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Engineering Notes

Approximate Optimization of Low-Thrust Transfers Between Low-Eccentricity Close Orbits

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I. Introduction

EDELBAUM [1] analyzed in a famous paper the flight of a spacecraft, capable of small thrust and moving on a low-eccentricity orbit. Under these assumptions, the orbit remains almost circular, and the optimal control laws to maximize changes of the semimajor axis, eccentricity, and inclination are determined. Edelbaum's analysis evaluates the changes that can be obtained during one revolution and the corresponding costs and is extended to multiple-revolution transfers. In particular, Edelbaum considered large simultaneous changes of the semimajor axis and inclination and determined his famous equation to compute the ΔV for low-thrust transfers between circular orbits with an inclination change [1].

Edelbaum's work has inspired authors [2–6] who have reformulated and extended his analysis, in particular, considering the variable specific impulse, the presence of eclipses, or Earth-oblateness perturbations. Optimal control laws that maximize changes of the semimajor axis, eccentricity, or inclination can be blended to obtain a prescribed orbit transfer [7–10] when a large number of revolutions around the main body is performed, while employing averaging techniques to evaluate the changes of the orbital elements. When the mission length is short, methods based on Edelbaum's analysis become inaccurate, as it uses the average changes that can be obtained during one revolution. As a matter of fact, the rate of change of eccentricity and inclination depends on the spacecraft position along its orbit, and the corresponding changes are not linear functions of time spent and angle flown. In addition, simultaneous changes of both the eccentricity and semimajor axis may be required, and this case has not yet been addressed in the literature; the relation between changes of the eccentricity and semimajor axis has been investigated in [11–13], but only with the purpose of orbit circularization in the presence of eclipses. For short transfers, shape-based methods [14,15] are usually employed, but difficulties, related to the unspecified thrust magnitude required by this approach, may arise.

The present work is motivated by the requirement of obtaining a fast and accurate cost estimation for electric propulsion transfers between close orbits, i.e., when small changes of orbital elements are required and the mission length is short (e.g., missions to near-Earth asteroids and geocentric missions for propellant refurbishment and debris removal). In these cases, existing methods could be inaccurate, and a new approach is presented here. Transfers between low-eccentricity orbits with small changes of the orbital elements are considered. Inclinations of the initial and final orbits are made small by properly choosing the reference plane (e.g., the initial orbit plane), but results have a general validity, as far as the inclination change is small. A suboptimal control law is introduced in order to allow for analytic integration of the differential equations that describe the change of the orbital elements. The numerical solution of an algebraic system provides the control law that is required to obtain the prescribed orbit change and the corresponding cost, transfer time, and angular length. Results are compared to the exact numerical solution, which is obtained by means of an indirect optimization method.

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II. Low-Eccentricity Low-Inclination Orbits

Gauss's form of Lagrange's planetary equations [16,17] expresses the differential equations that describe the time derivatives of orbital elements as functions of orthogonal components of the perturbing acceleration, in this case, the thrust. A simplified form of the equations is obtained in case of low eccentricity, that is, almost circular orbits, and low inclination, as supposed by Edelbaum [1], and derivatives of the semimajor axis a, eccentricity e, inclination i, right ascension of the ascending node Ω, argument of the periapsis ω, and longitude ϑ become

$$V\frac{\mathrm{d}a}{\mathrm{d}t} = 2rA_T \tag{1}$$

$$V\frac{\mathrm{d}e}{\mathrm{d}t} = 2\cos\nu A_T + \sin\nu A_R \tag{2}$$

$$V\frac{\mathrm{d}i}{\mathrm{d}t} = \cos(\omega+\nu)A_W \tag{3}$$

$$iV\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \sin(\omega+\nu)A_W \tag{4}$$

$$V\frac{\mathrm{d}\omega}{\mathrm{d}t} = -V\frac{\mathrm{d}\Omega}{\mathrm{d}t} + (2\sin\nu A_T - \cos\nu A_R)/e \tag{5}$$

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}t} = \sqrt{\mu/a^3} = V/r \tag{6}$$

where thrust acceleration components in the radial A_R (outward), tangential A_T (concurrent with the spacecraft motion), and out-of plane A_W (concurrent with the angular momentum) directions are introduced; on almost circular orbits, radius is $r\approx a$, velocity is $V\approx\sqrt{\mu/r}$, and true anomaly ν is related to longitude as $\vartheta\approx\Omega+\omega+\nu$.

