Life in the Rindler Reference Frame: Does an Uniformly Accelerated Charge Radiates? Is there a Bell 'Paradox'? Is Unruh Effect Real?

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Abstract

The determination of the electromagnetic field generated by a charge in hyperbolic motion is a classical problem for which the majority view is that the Liénard-Wiechert solution which implies that the charge radiates) is the correct one. However we analyze in this paper a less known solution due to Turakulov that differs from the Liénard-Wiechert one and which according to him does not radiate. We prove his conclusion to be wrong. We analyze the implications of both solutions concerning the validity of the Equivalence Principle. We analyze also two other issues related to hyperbolic motion, the so-called Bell's "paradox" which is as yet source of misunderstandings and the Unruh effect, which according to its standard derivation in the majority of the texts, is a correct prediction of quantum field theory. We recall that the standard derivation of the Unruh effect does not resist any tentative of any rigorous mathematical investigation, in particular the one based in the algebraic approach to field theory which we also recall. These results make us to align with some researchers that also conclude that the Unruh effect does not exist.

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1. Introduction

There are some problems in Relativity theory that are continuously source OF controversies, among them we discuss in this paper: (a) the problem of determining if an uniformly accelerated charge does or does not radiate [3]; (b) the so-called Bell's paradox and; (c) the Unruh effect.[4].

In order to obtain some light on the controversies we discuss in details in Section 2 the concept of (right and left) Rindler reference frames, Rindler observers and a chart naturally adapted to a given Rindler frame. These concepts are distinct and thus represented by different mathematical objects and having this in mind is a necessary condition to avoid misunderstandings, both OF mathematical as well as of physical nature.

In Section 3 we analyze Bell's "paradox" that even having a trivial solution seems to not been understood for some people even recently for it is confused with another distinct problem which if one does not pay the required attention seems analogous to the one formulated by Bell.

In Section 4 we discuss at length the problem of the electromagnetic field generated by a charge in hyperbolic motion. First we present the classical Liénard-Wiechert solu-

[3]This problem is important concerning one of the formulations of the Equivalence principle.

[4]We call the reader's attention that the references quoted in this paper are far from complete, so we apologize for papers not quoted.

tion, which implies that an observer at rest in an inertial reference frame observes that the charge radiates. Next we analyze (accepting that the Liénard-Wiechert solution is correct) if an observer comoving with the charge detects or no radiation. We argue with details that contrary to some views it is possible for a real observer living in a real laboratory[5] in hyperbolic motion to detect that the charge is radiating. Our conclusion is based (following [43]) on a careful analysis of different concepts of *energy* that are used in the literature, the one defined in an inertial reference frame and the other in the Rindler frame. In particular, we discuss in details the error in Pauli's argument.

But now we ask: is it necessary to accept the Liénard-Wiechert solutions as the true one describing the electromagnetic field generated by a charge in hyperbolic motion? To answer that question we analyzed the Turakulov [60] solution to this problem, which consisting in solving the wave equation for the electromagnetic potential in a special systems of coordinates where the equation gets separable. We have verified that Turakulov solution (which differs form the Liénard-Wiechert one) is correct (in particular, by using the Mathematica software). Turakulov claims that in his solution the charge does not radiate. However, we prove that his claim is wrong, i.e., we show that as in the case of the Liénard-Wiechert solution an observer comoving with the charge can detect that is is emitting radiation.

In Section 5 we discuss, taking into account that it seems a strong result the fact that a charge at rest in the Schwarzschild spacetime does not radiate [43], what the results of Section 4 implies for the validity or not of one of the forms in which Equivalence principle is presented in many texts.

Section 6 is dedicated to the Unruh effect. We first recall the standard presentation (emphasizing each one of the hypothesis used in its derivation) of the supposed fact that Rindler observers are living in a thermal bath with a Planck spectrum with temperature proportional to its local proper acceleration and thus such radiation may *excite* detectors on board. Existence of the Unruh radiation and Rindler particles seems to be the majority view. However, we emphasize that rigorous mathematical analysis of standard procedure (which is claimed to predict the Unruh effect) done by several authors shows clearly that such a procedure contain several inconsistencies. These rigorous analysis show that the Unruh effect does no exist, although it may be proved that detectors in hyperbolic notion can get excited, although the energy for that process comes form the source accelerating the detector and it is not (as some claims) due to fluctuations of the Minkowski vacuum. We recall in Appendix B a (necessarily resumed) introduction to the algebraic approach to quantum theory as applied to the Unruh effect in order to show how much we can trust each one of the suppositions used in the standard derivation of the Unruh effect. Detailed references are given at the appropriate places.

Section 7 presents our conclusions and in Appendix A we present our conventions and some necessary definitions of the concepts of reference frames, observers, instantaneous observers and naturally adapted charts to a given reference frame.

[5]This, of course, means that the laboratory (whatever its mathematical model) [10] must have finite spatial dimensions as determined by the observer at any instant of its propertime.

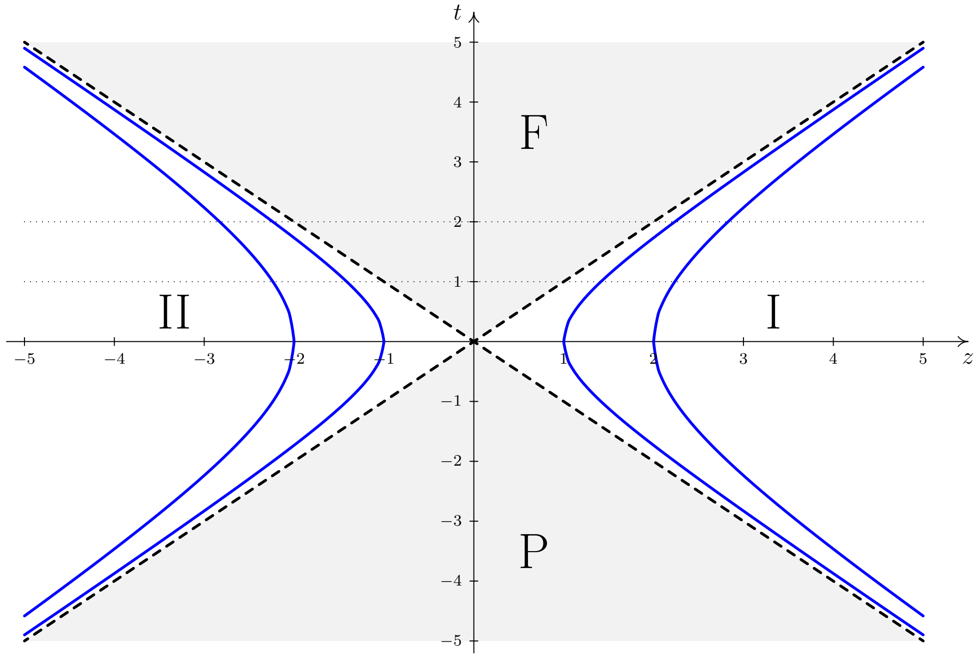

Figure 1: Some integral lines of the right $\boldsymbol{R}$ and left $\boldsymbol{L}$ Rindler reference frames

2. Rindler Reference Frame

A proper understanding of almost any problem in Relativity theory requires that we know (besides the basics of differential geometry[6]) exactly the meaning and the precise mathematical representation of the concepts of: (a) references frames and their classification; (b) a naturally adapted chart to a given reference frame; (c) observers and (d) instntaneous observers. The main results necessary for the understanding of the present paper and some other definitions are briefly recalled in Appendix A[7]. Essential is to have in mind that most of the possible reference frames used in Relativity theory are t*heoretical instruments*, i.e., they are not physically realizable as a material systems. This is particularly the case of the right and left Rindler reference frames and respective observers that we introduce next.

Let $\sigma : I \to M$, $s \mapsto \sigma(s)$ a timelike curve in M describing the motion of an accelerated observer (or an accelerated particle) where s is the proper time along σ. The coordinates of σ in ELP gauge (see Appendix A) are

$$x^{\mu}_{\sigma}(s) = \boldsymbol{x}^{\mu} \circ \sigma \ (s) \tag{1}$$

and for motion along the $x^3 = z$ axis it is

$$(x^{o}_{\sigma})^2 - (x^{3}_{\sigma})^2 = -\frac{1}{\mathrm{a}^2_{\sigma}}, \tag{2}$$

[6]Basics of differential geomety may be found in [13, 19, 21, 38]. Necessary concepts concerning Lorentzian manifods may be found in [41, 53].

[7]More details may be found in [48, 24].

where a_σ is a real constant for each curve σ. In Figure 1 we can see two curves σ and σ' for which $\frac{1}{\mathrm{a}_\sigma} = 1$ and $\frac{1}{\mathrm{a}_{\sigma'}} = 2$. To understand the meaning of the parameter a_σ in Eq.(2) we write

$$x^0_\sigma(s) = \frac{1}{\mathrm{a}_\sigma}\sinh(\mathrm{a}_\sigma s), \quad x^3_\sigma(s) = \frac{1}{\mathrm{a}_\sigma}\cosh(\mathrm{a}_\sigma s). \tag{3}$$

The unit velocity vector of the observer is

$$\boldsymbol{v}_\sigma(s) = \sigma_*(s) := v^\mu(s)\frac{\partial}{\partial x^\mu} = \cosh(\mathrm{a}_\sigma s)\frac{\partial}{\partial t} + \sinh(\mathrm{a}_\sigma s)\frac{\partial}{\partial \mathrm{z}}.$$

Now, the acceleration of σ is

$$\boldsymbol{a}_\sigma = \frac{d}{ds}\sigma_*(s) = \mathrm{a}_\sigma\left(\sinh(\mathrm{a}_\sigma s)\frac{\partial}{\partial t} + \cosh(a_\sigma s)\frac{\partial}{\partial z}\right)\Bigg|_\sigma \tag{4}$$

and of course, $\boldsymbol{a}_\sigma \cdot \boldsymbol{v}_\sigma = 0$ and $\boldsymbol{a}_\sigma \cdot \boldsymbol{a}_\sigma = -\mathrm{a}^2_\sigma$.

2.1. Rindler Coordinates

Introduce first the regions I,II, F and P of Minkowski spacetime

$$\mathcal{I} = \{(t,x,y,z) \ \mid \ -\infty < t < \infty, -\infty < x < \infty, , -\infty < y < \infty, 0 < z < \infty\}, \tag{5}$$

and two coordinate functions $(\boldsymbol{x}^0, \boldsymbol{x}^1, \boldsymbol{x}^2, \boldsymbol{x}^3)$ and $(\boldsymbol{x}'^0,\boldsymbol{x}'^1,\boldsymbol{x}'^2,\boldsymbol{x}'^3)$ covering such regions. For $\mathfrak{e} \in M$ it is $\{\boldsymbol{x}^0(\mathfrak{e}) = x^0 = t, \boldsymbol{x}^1(\mathfrak{e}) = x, \boldsymbol{x}^2(\mathfrak{e}) = y, \boldsymbol{x}^3(\mathfrak{e}) = z\}$ and $\{\boldsymbol{x}'^0(\mathfrak{e}) = \mathrm{t}, \boldsymbol{x}'^1(\mathfrak{e}) = \mathrm{x} = x, \boldsymbol{x}'^2(\mathfrak{e}) = \mathrm{y} = y, \boldsymbol{x}'^3(\mathfrak{e}) = \ \mathrm{z}\}$ with[8]

$$\begin{gathered}
\mathrm{z} = \pm\sqrt{z^2 - t^2}, \quad \mathrm{t} = \tanh^{-1}\left(\frac{t}{z}\right), \qquad |z| \geq |t|, \\
x^0 = t = \mathrm{z}\sinh\mathrm{t}, \quad x^3 = z = \mathrm{z}\cosh\mathrm{t} \qquad \text{in region I}, \\
x^0 = t = -\mathrm{z}\sinh\mathrm{t}, \quad x^3 = z = -\mathrm{z}\cosh\mathrm{t} \qquad \text{in region II}
\end{gathered} \tag{6}$$

and

$$\begin{gathered}
\mathrm{z} = \pm\sqrt{t^2 - z^2}, \quad \mathrm{t} = \tanh^{-1}\left(\frac{z}{t}\right), \qquad |t| \geq |z|, \\
x^0 = t = \mathrm{z}\cosh\mathrm{t}, \quad x^3 = z = \mathrm{z}\sinh\mathrm{t} \qquad \text{in region F}, \\
x^0 = t = -\mathrm{z}\cosh\mathrm{t}, \quad x^3 = z = -\mathrm{z}\sinh\mathrm{t} \qquad \text{in region P}.
\end{gathered} \tag{7}$$

The right Rindler *reference frame* $\boldsymbol{R} \in \sec T\mathrm{I}$ has support in region I and is defined by

$$\begin{gathered}
\boldsymbol{R} = \frac{z}{\sqrt{z^2 - t^2}}\frac{\partial}{\partial t} + \frac{t}{\sqrt{z^2 - t^2}}\frac{\partial}{\partial z} = \frac{1}{\mathrm{z}}\frac{\partial}{\partial \mathrm{t}}, \\
z > 0; \quad |z| \geq t.
\end{gathered} \tag{8}$$

[8] Of course the coordiantes (t,x,y,z) cover all M but the coordinates $(\mathrm{t},\mathrm{x},\mathrm{y},\mathrm{z})$ do not cover all M, they are singular at the origin.

The left reference Rindler frame $\boldsymbol{L} \in \sec T\mathrm{II}$ is defined by

$$\boldsymbol{L} = \frac{z}{\sqrt{z^2-t^2}}\frac{\partial}{\partial t} + \frac{t}{\sqrt{z^2-t^2}}\frac{\partial}{\partial z} = \frac{1}{\mathrm{z}}\frac{\partial}{\partial \mathrm{t}},$$
$$z<0: \quad |z| \geq t. \tag{9}$$

Then, we see that in I $\subset M$, $(t, x^1, x^2, \mathrm{z})$ as defined in Eq.(6) are a *naturally adapted coordinate system* to $\boldsymbol{R}$ [$(nacs|\boldsymbol{R})$] and $\boldsymbol{L}$ [$(nacs|\boldsymbol{L})$]. With D being the Levi-Civita connection of $\boldsymbol{g}$, the acceleration vector field associated to $\boldsymbol{R}$ is

$$\boldsymbol{a} = D_{\boldsymbol{R}}\boldsymbol{R} = \frac{1}{\mathrm{z}}\frac{\partial}{\partial \mathrm{z}}. \tag{10}$$

Also,

$$\boldsymbol{a}_\sigma = \frac{d}{ds}\sigma_*(s) = \mathrm{a}_\sigma \left.\frac{\partial}{\partial \mathrm{z}}\right|_\sigma \tag{11}$$

i.e., $\boldsymbol{a}_\sigma = D_{\boldsymbol{R}}\boldsymbol{R}|_\sigma = \frac{1}{\mathrm{z}}\frac{\partial}{\partial \mathrm{z}}\Big|_\sigma = \mathrm{a}_\sigma \frac{\partial}{\partial \mathrm{z}}\Big|_\sigma$. Moreover, recall that since σ is clearly an integral line of the vector field $\boldsymbol{R}$, it is $\boldsymbol{v}_\sigma = \boldsymbol{R}|_\sigma$.

Remark 1 *Note that in Eq.*(8) (*respectively* Eq.(9)) *it is necessary to impose* $z > 0$ (respectively, $z < 0$) *this being the reason for having defined the right and left Rindler reference frames.*

2.2. Decomposition of DR

Recall that the Minkowski metric field $\boldsymbol{g} = \eta_{\mu\nu}dx^\mu \otimes dx^\nu$reads in Rindler coordinates (in region I)

$$\boldsymbol{g} = g_{\mu\nu}dx^\mu \otimes dx^\upsilon = \mathrm{z}^2\mathrm{dt}\otimes\mathrm{dt} - \mathrm{dx}\otimes\mathrm{dx} - dy\otimes dy - dz\otimes dz$$
$$= \eta_{\mathbf{ab}}\boldsymbol{\gamma}^{\mathbf{a}}\otimes\boldsymbol{\gamma}^{\mathbf{b}} \tag{12}$$

where $\{\boldsymbol{\gamma}^0, \boldsymbol{\gamma}^0, \boldsymbol{\gamma}^2, \boldsymbol{\gamma}^3\} = \{\mathrm{z}dt, dx, dy, dz\}$ is an orthonormal coframe for $T^*\mathrm{I}$ which is dual to the orthonormal frame $\{e_0, e_1, e_2, e_3\} = \{\boldsymbol{R} = \frac{1}{\mathrm{z}}\frac{\partial}{\partial \mathrm{t}}, \frac{\partial}{\partial \mathrm{x}}, \frac{\partial}{\partial \mathrm{y}}, \frac{\partial}{\partial \mathrm{z}}\}$ for $T\mathrm{I}$. We write

$$D_{\frac{\partial}{\partial \mathrm{x}^\nu}}\mathrm{dx}^\mu = -\Gamma^{\mu\cdot\cdot}_{\cdot\nu\iota}\mathrm{dx}^\iota, \qquad D_{e_{\mathbf{b}}}\boldsymbol{\gamma}^a = -\boldsymbol{\Gamma}^{\mathbf{a}\cdot\cdot}_{\cdot\mathbf{bc}}\boldsymbol{\gamma}^c \tag{13}$$

and keep in mind that it is $\boldsymbol{\Gamma}^{\mathbf{a}\cdot\mathbf{c}}_{\cdot\mathbf{b}\cdot} = -\boldsymbol{\Gamma}^{\mathbf{c}\cdot\mathbf{a}}_{\cdot\mathbf{b}\cdot}$ (and of course, $\Gamma^{\mu\cdot\cdot}_{\cdot\nu\iota} = \Gamma^{\mu\cdot\cdot}_{\cdot\iota\nu}$)

Define the 1-form field (physically equivalent to $\boldsymbol{R}$)

$$R = \boldsymbol{g}(\boldsymbol{R},\ ) = R_\mu dx^\mu = \mathrm{z}\mathrm{dx}^0 = \boldsymbol{\gamma}^0. \tag{14}$$

Then, as well known[9] DR has the invariant decomposition

$$DR = a\otimes R + \omega_R + \varkappa + \frac{1}{3}\mathfrak{E}h, \tag{15}$$

[9]Se, e.g., [48].

with

$$
\begin{aligned}
a &:= \boldsymbol{g}(\boldsymbol{a},\ ),\\
\omega_R &:= \omega_{\mu\nu} dx^\mu \otimes dx^v = \frac{1}{2}\left(R_{\sigma;\tau} - R_{\tau;\sigma}\right) h^\sigma_\mu h^\tau_\nu dx^\mu \otimes dx^v\\
\varkappa &:= \varkappa_{\mu\nu} dx^\mu \otimes dx^v = \left[\frac{1}{2}\left(R_{\sigma;\tau} + R_{\tau;\sigma}\right) h^\sigma_\mu h^\tau_\nu - \frac{1}{3}\mathfrak{E} h_{\sigma\tau} h^\sigma_\mu h^\tau_\nu\right] dx^\mu \otimes dx^v\\
\mathfrak{E} &: = \operatorname{div} \boldsymbol{R} = R^\mu_{;\mu} = \delta R\\
h &:= \left(g_{\mu\nu} - R_\mu R_\nu\right) dx^\mu \otimes dx^v
\end{aligned} \tag{16}
$$

where a, ω, $\varkappa$ and $\mathfrak{E}$ are respectively the (form) *acceleration*, the *rotation tensor* (or vortex) of R, $\varkappa$ is the shear tensor of R and $\mathfrak{E}$ is the *expansion ratio* of R.

Now, $d\boldsymbol{\gamma}^0 = dz \wedge dx^0 = \frac{1}{z}\boldsymbol{\gamma}^3 \wedge \boldsymbol{\gamma}^0$ and thus $\boldsymbol{\gamma}^0 \wedge d\boldsymbol{\gamma}^0 = 0$ which implies that $\omega_R = 0$. See Appendix A and details in [48]

This means that the Rindler reference frame $\boldsymbol{R}$ is locally synchronizable, but since R is not an exact differential $\boldsymbol{R}$ is *not* proper time synchronizable, something that is obvious once we look at Figure 1 and see that for each time $t > 0$ of the inertial reference frame $\boldsymbol{I} = \partial/\partial t$ the Rindler observers following paths σ and σ' (which have of course, different proper accelerations) have also different speeds, so their clocks (according to an inertial observer) tic-tac at different ratios.

