

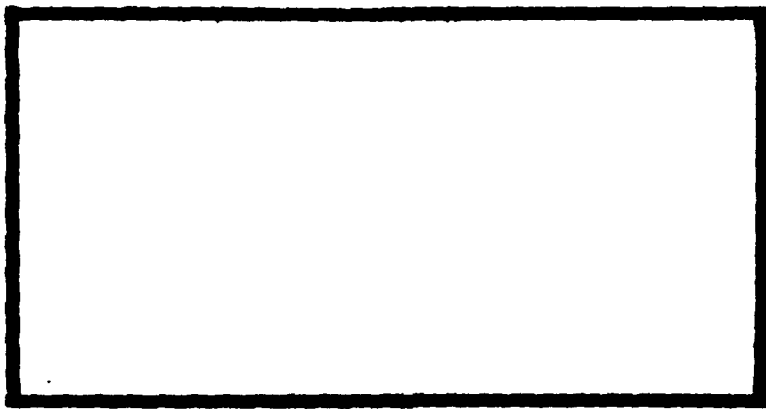
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LEVEL II



ANALYTICAL EQUATIONS FOR ORBITAL TRANSFER  
MANEUVERS OF A VEHICLE USING CONSTANT LOW THRUST

THESIS

AFIT/GA/AA/81D -3

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ANALYTICAL EQUATIONS FOR ORBITAL TRANSFER MANEUVERS  
OF A VEHICLE USING CONSTANT LOW THRUST

THESIS

Presented to the Faculty of the School of Engineering  
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in Partial Fulfillment of the  
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Master of Science

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### List of Symbols

$g$	- gravitational acceleration
$X$	- direction
$Y$	- direction
$Z$	- direction
$T$	- thrust/unit mass
$\theta$	- radius position angle
$r$	- radius
$E_c$	- specific energy of a circular orbit
$\mu$	- gravitational parameter
$\frac{d}{dt} ( )$	- time derivative
$F$	- force/unit mass
$V$	- velocity
$\bar{B}$	- vector quantity (B arbitrary)
$t$	- time
$( )_i$	- initial value
$( )_f$	- final value
$\Delta r$	- change in radius
$TP$	- orbital period
$( )_c$	- circular orbit value
$h$	- specific angular momentum
$\Delta h$	- change in specific angular momentum
$\dot{B}$	- $\frac{dB}{dt}$ (B arbitrary)
$i$	- inclination
$\Delta i$	- change in inclination
$M$	- moment



$dt$	- differential change in time
$d\theta$	- differential change in radius position angle
$\dot{\theta}$	- angular velocity
$J$	- Performance index
$A$	- unknown parameters
$\frac{d}{dA}(\ )$	- partial derivative with respect to A
$\nabla$	- gradient
$S$	- Conjugate - Gradient search direction
$\alpha$	- Conjugate - Gradient step size
$\beta$	- Conjugate - Gradient memory function
$\tau$	- psuedo throttle control
$\bar{B} \cdot \bar{C}$	- vector dot product (B and C arbitrary)
$\bar{B} \times \bar{C}$	- vector cross product (B and C arbitrary)
$\delta B$	- small change in B (B arbitrary)
$(\ )_{\max}$	- maximum value
$w$	- thrust time angle
$R$	- nondimensional radius
$(\ )_{\text{nom}}$	- specified values

Abstract

The object of this study is to derive a set of equations which predict the results of orbital maneuvers of vehicles using constant low thrust. These equations are developed by simplifying the problem to circular orbits in a two - body dynamic system. The results are presented in three parts. The first part solves for the equation of the coplanar radius change problem. The second finds the equation for the minimum fuel inclination change problem. The third part of this study puts the equations from the first two parts together to solve a minimum fuel transfer problem involving both radius and inclination changes.

The solution of the minimum fuel transfer problem is not to perform the total radius change and then perform the total inclination change. Instead, the solution is to perform both radius and inclination changes on a per orbit basis.

ANALYTICAL EQUATIONS FOR ORBITAL TRANSFER MANEUVERS  
OF A VEHICLE USING CONSTANT LOW THRUST

I Introduction

The launch of the Space Shuttle Columbia ushered in a new era in the exploration and utilization of space. Because of the reusability of this vehicle, the frequency and length of spaceflights will increase significantly. As more and more work and research is performed in Earth orbit, the need for new and more fuel efficient propulsion systems will arise. Since February 1970, one of these new propulsion systems has been orbiting the Earth. This system, a mercury bombardment ion thruster, is onboard the SERT II (Space Electric Rocket Test II) satellite. It was this satellite which proved that a constant low thrust could alter the orbit of a satellite (Ref 1). In the years since 1970, major advances in fuel efficiency and thruster service life have been made for electric propulsion systems (Ref 1,2).

But how can a low level of thrust change the orbit of a satellite? The answer lies in the fact that the thrust is constant over a long period of time. The velocity of a four metric ton satellite thrusting at 0.8 Newtons for 30 days can be changed by as much as 1,866.0 kilometers per hour. This velocity change can be achieved with substantial fuel savings using electric propulsion instead of a more conventional chemical propulsion system (Ref 1).

Changing the velocity of a satellite such as described above can be used to boost a satellite into a planetary mission orbit. This is only one of the many uses of electric propulsion systems. Some of the more immediate uses will be as stationkeeping, attitude, and repositioning jets

for satellites. Future uses include the placement and retrieval of satellites in geosynchronous orbit and the movement of large fragile space structures.

As mentioned above, one of the more immediate uses of constant low thrust will be in the repositioning of satellites. This repositioning maneuver will require changes in the radius and the inclination of the satellite's orbit. Reference 3 (chapter 3) derives a set of equations which give the change in velocity needed to achieve certain changes in radius and inclination. But these equations can not be used for a satellite using constant low thrust because they were derived with the assumption that the vehicle's velocity vector could be changed instantaneously. This is a valid assumption for high thrust chemical propulsion, but not for low thrust electric propulsion. Therefore, a set of equations must be derived for the constant low thrust case.

This paper has three objectives. The first objective is to derive an equation which will give the amount of radius change per orbit of a vehicle using constant low thrust. It will be assumed that the vehicle stays in the plane of its starting circular orbit. This makes the problem similar to the Hohmann Transfer solved in chapter 3 of Reference 3. The solution of the constant low thrust maneuver will be presented in chapter 2 of this paper.

The second objective is to derive an equation which will give the amount of inclination change of a vehicle, similar to the one mentioned above, using a minimum fuel control profile. It will be assumed that the vehicle will start and finish in a circular orbit and only change its inclination. Similar equations for the instantaneous high thrust case can be found in chapter 3 of Reference 3. The equations for the constant low thrust case will be derived in chapter 3 of this paper.

