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REPOSITIONING MANEUVERS FOR CIRCULAR ORBITS USING CONSTANT THRUST

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Abstract: An electric propulsion system supplying constant low thrust can perform station change maneuvers with significantly lower fuel costs than conventional propulsion systems. This paper introduces an analytical method of calculating the maneuver time for a given station change, assuming a tangential thrust profile and a constant mass flow rate. A non-tangential thrust angle profile is also developed which optimizes the station change rate while ensuring that the final orbit has zero eccentricity. Simulations of a common electric propulsion system show that the optimal profile can reduce final eccentricity by three orders in magnitude compared to the tangential profile.

Key Words: Aerospace Trajectories; Aerospace Control; Optimal Control; Station Change; Constant Thrust

1. INTRODUCTION

Applications for constant low thrust propulsion have appeared throughout the literature since the 1960's when practical electric propulsion systems were first demonstrated in the lab and in actual space operations (Friedlander, 1972). Many of the studies concern the use of electric propulsion for geosynchronous satellite station keeping, which was the first application of an electrothermal thruster in space operations on the VELA nuclear detection satellite in 1965 (Vondra *et al.*, 1984). Later examples of stationkeeping using electric propulsion are the Applied Technology Satellites (ATS) and the Synchronous Meteorological Satellite (SMS-C)(Isley and Duck, 1972). The other significant area of study has been low thrust orbit transfers, particularly for interplanetary and LEO-to-GEO transfers (Jasper, 1973; Wiesel and Alfano, 1985; Alfano and Thorne, 1993). These include numerous papers discussing the use of optimal control trajectories to minimize time or fuel usage of low thrust transfers.

One application that has not received extensive attention is orbit relocation maneuvers. However, the high specific impulse attainable with electric systems make them an attractive option for decreasing the fuel cost of station changes. Satellites could be relocated quickly without using a significant portion of the spacecraft's fuel budget. This would improve flexibility, increasing the competitiveness and efficiency of satellite constellations.

One of the earliest authors to consider station changes using continuous low thrust was Edelbaum (1961). He suggested that the optimal method was to thrust tangentially in one direction until half of the desired change was completed and then thrust in the opposite direction until the full change was complete. In his development of this solution, he assumes that mass flow is neglible and he does not consider the eccentricity changes caused by tangential thrusting. Isley and Duck also discuss simple station changes. They make the same assumptions as Edelbaum, but also allow a coast phase, creating a trade-off between transfer time and fuel use.

This paper proposes two improvements to Edelbaum's solution. First, an analytical solution to the relocation maneuver is presented which uses tangential thrusting, but includes the effect of propellant mass loss. Second, a control profile is developed using conventional optimal control techniques which varies the thrust angle to maximize station change while zeroing the final eccentricity. This aspect is important since many communications satellites in use today have strict stationkeeping requirements, and even a small eccentricity can cause variations in the satellite's station. Using the optimal control profile to perform relocation maneuvers, no additional maneuvers are necessary to eliminate the eccentricity introduced by a tangential profile.

2. NOMENCLATURE

r	radial position
u	radial velocity
v	"tangential" velocity
θ	true longitude

$\mathbf{x}$	state vector $(r, u, v)^T$
$\mathbf{f}(\mathbf{x},\phi,t)$	state vector derivative
σ	satellite station (longitude)
F	thruster force
ϕ	thrust vector angle
m	satellite mass
$\dot{m}$	mass flow rate
$\dot{m}_{sp}$	specific mass flow $(\dot{m}/m_0)$
g	gravitational acceleration
I_{sp}	specific impulse
t	time
T	total transfer time
J	performance index
H	Hamiltonian
λ	Lagrange multiplier vector
$\omega_{\oplus}$	Earth rotation rate ($7.292116 \times 10^{-5} \frac{\text{rad}}{\text{sec}}$)
P	orbital period
a	semi-major axis
μ	Earth gravitational parameter ($3.986012 \times 10^5 \text{km}^3/\text{sec}^2$)
A	satellite acceleration
τ	accumulated velocity change

SUBSCRIPTS

$-_0$	initial value
$-_f$	final value
$-_{geo}$	value for geostationary orbit
$-_t$	value for transfer orbit

