The fields and self-force of a constantly accelerating spherical shell

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We present a partial differential equation describing the electromagnetic potentials around a charge distribution undergoing rigid motion at constant proper acceleration, and obtain a set of solutions to this equation. These solutions are used to find the self-force exactly in a chosen case. The electromagnetic self-force for a spherical shell of charge of proper radius R undergoing rigid motion at constant proper acceleration a_0 is, to high order approximation, $(2e^2a_0/R)\sum_{n=0}^{\infty}(a_0R)^{2n}((2n-1)(2n+1)^2(2n+3))^{-1}$, and this is conjectured to be exact.

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1. Introduction

The problem of self-force, and the related problem of radiation reaction, has a long history; see Rohrlich (1997) for a brief review and Spohn (2004) for a longer survey. It is currently studied both because it is important in the extreme physical conditions now realised in experiments (Bulanov *et al.* (2011)) and because it continues to raise questions about the consistency and correct treatment of electromagnetism (Gralla *et al.* (2009); Cremaschini and Tessarotto (2011)).

Lorentz (1904, 1909), Abraham (1905), Poincaré (1905, 1906) and Schott (1912) carried out pioneering work on the understanding of self-force and performed approximate calculations for a rigid spherical shell whose centre undergoes arbitrary motion. They obtained the lowest order terms in a power series solution, expanded in powers of the radius R of the sphere, its acceleration and derivatives thereof, and combinations of all these. von Laue (1909) clarified and simplified the work of earlier authors by expressing it in a manifestly Lorentz-covariant form, see eqn (6.4). Dirac (1938) adopted a different strategy, starting from a manifestly covariant Lagrangian, and reproduced this equation; it is now commonly known as the Lorentz-Dirac or Abraham-Lorentz-Dirac (ALD) equation. To be precise, Dirac's version has a modified inertial term for reasons that we shall discuss below. The ALD equation includes terms of order R^{-1} and R^0 , and neglects terms of order Rand above.

Nodvik (1964) brought arguably greater rigour to the derivation using the Lagrangian formalism and presented the power series expansion to $O(R^2)$. Further work has clarified the interpretation of the force and the associated energy-momentum movements in the field, and has extended the calculation of the self-force, such that the 'linear' contribution can be obtained exactly, in a beautifully neat and well-behaved expression (eqn (6.5)). The word 'linear' here does not mean, however, that all the contribution at order R^1 is accounted for by this expression,

since it neglects terms involving R multiplied by non-linear combinations of the velocity or its derivatives (see Caldirola (1956), Yaghjian (1992) and Rohrlich (1999, 2000) for details).

Throughout this century-long period, the study of self-force has continued to raise doubts and questions concerning the correct interpretation or application of results. Many of these issues were helpfully clarified by Yaghjian (1992) and Rohrlich (1965, 1990). For example, Yaghjian showed that there is no 'preacceleration' in the classical electromagnetic theory of small charged bodies; its supposed existence (discussed by authors too numerous to list) was based on a misunderstanding of the approximations involved in the ALD equation. Numerous authors and textbooks continue to assert that the ALD equation has pathalogical runaway solutions, but this too is an artifact of its incorrect application, as we discuss in section 6, and as Gralla *et al.* (2009) and Rohrlich (2001), among others, have commented. A further, and different, source of confusion is the effect of Poincaré stresses in a body undergoing acceleration.

To have a concrete result with no approximation, for a non-trivial case, remains very helpful both for pedagogy and as a test case for more general but approximate formulae. Most previous work has been devoted to the case of arbitrary motion, endeavouring to extend the set of available approximate results and understand them better. The present work takes the complimentary path of restricting the motion to a simple case, and solving it exactly. We find the electromagnetic self-force for a spherical charged shell undergoing constant proper acceleration while maintaining constant proper size and shape. To our knowledge this has not previously been done. The method of calculation involves a useful general observation about the electromagnetic potentials in the case of arbitrary bodies undergoing rigid hyperbolic motion.

2. Strategy

The classical electromagnetic contribution to the self-force for an arbitrary distribution of charge, undergoing arbitrary motion, is

$$\mathbf{f}_{\text{self}} = \int (\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) \rho \, \mathrm{d}V \tag{2.1}$$

where ρ is the volume density of charge, the integral is taken over all space (though of course it suffices to restrict it to the region where $\rho \neq 0$) at some instant of time tin a chosen inertial reference frame, and the fields $\mathbf{E}(t, x, y, z)$, $\mathbf{B}(t, x, y, z)$ are those produced at each point by the set of charges in question (owing to their previous behaviour at some set of source events). Note that if ρ is everywhere finite, then the fields are also finite and continuous, and furthermore the contribution to the field at a given point (x, y, z) made by the charge within a distance d of that point vanishes in the limit $d \to 0$ (because the quantity of charge scales as ρd^3 but the fields of a point charge scale as $1/d^2$). Hence no ambiguity arises about whether the integral above correctly excludes the non-physical idea of a source interacting with itself: only pairwise interactions should be included, and only pairwise interactions contribute to the integral.

In the case of a distribution of charge that is rigid, there must be further forces acting (for example, the Poincaré stresses) and these further forces may be deemed

'internal' and yet have a non-vanishing sum over the body. Hence the total self-force includes both the electromagnetic part that is the main subject of this paper, and a further part that we shall discuss in section 6. The non-vanishing contribution from the 'internal' forces does not break Newton's third law but rather is required by it, since there is a continuous transfer of momentum between the shell and the surrounding electromagnetic field, mediated via the electric charge.