The third objective is to join the equations derived in this study for radius and inclination changes so that the total orbital transfer problem can be solved optimally. The problem is to be solved by finding the amount of each type of change which must be made per orbit to reach the end conditions in a specified number of orbits.

## II The Coplanar Radius Change Problem

The transfer between any two coplanar circular orbits is a very simple and useful maneuver. Today, this maneuver can be accomplished by a number of different ways. Among these, the most economical is the Hohmann transfer ellipse (Ref 3: 163-165). But these methods are based on two assumptions that are not valid when the vehicle in question is using low thrust. The first assumption is that the vehicle can change its velocity vector instantaneously. This assumption is violated since, with constant low thrust, it may take hours or days to significantly change the vehicle's velocity vector. The second assumption is that the vehicle can accelerate against the pull of the earth's steep gravity well. Since the acceleration of today's constant low thrust propulsion systems is on the order of  $10^{-4}$ g's, the second assumption is no longer valid. Therefore, with these two restrictions in mind, a new maneuver and a new equation describing the results of the maneuver must be sought.

The search for the new maneuver began by looking at how the radius of a circular orbit can be changed without violating the above restrictions. To overcome the strong pull of gravity, the vehicle must thrust in a direction which is along gravity's weakest lines. This direction happens to be perpendicular to the lines of force of the earth's gravitational field. This direction is parallel to the vehicle's velocity vector. It can be readily seen that if the vehicle thrusts in this direction the result is a spiral. This type of maneuver fulfills the objective of the above maneuvers while at the same time does not violate the above restrictions. Therefore, this will be the maneuver used to change the radius of a circular orbit using constant low thrust.

Now that the maneuver has been defined, an equation must be derived which will describe the results of such a maneuver. If the thrust is low enough, the spiral can be modeled as a series of concentric circular orbits with a discrete change in the radius at the point on the circular orbit where the thrusting was started. By looking at two of the concentric circular orbits and the curve that joins them (see Fig 1), the equation which gives the amount of radius change can be found.

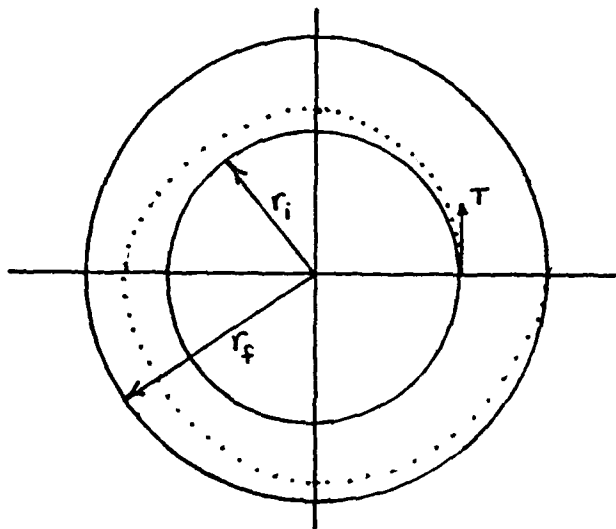


Fig 1: Spiral Approximation

The equation for the specific energy of a circular orbit is given by

$$E_c = - \frac{\mu}{2r} \quad (1)$$

where  $r$  is equal to the radius of the circular orbit. The time derivative of Eq 1 can be gotten by using elemental differential calculus. This derivative is simply

$$\frac{d(E_c)}{dt} = \frac{\mu}{2r^2} \frac{d(r)}{dt} \quad (2)$$

Since the energy of the circular orbit can only be increased by thrusting in the direction parallel to the velocity vector, the time derivative of  $E_c$  can be written as

$$\frac{d(E_c)}{dt} = \bar{F} \cdot \bar{V} \quad (3)$$

where  $\bar{F}$  is a specific force (force/unit mass) and  $\bar{V}$  is a velocity. Equations 1 and 2 can be equated and the  $\dot{r}$  term solved for to get the following equation:

$$\frac{d(r)}{dt} = (\bar{F} \cdot \bar{V}) \frac{2r^2}{\mu} \quad (4)$$

The first term of Eq 4 can be evaluated as the magnitude of the two vectors multiplied by the cosine of the angle between them. Since the thrust vector is aligned with the velocity vector, the angle between them is zero and the cosine term becomes one. Also the  $\bar{F}$  and  $\bar{V}$  terms can be evaluated as the thrust per unit mass and the velocity of a circular orbit, respectively. This can be done since the distance between the initial and final circular orbits is very small. Inserting these expressions into Eq 4 gives the following formula:

$$\frac{d(r)}{dt} = 2 T \sqrt{\frac{r^3}{\mu}} \quad (5)$$

The next step in the solution of this problem is to separate Eq 5 and integrate each side with the proper limits. This leads to the solution form which gives the change in radius using constant low thrust as a function of the thrust per unit mass and the time set to achieve the



change. This solution equation is

$$r_i^{-\frac{1}{2}} - r_f^{-\frac{1}{2}} = \frac{T}{\sqrt{\mu}} (t_f - t_i) \quad (6)$$

Since the spiral has been modeled as a circular orbit with a discrete change in the radius at the point where the thrusting was started,  $r_f$  can be written in the following form:

$$r_f = r_i + \Delta r \quad (7)$$

This allows Eq 6 to be written as

$$\Delta r = \frac{1}{\left[ \frac{T}{\sqrt{\mu}} (t_f - t_i) - \frac{1}{\sqrt{r_i}} \right]^2} - r_i \quad (8)$$

The  $(t_f - t_i)$  term can be interpreted as the period of a circular orbit because the  $\Delta r$  is very small. From Reference 3 (33), the period of a circular orbit is

$$T_p = \frac{2\pi}{\sqrt{\mu}} r^{\frac{3}{2}} \quad (9)$$

This allows Eq 8 to be written in its final form as

$$\Delta r = \frac{1}{\left[ \frac{2\pi r_i^{3/2}}{\mu} - \frac{1}{\sqrt{r_i}} \right]^2} - r_i \quad (10)$$

where  $\Delta r$  is the radius change per orbit using constant specific thrust,  $T$ , at radius  $r_i$ . Now that the equation for the coplanar radius change has been found the next step is to find the solution of the inclination change problem.

### III The Inclination Change Problem

The second chapter of this paper dealt with the transfer between two coplanar circular orbits. But a satellite may need more than a radius change, it may also need an inclination change. It is the object of this chapter to develop the equation which describes the inclination change.

#### Setting Up the Inclination Change Problem

In chapter 2, two restrictions were placed on the coplanar circular radius change problem. These two restrictions must be observed whenever constant low thrust is used. Just as in the radius change problem, the thrust must be oriented in a direction which is along a line perpendicular to the lines of force of the Earth's gravity gradient. In the radius change problem, this direction was chosen to be parallel to the velocity vector of the vehicle. However, in this problem the direction is perpendicular to both the gravity gradient and the velocity vector. Figure 2 shows the orientation of the thrust vector, as described above, to change the orbital plane in a counterclockwise direction about the x-axis.

