Parallax distance, time, and the twin "paradox"

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An extension to all space-time points of the definition of time and of distance for any observer, accelerating or not, by means of a parallax viewing of events is undertaken. The twin "paradox" is analyzed in terms of this definition, and it is shown that during the period of acceleration, the accelerated observer sees the other traveler recede and go backward in "time." This motion completely reconciles the calculations both observers make regarding the reading of each other's clocks when they meet again.

INTRODUCTION

One of the greatest barriers to the understanding of special relativity by students is the altered nature of time. In particular, in their introduction to the subject, it is often not made clear that there are really two different types of time for any one observer. There is the time at the position of the observer as measured by a clock which he carries with himself, and then there is the time of events which do not occur at his position, and which the observer must define by means of some convention. This conventional nature of times not at the observer's position is often hidden by asking the students to imagine all of space filled by friends of the observer, traveling at his velocity and each carrying a clock. These friends must then synchronize their clocks by means of some convention (e.g., by slow transport of their clocks or by interchanging light signals). Students tend to have difficulty imagining space filled by clocks and observers and friends, especially when more than one frame of reference is used, and especially when accelerations are involved (as in the twin "paradox").

The convention I will adopt in this paper is for the observer to measure the time of events at his own position by means of his own clock, to measure the distance to a faraway event by measuring the parallax of the light rays he receives from that event, and to measure the time of that event by measuring the time of reception of those light rays and correcting for the propagation delay by using the previously measured parallax distance and the velocity of light. This technique has three advantages:

(1) It emphasizes the conventional nature of distances to and times of events not at the observer's position.

(2) It is applicable to any observer, whether inertial or not.

(3) It requires measurements by the observer only of angles and times at his own position. Furthermore, unlike other such techniques, such as radar ranging, it requires measurement at one instant of time to determine the distance and time to any event.

Because students find the twin paradox genuinely puzzling and their confusion about the nature of time is most acutely displayed in this problem, I will develop the above ideas in the context of this problem.

TWIN PARADOX AND TIME

The twin paradox has generated one of the longest standing controversies in twentieth-century physics.[1] As it highlights most succinctly the difference between the traditional and the special relativistic views of time, and as it still presents a source of confusion to students, a careful analysis of the problems which arise would seem to be useful. In particular, the lack of symmetry between the two travellers has often seemed to be without cause. This paper will present a new way of examining the situation, which will highlight some of the unusual features of special relativity.

In my discussion of the twin paradox I will assume that special relativity is the correct description of the physics, and will show how the view of the situation by the two travelers can be fit consistently together.

The situation I wish to examine is the one in which two twins A and B start off together. In some frame, A remains at rest, suffering no accelerations throughout. B, on the other hand, travels away from A at velocity v for some period of time, suffers a brief period of acceleration which sends him back toward A with the same speed v. When the two are back together again, they discover that the elapsed time on B's clock is a factor $\sqrt{1 - v^2/c^2}$ smaller than that on A's.

The way in which this argument is often developed is to say that on the outward journey, A observes B's clock to be going a factor $\sqrt{1 - v^2/c^2}$ slower than his own. Similarly, on B's return trip his clock is going the same factor slower giving a factor of $\sqrt{1 - v^2/c^2}$ for the elapsed times for the whole trip.

The paradox is usually stated by asking why B cannot argue in exactly the same way about A's clock. B can regard A as travelling away with velocity v and thus regard A's clock as going slower by a factor of $\sqrt{1 - v^2/c^2}$. On the return portion, B can again regard A's clock as going slower. Why can B not argue that therefore A's clock will indicate an elapsed time a factor of $\sqrt{1 - v^2/c^2}$ smaller than B's elapsed time?

The usual explanation is to point out that B suffers acceleration while A does not. However, as the acceleration is assumed to occur for an arbitrarily short time interval as felt by B, and as the above argument is surely true at all times when B is not accelerating, the role played by the acceleration is often left unclear.

The crucial point is that the idea of "time," which is straightforward in classical physics, is no longer so in special relativity. In essence, there exist at least four different times referred to in the above discussions. Each of the twins, A and B, has the time as indicated on his own clock. This time is directly defined only for that point in space where the clock is located at that instant in time. However, each twin must also extend his definition of time to other points in space by means of some convention. Since the two clocks

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are not always at the same position, A must somehow extend his definition of time to the place where B's clock is, in order to compare that extended time with the reading on B's clock at that instant. Similarly, B must do the same in comparing A's clock time with his own extended definition of time.

By using the assumed constancy of the velocity of light, the extended definition of time can be linked with a concept of distance. Many such extensions are possible and have been used in the past. One such is a radar ranging, which, however, has the disadvantage of requiring two readings of a clock well separated in time. The method to be outlined here is local in the sense that the observer only needs information in an arbitrarily small region of space-time in order to be able to define the position and time of some other event in space-time.

