Simplified derivation of the Hawking–Unruh temperature for an accelerated observer in vacuum

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A detector undergoing uniform acceleration a in a vacuum field responds as though it were immersed in thermal radiation of temperature $T=\hbar a/2\pi kc$. An intuitive derivation of this result is given for a scalar field in one spatial dimension. The approach is extended to the case where the field detected by the accelerated observer is a spin 1/2 Dirac field.

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I. INTRODUCTION

Hawking[1] predicted that a black hole should radiate with a temperature $T=\hbar g/2\pi kc$, where g is the gravitational acceleration at the surface of the black hole, k is Boltzmann's constant, and c is the speed of light (the *Hawking effect*). The radiation results from the effect of the strong gravitation on the vacuum field. Shortly thereafter it was shown separately by Davies and Unruh that a uniformly accelerated detector moving through the usual flat space–time vacuum of convential quantum field theory responds as though it were in a thermal field of temperature[2–6]

$$T=\frac{\hbar a}{2\pi kc}, \tag{1}$$

where a is the acceleration in the instantaneous rest frame of the detector (the *Davies–Unruh effect*). We will refer to Eq. (1) as the *Hawking–Unruh temperature* because its applicability to flat space–time accelerated detectors or to stationary detectors situated outside the horizon of a black hole[7] depends only on the interpretation of the source of the acceleration a.

The results of Hawking, Davies, and Unruh suggest profound consequences for the merger of quantum field theory and general relativity and sparked intense debates over unresolved questions that are still being actively investigated. If black holes are not really "black," are naked singularities the ultimate fate of black holes, or will the long-sought fusion of quantum mechanics and general relativity into a coherent theory of quantum gravity prevent such occurrences? If a quantum mechanical pure state is dropped into a black hole and pure thermal (uncorrelated) radiation results, how does one explain the apparent nonunitary evolution of the pure state to a mixed state?

The intriguing consequence of quantum field theory for accelerated detectors indicated by Eq. (1) has not been derived in a physically intuitive way. Numerous explicit and detailed calculations have appeared in the scientific literature over the last 30 years for a wide variety of space–times. However, even for the simplest calculation involving a scalar field (bosons), the intricacies of field theory techniques, coupled with a forest of special function properties, make most derivations intractable for the nonspecialist. An investigation of the flat space–time Davies–Unruh effect for Dirac particles of spin 1/2 (fermions) brings in a whole host of new machinery, the least of which is the formulation of the Dirac equation in curvilinear coordinates (that is, in curved space–time). This formulation quickly goes beyond the expertise of most nonexperts. However, in both the boson and fermion cases, the beautiful and important results of Hawking, Davies, and Unruh can be stated quite simply: for a scalar field (bosons) a detector carried by an accelerated observer detects a Bose–Einstein (BE) number distribution of particles at temperature T given by Eq. (1), while for a spin 1/2 field (fermions) a detector carried by an accelerated observer detects a Fermi–Dirac number distribution at the same temperature.

The purpose of this paper is to present a simplified derivation of Eq. (1) that is suitable for advanced undergraduate or beginning graduate students and elucidates the essential underlying physics of the flat space–time Davies–Unruh effect. Once the simplest features of a quantized vacuum field are accepted, Eq. (1) emerges as a consequence of time-dependent Doppler shifts in the field detected by the accelerated observer.

In Sec. II the essential features of uniform acceleration are reviewed, and in Sec. III we use these results to obtain Eq. (1) in an almost trivial way based on the Doppler effect. In Sec. IV this simple approach to the derivation of the temperature in Eq. (1) is developed in more detail. In Sec. V we extend the previous calculations for scalar fields to spin 1/2 Dirac fields. We close with a brief summary and discussion.

II. UNIFORM ACCELERATION

We will refer to an observer moving with constant velocity in flat space–time as a *Minkowski observer* and refer to a *Rindler observer*[8] as one who travels with uniform acceleration in the positive z direction with respect to the former. Uniform acceleration is defined as a constant acceleration a in an instantaneous inertial frame in which the (Rindler) observer is at rest. The acceleration dv/dt of the Rindler observer as measured in the lab frame (that is, by the Minkowski observer) is given in terms of a by a Lorentz transformation formula which relates the acceleration in the two frames[9]

$$\frac{dv}{dt}=a\left(1-\frac{v^2}{c^2}\right)^{3/2}. \tag{2}$$

If we integrate Eq. (2) and take $v=0$ at $t=0$, we have $v(t)=at/\sqrt{1+a^2t^2/c^2}$. The relation $dt=d\tau/\sqrt{1-v^2/c^2}$ between the lab time, t, and the proper time, τ, for the accelerated observer gives $t(\tau)=(c/a)\sinh(a\tau/c)$ if we take $t(\tau=0)=0$. The velocity v of the accelerated observer as detected from the lab frame can be expressed in terms of the proper time τ as

$$v(\tau)=c\tanh\left(\frac{a\tau}{c}\right). \tag{3}$$

A straightforward integration of Eq. (3) using $v(t(\tau))=dz/dt=dz(\tau)/d\tau\cdot d\tau/dt$ yields the well-known hyperbolic orbit of the accelerated, Rindler observer in the z direction:[8]

$$t(\tau)=\frac{c}{a}\sinh\left(\frac{a\tau}{c}\right),\quad z(\tau)=\frac{c^2}{a}\cosh\left(\frac{a\tau}{c}\right). \tag{4}$$

We will consider $a>0$, that is, the observer accelerates in the positive z direction.