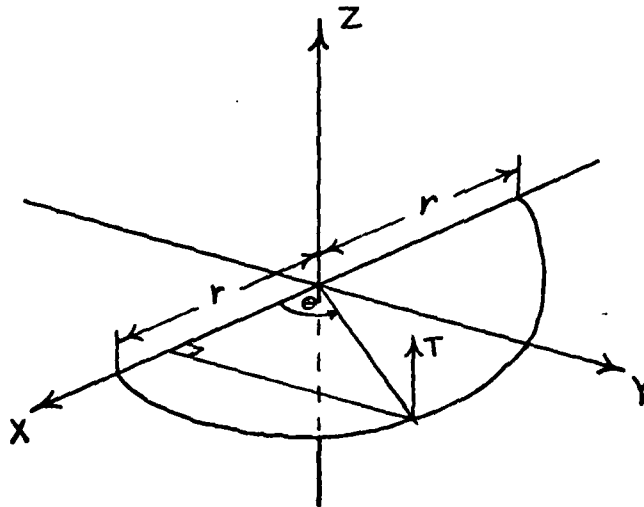


Fig 2: Thrust Vector Orientation

The equation for the inclination change problem can be found by looking at the specific angular momentum equations which describe a two - body circular orbit. Reference 3 (16-17) proves that for a two - body orbit the specific angular momentum of the orbit is constant and can be given by

$$\bar{h} = \bar{r} \times \bar{v} \quad (11)$$

It also proves that  $\bar{r}$  and  $\bar{v}$  form a plane that is fixed in space. This plane is called the orbital plane. Since  $\bar{h}$  is the vector cross product of  $\bar{r}$  and  $\bar{v}$ ,  $\bar{h}$  must be perpendicular to the orbital plane. The inclination of the orbital plane is defined in Reference 3 (58-59) as the angle,  $i$ , between  $\bar{k}$  axis of the Geocentric - equatorial coordinate system and the  $h$  vector (see Fig 3).

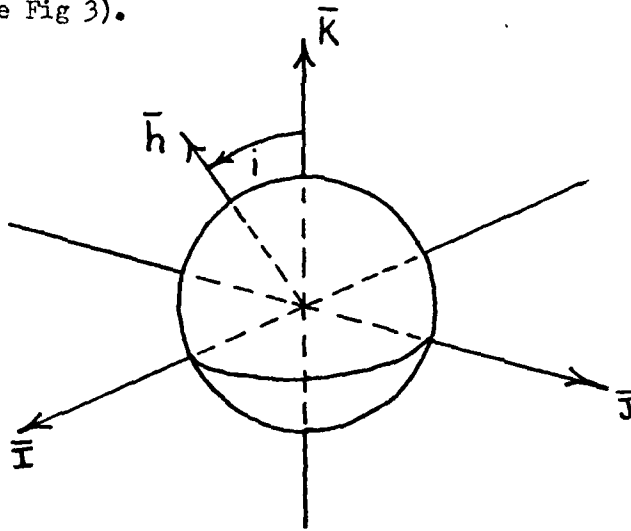


Fig 3: Geocentric - equatorial Reference Frame.

From Fig 4, it can be seen that the change in  $\bar{h}$  is defined as

$$\Delta \bar{h} = \bar{h}_2 - \bar{h}_1 \quad (12)$$

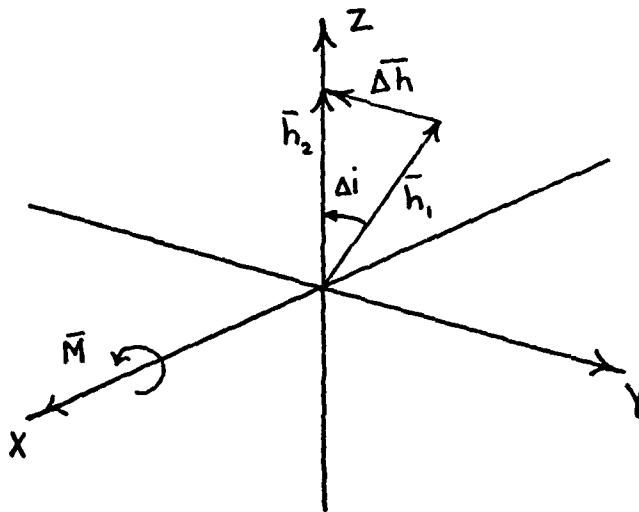


Fig 4: Inclination Change

The angle between  $\bar{h}_2$  and  $\bar{h}_1$  is the change in inclination between the two orbit planes. As this change in  $\bar{h}$  becomes small, the angle between  $\Delta \bar{h}$  and  $\bar{h}_1$  can be approximated by a right angle. Therefore, the angle between  $\bar{h}_1$  and  $\bar{h}_2$  becomes

$$\sin \Delta i = \frac{\Delta h}{h} \quad (13)$$

Since this equation holds only for  $\Delta \bar{h}$  very small, Eq 13 can be further simplified by using a small angle approximation for the sine of  $\Delta i$ .

Equation 13 can now be written as

$$\Delta i = \frac{\Delta h}{h} \quad (14)$$

From Fig 4, it can be seen that to change the inclination in the counterclockwise direction, a moment must be applied about the x-axis. From Reference 4 (23), the moment about the x-axis can be defined as the time derivative of the specific angular momentum. In equation form it is

$$\dot{\bar{h}} = \bar{M} \quad (15)$$

Equation 15 can be rewritten as

$$\frac{d(\bar{h})}{dt} = \bar{M} \quad (16)$$

This form can be separated and substituted into Eq 14 to get the following form for the change in inclination:

$$di = \frac{M}{h} dt \quad (17)$$

The next step in this derivation is to find the equation for the moment about the x-axis. From Fig 2, it is seen that the moment about the x-axis is produced by the component of the thrust vector perpendicular to the orbital plane. Therefore, the moment can be written as

$$\bar{M} = \bar{r} \times \bar{T} \quad (18)$$

From the notation in Fig 2 and the definition of the vector cross product, Eq 18 can be written as

$$M = r T \sin \theta \quad (19)$$

The substitution of Eq 19 into Eq 17 allows the change in inclination to be written as

$$di = \frac{rT}{h} \sin \theta dt \quad (20)$$

The change in time,  $dt$ , of Eq 20 can be written as

$$dt = \frac{d\theta}{\dot{\theta}} \quad (21)$$

or

$$dt = \frac{r}{v} d\theta \quad (22)$$

The substitution of Eq 22 into Eq 20 allows the change in inclination to be written as a function of  $\theta$  only since  $r$  and  $v$  are constants in a circular orbit. The new equation for the inclination change is

$$di = \frac{r^2 T}{hV} \sin \theta d\theta \quad (23)$$

The next step in this derivation is to integrate the right hand side of Eq 23 from 0 to  $2\pi$ . But, as stated above,  $r$ ,  $h$ , and  $v$  are all constant in a circular orbit. This leaves the specific thrust (thrust/unit mass) in the integral.