2.3. Constant Proper Distance Between σ and σ'

We can easily verify using the orthonormal coframe introduced above that since $d\boldsymbol{\gamma}^{\mathbf{i}} = 0$, $\mathbf{i} = 1, 2, 3$ it is $\boldsymbol{\Gamma}^{\mathbf{i}}_{\mathbf{ab}} = \boldsymbol{\Gamma}^{\mathbf{i}}_{\mathbf{ba}}$ for $\mathbf{i} = 1, 2, 3$ and $\mathbf{a}, \mathbf{b} = 0, 1, 2, 3$ and also from the form of $d\boldsymbol{\gamma}^0$ we realize that $\boldsymbol{\Gamma}^{\mathbf{0}\cdot\cdot}_{\cdot\mathbf{00}} = \boldsymbol{\Gamma}^{\mathbf{0}\cdot\mathbf{0}}_{\cdot\mathbf{0}\cdot} = -\boldsymbol{\Gamma}^{\mathbf{0}\cdot\mathbf{0}}_{\cdot\mathbf{0}\cdot} = 0$. Thus,

$$
\mathfrak{E} = \delta R = -\boldsymbol{\gamma}^{\mathbf{a}} \lrcorner D_{\boldsymbol{e}_{\mathbf{a}}}(\boldsymbol{\gamma}^{\mathbf{0}}) = \boldsymbol{\Gamma}^{\mathbf{0}\cdot\cdot}_{\cdot\mathbf{ab}} \boldsymbol{\gamma}^{\mathbf{a}} \lrcorner \boldsymbol{\gamma}^{\mathbf{b}} = \eta^{\mathbf{ab}} \boldsymbol{\Gamma}^{\mathbf{0}\cdot\cdot}_{\cdot\mathbf{ab}} = -\boldsymbol{\Gamma}^{\mathbf{a}\cdot\mathbf{0}}_{\cdot\mathbf{a}\cdot} = \boldsymbol{\Gamma}^{\mathbf{a}\cdot\cdot}_{\cdot\mathbf{a0}} = 0 \tag{17}
$$

and we realize that each observer following an integral line of $\boldsymbol{R}$, say σ_1 will maintain a *constant proper distance* to any of its neighbor observers which are following a different integral line of $\boldsymbol{R}$.

Of course, proper distance between an observer following σ and another one following σ' is operationally obtained in the following way: Using Rindler coordinates at an event, say $\mathfrak{e}_1 = (0, 0, 0, \mathrm{z}_1)$ the observer following σ send a light signal to σ' (in the direction $\boldsymbol{e}_3$) which arrives at the σ' worldline at the event $\mathfrak{e}_2 = (\mathrm{t}_2, 0, 0, \mathrm{z}_1 + \ell)$ where it is immediately reflected back to σ arriving at event $\mathfrak{e}_3 = (\mathrm{t}_3, 0, 0, \mathrm{z}_1)$. So, the total coordinate time for the two way trip of the light signal is t_3 and immediately we get (from the null geodesic equation followed by the light signal)

$$
\begin{aligned}
\mathrm{t}_2 &= \ln\left(1 + \frac{\ell}{\mathrm{z}_1}\right),\\
\mathrm{t}_3 - \mathrm{t}_2 &= \ln\left(1 + \frac{\ell}{\mathrm{z}_1}\right)
\end{aligned} \tag{18}
$$

and thus

$$t_3 = 2\ln\left(1+\frac{\ell}{z_1}\right). \tag{19}$$

Now, the observer at σ evaluates the total proper time for the total trip of the signal, it is $z_1 t_3$. The *proper distance* is by definition

$$d_{\sigma\sigma'} := \frac{1}{2} z_1 t_3 = z_1 \ln\left(1+\frac{\ell}{z_1}\right). \tag{20}$$

Eq.(20) shows that proper distance and coordinate distance are different in a Rindler reference frame.

Remark 2 *A look at* Figure 1 *shows immediately that inertial observers in* $\boldsymbol{I} = \partial/\partial t$ *will find that the distance between* σ *and* σ' *is shortening with the passage of* t *time. It is opportune to take into account that despite the fact that the Rindler coordinate times for the going and return paths are equal (the coordinate time being equal to proper time in* σ*) measured by the inertial observers are different and indeed as it is intuitive the return path is realized in a shorter inertial time.*

Remark 3 *Of course, if* $\boldsymbol{R} = \frac{1}{z}\partial/\partial t$ *is physically realizable by a rocket with the constraint that, e.g.,* $z_1 \leq z \leq (z_1+\ell)$ *then it needs to have a very special propulsion system, with its rear accelerating faster than the front. We do not see how such a rocket could be constructed.*[10]

3. Bell 'Paradox'

In [3] it is proposed the following question:

> Three small spaceships, A, B, and C, drift freely in a region of spacetime remote from other matter, without rotation and without relative motion, with B and C equidistant from A (Fig.1).

Figure 2: Figure 1 in Bell [3] (adapted)

> On reception of a signal from A the motors of B and C are ignited and they accelerate gently (Fig.2)

[10]Note that the original Rindler reference frame $\boldsymbol{R}$ for which $(0 < z < \infty)$ is only supposed to be a theoretical construct, it obviously cannot be realized by any material system.

Figure 3: Figure 3: Figure 2 in Bell [3] (adapted)

> Let ships B and C be identical, and have identical acceleration programmes. Then (as reckoned by an observer at A) they will have at every moment the same velocity, and so remain displaced one from the other by a fixed distance. Suppose that a fragile thread is tied initially between projections form B to C (Fig.3). If it is just long enough to span the required distance initially, then as the rockets speed up, it will become to short, because of its need to FitzGerald contract, and must finally break. It must break when at a sufficiently high velocity the artificial prevention to the natural contraction imposes intolerable stress.

Then Bell continues saying:

> Is this really so? This old problem came up for discussion once in the CERN canteen. A distinguished experimental physicists refused to accept that the thread would break, and regarded my assertion, that indeed it would, as a personal misinterpretation of special relativity. We decided to appeal to the CERN Theory Division for arbitration, and made a (not very systematic) canvas of opinion in it. there emerged a clear consensus that the tread would **not** break.
>
> Of course many people who give this wrong answer at first get the right answer on further reflection.

Recently Motl [37] wrote a note saying that Bell did not understand Special Relativity since the correct answer to his question is the CERN majority (first sight) view. Now, reading Motl's article one arrive at the conclusion that he did not understand correctly the *formulation* of Bell's problem. Indeed, the problem that is correctly analyzed in [37] was the one in each ships B and C are modelled as two distinct observers following two different integral lines of the Rindler reference frame $\boldsymbol{R}$ introduced in the previous section.

It is quite obvious to any one that read Section 1 that in this case (which is *not* the Bell's one)) B an C did not have the *same* acceleration programme as seem by observer A (represented by a particular integral line of the inertial frame $\boldsymbol{I} = \partial/\partial t$ the t axis in Figure 4).

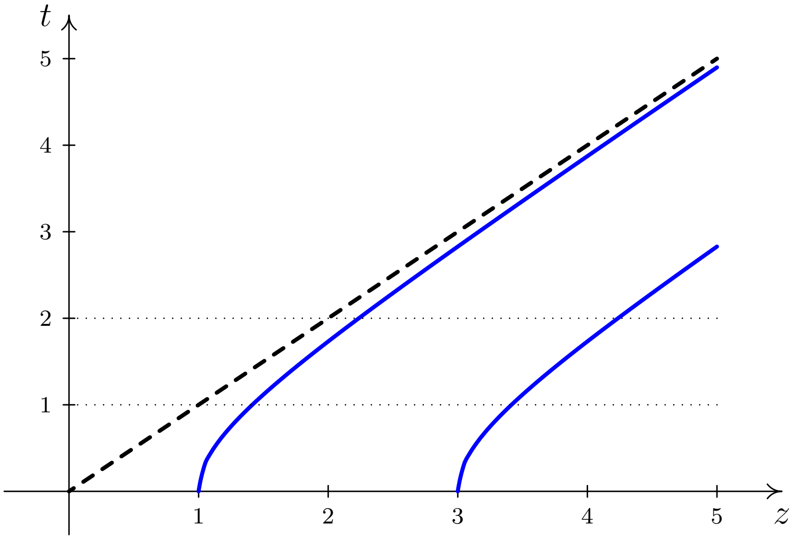

Figure 4: Spacetime diagram for Bell's question with ships B (tick line on the left) and C (tick line on the right) having the same acceleration relative to the inertial observer A.

In the case of Bell's question ships B and C are modelled (as a first approximation) as observers, i.e., as the timelike curves

$$t_B^2 - x_B^2 = -\frac{1}{a_B^2},$$
$$t_C^2 - (x_C - d)^2 = -\frac{1}{a_C^2} = -\frac{1}{a_B^2},$$

where to illustrate the situation we draw Figure 4 with $a_B = 1$ and $d = 2$. It is absolutely clear from Figure 4 that the distance between B and C any instant $t > 0$ as determined by the inertial observer is the same as it was at $t = 0$, when B and C start accelerating with the same accelerating programme.

A trivial calculation similar to the one in Subsection 2.3 above shows that proper distance between B and C as determined by B (or C) is *increasing* with the coordinate time t used by these observers which are modelled as integral lines of the Rindler reference frame $\boldsymbol{R}$. As a consequence of this fact we arrive at the conclusion that the thread cannot go during the acceleration period to its *natural* Lorentz deformed configuration and thus will break.

Bell's problem illustrate that bodies subject to special acceleration programs do not go to their Lorentz deformed configuration immediately. After the acceleration programme ends the body will acquire adiabatically its Lorentz deformed configuration. More on this issue is discussed in [47].

4. Does a Charge in Hyperbolic Motion Radiates?

4.1. The Answer Given by the Liénard-Wiechert Potential

It is usually assumed (see, e.g., [28, 32, 33, 42, 43, 44] that the electromagnetic potential $A = A_\mu(x)dx^\mu \in \sec T^*M$ generated by a charged particle in hyperbolic motion with world line given by $\sigma : \mathbb{R} \to M$, $s \mapsto \sigma(s)$, with parametric equations given by Eq.(3) and electric current given $J = J_\mu(x(s))dx^\mu|_\sigma = eV_\mu(s))dx^\mu|_\sigma \sec T^*M$ where

$$v^\mu(s) := \frac{d}{ds}x^\mu \circ \sigma(s), \quad v := (v^0, \mathbf{v}) = \left(\frac{1}{\sqrt{1-\mathbf{v}^2}}, 0, 0, \frac{\mathbf{v}^i}{\sqrt{1-\mathbf{v}^2}}\right), \tag{21}$$

$$J_\mu(x) = e\int ds v_\mu(s)\delta^{(4)}(x' - x\circ\sigma(s)) \tag{22}$$

is given by the solution of the differential equation

$$\Box A_\mu = J_\mu \tag{23}$$

through the well known formula

$$A_\mu(x) = e\int d^4x D_r(x-x')J_\mu(x') \tag{24}$$

where $D_r(x-x')$ is the retarded Green function[11] given by

$$\begin{aligned} D_r(x-x') &= \frac{1}{2\pi}\theta(x^0 - x'^0)\delta^{(4)}[(x-x')^2] \\ &= \frac{\theta(x^0 - x'^0)}{4\pi \mathrm{R}}\delta(x^0 - x'^0 - \mathrm{R}) \end{aligned} \tag{25}$$

with from the light cone constraint in Eq.(25)

$$\mathrm{R} = |\mathbf{x} - \mathbf{x}(\sigma(s)| = \left|x^0 - x^0(s)\right|. \tag{26}$$

Thus using Eq.(25) in Eq.(24) gives the famous Liénard-Wiechert formula,i.e.,

$$A_\mu(x) = \frac{e}{4\pi}\left.\frac{v_\mu(s)}{v\cdot[x - x(\sigma(s))]}\right|_{s=s_0} \tag{27}$$

and putting $\gamma = 1/\sqrt{1-\mathbf{v}^2}$, we have

$$v\cdot[x - x(\sigma(s))] = \gamma\mathrm{R}(1 - \mathbf{v}\bullet\mathbf{n}) \tag{28}$$

and thus

$$A^0(t,x) = \frac{e}{4\pi}\left.\frac{1}{(1-\mathbf{v}\bullet\mathbf{n})\mathrm{R}}\right|_{\mathrm{ret}}, \quad \mathbf{A}(t,x) = \frac{e}{4\pi}\left.\frac{\mathbf{v}}{(1-\mathbf{v}\bullet\mathbf{n})\mathrm{R}}\right|_{\mathrm{ret}} \tag{29}$$

[11] I.e., a solution of $\Box D_r(x-x') = \delta^{(4)}(x-x')$.

where ret means that that the value of the bracket must be calculated at the instant $x^0(s_0) = x^0 - \mathrm{R}$.

We also have for the components of the field $F = dA \in \sec \bigwedge^2 t^* M$

$$F_{\mu\nu}(x) = \frac{e}{4\pi} \frac{1}{v \cdot [x - x(\sigma(s))]} \frac{d}{ds} \left[\frac{[x - x_\sigma(s)]_\mu v_\nu - [x - x_\sigma(s)]_\nu v_\mu}{v \cdot [x - x(\sigma(s))]} \right]_{\mathrm{ret}} \tag{30}$$

and taking into account that $[x - x_\sigma(s)] = (\mathrm{R}, \mathrm{R}\mathbf{n})$, $\quad v_\mu = (\gamma, -\gamma \mathbf{v})$ and putting $\overset{\bullet}{\mathbf{v}} = d\mathbf{v}/dt$ it is

$$\frac{dv_\mu}{ds} = \gamma^2 \left(\gamma^2 \mathbf{v} \bullet \dot{\mathbf{v}}, -(\dot{\mathbf{v}} + \gamma^2 \mathbf{v}(\mathbf{v} \bullet \dot{\mathbf{v}})) \right) \tag{31}$$

and

$$\frac{d}{ds}[v \cdot (x - x(\sigma(s))] = -1 + (x - x(\sigma(s)))_\alpha \frac{dv^\alpha}{ds} \tag{32}$$

and thus we get

$$\mathbf{E}(t, \mathbf{x}) = \frac{e}{4\pi} \left[\frac{(\mathbf{n} - \mathbf{v})}{\gamma^2 (1 - \mathbf{v} \bullet \mathbf{n})^3 \mathrm{R}^2} \right]_{\mathrm{ret}} + \frac{e}{4\pi} \left[\frac{\mathbf{n} \times [(\mathbf{n} - \mathbf{v}) \times \dot{\mathbf{v}}}{\gamma^2 (1 - \mathbf{v} \bullet \mathbf{n})^3 \mathrm{R}} \right]_{\mathrm{ret}}, \tag{33}$$

$$\mathbf{B}(t, x) = \mathbf{n} \times \mathbf{E}(t, \mathbf{x}). \tag{34}$$

Since

$$\mathbf{n} \times [(\mathbf{n} - \mathbf{v}) \times \dot{\mathbf{v}} = (\mathbf{n} \bullet \dot{\mathbf{v}})(\mathbf{n} - \mathbf{v}) - \mathbf{n} \cdot (\mathbf{n} - \mathbf{v}) \dot{\mathbf{v}} \tag{35}$$

we see that for the hyperbolic motion where $\mathbf{v}$ is parallel to $\dot{\mathbf{v}}$ and

$$\mathbf{v}(t) = a_{\boldsymbol{\sigma}} \frac{t}{\sqrt{1 + a_{\boldsymbol{\sigma}}^2 t^2}} \hat{\mathbf{e}}_3, \quad \dot{\mathbf{v}}(t) = a_{\boldsymbol{\sigma}} \frac{1}{(1 + a_{\boldsymbol{\sigma}}^2 t^2)^{3/2}} \hat{\mathbf{e}}_3$$

the Liénard-Wiechert potential implies in a radiation field, i.e., a field that goes in the infinity (radiation zone) as $1/R$.

In Jackson's book [28] (page 667) one can read that when a charge is accelerated in a reference frame where its speed is $|\mathbf{v}| \ll 1$, the Poynting vector associated to the field given by Eqs.(33) and (34) is

$$\mathbf{S} = \mathbf{E} \times \mathbf{B} = |\mathbf{E}| \, \mathbf{n} \tag{36}$$

and the power irradiated per solid angle is [28]

$$\frac{dP}{d\Omega} = \frac{e^2}{(4\pi)^2} (\mathbf{n} \times \dot{\mathbf{v}}) \tag{37}$$

Thus the total instantaneous irradiated power (for a nonrelativistic accelerated charge) is

$$P = \frac{2}{3} \frac{e^2}{4\pi} \left| \dot{\mathbf{v}} \right|^2, \tag{38}$$

a result known as Larmor formula.

The correct formula valid for arbitrary speeds and with $P^\mu = mV^\mu$ (as one can verify after some algebra) is

$$
\begin{aligned}
P &= -\frac{2}{3}\frac{1}{4\pi}\frac{e^2}{m^2}\left(\frac{dP_\mu}{ds}\frac{dP^\mu}{ds}\right) \\
&= \frac{2}{3}\frac{1}{4\pi}e^2\gamma^6\left[|\dot{\mathbf{v}}|^2 - (\mathbf{v}\times\dot{\mathbf{v}})^2\right].
\end{aligned}
\tag{39}
$$

Remark 4 Eq.(37) *show that the radiated power in a linear accelerator is, of course, bigger for electrons than for, e.g., protons. However, as commented by Jackson* [28] *even for electrons in a linear accelerator with typical gain of 50 MeV/m the radiation loss is completely negligible In the case of circular accelerators like synchrotrons since the momentum* $\mathbf{p} = \gamma m\mathbf{v}$ *changes in direction rapidly we can show that the radiated power (predicted from the Liénard-Wiechert potential) is*

$$
P = \frac{2}{3}\frac{1}{4\pi}\frac{e^2}{m^2}\gamma^2\omega^2|\mathbf{p}|^2 \tag{40}
$$

where ω *is the angular momentum of the charged particle. This formula fits well the experimental results.*

4.2. Pauli's Answer

In this section we use the same parametrization as before for the coordinates of the charged particle in hyperbolic motion. Let $\mathfrak{e}$ (see Figure 5) be an arbitrary observation point with coordinates $x = (x^0 = t, x^1, x^2, x^3 = z)$. In what follows for simplicity of writing we denote the expression for the Lenard-Wiechert potential (Eq.(27)) as

$$
A_\mu(x) = \frac{e}{4\pi}\frac{v_\mu(s)}{v\cdot[x - x(\sigma(s))]}, \tag{41}
$$

but we cannot forget that at the end of our calculations we must put $s = s_0$. We have, explicitly for the velocity of the particle (moving in the x^3-direction with $a_\sigma = 1$)

$$
v^0(s) = \cosh s, \quad v^3(s) = \sinh s \tag{42}
$$

and so

$$
\begin{aligned}
v\cdot[x - x(\sigma(s))] &= x^0\cosh s - x^3\sinh s = \mathrm{x}^3\sinh(s - x^0) \\
&= \mathrm{z}\sinh(s - t).
\end{aligned}
\tag{43}
$$

Then, we have

$$
A^0(x) = \frac{e}{4\pi}\frac{\cosh s}{\mathrm{z}\sinh(s - \mathrm{t})}, \quad A^3(x) = \frac{e}{4\pi}\frac{\sinh s}{\mathrm{z}\sinh(s - \mathrm{t})} \tag{44}
$$

which are Eqs (249) in Pauli's book [45].