In the case of a spherical shell of charge, we treat a shell of finite thickness w, so that the volume density of charge is finite, and then explore the limit $w \to 0$, such that the surface charge density $\sigma = \rho w$ remains finite. We shall consider a shell which moves such that there exists a frame in which the whole shell is at rest at some moment, so the $\mathbf{v} \wedge \mathbf{B}$ term vanishes from the force. Let R be the inner radius of the shell in such a frame (the shell being spherical in that frame), and assume the charge density ρ is uniform between the inner and outer surfaces at r = R and r = R + w. For a sufficiently thin shell, the field within the material of the shell (where $\rho \neq 0$) can be approximated by the linear form

$$\mathbf{E}(r) = \mathbf{E}(R) + \frac{s}{w} \left(\mathbf{E}(R+w) - \mathbf{E}(R) \right)$$
(2.2)

where $\mathbf{E}(r, \theta, \phi)$ is the field at radius r from the centre of the sphere, s = r - R is the distance through the shell, and we suppressed explicit indication of the dependence of the fields on polar angles θ, ϕ . Hence

$$\mathbf{f}_{\text{self}} = \int \int \rho \frac{1}{w} ((w-s)\mathbf{E}(R) + s\mathbf{E}(R+w)) \mathrm{d}s \mathrm{d}S$$
(2.3)

where dS is an element of surface. Carrying out the integration over s gives, in the limit $w \to 0$,

$$\mathbf{f}_{\text{self}} = \int \sigma \frac{\mathbf{E}_{-} + \mathbf{E}_{+}}{2} \mathrm{d}S \tag{2.4}$$

where $\mathbf{E}_{-} = \lim_{\epsilon \to 0} \mathbf{E}(R - |\epsilon|)$ is the field on the interior surface of the shell, and $\mathbf{E}_{+} = \lim_{\epsilon \to 0} \mathbf{E}(R + |\epsilon|)$ is the field on the exterior surface of the shell. In other words, to calculate the self-force it suffices to use the average of these two fields. This deals with the discontinuity in \mathbf{E} in the limit $w \to 0$. Now, we know from Gauss' law what the size and direction of this discontinuity is: its size is σ/ϵ_0 and its direction is normal to the surface. For a spherically symmetric case, σ is the same everywhere over the shell and we have

$$\mathbf{E}_{+} = \mathbf{E}_{-} + \frac{\sigma}{\epsilon_0} \hat{\mathbf{r}}.$$
 (2.5)

Since the contribution from $\sigma \hat{\mathbf{r}}$ is radial and spherically symmetric, its integral gives zero contribution to the self-force. Therefore we can also write

$$\mathbf{f}_{\text{self}} = \sigma \int \mathbf{E}_{-} \mathrm{d}S = \sigma \int \mathbf{E}_{+} \mathrm{d}S.$$
(2.6)

In other words, one can use either the field on the interior surface or the field on the exterior surface to calculate the self-force. For convenience, we will use the field on the interior surface.

3. Simple calculation of the lowest order approximation

Consider a particle undergoing hyperbolic motion in the x direction, so that relative to some frame its position as a function of time is $y_s = z_s = 0$ and

$$x_s(t) = \sqrt{L^2 + t^2}$$
(3.1)

where

$$L = 1/a_0 \tag{3.2}$$

is a natural distance scale set by the proper acceleration a_0 . The potentials and fields of a point charge q undergoing such motion are easily calculated, see for example Fulton & Rohrlich (1960), Eriksen & Gron (2000), Steane (2012). Adopting rectangular coordinates and c = 1, one finds the scalar and vector potentials (Liénard-Wiechart potentials)

$$\phi = Q \frac{x\delta - t\zeta}{\zeta(x^2 - t^2)}, \quad A_x = Q \frac{t\delta - x\zeta}{\zeta(x^2 - t^2)}, \quad A_y = A_z = 0,$$
(3.3)

where $Q = q/4\pi\epsilon_0$ and

$$\delta = L^2 + x^2 + y^2 + z^2 - t^2, \quad \zeta = \sqrt{\delta^2 - 4L^2(x^2 - t^2)}.$$
(3.4)

In the following we will only need the solution at t = 0, which is the moment when the particle is at rest in the frame under consideration. At this moment one finds the magnetic field is $\mathbf{B} = 0$ and the electric field is (adopting now cylindrical coordinates)

$$E_x = -4QL^2(L^2 + \rho^2 - x^2)/\zeta^3, \quad E_\rho = 8QL^2\rho x/\zeta^3.$$
(3.5)

Now consider a spherical shell of charge undergoing rigid hyperbolic motion. By rigid motion we mean motion such that, at any moment, there is an instantaneous rest frame in which all parts of the body are at rest, and the physical dimensions of the body in each such frame are constant (i.e. the same in all successive instantaneous rest frames), see for example Steane (2012). Such rigid motion has the property that once the initial conditions and the worldline of one particle in the body are given, the worldlines of all other particles are fixed. For the case of constant proper acceleration (i.e. hyperbolic motion) of one part of a body, the resulting constraint takes a particularly simple form: every part of the body undergoes constant proper acceleration, but with a proper acceleration varying with distance from a plane normal to the acceleration. Such a body is at rest relative to the well-known "constantly accelerating reference frame"; the special plane is a horizon which can conveniently be placed at x = 0 by a suitable choice of origin. The worldline of each part of the shell is given again by (3.1), but now we must allow L to be chosen for each part such that that part comes to rest at the correct position at t = 0. This is easily done: one sets L equal to the location of the part at t = 0. For example, for a spherical shell of proper radius R centred at $(L_0, 0, 0)$ one must use $L = L_0 + R \cos \theta$ where θ is the azimuthal angle relative to the centre of the shell and the x axis.



Figure 1. Electric field lines in the xy plane, in the instantaneous rest frame, for a spherical charged shell of radius R = 1/2 undergoing rigid hyperbolic motion along the x axis with proper acceleration $a_0 = 1/L_0 = 1$.

The observation of the previous paragraph makes it possible to write down an integral expression for the field at the centre of the shell. A moment's thought about the geometry leads one to conclude that the integrand is obtained from Eq. (3.5) with the replacements $L \to x_s$, $\rho^2 \to R^2 - (x_s - L_0)^2$, $x \to L_0$, leading to

$$E_x(L_0, 0, 0) = \frac{Q}{4\pi R} \int_{L_0 - R}^{L_0 + R} \int_0^{2\pi} \mathrm{d}\phi \mathrm{d}x_s \frac{4x_s^2 \left(2L_0^2 - R^2 - 2L_0 x_s\right)}{R^3 \left(R^2 + 4L_0 x_s\right)^{3/2}}$$
(3.6)

and $E_y = E_z = 0$. The integration is straightforward. One obtains

$$E_x(L_0, 0, 0) = \frac{Q}{L_0^2} \left(-\frac{2}{3} \frac{L_0}{R} + \frac{2}{15} \frac{R}{L_0} \right).$$
(3.7)

