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Effect of Finite Thrusting Time in Orbital Maneuvers

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The effect of finite thrusting time in orbital maneuvers is investigated for Hohmann-type transfers. A closed form solution of the trajectory during thrusting is obtained by assuming that a constant thrust is applied normal to the focal radius and that the change of the radial position of the rocket is small. A numerical example is presented to show the thrusting time, thrust level, propellant consumption, and the conic trajectories following the powered flight paths, etc., in comparison with the ordinary impulsive thrust case. Finally, calculations of the lead angle and the lead time are introduced based on the analysis presented in this paper.

Nomenclature

A = $\dot{m}\bar{c}/M_0g_0$, dimensionless
B = $(\dot{m}/M_0)(r_0/g_0)^{1/2}$, dimensionless
C = $V_e/(g_0r_0)^{1/2}$, dimensionless
$\bar{c}$ = effective average exhaust velocity of the jet, fps
D = $V_r/(g_0r_0)^{1/2}$, dimensionless
E = energy per unit mass of rocket, ft-lb/slug
e = eccentricity of the elliptical transfer orbit, dimensionless or base of natural logarithms
F = $\dot{m}\bar{c}$ = total thrust, lb
g_0 = acceleration due to gravity at distance r_0 from the center of attraction, ft/sec²
g_e = acceleration due to gravity at earth surface, say, 32.2 fps
l = semilatus rectum of the transfer elliptical orbit, ft
$\dot{m}$ = constant flow rate of propellant mass, slug/sec
M_0 = mass of rocket at the beginning of thrusting, slug
M_p = propellant mass consumed, slug
ΔM_p = additional propellant mass consumed due to finite thrusting time, slug
n = number denoting initial acceleration or thrust level, dimensionless
r = distance from the center of attraction to the rocket at any time during thrusting, ft
r_1 = radius of final circular orbit, ft
r_f = distance from the center of attraction to the rocket at the end of thrusting, ft
r_0 = distance from the center of attraction to the rocket at the beginning of thrusting, ft
t = time measured from the beginning of thrusting, sec
t_f = finite thrusting time, sec
t_l = lead time (time required to travel the lead angle), sec
V_c = $(g_0r_0)^{1/2}$ = circumferential velocity of a circular orbit with radius r_0, fps
V_f = velocity of the rocket after the impulsive thrust or at the end of thrusting, fps
V_r = $(dr/dt)_0$ = radial velocity of the rocket at the beginning of thrusting, fps
V_θ = $r_0(d\theta/dt)_0$ = transverse velocity of the rocket at the beginning of thrusting, fps
θ = polar angle, measured from the initial line coincided with the radius vector r_0, rad
θ_f = angular displacement of the rocket at the end of thrusting, rad
ψ = lead angle, angle between the radius vector r_0 and the apsides line of transfer orbit, rad
ρ = r/r_0, dimensionless
ρ_f = r_f/r_0, dimensionless
τ = $(g_0/r_0)^{1/2}t$, dimensionless
μ = gravitational constant, ft³/sec²
$(\ )_0$ = denotes the quantity in the parentheses at $t = 0$

Subscript

I = impulsive thrust case

It is generally assumed that the thrusting time is zero for all impulsive thrusts in orbital maneuvers. However, for practical reasons, this case does not exist, and nonzero thrusting times must be considered. Transfer orbits then consist of powered flight paths, which are the trajectories of a rocket during the finite thrusting time intervals, and nonpowered flight or conic paths.

If, at a certain point in an orbit, a constant thrust is applied to a rocket, different thrust levels will result in different powered flight paths, even though the total propellant consumption (or the total impulse that is the product of the thrust and the thrusting time) remains the same. The conic paths following the powered flight paths will also change

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accordingly. Thus, the thrusting time is closely related to the various elements of orbital maneuvers.

For most cases of practical interest, the Hohmann-type transfer is often employed as the optimum orbital transfer by impulses (1,2).[2] If the effect of finite thrusting time on orbital transfers is conceived as that of perturbations, then the investigations on orbital maneuvers based on the impulsive thrust assumption should be valid to a certain degree, as has been demonstrated by Wang (3). Although Wang's results are derived from a constant thrust acceleration maneuver, the case of constant thrust seems to be of more practical importance. Thus, the present paper deals with the effect of finite thrusting time due to a constant thrust for Hohmann-type transfers, and this yields some basic information required for both rocket engine and guidance system design.