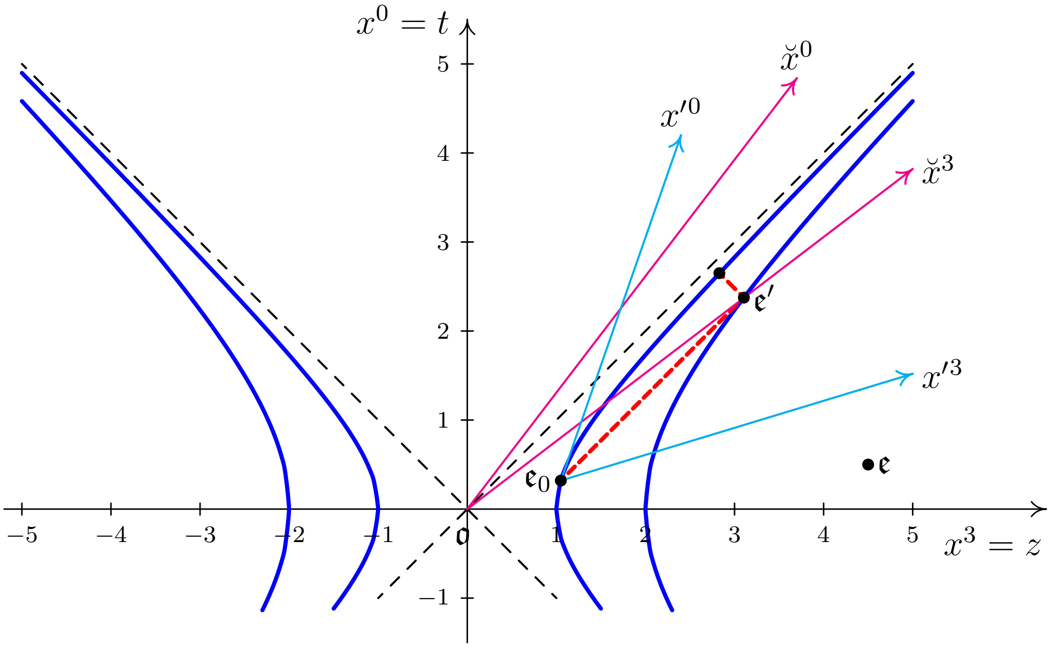

Figure 5: Graphic for presenting Pauli's argument

Pauli's argument for saying that a charge in hyperbolic motion does not radiate is the following:

(i) Consider the inertial reference frame $\boldsymbol{I}'$ where the charge is momentarily at rest at the instant $(x^0_{\mathfrak{e}'} - \mathrm{R}) = t_0$. This is the time coordinate (in the coordinates of the inertial frame $\boldsymbol{I}$) of the event $\mathfrak{e}_0$ in Figure 5.

A naturally adapted coordinate system for the reference frame $\boldsymbol{I}'$ is ($\mathfrak{v}{=}|\mathbf{v}|$)

$$\begin{aligned} x'^0 &= t_0 + \gamma(x^0 - \mathfrak{v}x^3), \\ x'^3 &= z_0 + \gamma(x^3 - \mathfrak{v}x^0), \\ x'^1 &= x^1, \quad x'^2 = x^2. \end{aligned} \tag{45}$$

and

$$\begin{aligned} \frac{\partial x'^0}{\partial x^0} &= \gamma = \cosh s, \quad \frac{\partial x'^0}{\partial x^3} = -\sinh s, \\ \frac{\partial x'^3}{\partial x^0} &= -\gamma\mathfrak{v} = -\sinh s, \quad \frac{\partial x'^3}{\partial x^3} = \cosh s. \end{aligned} \tag{46}$$

from where it follows that the components of the potential A in the new coordinates $\{x'^\mu\}$ are

$$A'^0(x') = \frac{e}{4\pi}\frac{1}{\mathrm{z}\sinh(s - \mathrm{t})}, \quad A'^3(x') = 0. \tag{47}$$

As a consequence of Eq.(47) it follows that the magnetic field $\mathbf{B}'$ as measured in the reference frame $\boldsymbol{I}'$ is null, thus the Poynting vector in this frame $\mathbf{S}'{=}\,\mathbf{E}' \times \mathbf{B}' = 0$ and thus (according to Pauli) an observer instantaneously at rest at event $\mathfrak{e}_0$ with respect to the charge will detect no radiation.

(ii) To conclude his argument Pauli consider a second inertial reference frame $\breve{\boldsymbol{I}}$ where the events $\mathfrak{o}$ and $\mathfrak{e}'$ are simultaneous and where $\mathfrak{e}'$ is an event on the world line of another observer at rest in the $\boldsymbol{R}$ frame which supposedly will receive —if it exists—the radiation field emitted by the charge at event $\mathfrak{e}_0$ (see Figure 5). A naturally adapted coordinate system to $\breve{\boldsymbol{I}}$ is

$$\begin{aligned}
&\breve{x}^0 = \breve{\gamma}(x^0 - \breve{\mathfrak{v}}x^3), \\
&\breve{x}^1 = x^1, \quad \breve{x}^2 = x^2, \\
&\breve{x}^3 = \breve{\gamma}(x^3 - \breve{\mathfrak{v}}x^0),
\end{aligned} \tag{48}$$

with

$$\breve{\mathfrak{v}} = \sinh \mathrm{t}/\cosh \mathrm{t}, \quad \breve{\gamma} = (1-\breve{\mathfrak{v}}^2)^{-1/2} = \cosh \mathrm{t}. \tag{49}$$

A trivial calculation gives

$$\breve{A}^0(\breve{x}) = \frac{e}{4\pi}\frac{\coth(s-\mathrm{t})}{\sqrt{(\breve{x}^3)^2 - ((\breve{x}^0)^2}}, \quad \breve{A}^3(\breve{x}) = \frac{e}{4\pi}\frac{1}{\sqrt{(\breve{x}^3)^2 - (\breve{x}^0)^2}}. \tag{50}$$

and since $\breve{\mathbf{B}} = (F_{32}, F_{13}, F_{21}) = 0$ it follows that the Poynting vector $\breve{\mathbf{S}} = \breve{\mathbf{E}} \times \breve{\mathbf{B}} = 0$. Thus an instantaneous observer $(\mathfrak{e}', \breve{\boldsymbol{I}}_{\mathfrak{e}'})$ in the $\breve{\boldsymbol{I}}$ frame momentously at rest relative to instantaneous observer $(\mathfrak{e}', \boldsymbol{R}_{\mathfrak{e}'})$ observer in the $\boldsymbol{R}$ frame at the considered event will also not detect any radiation emitted from $\mathfrak{e}_0$.

4.2.1. Calculation of Components of the Potentials in the R Frame

Using an obvious notation we write the components of the electromagnetic potential in the in the $\boldsymbol{R}$ frame as $A(\boldsymbol{x}'(\mathfrak{e})) = (\mathrm{A}^0(\mathrm{t},\mathrm{z}), 0, 0, -\mathrm{A}^3(\mathrm{t},\mathrm{z}))$ and we have

$$\begin{aligned}
\mathrm{A}_0 &= \frac{\partial x^0}{\partial \mathrm{x}^0}A_0 + \frac{\partial x^3}{\partial \mathrm{x}^0}A_3 = \frac{e}{4\pi}\coth(\mathrm{t}-s)\big|_{s=s_0}, \\
\mathrm{A}_3 &= \frac{\partial x^0}{\partial \mathrm{x}^3}A_0 + \frac{\partial x^3}{\partial \mathrm{x}^3}A_3 = -\frac{e}{4\pi \mathrm{z}}\tanh(s-\mathrm{t})\big|_{s=s_0}.
\end{aligned} \tag{51}$$

So,

$$\overrightarrow{\mathrm{E}}(\mathrm{t},\mathrm{z}) := (0, 0, \mathrm{F}_{03}(\mathrm{t},\mathrm{z})), \quad \overrightarrow{\mathrm{B}}(\mathrm{t},\mathrm{z}) = 0, \tag{52}$$

$$\mathrm{F}_{03}(\mathrm{t},\mathrm{z}) = \frac{\partial}{\partial \mathrm{t}}\mathrm{A}_3(\mathrm{t},\mathrm{z})\Big|_{s=s_0} - \frac{\partial}{\partial \mathrm{z}}\mathrm{A}_0(\mathrm{t},\mathrm{z})\Big|_{s=s_0} \tag{53}$$

and again the Poynting vector $\overrightarrow{\mathrm{E}} \times \overrightarrow{\mathrm{B}}$ is null. So, by Paui's argument the observers at rest in the $\boldsymbol{R}$ frame will detect no radiation.

4.3. Is Pauli Argument Correct?

In order to evaluate if Pauli's argument is correct we recall that the Liénard-Wiechert potential $A \in \sec \bigwedge^1 T^*M$ by construction is in Lorenz gauge, i.e., $\delta A = 0$ and moreover it satisfy the homogeneous wave equation for all spacetime points outside the worldline of the accelerated charge, i.e.,

$$\lozenge A = -d\delta A - \delta dA = -\delta dA = 0 \tag{54}$$

where $\lozenge$ is the Hodge Laplacian, and δ is the Hodge coderivative. Since $F = dA \in \sec \bigwedge^2 T^*M$ and

$$\lozenge F = -d\delta dA - \delta ddA = -d\delta dA = 0 \tag{55}$$

it follows that the electromagnetic field satisfies also a wave equation.

Remark 5 *Well, it is common practice to call an electromagnetic field satisfying the wave equation a electromagnetic wave. So, despite the fact that* $\overrightarrow{\mathrm{B}} = 0$ *observers outside the worldline of the accelerated charge (and living in the same accelerated laboratory) will perceive a pure electric wave.*

In our case

$$F = \mathrm{F}_{03} d\mathrm{x}^0 \wedge d\mathrm{x}^3 \tag{56}$$

and the energy momentum tensor of the electromagnetic field

$$\mathbf{T} = \mathrm{T}_{\mu\nu} d\mathrm{x}^\mu \otimes d\mathrm{x}^\nu \in \sec T_0^2 M \tag{57}$$

in the coordinates $\{\mathrm{x}^\mu\}$ (naturally adapted to the Rindler frame $\boldsymbol{R}$) has only the following non null component.

$$\mathrm{T}^{00}(\mathrm{t}, \mathrm{z}) = \frac{1}{2} \left|\mathrm{F}_{03}(\mathrm{t}, \mathrm{z})\right|^2 \tag{58}$$

So an observer, following the worldline σ' with $\mathrm{z} = \mathrm{z}_0 =$ constant ($\mathrm{z} > 1$) will detected a pseudo-energy density "wave" passing through the point where he is locate. Moreover, if this observer carries with him an electric charge say e' he will certainly detect that his charge is acted by the electromagnetic field with a (1-form) force

$$\mathfrak{F} = e' v_{\sigma'} \lrcorner F = v_{\sigma'}^0 \mathrm{F}_{03} d\mathrm{x}^3 \tag{59}$$

and he certainly will need more pseudo energy or better more Minkowski energy (fuel in his rocket) to maintain his charge (with mass m') at constant acceleration than the energy that he would have to use to maintain at a constant acceleration a particle with mass m' and null charge.

Also, since the energy arriving at the σ' worldline must be coming from energy radiated by the charge following σ, an observer maintaining the charge e (of mass m) at constant acceleration will expend more Minkowski energy than the one necessary for maintaining at a constant acceleration a particle with mass m and null charge.

4.4. The Rindler (Pseudo) Energy

It is a well known fact that outside the worldline σ of the accelerating charge the electromagnetic energy-momentum tensor has null divergence, i.e.,satisfy

$$D \cdot \mathbf{T} = \mathbf{0} \tag{60}$$

where D is the Levi-Civita connection of $\boldsymbol{g}$. Since $\boldsymbol{K} = \frac{\partial}{\partial \mathrm{t}}$ is a Killing vector field for the metric $\boldsymbol{g}$ as it is obvious looking at the representation of $\boldsymbol{g}$ in terms of the coordinates $\{\mathrm{x}^\mu\}$ adapted to the $\boldsymbol{R} = \frac{1}{\mathrm{z}}\boldsymbol{K}$ frame we have that the current

$$\mathcal{J}_R = \mathrm{K}^\nu \mathrm{T}_{\nu\mu} d\mathrm{x}^\mu \tag{61}$$

is conserved, i.e.,

$$\underset{\boldsymbol{g}}{\delta}\mathcal{J}_R = -\boldsymbol{\partial} \lrcorner \mathcal{J}_R = -\frac{1}{\sqrt{-\det \boldsymbol{g}}}\frac{\partial}{\partial \mathrm{x}^\mu}\left(\sqrt{-\det \boldsymbol{g}}\mathrm{K}^\nu \mathrm{T}_\nu^\mu\right) = 0. \tag{62}$$

Then, of course, the *scalar* quantity[12]

$$\mathcal{E} = \int_{\Sigma'} \star \mathcal{J}_R \tag{63}$$

is a conserved one. However, take notice that differently of the case of the similar current calculated with the Killing vector field $\partial/\partial t$ it does not qualify as the zero component of a momentum covector (*not* covector field). See details in [50].

In our case we have

$$\frac{\partial}{\partial \mathrm{x}^\mu}(\mathrm{z}\mathrm{T}_0^\mu) = 0 \tag{64}$$

Consider the accelerating charge following the σ worldline (for which $\mathrm{z} = 1$ and $s = \mathrm{t}$) surrounded by a 2-dimensional sphere Σ_t of constant radius $\mathrm{r} = \mathfrak{R}$ at time t. Now, from propertime $s_1 = \mathrm{t}_1$ to propertime $s_2 = \mathrm{t}_2$ the surface Σ_t moves producing a world tube in Minkowski spacetime.

Since

$$\frac{\partial}{\partial \mathrm{x}^0}\left(\mathrm{z}\mathrm{T}_0^0\right) = -\frac{\partial}{\partial \mathrm{x}^i}\left(\mathrm{z}\mathrm{T}_0^i\right) \tag{65}$$

the quantity $\mathcal{E}(\mathrm{t}_1 \mapsto \mathrm{t}_2)$ given by

$$\begin{aligned}\mathcal{E}(\mathrm{t}_1 \mapsto \mathrm{t}_2) &= \textstyle\int_{\mathrm{t}_1}^{\mathrm{t}_2} d\mathrm{t} \iiint \mathrm{r}^2 \sin\theta d\mathrm{r} d\theta d\varphi \frac{\partial}{\partial \mathrm{t}}\left(z\mathrm{T}_0^0\right) = -\int_{\mathrm{t}_1}^{\mathrm{t}_2} d\mathrm{t} \iiint \mathrm{r}^2 d\mathrm{r} d\Omega \frac{\partial}{\partial \mathrm{x}^i}\left(\mathrm{z}\mathrm{T}_0^i\right) \\ &= -\textstyle\int_{\mathrm{t}_1}^{\mathrm{t}_2} d\mathrm{t} \iint \left(\mathrm{z}\mathrm{T}_0^i\right) \mathrm{n}_i \, \mathfrak{R}^2 d\Omega\end{aligned} \tag{66}$$

(where $\{\mathrm{r}, \theta, \varphi\}$ are polar coordinates associated to $\{\mathrm{x}^1, \mathrm{x}^2, \mathrm{x}^3\}$ and n_i are the components of the normal vector to Σ_t) is null since $\mathrm{T}_0^i = 0$.

[12] If $N \subset M$ is the region where $\mathcal{J}_R$ has support then $\partial N = \Xi + \Xi' + \digamma$ where Ξ and Ξ' are spacelike surfaces and $\mathcal{J}_R$ is null in $\digamma$ (spatial infinity).

Thus if the observer following σ (of course, at rest relative to the accelerating charge) decide to call $\mathcal{E}(\mathrm{t}_1 \mapsto \mathrm{t}_2)$ the energy radiated by the charge he will arrive at the conclusion that he did not see any radiated energy.

But of course, $\mathcal{E}(\mathrm{t}_1 \mapsto \mathrm{t}_2)$ is not the extra Minkowski energy (calculated above) necessary for the observer to maintain the charge at constant acceleration. Parrott [44] quite appropriately nominate $\mathcal{E}(\mathrm{t}_1 \mapsto \mathrm{t}_2)$ the *pseudo-energy*, other people as authors of [15] call it Rindler energy.

Conclusion 6 *What seems clear at least to us is that whereas any one can buy Minkowski energy (e.g., in the form of fuel) for his rocket no one can buy the "magical" Rindler energy.*

4.5. The Turakulov Solution

In a paper published in the *Journal of Geometry and Physics* [59] Turakulov presented a solution for the problem of finding the electromagnetic field of a charge in uniformly accelerate motion by direct solving the wave equation for the potential $A \in \sec \bigwedge^1 T^*M$ using a separation of variables method instead of using the Liénard-Wiechert potential used in the previous discussion. Since this solution is not well known we recall and analyze it here with some details.

Turakulov started his analysis with the coordinates $(\mathrm{t}, \mathrm{x}, \mathrm{y}, \mathrm{z})$ introduced in Section 2 and proceeds as follows. In the $\mathrm{t} =$ constant Euclidean semi-spaces he introduced [13] *toroidal* coordinates (u, v, φ) by

$$\begin{aligned} &\mathrm{z} = \frac{a \sinh u}{\cosh u + \cos v}, \quad \rho = \frac{a \sin v}{\cosh u + \cos v}, \\ &u = \tanh^{-1}\left(\frac{2a\mathrm{z}}{\mathrm{z}^2 + \rho^2 + a^2}\right), \quad v = \tanh^{-1}\left(\frac{2a\mathrm{z}}{\mathrm{z}^2 + \rho^2 - a^2}\right). \end{aligned} \tag{67}$$

(where $\rho = +\sqrt{x^2 + y^2}$) and also introduce their pseudo Euclidean generalizations for the other domains, i.e.,

$$\begin{aligned} &\mathrm{z} = \frac{a \sin u}{\cos u + \cos v}, \quad \rho = \frac{a \sin v}{\cos u + \cos v}, \\ &u = \tan^{-1}\left(\frac{2a\mathrm{z}}{-\mathrm{z}^2 + \rho^2 + a^2}\right), \quad v = \tan^{-1}\left(\frac{2a\mathrm{z}}{-\mathrm{z}^2 + \rho^2 - a^2}\right). \end{aligned} \tag{68}$$

Let σ be the world line an uniformly accelerate charge, as we know it corresponds to $\mathrm{z} =$constant and thus the surfaces $u =$ constant forms a family of spheres defined by the equation

$$(z - a \coth u_0) + \rho^2 = a \sinh^{-1} u \tag{69}$$

[13] Toroidal coordinates (also caled bishperical coordinates) in discussed in Section 10.3 in volume II of the classical book by Morse and Feshbach [36].

involving the charge. The Minkowski metric metric in region I and II using the coordinates (t,u, v, ρ) reads

$$\boldsymbol{g} = \left(\frac{a}{\cosh u + \cos v}\right)^2 \left(\sinh^2 u \; \mathrm{d}t \otimes \mathrm{d}t - du \otimes du - dv \otimes dv - \sin^2 v d\varphi \otimes d\varphi\right) \quad (70)$$

and for regions F and P it is

$$\boldsymbol{g} = \left(\frac{a}{\cosh u + \cos v}\right)^2 \left(-\sin^2 u \; \mathrm{d}t \otimes \mathrm{d}t + du \otimes du - dv \otimes dv - \sin^2 v d\varphi \otimes d\varphi\right). \quad (71)$$

As we know the potential A^T in the Lorenz gauge $\delta A^T = 0$ satisfies the wave equation $\delta d A^T = 0$ Then supposing (as usual) that the potential is tangent to the the integral lines of $\boldsymbol{R}$ we can write[14]

$$A^T = \Theta(u, v)d\mathrm{t} \quad (72)$$

and the general solution of the wave equation is

$$\Theta(u, v) = \alpha_0(\cosh u - 1) + \textstyle\sum_{n=1}^{\infty} \alpha_n \sinh u \frac{d}{du} P_n(\cosh u) P_n(\cos \nu), \quad (73)$$

where P_n are Legendre polynomials and $\alpha_{0,}\alpha_n$ are constants. The field of a charge is simply specified only by the first term with $\alpha_0 = e$ the value of the charge generating the field. Thus, if the charge is at $u = \infty$ we have for regions I and II and P and F

$$A^T_{\mathrm{I,II}} = e(\cosh u - 1)d\mathrm{t}, \quad A^T_{\mathrm{P,F}} = e(\cos u - 1)d\mathrm{t}. \quad (74)$$

In terms of the coordinates (t, x, y, z), writing $A^T = A^T_\mu dx^\mu$ we have the following solution valid for all regions[15]:

$$\begin{aligned} A^T_0 &= -\frac{z}{z^2 - t^2}\left(\frac{t^2 - \rho^2 + z^2 - a^2}{\Lambda_+ \Lambda_-} - 1\right), \\ A^T_3 &= \frac{t}{z^2 - t^2}\left(\frac{t^2 - \rho^2 + z^2 - a^2}{\Lambda_+ \Lambda_-} - 1\right), \\ A^T_1 &= A^T_2 = 0, \\ \Lambda_\pm(t, x, y, z) &= \sqrt{(\sqrt{z^2 - t^2} \pm a)^2 + x^2 + y^2}. \end{aligned} \quad (75)$$

From these formulas we infer that

$$F^T = F_{\mathrm{t}u} d\mathrm{t} \wedge du = -e \sinh u d\mathrm{t} \wedge du \quad (76)$$

[14]Here the value of the charge is $e/4\pi = 1$.