It can be readily seen that the greatest amount of inclination change is achieved by allowing the thrust to be constant over the entire orbit. However, this wastes fuel because very little inclination change is achieved when the vehicle is close to the x-axis. If fuel usage is not

a problem, then this is the way to achieve the greatest amount of inclination change in the shortest amount of time. But on most spaceflights, fuel must be carefully conserved. Therefore thrust histories which are functions of  $\theta$  are of interest. This generalization allows Eq 23 to be written as

$$\Delta i = \frac{r^2}{hV} \int_0^{2\pi} T(\theta) \sin \theta \, d\theta \quad (24)$$

The substitution of the definitions of  $h$  and  $v$  from Reference 3 (29,34) allows Eq 24 to be written in its final form as

$$\Delta i = \frac{r^2}{\mu} \int_0^{2\pi} T(\theta) \sin \theta \, d\theta \quad (25)$$

If the form of  $T(\theta)$  is specified, Eq 25 can be integrated to find  $\Delta i$ . Since fuel has to be conserved, the logical form for  $T(\theta)$  needs to be chosen to minimize the fuel consumption. This form for  $T(\theta)$  can be found by solving an optimal control problem where the performance index, fuel consumption, to be minimized is given by

$$J = \int_0^{2\pi} |T(\theta)| \, d\theta \quad (26)$$

The minimum of Eq 26 without a constraint is obviously zero. Therefore Eq 25 with  $\Delta i$  specified is used as a constraint. This optimal control problem is solved in the following sections of this chapter.

### Suboptimal Control Approach

The solution of all but the simplest optimal control problems is very difficult to find. Therefore certain methods have been developed to approximate the solution of the optimal control problem (Ref 5). The method used to solve the inclination change problem is the suboptimal control method.

Suboptimal Control Problem. The suboptimal control problem is a parameter optimization problem. The transformation from optimal control to suboptimal control is achieved by assuming that the optimal controls of the problem in question can be approximated by a known mathematical form having a set of unknown parameters. When this transformation is used, the performance index,  $J$ , becomes

$$J = J(A) \quad (27)$$

where  $A$  is the set of unknown parameters associated with suboptimal approximation. The optimal  $A$ 's are found by solving

$$\frac{d(J)}{dA} = 0 \quad (28)$$

Solution Method. There are many methods used to solve the parameter optimization problem stated in the preceding section. The method chosen to solve the inclination problem is the Conjugate - Gradient method (Ref 6).

The Conjugate - Gradient method is a quasi - second order technique for solving parameter optimization problems. It is very similar to the Method of Steepest Descent otherwise known as the gradient method. The only major difference between the two is that the Conjugate - Gradient



method carries along previously calculated gradient directions in a fading memory form. It is this fading memory that gives this method its second order characteristics.

The iterative algorithm for this method is as follows:

- 1) Guess  $A^j$
- 2) Compute  $\nabla J(A^j)$
- 3) Let  $S^j = -\nabla J(A^j)$
- 4) Compute  $A^{j+1} = A^j + \alpha_*^j S^j$
- 5) Compute  $\nabla J(A^{j+1})$
- 6) If  $\nabla J(A^{j+1}) \leq \text{Tolerance}$ : Stop
- 7) Compute  $S^{j+1} = -\nabla J(A^{j+1}) + \beta^j S^j$
- 8) Increment  $j$ , go to step 4

There are several terms in the above algorithm that need to be defined. The first is the  $\alpha_*^j$  term of step 4. This term is a step size that is computed with a 1 - dimensional search method found in Reference 7 (55). This search method finds the minimum of a parabola in the  $J$  vs  $\alpha$  plane. The  $S$  terms are directions along which the algorithm searches for the optimal  $A$ 's. The  $\beta^j$  term is the fraction which tells the algorithm how much of the previous  $S$ 's should be used. This is the fading memory of the Conjugate - Gradient method. The  $\beta^j$  term is determined by the following formula.

$$\beta^j = \frac{|\nabla J(A^{j+1})|^2}{|\nabla J(A^j)|^2} \quad (29)$$

It should be noted that the same procedure which gives this technique its second order characteristics can also slow the convergence of the algorithm. Therefore, the  $\beta^j$  is set to zero periodically to prevent the algorithm from carrying too much past information. A more detailed explanation of the Conjugate - Gradient method can be found in Reference 6.

### Solving the Constrained Inclination Problem

Control Variable Form. It is obvious that the direction of the thrust vector must be in opposite directions on either side of the x-axis to achieve the inclination change represented in Fig 4. This means that  $T(\theta)$  is an odd function. Also the form of  $T(\theta)$  must be symmetric about the  $90^\circ$  point of the orbit. It is at this point that the greatest moment, and the corresponding greatest change in inclination, is achieved by the thrust. Because of these two qualifications, a Fourier sine series with the even numbered terms removed was chosen to be the approximate optimal control form (Ref 8: 35-36, 80; 9). This form is

$$T(\theta) = \sum_{k=1}^{\infty} (A_{2k-1} \sin(2k-1)\theta) \quad (30)$$

Now that the form for  $T(\theta)$  has been specified, the integration of Eq 25 can be performed. The evaluation of the constraint equation results in all the terms except the  $\sin \theta$  term being zero. This allows the first coefficient of the Fourier sine series to be solved for. This coefficient is

$$A_1 = \frac{\Delta i \mu}{r^2 \pi} \quad (31)$$

Thus for a given inclination change, the first coefficient of the Fourier sine series is known. The remaining coefficients, needed to give the minimum fuel consumption, are unknown. Since the first coefficient is known, it can be divided out. This results in  $T(\theta)$  having the following form.

$$T(\theta) = \frac{\Delta i \mu}{r^2 \pi} \tau \quad (32)$$

where  $\pi$  is given by

$$\pi = \sin \theta + \sum_{k=2}^{\infty} (A'_{2k-1} \sin (2k-1)\theta) \quad (33)$$

where

$$A'_{2k-1} = \frac{A_{2k-1}}{A_1} \quad (34)$$

With this form for  $T(\theta)$ , Eq 25 no longer needs to be used as a constraint. Now only the A's need to be found for minimum fuel usage. But because of this new form for  $T(\theta)$ , a new constraint must be used. The first term of Eq 25 can be looked at as the maximum thrust, while the second term looks like a throttle control. If no constraint is put on the throttle, the solution of the optimal control problem would be an infinite spike at the  $90^\circ$  point of the orbit. Therefore, Eq 33 has the following limits:

$$0 \leq \pi \leq 1 \quad (35)$$

Numerical Solution. The algorithm for the Conjugate - Gradient method requires the calculation of the gradient of the performance index with respect to the A's. These gradients are found using the central difference formula described in Reference 10. This formula has the following form:

$$\frac{d(J)}{dA_{2k-1}} = \frac{J(A_{2k-1} + \delta A_{2k-1}) - J(A_{2k-1} - \delta A_{2k-1})}{2 \delta A_{2k-1}} \quad (36)$$

where  $\delta A_{2k-1}$  was calculated by

$$\delta A_{2k-1} = 1E-04 A_{2k-1} \quad (37)$$

To find the perturbed J's, the integral of Eq 26 has to be evaluated. Since the integrand of this integral involves an absolute value, the areas above and below the x-axis of the T vs  $\theta$  curve have to be summed. This means that the roots of the control equation have to be found.