3. EQUATIONS OF MOTION

The dynamics of repositioning maneuvers are not strongly affected by perturbation effects, so two-body motion is a satisfactory model for the purposes of this paper. A constant thrust, acting in the orbital plane, provides the control input. Using a polar coordinate system, the equations of motion are

$$\begin{aligned} \dot{r} &= u \\ \dot{u} &= \frac{v^2}{r} - \frac{\mu}{r^2} + \frac{F\sin\phi}{m_0 - \dot{m}t} \\ \dot{v} &= -\frac{uv}{r} + \frac{F\cos\phi}{m_0 - \dot{m}t} \\ \dot{\theta} &= \frac{v}{r} \end{aligned} \tag{1}$$

An additional coordinate of interest is the longitude, or station of the satellite. The differential equation describing its time rate of change is determined by the geometry of the problem, as shown in Fig. 1. It can be written as

$$\dot{\sigma} = \dot{\theta} - \omega_{\oplus} \tag{2}$$

4. ANALYTIC SOLUTION FOR TANGENTIAL THRUSTING WITH MASS FLOW

Consider the repositioning maneuver in which a space vehicle in a circular orbit thrusts at a constant low level in a direction tangent to its orbit for a period of time, and then reverses the thrust direction until it returns to a circular orbit at its original altitude. In this maneuver, the position change is uniquely defined by the duration of the thrusting period. Edelbaum solved this problem assuming zero mass flow, or constant acceleration. In reality, thrust is likely to be constant, while the changing mass of the space vehicle will cause acceleration to vary slowly as

$$A(t) = \frac{A_0}{1 + \dot{m}_{sp}t} \tag{3}$$

Since the change in orbital position is an effect of changing the orbital period for a short drift phase, and orbital period is a function only of semi-major axis, the problem can be framed in terms of the change in semi-major axis. Introducing a new independent variable, τ, related to time by $d\tau = A(t)\,dt$, allows the derivative of the semi-major axis to be written as[7]

$$\frac{da}{d\tau} = \pm 2\sqrt{\frac{a^3}{\mu}} \tag{4}$$

where the positive sign applies when thrusting in the direction of orbital velocity (orbit raising) and the negative sign applies when thrusting in the direction opposite the orbital velocity. Now the position change can be found as the integral,

$$\Delta\sigma = \int_0^{t_f} (\dot{\theta}_0 - \dot{\theta}(t))\,dt \tag{5}$$

The first term of the integral is constant, so it can be computed and brought outside the integral sign. The second term of the integral is then written in terms of

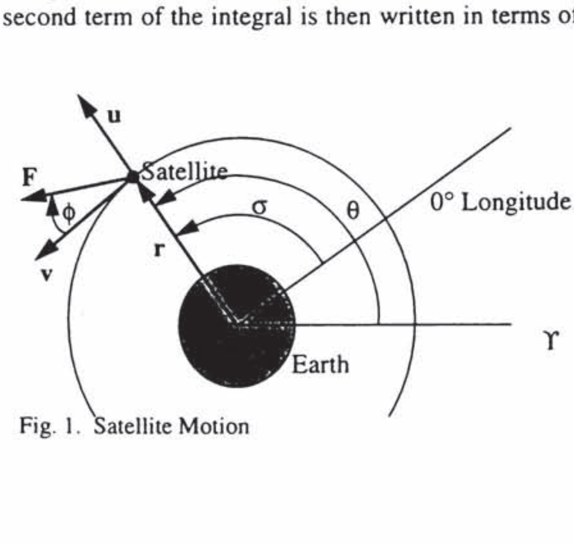

Fig. 1. Satellite Motion

semi-major axis, a, and the new independent variable, τ, as

$$\Delta\sigma = \sqrt{\frac{\mu}{a_0^3}}t_f - \left[\frac{1}{A_0}\int_0^{\tau_f}\sqrt{\frac{\mu}{a^3}}\exp\left(\frac{\dot{m}_{sp}\tau}{A_0}\right)d\tau\right] \tag{6}$$