[15]We have verified using the Mathematica software that indeed A_0 and A_3 satisfy the wave equation. Note that ther is are signal misprints in the formulas for A_0 and A_3 in [59] and the modulus $\sqrt{|z^2 - t^2|}$ in those formulas are not necessary.

and thus an observer comoving with the charge will see only an "electric field" which for him is in the u-direction and the *pseudo* energy evaluated beyond a given sphere $u = u_0$ of radius $\boldsymbol{r}$ is

$$\mathcal{E} = \frac{e^2}{2\boldsymbol{r}}. \tag{77}$$

Thus, Turakulov concludes as did Pauli did that there is no radiation. But is his conclusion correct?

4.5.1. Does the Turakulov Solution Implies that a Charge in Hyperbolic Motion does not Radiate?

Recall that in subsection 4.3 we showed that supposing that the Liénard-Wiechert solution is the correct one then Pauli's argument is incorrect since an observer following another integral line of $\boldsymbol{R}$ will see an electric "wave" (recall Eq.(58)) We now makes the same analysis as the one we did in the case of the Turakulov solution in order to find the correct answer to our question. We first explicitly calculate the electric and magnetic fields in the inertial frame $\boldsymbol{I} = \partial/\partial t$. We have

$$E_x = \frac{8a^2xz}{\Lambda_+^3\Lambda_-^3}, \qquad E_y = \frac{8a^2yz}{\Lambda_+^3\Lambda_-^3}, \qquad E_z = \frac{-4a^2[x^2+y^2+a^2-z^2+t^2]}{\Lambda_+^3\Lambda_-^3},$$

$$B_x = \frac{8a^2yt}{\Lambda_+^3\Lambda_-^3}, \quad B_y = \frac{-8a^2xt}{\Lambda_+^3\Lambda_-^3}, \quad B_z = 0. \tag{78}$$

The Poincaré invariants of the Turakulov solution $I_1 := \mathbf{E}^2 - \mathbf{B}^2$ and $I_2 := \mathbf{E} \bullet \mathbf{B}$ are

$$I_1 = \frac{16a^4}{\Lambda_+^6\Lambda_-^6}[(x^2+y^2-z^2+t^2)^2 + 4(x^2+y^2)(z^2+t^2)], \quad I_2 = 0. \tag{79}$$

This shows that an inertial observer at rest at (x, y, z) will detect a *time dependent* electromagnetic field configuration passing though his observation point. Of course, it is *not* a null field, but it certainly qualify as an electromagnetic wave. And what is important for our analysis is that the field carries energy and momentum from the accelerating charge to the point (x, y, z).

Indeed, consider a charge q at rest in the Rindler frame following an integral line σ' of $\boldsymbol{R}$ with constant Rindler coordinates $(\mathrm{t}, \mathrm{x} = \mathrm{x}_0, \mathrm{y} = \mathrm{y}_0\ \mathrm{z} = \mathrm{z}_0)$ and thus with inertial coordinates $(t, \mathrm{x}_0, y = \mathrm{y}_0,\ z = \sqrt{\mathrm{z}_0^2 + t^2})$.

As determined by the inertial observer the density of *real* energy and the Poynting vector arriving from the uniformly accelerated charge moving along the z-axis of the inertial frame to where the charge q is locate are:

$$\frac{1}{2}(\mathbf{E}^2+\mathbf{B}^2)$$
$$=\frac{1}{2}\mathring{\Lambda}_+^{-6}\mathring{\Lambda}_-^{-6}(128(\mathrm{x}_0^2+\mathrm{y}_0^2)t^2+64a^4(\mathrm{x}_0^2+\mathrm{y}_0^2)\mathrm{z}_0^2+16a^4(\mathrm{x}_0^2+\mathrm{y}_0^2+a^2-\mathrm{z}_0^2)^2),$$
$$\mathbf{S}=\mathbf{i}\frac{32a^4\mathrm{x}_0}{\mathring{\Lambda}_+^6\mathring{\Lambda}_-^6}(\mathrm{x}_0^2+\mathrm{y}_0^2+a^2-\mathrm{z}_0^2)t+\mathbf{j}\frac{-32a^4\mathrm{y}_0}{\mathring{\Lambda}_+^6\mathring{\Lambda}_-^6}(\mathrm{x}_0^2+\mathrm{y}_0^2+a^2-\mathrm{z}_0^2)t$$
$$+\mathbf{k}\frac{64a^4\sqrt{z_0^2+t^2}}{\mathring{\Lambda}_+^6\mathring{\Lambda}_-^6}(\mathrm{x}_0^2+\mathrm{y}_0^2)t,$$
$$\mathring{\Lambda}_\pm=\sqrt{(\mathrm{z}_0\pm a)^2+\mathrm{x}_0^2+\mathrm{y}_0^2} \tag{80}$$

Thus, we see that indeed there is a flux of *real* energy and momentum arriving at the charge q located at $(t,\mathrm{x}_0,y=\mathrm{y}_0,\ z=\sqrt{\mathrm{z}_0^2+t^2})$.

Moreover, the Lorentz force $\mathbf{F}_L$ acting on the charge q (according to the inertial observer) is

$$\mathbf{F}_L=q\mathbf{E}+\mathbf{q}\mathbf{v}_{\sigma'}\times\mathbf{B} \tag{81}$$

depends on t and is doing work on the charge q. So, an observer comoving with the charge q will need to expend more *real* energy to carry this charge than to carry a particle with zero charge.

More important: since the energy arriving at the charge q is the one produced by the charge e generating the field we arrive at the conclusion, as in the case of the Pauli solution that an observer carrying the charge e will speed more energy (fuel of its rocket) than when it carries a particle with zero charge.

Remark 7 *We already observed in* [34] *that the use of the retarded Green's function may result in non sequitur solutions in some cases. Most important is the fact that in* [61] *it is observed that the Green's function for a massless scalar field is the integral* $(\omega=k_0)$

$$G(x,x')=\frac{1}{(2\pi)^4}\int d^3\mathbf{k}\int d\omega\frac{e^{-i(\omega(t-t')-\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}'))}}{\mathbf{k}^2-\omega^2} \tag{82}$$

and the evaluation of the integral is done in all classical presentations in the complex ω*-plane and thus its result depends, as is well known from the path of integration chosen. But, contrary to what is commonly accepted this is not necessary for the integrand is not singular. This can be shown as follows. Recalling that* G *depends only on*

$$\tau^2-\mathfrak{r}^2=(t-t')^2-(\mathbf{x}-\mathbf{x}')^2$$

we can choose a coordinate system where $(\mathbf{x}-\mathbf{x}')^2=0$ *for the point under consideration, Then, introducing the coordinates*

$$\varkappa=\omega^2-\mathbf{k}^2,\quad \xi=\tanh^{-1}(|\mathbf{k}|/\omega),$$
$$\omega(t-t')-\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')=\varkappa\varsigma\cosh\xi \tag{83}$$

the Eq.(83) *becomes after some algebra*

$$G(\tau, \mathfrak{r}) = \frac{1}{4\pi^3} \int d\varkappa \int d\xi \int d\theta \int d\varphi \sin\theta \sinh^3 \xi \varkappa^2 e^{i\varkappa\varsigma \cosh \xi}. \tag{84}$$

This important result obtained in [61] *shows explicitly that it is possible to evaluate the Green's function without introducing the "famous"* $i\varepsilon$ *prescription! Turakulov also observed that putting* $\lambda = \varkappa\varsigma$ *the* Eq.(84) *gives*

$$G(\tau, \mathfrak{r}) = \frac{\pi^2}{\varsigma^2} \int d\lambda\lambda \int d\xi \sinh^2 \xi e^{i\lambda \cosh \xi}. \tag{85}$$

The conclusion is thus that integration only predetermines the factor $1/\ \varsigma^2$ *and it is now possible to select any path of integration in the complex plane, which means that the retarded Green's function is create by inserting a non-existence singularity into the integrand!*

Moreover, in it is shown in [61] *that the use of the retarded Green's function produces problems with energy-conservation when, e.g., a charge is accelerated in an external potential. Finally we observe that in* [60] *it is shown that when there are infinitesimally small changes of the acceleration there is emission of radiation.*

5. The Equivalence Principle

Consider first the statements (a) and (b):

(a) an observer (say Mary) living in a small constantly accelerated reference frame (e.g., a 'small' world tube, with non transparent walls of the reference frame $\boldsymbol{R}$) following an integral line σ of the $\boldsymbol{R}$ frame and for which $D_{\boldsymbol{R}}\boldsymbol{R}|_{\sigma} = \boldsymbol{a}_{|\sigma}$;

(b) an observer (say John) living in a 'small' reference frame, (e.g., a 'small' world tube, with non transparent walls of the reference frame $\boldsymbol{Z}$ in a Lorentzian spacetime structure $(M, \mathbf{g}, \mathbf{D}, \tau_{\mathbf{g}}, \uparrow)$ modelling a gravitational field (generated by some energy-momentum distribution) in General Relativity theory and such that $\mathbf{D}_{\boldsymbol{Z}}\boldsymbol{Z}|_{\lambda} = \boldsymbol{a}_{|\lambda} = \boldsymbol{a}_{|\sigma}$.

Then a common formulation of the *Equivalence Principle*[16] says that Mary or John cannot with *local*[17] experiments determine if she(he) lives in an uniformly accelerated frame in Minkowski spacetime or in the gravitational field modelled by $(M, \mathbf{g}, \mathbf{D}, \tau_{\mathbf{g}}, \uparrow)$.

Now, as well known (since long ago) and as proved rigorously (under well determined conditions) in [44] a charge in a static gravitational field in General Relativity theory does not radiated if it follows an integral line of a reference frame like $\boldsymbol{Z}$ in (b). An

[16] A thoughtful dicussion of the Equivalence Principle and the so-called Principle of Local Lorentz Invariance is given in [47]

[17] Of course, by local mathematicians means an (4-dimensional) open set U of the appropriate spacetime manifold. So, by doing experiments in U observers will detect using a gradiometer tidal force fields (proportional to the Riemann curvature tensor) if at rest in $\boldsymbol{Z}$ in a real gravitational field and will not detect any tidal force field if living in $\boldsymbol{R}$ in Minkowski spacetime. For more details see, e.g., [40, 47].

observer commoving with the charge will see only an electric field and thus will see no radiation since the Poynting vector is null.

Does this implies that the Equivalence Principle holds for local experiments with charged matter?

Well, if we accept that the Liénard-Wiechert solution the correct one, then the answer from the analysis given in the previous section is *no* (see also, [32, 33, 44]. In particular Parrot's argument is the following: since there is no radiation in the true gravitational field an observer at rest in the Schwarzschild spacetime following a worldline λ will spend the same amount of "energy" to maintain at constant acceleration $\boldsymbol{a}_{|\lambda} = \boldsymbol{a}_{|\sigma}$ a particle with mass m and null charge and one with mass m and charge $e \neq 0$.

Since we already know that in the $\boldsymbol{R}$ frame it is clear that an observer σ will spend *different* amounts (of Minkowski) energy to maintain at constant acceleration $\boldsymbol{a}_{|\lambda} = \boldsymbol{a}_{|\sigma}$ a particle with mass m and null charge and one with mass m and charge $e \neq 0$.

Of course, even supposing that the Liénard-Wiechert solution is the correct one many people does not agree with this conclusion and some of the arguments of the opposition is discussed in [44].

Remark 8 *From our point of view we think necessary to comment that Parrot' s argument would be a really strong one only if the concept of energy* (*and momentum*) *would be well defined in General Relativity, which is definitively not the case* [48, 49, 50]*. However, take notice that the quantity defined as "energy" by Parrot* (*the zero component of current of the form given by* Eq.(61)*, were in this case* $\boldsymbol{K}$ *is a timelike Killing vector field for the Schwarzschild metric is not the componet of any energy-momentum covector field, it looks more as the concept of energy in Newtonian physics. . Anyway, the quantity of the pseudo "energy" necessary to carry a particle in uniformily accelerated motion will certainly be different in the two cases of a charged and a non charged particle. In our opinion what is necessary is to construct an analysis of the problem charge in a gravitational theory where energy-momentum of a system can be defined and is a conserved quantity* [48, 49].

On the other hand if we accept that Turakulov solution as the correct one than again the Equivalence Principle is violated and for the same reason than in the case of the Liénard-Wiechert solution as discussed in Section 4.5.1.

So, which solution, Liénard-Wiechert or Turakulov is the correct one?

An answer can be given to the above question only with a clever experiment and for the best of our knowledge no such experiment has been done yet.

6. Some Comments on the Unhru Effect

6.1. Minkowski and Fulling-Unruh Quantization of the Klein-Gordon Field

(**u1**) To discuss the Unruh effect it is useful to introduce coordinates such that the solution of the Klein-Gordon equation in these variables becomes as simple as possible.

A standard choice is to take $(\mathfrak{t},\mathfrak{x},\mathfrak{y},\mathfrak{z})$ and $(\mathfrak{t}',\mathfrak{x}',\mathfrak{y}',\mathfrak{z}')$ for regions I and II defined by[18]

$$
\begin{gathered}
\mathfrak{t}=\frac{1}{a}\tanh^{-1}(\frac{t}{z}),\quad \mathfrak{z}=\frac{1}{2a}\ln[a(z^2-t^2)],\quad \mathfrak{x}=x,\quad \mathfrak{y}=y\\
t=\frac{1}{a}\exp(a\mathfrak{z})\sinh(a\mathfrak{t}),\quad z=\frac{1}{a}\exp(a\mathfrak{z})\cosh(a\mathfrak{t}),\quad |z|\geq t,\quad z>0,\\
\mathfrak{t}'=\frac{1}{a}\tanh^{-1}(\frac{t}{z}),\quad \mathfrak{z}'=\frac{1}{2a}\ln[a^2(z^2-t^2)],\quad \mathfrak{x}'=x,\quad \mathfrak{y}'=y.,\\
t=\frac{1}{a}\exp(a\mathfrak{z}')\sinh(a\mathfrak{t}'),\quad z=-\frac{1}{a}\exp(a\mathfrak{z}')\cosh(a\mathfrak{t}'),\quad |z|\geq t,\quad z<0,\\
\mathfrak{t},\mathfrak{z}\in(-\infty,\infty),\qquad a\in\mathbb{R}^+. \qquad (86)
\end{gathered}
$$

Take notice that in regions I and II the coordinates $\mathfrak{t}$ and $\mathfrak{z}$ are respectively timelike and spacelike and in region II the decreasing of $\mathfrak{t}$ corresponds to the increase of t.

The Minkowski metric in these coordinates (and in the regions I and II) reads

$$
\begin{gathered}
\boldsymbol{g}=\exp(2a\mathfrak{z})d\mathfrak{t}\otimes d\mathfrak{t}-d\mathfrak{x}\otimes d\mathfrak{x}-d\mathfrak{y}\otimes d\mathfrak{y}-\exp(2a\mathfrak{z})d\mathfrak{z}\otimes d\mathfrak{z}=\eta_{\mathbf{ab}}\mathfrak{g}^{\mathbf{a}}\otimes\mathfrak{g}^{\mathbf{b}},\\
\mathfrak{g}^{\mathbf{0}}=\exp(a\mathfrak{z})d\mathfrak{t},\quad \mathfrak{g}^{\mathbf{1}}=d\mathfrak{x},\quad \mathfrak{g}^{\mathbf{2}}=d\mathfrak{y},\quad \mathfrak{g}^{\mathbf{3}}=\exp(a\mathfrak{z})d\mathfrak{z}. \qquad (87)
\end{gathered}
$$

(**u2**) The right and left Rindler reference frames are represented by

$$
\begin{aligned}
\boldsymbol{R}&=\frac{1}{\exp(a\mathfrak{z})}\partial/\partial\mathfrak{t},\quad t\in(-\infty,\infty),\quad |z|\geq t,\quad z>0,\\
\boldsymbol{L}&=\frac{1}{\exp(a\mathfrak{z})}\partial/\partial\mathfrak{t},\quad t\in(-\infty,\infty),\quad |z|\geq t,\quad z<0. \qquad (88)
\end{aligned}
$$

and they are *not* Killing vector fields.[19]

Consider the integral line, say σ of $\boldsymbol{R}$ given by $\mathfrak{x},\mathfrak{y}=$ constant and $\mathfrak{z}=\mathfrak{z}_0=$ constant. We immediately find that its proper acceleration is

$$
a_\sigma=1/\sqrt{g_{00}(\mathfrak{z}_0)}. \qquad (89)
$$

(**u3**) However, the vector fields

$$
\begin{aligned}
\boldsymbol{I}&=\partial/\partial t,\\
\boldsymbol{Z}_{\mathrm{I}}&=\partial/\partial\mathfrak{t},\text{with } t\in(-\infty,\infty),|z|\geq t \text{ and } z>0,\\
\boldsymbol{Z}_{\mathrm{II}}&=\partial/\partial\mathfrak{t},\text{with } t\in(-\infty,\infty),|z|\geq t \text{ and } z<0, \qquad (90)
\end{aligned}
$$

are Killing vector fields, i.e., $\mathcal{L}_{\partial/\partial t}\boldsymbol{g}=\mathcal{L}_{\boldsymbol{Z}_{\mathrm{I}}}\boldsymbol{g}=\mathcal{L}_{\boldsymbol{Z}_{\mathrm{II}}}\boldsymbol{g}=0$. The inertial reference frame $\boldsymbol{I}$ besides being locally synchronizable is also propertime synchronizable, i.e., $\boldsymbol{g}(\boldsymbol{I},\ )=dt$ and the fields $\boldsymbol{Z}_{\mathrm{I}}$ and $\boldsymbol{Z}_{\mathrm{II}}$ although does not qualify as reference frames (according to

[18]Note that $(\mathfrak{t},\mathfrak{z})$ differs form the coodinates (t,z) introduced in Section 2.

[19]This can easily be verifed taking into account that $\mathcal{L}_{\boldsymbol{R}}\boldsymbol{g}=2\eta_{\mathbf{ab}}\mathcal{L}_{\boldsymbol{R}}\mathfrak{g}^{\mathbf{a}}\otimes\mathfrak{g}^{\mathbf{b}}$ and recalling that if $R=\boldsymbol{g}(\boldsymbol{R},\ )=\mathfrak{g}^{\mathbf{0}}$ we may evalute [48] as $\mathcal{L}_{\boldsymbol{R}}\mathfrak{g}^{\mathbf{a}}=d(\mathfrak{g}^{\mathbf{0}}\cdot\mathfrak{g}^{\mathbf{a}})+\mathfrak{g}^{\mathbf{0}}\lrcorner d\mathfrak{g}^{\mathbf{a}}$.

our definition) play an important role for our considerations of the Unruh effect. The reason is that both fields in the regions where they have support are such that

$$
\begin{aligned}
Z_{\mathrm{I}} &= \boldsymbol{g}(\boldsymbol{Z}_{\mathrm{I}},\ ) = \exp(2a\mathfrak{z})d\mathfrak{t}, \text{with } t \in (-\infty,\infty), |z| \geq t \text{ and } z > 0,\\
Z_{\mathrm{II}} &= \boldsymbol{g}(\boldsymbol{Z}_{\mathrm{II}},\ ) = \exp(2a\mathfrak{z})d\mathfrak{t}, \text{with } t \in (-\infty,\infty), |z| \geq t \text{ and } z > 0.
\end{aligned}
\tag{91}
$$

Thus the field $\boldsymbol{I}$ can be used to foliate all M as $M = \cup_t(\mathbb{R}\times\Sigma(t))$ where $\Sigma(t) \simeq \mathbb{R}^3$ is a Cauchy surface. Moreover, the field Z_{I} *(*respectively Z_{II}) can be used to foliate region I, (respectively region II) as $\mathrm{I} = \cup_{\mathrm{t}}(\mathbb{R}\times\Sigma_{\mathrm{I}}(\mathrm{t}))$ *(*respectively $\mathrm{II} = \cup_{\mathrm{t}}(\mathbb{R}\times\Sigma_{\mathrm{II}}(\mathfrak{t})))$ where $\Sigma_{\mathrm{I}}(\mathfrak{t}) \simeq \Sigma_{\mathrm{I}}$ and $\Sigma_{\mathrm{II}}(\mathrm{t}) \simeq \Sigma_{\mathrm{II}}$ are Cauchy surfaces.

