Additional contraction when a stick is accelerated

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According to the special theory of relativity a moving stick suffers Lorentz contraction. The contraction is given by

$$L = L_0\surd(1 - v^2/c^2), \tag{1}$$

where L_0 is the proper length[1] of the stick and L is the length of that stick measured in a system in which the stick is moving with velocity v *parallel to the axis of the stick*, c being the velocity of light. Consequently, the faster the stick is moving, the larger is the contraction. Now consider a stick that is moving with a velocity v_1 relative to an inertial system OXY (Fig. 1) initially and then it is accelerated to a larger velocity v_2. According to Eq. (1) the stick as seen by an observer stationary in OXY system should then appear to be shorter than it was. The problem of how this additional contraction comes into existence in the case that the acceleration takes place simultaneously at all points of the stick in a frame moving with the stick has been discussed by Atkinson.[2] Some aspects of a situation related to the present problem have also been discussed briefly by Evett.[3]

Now suppose the observer in OXY system insists on making an arrangement so that the whole stick is accelerated simultaneously according to his measurement. Since in the present situation, the observer sees that every point of the stick is accelerated at the same instant of time, the distance between every two points of the stick remains the same as before the acceleration and hence no additional contraction should be brought to the stick. However, according to the special theory of relativity, since the stick has been accelerated, it should suffer an additional shortening. Which argument is correct? The answer is both of them are correct and they are not contradictory.

It should be noted that Eq. (1) is in fact just a relation between the two lengths of a stick measured by two observers moving relative to each other; the equation says nothing besides this. In the present case, it is the proper length of the stick that has become lengthened and hence the paradox is resolved. The effect is expounded in the following argument: Suppose the observer in OXY system sees the whole stick start to move faster at an instant t of his time, an observer moving with the rear end A of the stick will find that he is accelerated only after the leading end B has attained the acceleration. If we use t'_A and t'_B to designate, respectively, the instant of time, observed by the observer moving with the rear end A, that the rear end and the leading end start to move faster, we have then by Lorentz transformation[4]

$$t'_B - t'_A = -(v/c^2)l/\surd(1 - v^2/c^2), \tag{2}$$

where v is the original velocity before the acceleration taking place and l is the length of the stick observed in the OXY system, which is a constant in the present case; the negative sign in the right-hand side of Eq. (2) indicates that t'_B is smaller than t'_A or, the leading end B is accelerated first and then the rear end A is accelerated. This accounts qualitatively for the result that the proper length is lengthened.

According to the special theory of relativity, the amount of lengthening in the proper length due to the increase of velocity should be

$$l^0(v_2) - l^0(v_1) = l\left[(1 - v_2^2/c^2)^{-1/2} - (1 - v_1^2/c^2)^{-1/2}\right], \tag{3}$$

where $l^0(v_2)$ and $l^0(v_1)$ stand for the proper length of the stick in the moving states of velocities v_2 and v_1, respectively, relative to the OXY system. That this right amount of lengthening in the proper length of the stick resulting from the acceleration is shown in the following argument: According to the preceding analysis, whenever the observer stationary in OXY system sees the whole stick simultaneously attain an infinitesimal amount of additional velocity Δv, observer stationary at the rear end A of the stick will also observe this additional velocity in the whole stick finally. However, he finds himself pick up the Δv only after the leading end has been accelerated for a period of time

$$\Delta t = (v/c^2)l/\surd(1 - v^2/c^2). \tag{4}$$

During this time interval, the leading end B then moves farther apart from him by a distance

$$\Delta l^0 = (\Delta v)'\Delta t, \tag{5}$$

where $(\Delta v)'$ is the velocity of the leading end B relative to the rear end A during the period of time Δt. It should be noted that $(\Delta v)'$ is not equal to Δv. The velocity transformation law implies that for point B to have velocity $v + \Delta v$ in frame OXY, it must have velocity

$$(\Delta v)' = \Delta v/[1 - v(v + \Delta v)/c^2] \tag{6}$$

in a frame moving with point A. In the limit of $\Delta v \to 0$, Eq. (6) becomes