To find the self-force of such a shell, to lowest order approximation, is now extremely easy. One simply multiplies this expression by q:

$$f_{\rm self} = -\frac{2}{3}\frac{e^2}{L_0R} + \frac{1}{L_0^2}O(R/L_0) = -\frac{2}{3}\frac{e^2a_0}{R} + O(a_0^3R)$$
(3.8)

where to reduce clutter we introduced $e^2 \equiv q^2/(4\pi\epsilon_0)$. To see that this expression is accurate to lowest order, as claimed, argue as follows. The self-force is in fact given by the either of the integrals in (2.6), of which we choose the first. The field at the interior surface, \mathbf{E}_- , may be obtained from a Taylor expansion of the field throughout the empty space in the interior of the shell, expanded about the point $(L_0, 0, 0)$. One finds that \mathbf{E}_- is given by Eq. (3.7) plus further terms forming a power series in $(x_s - L_0)/L_0$. Using the symmetry of the charge distribution, the linear terms cancel in the calculation of the self-force, and the next contribution is at order $(R/L_0)^2$ compared to the first. In summary, the field is, to good approximation, uniform throughout the interior (c.f. figure 1), and to find the self-force one simply multiplies this uniform field by the total charge of the shell. Thus we obtain Eq. (3.8). This is the simplest derivation of it of which I am aware.

4. The potentials

We now turn to the task of calculating the electric field, and hence the self-force, exactly. We will obtain the field from the potentials via $\mathbf{E} = -\nabla \phi - \partial \mathbf{A}/\partial t$. In

empty space $\boldsymbol{\nabla} \cdot \mathbf{E} = 0$, hence

$$\nabla^2 \phi = -\frac{\partial}{\partial t} \nabla \cdot \mathbf{A} = -\frac{\partial^2 A_x}{\partial x \partial t}$$
(4.1)

where we used (3.3) in the final step. This result is valid for the potentials of a single constantly accelerating point charge, and since for a distribution of charge the potentials simply add and the equation is linear, it remains valid for an arbitrary distribution of point charges, each undergoing constant proper acceleration in the x direction (not necessarily all having the same acceleration). Evaluating $\partial A_x/\partial t$ for a single point charge using (3.3), one finds at t = 0,

$$\left. \frac{\partial A_x}{\partial t} \right|_{t=0} = \frac{Q\delta}{\zeta x^2} = \frac{\phi}{x}.$$
(4.2)

It follows that the electric field can be obtained from

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{\phi}{x}\hat{\mathbf{x}} \tag{4.3}$$

and Eq (4.1) can be written

$$x^2 \nabla^2 \phi + x \frac{\partial \phi}{\partial x} - \phi = 0.$$
(4.4)

This is a linear homogeneous second order differential equation for the potential in free space. It may be regarded as a replacement for the Laplace equation; it applies in the free space around a point charge undergoing hyperbolic motion in the x direction.

Using the same argument concerning rigid motion as we presented after Eq (3.5), we will now show that Eq (4.4) has a wider range of validity. Consider a distribution of charge undergoing rigid hyperbolic motion in the x direction. Then in the instantaneous rest frame, the total scalar potential at any field point (x, y, z) is given by

$$\phi(x, y, z) = \int \int \int \mathrm{d}x_s \mathrm{d}y_s \mathrm{d}z_s \rho(x_s, y_s, z_s) \tilde{\phi}(x_s; x, y - y_x, z - z_s)$$
(4.5)

where ρ is the charge density and $\tilde{\phi}$ is the potential per unit charge obtained from Eqn (3.3):

$$\tilde{\phi}(L; x, y, z) = \frac{L^2 + x^2 + y^2 + z^2}{x[(L^2 + x^2 + y^2 + z^2)^2 - 4L^2 x^2]^{1/2}}.$$
(4.6)

Note that we are evaluating the potential quite correctly using the distribution of charge at its 'present' location (t = 0), not the location it had at the source events, because we can take advantage of the fact that we already know how to calculate what contribution each source event makes to the potential 'now' (at t = 0) for the simple type of motion under consideration.

The important property of Eq (4.5) for our purposes is that every source charge presents a contribution that is calculated by substituting the *same* value for x into the formula for $\tilde{\phi}$, and therefore every contribution satisfies (4.4) with a common

value of x, and therefore (4.4) applies to the net resulting ϕ (the one given by (4.5)). Hence Eq (4.4) describes the potential in the free space around, or in cavities within, any distribution of charge that is undergoing rigid hyperbolic motion. (In obtaining it we have made a gauge choice that makes $\phi \to 0$ at positions far from any finite charge contained in a finite region.)

The problem of finding the potential inside an accelerating spherical shell can now be addressed by seeking a potential function ϕ that satisfies (4.4) and that agrees with a suitable boundary condition.

(a) Solution of the differential equation

In order to solve (4.4), introduce another function $V(x,y,z)\equiv \phi/x.$ Then one finds that V satisfies

$$x\nabla^2 V + 3\frac{\partial V}{\partial x} = 0 \tag{4.7}$$

If V = V(x) (i.e. a solution with no dependence on y and z) this leads to a Cauchytype equation with general solution

$$V(x) = a_0 + a_1 \frac{1}{x^2} \tag{4.8}$$

More generally, the equation is not readily separable, but we can look for solutions of a form suitable to the type of problem we are treating. We would like solutions which are non-singular at $\rho = 0$ and which have axial symmetry, so we try the ansatz

$$V(x,\rho) = \sum_{n=0}^{N} f_n(x)\rho^{2(N-n)}$$
(4.9)

where $\rho = (y^2 + z^2)^{1/2}$ is the radial coordinate in a cylindrical coordinate system. Substituting this ansatz into (4.7) and using $\nabla^2 \rho^n = n^2 \rho^{n-2}$, one finds

$$\sum_{n=0}^{N} \left(x f_n'' + 3 f_n' \right) \rho^{2(N-n)} + 4(N-n)^2 x \rho^{2(N-n-1)} f_n = 0.$$
(4.10)

Since this is to be satisfied at all ρ , we can set the coefficient of $\rho^{2(N-k-1)}$ equal to zero:

$$xf_{k+1}'' + 3f_{k+1}' + 4(N-k)^2 xf_k = 0. ag{4.11}$$

Integrating, we have

$$xf'_{k+1} + 2f_{k+1} = -4(N-k)^2 \int xf_k dx$$
(4.12)

which applies for values of k between 0 and N-1. The coefficient of ρ^{2N} in (4.10) gives

$$xf_0'' + 3f_0' = 0. (4.13)$$

$$\begin{array}{ll} 1, & 1/x^2, \\ 2\rho^2 - x^2, & \rho^2/x^2 - 2\log x, \\ 3\rho^4 - 6\rho^2 x^2 + x^4, & \rho^4/x^2 - 3x^2 - 4(2\rho^2 - x^2)\log x, \\ 4\rho^6 - 18\rho^4 x^2 + 12\rho^2 x^4 - x^6, & \rho^6/x^2 - 27\rho^2 x^2 + 7x^4 - 6(3\rho^4 - 6\rho^2 x^2 + x^4)\log x \end{array}$$

Table 1. The first few solutions of Eq (4.7) having the form (4.9).

