. 509–515.
- [16] Yang, G., "Direct Optimization of Low-thrust Many-revolution Earth-orbit Transfers," Chinese Journal of Aeronautics, Vol. 22, No. 4, August 2009, pp. 426–433.
- [17] Scheel, W. A. and Conway, B. A., "Optimization of Very-Low-Thrust, Many-Revolution Spacecraft Trajectories," *Journal of Guidance, Control, and Dynamics*, Vol. 17, No. 6, November-December 1994, pp. 1185–1192.
- [18] Haberkorn, T., Martinon, P., and Gergaud, J., "Low-Thrust Minimum-Fuel Orbital Transfer: A Homotopic Approach," Journal of Guidance, Control, and Dynamics, Vol. 27, No. 6, 2004.
- [19] Betts, J. T., "Very Low-Thrust Trajectory Optimization Using a Direct SQP Method," Journal of Computational and Applied Mathematics, Vol. 120, No. 1-2, August 2000, pp. 27–40.
- [20] Ross, I. M., Gong, Q., and Sekhavat, P., "Low-Thrust, High-Accuracy Trajectory Optimization," Journal of Guidance, Control, and Dynamics, Vol. 30, No. 4, July-August 2007, pp. 921–933.
- [21] Schubert, K. F. and Rao, A. V., "Minimum-Time Low-Earth Orbit to High-Earth Orbit Low-Thrust Trajectory Optimization," AAS/AIAA Astrodynamics Specialist Conference, No. 12-926, Hilton Head, SC, August 2013.
- [22] Benson, D. A., Huntington, G. T., Thorvaldsen, T. P., and Rao, A. V., "Direct Trajectory Optimization and Costate Estimation via an Orthogonal Collocation Method," *Journal of Guidance, Control,* and Dynamics, Vol. 29, No. 6, November-December 2006, pp. 1435–1440.
- [23] Huntington, G. T., Benson, D. A., and Rao, A. V., "Optimal Configuration of Tetrahedral Spacecraft Formations," *Journal of the Astronautical Sciences*, Vol. 55, No. 2, April-June 2007, pp. 141–169.
- [24] Huntington, G. T. and Rao, A. V., "Optimal Reconfiguration of Tetrahedral Spacecraft Formations Using the Gauss Pseudospectral Method," *Journal of Guidance, Control, and Dynamics*, Vol. 31, No. 3, May-June 2008, pp. 141–169.
- [25] Rao, A. V., Benson, D. A., Darby, C. L., Francolin, C., Patterson, M. A., Sanders, I., and Huntington, G. T., "Algorithm 902: GPOPS, A Matlab Software for Solving Multiple-Phase Optimal Control Problems Using the Gauss Pseudospectral Method," ACM Transactions on Mathematical Software, Vol. 37, No. 2, April–June 2010, Article 22, 39 pages.