These equations are ill defined for small values of e and i, which are instead of interest in the present case. A new set of variables, similar to the equinoctial elements set [18] and more suited to deal with the problem considered here, is adopted, and $e_x = e\cos(\Omega+\omega)$, $e_y = e\sin(\Omega+\omega)$, $i_x = i\cos\Omega$, $i_y = i\sin\Omega$ are introduced; e_x and e_y are related to the change of the eccentricity magnitude and rotation of the line of apsides; i_x and i_y concern the change of the inclination magnitude and rotation of the line of nodes. Also, longitude ϑ is used as the independent variable, replacing time. Finally, angles between thrust projection on orbit plane and tangential direction (α) and thrust and orbit plane (β) are introduced, and the thrust (T) acceleration components are written as

$$A_R = (T/m)\sin\alpha\cos\beta \qquad A_T = (T/m)\cos\alpha\cos\beta$$
$$A_W = (T/m)\sin\beta \tag{7}$$

where m is the spacecraft mass. The set of differential equations becomes

$$\frac{\mathrm{d}a}{\mathrm{d}\vartheta} = 2r\frac{T/m}{\mu/r^2}\cos\alpha\cos\beta \tag{8}$$

$$\frac{\mathrm{d}e_x}{\mathrm{d}\vartheta} = (2\cos\vartheta\cos\alpha\cos\beta + \sin\vartheta\sin\alpha\cos\beta)\frac{T/m}{\mu/r^2} \tag{9}$$

$$\frac{\mathrm{d}e_y}{\mathrm{d}\vartheta} = (2\sin\vartheta\cos\alpha\cos\beta - \cos\vartheta\sin\alpha\cos\beta)\frac{T/m}{\mu/r^2} \tag{10}$$

$$\frac{\mathrm{d}i_x}{\mathrm{d}\vartheta} = \cos\vartheta\sin\beta\frac{T/m}{\mu/r^2} \tag{11}$$

$$\frac{\mathrm{d}i_y}{\mathrm{d}\vartheta} = \sin\vartheta\sin\beta\frac{T/m}{\mu/r^2} \tag{12}$$

$$\frac{\mathrm{d}t}{\mathrm{d}\vartheta} = \sqrt{r^3/\mu} \tag{13}$$

Radius and thrust acceleration are considered to be constant in the present analysis, with suitable average values for r and m; the angles α and β are the problem control variables. In addition to the removal of singularities, this set of variables has the additional advantage that differential equations do not depend on state variables but are only functions of the independent variable ϑ.

III. Edelbaum's Optimal Control Laws

The problem considered here concerns the minimum-time all-propulsive transfer between given orbits for fixed initial time (i.e., position on the starting orbit); prescribed changes of orbital elements (that is, a, e_x, e_y, i_x, and i_y) are therefore to be obtained. The time equation is actually neglected, and the equivalent minimization of the transfer angular length is sought, as the angular velocity is considered to be constant. A maximization problem is preferred, and $-\vartheta_f$ is maximized, given ϑ_0 (subscripts 0 and f indicate the starting and final points, respectively). The theory of optimal control [19,20] can be applied to determine the optimal control law. The Hamiltonian is defined by associating an adjoint variable λ to each differential equation,

$$\begin{aligned} H = A[&\lambda_a 2r\cos\alpha\cos\beta + \lambda_{ex}(2\cos\vartheta\cos\alpha\cos\beta \\ &+ \sin\vartheta\sin\alpha\cos\beta) + \lambda_{ey}(2\sin\vartheta\cos\alpha\cos\beta \\ &- \cos\vartheta\sin\alpha\cos\beta) + \lambda_{ix}\cos\vartheta\sin\beta + \lambda_{iy}\sin\vartheta\sin\beta] \end{aligned} \tag{14}$$

where the nondimensional acceleration $A = (T/m)/(\mu/r^2)$ has been introduced. Euler–Lagrange equations [19] state that the adjoint variables λ are actually adjoint constants since H does not depend on the state variables.