Analysis

For Hohmann-type transfers, it can be assumed that, at the beginning of thrusting, a rocket is near the apogee or the perigee of an elliptical orbit or at any point on a circular orbit, and the following assumptions are justified:

1 The direction of the thrust is normal to the focal radius of the trajectory.

2 The change of focal radius of the trajectory is very small during the thrusting time interval.

It was further assumed that:

1 The magnitude of thrust is constant during the entire thrusting time interval, as is the usual case of a liquid chemical rocket.

2 The rocket is under the influence of the gravitational attraction of a single body, say, the earth.

Based on these assumptions, the equations of motion of a rocket in polar coordinates are (see Fig. 1)

$$\frac{d^2r}{dt^2} = r\left(\frac{d\theta}{dt}\right)^2 - \frac{g_0r_0^2}{r^2} \qquad [1]$$

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = \pm\frac{\dot{m}\bar{c}}{M_0 - \dot{m}t}\,r \qquad [2]^3$$

Let

$$\rho = r/r_0 \qquad \tau = (g_0/r_0)^{1/2}t$$

$$A = \dot{m}\bar{c}/M_0g_0 \qquad B = (\dot{m}/M_0)(r_0/g_0)^{1/2}$$

and Eqs. [1] and [2] become

$$\frac{d^2\rho}{d\tau^2} = \rho\left(\frac{d\theta}{d\tau}\right)^2 - \frac{1}{\rho^2} \qquad [3]$$

$$\frac{d}{d\tau}\left(\rho^2\frac{d\theta}{d\tau}\right) = \pm\frac{A}{1 - B\tau}\,\rho \qquad [4]$$

The initial conditions for Eqs. [1] and [2] are

$$\left(\frac{dr}{dt}\right)_0 = V_r \qquad \left(\frac{d\theta}{dt}\right)_0 = \frac{V_\theta}{r_0} \text{ at } r = r_0 \qquad t = 0$$

And the corresponding initial conditions for Eqs. [3] and [4] are

$$\left(\frac{d\rho}{d\tau}\right)_0 = \frac{V_r}{(g_0r_0)^{1/2}} = D \qquad \left(\frac{d\theta}{d\tau}\right)_0 = \frac{V_\theta}{(g_0r_0)^{1/2}} = C$$

[2] Numbers in parentheses indicate References at end of paper.

[3] The plus sign in Eq. [2] shows that the thrust is in the direction of increasing θ (positive thrust), and the minus sign has the opposite sense (negative thrust). From here on throughout this paper, whenever a term has two signs in front of it, the "upper sign" represents the positive thrust case, and the "lower sign" is for the negative thrust case.

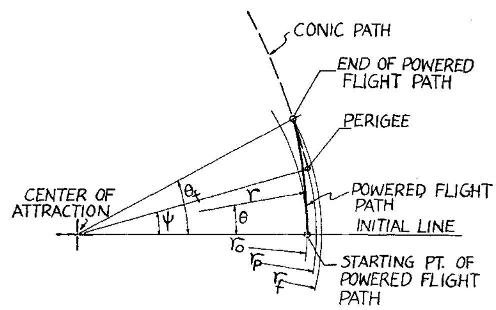

Fig. 1 Notation of powered flight path

at $\rho = 1$, $\tau = 0$. Another initial condition is obtained from Eq. [3], i.e.,

$$(d^2\rho/d\tau^2)_0 = C^2 - 1$$

By combining Eqs. [3] and [4], one gets

$$\frac{d}{d\tau}\left(\rho^3\frac{d^2\rho}{d\tau^2} + \rho\right)^{1/2} = \pm\frac{A}{1 - B\tau}\,\rho \qquad [5]$$

Based on assumption 2, that ρ is nearly equal to unity, Eq. [5] can be simplified as

$$\frac{d}{d\tau}\left(\frac{d^2\rho}{d\tau^2} + 1\right)^{1/2} = \pm\frac{A}{1 - B\tau} \qquad [6]$$

Integrating Eq. [6] with the initial condition $(d^2\rho/d\tau^2)_0 = C^2 - 1$ at $\tau = 0$, the resulting equation is

$$\left(\frac{d^2\rho}{d\tau^2} + 1\right)^{1/2} = \mp\frac{A}{B}\ln(1 - B\tau) + C \qquad [7]$$