- [26] Garg, D., Patterson, M. A., Hager, W. W., Rao, A. V., Benson, D. A., and Huntington, G. T., "A Unified Framework for the Numerical Solution of Optimal Control Problems Using Pseudospectral Methods," *Automatica*, Vol. 46, No. 11, November 2010, pp. 1843–1851.
- [27] Garg, D., Patterson, M. A., Darby, C. L., Francolin, C., Huntington, G. T., Hager, W. W., and Rao, A. V., "Direct Trajectory Optimization and Costate Estimation of Finite-Horizon and Infinite-Horizon Optimal Control Problems via a Radau Pseudospectral Method," *Computational Optimization and Applications*, Vol. 49, No. 2, June 2011, pp. 335–358.
- [28] Patterson, M. A., Hager, W. W., and Rao, A. V., "A ph Mesh Refinement Method for Optimal Control," Optimal Control Applications and Methods, January 2014, Online in Early View, DOI: 10.1002/oca2114.
- [29] Patterson, M. A. and Rao, A. V., "GPOPS-II: A MATLAB Software for Solving Multiple-Phase Optimal Control Problems Using hp-Adaptive Gaussian Quadrature Collocation Methods and Sparse Nonlinear Programming," ACM Transactions on Mathematical Software, Vol. 41, No. 1, October 2014, pp. 1:1–1:37.
- [30] Walker, M. J. H., Owens, J., and Ireland, B., "A Set of Modified Equinoctial Orbit Elements," *Celestial Mechanics*, Vol. 36, No. 4, 1985, pp. 409–419.
- [31] Prussing, J. E. and Conway, B. A., Orbital Mechanics, Oxford University Press, 2nd ed., 2013.
- [32] Patterson, M. A. and Rao, A. V., "Exploiting Sparsity in Direct Collocation Pseudospectral Methods for Solving Optimal Control Problems," *Journal of Spacecraft and Rockets*, Vol. 49, No. 2, March–April 2012, pp. 364–377.
- [33] Darby, C. L., Hager, W. W., and Rao, A. V., "Direct Trajectory Optimization Using a Variable Low-Order Adaptive Pseudospectral Method," *Journal of Spacecraft and Rockets*, Vol. 48, No. 3, May-June 2011, pp. 433–445.
- [34] Darby, C. L., Hager, W. W., and Rao, A. V., "An hp-adaptive pseudospectral method for solving optimal control problems," *Optimal Control Applications and Methods*, Vol. 32, August 2011, pp. 476– 502.
- [35] Biegler, L. T. and Zavala, V. M., "Large-Scale Nonlinear Programming Using IPOPT: An Integrating Framework for Enterprise-Wide Optimization," *Computers and Chemical Engineering*, Vol. 33, No. 3, March 2008, pp. 575–582.
- [36] Weinstein, M. J. and Rao, A. V., "A Source Transformation via Operator Overloading Method for the Automatic Differentiation of Mathematical Functions in MATLAB," ACM Transactions on Mathematical Software, September 2014, Accepted for Publication.
- [37] Chobotov, V. A., Orbital Mechanics, American Institute of Aeronautics and Astronautics, Inc., 1991.
- [38] Wackerly, D. D., Mendenhall, W., and Scheaffer, R. L., Mathematical Statistics with Applications, Thomson Brooks/Cole, 7th ed., 2008.
- [39] Hou, H., Convergence Analysis of Orthogonal Collocation Methods for Unconstrained Optimal Control, Ph.d. thesis, University of Florida, Gainesville, Florida, May 2013.

# 6 Appendix

In this Appendix we derive expressions for the components of the costate of the optimal control problem given Section 2 in terms of the components of the costate in terms of the modified equinoctial elements. First, the augmented Hamiltonian,  $\mathcal{H}$ , of the minimum-time optimal control problem described in Section 2 is given in terms of the differential equations in modified equinoctial elements as

$$\mathcal{H} = \lambda_p G_p + \lambda_f G_f + \lambda_g G_g + \lambda_h G_h + \lambda_k G_k + \lambda_m G_m + \lambda_t G_t - \mu (u_r^2 + u_\theta^2 + u_h^2 - 1), \tag{39}$$

where  $(\lambda_p, \lambda_f, \lambda_g, \lambda_h, \lambda_k, \lambda_m, \lambda_t)$  is the costate associated with the differential equations of Eqs. (3) and (4) and  $\mu$  is the Lagrange multiplier associated with the path constraint of Eq. (17). The Hamiltonian can be expressed equivalently in terms of the classical orbital elements as

$$\mathcal{H} = \lambda_a G_a + \lambda_e G_e + \lambda_i G_i + \lambda_\Omega G_\Omega + \lambda_\omega G_\omega + \lambda_m G_m + \lambda_t G_t, \tag{40}$$

where  $(G_a, G_e, G_i, G_\Omega, G_\omega)$  define the right-hand sides of those components of the equations of motion given in Eqs. (3) and (4) that correspond to the dynamics for the orbital elements  $a, e, i, \Omega, \omega$ , that is,

$$\frac{da}{dL} = G_a,$$

$$\frac{de}{dL} = G_e,$$

$$\frac{di}{dL} = G_i,$$

$$\frac{d\Omega}{dL} = G_\Omega,$$

$$\frac{d\omega}{dL} = G_\omega.$$
(41)