III. INDICATION OF THERMAL EFFECT OF ACCELERATION

Consider a plane wave field of frequency ω_K and wave vector $\mathbf{K}$ parallel or anti-parallel to the z direction along which the observer is accelerated. In the instantaneous rest frame of the observer, the frequency ω_K' of this field is given by the Lorentz transformation

$$\omega_K'(\tau)=\frac{\omega_K-Kv(\tau)}{\sqrt{1-v^2(\tau)/c^2}}=\frac{\omega_K\left[1-\tanh\left(\frac{a\tau}{c}\right)\right]}{\sqrt{1-\tanh^2\left(\frac{a\tau}{c}\right)}}=\omega_K e^{-a\tau/c} \tag{5}$$

for $K=+\omega_K/c$, that is, for plane wave propagation along the z direction of the observer's acceleration. For propagation in the $-z$ direction,

$$\omega_K'(\tau)=\omega_K e^{a\tau/c}, \tag{6}$$

where $K=-\omega_K/c$. Note that for small values of $a\tau$, $\omega_K'\cong\omega_K(1\mp a\tau/c)$, the familiar Doppler shift. Equations (5) and (6) involve *time-dependent* Doppler shifts detected by the accelerated observer.

Because of these Doppler shifts, the accelerated observer sees waves with a time-dependent phase $\varphi(\tau)\equiv\int^\tau\omega_K'(\tau')d\tau'=(\omega_K c/a)\exp(a\tau/c)$. We suppose therefore that, for a wave propagating in the $-z$ direction, for which $\int^\tau\omega_K'(\tau')d\tau'=(\omega_K c/a)\exp(a\tau/c)$, the observer sees a frequency spectrum $S(\Omega)$ proportional to

$$\left|\int_{-\infty}^{\infty}d\tau\, e^{i\Omega\tau}e^{i(\omega_K c/a)e^{a\tau/c}}\right|^2. \tag{7}$$

If we change variables to $y=e^{a\tau/c}$, we have

$$\int_{-\infty}^{\infty}d\tau\, e^{i\Omega\tau}e^{i(\omega_K c/a)e^{a\tau/c}}=\frac{c}{a}\int_0^\infty dy\, y^{(i\Omega c/a-1)}e^{i(\omega_K c/a)y}=\frac{c}{a}\Gamma\left(\frac{i\Omega c}{a}\right)\times\left(\frac{\omega_K c}{a}\right)^{-i\Omega c/a}e^{-\pi\Omega c/2a}, \tag{8}$$

where Γ is the gamma function.[10] Then, because $|\Gamma(i\Omega c/a)|^2=\pi/[(\Omega c/a)\sinh(\pi\Omega c/a)]$,[11] we obtain

$$\left|\int_{-\infty}^{\infty}d\tau\, e^{i\Omega\tau}e^{i(\omega_K c/a)e^{a\tau/c}}\right|^2=\frac{2\pi c}{\Omega a}\frac{1}{e^{2\pi\Omega c/a}-1}. \tag{9}$$

The time-dependent Doppler shift detected by the accelerated observer therefore leads to the Planck factor $(e^{\hbar\Omega/kT}-1)^{-1}$ indicative of a Bose–Einstein distribution for scalar (boson) particles with $T\equiv\hbar a/2\pi kc$, which is just Eq. (1). We obtain the same result for a wave propagating in the $+z$ direction.

Note that the time-dependent phase can also be obtained directly by considering the standard nonaccelerated Minkowski plane wave $\exp[i\varphi_\pm]\equiv\exp[i(Kz\pm\omega_K t)]$ and substituting Eq. (4): $\varphi_\pm(\tau)=Kz(\tau)\pm\omega_K t(\tau)=(\omega_K c/a)\times\exp(\pm a\tau/c)$ with $K=\omega_K/c$.[12,13]

IV. A MORE FORMAL DERIVATION

The derivation of the temperature in Eq. (1) leaves much to be desired. We have restricted ourselves to a single field frequency ω_K, whereas a quantum field in vacuum has components at all frequencies. Moreover, we have noted the appearance of the Planck factor, but have not actually compared our result to that appropriate to an observer at rest in a *thermal* field (that is, a field in which the average number of particles is given by a BE distribution for bosons or a Fermi–Dirac distribution for fermions at a fixed temperature T).

To rectify these deficiencies, let us consider a massless scalar field in one spatial dimension (z), quantized in a box of volume V:[14]

$$\hat{\phi}=\sum_K\left(\frac{2\pi\hbar c^2}{\omega_K V}\right)^{1/2}[\hat{a}_K e^{-i\omega_K t}+\hat{a}_K^\dagger e^{i\omega_K t}]. \tag{10}$$

Here $K=\pm\omega_K/c$, and $\hat{a}_K$ and $\hat{a}_K^\dagger$ are, respectively, the annihilation and creation operators for mode K ($[\hat{a}_K,a_{K'}^\dagger]=\delta_{KK'}$, $[\hat{a}_K,a_{K'}]=0$). We use ˆ to denote quantum mechanical operators. The expectation value $\langle(d\hat{\phi}/dt)^2\rangle/4\pi c^2$ of the energy density of this field is $V^{-1}\Sigma_K\hbar\omega_K[\langle\hat{a}_K^\dagger\hat{a}_K\rangle+1/2]$. For simplicity, we consider the field at a particular point in space (say, $z=0$), because spatial variations of the field will be of no consequence for our purposes.