This has the general solution $f_0 = a_0 + a_1/x^2$, which can be substituted into the right hand side of (4.12) in order to find f_1 , and hence f_2 and so on. Table 1 shows the first few solutions found this way.

In the rest of this paper we will restrict attention to solutions having the form

$$f_k(x) = \sum_{m=0}^k c_m^k x^{2m}$$
(4.14)

where the c_m^k are constants and $k \ge 0$. This restriction is merely for convenience: one can expand $V(x, \rho)$ for any given charge distribution in terms of any sufficiently complete set of basis solutions, and this is one such set for our problem. Substituting (4.14) into (4.12) we obtain

$$\sum_{m=0}^{k+1} 2(m+1)c_m^{k+1}x^{2m} = -4(N-k)^2 \sum_{m=0}^k c_m^k \frac{x^{2(m+1)}}{2(m+1)}$$
(4.15)

Hence

$$c_{m+1}^{k+1} = \frac{-(N-k)^2}{(m+1)(m+2)} c_m^k \tag{4.16}$$

and c_0^k are undetermined constants to be found from boundary conditions. For example, with N = 2 one finds c_0^0 is undetermined, then $c_1^1 = -2c_0^0$ and c_0^1 is undetermined. These are in turn used to find $c_2^2 = (-1/6)c_1^1 = c_0^0/3$ and $c_1^2 = (-1/2)c_0^1$. Finally, c_0^2 is undetermined. For any given value of N we thus find a solution involving a set of N + 1 undetermined constants, each of which multiplies a polynomial function of x and ρ . The polynomial function which appears with the factor c_0^n is the one involving $\{c_0^n, c_1^{n+1}, c_2^{n+2}, \ldots, c_{N-n}^N\}$. To be precise, it is $\sum_{m=0}^{N-n} c_m^{(n+m)} x^{2m} \rho^{2(N-n-m)}$. By using the above recurrence relation, the terms in the sum are easily found, and one obtains

$$V_n(x,\rho) = \sum_{m=0}^n \frac{(-1)^m}{m+1} \binom{n}{m}^2 x^{2m} \rho^{2(n-m)} = \rho^{2n} {}_2F_1(-n,-n,2,-x^2/\rho^2) \,. \tag{4.17}$$

where $\binom{n}{m} = n!/n!(n-m)!$ is a binomial coefficient and $_2F_1$ is the hypergeometric function. For $n = 0...\infty$ we thus obtain an infinite set of linearly independent solutions of Eqn (4.7). The first few of these functions are shown in the left hand column of table 1, in each case multiplied by (n + 1) to make the coefficients all integer.

The electric field given by these solutions is (using Eqn (4.3) and $\phi = xV_n$)

$$E_{x,n} = -x\frac{\partial V_n}{\partial x} - 2V_n = 2\sum_{m=0}^n (-1)^{m+1} \binom{n}{m}^2 x^{2m} \rho^{2(n-m)}$$
(4.18)

$$= -2\rho^{2n}{}_{2}F_{1}\left(-n, -n, 1, -x^{2}/\rho^{2}\right), \qquad (4.19)$$

and

$$E_{\rho,n} = -x \frac{\partial V_n}{\partial \rho} = 2 \sum_{m=0}^{n-1} (-1)^{m+1} \frac{n-m}{m+1} \left(\frac{n}{m}\right)^2 x^{2m+1} \rho^{2(n-m-1/2)}$$
(4.20)

$$= -2nx\rho^{2n-1}{}_{2}F_{1}(1-n,-n,2,-x^{2}/\rho^{2}).$$
(4.21)

5. Interior field and self-force of the accelerating spherical shell

We now apply the ideas of the previous section to the case of the spherical shell of charge undergoing rigid hyperbolic motion. Without loss of generality, choose distance units such that the centre of the shell comes to rest at x = 1 (this amounts to setting $L_0 = 1$ in Eqn (3.8)). Then the surface of the shell is at $\rho^2 = R^2 - (x-1)^2$. By substituting this into (4.19) one finds the contribution to the electric field at the (interior surface of the) shell for each of the potential solutions identified in the previous section. For a uniformly charged spherical shell with unit total charge, the contribution to the self-force is then given by

$$f_{\text{self},n} = \frac{1}{2R} \int_{1-R}^{1+R} E_{x,n}(x,\rho(x)) dx = 2 \sum_{m=0}^{n} \frac{(-1)^{m+n+1}}{2m+1} R^{2m} \binom{n}{m}$$
(5.1)

$$= 2(-1)^{n+1} {}_{2}F_{1}(1/2, -n, 3/2, R^{2}).$$
(5.2)

For example, the first few values of this expression are given by

$$\frac{1}{2}f_{\text{self},n} = -1, \ \left(1 - \frac{R^2}{3}\right), \ \left(-1 + \frac{2R^2}{3} - \frac{R^4}{5}\right), \ \left(1 - \frac{3R^2}{3} + \frac{3R^2}{5} - \frac{R^6}{7}\right), \ \dots (5.3)$$