The optimal controls are obtained by nullifying the partial derivatives of H with respect to α and β. It is useful to introduce the quantities $\lambda_e = \pm\sqrt{\lambda_{ex}^2 + \lambda_{ey}^2}$ (the sign must be the same as λ_a) and $\vartheta_e = \tan^{-1}(\lambda_{ey}/\lambda_{ex})$ with the quadrant chosen properly in order to have $\lambda_{ex} = \lambda_e\cos\vartheta_e$ and $\lambda_{ey} = \lambda_e\sin\vartheta_e$; in a similar way, $\lambda_{ix} = \lambda_i\cos\vartheta_i$ and $\lambda_{iy} = \lambda_i\sin\vartheta_i$ are defined. The optimal controls maximize the Hamiltonian in agreement with Pontryagin's maximum principle, and α and β are determined from

$$\cos\alpha = \frac{2[\lambda_a r + \lambda_e\cos(\vartheta - \vartheta_e)]}{X_\alpha} \qquad \sin\alpha = \frac{\lambda_e\sin(\vartheta - \vartheta_e)}{X_\alpha} \tag{15}$$

$$\cos\beta = \frac{\sqrt{4[\lambda_a r + \lambda_e\cos(\vartheta - \vartheta_e)]^2 + [\lambda_e\sin(\vartheta - \vartheta_e)]^2}}{X_\beta}$$
$$\sin\beta = \frac{\lambda_i\cos(\vartheta - \vartheta_i)}{X_\beta} \tag{16}$$

with $X_\alpha^2 = 4[\lambda_a r + \lambda_e\cos(\vartheta - \vartheta_e)]^2 + [\lambda_e\sin(\vartheta - \vartheta_e)]^2$ and $X_\beta^2 = 4[\lambda_a r + \lambda_e\cos(\vartheta - \vartheta_e)]^2 + [\lambda_e\sin(\vartheta - \vartheta_e)]^2 + [\lambda_i\cos(\vartheta - \vartheta_i)]^2$.

The Hamiltonian is homogeneous with respect to the adjoint constants, which can be arbitrarily scaled by fixing the value of one of them, without changing the control law. The problem is therefore characterized by five unknowns (four adjoint constants and the final longitude ϑ_f), which must be chosen in order to obtain the prescribed changes of state variables a, e_x, e_y, i_x, and i_y.

Edelbaum [1] considered three special cases. The change of the semimajor axis a corresponds to $\lambda_{ex} = \lambda_{ey} = \lambda_{ix} = \lambda_{iy} = 0$, and $\beta = 0$, and $\alpha = 0$ (for positive λ_a and Δa) or $\alpha = \pi$ (when λ_a and Δa are negative) are obtained, that is, the thrust is tangential. The change of eccentricity occurs for $\lambda_a = \lambda_{ix} = \lambda_{iy} = 0$, and thrust angles are $\beta = 0$ and

$$\alpha = \tan^{-1}[(1/2)\tan(\vartheta - \vartheta_e)] \tag{17}$$

with $\tan\vartheta_e = \Delta e_y/\Delta e_x$. This optimal control law is very well approximated [1] by constant-direction thrust with $\alpha = \vartheta - \vartheta_e$. It should be noted that the thrust direction must be approximately perpendicular to the desired change of eccentricity vector.

Finally, the simultaneous change of a and i requires $\lambda_{ex} = \lambda_{ey} = 0$, with $\alpha = 0$ or π (depending on the sign of λ_a and Δa) and

$$\beta = \tan^{-1}\left[\frac{\lambda_i\cos(\vartheta - \vartheta_i)}{2\lambda_a r}\right] \tag{18}$$

In this case, only a small penalty [1] occurs if constant-β thrusting is used instead of the optimal control law; actually, β must change sign every half-revolution, according to the sign of $\cos(\vartheta - \vartheta_i)$. Edelbaum has evaluated the ΔV for each case [1],

$$\Delta V/V = 0.5\Delta a/r \qquad \Delta V/V \approx 0.649\Delta e \qquad \Delta V/V = (\pi/2)\Delta i \tag{19}$$

with the constant-β law used instead of the optimal one for the plane change.