For a (bosonic) thermal state the number operator $\hat{a}_K^\dagger\hat{a}_K$ has the expectation value $(e^{\hbar\omega_K/kT}-1)^{-1}$. Consider the Fourier transform operator

$$\hat{g}(\Omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dt\,\hat{\phi}e^{i\Omega t}=\sum_K\left(\frac{2\pi\hbar c^2}{\omega_K V}\right)^{1/2}\hat{a}_K\delta(\omega_K-\Omega)\quad(\Omega>0). \tag{11}$$

The expectation value $\langle \hat{g}^\dagger(\Omega)\hat{g}(\Omega')\rangle$ in thermal equilibrium is therefore

$$\begin{aligned}\langle \hat{g}^\dagger(\Omega)\hat{g}(\Omega')\rangle &= \sum_K \left(\frac{2\pi\hbar c^2}{\omega_K V}\right)\langle \hat{a}_K^\dagger \hat{a}_K\rangle \delta(\Omega-\Omega') \\ &\quad \times \delta(\omega_K-\Omega) \\ &= \sum_K \left(\frac{2\pi\hbar c^2}{\omega_K V}\right)\frac{1}{e^{\hbar\omega_K/kT}-1} \\ &\quad \times \delta(\Omega-\Omega')\,\delta(\omega_K-\Omega). \end{aligned} \tag{12}$$

We go to the limit where the volume of the quantization box becomes very large, $V\to\infty$, so that we can replace the sum over K by an integral: $\Sigma_K \to (V/2\pi)\int dK$.[15] Thus

$$\begin{aligned}\langle \hat{g}^\dagger(\Omega)\hat{g}(\Omega')\rangle &= \hbar c^2 \int_{-\infty}^{\infty} dK \frac{1}{\omega_K}\frac{1}{e^{\hbar\Omega/kT}-1} \\ &\quad \times \delta(|K|c-\Omega)\,\delta(\Omega-\Omega') \\ &= \frac{2\hbar c/\Omega}{e^{\hbar\Omega/kT}-1}\,\delta(\Omega-\Omega'). \end{aligned} \tag{13}$$

Now consider an observer in uniform acceleration in the quantized *vacuum* field, that is, the particle free vacuum appropriate for the accelerated Rindler observer. This observer sees each field frequency Doppler-shifted according to Eqs. (5) and (6), and so for him/her the operator $\hat{g}(\Omega)$ has the form

$$\begin{aligned}\hat{g}(\Omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau\, e^{i\Omega\tau}\sum_K \left(\frac{2\pi\hbar c^2}{\omega_K V}\right)^{1/2} \\ &\quad \times [\hat{a}_K e^{-i\int^{\tau} d\tau'\,\omega_K'(\tau')} + \hat{a}_K^\dagger e^{i\int^{\tau} d\tau'\,\omega_K(\tau')}] \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau\, e^{i\Omega\tau}\sum_K \left(\frac{2\pi\hbar c^2}{\omega_K V}\right)^{1/2} \\ &\quad \times [\hat{a}_K e^{i(\epsilon_K\omega_K c/a)e^{-\epsilon_K a\tau/c}} + \hat{a}_K^\dagger e^{-i(\epsilon_K\omega_K c/a)e^{-\epsilon_K\omega_K\tau/c}}], \end{aligned} \tag{14}$$

where $\epsilon_K = |K|/K$. Because $\hat{a}_K|\text{vacuum}\rangle = 0$, only the $\hat{a}_K^\dagger$ terms in Eq. (14) contribute to the vacuum expectation value $\langle \hat{g}^\dagger(\Omega)\hat{g}(\Omega')\rangle$. If we perform the integrals over τ as before and use $\langle \hat{a}_K a_{K'}^\dagger\rangle = \delta_{KK'}$, we obtain

$$\begin{aligned}\langle \hat{g}^\dagger(\Omega)\hat{g}(\Omega')\rangle &= \left(\frac{c}{2\pi a}\right)^2\left(\frac{2\pi\hbar c^2}{V}\right)\left|\Gamma\left(\frac{i\Omega c}{a}\right)\right|^2 \\ &\quad \times e^{-\pi c\Omega/a}\sum_K \frac{1}{\omega_K}\left(\frac{\omega_K c}{a}\right)^{i\epsilon_K(\Omega-\Omega')c/a}, \end{aligned} \tag{15}$$

where we have used the fact that the sum over K vanishes unless $\Omega=\Omega'$. We show in the Appendix that the sum over K is $(2Va/c^2)\,\delta(\Omega-\Omega')$, so that

$$\begin{aligned}\langle \hat{g}^\dagger(\Omega)\hat{g}(\Omega')\rangle &= \frac{\hbar c^2}{\pi a}\left|\Gamma\left(\frac{i\Omega c}{a}\right)\right|^2 e^{-\pi c\Omega/a}\,\delta(\Omega-\Omega') \\ &= \frac{2\hbar c/\Omega}{e^{2\pi\Omega c/a}-1}\,\delta(\Omega-\Omega'), \end{aligned} \tag{16}$$

which is identical to the thermal result Eq. (13) if we define the temperature by Eq. (1).