To find the self-force of the shell, it remains to identify what linear combination of these solutions matches the potential, and hence the field, in the interior of a uniformly charged shell undergoing rigid hyperbolic motion. To find this we first evaluate the potential on the axis of the shell, by performing the integral in (4.5) for y = z = 0:

$$\phi(x,0,0) = \frac{1}{2R} \int_{1-R}^{1+R} \frac{R^2 + 2x_s + x^2 - 1}{x\sqrt{(R^2 + 2x_s + x^2 - 1)^2 - 4x_s^2 x^2}} \mathrm{d}x_s$$
(5.4)

$$= \frac{|\beta_{-}| - |\beta_{+}|}{4Rs^{2}x} + \frac{ix(R^{2} + s^{2})}{4Rs^{3}} \log\left[\frac{s|\beta_{+}| + i(\beta_{+} - 2Rx^{2})}{s|\beta_{-}| + i(\beta_{-} + 2Rx^{2})}\right] (5.5)$$

where $s = \sqrt{x^2 - 1}$ and $\beta_{\pm} = (1 \pm R)^2 - x^2$. This function is plotted in figure 2 (note, the function is real since the argument of the log has unit modulus when s



Figure 2. Electric field E (thick line) and potential ϕ (thin line) on the x axis for a sphere of radius R = 1/2 with unit charge accelerating in the x direction with proper acceleration $a_0 = 1/L_0 = 1$.

is real). For 1 - R < x < 1 + R (that is, for points inside the sphere), it takes the form

$$\phi(x,0,0) = \frac{-R^2 + x^2 - 1}{2Rx(x^2 - 1)} + \frac{x(R^2 + x^2 - 1)}{4R(x^2 - 1)^{3/2}} \tan^{-1} \left[\frac{2\sqrt{x^2 - 1}}{2 - x^2}\right]$$
(5.6)

We will now find the potential throughout the interior of the spherical shell by finding a function that satisfies (4.7) and that matches the known potential and all its derivatives with respect to x at the point (x, y, z) = (1, 0, 0) (the centre of the sphere). This may be compared with matching a boundary condition along an infinite line, since for an analytic function, the value of the function and all its derivatives at a point suffice to define the function along a line by Taylor expansion. Of course the potential given by the shell actually has a discontinuous change in slope at $x = 1 \pm R$, but this is irrelevant to the method. By using ϕ and all its derivatives at one point we gain sufficient information to fix the potential throughout the interior of the shell.

We employ the solution

$$V(x,\rho) = \sum_{n=0}^{N} a_n V_n(x,\rho)$$
(5.7)

where $V_n(x, \rho)$ is given by (4.17), the constants a_n are to be determined, and the solution becomes exact in the limit $N \to \infty$. We find the constants by writing down the Taylor expansion of both V(x, 0) and $\phi(x, 0, 0)$ about the point x = 1, and equating coefficients of powers of (x - 1). This gives a set of N + 1 simultaneous equations for the a_n which can be solved by matrix inversion. For example, for N = 2 one finds

$$a_0 = \frac{34}{15R} + \frac{122}{105}R, \ a_1 = \frac{56}{15R} + \frac{88}{35}R, \ a_2 = \frac{9}{5R} + \frac{9}{7}R.$$
 (5.8)

The self-force is then given by

$$f_{\text{self}} \simeq \sum_{n=0}^{N} a_n f_{\text{self},n} \tag{5.9}$$

where the expression becomes exact in the limit $N \to \infty$ and for given finite N one obtains an exact expression for the terms up to order R^{2N-3} (see appendix). The above values for a_0, a_1, a_2 when N = 2 yield, for example,

$$f_{\rm self} \simeq -\frac{2}{3R} + \frac{2}{45}R - \frac{358}{525}R^3 - \frac{18}{35}R^5$$
 (5.10)

in which the first two terms are exact. Further details of the Taylor expansions and the solution coefficients are given in the appendix. The end result is that the selfforce on the uniformly charged spherical shell undergoing rigid hyperbolic motion is

$$f_{\rm self} = \frac{2e^2}{RL_0} \sum_{n=0}^{\infty} \frac{(R/L_0)^{2n}}{(2n-1)(2n+1)^2(2n+3)},\tag{5.11}$$

where I have calculated this result for terms up to order R^{101} , and I conjecture its validity at all orders. This is the main result of this paper. At $L_0 = 1$ and $e^2 = 1$ the first few terms in this series are

$$f_{\text{self}} \simeq -\frac{2}{3R} + \frac{2}{45}R + \frac{2}{525}R^3 + \frac{2}{2205}R^5 + \frac{2}{6237}R^7 + \cdots$$
 (5.12)

It is remarkable that the result takes such a simple form. One expects a power series in odd powers of R, but it is striking that the coefficients have such a simple expression. The coefficients in the Taylor series for $\phi(x, 0, 0)$ are, by contrast, much more complicated.

The lowest order terms in the expression (5.12) for f_{self} can be found by hand. The series expansion and matrix inversion method described above is essentially simple if laborious; it can easily be executed in a computer algebra system such as *Mathematica*. This is how the series up to order R^{101} was calculated. The problem of finding the expression for an arbitrary coefficient in the series, and thus confirming the (eminently reasonable) conjecture that the result is valid at all orders, is more difficult and remains open. As a further check, the field inside the shell was calculated by direct numerical integration of the fields of a point charge given in Eqn (3.5), and compared with the one obtained from equations (4.19) and (4.21), multiplied by the coefficients a_n and summed. This confirms, to within numerical precision, that one obtains the field correctly at points away from the x axis (as well as on it).

Numerical integration was also used to check the overall result (5.11). The field at the interior surface of the shell was obtained by calculating the field (numerically) at four points near the surface and then extrapolating to the surface using a cubic fit; this enables the singular behaviour to be avoided. The numerically-calculated self-force is compared with the prediction (5.11) in figure 3. By fitting a polynomial in R^2 to the numerical calculation of Rf_{self} , the first few coefficients in (5.11) were confirmed to good approximation (one part in 10^{9,6,3} for the term of order $R^{-1,1,3}$ respectively). However the numerical calculation rapidly runs out of precision in obtaining the higher order terms by this 'brute force' method, whereas the analytical result is easy to obtain for many more orders.



Figure 3. Top: self-force as a function of R, for $L_0 = 1$, as calculated by numerical integration (points) and by eqn (5.11) (line). Bottom: the difference between the numerical and analytical results.