To integrate this expression, semi-major axis must be written in terms of τ. This can be done by dividing the integral into the two parts of the repositioning maneuver, and using the differential relationship given in equation (4). For a positive reposition, the initial thrusting is opposite the orbital velocity, so the relationship is

$$\tau = \sqrt{\frac{\mu}{a}} - \sqrt{\frac{\mu}{a_0}} \tag{7}$$

if $\tau(a_0) = 0$. This can be rewritten as

$$\sqrt{\frac{\mu}{a^3}} = \frac{1}{\mu}\left(\sqrt{\frac{\mu}{a}}\right)^3 = \frac{1}{\mu}\left(\tau + \sqrt{\frac{\mu}{a_0}}\right)^3 \tag{8}$$

For the second half of the maneuver, a similar function can be constructed,

$$\sqrt{\frac{\mu}{a^3}} = \frac{1}{\mu}\left(\sqrt{\frac{\mu}{a}}\right)^3 = \frac{1}{\mu}\left(2\tau_1 + \sqrt{\frac{\mu}{a_0}} - \tau\right)^3 \tag{9}$$

where τ_1 is the value of the independent variable at the point where the thrust is reversed. Substituting these expressions into equation (6), the integral becomes

$$\Delta\sigma = \sqrt{\frac{\mu}{a_0^3}}t_f - \frac{1}{\mu A_0}\left[\int_0^{\tau_1}\left(\tau + \sqrt{\frac{\mu}{a_0}}\right)^3\exp\left(\frac{\dot{m}_{sp}\tau}{A_0}\right)d\tau + \int_{\tau_1}^{\tau_2}\left(2\tau_1 + \sqrt{\frac{\mu}{a_0}} - \tau\right)^3\exp\left(\frac{\dot{m}_{sp}\tau}{A_0}\right)d\tau\right] \tag{10}$$

Integrating by parts, the first integral becomes

$$\frac{1}{\mu A_0}\exp\left(\frac{\dot{m}_{sp}\tau}{A_0}\right)\left[\left(\frac{A_0}{\dot{m}_{sp}}\right)(\ldots)^3 - 3\left(\frac{A_0}{\dot{m}_{sp}}\right)^2(\ldots)^2 + 6\left(\frac{A_0}{\dot{m}_{sp}}\right)^3(\ldots) - 6\left(\frac{A_0}{\dot{m}_{sp}}\right)^4\right] \tag{11}$$

where

$$(\ldots) = \left(\tau + \sqrt{\frac{\mu}{a_0}}\right) \tag{12}$$

After integration by parts, the second integral is identical to equation (11) except that sign changes on the second and fourth terms in the brackets and the ellipsis now represents

$$(\ldots) = \left(2\tau_1 + \sqrt{\frac{\mu}{a_0}} - \tau\right) \tag{13}$$

Now the only unknowns in the $\Delta\sigma$ equation are the limits of integration, τ_1 and τ_2. These can be related to the final time t_f, by the original differential relation,

$$d\tau = \frac{A_0}{1 + \dot{m}_{sp}t}dt \tag{14}$$

Integrating, this becomes

$$\tau = \frac{A_0}{\dot{m}_{sp}}\ln\left(1 + \dot{m}_{sp}t\right) \tag{15}$$

when $\tau(0) = 0$. In this form, it can be seen that τ represents the accumulated velocity change. Since the velocity change from the beginning of the transfer to the point of thrust reversal must equal the velocity change from thrust reversal to the end of the transfer, $\tau_2 = 2\tau_1$. Then, since $\tau(t_f) = \tau_2$, the value of τ_1 becomes

$$\tau_1 = \frac{A_0}{2\dot{m}_{sp}}\ln\left(1 + \dot{m}_{sp}t_f\right) \tag{16}$$

Using equation (16) with equations (10) - (13), the total positive position change can be calculated for any given transfer time. A similar derivation produces the formulas required for determining a negative position change. These formulas can be used to quickly estimate the time needed for any station change where tangential thrusting is used, and gives improved accuracy over the original formulas given by Edelbaum or Isley and Duck. A computer algorithm using the positive position change formulas should also check for consequences of lowering the orbit, such as increased atmospheric drag, or even worse, collision with the earth. The formulas can also include a coast phase, with a little revision. A further refinement is to use an optimal control profile to not only include the effects of mass loss, but also eliminate any eccentricity that may build up during the transfer as a result of tangential thrusting.