Two methods from Reference 10 (65-71) are used to find the roots. The first method is the Bisection method. This is a brute force method which is used to isolate the root. When the root is isolated, the Secant method is used to improve the accuracy of the answer. The Secant method is a Newton - Raphson method where the derivatives are replaced with difference approximations.

Included in the root searching algorithm is an algorithm which finds the points where the value of  $T(\theta)$  is greater than  $T_{\max}$ . Since the thrust cannot be greater than  $T_{\max}$ , the area above this line cannot be included in the calculation of the performance index. Therefore, when the integral is evaluated, the value of the integrand is constant between the points where the  $T(\theta)$  curve lies above the  $T_{\max}$  line. The area where the integrand is constant is calculated by using the formula for the area of a rectangle. The areas on either side of the rectangle are calculated using a Runge - Kutta numerical integrator with a variable step size. Since the areas on either side of the rectangle are the same, only one of them has to be calculated. The total area is twice this value plus the area of the rectangle.

The algorithm was started with only one A in Eq 33. The A was changed until the gradient of the performance index was less than or equal to a set tolerance. The solution was then said to be converged and another A was added. A's were to be added to Eq 33 until the performance index failed to change with the new addition. But before

this happened, the solution to the optimal control problem was suggested from the partial results of the program.

Solution Results. The Conjugate - Gradient program described above was run until nine  $\Lambda$ 's had been added to the optimal control equation. The plot of this equation can be seen in Fig 5, but the program did not use the total equation. It used the equation until it went above the  $T_{\max}$  line and then followed the  $T_{\max}$  line until it dropped off to follow the  $T(\theta)$  equation again. This form can be seen in Fig 6. If the small wiggles are considered to be noise, then the form which emerges is a square wave centered about the  $90^\circ$  point of the orbit. To test this idea, a nine term square wave approximation was used to compute a performance index. The value of the performance index of the square wave was substantially lower than the performance index of the nine term Fourier sine series. Since it takes up to forty terms to accurately describe a square wave, the performance index can be drastically lowered by using the square wave instead of the Fourier sine series. Therefore, it was concluded that the solution of the constrained inclination change optimal control problem is a square wave with  $T_{\max}$  as its maximum.

#### Bang - Coast - Bang Solution

The square wave solution of the inclination change problem was suggested by the results of the suboptimal control problem outlined in the preceding sections. A plot of the square wave can be seen in Fig 7.