$$(\Delta v)' = \Delta v/(1 - v^2/c^2). \tag{7}$$

Upon combining Eqs. (4), (5), and (7), the infinitesimal addi-

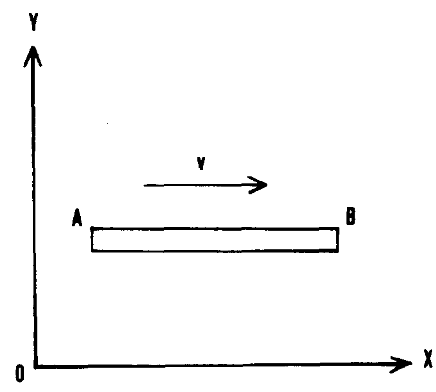

Fig. 1. Stick AB moves with velocity v relative to OXY system.

tional lengthening in the stick takes on the form

$$\Delta l^0 = (v/c^2) l \Delta v/(1 - v^2/c^2)^{3/2}. \quad (8)$$

Consequently the total lengthening resulting from the acceleration from v_1 to v_2 is

$$l^0(v_2) - l^0(v_1) = \int_{v_1}^{v_2} (v/c^2) l/(1 - v^2/c^2)^{3/2}\, dv. \quad (9)$$

Remembering l is a constant in the present case, we obtain finally

$$l^0(v_2) - l^0(v_1) = l(1 - v_2^2/c^2)^{-1/2} - l(1 - v_1^2/c^2)^{-1/2}, \quad (10)$$

which is identical to Eq. (3).

Finally, it should be mentioned that the lengthening in the proper length of the stick is, of course, due to the force acting along the stick, which produces the acceleration.

[1]See, e.g., Panofsky and Phillips, *Classical Electricity and Magnetism* (Addison-Wesley, Reading, MA, 2nd ed., 1962), p. 291.
[2]R. d'E. Atkinson, Am. J. Phys. **48**, 581 (1980).
[3]A. A. Evett, Am. J. Phys. **40**, 1170 (1972).
[4]The equation is directly obtained from Lorentz transformation by setting $t_A = t_B$ in the OXY system.

Cosmological and quantum constraint on particle masses

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In his well-known book *Gravitation and Cosmology*, Steven Weinberg[1] has drawn attention to a curious relation involving the Hubble constant H_0, the gravitational constant G, Planck's constant $\hbar$, the velocity of light c, and the mass of a typical elementary particle m. The relation is[2] [Eq. (16.4.2) of Weinberg]

$$m = (\hbar^2 H_0/Gc)^{1/3} \approx m_\pi. \quad (1)$$

He considers this as a clue pointing to the fact that parameters pertinent to particle physics are not determined solely by considerations of microphysics, but in part by the influence of the whole universe. He also suggests that in considering the possible interpretation of Eq. (1), one must note the remarkable fact that it relates a single cosmological parameter, H_0, to the fundamental constants $\hbar$, G, C, and m_π. He also points out that Eq. (1) is so far unexplained. In the following discussion we shall attempt to understand the hitherto unexplained relation (1), as a simple constraint imposed on the mass of an elementary particle by combination of the uncertainty principle with H_0. We first ask the question whether the gravitational self-energy of a single particle has any meaning in the quantum sense of measurability. Is it a measurable quantity? Consider an elementary particle of mass m. By quantum mechanics we have to localize the wave packet representing it over a region of dimension $(\hbar/mc)$. The gravitational self-energy of the particle corresponding to this localization would be

$$E_p = \frac{Gm^2}{\hbar/mc} = \frac{Gm^3c}{\hbar}. \quad (2)$$

This has to be measurable at least over the Hubble age of the universe given by $(1/H_0)$. The uncertainty principle would then constrain E_p and therefore m through the relation

$$(Gm^3c/\hbar)(1/H_0) \approx \hbar, \quad (3)$$

giving

$$m \approx (\hbar^2 H_0/Gc)^{1/3}, \quad (4)$$

which is the same as Eq. (1). Weinberg's relation may then be understood as the operational requirement that the mass of an elementary particle be such that its gravitational self-energy be at least measurable over a Hubble period. The notion of the gravitational self-energy of a single particle and its measurability is usually ignored in discussions on quantum gravity.

[1]S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).
[2]Reference 1, Chap. 16, p. 619.