6. Discussion

Equation (5.11) is the result of a fully relativistic calculation, and is believed to be exact. It gives the spatial part of the self-four-force in the instantaneous rest frame. The temporal part is zero in that frame. Since the resulting four-vector is parallel to the four-acceleration, it may be written

$$\mathsf{F}_{\text{self}} = \frac{2e^2}{R} \dot{\mathsf{v}} \sum_{n=0}^{\infty} \frac{(Ra_0)^{2n}}{(2n-1)(2n+1)^2(2n+3)} \tag{6.1}$$

where v is the four-velocity (in an index-free notation) and the dot signifies the derivative with respect to proper time. The equation of motion of the spherical shell of charge is

$$\mathsf{F}_{\text{other}} + \mathsf{F}_{\text{self}} = m_0 \dot{\mathsf{v}} \tag{6.2}$$

where F_{other} is the four-force applied to the shell by some other influence, which must, in this case, take such a form that, when combined with F_{self} , it gives to the shell the motion that has been assumed. m_0 is the 'bare' rest mass of the shell—that is, the rest mass which enters on the right hand side of the equation of motion of the material of the shell itself. This does not include a contribution from the energymomentum density of the electromagnetic field sourced by the charge of this shell, which is quite correct. To find out about the dynamics of the field sourced by the shell, one can appeal to the conservation of energy-momentum which is perfectly obeyed in classical electromagnetism. The force from the field on the charge (i.e. the electromagnetic self-force f_{self}) implies the presence of an equal and opposite 'force on' (i.e. rate of injection of momentum into) the electromagnetic field. Hence the field sourced by the spherical shell must be acquiring energy-momentum at the rate $-F_{self}$.

Now, we assumed that the motion of the shell was rigid. Therefore, the 4-force $F_{\rm other}$ must both oppose the tendency of the electromagnetic force to explode or otherwise deform the proper shape of the shell, and it must also provide the correct amount of force to result in the assumed net acceleration of the shell. It makes

sense, therefore, to write

$$\mathsf{F}_{\text{other}} = \mathsf{F}_{\text{ext}} + \mathsf{F}_{\text{P}} \tag{6.3}$$

where F_{ext} is provided by an external force-producing entity such as a rod pushing on the sphere, and F_P is deemed 'internal', it may be regarded as the integral of the Poincaré stresses.

To clarify the separation into F_{ext} , F_P and F_{self} it helps to imagine a deliberately artificial scenario. Consider a non-rigid latex balloon with a uniform coating of charge attached to it by glue, and suppose that inside the balloon there is a programable device which can push out or pull in the interior surface of the balloon (the latex itself having negligible tension). We plan to push this balloon from the outside using a wooden rod, so as to cause hyperbolic motion of its centre. Before doing so, we first program the device to produce, starting from some preset starting time, whatever forces are calculated to maintain rigid motion of the balloon, given that it is going to be pushed externally by our rod. The internal device cannot, of course, produce any arbitrary combination of forces but must respect Newton's third law. Therefore we shall have to plan for the external force to have a suitable form in order to obtain a self-consistent solution. Once this is done, we can launch the experiment. Then the three terms F_{ext} , F_P , F_{self} are provided by the rod, the internal device and the electromagnetic field sourced by the balloon, respectively.

In this separation of forces, F_{P} may or may not be called a self-force. From the point of view of the surface of the balloon, F_{P} is an 'external' force, i.e. one not sourced by the rubber of the balloon's surface, nor by the charge and its associated electromagnetic field. However, from the point of view of an outside observer observing the sphere has a whole, F_{P} is a self-force, because it contributes something in addition to the force $\mathsf{F}_{\mathrm{ext}}$ that 'the rest of the world' is providing.

The assumed motion can also be obtained to good approximation in a less artificial scenario. For a very stiff sphere, for example one made of diamond and given a uniform coating of charge, F_{ext} could, for example, be the force from a uniform external electric field applied in the x direction. Such an applied field will tend to stretch the sphere (since it gives to all parts of the sphere the same acceleration in the instantaneous rest frame, whereas we want the part momentarily at rest at given x to have acceleration proportional to 1/x). The self-force is somewhat larger at the back of the sphere, and therefore also tends to stretch the sphere. If the stiffness of the sphere is sufficient to make this stretching negligible, then rigid hyperbolic motion will result in a uniform applied electric field. Another way to realize the conditions approximately is to hold the sphere fixed in a nearly-uniform gravitational field.

In the limit of a small sphere, one expects to obtain the ALD equation

$$\mathsf{F}_{\text{self}} \simeq \frac{2}{3} e^2 \left(-\frac{\dot{\mathsf{v}}}{R} + \ddot{\mathsf{v}} - \dot{\mathsf{v}}^2 \mathsf{v} \right) \tag{6.4}$$

where we use the metric signature Tr(g) = +2 so $\dot{v}^2 = a_0^2$. The three terms on the right hand side are called the inertial, Schott and loss (or radiation) terms, associated respectively with the provision of four-momentum to the field bound to the charge, its redistribution within the bound field, and the provision of fourmomentum to the radiated field. Hyperbolic motion is characterized by $\ddot{v} = \dot{v}^2 v$

so exhibits a well-known balance between the last two terms. Hence we expect $F_{self} \simeq -(2/3)e^2\dot{v}/R$, which is the first term (the one given by n = 0) in our series (5.11). The next term in the series is of order Ra_0^3 and is not accounted for in the approximations leading to Eqn (6.4).

An aside on point-like particles. For the avoidance of all confusion, it should be emphasized that the ALD equation is quite correct, and when used in the equation of motion (6.2) it gives correct solutions with no pathalogical behaviour, within the approximations it assumes. Those approximations include that the applied force should not vary significantly on the timescale 2R (the time for light to cross the sphere)—this gets rid of the 'pre-acceleration' phenomenon (see Yaghjian (1992)). One must also insist that $m_0 \ge 0$, of course—this gets rid of the runaway solutions. The introduction of negative mass into classical physics would produce pathalogical predictions in all sorts of scenarios; the fact that it would do so here has nothing particularly to do with electromagnetism. However, what one can correctly infer is that the ALD equation only describes a small spherical shell if the 'observed mass', defined by $m = m_0 + 2e^2/3R$, is larger than $2e^2/3R$. One cannot by this route obtain an equation of motion for a pointlike particle with finite charge and finite observed mass. In classical physics such an entity is simply impossible (it has infinite field energy and infinitely negative bare mass); in quantum physics it is also impossible, in the sense that a delta-function-like wavefunction is never achieved in practice. Gralle et al. (2009) have shown how to obtain the equation of motion, in classical electromagnetism, for a small object whose total mass and charge tend to zero in a suitably smooth way as the size of the object tends to zero. They achieve this by asserting energy-momentum conservation for the total stress tensor, thus making the Lorentz force equation a derived not an axiomatic part of the theory (c.f. Steane (2012) section 16.4). The ALD equation then emerges as the lowest-order correction to the force on a finite-sized body; see their paper for details. The connection to the present work is twofold. First, both here and in Gralle et al. the calculation avoids unphysical (and mathematically dubious) properties such as infinite field energy. Secondly, their method should reproduce our result to all orders; this would be a useful further check (to all orders in R if possible, or failing that then to a few orders).