Squaring Eq. [7] and simplifying, one has

$$\frac{d^2\rho}{d\tau^2} = \frac{A^2}{B^2}[\ln(1 - B\tau)]^2 \mp 2\frac{AC}{B}\ln(1 - B\tau) + C^2 - 1 \qquad [8]$$

Integrating Eq. [8] with the initial condition $(d\rho/d\tau)_0 = D$ at $\tau = 0$ yields

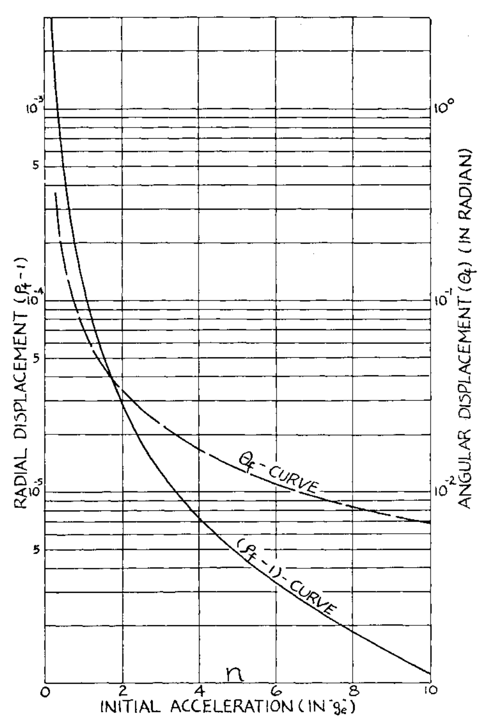

Fig. 2 Position of rocket at end of powered flight path

$$\frac{d\rho}{d\tau} = -\frac{A^2}{B^3}(1 - B\tau)[\ln(1 - B\tau)]^2 + \frac{2A}{B^2}\left(\frac{A}{B} \pm C\right)(1 - B\tau)[\ln(1 - B\tau) - 1] + (C^2 - 1)\tau + \left[D + \frac{2A}{B^2}\left(\frac{A}{B} \pm C\right)\right] \quad [9]$$

Integrating Eq. [9] and making use of the initial condition $\rho = 1$ at $\tau = 0$ gives

$$\rho = \frac{A^2}{2B^4}(1 - B\tau)^2[\ln(1 - B\tau)]^2 + \frac{A}{B^3}\left(\frac{7A}{4B} \pm \frac{3}{2}C\right) \times (1 - B\tau)^2 - \frac{A}{B^3}\left(\frac{3A}{2B} \pm C\right)(1 - B\tau)^2 \ln(1 - B\tau) + \frac{1}{2}(C^2 - 1)\tau^2 + \left[D + \frac{2A}{B^2}\left(\frac{A}{B} \pm C\right)\right]\tau + 1 - \frac{A}{B^3}\left(\frac{7A}{4B} \pm \frac{3}{2}C\right) \quad [10]$$

Combining Eqs. [10] and [4] and integrating once with respect to τ results in

$$\rho^2 \frac{d\theta}{d\tau} = \mp \frac{A^3}{4B^5}(1 - B\tau)^2[\ln(1 - B\tau)]^2 \pm \frac{A^2}{B^4}\left(\frac{A}{B} \pm \frac{1}{2}C\right)(1 - B\tau)^2 \ln(1 - B\tau) \mp \frac{A^2}{B^4}\left(\frac{11A}{8B} \pm C\right)(1 - B\tau)^2 \mp \frac{A}{B}\left[\frac{1}{2B^2}(C^2 - 1) + \frac{D}{B} + 1 + \frac{A^2}{4B^4} \pm \frac{AC}{2B^3}\right]\ln(1 - B\tau) \mp \frac{A}{4B}(C^2 - 1)\tau^2 \mp \frac{A}{B}\left[\frac{1}{2B}(C^2 - 1) + D + \frac{2A}{B^2}\left(\frac{A}{B} \pm C\right)\right]\tau + \left[C \pm \frac{A^2}{B^4}\left(\frac{11A}{8B} \pm C\right)\right] \quad [11]$$