Note that the expectation value $\langle \hat{a}_K a_{K'}^\dagger\rangle = \delta_{KK'}$ involves the creation and annihilation operators of the accelerated observer and is taken with respect to the accelerated observer's vacuum, which is different from the vacuum detected by the nonaccelerated observer. This point is discussed more fully in Sec. VI.

V. FERMI–DIRAC STATISTICS FOR DIRAC PARTICLES

We have considered a scalar field and derived the Planck factor $(e^{\hbar\Omega/kT}-1)^{-1}$ indicative of Bose–Einstein (BE) statistics. We began with the standard plane-wave solutions of the form $\exp[i(Kz\pm\omega_K t)]$ for the nonaccelerated Minkowski observer and considered the time-dependent Doppler shifts as detected by the accelerated observer. For spin 1/2 Dirac particles we would expect an analogous derivation to obtain $(e^{\hbar\Omega/kT}+1)^{-1}$ indicative of Fermi–Dirac (FD) statistics.

Mathematically, the essential point involves the replacement $i\Omega c/a \to i\Omega c/a + 1/2$ in the integrals in Eqs. (7)–(9),[16] and the relation $|\Gamma(i\Omega c/a+1/2)|^2 = \pi/\cosh(\pi\Omega c/a)$.[11] Physically, this replacment arises from the additional spinor nature of the Dirac wave function over that of the scalar plane wave. For a scalar field, only the phase had to be instantaneously Lorentz-transformed to the comoving frame of the accelerated observer. For non-zero spin, the spinor structure of the particles must also be transformed,[17] or Fermi–Walker transported[18] along a particle's trajectory to ensure that it does not "rotate" as it travels along the accelerated trajectory. Ensuring this "nonrotating" condition in the observer's instantaneous rest frame leads to a time-dependent Lorentz transformation of the Dirac bispinor of the form[19] $\hat{S}(\tau) = \exp(\gamma^0\gamma^3 a\tau/2c) = \cosh(a\tau/2c) + \gamma^0\gamma^3\sinh(a\tau/2c)$, where the 4×4 constant Dirac matrices are given by

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix}, \tag{17}$$

and $\sigma_z = \text{diagonal}(1,-1)$ is the usual 2×2 Pauli spin matrix in the z direction. If $\hat{S}(\tau)$ acts on a spin up state $|\uparrow\rangle = [1,0,1,0]^T$,[20] $\hat{S}(\tau)|\uparrow\rangle = \exp(a\tau/2c)|\uparrow\rangle$. Thus, for the spin up Dirac particle we should replace the plane wave scalar wave function $\exp[i\varphi(\tau)]$ used in Eq. (7) by $\exp(a\tau/2c)\exp[i\varphi(\tau)]$.[21] This replacement leads to $i\Omega c/a \to i\Omega c/a + 1/2$ in Eq. (8), and therefore the result

$$\left|\int_{-\infty}^{\infty} d\tau\, e^{i\Omega\tau} e^{a\tau/2c} e^{i(\omega_K c/a)e^{a\tau/c}}\right|^2 = \frac{2\pi c}{\omega_K a}\frac{1}{e^{2\pi\Omega c/a}+1}. \tag{18}$$

If we compare Eq. (18) with Eq. (9), we note the crucial change of sign in the denominator from -1 for BE statistics to $+1$ for FD statistics. We also note that the prefactor in Eq. (9) involves the dimensionless frequency $\Omega c/a$ while in Eq. (18) the prefactor involves the factor $\omega_K c/a$ (the argument of the exponential in the distribution function is still $\hbar\Omega/kT$ with the same Hawking–Unruh temperature $T=\hbar a/2\pi kc$). This difference in the prefactor is no cause for concern, because in fact a single frequency ω_K detectable by a Minkowski observer is actually spread over a continuous range of frequencies Ω detectable by the accelerated Rindler observer, with a peak centered at $\Omega=\omega_K$.[22] This fact allows us to replace ω_K by Ω in the final result. [This frequency replacement is explicitly evidenced by the delta function $\delta(\omega_K-\Omega)$ in Eqs. (11)–(16) in the comparison of the thermal and accelerated correlation functions.]

For the spin up Dirac particle, the more formal field-theoretic derivation of Sec. IV proceeds in exactly the same fashion, with the modification of the accelerated wave function from $\exp[i\varphi(\tau)]\rightarrow\exp(a\tau/2c)\exp[i\varphi(\tau)]$ and the use of anti-commutators $\{\hat{a}_K,a^{\dagger}_{K'}\}=\delta_{KK'}$ for the quantum mechanical creation and annihilation operators instead of the commutators appropriate for scalar BE particles. For the correlation function we find

$$\langle\hat{g}^{\dagger}(\Omega)\hat{g}(\Omega')\rangle=\frac{2\hbar c/\omega_K}{e^{2\pi\Omega c/a}+1}\,\delta(\Omega-\Omega'), \tag{19}$$

the FD analogue of Eq. (16).