These control laws produce the expected results only when an integer number of revolutions is performed; tangential thrust produces only changes of a with $\beta = 0$ and changes of a and i with the constant-β law, whereas thrusting according to Eq. (17) (or to the approximate law $\alpha = \vartheta - \vartheta_e$) with $\beta = 0$ only changes e_x and e_y. These considerations do not hold when a fraction of a revolution is performed, as tangential thrust also changes eccentricity, and the control law of Eq. (17) also changes a, depending on the instantaneous position along the orbit. Also, variables do not change uniformly along the orbit, whereas Eq. (19) are based on averages over one revolution.

In-plane thrusting modifies the semimajor axis and eccentricity. The optimal thrust direction (i.e., α) results to be a compromise between those required to change a only and e only; the latter, given by Eq. (15) with $\lambda_a = 0$, depends on ϑ_e, which is determined by Δe_x and Δe_y (it must be noted that vector Δe points toward ϑ_e when $\lambda_e > 0$ and toward the opposite direction otherwise). For $-\pi/2 < \vartheta - \vartheta_e < \pi/2$, the thrust to change eccentricity has a positive component along the spacecraft velocity when $\Delta a > 0$ (that is, positive λ_a and λ_e), and a negative one when $\Delta a < 0$; this part of the trajectory is therefore favorable for simultaneous changes of a and e. In contrast, the portion of the trajectory with $\pi/2 < \vartheta - \vartheta_e < 3\pi/2$ results to be unfavorable. As a general rule, when a is increased, thrusting is efficient for both changes of the semimajor axis and eccentricity when ϑ is close to the direction to which the vector Δe points and less useful on the other side. The opposite occurs when a must be reduced.

When a long mission with many revolutions is performed, such as the missions considered by Edelbaum, the contribution of the periodical variations over one revolution can be neglected, and Edelbaum's formula can be (and actually are) adopted with very good accuracy. Results may instead be misleading when short missions, as those treated in the present Note, are analyzed, depending on whether the spacecraft flies in favorable or unfavorable zones when it moves on the fraction of the revolution that exceeds the integer.

IV. Suboptimal Approximate Control Law

The problem considered in this Note concerns the transfer between two orbits (subscripts 1 and 2) starting from a given initial position ϑ_0; small changes of the semimajor axis and small values of eccentricity and inclination are considered. The following quantities are defined as

$$\Delta a = a_2 - a_1 \tag{20}$$

$$\Delta e = \sqrt{\Delta e_x^2 + \Delta e_y^2} = \sqrt{(e_{x2} - e_{x1})^2 + (e_{y2} - e_{y1})^2} \tag{21}$$

$$\Delta i = \sqrt{\Delta i_x^2 + \Delta i_y^2} = \sqrt{(i_{x2} - i_{x1})^2 + (i_{y2} - i_{y1})^2} \tag{22}$$

It should be noted that Δe and Δi do not represent changes of eccentricity and inclination magnitudes only but also take changes of ω and Ω into account.

Edelbaum did not treat the case of the simultaneous changes of a, e, and i, but a simple way of estimating the transfer cost can be obtained if the changes of a and e are considered to be independent and obtained with perpendicular acceleration components, just as in the case of a and i. With this assumption, the mission ΔV can be estimated as the vectorial sum of the ΔV for the basic changes Δa, Δe, and Δi. According to Eq. (19),

$$\Delta V = \sqrt{(k_a \Delta a)^2 + (k_e \Delta e)^2 + (k_i \Delta i)^2} \tag{23}$$

is assumed, where the cost coefficients [1]

$$k_a = \frac{V_{\text{avg}}}{2a_{\text{avg}}} \qquad k_e = 0.649 V_{\text{avg}} \qquad k_i = \frac{\pi}{2} V_{\text{avg}} \tag{24}$$

are evaluated using average values between the starting and final orbits:

$$a_{\text{avg}} = (a_1 + a_2)/2 \qquad V_{\text{avg}} = \sqrt{\mu/a_0} \tag{25}$$

This approximation implicitly assumes a linear change of the variables along the orbit and can be considered as a direct extension of Edelbaum's analysis.