PARALLAX DISTANCE AND TIME

When seeing an event, one can determine both the time when the event occurred, and the point at which that event occurred by determining the distance to that event by means of parallax. If two receptors are separated by a distance ϵ in a direction perpendicular to the direction of travel of the light from that event, and if the difference in angle of the receipt of the light rays at the two receptors is $\delta\phi$, then the distance to that event may be defined to be

$$r = \epsilon/\delta\phi,$$

where I have assumed that ϵ has been chosen so that $\delta\phi \ll 1$.

The time at which that event occurred can be defined to be

$$t_e = t_r - |r|/c,$$

where t_r is the time of receipt of the light rays and t_e is the defined time of occurrence of that event. Note that t_e is a conventional time, and not a time as measured on any clock.

I will use the above extension of the definition of time and distance to examine the situation seen by the two observers A and B.

One relation which will be important in this work is the the aberration formula.[2] As the velocity of light is finite, the angle of a light ray with some given direction will not be the same to moving observers. This is an effect which is already present in nonrelativistic physics. The relation between the angle ϕ of a light ray with the positive x axis, say, as measured by a (by definition) stationary observer, and that angle $\tilde{\phi}$ measured by an observer moving with velocity v in the positive x direction is given in special relativity by

$$\tan\tilde{\phi} = \frac{\sqrt{1 - v^2/c^2}\,\sin\phi}{\cos\phi + (v/c)}.$$

If ϕ is very small, we obtain

$$\tilde{\phi} \approx \sqrt{\frac{1 - v/c}{1 + v/c}}\,\phi,$$

while if ϕ is very near π we have

$$\tilde{\phi} - \pi = \sqrt{\frac{1 + v/c}{1 - v/c}}\,(\phi - \pi).$$

For two observers at one point, one moving with instantaneous velocity v in the positive x direction, events behind

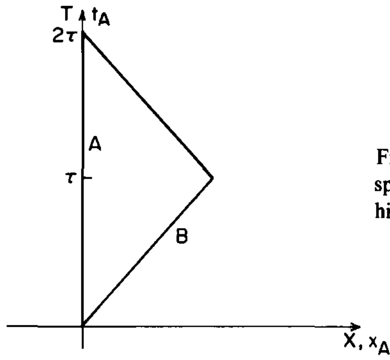

Fig. 1. Paths of A and B in space-time and as seen by A in his extended coordinates.

him will look a factor

$$\sqrt{\frac{1 - v/c}{1 + v/c}}$$

nearer than they do for the stationary observer. Similarly, if an observer suddenly changes his velocity, events in the direction of his acceleration suddenly appear to be further away, while events in the opposite direction appear to be nearer. It is this effect which will be important for elucidating the twin paradox.

Let us set up a detailed situation. We set up an inertial coordinate system T, X, Y, Z. In this coordinate system A travels along the path

$$X = Y = Z = 0.$$

As we are assuming the validity of special relativity, the time t_A as measured by A's clock will be given by

$$t_A = T.$$

The path travelled by B will be given by (see Fig. 1)

$$Y = Z = 0,$$

$$X = vT \qquad 0 < T < \tau,$$

$$X = v(2\tau - T) \qquad \tau < T < 2\tau.$$

The time $\hat{t}_B$ as measured by B's clock will be given by

$$\hat{t}_B = \sqrt{1 - v^2/c^2}\,T$$

throughout the trip.

We must now let A and B set up their extended definition of times t_A and t_B and of spatial coordinates x_A and x_B (we will concern ourselves only with events in the $Y = Z = 0$ plane and for these events $y_A = z_A = y_B = z_B = 0$). See Table I for the definition of the various times and coordinates.

Let us say that an event occurs at a point with coordinates T, X in the space-time. The light from this event will reach A at time

$$T' = T + |X - 0|/c,$$

which will correspond to the time on A's clock of

$$\hat{t}'_A = T' = T + |X|/c.$$

Let us assume that A's receptors are a small distance $\epsilon/2$ above and below the $Y = Z = 0$ plane in the Y direction. (See Fig. 2.) The light will reach the upper receptor at an angle $\epsilon/2|X|$ below the X direction, while the light will reach the other receptor at an angle $\epsilon/2|X|$ above to give a total angular difference of

$$\delta\phi = \epsilon/|X|.$$

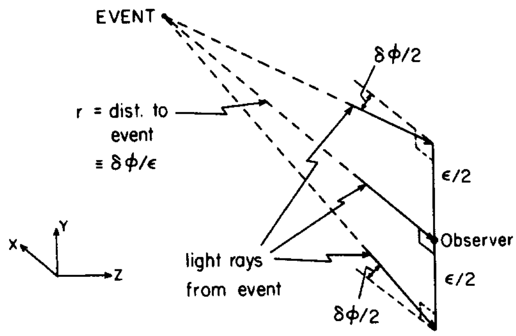

Fig. 2. Parallax definition of distance.

