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Brief Communication

Electromagnetic field of a charge moving with constant acceleration

Z.Ya. Turakulov [1]

Institute of Nuclear Physics, Ulugbek, Tashkent 702132, Uzbekistan

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Abstract

The field of a charge moving with constant acceleration is constructed analytically as an exact solution of the Maxwell equations obtained by the separation of variables method. It is shown that these fields do not contain any radiative part. The field of a moving charge is purely static and its energy evaluated beyond a sphere of any given radius does not depend on the acceleration.

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In this article we construct the electromagnetic field of a charge moving with constant acceleration as an exact solution of the Maxwell equations. For this purpose we introduce a curvilinear coordinate system for the Minkowskian space-time in which the charge world line coincides with a curved time-like coordinate line. The system may be introduced as follows. Let $\{t, z, \rho, \varphi\}$ be the standard circular cylinder coordinate system. The substitutions

$$\xi = \pm\,\mathrm{artanh}(t/z), \quad \zeta = \sqrt{z^2 - t^2}, \quad t = \zeta \sinh \xi, \quad z = \zeta \cosh \xi \tag{1}$$

for $|z| \geq |t|$ and

[1] E-mail: ULUG@hq.iae.tashkent.su.

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$$\xi = \pm \operatorname{artanh}(z/t), \quad \zeta = \sqrt{t^2 - z^2}, \quad t = \zeta \cosh \xi, \quad z = \zeta \sinh \xi \tag{2}$$

for $|z| \leq |t|$ define a coordinate system consisting of two charts which cover the corresponding space–time domains. Hereafter we assume that $t > -z$ because the remaining area of the space–time lies in the past from the charge world line chosen as $\zeta = \text{const.}$, $\rho = 0$. The coordinate surfaces $\xi = \text{const.}$ are three-dimensional Euclidean ($|z| > |t|$) or pseudo-Euclidean ($|z| < |t|$) semispaces equipped with ordinary circular cylinder coördinate systems $\{\zeta, \rho, \varphi\}$. The displacements $\xi \to \xi + c$ form a one-parametric group of Lorentz transformations such that the coordinate ξ parametrizes the charge world line and labels inertial frames in which the charge is at rest for any given value of the parameter. As the charge world line is nothing but an orbit of the group, it is natural to assume that its field remains invariant under the transformations, i.e., the field does not depend on the coordinate ξ. We assume also the vector potential of its electromagnetic field to be tangent to the orbits.

Let us introduce the well-known bispherical coordinate system $\{u, v, \varphi\}$ [1] in the $\xi = \text{const.}$ Euclidean semispaces:

$$\tanh u = \frac{2a\zeta}{\zeta^2 + \rho^2 + a^2}, \qquad \tan v = -\frac{2a\rho}{\zeta^2 - a^2 + \rho^2},$$
$$\zeta = \frac{a \sinh u}{\cosh u + \cos v}, \qquad \rho = \frac{a \sin v}{\cosh u + \cos v}, \tag{3}$$

and their pseudo-Euclidean generalizations for the other domain,

$$\tan u = \frac{2a\zeta}{-\zeta^2 + \rho^2 + a^2}, \qquad \tan v = -\frac{2a\rho}{-\zeta^2 - a^2 + \rho^2},$$
$$\zeta = \frac{a \sin u}{\cos u + \cos v}, \qquad \rho = \frac{a \sin v}{\cos u + \cos v}, \tag{4}$$

respectively. In the charge rest frame the surfaces $u = \text{const.}$ form a family of spheres surrounding the charge:

$$(\zeta - a \coth u_0)^2 + \rho^2 = a \sinh^{-1} u\,. \tag{5}$$

The extremal values $u = 0$ and $u = \infty$ correspond to the plane $\zeta = 0$ bounding the semispace and the pole of the coordinate system [1], respectively. The charge is assumed to lie in the pole. The metric in these coordinates has the following form:

$$ds^2 = a^2 \frac{\sinh^2 u \, d\xi^2 - du^2 - dv^2 - \sin^2 v \, d\varphi^2}{(\cosh u + \cos v)^2}, \quad |z| > |t|\,,$$

$$ds^2 = a^2 \frac{du^2 - dv^2 - \sin^2 u \, d\xi^2 - \sin^2 v \, d\varphi^2}{(\cos u + \cos v)^2}, \quad |z| < |t| \, .$$