Another important approximate expression for the self-force of a spherical shell undergoing arbitrary rigid motion, (that is, relativistically rigid motion in which the proper acceleration may vary) is (Caldirola (1956), Yaghjian (1992))

$$\mathsf{F}_{\text{self}} \simeq \frac{2}{3} \frac{e^2}{R} \left(\frac{\mathsf{v}(\tau - \tau_R)}{\tau_R} + \frac{\mathsf{v}(\tau) \cdot \mathsf{v}(\tau - \tau_R)}{\tau_R} \mathsf{v}(\tau) \right)$$
(6.5)

where $\mathbf{v}(\tau)$ is the four-velocity at proper time τ and $\tau_R = 2R$ is the time taken for light to cross the sphere. This result is more accurate than (6.4) in that it evaluates a Taylor expansion for the field to all orders in the derivatives of \mathbf{v} , but it still makes a linear approximation in which all nonlinear powers of the acceleration and its derivatives are neglected. In the instantaneous rest frame, the spatial part of this equation gives the three-force

$$\mathbf{f}_{\text{self}} \simeq \frac{2}{3} \frac{e^2}{2R^2} \mathbf{v}(\tau - \tau_R). \tag{6.6}$$

which for hyperbolic motion $(v(\tau) = \tanh(a_0\tau))$ leads to

$$f_{\text{self}} \simeq e^2 \left(-\frac{2}{3} \frac{a_0}{R} + \frac{8}{9} R a_0^3 + \cdots \right).$$
 (6.7)

Comparing this with (5.11), we find that for hyperbolic motion Eqn (6.5) correctly gives the inertial term but it should not be expected to, and does not, correctly give the term of order Ra_0^3 , which is the leading term after the inertial term is accounted for (in the special case of hyperbolic motion).

In an important and extensive paper, Nodvik (1964) gave a treatment of selfforce in electromagnetism using the calculus of variations to obtain covariant Euler-Lagrange equations for a body undergoing relativistically rigid motion. He then obtains a perturbative expansion for the self-force up to order R^2 in our terminology. His general expression (eqns (7.21)-(7.24) of Nodvik (1964)) allows for rotation and an arbitrary distribution of charge. In the case of a non-rotating spherical shell it yields

$$\begin{aligned} \mathsf{F}_{\text{self}}^{\text{Nodvik}} &= \frac{2}{3}e^2 \left\{ -\frac{3}{4}\frac{\dot{\mathsf{v}}}{R} + \ddot{\mathsf{v}} - \dot{\mathsf{v}}^2\mathsf{v} + R \left[2(\dot{\mathsf{v}}\cdot\ddot{\mathsf{v}})\mathsf{v} + \dot{\mathsf{v}}^2\dot{\mathsf{v}} - \frac{2}{3}\ddot{\mathsf{v}} \right] \\ &+ \frac{R^2}{3} \left[\mathsf{v}^{(4)} + (\mathsf{v}\cdot\mathsf{v}^{(4)})\mathsf{v} - 2\dot{\mathsf{v}}^2(\ddot{\mathsf{v}} - \dot{\mathsf{v}}^2\mathsf{v}) - 6(\dot{\mathsf{v}}\cdot\ddot{\mathsf{v}})\dot{\mathsf{v}} \right] \right\} + O(R^3) \ (6.8) \end{aligned}$$

where $\mathbf{v}^{(n)} = d^n \mathbf{v} / d\tau^n$. Applied to the case of constant proper acceleration, and setting $e^2 = a_0 = 1$, this yields

$$\mathsf{F}_{\text{self}}^{\text{Nodvik}} = -\frac{1}{2R} + \frac{2}{9}R + O(R^3), \tag{6.9}$$

which should be compared to our eqn (5.12). The two results are in agreement for the even powers of R, but not for the odd powers. This is because eqn (5.12) gives the contribution purely from the electromagnetic field sourced by the shell, whereas (6.9) is the sum of this and the contribution from Poincaré stresses. This fact is, however, less than clear in Nodvik's presentation which lumps the selfelectromagnetic and the other part together. The same issue was already noted by Dirac (1938) and is exhibited, for example, in the lowest order term (the inertial term). Since the inertial term can be lumped together with the mass term in the equation of motion and called 'renormalisation', this difference has no easily observable consequence. However, this should not be taken to mean that there is any doubt about the calculation. The electric field predicted by Maxwell's equations is certainly the one we have calculated, and there is no need to abandon the Lorentz force equation when calculating the force on a continuous distribution of charge; it is the simplest equation consistent with energy-momentum conservation. Therefor eqn (3.8) is also correct. In the method adopted first by Dirac and then by Nodvik and many later workers, the rigidity of the charge distribution is accounted for by including it as a constraint in the variational procedure, while insisting that all equations are manifestly covariant. This means that when the electromagnetic forces in the physical scenario do not on their own give the assumed motion (such as rigid motion), the application of the constraint automatically gives rise to a further force which, although it may have some other physical origin, such as a strong

nuclear force, or the programmable device inside our imagined balloon, appears in the equations as a modification of the force called 'electromagnetic'. Having said that, it is appropriate to remark that what one usually wants to know in practice is the total self-force for a given rigid body, i.e. the sum $F_{\rm self}+F_{\rm P}$ in our notation, and this is what Nodvik has provided.