We now briefly describe how the Unruh effect for a complex Klein-Gordon field is presented in almost all texts[20] dealing with the issue.

(u4) Let $\phi \in \sec(\mathbb{C}\otimes\bigwedge^0 T^*M)$. Our departure point is to first solve the Klein-Gordon equation

$$
-\delta d\phi + \mu^2\phi = 0 \tag{92}
$$

valid for all M, in the global naturally adapted coordinates (in ELP gauge) to $\boldsymbol{I}$ and next to solve it in regions I and II using the coordinates defined in Eq.(86) (and then extend this new solution for all M). In the first case we use the $t = 0$ as Cauchy surface to given initial data. In the second case we use the $\mathfrak{t} = 0$ Cauchy surface to give initial data (see below).

The positive energy solutions will be called *Minkowski modes* for the first case and *Fulling-Unruh modes* for the second case (i.e., the solutions in regions I and II). In order to simplify the writing of the formulas that follows we introduce the notations

$$
\begin{gathered}
\phi_M(x) = \phi_M(t,x,y,z), \quad \phi_{\mathrm{I}}(\mathfrak{l}) = \phi_{\mathrm{I}}(\mathfrak{t},\mathfrak{x},\mathfrak{y},\mathfrak{z}), \quad \phi_{\mathrm{II}}(\mathfrak{l}') = \phi_{\mathrm{II}}(\mathfrak{t},\mathfrak{x},\mathfrak{y},\mathfrak{z}),,\\
k\cdot x = k_\alpha x^\alpha, \quad \omega_{\mathbf{k}} = k_0 = +\sqrt{\mathbf{k}^2+\mu^2}, \quad k\cdot k = (k_0)^2 - \mathbf{k}^2 = \mu^2, \quad \mathbf{k}^2 = \mathbf{k}\bullet\mathbf{k},\\
\mathbf{q} = (k_1,k_2), \quad \mathbf{r} = (x^1,x^2) = (x,y) \text{ and } \mathbf{q}\bullet\mathbf{r} = k_1x^1 + k_2x^2, \quad \nu = +\sqrt{\mathbf{q}^2+\mu^2}.
\end{gathered}
\tag{93}
$$

Observing that in region II the timelike coordinate $\mathfrak{t}'$ decreases when t increases we have that the elementary modes (of positive energy) which are solutions of the Klein-Gordon equation in the three regions:

$$
\begin{aligned}
\phi_{M\mathbf{k}}(x) &= [(2\pi)^3 2\omega_{\mathbf{k}})]^{-1/2} e^{-ik\cdot x},\\
\phi_{\mathrm{I}\nu\mathbf{q}}(\mathfrak{l}) &= [(2\pi)^2 2\nu)]^{-1/2} F_{\mathrm{I}\nu\mathbf{q}}(\mathfrak{z}) e^{-i(\nu\mathfrak{t}-\mathbf{q}\bullet\mathbf{r})},\\
\phi_{\mathrm{I}\nu\mathbf{q}}(\mathfrak{l}') &= [(2\pi)^2 2\nu)]^{-1/2} F_{\mathrm{II}\nu\mathbf{q}}(\mathfrak{z}) e^{+i(\nu\mathfrak{t}'+\mathbf{q}\bullet\mathbf{r})},
\end{aligned}
\tag{94}
$$

with

$$
\begin{aligned}
F_{\mathrm{I}\nu\mathbf{q}}(\mathfrak{z}) &= (2\pi^{-1})^{1/2} C_{\mathrm{I}\mathbf{q}} \frac{1}{\Gamma(i\nu)} (\frac{\nu}{2a})^{i\nu} K_{i\nu}(\nu\mathfrak{z}),\\
F_{\mathrm{II}\nu\mathbf{q}}(\mathfrak{z}') &= (2\pi^{-1})^{1/2} C_{\mathrm{II}\mathbf{q}}(a) \frac{1}{\Gamma(i\nu)} (\frac{\nu}{2a})^{i\nu} K_{i\nu}(\nu\mathfrak{z}'),
\end{aligned}
\tag{95}
$$

---

[20] E.g., in [15, 18, 26, 55, 56, 62, 66]. The presentations eventually differ in the use of other coordinate systems.

where $C_{\mathrm{I}\mathbf{q}}$ are arbitrary "phase factor", Γ is the gamma function and $K_{i\nu}$ are the modified Bessel functions of second kind.

Remark 9 *Before we continue it is important to emphasize that the concept of energy defined in regions* I *and* II *are indeed the pseudo-energy concept that we discussed in previous section.*

(**u5**) We use the positive frequencies in standard way in order construct Hilbert spaces $\mathcal{H}$, $\mathcal{H}_{\mathrm{I}}$ and $\mathcal{H}_{\mathrm{II}}$ by defining the well known scalar products for the spaces of positive energy-solutions. This is done by introducing the spaces of square integrable functions $\mathcal{K}_M, \mathcal{K}_{\mathrm{I}}$ and $\mathcal{K}_{\mathrm{II}}$ respectively of the forms

$$\begin{aligned}\Phi_M(x) &= \int d^3\mathbf{k}[a(\mathbf{k})\phi_{M\mathbf{k}}(x) + \bar{a}^*(\mathbf{k})\phi^*_{M\mathbf{k}}(x)]\\ \Phi_{\mathrm{I}}(\mathfrak{l}) &= \int_0^\infty d\nu \int d^2\mathbf{q}[b_{\mathrm{I}\nu}(\mathbf{q})\phi_{\mathrm{I}\nu\mathbf{q}}(\mathfrak{l}) + \bar{b}^*_{\mathrm{I}\nu}(\mathbf{q})\phi^*_{\mathrm{I}\nu\mathbf{q}}(\mathfrak{l})]\\ \Phi_{\mathrm{II}}(\mathfrak{l}') &= \int_0^\infty d\nu \int d^2\mathbf{q}[b_{\mathrm{II}\nu}(\mathbf{q})\phi_{\mathrm{II}\nu\mathbf{q}}(\mathfrak{l}') + \bar{b}^*_{\mathrm{II}\nu}(\mathbf{q})\phi^*_{\mathrm{II}\nu\mathbf{q}}(\mathfrak{l}')]\end{aligned} \tag{96}$$

where $a, b_{\mathrm{I}\nu}, b_{\mathrm{II}\nu}, \bar{a}, \bar{b}_{\mathrm{I}\nu}, \bar{b}_{\mathrm{II}\nu}$ are arbitrary square integrable functions (elements of $\mathfrak{L}(\mathbb{R}^3)$).

Take notice that $\hat{\phi}_{\mathrm{I}} + \hat{\phi}_{\mathrm{II}}$ can be extended to all M by extending $\phi_{\mathrm{I}\nu\mathbf{q}}$ and $\phi_{\mathrm{II}\nu\mathbf{q}}(\mathfrak{l})$ to all M.

Now, we construct in the space of these functions the usual inner products ($J = M, \mathrm{I}, \mathrm{II}$)

$$\langle \Phi_J, \Psi_J \rangle_J = i \int_\Sigma d\Sigma n^a (\Phi^*_J \frac{\partial}{\partial x^a_J}\Psi_J - \Phi_J \frac{\partial}{\partial x^a_J}\Psi^*_J) \tag{97}$$

where $J = M, \mathrm{I}, \mathrm{II}$ and x^a_J denotes the appropriate variables for each domain and finally we construct as usual the Hilbert spaces $\mathcal{H}, \mathcal{H}_{\mathrm{I}}$ and $\mathcal{H}_{\mathrm{II}}$ by completion of the respective $\mathcal{K}$ spaces and n^a are the components of the normal to the spacelike surface Σ.

In particular, choosing Σ to be hypersurface $t = 0$ for the Minkowski modes and $\mathfrak{t} = 0$ for the Rindler modes we have

$$\begin{gathered}\langle \phi_{M\mathbf{k}}, \phi_{M\mathbf{k}'} \rangle_M = \delta(\mathbf{k} - \mathbf{k}'), \quad \langle \phi^*_{M\mathbf{k}}, \phi^*_{M\mathbf{k}'} \rangle_M = -\delta(\mathbf{k} - \mathbf{k}'),\\ \langle \phi_{\mathrm{I}\nu\mathbf{q}}, \phi_{\mathrm{I}\nu'\mathbf{q}'} \rangle_{\mathrm{I}} = \delta(\nu - \nu')\delta(\mathbf{q} - \mathbf{q}'), \quad \langle \phi_{\mathrm{I}\nu\mathbf{q}}, \phi_{\mathrm{I}\nu'\mathbf{q}'} \rangle_{\mathrm{I}} = -\delta(\nu - \nu')\delta(\mathbf{q} - \mathbf{q}'),\\ \langle \phi_{\mathrm{II}\nu\mathbf{q}}, \phi_{\mathrm{I}\nu'\mathbf{q}'} \rangle_{\mathrm{II}} = \delta(\nu - \nu')\delta(\mathbf{q} - \mathbf{q}'), \quad \langle \phi_{\mathrm{II}\nu\mathbf{q}}, \phi_{\mathrm{II}\nu'\mathbf{q}'} \rangle_{\mathrm{II}} = -\delta(\nu - \nu')\delta(\mathbf{q} - \mathbf{q}'),\\ \langle \phi_{M\mathbf{k}}, \phi^*_{M\mathbf{k}'} \rangle_M = 0, \quad \langle \phi_{\mathrm{I}\nu\mathbf{q}}, \phi^*_{\mathrm{I}\nu'\mathbf{q}'} \rangle_{\mathrm{I}} = 0, \quad \langle \phi_{\mathrm{II}\nu\mathbf{q}}, \phi^*_{\mathrm{II}\nu'\mathbf{q}'} \rangle_{\mathrm{II}} = 0.\end{gathered} \tag{98}$$

(**u6**) From $\mathcal{H}, \mathcal{H}_{\mathrm{I}}$ and $\mathcal{H}_{\mathrm{II}}$ we construct the Fock-Hilbert space $\mathcal{F}(\mathcal{H})$, $\mathcal{F}(\mathcal{H}_{\mathrm{I}})$ and $\mathcal{F}(\mathcal{H}_{\mathrm{II}})$ which describe all possible physical states of the quantum fields

$$\hat{\phi}_M(x) = \int d^3\mathbf{k}\left[\mathbf{a}(\mathbf{k})\,\phi_{M\mathbf{k}} + \bar{\mathbf{a}}^\dagger(\mathbf{k})\,\phi^*_{M\mathbf{k}}\right], \tag{99a}$$

$$\hat{\phi}_{\mathrm{I}}(\mathfrak{l}) = \int_0^\infty d\nu \int d^2\mathbf{q}\left[\mathbf{b}_{\mathrm{I}\nu}(\mathbf{q})\,\phi_{\mathrm{I}_\nu\mathbf{q}}(\mathfrak{l}) + \bar{\mathbf{b}}^\dagger_{\mathrm{I}\nu}(\mathbf{q})\,\phi^*_{\mathrm{I}_\nu\mathbf{q}}(\mathfrak{l})\right], \tag{99b}$$

$$\hat{\phi}_{\mathrm{II}}(\mathfrak{l}') = \int_0^\infty d\nu \int d^2\mathbf{q}\left[\mathbf{b}_{\mathrm{II}\nu}(\mathbf{q})\,\phi_{\mathrm{II}_\nu\mathbf{q}}(\mathfrak{l}') + \bar{\mathbf{b}}^\dagger_{\mathrm{II}\nu}(\mathbf{q})\,\phi^*_{\mathrm{I}_\nu\mathbf{q}}(\mathfrak{l}')\right], \tag{99c}$$

which are operator valued distributions acting respectively on $\mathcal{F}(\mathcal{H}), \mathcal{F}(\mathcal{H}_{\rm II})$ $\mathcal{F}(\mathcal{H})$ and where the $\mathbf{a}$,$\mathbf{a}^{\dagger}$, $\mathbf{b}_{\mathrm{I}\nu}, \mathbf{b}_{\mathrm{I}\nu}^{\dagger}$ and $\mathbf{b}_{\mathrm{II}\nu}, \mathbf{b}_{\mathrm{II}\nu}^{\dagger}$ (respectively $\bar{\mathbf{a}}$,$\bar{\mathbf{a}}^{\dagger}$, $\bar{\mathbf{b}}_{\mathrm{I}\nu}, \bar{\mathbf{b}}_{\mathrm{I}\nu}^{\dagger}$ and $\bar{\mathbf{b}}_{\mathrm{II}\nu}, \bar{\mathbf{b}}_{\mathrm{II}\nu}^{\dagger}$) are destruction and creation operators for *positive* (respectively negative) charged particles. We have for the *non null* commutators:

$$\begin{aligned} \left[\bar{\mathbf{a}}\left(\mathbf{k}\right), \bar{\mathbf{a}}^{\dagger}\left(\mathbf{k}'\right)\right] &= \left[\mathbf{a}\left(\mathbf{k}\right), \mathbf{a}^{\dagger}\left(\mathbf{k}'\right)\right] = \delta(\mathbf{k}-\mathbf{k}'), \\ \left[\bar{\mathbf{b}}_{\mathrm{I}\nu}\left(\mathbf{q}\right), \bar{\mathbf{b}}_{\mathrm{I}\nu'}\left(\mathbf{q}'\right)\right] &= \left[\mathbf{b}_{\mathrm{I}\nu}\left(\mathbf{q}\right), \mathbf{b}_{\mathrm{I}\nu'}\left(\mathbf{q}'\right)\right] = \delta(\nu-\nu')\delta(\mathbf{q}-\mathbf{q}'), \\ \left[\bar{\mathbf{b}}_{\mathrm{II}\nu}\left(\mathbf{q}\right), \bar{\mathbf{b}}_{\mathrm{II}\nu'}\left(\mathbf{q}'\right)\right] &= \left[\mathbf{b}_{\mathrm{II}\nu}\left(\mathbf{q}\right), \mathbf{b}_{\mathrm{II}\nu'}\left(\mathbf{q}'\right)\right] = \delta(\nu-\nu')\delta(\mathbf{q}-\mathbf{q}'). \end{aligned} \tag{100}$$

We suppose that we have a second quantum field construction for all Minkowski spacetime (with eigenfunctions properly extended for all domains) once we choose as the one-particle Hilbert space $\mathcal{H}_{\rm II} \oplus \mathcal{H}_{\rm I}$. Now, take notice that [66]

$$\mathcal{F}(\mathcal{H}_{\rm II} \oplus \mathcal{H}_{\rm I}) \simeq \mathcal{F}(\mathcal{H}_{\rm II}) \otimes \mathcal{F}(\mathcal{H}_{\rm I}). \tag{101}$$

(**u7**) The Minkowski vacuum and the vacua for regions I, II are defined respectively by the states $|0\rangle_M \in \mathcal{F}(\mathcal{H})$, $|0\rangle_{\rm I} \in \mathcal{F}(\mathcal{H}_{\rm I})$, $|0\rangle_{\rm II} \in \mathcal{F}(\mathcal{H}_{\rm II})$ such that

$$\begin{gathered} \mathbf{a}\left(\mathbf{k}\right)|0\rangle_M = \bar{\mathbf{a}}\left(\mathbf{k}\right)|0\rangle_M = 0 \ \forall \mathbf{k}, \\ \mathbf{b}_{\mathrm{I}\nu}(\mathbf{q})|0\rangle_{\rm I} = \bar{\mathbf{b}}_{\mathrm{I}\nu}(\mathbf{q})|0\rangle_{\rm I} = 0, \text{and } \mathbf{b}_{\mathrm{II}\nu}\left(\mathbf{q}\right)|0\rangle_{\rm II} = \bar{\mathbf{b}}_{\mathrm{II}\nu}\left(\mathbf{q}\right)|0\rangle_{\rm II} = 0, \forall \mathbf{q},\nu. \end{gathered} \tag{102}$$

The respective particle number operators for modes $\mathbf{k}$, $\mathrm{I}\nu$ and $\mathrm{II}\nu$ are $N_{\mathbf{k}} = \mathbf{a}^{\dagger}\left(\mathbf{k}\right)\mathbf{a}\left(\mathbf{k}\right)$, $\bar{N}_{\mathbf{k}} = \mathbf{a}^{\dagger}\left(\mathbf{k}\right)\mathbf{a}\left(\mathbf{k}\right)$, $N_{\mathrm{I}\nu\mathbf{q}} = \mathbf{b}_{\mathrm{I}\nu}^{\dagger}\left(\mathbf{q}\right)\mathbf{b}_{\mathrm{I}\nu}\left(\mathbf{q}\right)$, $\bar{N}_{\mathrm{I}\nu\mathbf{q}} = \mathbf{b}_{\mathrm{I}\nu}^{\dagger}\left(\mathbf{q}\right)\mathbf{b}_{\mathrm{I}\nu}\left(\mathbf{q}\right)$ and $N_{\mathrm{II}\nu\mathbf{q}} = \mathbf{b}_{\mathrm{II}\nu}^{\dagger}\left(\mathbf{q}\right)\mathbf{b}_{\mathrm{II}\nu}\left(\mathbf{q}\right)$, $\bar{N}_{\mathrm{II}\nu\mathbf{q}} = \mathbf{b}_{\mathrm{II}\nu}^{\dagger}\left(\mathbf{q}\right)\mathbf{b}_{\mathrm{II}\nu}\left(\mathbf{q}\right)$. Of course,

$$\begin{aligned} &{}_M\langle 0|N_{\mathbf{k}}|0\rangle_M = 0, \ {}_{\rm I}\langle 0|N_{\mathrm{I}\nu\mathbf{q}}|0\rangle_{\rm I} = 0, \ {}_{\rm II}\langle 0|N_{\mathrm{II}\nu\mathbf{q}}|0\rangle_{\rm II} = 0, \\ &{}_M\langle 0|\bar{N}_{\mathbf{k}}|0\rangle_M = 0, \ {}_{\rm I}\langle 0|\bar{N}_{\mathrm{I}\nu\mathbf{q}}|0\rangle_{\rm I} = 0, \ {}_{\rm II}\langle 0|\bar{N}_{\mathrm{II}\nu\mathbf{q}}|0\rangle_{\rm II} = 0. \end{aligned} \tag{103}$$

(**u8**) In some presentations it is supposed that the quantum field in regions I + II obtained through the above quantization procedures can be described by

$$\hat{\phi}_{\rm I} + \hat{\phi}_{\rm II} \tag{104}$$

acting on $\mathcal{F}(\mathcal{H}_{\rm II} \oplus \mathcal{H}_{\rm I})$. However, here we suppose that the quantum field $\hat{\phi}'$ in regions I + II is described by an "entangled field" made from $\hat{\phi}_{\rm I}(x)$ and $\hat{\phi}_{\rm II}(x)$ acting on $\mathcal{F}(\mathcal{H}_{\rm II}) \otimes \mathcal{F}(\mathcal{H}_{\rm I})$, i.e., described by

$$\hat{\phi}' = \mathbf{1}_{\rm II} \otimes \hat{\phi}_{\rm I} + \hat{\phi}_{\rm II} \otimes \mathbf{1}_{\rm I} \tag{105}$$

acting (see Eq.(101)) on the Fock-Hilbert space $\mathcal{F}(\mathcal{H}_{11}) \otimes \mathcal{F}(\mathcal{H}_{1})$

Moreover, it is taken as obvious that (see e.g., [62]) that it is not necessary to analyze what happens in regions F and P.