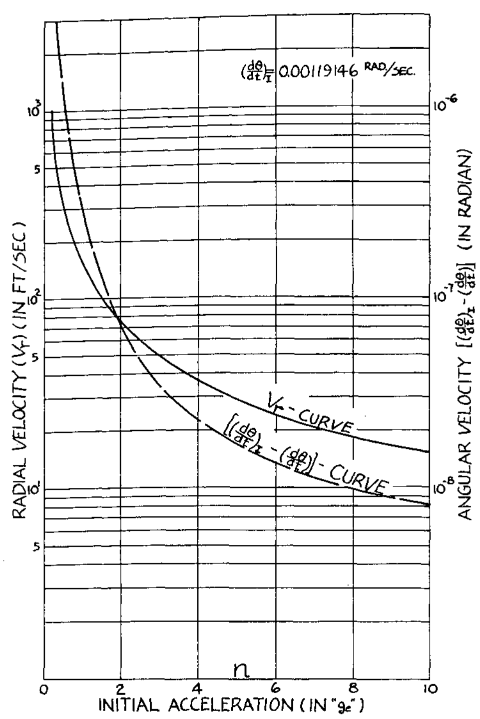

Fig. 3 Angular and radial velocities at end of powered flight path

where ρ is calculated from Eq. [10], and the constant term is obtained by applying the initial condition $(d\theta/d\tau)_0 = C$ at $\rho = 1$, $\tau = 0$.

Based on the assumption that ρ is nearly equal to unity and the initial condition $\theta = 0$ at $\tau = 0$, after integrating, Eq. [11] becomes

$$\theta = \pm \frac{A^3}{12B^6}(1 - B\tau)^3[\ln(1 - B\tau)]^2 \mp \frac{A^2}{B^5}\left(\frac{7A}{18B} \pm \frac{1}{6}C\right)(1 - B\tau)^3 \ln(1 - B\tau) \pm \frac{A^2}{B^5}\left(\frac{127A}{216B} \pm \frac{7}{18}C\right)(1 - B\tau)^3 \pm \frac{A}{B^2}\left[\frac{1}{2B^2}(C^2 - 1) + \frac{D}{B} + 1 + \frac{A^2}{4B^4} \pm \frac{AC}{2B^3}\right] \times [\ln(1 - B\tau) - 1](1 - B\tau) \mp \frac{A}{12B}(C^2 - 1)\tau^3 \mp \frac{A}{B}\left[\frac{1}{4B}(C^2 - 1) + \frac{D}{2} + \frac{A}{B^2}\left(\frac{A}{B} \pm C\right)\right]\tau^2 + C\tau \pm \frac{A^2}{B^4}\left(\frac{11A}{8B} \pm C\right)\tau - \frac{A}{B^2}\left[\pm \frac{73A^2}{216B^4} - \frac{AC}{9B^3} \mp \frac{1}{2B^2}(C^2 - 1) \mp \frac{D}{B} \mp 1\right] \quad [12]$$

It is noted that A, B, C, and D in the previous equations are known values from the given data. With the results of Eqs. [9–12], the location and the velocity of the rocket at the end of thrusting can be found. Eqs. [10] and [12] describe the powered flight path of the rocket during the finite thrusting time, and the conic path of the rocket following the powered flight path can be determined by usual means (4).

Example

In order to bring out the effect of finite thrusting time in orbital maneuvers, the following example is studied. It is