5. OPTIMAL CONTROL FORMULATION

To obtain the optimal station change transfer, the thrust angle must be controlled so as to maximize the total station change, while ensuring that the final orbit is recircularized at geosynchronous altitude. Optimization techniques based on variational calculus are well suited to solving this problem, and have been discussed extensively in the literature. The formulation below generally follows the development of the theory given in Bryson and Ho (1975).

The performance index to be maximized is the station change, integrated over the total transfer time,

$$J = \int_{t_o}^{t_f} \dot{\sigma} dt \tag{17}$$

The equations of motion, $\mathbf{f}(t)$, can be adjoined to the performance index with Lagrange multiplier functions, $\lambda(t)$, and the Hamiltonian, H, is then defined as

$$H = \dot{\sigma} - \lambda^T \mathbf{f} \tag{18}$$

It can be shown that the performance index is optimized by the optimality condition

$$\frac{\partial H}{\partial \phi} = 0 \tag{19}$$

which leads to the control law

$$\tan\phi = \frac{\lambda_u}{\lambda_v} \tag{20}$$

where λ_r, λ_u, λ_v, are determined by the differential equations

$$\dot{\lambda}_r = \frac{1}{r^2}\left(\lambda_u\left(v^2 - \frac{2\mu}{r}\right) - \lambda_v uv + v\right)$$

$$\dot{\lambda}_u = \lambda_v \frac{v}{r} - \lambda_r \tag{21}$$

$$\dot{\lambda}_v = \frac{1}{r}(\lambda_v u - 2\lambda_u v - 1)$$

Combining the first three state equations from (1), with the costate equations (21), and the control law from (20), there are six first order non-linear ODE's that describe the optimal trajectory. The equations for θ and λ_θ can be dropped to decrease the computational load, since they do not affect the system dynamics. Six boundary conditions are also needed to define the system. These boundary conditions are

$$r(t_o) = r(t_f) = r_{geo} = 42164.2 \text{ km}$$

$$u(t_o) = u(t_f) = u_{geo} = 3.07466 \text{ km/sec} \tag{22}$$

$$v(t_o) = v(t_f) = v_{geo} = 0.0 \text{ km/sec}$$

The system can be solved numerically using the "shooting" method for boundary-value problems (Press *et al.*, 1989). Using this method, the initial values for λ_r, λ_u, λ_v are guessed and then the ODE's are integrated numerically to the final time. Next, the errors in the final conditions of r, u, and v are computed. Then a Jacobian matrix (relating small changes in the initial lambdas to small changes in the final states) is created numerically. Using this matrix, new guesses for λ_r, λ_u, λ_v are made, and the process is repeated until the final state errors are acceptable.

6. EVALUATION OF THE OPTIMAL STATION CHANGE

The performance of the optimal control method was evaluated using simulations to predict the behavior of a satellite during a station change maneuver, first using optimal control, and then using the tangential thrust suggested by Edelbaum. The dynamics were modeled using the equations of motion (1) and propagated with a fourth order Runge-Kutta integration scheme. The constants in the equations were chosen to represent a typical communications satellite and typical form of electric propulsion. Specifically, the following figures were developed assuming a geostationary satellite with an initial mass of 1000 kilograms and a propulsion system using an arcjet with a specific impulse of 1000 seconds. The total thrust would depend primarily on the power available, so various thrust levels for the arcjet were considered: 2.24 N, 0.224 N, and 0.0224 N.

Figure 2 compares the minimum transfer time for a station change using each method. The results for the tangential case agree well with those given in Isley and Duck, which indicates that the mass loss during the transfer is indeed largely negligible. It is also evident that the two methods produce very similar results. For the lowest thrust case, the two methods are virtually indistinguishable, while an examination of the highest thrust case reveals a slight periodic variation in which the fastest transfer rate alternates between the optimal and tangential control. This variation relates to the number of revolutions that occur during the transfer. For transfers completed over an even number of revolutions, the optimal method results in a marginally faster transfer, while for transfers with an odd number of revolutions, the tangential method is faster.