6.2. "Deduction" of the Unruh Effect

(**u9**) As it is well known the delta functions in Eqs (98) and (100) leads to problems and so to continue the analysis it is usual to introduce in the Hilbert spaces[21] $\mathcal{H}, \mathcal{H}_{\mathrm{I}}$ and $\mathcal{H}_{\mathrm{II}}$ countable basis, which we denote in Fourier space by

$$f_{\mathbf{m},\mathbf{l},\varrho}(\mathbf{k}) = \varrho^{-\frac{3}{2}} \exp\left(-\frac{2\pi i \mathbf{k} \bullet \mathbf{l}}{\varrho}\right) \chi_{[(|\mathbf{m}|-1/2)\varrho,(|\mathbf{m}|+1/2)\varrho]}(\mathbf{k}), \tag{106}$$

where $\varrho \in \mathbb{R}^+$ (has inverse length dimension) and χ_S is the characteristic function of the set S[22]. The functions $f_{\mathbf{m},\mathbf{l},\varrho}(\mathbf{k})$ are localized in Fourier space around[23] $\mathbf{m} = (m_1, m_2, m_3)$ and have wave number vector $\mathbf{l} = (\ell_1, \ell_2, \ell_3)$, and thus in $\mathbb{R}^3$ they are localized around $\mathbf{l}$ with wave number vector $\mathbf{m}$. We immediately have that[24]

$$\begin{aligned} &\int d\mathbf{k} f^*_{\mathbf{m},\mathbf{l},\varrho}(\mathbf{k}) f_{\mathbf{m}',\mathbf{l}',\varrho}(\mathbf{k}) \\ &:= \frac{1}{\varrho^3}\delta_{\mathbf{m}\mathbf{m}'} \prod_i \int_{(m_i-1/2)\varrho}^{(m_i+1/2)\varrho} dk_i \exp\left(-\frac{2\pi i k_i(\ell_i - \ell'_i)}{\varrho}\right) = \delta_{\mathbf{m}\mathbf{m}'}\delta_{\ell\ell'}. \end{aligned} \tag{107}$$

and

$$\begin{aligned} \textstyle\sum_{\mathbf{l}\in\mathbb{Z}^3} f_{\mathbf{m},\mathbf{l},\varrho}(\mathbf{k}) f_{\mathbf{m},\mathbf{l},\varrho}(\mathbf{k}') &= \chi_{[(|\mathbf{m}|-1/2)\varrho,(|\mathbf{m}|+1/2)\varrho]}(\mathbf{k})\delta(\mathbf{k}-\mathbf{k}'), \\ \textstyle\sum_{\mathbf{l},\mathbf{m}\in\mathbb{Z}^3} f_{\mathbf{m},\mathbf{l},\varrho}(\mathbf{k}) f_{\mathbf{m},\mathbf{l},\varrho}(\mathbf{k}') &= \delta(\mathbf{k}-\mathbf{k}'). \end{aligned} \tag{108}$$

(**u10**) Now, in the Hilbert spaces $\mathcal{H}$, $\mathcal{H}_{\mathrm{I}}$ and $\mathcal{H}_{\mathrm{II}}$ we construct the *positive frequencies* solutions of the Klein-Gordon equation, i.e.,

$$\begin{aligned} \Phi_{M,\mathbf{m},\mathbf{l},\varrho}(x) &= \int d^3\mathbf{k} f_{\mathbf{m},\mathbf{l},\varrho}(\mathbf{k})\phi_{M\mathbf{k}}(x), \\ \Phi_{\mathrm{I},\mathbf{m},\mathbf{l},\varrho}(\mathfrak{l}) &= \int_0^\infty d\nu \int d^2\mathbf{q} f_{\mathbf{m},\mathbf{l},\varrho}(\mathbf{k})\phi_{\mathrm{I}\nu\mathbf{q}}(\mathfrak{l}), \\ \Phi_{\mathrm{II},\mathbf{m},\mathbf{l},\varrho}(\mathfrak{l}') &= \int_0^\infty d\nu \int d^2\mathbf{q} f_{\mathbf{m},\mathbf{l},\varrho}(\mathbf{k})\phi_{\mathrm{II}\nu\mathbf{q}}(\mathfrak{l}'). \end{aligned} \tag{109}$$

We have

$$\begin{aligned} \langle\Phi_{M,\mathbf{m},\mathbf{l},\varrho}, \Phi_{M,\mathbf{n},\mathbf{l}',\varrho}\rangle_M = \delta_{\mathbf{m}\mathbf{m}'}\delta_{\ell\ell'}, &\quad \langle\Phi^*_{M,\mathbf{m},\mathbf{l},\varrho}, \Phi^*_{M,\mathbf{n},\mathbf{l}',\varrho}\rangle_M = -\delta_{\mathbf{m}\mathbf{m}'}\delta_{\ell\ell'}, \\ \langle\Phi_{\mathrm{I},\mathbf{m},\mathbf{l},\varrho}, \Phi_{\mathrm{I},\mathbf{m}',\mathbf{l}',\varrho}\rangle_{\mathrm{I}} = \delta_{\mathbf{m}\mathbf{m}'}\delta_{\ell\ell'}, &\quad \langle\Phi^*_{\mathrm{I},\mathbf{m},\mathbf{l},\varrho}, \Phi^*_{\mathrm{I},\mathbf{m}',\mathbf{l}',\varrho}\rangle_{\mathrm{I}} = -\delta_{\mathbf{m}\mathbf{m}'}\delta_{\ell\ell'}, \\ \langle\Phi_{\mathrm{II},\mathbf{m},\mathbf{l},\varrho}, \Phi_{\mathrm{II},\mathbf{m}',\mathbf{l}',\varrho}\rangle_{\mathrm{II}} = \delta_{\mathbf{m}\mathbf{m}'}\delta_{\ell\ell'}, &\quad \langle\Phi^*_{\mathrm{II},\mathbf{m},\mathbf{l},\varrho}, \Phi^*_{\mathrm{II},\mathbf{m}',\mathbf{l}',\varrho}\rangle_{\mathrm{II}} = -\delta_{\mathbf{m}\mathbf{m}'}\delta_{\ell\ell'}, \\ \langle\Phi_{M,\mathbf{m},\mathbf{l},\varrho}, \Phi^*_{M,\mathbf{n},\mathbf{l}',\varrho}\rangle_M = 0, \quad \langle\Phi_{\mathrm{I},\mathbf{m},\mathbf{l},\varrho}, &\Phi^*_{\mathrm{I},\mathbf{m}',\mathbf{l}',\varrho}\rangle_{\mathrm{I}} = 0, \quad \langle\Phi_{\mathrm{II},\mathbf{m},\mathbf{l},\varrho}, \Phi^*_{\mathrm{II},\mathbf{m}',\mathbf{l}',\varrho}\rangle_{\mathrm{II}} = 0 \end{aligned} \tag{110}$$

---

[21] Note that $\mathcal{H}, \mathcal{H}_{\mathrm{I}}$ and $\mathcal{H}_{\mathrm{II}}$ are isomorphic to $\mathfrak{L}^2(\mathbb{R}^3)$.

[22] For each $\mathbf{m} = (m_1, m_2, m_3)$ it is $S = \{(x^1, x^2, x^3) \mid (m_i - 1/2)\varrho < x^i < (m_i + 1/2)\varrho, \; x^i \in \mathbb{R}, i = 1, 2, 3\}$.

[23] The $m_i, \ell_i \in \mathbb{Z}$, $i = 1, 2, 3$.

[24] Take notice that in the term $\exp\left(-\frac{2\pi i k_i(\ell_i - \ell'_i)}{\varrho}\right)$ in Eq.(107) $k_i\ell_i$ does not means that we are summing in the indice i.

and so

$$\phi_{M\mathbf{k}}(x) = \sum\nolimits_{\mathbf{l},\mathbf{m}\in\mathbb{Z}^3} f_{\mathbf{m},\mathbf{l},\varrho}(\mathbf{k})\Phi_{M,\mathbf{m},\mathbf{l},\varrho}(x),$$
$$\phi_{\mathrm{I}\nu\mathbf{q}}(\mathfrak{l}) = \sum\nolimits_{\mathbf{l},\mathbf{m}\in\mathbb{Z}^3} f_{\mathbf{m},\mathbf{l},\varrho}(\mathbf{k})\Phi_{\mathrm{I},\mathbf{m},\mathbf{l},\varrho}(\mathfrak{l}),$$
$$\phi_{\mathrm{II}\nu\mathbf{q}}(\mathfrak{l}') = \sum\nolimits_{\mathbf{l},\mathbf{m}\in\mathbb{Z}^3} f_{\mathbf{m},\mathbf{l},\varrho}(\mathbf{k})\Phi_{\mathrm{II},\mathbf{m},\mathbf{l},\varrho}(\mathfrak{l}'). \tag{111}$$

The field operators are then written as

$$\hat{\phi}_M(x) = \sum\nolimits_{\mathbf{l},\mathbf{m}\in\mathbb{Z}^3}\left[\mathbf{a}_{\mathbf{m},\mathbf{l},\varrho}\phi_{M,\mathbf{m},\mathbf{l},\varrho}(x) + \bar{\mathbf{a}}^\dagger\phi^*_{M,\mathbf{m},\mathbf{l},\varrho}(x)\right], \tag{112a}$$
$$\hat{\phi}_{\mathrm{I}}(\mathfrak{l}) = \sum\nolimits_{\mathbf{l},\mathbf{m}\in\mathbb{Z}^3}\left[\mathbf{b}_{\mathrm{I}\mathbf{m},\mathbf{l},\varrho}\phi_{\mathrm{I}_\nu\mathbf{q}}(\mathfrak{l}) + \bar{\mathbf{b}}^\dagger_{\mathrm{I}\mathbf{m},\mathbf{l},\varrho}\phi^*_{\mathrm{I}_\nu\mathbf{q}}(\mathfrak{l})\right], \tag{112b}$$
$$\hat{\phi}_{\mathrm{II}}(\mathfrak{l}') = \sum\nolimits_{\mathbf{l},\mathbf{m}\in\mathbb{Z}^3}\left[\mathbf{b}_{\mathrm{II}\nu}\phi_{\mathrm{II}_\nu\mathbf{q}}(\mathfrak{l}) + \bar{\mathbf{b}}^\dagger_{\mathrm{II}\nu}\phi^*_{\mathrm{I}_\nu\mathbf{q}}(\mathfrak{l})\right], \tag{112c}$$

with

$$\mathbf{a}_{\mathbf{m},\mathbf{l},\varrho} = \int d^3\mathbf{k} f^*_{\mathbf{m},\mathbf{l},\varrho}(\mathbf{k})\mathbf{a}(\mathbf{k}),$$
$$\mathbf{b}_{\mathrm{I}\mathbf{m},\mathbf{l},\varrho} = \int_0^\infty d\nu \int d^2\mathbf{q} f^*_{\mathbf{m},\mathbf{l},\varrho}(\mathbf{k})\mathbf{b}_{\mathrm{I}\nu}(\mathbf{q}),\quad \mathbf{b}_{\mathrm{II}\mathbf{m},\mathbf{l},\varrho} = \int_0^\infty d\nu \int d^2\mathbf{q} f^*_{\mathbf{m},\mathbf{l},\varrho}(\mathbf{k})\mathbf{b}_{\mathrm{II}\nu}(\mathbf{q}) \tag{113}$$

and analogus equations for the operators $\bar{\mathbf{a}}_{\mathbf{m},\mathbf{l},\varrho}, \bar{\mathbf{b}}_{\mathrm{I}\mathbf{m},\mathbf{l},\varrho}$ and $\bar{\mathbf{b}}_{\mathrm{II}\mathbf{m},\mathbf{l},\varrho}$. The non null commutators are

$$[\mathbf{a}_{\mathbf{m},\mathbf{l},\varrho}, \mathbf{a}^\dagger_{\mathbf{m}',\mathbf{l}'_\varrho}] = \delta_{\mathbf{m}\mathbf{m}'}\delta_{\mathbf{l}\mathbf{l}'}, [\mathbf{b}_{\mathrm{J}\mathbf{m},\mathbf{l},\varrho}, \mathbf{b}_{\mathrm{J}'\mathbf{m}',\mathbf{l}'_\varrho}] = \delta_{\mathrm{J}\mathrm{J}'}\delta_{\mathbf{m}\mathbf{m}'}\delta_{\mathbf{l}\mathbf{l}'},$$
$$[\mathbf{b}_{\mathrm{J}\mathbf{m},\mathbf{l},\varrho}, \mathbf{b}_{\mathrm{J}\mathbf{m}',\mathbf{l}'_\varrho}] = \delta_{\mathrm{J}\mathrm{J}'}\delta_{\mathbf{m}\mathbf{m}'}\delta_{\mathbf{l}\mathbf{l}'} \tag{114}$$

with $\mathrm{J} = \mathrm{I}, \mathrm{II}$ (and analogous equations involving the operators $\bar{\mathbf{a}}_{\mathbf{m},\mathbf{l},\varrho}, \bar{\mathbf{b}}_{\mathrm{I}\mathbf{m},\mathbf{l},\varrho}$ and $\bar{\mathbf{b}}_{\mathrm{II}\mathbf{m},\mathbf{l},\varrho}$). Of course,

$${}_M\langle 0|\mathbf{a}_{\mathbf{m},\mathbf{l},\varrho}\mathbf{a}^\dagger_{\mathbf{m}',\mathbf{l}'_\varrho}|0\rangle_M = 1,\ {}_{\mathrm{I}}\langle 0|\mathbf{b}_{\mathbf{m},\mathbf{l},\varrho}\mathbf{b}^\dagger_{\mathbf{m}',\mathbf{l}'_\varrho}|0\rangle_{\mathrm{I}} = 1,\ {}_{\mathrm{II}}\langle 0|\mathbf{b}_{\mathbf{m},\mathbf{l},\varrho}\mathbf{b}^\dagger_{\mathbf{m}',\mathbf{l}'_\varrho}|0\rangle_{\mathrm{II}} = 1 \tag{115}$$

and analogous equations involving the operators $\bar{\mathbf{a}}_{\mathbf{m},\mathbf{l},\varrho}, \bar{\mathbf{b}}_{\mathrm{I}\mathbf{m},\mathbf{l},\varrho}$ and $\bar{\mathbf{b}}_{\mathrm{II}\mathbf{m},\mathbf{l},\varrho}$.

(**u11**) The Fulling-Rindler vacuum $|0\rangle_F := |0\rangle_{\mathrm{II}} \otimes |0\rangle_{\mathrm{I}} \in \mathcal{F}(\mathcal{H}')$ is then defined by

$$\mathbf{1}_{\mathrm{II}} \otimes \mathbf{b}_{\mathrm{I}\mathbf{m},\mathbf{l},\varrho}|0\rangle_F = \mathbf{1}_{\mathrm{II}} \otimes \bar{\mathbf{b}}_{\mathrm{I}\mathbf{m},\mathbf{l},\varrho}|0\rangle_F = 0,\ \mathbf{b}_{\mathrm{II}\mathbf{m},\mathbf{l},\varrho} \otimes \mathbf{1}_{\mathrm{I}}|0\rangle_F = \bar{\mathbf{b}}_{\mathrm{II}\mathbf{m},\mathbf{l},\varrho} \otimes \mathbf{1}_{\mathrm{I}}|0\rangle_F = 0. \tag{116}$$

(**u12**) Let $\hat{\phi}_{M,\mathrm{I+II}}$ be the representation in $\mathcal{F}(\mathcal{H}_{\mathrm{II}})\otimes\mathcal{F}(\mathcal{H}_{\mathrm{I}})$ of the restriction of the field $\hat{\phi}_M$ given by Eq.(99a) to regions $\mathrm{I} + \mathrm{II}$. It is a well known fact [22] that the Minkowski quantization of the Klein-Gordon field and the Unruh quantization producing $\hat{\phi}'$ are *not* unitary equivalent[25].

Anyhow, it is supposed that we can identify

$$\mathcal{F}(\mathcal{H})|_{\mathcal{H}'} = \mathcal{F}(\mathcal{H}') = \mathcal{F}(\mathcal{H}_1) \otimes \mathcal{F}(\mathcal{H}_{11}) \tag{117}$$

[25]See Appendix B to know how this reult is obtained in the algebraic approach to quantum theory..

and writing

$$\hat{\phi}_{M,\mathrm{I+II}} = \mathbf{1}_{\mathrm{II}} \otimes \hat{\phi}_{M,\mathrm{I}} + \hat{\phi}_{M,\mathrm{II}} \otimes \mathbf{1}_{\mathrm{I}}$$

we thus put

$$\hat{\phi}_{M,\mathrm{I+II}} = \hat{\phi}'. \tag{118}$$

(**u13**) Under these conditions the relation between those representations is supposed to be given by the well known Bogolubov transformations which express the operators $\mathbf{b}, \mathbf{b}^\dagger$ as functions of the operators $\mathbf{a}, \mathbf{a}^\dagger$. We have $(\mathrm{J} = \mathrm{I}, \mathrm{II})$

$$\begin{aligned} \mathbf{b}_{\mathbf{Jm},\mathbf{l},_\varrho} &= \sum\nolimits_{\mathbf{l},\mathbf{m}\in\mathbb{Z}^3} \mathbf{a}_{\mathbf{m},\mathbf{l},_\varrho} \Xi_{\mathbf{Jm},\mathbf{l},\mathbf{m}',\mathbf{l}',\varrho} + \bar{\mathbf{a}}^\dagger_{\mathbf{m},\mathbf{l},_\varrho} \Upsilon_{\mathbf{Jm},\mathbf{l},\mathbf{m}',\mathbf{l}',\varrho}, \\ \bar{\mathbf{b}}_{\mathbf{Jm},\mathbf{l},_\varrho} &= \sum\nolimits_{\mathbf{l},\mathbf{m}\in\mathbb{Z}^3} \mathbf{a}\dagger_{\mathbf{m},\mathbf{l},_\varrho} \Upsilon_{\mathbf{Jm},\mathbf{l},\mathbf{m}',\mathbf{l}',\varrho} + \bar{\mathbf{a}}_{\mathbf{m},\mathbf{l},_\varrho} \Xi_{\mathbf{Jm},\mathbf{l},\mathbf{m}',\mathbf{l}',\varrho}. \end{aligned} \tag{119}$$

The explicit calculation of the operators $\mathbf{b}_{\mathbf{Jm},\mathbf{l},_\varrho}$ and $\bar{\mathbf{b}}_{\mathbf{Jm},\mathbf{l},_\varrho}$ is done by first evaluating $\Xi_{\mathbf{Jm},\mathbf{l},\mathbf{m}',\mathbf{l}',\varrho}$ and $\Upsilon_{\mathbf{Jm},\mathbf{l},\mathbf{m}',\mathbf{l}',\varrho}$. The well known result is [57]

$$\begin{aligned} \Xi_{\mathbf{Jm},\mathbf{l},\mathbf{m}',\mathbf{l}',\varrho} = & \int_0^\infty d\nu \int_{-\infty}^\infty dp_1 \int\int\int\int dk_1 dk_2 dp_2 dp_3 [f^*_{m_1,\ell_1,\varrho}(\nu) f_{m'_1,\ell'_1,\varrho}(p_1) \\ & \times f^*_{m_2,\ell_2,\varrho}(k_1) f^*_{m_3,\ell_3,\varrho}(k_2) f_{m_2,\ell_2,\varrho}(p_2) f_{m_3,\ell_3,\varrho}(p_3) \Xi_{\mathrm{J}\nu,\mathbf{pk}} \end{aligned} \tag{120}$$

(with analogous expression for $\Upsilon_{\mathbf{Jm},\mathbf{l},\mathbf{m}',\mathbf{l}',\varrho}$ where $\Xi_{\mathrm{J}\nu,\mathbf{pk}}$ is substituted by $\Upsilon_{\mathrm{I}\nu,\mathbf{pk}}$) with

$$\begin{aligned} \Xi_{\mathrm{I}\nu,\mathbf{pk}} &= \frac{1}{2\pi}\delta(p_1 - k_1)\delta(k_2 - p_2) e^{\frac{\pi\nu}{2}} \left|\Gamma(i\nu)\right| \left(\frac{\nu}{\omega_{\mathbf{k}}}\right)^{\frac{1}{2}} \left(\frac{\omega_{\mathbf{k}} + p_3}{\omega_{\mathbf{k}} - p_3}\right)^{\frac{i\nu}{2}}, \\ \Upsilon_{\mathrm{I}\nu,\mathbf{pk}} &= \frac{1}{2\pi}\delta(p_1 - k_1)\delta(k_1 - p_1) e^{-\frac{\pi\nu}{2}} \left|\Gamma(i\nu)\right| \left(\frac{\nu}{\omega_{\mathbf{k}}}\right)^{\frac{1}{2}} \left(\frac{\omega_{\mathbf{k}} + p_3}{\omega_{\mathbf{k}} - p_3}\right)^{\frac{i\nu}{2}} \end{aligned} \tag{121}$$

Next $\mathbf{b}_{\mathbf{Jm},\mathbf{l},_\varrho}$ and $\bar{\mathbf{b}}_{\mathbf{Jm},\mathbf{l},_\varrho}$ are approximated for the case where ϱ is very small and such that $\varrho m_3 \approx 1$ by the corresponding $\mathbf{b}_{\mathrm{J}\nu}(\mathbf{q})$. We have that

$$\nu \mapsto \nu_{m_3} : m_3\varrho, \quad \omega_{\mathbf{k}} \mapsto \omega_{\mathbf{m}'} := \sqrt{\varrho^2 \sum_i (m'_i)^2 + \mu^2} \tag{122}$$

and thus using this approximation we write

$$\begin{aligned} \Xi_{\mathbf{Im},\mathbf{l},\mathbf{m}',\mathbf{l}',\varrho} &= \frac{\varrho}{\sqrt{2\pi}} \Theta(m_3 + \frac{1}{2}) \delta_{m_1,m'_1} \delta_{\ell'_1,0} \delta_{m_2,m'_2} \delta_{m'_3,0} \delta_{\ell_2,\ell'_2} \delta_{\ell_3,\ell'_3} \\ & \times \frac{1}{\sqrt{\omega_{\mathbf{n}}}} \frac{1}{\sqrt{1 - e^{-2\pi\nu_{m_3}}}} \left(\frac{\omega_{\mathbf{m}'} + m'_3\varrho}{\omega_{\mathbf{m}'} - m'_3\varrho}\right)^{\frac{i\nu_{m_3}}{2}}, \\ \Upsilon_{\mathbf{Im},\mathbf{l},\mathbf{m}',\mathbf{l}',\varrho} &= \frac{\varrho}{\sqrt{2\pi}} \Theta(m_3 + \frac{1}{2}) \delta_{m_1,m'_1} \delta_{\ell'_1,0} \delta_{m_2,-m'_2} \delta_{m'_3,0} \delta_{\ell_2,-\ell'_2} \delta_{\ell_3,-\ell'_3} \\ & \times \frac{1}{\sqrt{\omega_{\mathbf{n}}}} \frac{1}{\sqrt{1 - e^{-2\pi\nu_{m_3}}}} \left(\frac{\omega_{\mathbf{m}'} + m'_3\varrho}{\omega_{\mathbf{m}'} - m'_3\varrho}\right)^{\frac{i\nu_{m_3}}{2}}. \end{aligned} \tag{123}$$

where the errors $\triangle\Xi_{\mathbf{Im,l,m',l'},\varrho}$ and $\triangle\Upsilon_{\mathbf{Im,l,m',l'},\varrho}$ are estimated to be of order ϱ.