VI. SUMMARY AND DISCUSSION

In the usual derivation of the Hawking-Unruh temperature Eq. (1),[2–6] one solves the wave (or Dirac) equation for the field mode functions in the Rindler coordinates Eq. (4), and then quantizes them. Because the hyperbolic orbit of the accelerated observer Eq. (4) is confined to the region of Minkowski space–time $z>0$, $z>|t|$ bounded by the asymptotes $t=\pm z$ called the *right Rindler wedge* [with mirror orbits confined to the *left Rindler wedge* defined by $z<0$, $|z|>|t|$ obtained from defining the accelerated observer's coordinates as $t(\tau)=-(c/a)\sinh(a\tau/c)$ and $z(\tau)=-(c^2/a)\cosh(a\tau/c)$], it turns out that the vacuum detected by the accelerated observer in say, the right Rindler wedge is different than the usual Minkowski vacuum (defined for all z and t) detected by the unaccelerated observer. The inequivalence of these vacua (and hence the Minkowski versus Rindler quantization procedures[23]) is due to the fact that the right and left Rindler wedges are causally disconnected from each other. Readers can easily convince themselves of the causal disconnectedness of the right and left Rindler wedges by drawing a Minkowski diagram in (z,t) coordinates and observing that light rays at $\pm 45°$ emanating from one wedge do not penetrate the other wedge. Hence the Minkowski vacuum that the accelerated observer moves through appears to her/him as an excited state containing particles, and not as the particle free vacuum appropriate for the right Rindler wedge. The Bose–Einstein distribution with the Hawking–Unruh temperature T for scalar fields (Fermi–Dirac for Dirac fields) is usually derived by considering the expectation value of the number operator $\hat{a}^{\dagger}_R\hat{a}_R$ for the accelerated observer (in the right Rindler wedge) in the unaccelerated Minkowski vacuum $|0_M\rangle$, that is, $\langle 0_M|\hat{a}^{\dagger}_R\hat{a}_R|0_M\rangle\sim[\exp(\hbar\Omega/kT)\pm 1]^{-1}$ (with the upper sign for scalar fields and lower sign for Dirac fields). The proportionality of the particle number spectrum detected by the accelerated Rindler observer moving through the Minkowski vacuum to a thermal spectrum is referred to as the *thermalization theorem* by Takagi.[5]

In this work we have taken a slightly different viewpoint.[24] For a scalar field, we first consider an unaccelerated Minkowski observer in a thermal state and find that the expectation value of a field correlation function is proportional to the Bose–Einstein distribution. We then consider the calculation of this correlation function again, but this time for an accelerated observer in his particle free Rindler vacuum[25] state $|0_R\rangle$, such that for a single mode, $\langle 0_R|\hat{a}_R\hat{a}^{\dagger}_R|0_R\rangle=1$. The new feature is that from his local stationary perspective, the accelerated observer detects all Minkowski frequencies (arising from the the usual plane waves associated with Minkowski states) as time-dependent Doppler shifted frequencies.

The derivation presented here shows why quantum field fluctuations in the vacuum state are crucial for the thermal effect of acceleration: $\langle\hat{g}^{\dagger}(\Omega)\hat{g}(\Omega')\rangle$ is nonvanishing because the vacuum expectation $\langle\hat{a}_K\hat{a}^{\dagger}_K\rangle\neq 0$. But there is more to it than that, because $\langle\hat{a}_K\hat{a}^{\dagger}_K\rangle$ also is nonvanishing for an observer with $a=0$. For such an observer, however,

$$\begin{aligned}\int_{-\infty}^{\infty}d\tau\,e^{i\Omega\tau}e^{i\int^{\tau}d\tau'\,\omega_K(\tau')}&=\int_{-\infty}^{\infty}d\tau\,e^{i(\Omega+\omega_K)\tau}\\&=2\pi\delta(\Omega+\omega_K)\\&=0\end{aligned} \tag{20}$$

for scalar particles, because both Ω and ω_K are positive. In other words, the thermal effect of acceleration in our model arises because of the nontrivial nature of the quantum vacuum and the time-dependent Doppler shifts detected by the accelerated observer. For Dirac particles, the essential new feature is the additional spinor structure of the wave function over that of the scalar plane wave. To keep the spin nonrotating in the comoving frame of the accelerated observer, the Dirac bispinor must be Fermi–Walker transported along the accelerated trajectory, resulting in an additional time-dependent Lorentz transformation. Formally, this transformation induces a shifting of $i\Omega c/a\rightarrow i\Omega c/a+1/2$ in the calculation of relevant gamma function-like integrals, leading to the FD Planck factor.

In the following we briefly discuss the relationship of our correlation function to those used in the usual literature on this subject and point out a not widely appreciated subtlety relating details of the spatial Rindler mode functions (which we have ignored in our model) to the statistics of the noise spectrum detected by the accelerated observer.

In our model, we have not motivated the use of the correlation function $\langle\hat{g}^{\dagger}(\Omega)\hat{g}(\Omega')\rangle$ aside from the fact that we could calculate it for a nonaccelerated observer in a thermal field and for a uniformly accelerated observer in vacuum and compare the results. It is easy to show that a harmonic oscillator with frequency ω_0 and dissipation coefficient γ, linearly coupled to the field Eq. (10), reaches a steady-state energy expectation value

$$\langle E\rangle\propto\int_0^{\infty}d\Omega\int_0^{\infty}d\Omega'\frac{\langle\hat{g}^{\dagger}(\Omega)\hat{g}(\Omega')\rangle}{(\Omega-\omega_0-i\gamma)(\Omega'-\omega_0+i\gamma)}, \tag{21}$$

which offers some motivation for considering $\langle\hat{g}^{\dagger}(\Omega)\hat{g}(\Omega')\rangle$. In fact, it can be shown that $\langle E\rangle=[e^{\hbar\omega_0/kT}-1]^{-1}$, which shows again that our accelerated observer acquires the characteristics appropriate to being in a thermal field at the temperature $T=\hbar a/2\pi kc$.