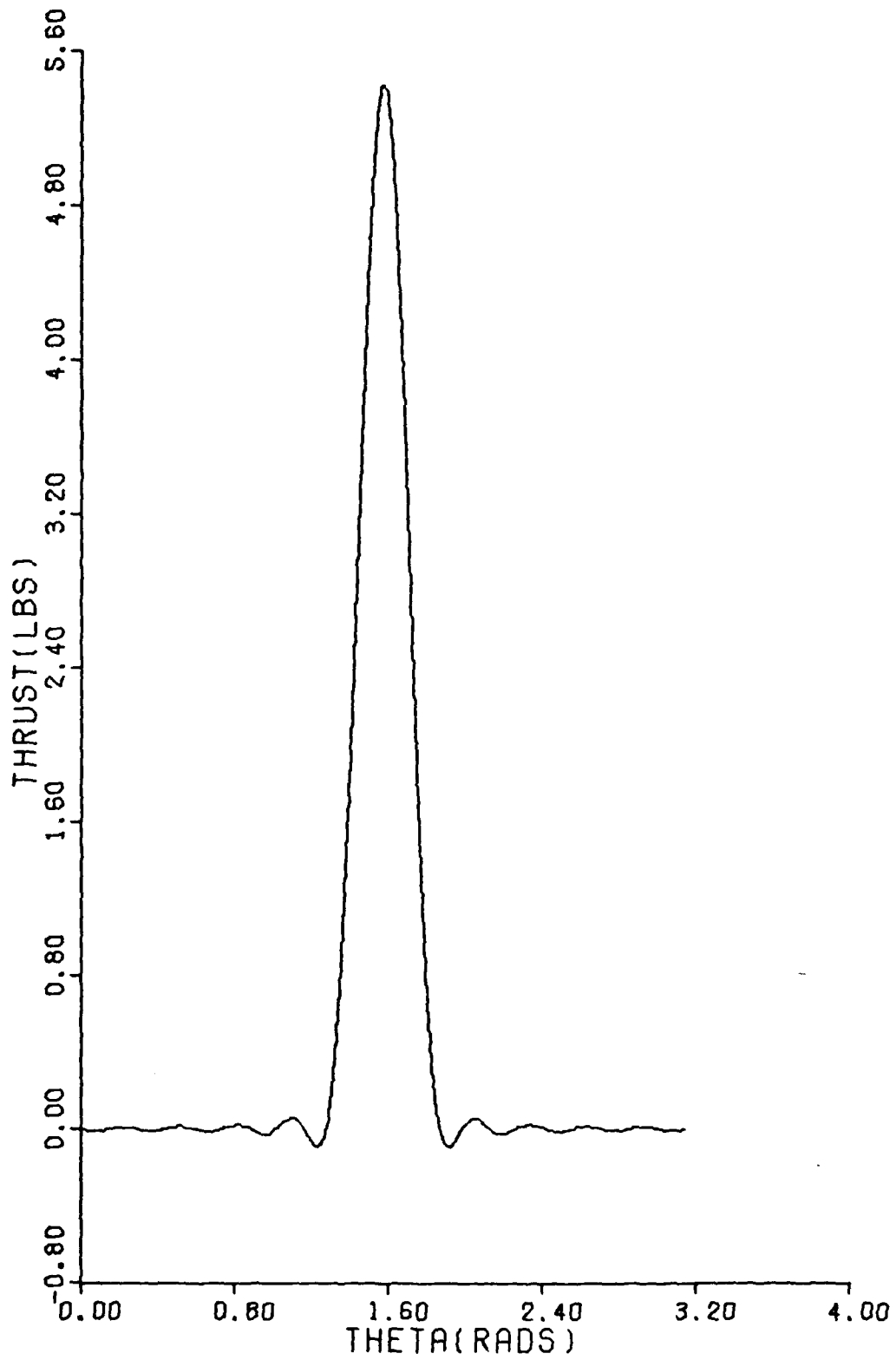


Fig 5: Optimum Nine Term Fourier Sine Series

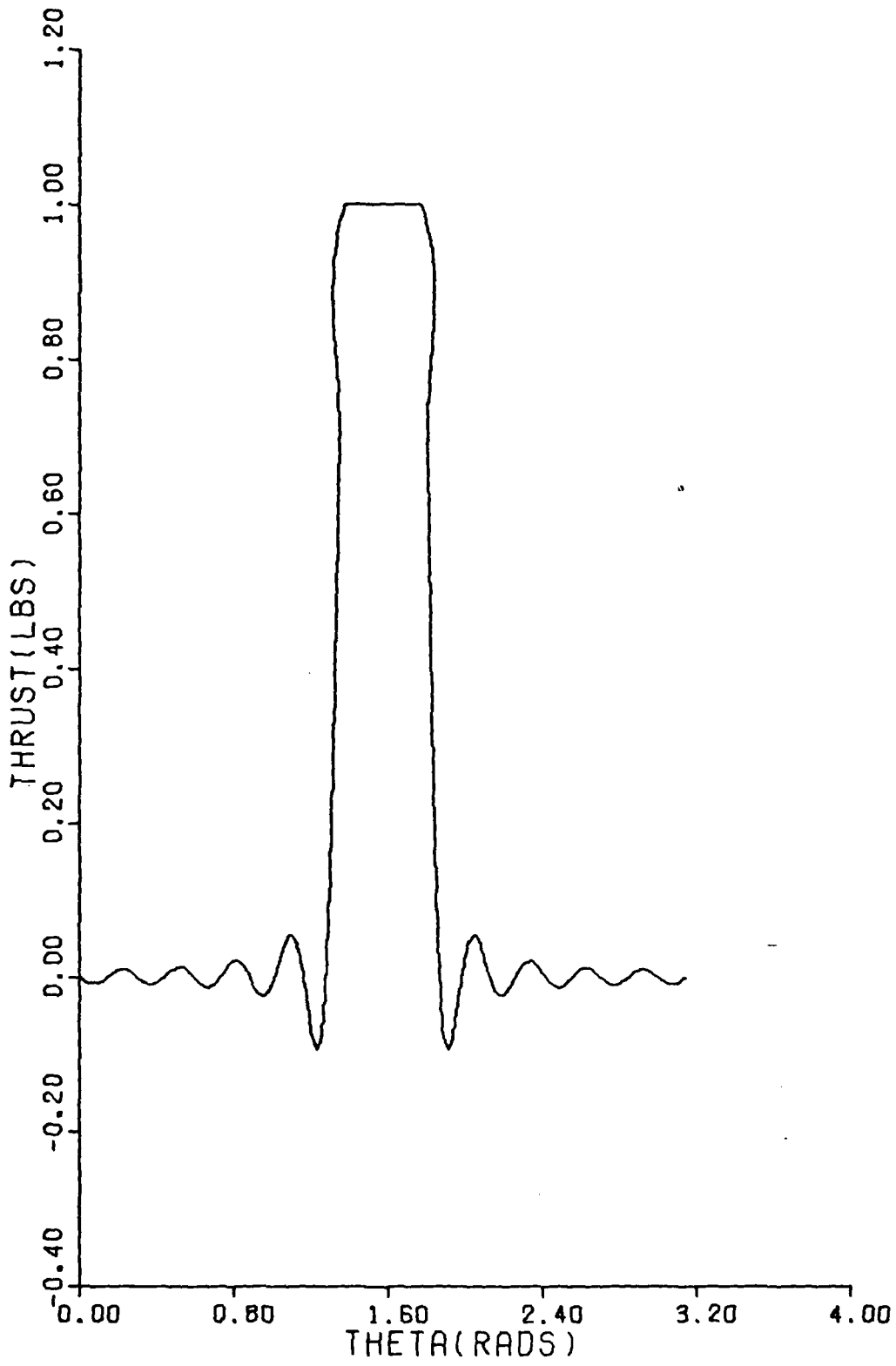


Fig 6: Constrained Optimum Control Profile

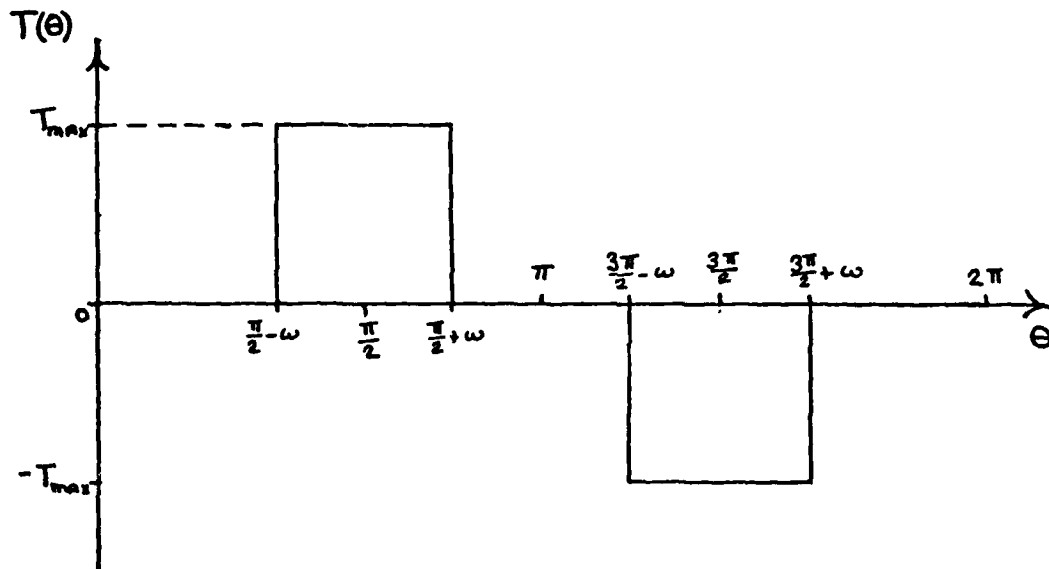


Fig 7: Square Wave

To find the amount of inclination change this control profile gives, Eq 25 must be integrated between the limits set forth in the graph with  $T(\theta) = T_{max}$ . Equation 25 can be rewritten as

$$\Delta i = \frac{r^2 T_{max}}{\mu} \left[ \int_{\frac{\pi}{2} - \omega}^{\frac{\pi}{2} + \omega} \sin \theta \, d\theta - \int_{\frac{3\pi}{2} - \omega}^{\frac{3\pi}{2} + \omega} \sin \theta \, d\theta \right] \quad (38)$$

To evaluate Eq 38, two integrals must be solved. Since thrusting on either side of the orbit achieves a change in the inclination, the integral can be simplified to the following form:

$$\Delta i = \frac{2r^2 T_{max}}{\mu} \int_{\frac{\pi}{2} - \omega}^{\frac{\pi}{2} + \omega} \sin \theta \, d\theta \quad (39)$$

The evaluation of this integral and the use of trigometric formulas to simplify the integrated form, leads to

$$\Delta i = \frac{4r^2 T_{max}}{\mu} \sin \omega \quad (40)$$



where  $i$  is the inclination change per orbit using constant specific thrust,  $T_{\max}$ , at radius  $r$  for  $w$ , the thrust time angle.