Finally, in view of the fact that for hyperbolic motion $\ddot{v} = a_0^2 v$ and using $v^2 = -1$, we have $\ddot{v} \cdot v = -a_0^2$. By repeatedly differentiating $\ddot{v} = a_0^2 v$ one finds

$$a_0^{2n} = -\mathbf{v} \cdot \mathbf{v}^{(2n)} \tag{6.10}$$

Hence another way to write Eqn (6.1) is

$$\mathsf{F}_{\text{self}} = \frac{2e^2}{R} \dot{\mathsf{v}} \sum_{n=0}^{\infty} \frac{-R^{2n} \mathsf{v} \cdot \mathsf{v}^{(2n)}}{(2n-1)(2n+1)^2(2n+3)}.$$
(6.11)

Appendix A. Series solution for the field

We would like to find the derivatives of $\phi(x, 0, 0)$ with respect to x at x = 1. This amounts to finding the Taylor expansion of $\phi(x, 0, 0)$ about x = 1. To this end, introduce $h \equiv x - 1$, then eqn. (5.6) can be written

$$\phi(x,0,0) = \phi_{-}(h)R^{-1} + \phi_{+}(h)R \tag{A1}$$

where

$$\phi_{-}(h) = \frac{1}{2(1+h)} + T(h), \quad \phi_{+}(h) = \frac{1}{h(2+h)} \left(\frac{-1}{2(1+h)} + T(h)\right)$$
(A2)

with

$$T(h) = \frac{h(1+h)(2+h)}{4(h(2+h))^{3/2}} \tan^{-1}\left(\frac{2\sqrt{h(2+h)}}{1-h(2+h)}\right).$$
 (A 3)

The first eleven terms in the Taylor expansions of ϕ_{-} and ϕ_{+} are

$$\begin{split} \phi_{-} &\simeq 1 - \frac{h}{3} + \frac{2h^2}{5} - \frac{46h^3}{105} + \frac{29h^4}{63} - \frac{547h^5}{1155} + \frac{620h^6}{1287} - \frac{21932h^7}{45045} + \frac{5959h^8}{12155} - \frac{204739h^9}{415701} + \frac{72614h^{10}}{146965} \\ \phi_{+} &\simeq \frac{1}{3} - \frac{7h}{15} + \frac{18h^2}{35} - \frac{166h^3}{315} + \frac{365h^4}{693} - \frac{523h^5}{1001} + \frac{3332h^6}{6435} - \frac{56204h^7}{109395} + \frac{117927h^8}{230945} - \frac{164335h^9}{323323} + \frac{342430h^{10}}{676039} \\ \end{array}$$

To find the self-force, we use the form (5.7), choosing a finite value for N and finding the coefficients a_n by matching the above Taylor series. To illustrate, we display the working for the case N = 3. One finds the set of equations

$$\begin{pmatrix} 1 & -1/2 & 1/3 & -1/4 \\ 1 & -3/2 & 5/3 & -7/4 \\ 0 & -3/2 & 10/3 & -21/4 \\ 0 & -1/2 & 10/3 & -35/4 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \frac{1}{R} \begin{pmatrix} 1 \\ -1/3 \\ 2/5 \\ -46/105 \end{pmatrix} + R \begin{pmatrix} 1/3 \\ -7/15 \\ 18/35 \\ -166/315 \end{pmatrix}.$$

The solution is

$$a_0 = \frac{298}{105R} + \frac{506R}{315}, \ a_1 = \frac{752}{105R} + \frac{544R}{105}, \ a(2) = \frac{243}{35R} + \frac{37R}{7}, \ a(3) = \frac{16}{7R} + \frac{16R}{9}$$

One can now find an approximation to f_{self} by using these values in (5.9). One finds

$$f_{\text{self}} \simeq -\frac{2}{3R} + \frac{2R}{45} + \frac{2R^3}{525} - \frac{466R^5}{735} - \frac{32R^7}{63}.$$
 (A 4)

It is straightforward to perform a similar calculation at higher values of N. One thus obtains both an increasingly accurate estimate of the self force, and also a power series approximation to the electric field inside the spherical shell. The terms in the power series for $V(h, \rho)$ are correct up to order $h^k \rho^p$ with k + p = N, and consequently the terms in the power series for the electric field are correct up to order $h^k \rho^p$ with k + p = N - 1. Therefore the series for f_{self} is guaranteed to be correct up to order R^{N-2} (not N - 1 because each a_n coefficient involves a 1/R term).

Comparing (A 4) with the expression (5.12) which was obtained using a higher value of N so as to obtain a higher order approximation to the potential, we observe that the solution with N = 3 has correctly reproduced the O(1/R) term and the O(R) term as expected, and it has also given the $O(R^3)$ term correctly. Is this just a coincidence? Perhaps not, because a similar observation can be made about the O(R) term in eqn (5.10). By examining the results for values of N in the range 2 to 20, we found that the method at given N yields the self force correctly up to order R^{2N-3} . In other words it is surprisingly effective, converging more rapidly than was expected. However I have not been able to discover why. This question might be explored by expanding the potential directly in terms of the V_n functions, rather than going via the polynomial series. That is one possible direction for future work.

We will finish by illustrating this point in its simplest form, which is found when N = 2. The case N = 2, calculated as above by matching the Taylor series on the x axis, leads to the following approximation for the x-component of the electric field:

$$E_x \simeq \left(-\frac{2}{3} + \frac{8}{15}h - \frac{212}{15}h^2 + \frac{104}{15}\rho^2\right)R^{-1} + \frac{2}{15}R\tag{A5}$$

whereas the correct power series expansion to this order is

$$E_x \simeq \left(-\frac{2}{3} + \frac{8}{15}h - \frac{44}{105}h^2 + \frac{8}{105}\rho^2\right)R^{-1} + \frac{2}{15}R.$$
 (A 6)

The N = 2 approximation does not give the terms of order h^2/R and ρ^2/R correctly. However, to find the self force we evaluate the field at $\rho^2 + h^2 = R^2$ and then integrate over h between -R and +R. The calculation using the correct expression for E_x yields, for the coefficient of the term of order R in f_{self} :

$$-\frac{44}{105} \times \frac{1}{3} + \frac{8}{105} \times \frac{2}{3} + \frac{2}{15} = \frac{2}{45}.$$

and the calculation using the N = 2 approximation to E_x yields

$$-\frac{212}{15} \times \frac{1}{3} + \frac{104}{15} \times \frac{2}{3} + \frac{2}{15} = \frac{2}{45}.$$

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