For short transfers, however, results are affected by the mutual influence between a and e and by the nonlinear dependence of variable changes on ϑ. In the following, a new approach to deal with these issues will be described. The general optimal control problem consists of obtaining prescribed changes of five state variables with the shortest transfer. When the optimal control laws outlined in Sec. III are used, Eqs. (8–13) are analytically integrable only in few special cases; numerical integration is not attractive, as it is time consuming, and a suboptimal control law that allows for analytical integration is instead adopted. After the problem has been solved and the transfer angular length $\Delta\vartheta = \vartheta_f - \vartheta_0$ is obtained, the mission time length, final mass, and ΔV are evaluated as

$$\Delta t = \Delta\vartheta/(V/r) \qquad m_f = m_0 - (T/c)\Delta t \qquad \Delta V = \Delta t (T/m) \tag{26}$$

with proper average values for r, V (e.g., $r = a_{\text{avg}}$, $V = V_{\text{avg}}$) and $m = (m_0 + m_f)/2$.

Only the coplanar case ($\Delta i_x = \Delta i_y = 0$), which requires $\lambda_{ix} = \lambda_{iy} = 0$ and consequently $\beta = 0$, is considered in this Note. The optimal control law for the in-plane thrust angle α, that is, Eq. (15), takes different shapes, which are plotted in Fig. 1, according to various values of λ_a/λ_e. The optimal law is approximated to make the differential system analytically integrable by assuming, for $-\pi \le \vartheta - \vartheta_e \le \pi$,

$$\alpha = \begin{cases} \Lambda(\vartheta - \vartheta_e) & \text{for } \lambda_a, \lambda_e > 0 \\ \Lambda(\vartheta - \vartheta_e) + \pi & \text{for } \lambda_a, \lambda_e < 0 \end{cases} \tag{27}$$

with $0 \le \Lambda \le 1$ chosen properly in dependence of the ratio λ_a/λ_e. It should be noted that $\Lambda = 0$ corresponds to tangential thrusting, whereas $\Lambda = 1$ corresponds to constant-direction thrusting.

When the approximate control law is used, the relevant equations become, for positive λ_a and λ_e,

$$\frac{\mathrm{d}a}{\mathrm{d}\vartheta} = 2Ar \cos[\Lambda(\vartheta - \vartheta_e)] \tag{28}$$

$$\frac{\mathrm{d}e_x}{\mathrm{d}\vartheta} = A\{2 \cos\vartheta \cos[\Lambda(\vartheta - \vartheta_e)] + \sin\vartheta \sin[\Lambda(\vartheta - \vartheta_e)]\} \tag{29}$$

$$\frac{\mathrm{d}e_y}{\mathrm{d}\vartheta} = A\{2 \sin\vartheta \cos[\Lambda(\vartheta - \vartheta_e)] - \cos\vartheta \sin[\Lambda(\vartheta - \vartheta_e)]\} \tag{30}$$

The right-hand side terms change their sign for negative λ_a and λ_e. With the help of trigonometric equivalences, Eqs. (29) and (30) are rewritten as

$$\frac{\mathrm{d}e_x}{\mathrm{d}\vartheta} = (A/2)\{3\cos[(1-\Lambda)\vartheta + \Lambda\vartheta_e] + \cos[(1+\Lambda)\vartheta - \Lambda\vartheta_e]\} \tag{31}$$

$$\frac{\mathrm{d}e_y}{\mathrm{d}\vartheta} = (A/2)\{3\sin[(1-\Lambda)\vartheta + \Lambda\vartheta_e] + \sin[(1+\Lambda)\vartheta - \Lambda\vartheta_e]\} \tag{32}$$

Equations (28), (31), and (32) are integrated from ϑ_0 to ϑ_f, producing

$$\Delta a = \frac{2Ar}{\Lambda}\{\sin[\Lambda(\vartheta_f - \vartheta_e)] - \sin[\Lambda(\vartheta_0 - \vartheta_e)]\} \tag{33}$$

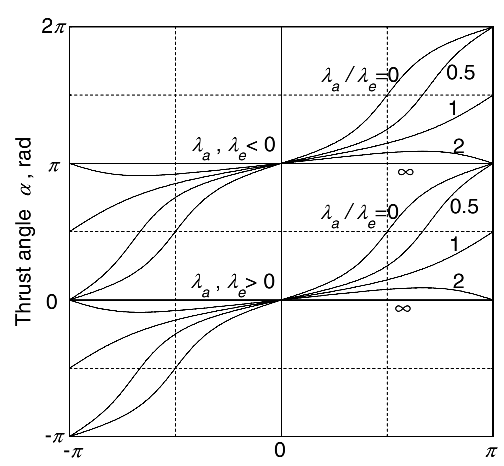

Fig. 1 Optimal thrust angle.