Now that the equations for the radius and inclination changes have been found, the two separate equations must be joined to form an equation which can be used to describe the total orbital transfer problem.

#### IV The Combined Orbital Transfer Problem

In the preceding two chapters, the problems of the coplanar circular radius change and the inclination change were solved. Although the general combined orbital transfer problem could not be solved, the solution of a specific example problem was found. It is the purpose of this chapter to define and solve this problem.

##### Defining the Orbital Transfer Problem

The orbital transfer problem can be stated as follows: Given the initial and final values of the radius and inclination and the number of orbits in which to achieve these changes, what amounts of radius and inclination changes must be attained per orbit to match the final values? To simplify the problem, two assumptions were made. The first was that the vehicle is using full thrust in either the radius change direction or the inclination change direction, but never dividing the thrust between the two directions. The second assumption was that the direction of the thrust vector can be changed instantaneously.

There are two methods which can be used to achieve the final radius and inclination. The first is to thrust parallel to the velocity vector, changing the radius, until the final radius is achieved. Then the thrust is directed perpendicular to the orbital plane, changing the inclination, until the final inclination is achieved. The second method is to change both the radius and the inclination in a given orbit. It is the second method which is the most fuel and time efficient. The inclination change equation, Eq 40, is in the ideal form to handle this situation. But the radius change equation, Eq 10, is not. Therefore, Eq 10 must be modified to allow for this situation.

### Modification of the Radius Change Equation

Equation 10 was derived with the assumption that the entire period of the circular orbit would be used for the radius change. But for the total orbital transfer problem, the time spent changing the inclination must be subtracted from the period of the orbit. Since the inclination thrust time angle,  $w$ , of Eq 40 is in radians, the equation for the angular velocity of a circular orbit must be used to make the change to seconds. Also it must be realized that this change has to be multiplied by four since there are four such times per orbit. The new equation for the time to change the radius is

$$TP = TP_c - 4\omega \sqrt{\frac{r^3}{\mu}} \quad (41)$$

Equation 10 can now be written in the more usable form of

$$\Delta r = \frac{1}{\left[ \frac{T}{\sqrt{\mu}} \left( TP_c - 4\omega \sqrt{\frac{r_i^3}{\mu}} \right) - \frac{1}{\sqrt{r_i}} \right]^2} - r_i \quad (42)$$

Now that the radius change equation has been modified, the total orbital transfer problem can be solved.

### Solution of the Orbital Transfer Problem

As stated in the first section of this chapter, the amount of radius and inclination change per orbit must be found to match the final radius and inclination,  $r_{nom}$  and  $i_{nom}$  respectively, after a fixed number of orbits. This will be done by setting up a suboptimal control computer program similar to the one in chapter 3 of this paper.

Solution Method. The algorithm given in chapter 3 for a suboptimal control Conjugate - Gradient computer program will still be used for this problem. But two changes have to be made to the complete program. The first is to change the form of the control equation. The following form will be used:

$$\omega(r) = A_0 + \sum_{k=1}^{\infty} (A_k R)^k \quad (43)$$

where

$$R = (r - r_i) / (r_{Nom} - r_i) \quad (44)$$

The second change is in the form of the performance index. The new performance index is

$$J = (r_f - r_{Nom})^2 + (i_f - i_{Nom})^2 \quad (45)$$

The  $r_f$  and  $i_f$  terms in Eq 45 are the calculated final radius and inclination achieved using Eqs 40, 42, and 43 in an iterative loop for the specified number of orbits.

Solution Results. The suboptimal control program described in the preceding section was run with the following set of specified initial and final conditions. The initial radius and inclination were:

$$r_i = 4263.0 \text{ miles}$$

$$i_i = 10.0 \text{ degrees}$$

The final radius and inclination were:

$$r_{nom} = 4433.057 \text{ miles}$$

$$i_{nom} = 10.746 \text{ degrees}$$

The specified final radius and inclination are the results of a computer

program which calculated the  $\Delta r$  and the  $\Delta i$  of 1000 orbits with the following  $w(r)$  control profile.

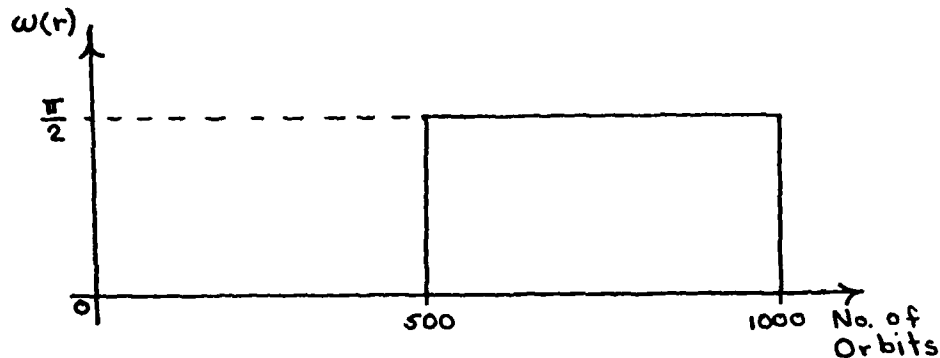


Fig 8: Control Profile for Specified Final Conditions

This control profile corresponds to the case where the vehicle does 500 radius change orbits and then 500 inclination change orbits. This is called the bang-bang orbital transfer maneuver.

The suboptimal control program was started at 900 orbits with a quadratic control equation. When the program converged on a set of coefficients which matched the end conditions,  $r_f$  equal to  $r_{nom}$  and  $i_f$  equal to  $i_{nom}$ , and satisfied Eq 28, the number of orbits was decreased by one and the program started again. The rationale behind this strategy was that the coefficients of the  $n$  orbit case would be good initial guesses for the  $(n - 1)$  orbit case. It was discovered that after a few orbits had been subtracted from the 900, the program refused to converge. At this point another coefficient was added to the control equation. Again the program ran smoothly until a number of orbits had been subtracted. Since the object of this program was to prove the existence of a solution to the example problem stated in the first section of this chapter, the program was terminated after the coefficients for 880 orbits had converged. The converged coefficients for 880, 890, and 900 orbits were chosen to show the results of the program.