In an extensive review of the Davies–Unruh effect, Takagi[5] utilizes the quantum two-point correlation (Wightman) function $g_W(\tau,\tau')\equiv\langle 0_M|\hat{\phi}(\tau)\hat{\phi}^\dagger(\tau')|0_M\rangle$ to determine the power spectrum of the vacuum noise detected by the accelerated observer for a scalar field,

$$S(\Omega)\equiv\lim_{s\downarrow 0}\int_{\infty}^{\infty}e^{-i\Omega\tau-s|\tau|}g_W(\tau), \tag{22}$$

which is very much in the spirit of our calculation in Sec. IV. Here the field operator $\hat{\phi}(\tau)$ is expanded in terms of the Rindler mode functions and involves the creation and annihilation operators for both the right and left Rindler wedges. Takagi shows the remarkable, although not widely known result, that for a scalar field in a Rindler space–time of dimension n, $S_n(\Omega)\sim f_n(\Omega)/[\exp(\hbar\Omega/kT)-(-1)^n]$. For even-dimensional space–times (for example, $n=2$ as considered in this work or the usual $n=4$) $S_n(\Omega)$ is proportional to the Bose–Einstein distribution function $[\exp(\hbar\Omega/kT)-1]^{-1}$, and essentially reproduces our Eq. (16) [up to powers of $\Omega c/a$, contained in the function $f_n(\Omega)$]. However, for odd n, $S_n(\Omega)$ is proportional to the Fermi–Dirac distribution $[\exp(\hbar\Omega/kT)+1]^{-1}$. For Dirac particles the opposite is true, namely for even space–time dimensions $S_n(\Omega)$ is proportional to the FD distribution and for odd space–time dimensions $S_n(\Omega)$ is proportional to the BE distribution. This curious fact arises from the dependence of $S_n(\Omega)$ on two factors in its calculation. The first is the previously mentioned thermalization theorem, that is, the number spectrum of accelerated (Rindler) particles in the usual nonaccelerated Minkowski vacuum is proportional to the BE distribution function for scalar fields and is proportional to the FD distribution function for Dirac fields. The second factor that switches the form of $S_n(\Omega)$ from BE to FD depends on the detailed form of the Rindler mode functions.[5] Although the trajectory of the accelerated observer takes place in $1+1$ dimensions [the (z,t) plane], the quantum field exists in the full n-dimensional space–time, and thus $S_n(\Omega)$ ultimately depends on the form of the mode functions in the full space–time. In space–times of even dimensions the number spectrum of Rindler particles in the Minkowski vacuum and the noise spectrum of the vacuum fluctuations (that is, the response of the accelerated particle detector) both depend on the same distribution function, and these two effects often are incorrectly equated.

In our simplified derivation we have bypassed this technicality by performing our calculations in $1+1$ dimensions (that is, $n=2$). We have concentrated on the power spectrum of vacuum fluctuations as seen by a particle detector carried by the accelerated observer. We have shown that in $1+1$ dimensions the spectrum of fluctuations is proportional to the Bose–Einstein distribution function for scalar fields and to the Fermi–Dirac distribution for spin 1/2 fields, with the Hawking–Unruh temperature defined by Eq. (1). The dependence of the noise spectrum on these distribution functions is ultimately traced back to the time-dependent Doppler shifts as detected by the accelerated observer as he/she moves through the usual nonaccelerated Minkowski vacuum.

We hope that the calculations exhibited here are sufficiently straightforward to give an intuitive understanding of the essential physical origin of the Hawking–Unruh temperature experienced by a uniformly accelerated observer.

Suggested Problem. Discuss when the Hawking–Unruh temperature from Eq. (1) would become physically detectable by utilizing the expression $a=GM/r^2$ for the gravitational acceleration of a test mass at a distance r from a mass M, and determine T at the surface of the earth, the Sun, and a Schwarzschild black hole.

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APPENDIX: MODE SUM CALCULATION IN EQ. (15)

If we convert the sum over K to an integral, we have, for $K>0$,

$$\sum_{K>0}\frac{1}{\omega_K}\left(\frac{\omega_K c}{a}\right)^{i\epsilon_K(\Omega-\Omega')c/a}=\frac{V}{2\pi}\int_0^\infty dK\,\frac{1}{\omega_K}\times\left(\frac{\omega_K c}{a}\right)^{i(\Omega-\Omega')c/a}. \tag{A1}$$

We let $x=\log(\omega_K c/a)$ and write Eq. (A1) as

$$\frac{V}{2\pi c}\int_{-\infty}^{\infty}dx\,e^{-ix(\Omega-\Omega')c/a}=\frac{Va}{c^2}\,\delta(\Omega-\Omega'). \tag{A2}$$

The same result is obtained for the sum over $K<0$, so that the sum over all K is $(2Va/c^2)\,\delta(\Omega-\Omega')$.