$$\Delta e_x = \frac{3A}{2(1-\Lambda)}\{\sin[(1-\Lambda)\vartheta_f + \Lambda\vartheta_e] - \sin[(1-\Lambda)\vartheta_0 + \Lambda\vartheta_e]\} + \\ + \frac{A}{2(1+\Lambda)}\{\sin[(1+\Lambda)\vartheta_f - \Lambda\vartheta_e] - \sin[(1+\Lambda)\vartheta_0 - \Lambda\vartheta_e]\} \quad (34)$$

$$\Delta e_y = \frac{3A}{2(1-\Lambda)}\{-\cos[(1-\Lambda)\vartheta_f + \Lambda\vartheta_e] + \cos[(1-\Lambda)\vartheta_0 + \Lambda\vartheta_e]\} + \\ + \frac{A}{2(1+\Lambda)}\{-\cos[(1+\Lambda)\vartheta_f - \Lambda\vartheta_e] + \cos[(1+\Lambda)\vartheta_0 - \Lambda\vartheta_e]\} \quad (35)$$

Special cases are $\Lambda = 0$, which gives

$$\Delta a = 2Ar(\vartheta_f - \vartheta_0) \quad (36)$$

and $\Lambda = 1$, which provides

$$\Delta e_x = (3A/2)(\vartheta_f - \vartheta_0)\cos\vartheta_e \\ + (A/2)[\sin(2\vartheta_f - \vartheta_e) - \sin(2\vartheta_0 - \vartheta_e)] \quad (37)$$

$$\Delta e_y = (3A/2)(\vartheta_f - \vartheta_0)\sin\vartheta_e \\ - (A/2)[\cos(2\vartheta_f - \vartheta_e) - \cos(2\vartheta_0 - \vartheta_e)] \quad (38)$$

To evaluate changes that correspond to one complete revolution (subscript 2π), $\vartheta_0 = \vartheta_e - \pi$ and $\vartheta_f = \vartheta_e + \pi$ are used to obtain, after simple passages exploiting trigonometric equivalences,

$$(\Delta a)_{2\pi} = \frac{4Ar}{\Lambda}\sin(\Lambda\pi) \quad (39)$$

$$(\Delta e_x)_{2\pi} = \left[\frac{3A}{(1-\Lambda)} - \frac{A}{(1+\Lambda)}\right]\sin(\Lambda\pi)\cos\vartheta_e \quad (40)$$

$$(\Delta e_y)_{2\pi} = \left[\frac{3A}{(1-\Lambda)} - \frac{A}{(1+\Lambda)}\right]\sin(\Lambda\pi)\sin\vartheta_e \quad (41)$$

For $\Lambda = 0$, $(\Delta a)_{2\pi} = 4\pi Ar$ and $(\Delta e_x)_{2\pi} = (\Delta e_y)_{2\pi} = 0$. For $\Lambda = 1$, $(\Delta a)_{2\pi} = 0$, $(\Delta e_x)_{2\pi} = 3A\pi\cos\vartheta_e$, and $(\Delta e_y)_{2\pi} = 3A\pi\sin\vartheta_e$; the change of the eccentricity magnitude is in this case $(\Delta e)_{2\pi}/A = 3\pi \approx 9.4248$, slightly lower than the optimal value 9.6888, corresponding to Eq. (17), taken from [1].

It must be remembered that Eqs. (33–35), (37), and (38) only hold for both $-\pi \le \vartheta_0 - \vartheta_e \le \pi$ and $-\pi \le \vartheta_f - \vartheta_e \le \pi$. In other cases, integration must be properly split to remain within these limits. Given the orbital elements of initial and final orbits, three unknown quantities, namely, Λ, ϑ_e, and ϑ_f, must be determined in order to obtain the prescribed changes Δa, Δe_x, and Δe_y. An iterative procedure, which is based on Newton's method, is used to solve the problem; tentative values for the unknowns are adopted, and the error on the prescribed changes is evaluated. The tentative values are then varied by a small quantity (e.g., 10^{-4}) to evaluate the derivatives of the errors with respect to the parameters, according to a finite-difference scheme; error derivatives are then used to correct the tentative values to take errors to zero.