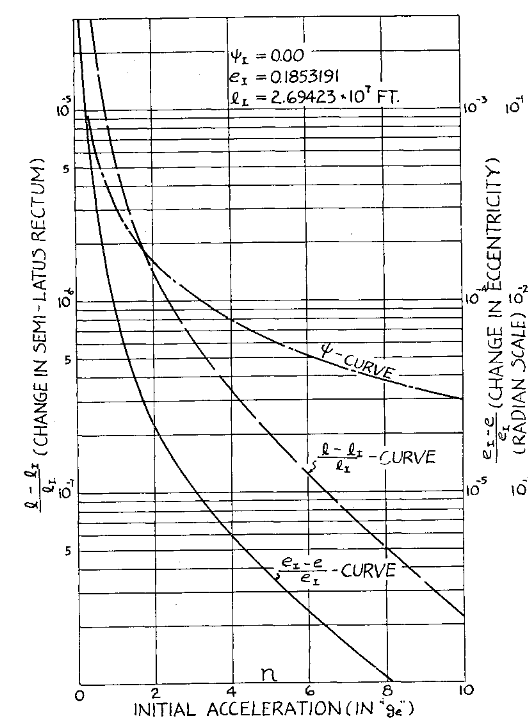

Fig. 4 Variation of geometric elements of conic path

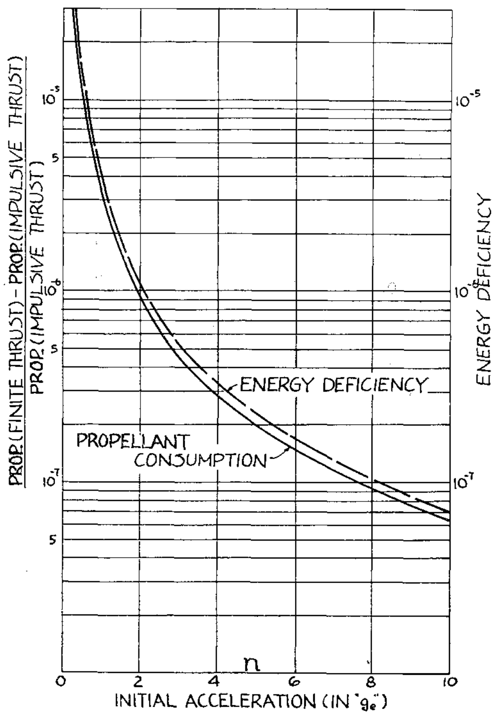

Fig. 5 Propellant consumption and energy deficiency

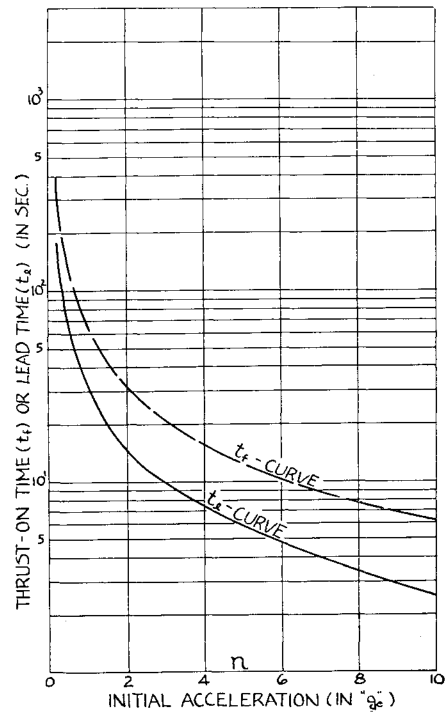

Fig. 6 Thrust-on time and lead time

noted that the setup of this example is to form a skeleton for the general discussion of the problem rather than the particular problem of the example itself.

A rocket is originally in a 300-naut mile curcular orbit around the earth, and it is desired to transfer the rocket to a coplanar 2000-naut mile circular orbit. The powered flight path at the point where the first impulse would be applied will be analyzed.

The given data are as follows:

r_0(300-naut mile orbit) $= 2.273 \times 10^7$ ft
r_1(2000-naut mile orbit) $= 3.307 \times 10^7$ ft
μ(earth gravity constant) $= 1.40643 \times 10^{16}$ ft^3/sec^2
$\bar{c} = 10^4$ fps
$V_c = (\mu/r_0)^{1/2} = 24{,}875$ fps

For the impulsive thrust case, the following figures are computed according to well-known formulas in orbital mechanics:

$$(V_f)_I = \left[2\mu\left(\frac{1}{r_0} - \frac{1}{r_1 + r_0}\right)\right]^{1/2} = 27{,}082 \text{ fps}$$
$$\Delta V = (V_f)_I - V_c = 2207 \text{ fps}$$
$$E_I = \tfrac{1}{2}(V_f)_I{}^2 - (\mu/r_0) = 0.40734044$$
$$l_I = 2.69423 \times 10^7 \text{ ft}$$
$$e_I = 0.1853191$$
$$\left(\frac{d\theta}{dt}\right)_I = \frac{(V_f)_I}{r_0} = 0.00119146 \text{ rad/sec}$$
$$M_p = M_0[1 - e^{-(\Delta V/\bar{c})}] = 0.1981 M_0$$

Based on the same amount of propellant consumption, the resulting powered flight paths due to different thrust levels are investigated.

For

$$F = \dot{m}\bar{c} = n g_e M_0$$

and

$$t_f = \frac{M_p}{\dot{m}} = \frac{0.1981\,\bar{c}}{n g_e} = \frac{61.5}{n}$$

the finite thrusting time t_f can be calculated according to the different thrust levels n (or initial acceleration).

Fig. 2 shows the angular and radial displacements of the rocket at the end of the powered flight paths at different thrust levels n. Fig. 3 gives the corresponding angular and radial velocities.

