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Optimal Many-Revolution Orbit Transfer

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Introduction

THE problem of low-thrust, minimum-time orbit transfer has been studied analytically by several authors in the last decades. The method of two time variables has been used by Levin,' Shi and Eckstein,[2] and Eckstein and Shi[3] to study the effect of ad hoc thrust programs on the orbit without considering optimization. In an important paper, Edelbaum[4] solved the optimal one-orbit control problem for the first time, and obtained a suboptimal solution for the long time scale problem.

In this Note we shall also use the method of two time scales, separating the optimal control problem into a "fast" time scale problem over one orbit, and a "slow" time scale problem over the entire transfer. We shall recapitulate Edelbaum's solution of the fast time scale problem, arid then we shall solve, for the first time, the optimal control problem for the overall transfer. The optimal control law for the slow time scale problem will be found in explicit form. This reduces the slow time scale problem to a two-point boundary value problem in semimajor axis inclination space. The solution space for this problem has been globally mapped, and explicit total velocity change requirements for any desired transfer can be easily obtained. The solution assumes constant thrust with decreasing mass of the vehicle, but is also optimum for any slowly varying throttle program.

The Short Time Scale Problem

The optimal control problem over the fast time scale of one orbit was first solved by Edelbaum. However, we quickly review those portions pertinent to our later discussion. For a low-thrust vehicle in a nearly circular orbit, the two Lagrange planetary equations we shall need (in their acceleration component form) are (Danby[5]):

$$\frac{da}{dt}=\frac{2Aa^{3/2}\cos\vartheta}{\sqrt{\mu}} \tag{1}$$

$$\frac{di}{dt}=\frac{Aa^{1/2}\sin\vartheta\cos f}{\sqrt{\mu}} \tag{2}$$

where μ is the gravitational parameter, a the semimajor axis, i the inclination, and f the true anomaly, measured from the node since the eccentricity is small. The vehicle acceleration is A, and the vehicle pitch angle from the orbital plane is 8. For small eccentricity there is no dependence on any vehicle yaw component of acceleration.

Assuming the control $\vartheta(f)$ to be a function of true anomaly, we may pose the problem of maximizing the inclination change over one orbit while still achieving a given semimajor axis change Aa. Assuming small changes over the orbit leads to the optimization problem

$$\delta\int_0^{2\pi}\left[\frac{a^2A}{\mu}\sin\vartheta(f)\cos f+\lambda\left(\frac{2a^3A}{\mu}\cos\vartheta(f)-\frac{\Delta a}{2\pi}\right)\right]df=0 \tag{3}$$

Simple techniques yield the control law

$$\vartheta(f)=\tan^{-1}\left(\frac{\cos f}{\sqrt{1/u-1}}\right) \tag{4}$$

and the changes in a and i per orbit are

$$da=\frac{8a^3A}{\mu}\sqrt{1-u}\,K(u) \tag{5}$$

$$di=\frac{4a^2A}{\mu}\left[\frac{1}{\sqrt{u}}E(u)+\left(\sqrt{u}-\frac{1}{\sqrt{u}}\right)K(u)\right] \tag{6}$$

Here, in anticipation of the next section, we have introduced the control variable

$$u\equiv\frac{1}{4\lambda^2a^2+1} \tag{7}$$

The operative range of u is from 0 to 1, and K and E are the complete elliptic integrals of the first and second kinds, respectively. Very convenient approximation formulas for these functions are given by Abramowitz and Segun.[6]

Several other results must be mentioned. The net change in eccentricity and node per orbit is zero with this control program, so initially circular obits stay nearly circular. The control law $\vartheta(f)$ varies from a pure in-track acceleration for $u=0$ when only a changes arid to a square wave for $u=1$ when only the inclination changes.