Denoting by $|0,\mathrm{II},\mathrm{I}\rangle_M$ the restriction of the Minkowski vacuum state $|0\rangle_M$ to the region $\mathrm{II}+\mathrm{I}$ we have putting $\nu_{m_3}=\nu_j/a$ that, e.g., the expectation value of particles of type $\mathbf{b}^{\dagger}_{\mathbf{Im,l},\varrho}$ in the state $|0,\mathrm{II},\mathrm{I}\rangle_M$ is:

$$
\begin{aligned}
&{}_M\langle 0,\mathrm{II},\mathrm{I}|\mathbf{1}_{\mathrm{II}}\otimes\mathbf{b}^{\dagger}_{\mathbf{Im,l},\varrho}\mathbf{b}_{\mathbf{Im,l},\varrho}|0,\mathrm{II},\mathrm{I}\rangle_M \\
&=\frac{\varrho^2}{2\pi}\delta_{\ell_1 0}\ {}_M\langle 0,\mathrm{II},\mathrm{I}||0,\mathrm{II},\mathrm{I}\rangle_M\frac{1}{e^{2\pi\nu_j/a}-1}\sum\nolimits_{j\in\mathbb{Z}}\frac{1}{\omega_j}
\end{aligned}
\tag{124}
$$

Eq.(126) shows that even if we suppose that ${}_M\langle 0,\mathrm{I},\mathrm{II}||0,\mathrm{II},\mathrm{I}\rangle_M=\ {}_M\langle 0|0\rangle_M=1$, the vector $\mathbf{b}_{\mathbf{Im,l},\varrho}|0,\mathrm{II},\mathrm{I}\rangle_M\in\mathcal{F}(\mathcal{H}_1)\otimes\mathcal{F}(\mathcal{H}_{\mathrm{II}})$ has not a finite norm, thus showing that the procedure we have been using until now is not a mathematical legitimate one.

(**u14**) Nevertheless, taking the above approximation for the Bogolubov transformation as a good one for at least a region where $\varrho m_3\approx 1$, the state $|0,\mathrm{II},\mathrm{I}\rangle_M$ is written

$$
\begin{aligned}
&|0,\mathrm{II},\mathrm{I}\rangle_M \\
&=\Omega^{-1}\exp\left\{\sum\nolimits_{j,m_1}e^{-2\pi\nu_{m_1}}\left((\mathbf{b}^{+}_{\mathbf{IIm,l},\varrho})^{n_j}\otimes\mathbf{1}_{\mathrm{I}}+\mathbf{1}_{\mathrm{II}}\otimes(\mathbf{b}^{+}_{\mathbf{Im,l},\varrho})^{n_j}\right)\right\}|0\rangle_{\mathrm{II}}\otimes|0\rangle_{\mathrm{I}} \\
&=\Omega^{-1}\prod_j\sum\nolimits_n e^{-\pi n\nu_j/a}|\check{n}_j\rangle_{\mathrm{II}}\otimes|\check{n}_j\rangle_{\mathrm{I}},
\end{aligned}
\tag{125}
$$

where Ω is a normalization constant and $|\check{n}_j\rangle_{\mathrm{J}}=|\check{n}_j\rangle_{\mathrm{J}}+|0\rangle_{\mathrm{J}}$, $\mathrm{J}=\mathrm{I},\mathrm{II}$.

(**u15**) Using the fact that regions I and II are causally disconnected, i.e., observers following integral lines of $\boldsymbol{R}$, can only detect *right* Rindler particles it is supposed that these observers can only describe (according to standard quantum mechanics prescription) the state of the Minkowski quantum vacuum by a mixed state [66], i.e., a density matrix obtained by tracing over the states of the region II the pure state density matrix $\hat{\rho}=|0,\mathrm{I},\mathrm{II}\rangle_M\langle 0,\mathrm{I},\mathrm{II}|_M$. The result is

$$
\hat{\rho}_{\mathrm{I}}=\mathrm{tr}_{\mathrm{I}}(\hat{\rho})=\Omega^{-1}\prod_j\sum\nolimits_n e^{-2\pi n\nu_j/a}|n_j\rangle_{\mathrm{I}}\otimes\ {}_{\mathrm{I}}\langle n_j|,
\tag{126}
$$

which looks like a thermal spectrum with temperature parameter $a/2\pi$.

Remark 10 *Take notice that for an observer following the worldline σ with $\mathfrak{z}=$constant in region I the local temperature of the thermal radiation is* [62]

$$
T(\mathfrak{z})=\frac{1}{\sqrt{g_{00}(\mathfrak{z})}}\frac{a}{2\pi}
\tag{127}
$$

and thus $T(\mathfrak{z})\sqrt{g_{00}(\mathfrak{z})}$ is a constant. This is extremely important for otherwise thermodynamical equilibrium (*according to Tolman's version* [58]) *would not be possible in the $\boldsymbol{R}$ frame.*

(**u16**) Given Eq.(126) since $n\nu_j$ is the value of the *pseudo energy* in the $|n_j\rangle_{\rm I}$ state and since $\hat{\rho}_{\rm I}$ looks like a thermal density matrix $\rho_T = e^{-H/T}$ it is claimed that:

T*he Minkowski vacuum in region I is seem by observers living there as a thermal bath at temperature $a/2\pi$ of the so-called Rindler particles, which can excite well designed detectors.* [25, 26, 62, 63, 52, 56, 66] *Even more, it is claimed,(e.g., in* [56]) *that the Rindler particles are irradiated from the boundary of the region I* (*which is supposed to be "analogous" to the horizon of a blackhole which is supposed to radiate due to the so-called Hawking effect*).

(**u17**) The fact is that a rigorous mathematical analysis of the problem, based on the algebraic approach to field theory[26] (which for completeness, we recall in Appendix B), it is possible to show that the hypothesis given by Eq.(117) and thus Eq.(124) are *not* correct. Indeed, there we recall that strictly speaking the density matrix $\hat{\rho}$ and thus $\hat{\rho}_{\rm I}$ are meaningless. Also, many people has serious doubts if Fulling-Rindler vacuum $|0\rangle_F := |0\rangle_{\rm II} \otimes |0\rangle_{\rm I}$.can be physically realizable. These arguments are, in our opinion) stronger ones and the reader is invited to at least give a look in Appendix B (where the main references on original papers dealing with the issue of the algebraic approach to the Unruh effect may be found) in order to have an idea of the truth of what has just been stated.

(**u18**) As it is the case of the problem of the electromagnetic field generated by a charge in hyperbolic motion, there are several researchers that are convinced that the Unruh effect does *not* exist.

Besides the inconsistencies recalled in Appendix B several others are discussed, e.g., in [14, 20, 1, 10] The most important one in our opinion, has been realized in [20] where it is shown that both in the conventional approach as well as in the algebraic approach to quantum field theory it is impossible to perform the quantization of Unruh modes in Minkowski spacetime. Authors claim (and we agree with them) that Unruh quantization in a Rindler frame implies setting a *boundary condition* for the quantum field operator which changes the topological properties and symmetry group of the spacetime (where the Rindler reference frame has support) and leads to a field theory in the two disconnected regions I and II. They concluded that the Rindler effect does not exist.

(**u19**) Despite this fact, in a recent publication [12] authors that pertain to the majority view (i.e., those that believe in the existence of the thermal radiation) state:

> "Then, instead of waiting for experimentalists to perform the experiment, we use standard classical electrodynamics to anticipate its output and show that it reveals the presence of a thermal bath with temperature T_U in the accelerated frame. Unless one is willing to question the validity of classical electrodynamics, this must be seen as a virtual observation of the Unruh effect".

Well, authors of [12] also believe that a charge in hyperbolic motion radiates, and that the correct solution to the problem is the one given by the Liénard-Wiechert potential.

[26] First applied to the Unruh effect problem in [29].

But what will be of the statement that we cannot doubt classical electrodynamics if turns out that the Turakulov solution is the correct one (i.e., experimentally confirmed)?

Another important question is the following one: does a detector following an integral line of $\boldsymbol{R}$ get excited?

(**u20**) Several thoughtful analysis of the problem done from the point of view of an inertial reference frame shows that the detector get excited. This is discussed in [15] and a very simple model of a detector showing that the statement is correct may be found in [39]. But, of course, it is necessary to leave clear that this excitation energy can only come from the source that maintains the detector accelerated and it is not an excitation due to fluctuations of the zero point of the field as claimed, e.g. in [1].

7. Conclusions

There are some problems in Relativity Theory that are source of controversies since a long time. One of them has to do with the question if a charge in uniformly accelerated motion radiates. This problem is important, in particular, in its connection with one of the forms of the Equivalence Principle. In this paper we recalled that there are two different solutions for the electromagnetic field generated by a charge in hyperbolic motion, the Liénard-Wiechert (LW) one (obtained by the retarded Green function) and the less known one discovered by Turakulov in 1994 (and which we have verified to be correct, in particular using the software Mathematica). According to the LW solution the charge radiates and claims that an observer comoving with the charge does not detect any radiation is shown to be wrong. This is done by analyzing the different concepts of energy used by people that claims that no radiation is detected. Turakulov claims in [59] that his solution implies that there are no radiation. However, we have proved that he is also wrong, the reason being essentially the same as in the case of the Liénard-Wiechert solution. On the other hand we recalled that a charge at rest in Schwarzschild spacetime does not radiate. Thus, if the LW or the Turakulov solution is the correct one, then experiment with charges may show that the Equivalence Principle is false.

Another problem which we investigate is the so-called Bell's "paradox". We discussed it in details since it is, as yet, a source of misunderstandings.

Finally, we briefly recall how the so-called Unruh effect is obtained in almost all texts using some well ideas of quantum field theory. We comment that this standard approach seems to imply that an observer in hyperbolic motion is immersed in a thermal bath with temperature proportional to its proper acceleration. Acceptance that this is indeed the case is almost the majority view among physicists. However, fact is that the standard approach does not resist a rigorous mathematical analysis, in particular when one use the algebraic approach to quantum field theory. Thus as it is the case with the problem of determining the electromagnetic field of a charge in hyperbolic motion there are dissidents of the majority view. Having studied the arguments of several papers we presently agree with [10, 20] that there is no Unruh effect. However, it is not hard to show that a detector in hyperbolic motion on the Minkowski vacuum gets excited, but the energy producing such excitation, contrary to some claims (as, e.g., in [1]) does not come from the fluctuations of the zero point field, but comes from the source pushing the charge.

A. Some Notations and Definitions

(**a1**) Let M be a four dimensional, real, connected, paracompact and non-compact manifold. We recall that a *Lorentzian manifold* as a pair $(M, \mathbf{g})$, where $\mathbf{g} \in \sec T_2^0 M$ is a Lorentzian metric of signature $(1,3)$, i.e., $\forall \mathfrak{e} \in M, T_x M \simeq T_{\mathfrak{e}}^* M \simeq \mathbb{R}^4$. Moreover, $\forall x \in M, (T_x M, \mathbf{g}_x) \simeq \mathbb{R}^{1,3}$, where $\mathbb{R}^{1,3}$ is the Minkowski *vector* space We define a *Lorentzian spacetime* M as pentuple $(M, \mathbf{g}, \boldsymbol{D}, \tau_{\mathbf{g}}, \uparrow)$, where $(M, \mathbf{g}, \tau_{\mathbf{g}}, \uparrow))$ is an *oriented* Lorentzian manifold (oriented by $\tau_{\boldsymbol{g}}$) and *time* oriented[27] by $\uparrow$, and D is the Levi-Civita connection of $\boldsymbol{g}$. Let $U \subseteq M$ be an open set covered, say, by coordinates (y^0, y^1, y^2, y^3). . Let $U \subseteq M$ be an open set covered by coordinates $\{x^\mu\}$. Let $\{e_\mu = \partial_\mu\}$ be a coordinate basis of $T\mathcal{U}$ and $\{\boldsymbol{\vartheta}^\mu = dx^\mu\}$ the dual basis on $T^*\mathcal{U}$, i.e., $\boldsymbol{\vartheta}^\mu(\partial_\nu) = \delta_\nu^\mu$. If $\mathbf{g} = g_{\mu\nu}\boldsymbol{\vartheta}^\mu \otimes \boldsymbol{\vartheta}^\nu$ is the metric on $T\mathcal{U}$ we denote by $\mathtt{g} = g^{\mu\nu}\boldsymbol{\partial}_\mu \otimes \boldsymbol{\partial}_\nu$ the metric of $T^*\mathcal{U}$, such that $g^{\mu\rho}g_{\rho\nu} = \delta_\nu^\mu$. We introduce also $\{\boldsymbol{\partial}^\mu\}$ and $\{\boldsymbol{\vartheta}_\mu\}$, respectively, as the reciprocal bases of $\{e_\mu\}$ and $\{\boldsymbol{\vartheta}_\mu\}$, i.e., we have

$$\mathbf{g}(\boldsymbol{\partial}_\nu, \boldsymbol{\partial}^\mu) = \delta_\nu^\mu, \quad g(\boldsymbol{\vartheta}^\mu, \boldsymbol{\vartheta}_\nu) = \delta_\nu^\mu. \tag{128}$$

(**a2**) Call $(M \simeq \mathbb{R}^4, \boldsymbol{g}, D, \tau_{\boldsymbol{g}}, \uparrow)$ the *Minkowski spacetime structure*. When $M \simeq \mathbb{R}^4$ there is (infinitely) global charts. Call (x^0, x^1, x^2, x^3) the coordinates of one of those charts. These coordinates are said to be in *Einstein-Lorentz-Poincaré* (ELP) gauge. In these coordinates

$$\boldsymbol{g} = \eta_{\mu\nu} dx^\mu \otimes dx^\nu \text{and } \mathtt{g} = \eta^{\mu\nu} \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu} \tag{129}$$

where the matrix with entries $\eta_{\mu\nu}$ and also the matrix with entries $\eta^{\mu\nu}$ are diagonal matrices $\mathrm{diag}(1, -1, -1, -1)$.

(**a3**) In a general Lorentzian structure if $\mathbf{Q} \in \sec TU \subset \sec TM$ is a time-like vector field such that $\boldsymbol{g}(\mathbf{Q}, \mathbf{Q}) = 1$, then there exist, in a coordinate neighborhood U, three space-like vector fields $\boldsymbol{e}_\mathbf{i}$ which together with $\mathbf{Q}$ form an orthogonal moving frame for $x \in U$ [13, 48].

(**a4**) A *moving frame* at $x \in M$ is a basis for the tangent space $T_x M$. An orthonormal (moving) frame at $x \in M$ is a basis of orthonormal vectors for $T_x M$.

(**a5**) An *observer* in a general Lorentzian spacetime is a future pointing time-like curve $\sigma : \mathbb{R} \supset I \to M$ such that $\boldsymbol{g}(\sigma_*, \sigma_*) = 1$. The timelike curve σ is said to be the worldline of the observer.

(**a6**) An *instantaneous observer* is an element of TM, i.e., a pair $(x, \mathcal{Q})$, where $x \in M$, and $\mathcal{Q} \in T_x M$ is a future pointing unit timelike vector. $\mathrm{Span}\mathcal{Q} \subset T_x M$ is the *local time axis* of the observer and $\mathcal{Q}^\perp$ is the *observer rest space.*

(**a7**) Of course, $T_x M = \mathrm{Span}\mathcal{Q} \oplus \mathcal{Q}^\perp$, and we denote in what follows $\mathrm{Span}\mathcal{Q} = T$ and $\mathcal{Q}^\perp = H$, which is called the *rest space* of the instantaneous observer. If $\sigma : \mathbb{R} \supset I \to M$

[27] Please, consult, e.g., [48].

is an observer, then $(\sigma u, \sigma_* u)$ is said to be the local observer at u and write $T_{\sigma u}M = T_u \oplus H_u$, $u \in I$.

(**a8**) The *orthogonal projections* are the mappings

$$\mathbf{p}_u = T_{\sigma u}M \to H_u \ , \ \mathbf{q}_u : T_{\sigma u}M \to T_u. \tag{130}$$

Then if $\mathbf{Y}$ is a vector field over σ then $\mathbf{pY}$ and $\mathbf{qY}$ are vector fields over σ given by

$$(\mathbf{pY})_u = \mathbf{p}_u(\mathbf{Y}_u), \ (\mathbf{qY})_u = \mathbf{q}_u(\mathbf{Y}_u). \tag{131}$$

(**a9**) Let $(x, \mathcal{Q})$ be a instantaneous observer and $\mathbf{p}_x : T_xM \to H$ the orthogonal projection. The *projection tensor* is the symmetric bilinear mapping $\mathbf{h} : \sec(\ TM \times TM) \to \mathbb{R}$ such that for any $\mathbf{U}, \mathbf{W} \in T_xM$ we have:

$$\mathbf{h}_x(\mathbf{U}, \mathbf{W}) = \boldsymbol{g}_x(\mathbf{pU}, \mathbf{pW}) \tag{132}$$

Let $\{x^\mu\}$ be coordinates of a chart covering $U \subset M$, $x \in U$ and $\alpha_{\mathcal{Q}} = \boldsymbol{g}_x(\mathcal{Q}, \ )$. We have the properties:

(a)	$\mathbf{h}_X = \boldsymbol{g}_X - \alpha_{\mathcal{Q}} \otimes \alpha_{\mathcal{Q}}$	
(b)	$\mathbf{h}\vert_{\mathcal{Q}^\perp} = \boldsymbol{g}_x\vert_{\mathcal{Q}^\perp}$	
(c)	$\mathbf{h}(\mathcal{Q},) = 0$	
(d)	$\mathbf{h}(\mathbf{U},) = \boldsymbol{g}(\mathbf{U},) \Leftrightarrow \ \boldsymbol{g}(\mathbf{U}, \mathcal{Q}) = 0$	(133)
(e)	$\mathbf{p} = h^\mu_\nu \frac{\partial}{\partial x^\mu}\Big\vert_x \otimes dx^\nu\vert_x$	
(f)	$\mathrm{trace}(h^\mu_\nu \frac{\partial}{\partial x^\mu}\Big\vert_x \otimes dx^\nu\vert_x) = -3$	

The result quote in (**a3**) together with the above definitions suggest to introduce the following notions:

(**a10**) A *reference frame* for $U \subseteq M$ in a spacetime structure $(M, \boldsymbol{g}, D, \tau_{\boldsymbol{g}}, \uparrow)$ is a time-like vector field which is a section of TU such that each one of its integral lines is an observer.