As A is at rest in this frame, this angle will be the parallax angle seen by A. A defines himself to be at coordinate $x_A = y_A = z_A = 0$. The event will be defined to be at a distance

$$|r| = \epsilon/\delta\phi = |X|$$

away and in the x_A direction, giving it coordinates

$$x_A = X, \qquad y_A = 0, \qquad z_A = 0.$$

The time t_A at which this event occurred is defined to be

$$t_A = \hat{t}'_A - |r|/c = T + |X|/c - |X|/c = T.$$

For A therefore, his time extended to all events, and his spatial coordinates of any event—both defined by parallax as described previously—coincide with the space-time coordinates T, X. (This is in fact true for all directions.)

We can now ask about the relation between B's extension of his definition of time and space to all events. Again consider an event occurring at $T, X, 0, 0$. The time of arrival at B, T'', will depend on whether the light arrives at B before or after $T'' = \tau$. It will arrive before this time if

$$T - X/c < \tau(1 - v/c)$$

for events such that $X < vT$ (i.e., events which lie on the same side of B as does A). The time of arrival at B of light from such events is

$$T'' = (X - cT)/(c + v).$$

The time $\hat{t}''_B$ on B's clock of the arrival of the light from this event is then

$$\hat{t}''_B = \sqrt{1 - v^2/c^2}\, T''$$
$$= \sqrt{1 - v^2/c^2}\,(X - cT)/(c + v).$$

Again B's receptors are a distance $\epsilon/2$ above and below the $Y = Z = 0$ plane and the angular difference between the rays at the two receptors in the Minkowsky coordinate system is

$$\delta\phi''_B = \epsilon/|vT'' - X|$$
$$= \epsilon(1 + v/c)/(X - vT).$$

This, however, is not the angle seen by B, due to the aberration effect. The angle $\delta\hat{\phi}''_B$ seen by B will be given by

$$\delta\hat{\phi}''_B = \epsilon\sqrt{\frac{1 - v/c}{1 + v/c}}\frac{1 + v/c}{X - vT}$$
$$= \epsilon\sqrt{1 - v^2/c^2}/(X - vT).$$

B will therefore say that the event is at a distance

$$|r''| = \frac{\epsilon}{\delta\hat{\phi}''_B} = \frac{X - vT}{\sqrt{1 - v^2/c^2}}$$

away. In B's coordinate system he himself is at

$$x_B = y_B = z_B = 0$$

and he chooses the direction away from A as the $+x_B$ axis. The event at X, T therefore occurs at the coordinate

$$x_B = 0 - |r_B| = \frac{X - vT}{\sqrt{1 - v^2/c^2}}.$$

The time t_B when this event occurs is defined by

$$t_B = \hat{t}''_B - r_B/c$$

$$= \sqrt{1 - v^2/c^2}\,(X - cT)/(c + v) - \frac{(X - vT)}{\sqrt{1 - v^2/c^2}}$$

$$= \frac{T - vX/c^2}{\sqrt{1 - v^2/c^2}}.$$

Not surprisingly, this is the Lorentz transformation relation. A more detailed calculation would show that the full Lorentz relations would apply to all X, Y, Z, T such that

$$(T - \tau) - |(X - vT)^2 + Y^2 + Z^2|^{1/2} < 0$$

(i.e., inside the past null cone from the turnaround point at

$$T = \tau, \qquad X = v\tau, \qquad Y = Z = 0).$$

For events outside this null cone an identical calculation shows that the relation between the parallax extended coordinates t_B, x_B and T, X for points with $Y = Z = 0$ are given by

$$t_B = \frac{T + vX/c^2 - 2v^2\tau/c^2}{\sqrt{1 - v^2/c^2}},$$

$$x_B = \frac{X + v(T - 2\tau)}{\sqrt{1 - v^2/c^2}}.$$

Again these are simply the Lorentz relations one would expect.

We can now examine the path of A in this extended coordinate system of B. A follows the path $X = Y = Z = 0$ with $t_A = T$. When A's clock reads $\hat{t}_A$, B will place A at the point $\tilde{t}$, $\tilde{x}$ where

Table I. Definition of coordinates and times for various observers.