The Long Time Scale Problem

In the previous section we reviewed the optimal way to produce small changes in the orbital elements over one orbit. On the long time scale of the entire transfer, these expressions can be divided by the Keplerian period $dt=2\pi a^{3/2}/\sqrt{\mu}$ (which is to be regarded as a "short" time) to yield equations of motion on the long time scale. Also on the long time scale we must include the effects of depletion of fuel which (for constant thrust) causes the vehicle acceleration to vary as

$$A(t)=\frac{A_0}{1-\dot{m}t} \tag{8}$$

Here $\dot{m}$ is the specific fuel usage rate, and A, is the initial acceleration. This last effect introduces explicit time

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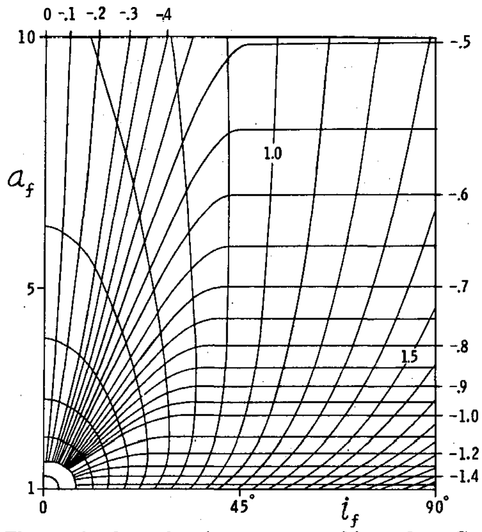

Fig. 1 The a_f, i_f plane showing contours of λ_i and τ_f. Contours of λ_i represent trajectories, and are labeled with the λ value along the top and right. The contour interval for τ is 0.1.

dependence into the control Hamiltonian, but this may be eliminated if we introduce the new independent variable τ by placing $d\tau = A(t)dt$, which leads to the time transformation

$$\tau = -\frac{A_0}{\dot{m}} \ln(1-\dot{m}t) \tag{9}$$

Equation (9) can be recognized as a form of the rocket equation. Physically, τ is the total accumulated velocity change of the vehicle.

The equations of motion on the long time scale then become

$$\frac{da}{d\tau} - \frac{4a^{3/2}}{\pi\sqrt{\mu}} \sqrt{1-u}\, K(u) \equiv \frac{4a^{3/2}}{\pi\sqrt{\mu}} P(u) \tag{10}$$

$$\frac{di}{d\tau} = \frac{2a^{1/2}}{\pi\sqrt{\mu}} \left[\frac{1}{u}E(u) + \left(\sqrt{u} - \frac{1}{\sqrt{u}}\right)K(u)\right] \equiv \frac{2a^{1/2}}{\pi\sqrt{\mu}} R(u) \tag{11}$$

In this form the equations of motion are free of the values of A, and $\dot{m}$ for a particular vehicle, and are also free of the time variable τ.

We are now in a position to pose the long time scale problem. We wish to effect a transfer from given initial values of a, and i_0 to given final values of a_f and i_f in the minimum τ interval, τ_f, simultaneously minimizing elapsed time and fuel consumption. This is a standard minimum time-optimal control problem treated, for example, in Bryson and Ho.[7] The control Hamiltonian is

$$H = 1 + \lambda_a \frac{4a^{3/2}}{\pi\sqrt{\mu}} P(u) + \lambda_i \frac{2a^{1/2}}{\pi\sqrt{\mu}} R(u) \tag{12}$$

where two Lagrange multipliers have been introduced. The optimal control program is the solution of

$$\frac{da}{d\tau} = \frac{\partial H}{\partial \lambda_a} \qquad \frac{di}{d\tau} = \frac{\partial H}{\partial \lambda_i} \tag{13}$$

$$\frac{d\lambda_a}{d\tau} = -\frac{\partial H}{\partial a} \qquad \frac{d\lambda_i}{d\tau} = -\frac{\partial H}{\partial i} \tag{14}$$

$$\frac{\partial H}{\partial u} = 0 \tag{15}$$

$$H(\tau_f) = 0 \tag{16}$$

Table 1 Expansion coefficients for $\varphi^{-1}(Z)$

i	α_i	β_i
0	0.0	1.0
1	2.467410607	0.4609698838
2	− 1.907470562	13.7756315324
3	35.892442177	− 69.1245316678
4	− 214.672979624	279.0671832500
5	947.773273608	− 397.6628952136
6	− 2114.861134906	− 70.0139935047
7	2271.240058672	528.0334266841
8	− 1127.457440108	− 324.9303836520
9	192.953875268	20.5838245170
10	8.577733773	18.8165370778