Because the components of the costate  $\lambda_m$  and  $\lambda_t$  are the same using either modified equinoctial elements or orbital elements and the control is the same in both formulations, the Hamiltonian given in either (39) or (39) can be replaced with the reduced Hamiltonian,

$$\mathcal{H}_{r} = \lambda_{p}G_{p} + \lambda_{f}G_{f} + \lambda_{g}G_{g} + \lambda_{h}G_{h} + \lambda_{k}G_{k},$$

$$= \lambda_{a}G_{a} + \lambda_{e}G_{e} + \lambda_{i}G_{i} + \lambda_{\Omega}G_{\Omega} + \lambda_{\omega}G_{\omega}.$$
(42)

Next, the relationship between the modified equinoctial elements and the classical orbital elements are given as

$$a = a(p, f, g) = \frac{p}{1 - f^2 - g^2}$$

$$e = e(f, g) = \sqrt{f^2 + g^2}$$

$$i = i(h, k) = \tan^{-1}\left(\frac{2\sqrt{h^2 + k^2}}{1 - k^2 - h^2}\right)$$

$$\Omega = \Omega(h, k) = \tan^{-1}\left(\frac{k}{h}\right)$$

$$\omega = \omega(f, g, h, k) = \tan^{-1}\left(\frac{gh - fk}{fh + gk}\right).$$
(43)

The expressions for da/dL, de/dL, di/dL,  $d\Omega/dL$ , and  $d\omega/dL$  are then given in terms of modified equinoctial elements as

$$\frac{da}{dL} = \frac{\partial a}{\partial p} \frac{dp}{dL} + \frac{\partial a}{\partial f} \frac{df}{dL} + \frac{\partial a}{\partial g} \frac{dg}{dL} = \frac{\partial a}{\partial p} G_p + \frac{\partial a}{\partial f} G_f + \frac{\partial a}{\partial g} G_g,$$

$$\frac{de}{dL} = \frac{\partial e}{\partial f} \frac{df}{dL} + \frac{\partial e}{\partial g} \frac{dg}{dL} = \frac{\partial e}{\partial f} G_f + \frac{\partial e}{\partial g} G_g,$$

$$\frac{di}{dL} = \frac{\partial i}{\partial h} \frac{dh}{dL} + \frac{\partial i}{\partial k} \frac{dk}{dL} = \frac{\partial i}{\partial h} G_h + \frac{\partial i}{\partial k} G_k,$$

$$\frac{d\Omega}{dL} = \frac{\partial \Omega}{\partial h} \frac{dh}{dL} + \frac{\partial \Omega}{\partial g} \frac{dg}{dL} = \frac{\partial \Omega}{\partial h} G_h + \frac{\partial \Omega}{\partial k} G_k,$$

$$\frac{d\omega}{dL} = \frac{\partial \omega}{\partial f} \frac{df}{dL} + \frac{\partial \omega}{\partial g} \frac{dg}{dL} + \frac{\partial \omega}{\partial h} \frac{dh}{dL} + \frac{\partial \omega}{\partial k} \frac{dk}{dL} = \frac{\partial \omega}{\partial f} G_f + \frac{\partial \omega}{\partial g} G_g + \frac{\partial \omega}{\partial h} G_h + \frac{\partial \omega}{\partial k} G_k.$$
(44)

Substituting (44) into (42), the reduced Hamiltonian can be expressed as

$$\mathcal{H}_{r} = \lambda_{a} \left[ \frac{\partial a}{\partial p} G_{p} + \frac{\partial a}{\partial f} G_{f} + \frac{\partial a}{\partial g} G_{g} \right] + \lambda_{e} \left[ \frac{\partial e}{\partial f} G_{f} + \frac{\partial e}{\partial g} G_{g} \right] + \lambda_{i} \left[ \frac{\partial \Omega}{\partial h} G_{h} + \frac{\partial \Omega}{\partial k} G_{k} \right]$$