The three cases, that were chosen, differed by the number of orbits and the number of coefficients in the control equation. The three cases are summarized in Table 1. The coefficients for each of the three cases can be found in Table 2. Figure 9 shows the plots of  $w(r)$  vs  $r$  for the above three cases. Table 3 shows the savings in time between the three cases chosen to represent the solution of the total orbital transfer problem and the nominal values for the 1000 orbits. These savings in time are equal to fuel savings since the thrust of the vehicle was assumed to be at full thrust throughout the entire problem.

<u>Case</u>	<u>No. of Orbits</u>	<u>No. of Coefficients</u>
1	900	3
2	890	4
3	880	7

Table 1: Sample Results

Case	$A_0$	$A_1$	$A_2$	$A_3$	$A_4$	$A_5$	$A_6$
1	-.23051	.24129	1.41085	0	0	0	0
2	-.05269	.183179	1.23758	$-4.5 \times 10^{-6}$	0	0	0
3	.073167	.13706	1.1349	-.0266	-.01071	-.0044	-.0045

Table 2: Control Equation Coefficients

<u>Case</u>	<u>% decrease</u>
1	10.44
2	11.53
3	12.64

Table 3: Fuel Savings

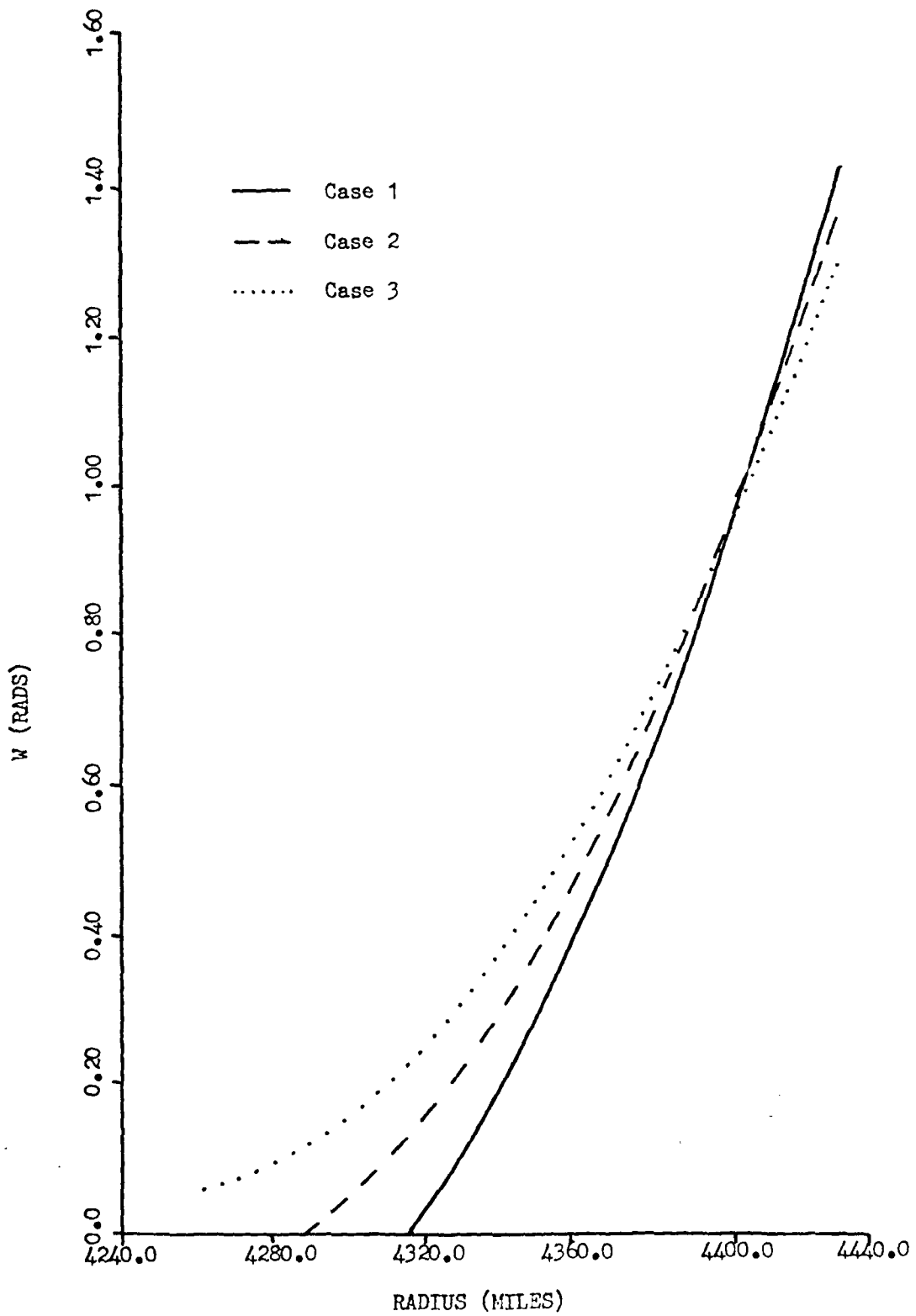


Fig 9: Control Profile for Cases 1, 2, 3



## V Conclusions and Recommendations

The following conclusions and recommendations are based on the results of this study.

The equation derived for the coplanar circular radius change, Eq 10, is no more complex than the equation given in Reference 3 (169) for the instantaneous high thrust case. Equation 10 lends itself, very easily, to computer use.

The equation derived for the inclination change problem, Eq 39, is also no more complex than its counterpart in Reference 3 (169). This equation also is easily implemented on the computer.

The third conclusion of this study is that significant savings (10% to 12%) in fuel and time can be achieved, over the 1000 orbit bang-bang maneuver, if a simple control profile is followed in the total orbital transfer problem. This savings in fuel can be translated directly into an increase in payload weight. Therefore increases in mission capabilities can be achieved without a loss of vehicle performance or an increase in mission time.

There are three areas which can be expanded on from this study. The first is to find an analytical equation for  $w(r)$  so that the general orbit transfer problem can be solved. The second is the use of a thrust angle which allows the full thrust to do both radius and inclination changes at the same time. This paper dealt only with an angle that allowed either radius or inclination changes, but never both. Probably the most important area in which to investigate is the use of complex orbital dynamics. This paper dealt only with two - body dynamics. Two - body dynamics is a good approximation for low earth orbits, but as the radius of the orbit is increased, the accuracy of this approximation decreases. Therefore, for missions to geosynchronous orbit, complex orbital dynamics must be used.

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changes.

The solution of the minimum fuel transfer problem is not to perform the total radius change and then perform the total inclination change. Instead, the solution is to perform both radius and inclination changes on a per orbit basis.

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