Equations (13) repeat the state equations of motion. Since the Hamiltonian is independent of i, the first of Eqs. (14) states that λ_i is a constant. Also, since H is not an explicit function of τ, H itself is constant, and the transversality condition (16) tells us that this value of H is zero.

The explicit control law for this problem can be found. The optimality condition (15) and the statement that $H=0$ supply us with two linear equations for λ_a and λ_i. These can be solved for the constant value of λ_i and the result inverted to yield the control law for the long-term problem

$$u = \varphi^{-1}\left[\frac{\sqrt{\mu}\pi}{2\lambda_i\sqrt{a}}\right] \tag{17}$$

in terms of the semimajor axis and the constant value of λ_i. The function φ is defined as

$$\varphi(u) \equiv \frac{R'(u)P(u)}{P'(u)} - R(u) \tag{18}$$

where primes refer to differentiation with respect to u. This function is monotonic, and, hence, invertible. An approximation formula for φ^{-1} is included in the Appendix.

With the discovery of the explicit control law, the complete minimum-time control problem has been reduced to the solution to the state equations of motion (13), subject to the control law (17), with given boundary conditions. This is a very significant reduction of order compared to the original problem. Starting from given initial conditions a, and i_0, two unknown constants, h, and τ_f, are to be determined by requiring that the final conditions a_f and i_{fd} are met. This two-variable boundary value problem is easily surveyed by numerical techniques, and the results globally mapped.

We now introduce nondimensional units by setting $a_0 = 1$, $i_0 = 0$, and setting the gravitational parameter $\mu = 1$, which implicitly sets the time unit via Kepler's third law. Numerically integrating the state equations of motion with the control law (17) produces Fig. 1, in which contours of h, and τ_f are plotted on the a_f, i_f plane. Curves of constant λ_i are also trajectories across the plane. It is gratifying to note that large inclination changes are performed at the largest value of i in the transfer being executed. This behavior parallels the behavior of impulsive maneuvers. The values of λ_i are in the range from 0 to $-\pi/2$, while the τ contour interval is 0.1 in the dimenisionless units. The figure permits simple interpolation of approximate values of the constants τ_f and λ_i to solve any particular long time scale boundary value problem.

Discussion

We have formulated the low-thrust, many-revolution circle to circle orbit transfer problem, and under very general conditions reduced this optimal control problem to quadratures. Knowing the desired initial and final orbits, the slow time scale boundary value problem would be solved first. The vehicle would then be commanded to execute a periodic pitch angle program given by Eq. (4), where the slow control variable u is given by Eq. (17). Whenever the fundamental assumptions of the two time scale method are met, the fast time variable f and the slow time variable τ (or a) are completely independent of one another. As long as very many resolutions elapse the vehicle will arrive in the final orbit when $\tau=\tau_f$. The vehicle need not have the capability of solving the optimal control problem. Given the values of λ_i and τ_f for the desired transfer, the vehicle only needs access to the current values of a and f to calculate the control functions.

From an operational point of view, the separation of the two time variables has several advantages. Should a thruster fail, for example, the optimal solution *does not change.* Only the τ to t conversion changes as vehicle parameters are altered. Should the vehicle need to return to its original orbit after depositing a payload, the optimal return trajectory will be the simple r-time reverse of the outbound transfer, as long as very many revolutions elapse on the return leg. In fact, the time transformation is not limited to the constant thrust vehicle we have assumed. *Any arbitrary throttle program $A(t)$ may be used in the time transformation and the same optimal r-time trajectory results.* The only restriction is that $A(f)$ can vary only "slowly" over the period of one orbit.