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[1]S. W. Hawking, ''Black hole explosions,'' Nature (London) **248**, 30–31 (1974); ''Particle creation by black holes,'' Commun. Math. Phys. **43**, 199–220 (1975).

[2]W. G. Unruh, ''Notes on black hole evaporation,'' Phys. Rev. D **14**, 870–892 (1976).

[3]P. C. W. Davies, ''Scalar production in Schwarzschild and Rindler metrics,'' J. Phys. A **8**, 609–616 (1975).

[4]The literature on this subject is vast. For some articles directly relevant for the work presented here see, for instance, P. Candelas and D. W. Sciama, ''Irreversible thermodynamics of black holes,'' Phys. Rev. Lett. **38**, 1372–1375 (1977); T. H. Boyer, ''Thermal effects of acceleration through random classical radiation,'' Phys. Rev. D **21**, 2137–2148 (1980) and ''Thermal effects of acceleration for a classical dipole oscillator in classical electromagnetic zero-point radiation,'' **29**, 1089–1095 (1984); D. W. Sciama, P. Candelas, and D. Deutsch, ''Quantum field theory, horizons, and thermodynamics,'' Adv. Phys. **30**, 327–366 (1981).

[5]An extensive review is given by S. Takagi, ''Vacuum noise and stress induced by uniform acceleration,'' Prog. Theor. Phys. **88**, 1–142 (1986). See Chap. 2 for a review of the Davies–Unruh effect.

[6]N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge U.P., New York, 1982).

[7]Note that in order for a detector to remain at a fixed location outside the horizon of a black hole, it must undergo constant acceleration just to remain in place.

[8]W. Rindler, ''Kruskal space and the uniformly accelerated observer,'' Am. J. Phys. **34**, 1174–1178 (1966).

[9]P. W. Milonni, *The Quantum Vacuum* (Academic, New York, 1994), pp. 60–64.

[10]I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic, New York, 1980). We have used integrals 3.761.4 and 3.761.9, $\int_0^\infty x^{\mu-1}\sin(ax)dx=[\Gamma(\mu)/a^\mu]\sin(\mu\pi/2)$ and $\int_0^\infty x^{\mu-1}\cos(ax)dx=[\Gamma(\mu)/a^\mu]\cos(\mu\pi/2)$ respectively, in the combination of the second plus i times the first. Taken together, both integrals have a domain of definition $a>0$, $0<\mathrm{Re}(\mu)<1$. In Eq. (8) we have $\mu=i\Omega c/a$ with $\mathrm{Re}(\mu)=0$. The integrals can be regularized and thus remain valid in the limit $\mathrm{Re}(\mu)\to 0$ as

can be seen by adding a small imaginary part $-i\epsilon a/c$, $\epsilon>0$ to the frequency Ω so that $\mu\rightarrow\mu'\equiv\mu+\epsilon$ and $0<\mathrm{Re}(\mu')=\epsilon<1$ is strictly in the domain of definition of the integrals. In the limit of $\epsilon\rightarrow 0$ we have $[\Gamma(\mu+\epsilon)/a^{\mu+\epsilon}]e^{i(\mu+\epsilon)\pi/2}\rightarrow[\Gamma(\mu)/a^{\mu}]e^{i\mu\pi/2}$ and thus obtain the same values for the integrals as if we had just set $\mathrm{Re}(\mu')=0$ initially. The equivalence of these two approaches to evaluate the integrals occurs because the standard integral form of the gamma function $\Gamma(z)=\int_0^\infty dt\, e^{-z}t^{z-1}$, $\mathrm{Re}(z)>0$ can be analytically continued in the complex plane and in fact remains well defined, in particular, for $\mathrm{Re}(z)\rightarrow 0$, $\mathrm{Im}(z)\neq 0$, which is the case in Eq. (8). See for for example, J. T. Cushing, *Applied Analytical Mathematics for Physical Scientists* (Wiley, New York, 1975), p. 343.

[11]Reference 10, Sec. 8.332.

[12]Related, although much more involved derivations of the Davies–Unruh effect based on a similar substitution can be found in L. Pringle, ''Rindler observers, correlated states, boundary conditions, and the meaning of the thermal spectrum,'' Phys. Rev. D **39**, 2178–2186 (1989); U. H. Gerlach, ''Minkowski Bessel modes,'' *ibid.* **38**, 514–521 (1988), gr-qc/9910097 and ''Quantum states of a field partitioned by an accelerated frame,'' **40**, 1037–1047 (1989). We can justify this substitution from the form $\varphi=\int k_\mu(x)dx^\mu$ of the phase of a quantum mechanical particle in curved space–time. See L. Stodolsky ''Matter and light wave interferometry in gravitational fields,'' Gen. Relativ. Gravit. **11**, 391–405 (1979) and P. M. Alsing, J. C. Evans, and K. K. Nandi, ''The phase of a quantum mechanical particle in curved spacetime,'' *ibid.* **33**, 1459–1487 (2001), gr-qc/0010065.

[13]A similar derivation in terms of Doppler shifts appears in the appendix of H. Kolbenstvedt, ''The principle of equivalence and quantum detectors,'' Eur. J. Phys. **12**, 119–121 (1991). T. Padmanabhan and coauthors also have derived Eq. (9) by similarly considering the power spectrum of Doppler shifted plane waves as detected by the accelerated observer. See K. Srinivasan, L. Sriramkumar, and T. Padmanabhan, ''Plane waves viewed from an accelerated frame: Quantum physics in a classical setting,'' Phys. Rev. D **56**, 6692–6694 (1997); T. Padmanabhan, ''Gravity and the thermodynamics of horizons,'' gr-qc/0311036.