Fig. 4 shows the variations of three geometric elements (semilatus rectum, eccentricity, and the angle between the line of apsides and the initial line) of the conic paths following the powered flight paths resulting from different thrust levels.

The characteristic velocity, which is basically a function of geometry of the related orbits, is quite familiar to the readers in the investigations of orbital transfer by impulses. During the application of impulses, no change in r (distance between the rocket and the center of attraction) is involved. However, for the case of finite thrusting time, the rocket changes its velocity as well as its distance r from the center of attraction continuously along the powered flight path. If one realizes the effect of changes in velocity at different r by impulses, it is naturally concluded that the energy level E (which includes both the characteristics of the velocity and the distance r) rather than the characteristic velocity (which is only a measure of change in velocity) should be used in the study of the effect of finite thrusting time in orbital maneuvers.

During the power flight the rocket moves against gravity, and a certain amount of energy is spent to overcome this gravitational action; therefore, the resulting energy level would be lower in comparison with the impulsive thrust case if the same amount of propellant is consumed. Fig. 5 shows the relationship between the energy deficiency defined as

$$\frac{\text{energy level (impulsive thrust)} - \text{energy level (finite thrust)}}{\text{energy change (impulsive thrust)}}$$

and the thrust level.

In order to reach the same energy level as the impulsive thrust case, a rocket using finite thrusting time will require additional propellant, which is computed by the following equations:

$$\Delta E = V_f \cdot \Delta V_f + (\mu/r_f{}^2) \cdot \Delta r_f \sim V_f \cdot \Delta V_f$$

$$M_p = (1 - 0.1981)M_0(1 - e^{-(\Delta V_f/\bar{c})}) = 0.8019M_0(1 - e^{-(\Delta V_f/\bar{c})})$$

where ΔE is energy level difference. The relationship between $\Delta M_p/M_p$ and various values of n is also plotted in Fig. 5.

As was mentioned, investigations on orbital transfer by impulses are valid to a certain degree, and the present study on the effect of finite thrusting time may be conceived as a refinement. If a certain scheme of orbital transfer is to be maintained as a result of the investigations based on the assumptions of impulsive thrust, which is particularly important from the guidance system point of view, then the effect of finite thrusting time in orbital maneuvers based on this scheme can be brought out.

It is recalled that, for Hohmann-type transfers, the lines of apsides for original, transfer, and final orbits are colinear. For finite thrusting time assumptions, a rocket must commence thrusting before it reaches the apsides. The angle between the radius vector at start of thrusting and the line of apsides is known as "lead angle," which is ψ in the present example. However, if the original orbit is an ellipse, the "lead angle" must be found by cut-and-try.

The time required for the rocket traveling the angular displacement ψ is known as "lead time (t_l)," which is the time prior to reaching the apsides of the original and transfer orbits. In the present example, one apside is the perigee of the transfer orbit. The "lead time" is computed by Eq. [12]. The finite thrusting time and the "lead time" for various thrust levels are plotted in Fig. 6.

Conclusion

The effect of finite thrusting time in orbital maneuvers for Hohmann-type transfers can be studied by the analytical approach presented in this paper. The characteristics of the powered flight paths resulting from various thrust levels may be examined according to different practical purposes. As a result of the typical example studied in this paper, the following conclusions could be drawn:

1 Any error analysis based on impulsive thrust cases is of doubtful value, for the effect of finite thrusting time does yield different conic paths at different thrust levels as shown in Fig. 4.

2 Finite thrusting time has little effect on propellant consumption for Hohmann-type transfers in comparison with the impulsive thrust case.

3 Because of the existence of finite thrusting time, the lead angle and the lead time will play an important role in maintaining the orientation of the line of apsides of orbits which should be particularly prominent when transferring between highly elliptical orbits.

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New Journal: *Reviews of Geophysics*

The American Geophysical Union announces publication of a new quarterly, *Reviews of Geophysics*. The first issue will be published in February 1963. The new journal is intended to provide a link between the active specialist working on the frontiers of geophysics and the student and mature scientist undertaking research in the field for the first time. It will carry articles that are readable and that present an overall view of a particular area of current interest, including critical evaluation and adequate bibliographies of source material. It will be edited by Gordon J. F. MacDonald of the Institute of Geophysics and Planetary Physics, University of California, Los Angeles 24, California.

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