V. Results

Missions toward near-Earth asteroids are considered to test the proposed approach; the sun is the main body, and a circular initial orbit with 1 astronomical unit (AU) radius is assumed in the following as the starting orbit. The initial orbit lies on the reference plane, and the initial longitude is $\vartheta_0 = 0$. The spacecraft initial mass is 4000 kg, and the thrust is 300 mN with a 3000 s specific impulse. The mission cost is estimated both with Eq. (23) and with Eq. (26), after the transfer that uses the approximate control law has been computed and $\Delta\vartheta$ has been evaluated. These estimations are compared to the optimal numerical solution, which is obtained by means of an indirect optimization procedure [20,21] applied to the spacecraft motion equations with two-body problem dynamics. The indirect method is efficient and relatively fast, as roughly 15 s on a i7-2600 CPU at 3.40 GHz are required to obtain convergence to the optimal solution; however, a suitable tentative solution is required, and an experienced user's intervention may be required. On the contrary, the approximate method requires on average 3/1000 of a second, and convergence (with 10^{-6} tolerance) does not require the user's action; therefore, it is well suited when a large number of transfers must be evaluated. In exceptional cases, convergence to the required accuracy may not be obtained, but the best estimation (i.e., the one corresponding to the minimum error) is instead used.

Only the coplanar case ($i_1 = i_2 = 0$) is considered here. Results for different coplanar transfers are shown in Figs. 2–4; ΔV is normalized by using the circular velocity at 1 AU as a reference value. Changes of a only are initially considered, and results are shown in Fig. 2. The spacecraft performs one revolution for $\Delta a \approx 0.21$. The basic Edelbaum solution, that is, Eq. (23), is very accurate around this value but underestimates the transfer cost when the number of revolutions is not an integer (the error reaches 50% for the shortest transfers considered here); tangential thrusting cannot be used in these cases, and the thrust must be misaligned to contrast the undesired eccentricity change that the tangent thrust would produce. The proposed modified approach is instead very accurate over the whole range of values explored, with an overestimation of the cost always below 3% except in one case in which it reaches 7%. The larger cost with respect to the optimum is related to the use of the approximate law $\beta = \Lambda(\vartheta - \vartheta_e)$ instead of the optimal control law.

Figure 3 considers changes of eccentricity only, with constant a; changes $\Delta e = 0.05$, 0.1, and 0.15 are evaluated, and transfer cost is plotted as a function of the longitude of periapsis of the final orbit $\Omega_2 + \omega_2$. Equation (23) does not consider effects of the orientation of the line of apsides and therefore provides a constant cost. On the contrary, $\Omega_2 + \omega_2$ remarkably influences the results, in particular, for the smallest eccentricity change, which corresponds to transfers about a half-revolution long. The use of $\Lambda = 1$ makes Δa return to zero after the half-revolution for $\vartheta_e = 0$ (and, in a symmetrical way, for $\vartheta_e = \pi$); constant-direction thrusting can therefore be adopted when $\Delta_e = 0.05$ and $\Omega_2 + \omega_2$ is close to 0 or π, and the mission cost is minimum; in these cases, Eq. (23) provides almost exactly the correct cost. For different positions of the line of nodes, constant-direction thrusting would also produce variations of the semimajor axis after a half-revolution, and it is necessary to modify the control law, increasing the mission cost. In these cases, the proposed approximate solution provides an excellent estimation of the actual ΔV (error always below 8%), correctly following the dependence on the line of apsides position; on the other hand, Eq. (23) exhibits errors that reach 30% for $\Omega_2 + \omega_2$ close to $\pi/2$ and $3\pi/2$. For $\Delta e = 0.1$,

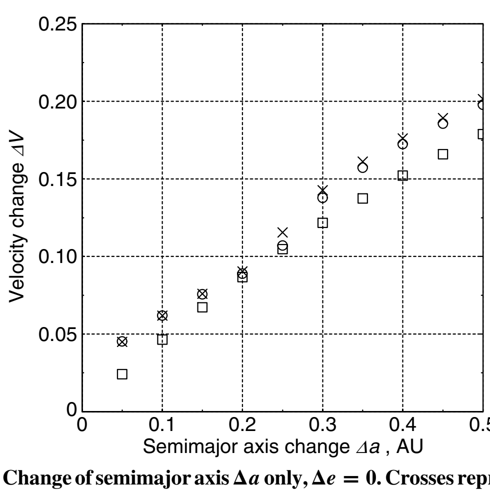

Fig. 2 Change of semimajor axis Δa only, $\Delta e = 0$. Crosses represent the exact solution, squares represent Eq. (23), and circles represent the approximate solution.