Edelbaum, besides being the first to solve the short time scale problem, also addressed the long time scale problem and found a suboptimal solution. In order to find this solution, he assumed that the fast control variable $\vartheta(f)$ was a square wave for any value of u, and, more critically, he assumed that the vehicle *acceleration* (not its thrust) was constant. The first assumption may be defendable as differing only slightly from the optimal ϑ program, but modern attitude control systems are quite capable of executing the optimal $\vartheta(f)$ program. The restriction of Edelbaum's result to constant acceleration vehicles is much more severe. Our work has eliminated this restriction completely.

We have simulated several transfers numerically by using our control law in the full set of Lagrange planetary equations. The method of two time scales is, of course, exact only when an infinite number of revolutions elapse during the transfer, and our solution is exact only in this case. In all simulated transfers, the error in obtaining the desired final conditions a_f and i_f show a first-power law dependence on $1/N$, where N is the number of revolutions which elapse in the transfer. This strongly indicates that the problem of optimal *finite revolution* transfer should be solvable by a perturbation theory attack, starting from our infinite revolution solution. This is a topic of current research.

Conclusions

In this work we have detailed, for the first time, a closed-form solution to the low-thrust circle to circle orbit transfer problem. The solution is exact for an infinite number of revolutions during the transfer. The solution is simple enough to be implemented in the onboard control system of an orbit transfer vehicle. It can also tolerate changes in thrust programs over a very wide range, including loss of thrusters. Finally, this solution holds the promise of success of a perturbation theory approach to finite revolution orbit transfer.

Appendix

To implement our solution the user will need methods to easily generate values of the function φ^{-1}. The function $u=\varphi^{-1}(X)$ is best approximated as a rational function of the variable $Z=X^{-2}$ in the form

$$u=\varphi^{-1}(Z=X^{-2})\approx \sum_{i=0}^{10}\alpha_i Z^i \Big/ \sum_{i=0}^{10}\beta_i Z^i \tag{A1}$$

The variable Z eliminates a singularity in φ. The coefficients α_i and β_i listed in Table A1, yield values for φ^{-1} which are accurate to at least seven figures. This is a Chebyshev fit, not a Taylor series, and should not be truncated.

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The Number of Multiplications Required to Chain Coordinate Transformations

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Introduction

MANY applications in guidance and control require transformations of coordinates and/or the rotation of vectors. When these computations are performed in the inner loop of a real-time guidance system, the computations can be quite demanding, even allowing for the great advances in hardware in recent years. In dealing with this problem, a number of investigators (for example, Ickes[1]) have rediscovered the eminent suitability of quaternions for this application.

Hamilton discovered his celebrated quaternions in 1843. Following the lead of a then recent discovery that complex numbers can be interpreted as rotations in the plane, Hamilton was able to interpret quaternions as, among other things, three-dimensional spatial rotations. The high hopes Hamilton had for the quaternions were not all realized (Ref. 2).† With Gibb's development of his vector analysis (c. 1901), the quaternions experienced a steady erosion in popularity, until rediscovered for guidance and control.

As shown by Ickes,[1] the straightforward multiplication of 3×3 matrices requires 27 multiplications and 18 additions, while straightforward multiplication of quaternions requires 16 and 12, respectively. The principal results of the present paper show that the previous numbers are not minimal: adaptations of Strassen-Winograd algorithms reduce the required number of multiplications to as few as ten. The achievable relief in computational requirements over the conventional quaternion algorithm is about 15%. Of course, the improvement over conventional matrix multiplication is much higher (approximately a factor of two).

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†Written by a competent mathematician, this work contains valuable mathematical exposition besides the obligatory biographical material. Chapter VII deals with the quaternions: the motivation behind them, their genesis, Hamilton's very high expectations for them, and their somewhat disappointing fate and the reasons thereof.