$$+ \lambda_{\Omega} \left[ \frac{\partial \Omega}{\partial h} G_{h} + \frac{\partial \Omega}{\partial k} G_{k} \right] + \lambda_{\omega} \left[ \frac{\partial \omega}{\partial f} G_{f} + \frac{\partial \omega}{\partial g} G_{g} + \frac{\partial \omega}{\partial h} G_{h} + \frac{\partial \omega}{\partial k} G_{k} \right]$$

$$(45)$$

and rearranged to yield

$$\mathcal{H}_{r} = \left[\lambda_{a}\frac{\partial a}{\partial p}\right]G_{p} + \left[\lambda_{a}\frac{\partial a}{\partial f} + \lambda_{e}\frac{\partial e}{\partial f} + \lambda_{\omega}\frac{\partial \omega}{\partial f}\right]G_{f} + \left[\lambda_{a}\frac{\partial a}{\partial g} + \lambda_{e}\frac{\partial e}{\partial g} + \lambda_{\omega}\frac{\partial \omega}{\partial g}\right]G_{g} + \left[\lambda_{i}\frac{\partial i}{\partial h} + \lambda_{\Omega}\frac{\partial \Omega}{\partial h} + \lambda_{\omega}\frac{\partial \omega}{\partial h}\right]G_{h} + \left[\lambda_{i}\frac{\partial i}{\partial k} + \lambda_{\Omega}\frac{\partial \Omega}{\partial k} + \lambda_{\omega}\frac{\partial \omega}{\partial k}\right]G_{k}$$

$$= \lambda_{p}G_{p} + \lambda_{f}G_{f} + \lambda_{g}G_{g} + \lambda_{h}G_{h} + \lambda_{k}G_{k}.$$

$$(46)$$

Equating terms in (46) leads to the following system of five linear equations that relate  $(\lambda_a, \lambda_e, \lambda_i, \lambda_\Omega, \lambda_\omega)$ to  $(\lambda_p, \lambda_f, \lambda_g, \lambda_h, \lambda_k)$ :

$$\begin{bmatrix} \lambda_{p} \\ \lambda_{f} \\ \lambda_{g} \\ \lambda_{h} \\ \lambda_{k} \end{bmatrix} = \begin{bmatrix} \frac{\partial a}{\partial p} & 0 & 0 & 0 & 0 \\ \frac{\partial a}{\partial f} & \frac{\partial e}{\partial f} & 0 & 0 & \frac{\partial \omega}{\partial f} \\ \frac{\partial a}{\partial g} & \frac{\partial e}{\partial g} & 0 & 0 & \frac{\partial \omega}{\partial g} \\ 0 & 0 & \frac{\partial i}{\partial h} & \frac{\partial \Omega}{\partial h} & \frac{\partial \omega}{\partial h} \\ 0 & 0 & \frac{\partial i}{\partial k} & \frac{\partial \Omega}{\partial k} & \frac{\partial \omega}{\partial k} \end{bmatrix} \begin{bmatrix} \lambda_{a} \\ \lambda_{e} \\ \lambda_{i} \\ \lambda_{a} \\ \lambda_{0} \\ \lambda_{\omega} \end{bmatrix}$$
(47)

Assuming that the system matrix

$\boxed{\frac{\partial a}{\partial p}}$	0	0	0	0
$\left  \frac{\partial a}{\partial f} \right $	$rac{\partial e}{\partial f}$	0	0	$rac{\partial \omega}{\partial f}$
$\frac{\partial a}{\partial g}$	$rac{\partial e}{\partial g}$	0	0	$rac{\partial \omega}{\partial g}$
0	0	$rac{\partial i}{\partial h}$	$rac{\partial\Omega}{\partial h}$	$rac{\partial \omega}{\partial h}$
0	0	$rac{\partial i}{\partial k}$	$rac{\partial\Omega}{\partial k}$	$\frac{\partial \omega}{\partial k}$

is invertible, Eq. (47) can be solved to obtain  $(\lambda_a, \lambda_e, \lambda_i, \lambda_\Omega, \lambda_\omega)$ .