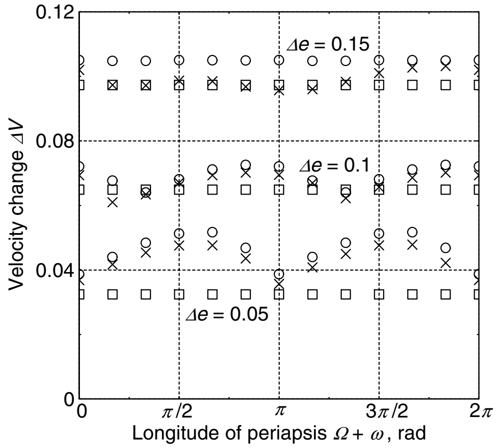

Fig. 3 Change of eccentricity Δe only, $\Delta a = 0$. Crosses represent the exact solution, squares represent Eq. (23), and circles represent the approximate solution.

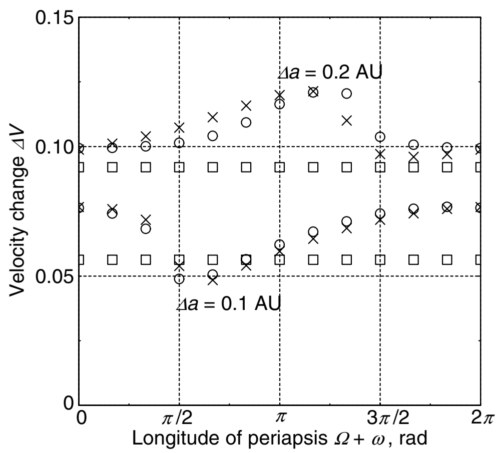

Fig. 4 Change of the semimajor axis and eccentricity, $\Delta e = 0.05$. Crosses represent the exact solution, squares represent Eq. (23), and circles represent the approximate solution.

both Eq. (23) and the proposed approximation show a good behavior, as, in this case, the transfer length becomes larger than $3/4$ of revolution and gets closer to 2π; errors are below 2% for the approximate solution and below 8% with Eq. (23). The effect of larger eccentricity influences the results for $\Delta e = 0.15$, and the optimal solution takes advantage of the radius change during the transfer; this behavior cannot be taken into account by the approximate solutions; errors may reach 10% with the approximate control law. On the other hand, errors are smaller with Eq. (23), which fortuitously provides good results, as this formula usually underestimates the transfer cost.

Simultaneous changes of a and e are considered in Fig. 4, and the results of the previous case are confirmed. The approximate solution proposed here correctly replicates the behavior of the optimal solution, which is remarkably influenced by $\Omega_2 + \omega_2$; errors exceed 6% in just four cases but always remain below 10%. Equation (23) only occasionally provides an accurate estimation of the actual cost.

VI. Conclusions

An approximate method for the evaluation of low-thrust transfers between close, low-eccentricity orbits has been presented and thoroughly tested. The method is based on Edelbaum's analysis and exploits an approximate control law that allows for an analytical integration of the relevant differential equations. Numerical integration is therefore not required, and the evaluation of the transfer costs is orders of magnitude faster with respect to the classical optimization method and more precise with respect to the original Edelbaum analysis when short transfers are considered. The method could therefore be very useful when a large number of transfer opportunities must be computed and compared to each other.

The proposed approximate solution provides an excellent estimation of the transfer cost in the planar case, at least until the eccentricity does not become too large. Improvements are instead required to consider the peculiarities of three-dimensional transfers. In the present analysis, the orbit transfer problem has been considered; the proposed method provides an estimation of transfer time and angular length and is therefore also suited to deal with rendezvous missions, for which the phasing of the target object comes into play.

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