[14]When we later convert a sum over K to an integral, we obtain the Lorentz-invariant measure dK/ω_K as a consequence of the $1/\sqrt{\omega_K}$ in Eq. (10). The use of this invariant measure eliminates the need to explicitly transform the frequency term $1/\sqrt{\omega_K}$ in Eq. (14), for instance.

[15]Because we use a one-dimensional model, the factor $V/2\pi$ appears instead of the more familiar $V/(2\pi)^3$. In other works, our volume V here is really just a length.

[16]If we use the same gamma function integrals as in Ref. 10, we will have for the Dirac case $\mu=i\Omega c/a+1/2$, with $\mathrm{Re}(\mu)=1/2$ clearly in their domain of definition $0<\mathrm{Re}(\mu)<1$.

[17]S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972), pp. 365–370.

[18]C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), Chap. 6, pp. 163–176.

[19]Reference 5, p. 101, and P. Candelas and D. Deutsch, ''Fermion fields in accelerated states,'' Proc. R. Soc. London, Ser. A **362**, 251–262 (1978). This result can be understood as follows. If the observer was traveling at constant velocity v in the positive z direction, we would Lorentz transform the spinor in the usual special relativistic way via the operator $\hat{S}(v)=\exp(\gamma^0\gamma^3\xi/2)$ where $\tanh\xi=v/c$. See J. D. Bjorkin and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw–Hill, New York, 1964), pp. 28–30. From Eq. (3) we have $v/c=\tanh(a\tau/c)$ so that $\xi=a\tau/c$, yielding the spinor Lorentz transformation to the instantaneous rest frame of the accelerated observer.

[20]For simplicity, we have chosen the spin up wave function as an eigenstate of $\hat{S}(\tau)$ with eigenvalue $\exp(a\tau/2c)$. The exact spatial dependence of the accelerated (Rindler) spin up wave function is more complicated than this simple form, although both have the same zero bispinor components. See W. Greiner, B. Müller, and J. Rafelski, *Quantum Electrodynamics in Strong Fields* (Springer, New York, 1985), Chap. 21.3, pp. 563–567; M. Soffel, B. Müller, and W. Greiner, ''Dirac particles in Rindler spacetime,'' Phys. Rev. D **22**, 1935–1937 (1980).

[21]The spinor Lorentz transformation $\hat{S}(\tau)$ does not mix spin components. Thus, for example, a spin up Minkowski state remains a spin up accelerated (Rindler) state. We can therefore drop the constant spinor $|\uparrow\rangle$ from our calculations and retain the essential, new time-dependent modification $\exp(a\tau/2c)$ to the plane wave for our Dirac ''wave function.''

[22]Reference 5, Sec. 2, in particular Eqs. (2.7.4) and (2.8.8).

[23]S. A. Fulling, ''Nonuniqueness of canonical field quantization in Riemannian spacetime,'' Phys. Rev. D **7**, 2850–2862 (1973) and *Aspects of Quantum Field Theory in Curved Space-Time* (Cambridge U.P., New York, 1989).

[24]This viewpoint is also taken in a different derivation of the Unruh–Davies effect in Ref. 9.

[25]The unaccelerated Minkowski vacuum $|0_M\rangle$ is unitarily related to the Rindler vacuum $|0_R\rangle\otimes|0_L\rangle$ via $|0_M\rangle=\hat{S}(r)|0_R\rangle\otimes|0_L\rangle$, where $\hat{S}(r)$ is the *squeezing* operator $\hat{S}(r)=\exp[r(\hat{a}_R\hat{a}_L-\hat{a}_R^\dagger\hat{a}_L^\dagger)]$. The subscripts R and L denote the right $(z>0,z>|t|)$ and left $(z<0,|z|>|t|)$ Rindler wedges, respectively, which are regions of Minkowski space–time bounded by the asymptotes $t=\pm z$. $|0_R\rangle$ is the Fock state of zero particles in the right Rindler wedge and $|0_L\rangle$ is the Fock state of zero particles in the left Rindler wedge. Note that the orbit of the accelerated Rindler observer given by Eq. (4) is confined to the right Rindler wedge. Because the right and left Rindler wedges of Minkowski space–time are causally disconnected from each other, the creation and annihilation operators $\hat{a}_R^\dagger$, $\hat{a}_R$ and $\hat{a}_L^\dagger$, $\hat{a}_L$ live in the right and left wedges, respectively, and mutually commute with each other, that is, $[\hat{a}_R,\hat{a}_L^\dagger]=0$, etc. Because, physical states that live in the right wedge have zero support in the left wedge (and vice versa), they are described by functions solely of the operators $\hat{a}_R^\dagger$, $\hat{a}_R$ appropriate for the right wedge, that is, $|\Psi_R\rangle=f(\hat{a}_R,\hat{a}_R^\dagger)|0_R\rangle\otimes|0_L\rangle=|\psi_R\rangle\otimes|0_L\rangle$. It is in this sense that we can speak of $|0_R\rangle$ as the vacuum for the right Rindler wedge, and similarly $|0_L\rangle$ as the vacuum for the left Rindler wedge.