The Maxwell equations for a source-free electromagnetic field specified as a 1-form α is $d^*d\alpha = 0$. Then, if a field is tangent to the orbits, i.e., $\alpha = \Phi(u, v) \, d\xi$, the equations have the following form:

$$0 = d^*d\alpha$$
$$= a^2 \left\{ -\frac{\sin v}{\sinh u} \left[\sinh u \left(\frac{\Phi_u}{\sinh u} \right)_u + \frac{1}{\sin v} (\Phi_v \sin v)_v \right] \right\} du \wedge dv \wedge d\varphi \, ,$$

where we used the Levi-Civita symbol which is $\varepsilon_{\xi u v \varphi} = a^4 (\cosh u + \cos v)^{-4} \sinh u \sin v$. The general solution of this equation,

$$\Phi(u, v) = a_0 (\cosh u - 1) + \sum_{n=1}^{\infty} a_n \sinh u \frac{d}{du} p_n(\cosh u) P_n(\cos v) \, ,$$

where $p_n(x)$ and $P_n(x)$ are Legendre polynomials, a_n are constant coefficients [1], represents the complete multipole expansion. The field of a charge is specified by the zeroth term with a_0 being the charge value. Indeed, if $\alpha = q(\cosh u - 1) \, d\xi$ then the integral $\oint {}^*d\alpha$ taken on any sphere $u = \text{const.}$ is equal to the flow of the electric field strength through it and, hence, to the charge value multiplied by 4π. Since ${}^*d\alpha = a^2 q \sin v \, dv \wedge d\varphi$ the integration is trivial and the factor q is the charge value. Consequently, the charge q placed in the point $u = \infty$ produces the field

$$\alpha = q(\cosh u - 1) \, d\xi, \qquad \alpha = q(\cos u - 1) \, d\xi \, ,$$

for the $|z| > |t|$ and $|z| < |t|$ domains, respectively (the derivation of the second expression is similar). In the charge rest frame the strength of the magnetic field is zero and that of the electric field has only a u-component equal to $a^{-1} q(\cosh u + \cos v)$. Its energy evaluated beyond a sphere $u = u_0$ does not depend on the charge acceleration:

$$m_e = \frac{1}{8\pi} \int E^2 \, d^3x = \frac{q^2}{8\pi a} \int_0^{u_0} du \int_0^{\pi} dv \int_0^{2\pi} d\varphi \, (\cosh u + \cos v) \sin v$$
$$= \frac{q^2}{2a} \sinh u_0 = (2a^{-1} q^2 \sinh u_0) \equiv \frac{q^2}{2R_e} \, ,$$

where, due to Eq. (5), R_e is the radius of the sphere. It is seen explicitly that the field does not contain any radiative part.

To rewrite the field in standard Lorentzian coordinates it suffices to return to the coordinates $\{t, z, \rho, \varphi\}$. It turns out that the vector potential components are

$$A_t = \frac{z}{z^2-t^2}\left(\frac{z^2-t^2+\rho^2+a^2}{\sqrt{[(\sqrt{|z^2-t^2|}-a)^2+\rho^2][(\sqrt{|z^2-t^2|}+a)^2+\rho^2]}}-1\right),$$

$$A_z = \frac{t}{z^2-t^2}\left(\frac{z^2-t^2+\rho^2+a^2}{\sqrt{[(\sqrt{|z^2-t^2|}+a)^2+\rho^2][(\sqrt{|z^2-t^2|}+a)^2-\rho^2]}}-1\right),$$

for both domains.

References

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