The final eccentricity for the lowest thrust (0.0224 N) family of transfers is shown in Fig. 3. The final eccentricity resulting from the tangential method is periodic, and on average, about three orders of magnitude greater than the final eccentricity resulting from the optimal method. By design, the optimal method should produce zero final eccentricity, but it is a function of the numerical accuracy of the method and machine used to compute the optimal control func-

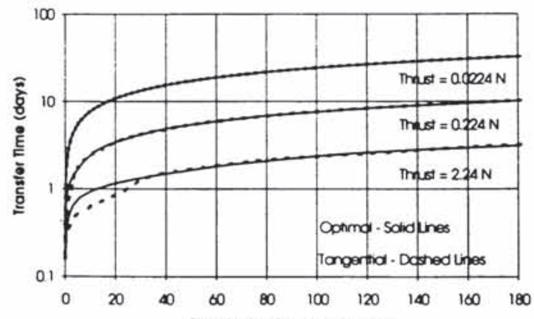

Fig. 2. Optimal Station Change Maneuver

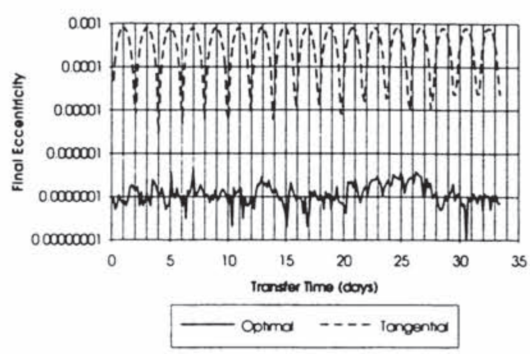

Fig. 3. Transfer Recircularization

tion. In these simulations, the final values of r, u, and v were computed to six significant figures, so the final eccentricities are zero only to six significant figures.

The eccentricity build-up caused by the tangential method can have significant ramifications for the successful operation of a communications satellite. Consider the worst case from Fig. 3, $e = 0.001$. A geosynchronous satellite with this eccentricity would have a daily station oscillation of about ±0.11°. Many communications satellites, such as INTELSAT-V and DSCS III (Martin, 1991), have station tolerance requirements of ±0.1°, so even a small eccentricity can cause the satellite to exceed its station tolerances. Thrust levels higher than the 0.0224 N used in generating Fig. 3 would result in even higher maximum eccentricities.

The eccentricity build-up of the tangential method suggests an explanation for the slight periodic variation seen in Fig. 2. The final eccentricity reaches a maximum when the duration of the transfer spans an odd number of revolutions and a minimum when the duration is an even number of revolutions. This period corresponds to a two-day cycle for short transfers, but shrinks for long transfers since the average orbital period begins to differ from the geosynchronous orbit significantly in these cases. Transfers spanning an even number of revolutions require little energy to recircularize so the optimal method outperforms the tangential method in these cases, whereas transfers spanning an odd number of revolutions build up a relatively large eccentricity so that the optimal method must divert energy from the station change to the recircularization of the final orbit. This "cost of recircularization" phenomenon is also seen in orbit raising problems using constant thrust, and is discussed in Alfano and Thorne.

Electric propulsion has been widely accepted as offering large fuel savings when compared to conventional methods, and the following simple derivation quantifies these benefits in the problem of orbit relocation, by comparing continuous thrust electric propulsion systems to conventional hydrazine propulsion systems. For satellites with hydrazine thrusters, the typical maneuver consists of two impulsive burns. The first burn establishes a drift orbit, and the second burn stops the drift and recircularizes the orbit. By simply drifting, any size of station change can be achieved with very little fuel. However, the drift *rate* is directly proportional to the ΔV used. Therefore an unbiased metric for the comparison of the two-burn and continuous thrust methods is the cost associated with producing equal, *average* relocation rates.

For the two-burn transfer, the relocation rate, $\dot{\sigma}$, depends on the difference of the angular velocity of the geosynchronous and transfer orbits.

$$\dot{\sigma} = \omega_t - \omega_{\oplus} \tag{23}$$

Or in terms of the difference in semi-major axis

$$\dot{\sigma} = -\frac{3}{2}\frac{\Delta a}{a_{geo}}\omega_{\oplus} \tag{24}$$

where a binomial expansion truncated to first order has been used to simplify the expression. Then, using simple relationships obtained by differentiating the energy equation, equation (24) becomes

$$\Delta V = -\frac{r_{geo}}{3}\dot{\sigma} \tag{25}$$

This equation gives the ΔV needed to generate a given drift rate. The total two-burn transfer would require a stopping maneuver as well, so the total ΔV is twice is twice this expression or

$$\Delta V_{\text{two-burn}} = -\frac{2}{3} r_{geo}\dot{\sigma} \tag{26}$$

Note that the rate, $\dot{\sigma}$, is the average rate, since it is constant throughout the transfer.