Coordinates	
T, X, Y, Z	Background Minkowski space-time coordinates
t_A, x_A, y_A, z_A	Parallax coordinates defined by A
t_B, x_B, y_B, z_B	Parallax coordinates defined by B
$\hat{t}_A$	Time as read on A's clock = t_A at $x_A = y_A = z_A = 0$
$\hat{t}_B$	Time as read on B's clock = t_B at $x_B = y_B = z_B = 0$
$\tilde{t}_A(\hat{t}_B)$	Time in A's coordinate system at position of B when B's clock reads $\hat{t}_B$
$\tilde{t}_B(\hat{t}_A)$	Time in B's coordinate system at position of A when A's clock reads $\hat{t}_A$

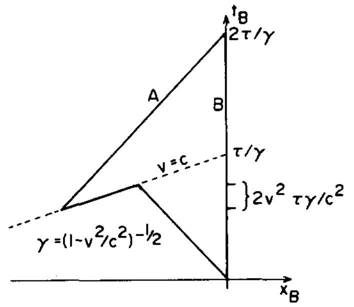

Fig. 3. Path of A and B as seen by B in his extended coordinates.

$$\tilde{t}_B = \frac{t_A}{\sqrt{1 - v^2/c^2}},$$

$$\tilde{x}_B = \frac{-vt_A}{\sqrt{1 - v^2/c^2}},$$

for $t_A < \tau(1 - v/c)$; while for $t_A > \tau\,(1 - v/c)$,

$$\tilde{t}_B = \frac{t_A - 2v^2\tau/c^2}{\sqrt{1 - v^2/c^2}},$$

$$\tilde{x}_B = \frac{v(t_A - 2\tau)}{\sqrt{1 - v^2/c^2}}.$$

Before B feels himself accelerate, he does see A's clock running slower than his own extended definition of time:

$$dt_A = \sqrt{1 - v^2/c^2}\, d\tilde{t}.$$

Similarly, after B feels his acceleration, he again sees A's clock go slower by the same factor. However, during the brief acceleration, B sees A act rather strangely. Just before the acceleration, A is at

$$\tilde{t}_B = \sqrt{\frac{1 - v/c}{1 + v/c}}, \qquad \tilde{x}_B = -v\tau\sqrt{\frac{1 - v/c}{1 + v/c}}.$$

Just after the acceleration, B sees A at

$$\tilde{t}_B = \sqrt{\frac{1 - v/c}{1 + v/c}} - \frac{2v^2\tau/c^2}{\sqrt{1 - v^2/c^2}}, \; \tilde{x}_B = -v\tau\sqrt{\frac{1 + v/c}{1 - v/c}}.$$

The acceleration felt by B will seem to him to have had the effect of pushing A further away a distance of

$$|\Delta\tilde{x}_B| = \frac{2v^2\tau}{c\sqrt{1 - v^2/c^2}}$$

and back in time a distance of

$$|\Delta\tilde{t}_B| = \frac{2v^2\tau}{\sqrt{1 - v^2/c^2}}.$$

During this strange behavior of A, A's clock will not be seen by B to run backwards (see Fig. 3). The time seen on A's clock just after B feels his acceleration will be the same as that seen just before.

Although on both the outward and inward legs of A's journey, as seen by B, A's clock will go slower by a factor of $\sqrt{1 - v^2/c^2}$, the additional "time" (as defined by B) that A has to cover due to the setback at the instant B feels the acceleration exactly compensates for the slowdown of A's clock. The time on A's clock at their moment of meeting will be 2τ, while that on B's will be $\sqrt{1 - v^2/c^2}\; 2\tau$, as required.

We see therefore, that although the standard argument resulting in the twin paradox is correct in all particulars, it fails to take into account the rather strange behavior of A as seen by B during the brief moment of B's acceleration. B does not see A as simply travelling out with velocity v, turning around, and returning. Rather B sees A traveling outward with velocity v, then suddenly traveling away and backward in time, and then returning with velocity v.

The above indicates the care that one must at all times exercise when talking about time and space in the context of special relativity. It also demonstrates the conventional (i.e., defined by convention) character of times defined at locations not at the clock associated with the observer himself.

The convention suggested here for defining distances and times not located at the observer by means of parallax offers the advantage over the more traditional radar ranging techniques that it is a purely local process, and does not require the observation at widely separated times. It requires only that times and angles in an arbitrarily small region of spacetime near the observer be used. Its only disadvantage (if that is what it is) is the rather strange behavior of bodies as seen by an accelerating observer.

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[1]A. Einstein, Ann. Phys. (Leipzig) **17**, 891 (1905). The literature generated on this topic is immense. See, for example, L. Marder, *Time and the Space Traveller* (Allen and Unwin, London, 1971) and references therein.

[2]See, for example, A. P. French, *Special Relativity* (Norton, New York, 1968), pp. 132–134.