For the continuous thrust case, determining the appropriate drift rate is slightly more complex, since it cannot be considered constant during the transfer. One approach is to integrate the drift rate over the entire transfer to compute the total station change, and then divide by the transfer time to obtain an average drift rate. Start with equation (25), and substitute $(F\Delta t)/m_0$ for ΔV, giving

$$\dot{\sigma} = -\frac{3F\Delta t}{a_{geo} m_0} \tag{27}$$

If the semi-major axis is assumed to be approximately constant during the transfer (which is reasonable for drift rates up to about 30°/day), then this equation can be integrated to obtain

$$|\sigma| = \frac{3}{2}\frac{|F|}{a_{geo} m_0} T^2 \tag{28}$$

This expression considers only one-half of the transfer. In reality, the vehicle would thrust in two opposite directions, each for one half of the transfer. For low mass loss, this is essentially a symmetric process, so the total station change is twice the station change achieved in the first half of the transfer. In the equation this becomes

$$\sigma = 2\left(\frac{3}{2}\frac{|F|}{a_{geo}m_0}\left(\frac{T}{2}\right)^2\right) = \frac{3}{4}\frac{|F|}{a_{geo}m_0}T^2 \qquad (29)$$

Substituting $\dot{\sigma}_{average} = \sigma/T$ and $\Delta V_{continuous} = FT/m_0$, this becomes

$$|\Delta V|_{continuous} = \frac{4}{3}r_{geo}|\dot{\sigma}|_{average} \qquad (30)$$

A comparison of equations (25) and (30) shows that a high thrust system is twice as efficient as low thrust system in terms of ΔV. The advantage of electric propulsion is only seen when the ΔV is written in terms of the fuel used. The impulse can be approximated by

$$\Delta V \approx \frac{F\Delta t}{m_0} = \frac{I_{sp}g\dot{m}\Delta t}{m_0} = \frac{I_{sp}g\Delta m}{m_0} \qquad (31)$$

Substituting this expression for ΔV into equations (25) and (30) and solving for Δm (fuel used) gives

$$\Delta m_{two\text{-}burn} = \frac{2r_{geo}m_0}{3I_{sp}g}\dot{\sigma}_{average} \qquad (32)$$

$$\Delta m_{continuous} = \frac{4r_{geo}m_0}{3I_{sp}g}\dot{\sigma}_{average} \qquad (33)$$

These expressions indicate that the continuous system needs a specific impulse only twice that of the high thrust system to eliminate its advantage, a figure which is easily attainable. For example, the most common type of propulsion system today uses monopropellant hydrazine with a specific impulse of about 200-225 seconds, while a typical arcjet can have a specific impulse of more than 1000 seconds (Larson and Wertz, 1992). The inefficiency of low thrust maneuvers is overshadowed by the efficiency of the low thrust propulsion systems, clearly demonstrating their value for use on satellites for a variety of tasks.

7. CONCLUSION

Constant low thrust should be used to perform station change maneuvers on future satellite systems. The derivation presented here highlights the fuel savings achievable by low thrust electric propulsion systems compared with conventional propulsion systems. This reduction in fuel costs for station changes translate to longer life, and more frequent or more rapid station changes. This allows greater flexibility and improved overall performance of satellite systems.

The application of optimal control techniques to the problem of relocating satellites in circular orbit with constant thrust has a significant advantage over tangential thrust control. The optimal method ensures the new orbit has zero eccentricity, while achieving a an essentially identical rate of relocation. By constraining the final eccentricity, significant daily variations in the station are avoided. This is an important factor in minimizing stationkeeping effort following a station change maneuver. While the optimal control method is an improvement over tangential thrusting, the analytic solution quickly determines maneuver time for any required station change. This estimate is excellent for planning, since the optimal maneuver will involve nearly tangential thrusting in most cases.

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