(**a11**) Let $\mathbf{Q} \in \sec TM$, be a reference frame. A chart in $U \subseteq M$ of an oriented atlas of M with coordinate functions $(\boldsymbol{y}^\mu)$ and coordinates $(\boldsymbol{y}^0(\mathfrak{e}) = y^0, \boldsymbol{y}^1(\mathfrak{e}) = y^1, \boldsymbol{y}^2(\mathfrak{e}) = y^2, \boldsymbol{y}^3(\mathfrak{e}) = y^3)$ such that $\partial/\partial y^0 \in \sec TU$ is a timelike vector field and the $\partial/\partial y^i \in \sec TU$ $(i = 1, 2, 3)$ are spacelike vector fields is said to be a possible naturally adapted coordinate chart to the frame $\mathbf{Q}$ *(*denoted *(nacs-*$\mathbf{Q}$*)* in what follows*)* if the space-like components of $\mathbf{Q}$ are null in the natural coordinate basis $\{\partial/\partial x^\mu\}$ of TU associated with the chart. We also say that (y^0, y^1, y^2, y^3) are naturally adapted coordinates to the frame $\mathbf{Q}$.

Remark 11 *It is crucial, in order to avoid misunderstandings, to have in mind that most of the reference frames used in the formulation of physical theories are theoretical objects, i.e., a reference frame does not need to have material support in the region were it has mathematical support.*

(**a12**) References frames in Lorentzian spacetimes can be *classified* according to the decomposition of $D\mathbf{Q}$ and according to their *synchronizability.* Details may be found in [48]. We analyze in detail the nature of the right Rindler reference frame in Section 2. Here we only recall that $\mathbf{Q}$ is locally synchronizable if it rotation tensor ω (coming form the decomposition of $Q = \boldsymbol{g}(\mathbf{Q},\ )$ and we can show $\omega = 0 \Longleftrightarrow Q \wedge dQ = 0$. Also, $\mathbf{Q}$ is synchronizable if besides being irrotational also there exists a function H on U and a timelike coordinate, say u (part of a naturally adapted coordinate system to $\mathbf{Q}$) such that $Q = Hdu$. Finally, $\mathbf{Q}$ is said to be propertime synchronizable if $Q = du$.

(**a13**) We also used in the main text the following conventions:

$$\begin{gathered}\boldsymbol{g}(\boldsymbol{A},\boldsymbol{B}) = \boldsymbol{A}\cdot\boldsymbol{B}, \quad \mathrm{g}(C,D) = C \cdot D, \\ \boldsymbol{A},\boldsymbol{B} \in \sec TM, \quad C,D \in \sec {\textstyle\bigwedge^1} T^*M. \end{gathered} \tag{134}$$

and the scalar product of Euclidean vector fields is denoted by $\bullet$.

(**a14**) Moreover, d and δ denotes the differential and Hodge codifferential operators acting on sections of $\bigwedge T^*M$ and $\lrcorner$ denotes the left contraction operator of form fields [48].

B. $C^\star$ Algebras and the Unruh "Effect"

The reason for including this Appendix in this paper is for the interested reader to have an idea of how much he can *trust* the standard approach recalled in the main text which result in the claim that Rindler observers live in a thermal bath. The algebraic approach to quantum field theory is based on $C^\star$-algebras[28] which are now briefly recalled.

(**b1**) Let then be $\mathcal{A}$ a C^*-algebra over $\mathbb{C}$ whose some of its elements may be associated to the observables[29] (associated to the quantum field $\hat{\phi}$). We recall that a *representation* of a C^*-algebra is a linear mapping

$$f : \mathcal{A} \to \mathcal{B}(\mathfrak{H}), \quad A \mapsto f(A), \quad f(A^\star) = f(A)^\dagger. \tag{135}$$

where $\mathcal{B}(\mathfrak{H})$ is an algebra of bounded linear operators on a Hilbert space $\mathfrak{H}$. The observables are associated with elements $A = A^*$, where * denotes the *involution* operation in $\mathcal{A}$, i.e., $\mathcal{A}\mathcal{A}^\star = 1$ and † denotes the Hermitian conjugate in $\mathcal{B}(\mathfrak{H})$

(**b2**) A representation $(f,\mathfrak{H})$ of $\mathcal{A}$ is said *faithful* if $f(A) = 0$ if $A = 0$ and $(f,\mathcal{H})$ is irreducible if the only closed subspaces of $\mathfrak{H}$ invariant under f are $\{0\}$ and $\mathfrak{H}$.

(**b3**) Let $\mathfrak{L} \subset \mathfrak{H}$ be a non zero closed subspace of invariant under f. Let $\hat{\mathbf{P}}_{\mathcal{L}}$ be the *orthogonal projection* operator on $\mathcal{L}$. A *subrepresentation* of $f_{\mathcal{L}}$ is the mapping

$$f_{\mathcal{L}} : \mathcal{A} \to \mathcal{B}(\mathfrak{H}), \quad A \mapsto f(A)\hat{\mathbf{P}}_{\mathcal{L}}. \tag{136}$$

[28] For a susccint presentation of C^*-algebras, enough for the understanding of the following see, e.g., [17]. There the reader will find there the main references on the algebraic (and axiomatic) approach to quantum field theory. Also, the reader who wants to know all the details concerning the algebraic approach to the Unruh effect must study the texts quoted below which has been heavily used in the writing of this Appendix B.

[29] I.e., the self-adjoints elements of $\mathcal{A}$

(**b3**) Two representations, say $(f_1, \mathfrak{H}_1)$ and $(f_2, \mathfrak{H}_2)$ of $\mathcal{A}$ are said to be unitarily equivalent is there exists an isomorphism $\mathbf{U} : \mathfrak{H}_1 \to \mathfrak{H}_2$, such that

$$\mathbf{U} f_1(\mathcal{A}) \mathbf{U}^{-1} = f_2(\mathcal{A}). \tag{137}$$

(**b4**) A *state* on $\mathcal{A}$ is a mapping

$$\begin{aligned} &\omega : \mathcal{A} \to \mathbb{R}, \\ &\omega(1) = 1, \quad \omega(A^\star A) \geq 0, \forall A \in \mathcal{A}. \end{aligned} \tag{138}$$

(**b5**) A *pure state* ω on $\mathcal{A}$ is one that cannot be written as a non-trivial convex linear combination other states. On the other hand a *state* ω on $\mathcal{A}$ is said to be *mixed* if it can be written as a non-trivial convex linear combination other states.

(**b6**) It is important to recall that a result (theorem) due to Gel'fand, Naimark and Segal (GNS) [23, 51] establishes that for any ω on $\mathcal{A}$ there always exists a representation $(f_\omega, \mathfrak{H}_\omega)$ of $\mathcal{A}$ and $\Phi_\omega \in \mathfrak{H}_\omega$ (usually called a *cyclic vector*) such that $f_\omega(\mathcal{A})\Phi_\omega$ is dense in $\mathfrak{H}_\omega$ and

$$\omega(A) = \langle \Phi_\omega | f_\omega(A) | \Phi_\omega \rangle. \tag{139}$$

Moreover the GNS result warrants that up to unitary equivalence, $(f_\omega, \mathfrak{H}_\omega)$ is the unique *cyclic* representation of $\mathcal{A}$.

(**b7**) The *folium* $\mathfrak{F}(\omega)$ of ω on $\mathcal{A}$ is the set of all abstract states that can be expressed as density matrices on the Hilbert space of the GNS representation determined by $\mathfrak{H}_\omega$.

(**b8**) Given states ω_1, ω_2 on $\mathcal{A}$ they are said *quasi-equivalent* if and only if $\mathfrak{F}(\omega_1) = \mathfrak{F}(\omega_2)$. The states ω_1, ω_2 on $\mathcal{A}$ are said to be *disjoint* if $\mathfrak{F}(\omega_1) \cap \mathfrak{F}(\omega_2) = \varnothing$.

(**b9**) It is possible to show that:

(i) *Any irreducible representation have no proper subrepresentations and in this case if ω_1 and ω_2 are pure states, quasi-equivalence reduces to unitary equivalence and disjointness reduces to non-unitary equivalences;*

(ii) *When ω_1 and ω_2 are mixed states they in general are not quasi equivalent or disjoint.*

This happens when, e.g., ω_1 has disjoint representations and one of then is unitarily equivalent to ω_2.

(**b10**) For our considerations it is important to recall the following result [9]:

The states ω_1 and ω_2 are disjoint if and only if the GNS representation of $f_{\omega_1+\omega_2}$ determined by $\omega_1 + \omega_2$ satisfies

$$(f_{\omega_1+\omega_2}, \mathfrak{H}_{\omega_1+\omega_2}) = (f_{\omega_1} \oplus f_{\omega_2}, \mathfrak{H}_{\omega_1} \oplus \mathfrak{H}_{\omega_2}), \tag{140}$$

i.e., the direct sum of the representations f_{ω_1} and f_{ω_2}. Elements of $\mathfrak{H}_{\omega_1+\omega_2}$ are *denoted by*

$$|\Phi_{\omega_1+\omega_2}\rangle = |\Phi_{\omega_1}\rangle \oplus |\Phi_{\omega_2}\rangle \tag{141}$$

(**b11**) To continue the presentation it is necessary to use a particular C^*-algebra, namely the Weyl algebra[30] $\mathcal{A}_W(M)$ which encodes (see, e.g., [11]), in particular an

[30]Also called Symplectic Clifford Algebra [16, 67].

exponential version of the canonical commutation relations for the Klein-Gordon field used in the analysis of the Unruh effect in this paper. Use of the Weyl algebras is opportune because in a version appearing in [31] it leads to a net of algebras $\{\mathcal{A}(U)\}$ where if $U \subset M$ is an open set of compact closure which qualifies as a globally hyperbolic spacetime structure $(U, \boldsymbol{g}|_U, D|_U, \tau_{\boldsymbol{g}}|_U, \uparrow)$ then if $U \subset U' \subset M$ it is $\mathcal{A}(U) \subset \mathcal{A}(U')$.

(**b12**) It is also necessary to know the following result [7, 8, 9]:

Let $\boldsymbol{Z} \in \sec TU$ *where* U *qualifies as a globally hyperbolic spacetime which is foliated with Cauchy surfaces*[31] $\Sigma(u)$. *Let* $n \in \sec TM$ *be the unit normal to* Σ, *a member of the foliation. Only if for some* $\varepsilon \in R$, Z *satisfies*

$$\boldsymbol{Z} \cdot \boldsymbol{Z} \geq \varepsilon \boldsymbol{Z} \cdot \boldsymbol{n} \geq \varepsilon^2 \tag{142}$$

there exists a procedure that associates with Σ *a so-called quasi-free state* ω_Σ *on* $A(M)$.

(**b13**) *Quasi-free states* are the ones for which the n-point functions of quantum field theory are determined by the two point functions and their importance here lies in the fact that it can be shown that the GNS representation of ω_Σ has a natural Fock-Hilbert space structure $\mathcal{F}(\Sigma)$ where ω_Σ is represented by the vacuum state $|0\rangle_\Sigma \in \mathcal{F}(\Sigma)$. Thus, ω_Σ qualifies as a candidate for the vacuum state.

Remark 12 *Note that if we take* $\boldsymbol{Z}$ *equal to* $\boldsymbol{I}$ *since it is irrotational (and a Killing vector field), it can be used to foliate* M *and for* $\boldsymbol{I}$ Eq.(142) *is satisfied. Then we naturally can construct* ω_M *on* $\mathcal{A}$ *representing the state* $|0\rangle_M \in \mathcal{F}(\mathcal{H})$. *Also, if we take* $\boldsymbol{Z} = \boldsymbol{Z}_{\mathrm{I}}$ or $\boldsymbol{Z} = \boldsymbol{Z}_{\mathrm{II}}$ *(as defined in* Eqs.(90)*) since these fields besides being Killing vector fields are also irrotational, they can be used to foliate regions I and II where the respective Cauchy surfaces are of course, spacelike surfaces orthogonal respectively to* $\boldsymbol{Z}_{\mathrm{I}}$ *and* $\boldsymbol{Z}_{\mathrm{Ii}}$. *In these cases,* Eq.(142) *is violated near the "horizon".and it is not possible to construct*[32] ω_{I} *on* $\mathcal{A}(\mathrm{I})$ *and* ω_{II} *on* $\mathcal{A}(\mathrm{II})$.*These states are the ones associate with the vacuum states* $|0\rangle_{\mathrm{I}}$ *and* $|0\rangle_{\mathrm{II}}$ *described above.*

(**b14)** We have now the fundamental result:

The states $\omega_M|_{\mathcal{A}(\mathrm{I})}$ *(respectively* $\omega_M|_{\mathcal{A}(\mathrm{II})}$*) and* ω_{I} *(respectively* ω_{II}*) are disjoint.*

(b15) To understand what is the meaning of this statement it is necessary to recall the definition of a *von Neumann algebra* [65].(denoted W^*-algebra). It is a special type of a C^*-algebra of bounded operators on a Hilbert space that is closed in the weak operator topology and contains the identity operator.

(**b16**) What is important for us here is that if $\mathcal{A}$ is a C^*-algebra identified with the space of bound operators $\mathfrak{B}(\mathcal{H})$ of an appropriate Hilbert space then $\mathcal{A}$ is a W^*-algebra if and only if

$$\mathcal{A} = \mathcal{A}'', \tag{143}$$

where $\mathcal{A}'$ denotes the so called *commutant* of $\mathcal{A}$, i.e., the set of operators that commute with all elements of $\mathcal{A}$. Of course, $\mathcal{A}''$ denotes the commutant of the commutant and is called bicommutant.

[31] u is a parameter indexing the foliation.

[32] The states ω_{I} on $\mathcal{A}(\mathrm{I})$ and ω_{II} on $\mathcal{A}(\mathrm{II})$ are called Boulware vacuum states[5].

(**b17**) Given a representation $(f, \mathfrak{H})$ of $\mathcal{A}$ we denote $f''(\mathcal{A})$ the so-called *double commutant* of $f(\mathcal{A})$. It is called the von Neumann algebra and denoted $W_f(\mathcal{A})$. If the commutant $f'(\mathcal{A})$ is an Abelian algebra $W_f(\mathcal{A})$ is called type **I** and it is the case given von Neumann theorem that if ω is an state on $\mathcal{A}$ then $W_f(\mathcal{A})$ can be identified with $\mathfrak{B}(\mathcal{H}_\omega)$ for a GNS representation $(f_\omega, \mathcal{H}_\omega)$.

(**b18**) A *factorial state* ω on $\mathcal{A}$ (and their GNS representation $\Phi_\omega \in \mathcal{H}_\omega$) is one for which the only multiples of the identity are elements of $W_{f\omega}(\mathcal{A}) \cap W_{f\omega}(\mathcal{A})'$.

(**b19**) A *normal state* ω on $\mathcal{A}$ (and their GNS representation $\Phi_\omega \in \mathcal{H}_\omega$) is one whose canonical extension to a state $\breve{\omega} \in W_{f\omega}(\mathcal{A})$ is countably additive.

(**b20**) Von Neumann algebras can also be of types [4] **II** and **III**. Type **III** are important for the sequel and it is one where factors are factors that do not contain any nonzero finite projections at all.

(**b21**) Given these definitions it is possible to show the following results concerning C^*-algebras:

(**b21a**) If f and f' are non degenerate representations of $\mathcal{A}$, then they are quasi-equivalent if and only if there is a *-isomorphism

$$\begin{aligned} \boldsymbol{i} &: W_f(\mathcal{A}) \to W_{f'}(\mathcal{A}), \\ \boldsymbol{i}(f(A)) &= f'(A) \end{aligned} \tag{144}$$

(**b21b**) The representations f and f' are quasi equivalent if an only if f has no subrepresentation disjoint from f' and vice-versa.

(**b21c**) A representation of a $\mathcal{A}$ is factorial if and only if every subrepresentation of f is quasi equivalent to f'.

From (**b21a**) it follows (see, e.g., [11]) that $f_{\omega_{\mathrm{I}}}$ (respectively $f_{\omega_{\mathrm{II}}}$) and $f_{\omega_M|_{\mathcal{A}(\mathrm{I})}}$ (respectively $f_{\omega_M|_{\mathcal{A}(\mathrm{II})}}$) are not isomorphic since $W_{f_{\omega\mathrm{I}}}(\mathcal{A})$ (respectively $W_{f_{\omega\mathrm{I}}}(\mathcal{A})$) is a von Neumann algebra of type **I** whereas $W_{f_{\omega_M|_{\mathcal{A}(\mathrm{I})}}}(\mathcal{A})$ (respectively $W_{f_{\omega_M|_{\mathcal{A}(\mathrm{II})}}}(\mathcal{A})$) is a von Neumann algebra of type **III** [2].

(**b22**) It is the case that in general not to be quasi equivalent does not implies being disjoint., but in our particular case ω_{I} (respectively ω_{II}) is a pure state which is irreducible and as such has no no trivial representation. Also, $\omega_M|_{\mathcal{A}(\mathrm{I})}$ (respectively $\omega_M|_{\mathcal{A}(\mathrm{II})}$) is factorial and (c) implies that it is equivalent to each one of its subrepresentation. Finally, from (**a**) it follows that $f_{\omega_{\mathrm{I}}}$ (respectively $f_{\omega_{\mathrm{II}}}$) and $f_{\omega_M|_{\mathcal{A}(\mathrm{I})}}$ (respectively $f_{\omega_M|_{\mathcal{A}(\mathrm{II})}}$) is disjoint if and only if they are not quasi equivalent.

Now, what does it means that $f_{\omega_{\mathrm{I}}}$ (respectively $f_{\omega_{\mathrm{II}}}$) and $f_{\omega_M|_{\mathcal{A}(\mathrm{I})}}$ (respectively $f_{\omega_M|_{\mathcal{A}(\mathrm{II})}}$) is disjoint?

(**b23**) Recall, e.g., that what ω_M has to say about region I is given by $\omega_M|_{\mathcal{A}(\mathrm{I})}$ and from what we already recalled above cannot be represented by a density matrix in the representation $f_{\omega_{\mathrm{I}}}$, in particular for any representation on $\mathcal{A}(\mathrm{I})$.This happens because it is impossible to write $\mathcal{A}(M)$ as a tensor product $\mathcal{A}' \otimes \mathcal{A}(\mathrm{I})$ for some $\mathcal{A}'$. This result is called *expressive incompleteness.*

(**b24**) Despite expressive incompleteness we have the following result by Verch [64]:

On $U \subset I \subset M$ (which is open and of compact closure) *let* $f_{\omega_M} | \mathcal{A}(U)$ be the GNS representation constructed from ω_M restrict to the image $\omega_M|\mathcal{A}(U)$ under f_{ω_M} of $\mathcal{A}(U)$ (and completing in the natural topology of $\mathcal{H}_{\omega_M}$) and analogous construct[33] $\omega_{\mathrm{I}}|\mathcal{A}(U)$ the image of ω_{I} under $f_{\omega_{\mathrm{I}}}|\mathcal{A}(U)$[34]*. Then,*$f_{\omega_M} | \mathcal{A}(U)$ and $f_{\omega_{\mathrm{I}}}|\mathcal{A}(U)$ *are quasi equivalent.*

(**b25**) The result presented in (**b24**) is the only one that would permit legitimately to physicists to talk about ω_M and ω_{I} as being quasi equivalents, for indeed as already recalled f_{ω_M} and $f_{\omega_{\mathrm{I}}}$ are indeed disjoint representations of the algebra of observables $\mathcal{A}$ and thus not unitarily equivalents.

(**b26**) Anyway, the above result implies that only if we do measurements on observables of the algebra $\mathcal{A}$ in regions of non compact closure can distinguish the representations f_{ω_M} and $f_{\omega_{\mathrm{I}}}$.

(**b27**) Finally one can ask the question: is $f_{\omega_M|_{\mathcal{A}(U)}}$ and $f_{\omega_M|_{\mathcal{A}(U)}}$ where again $U \subset I \subset M$ (open and of compact closure) quasi equivalent?

The answer to this question is (for the best of our knowledge) *not* known and this is another hindrance that makes one to affirm that no convincing theoretical proof that the Unruh effect is a real effect exists.

(**b28**) In the standard "deduction" (Section 6.1) of the Unruh effect it is claimed that the uniformly accelerated observer detects a thermal bath. Supports that the effect is a real one try to endorse their claim by using the notion of KMS states[35] (which as well known generalizes the notion of equilibrium state) [30, 35, 7, 8, 9]. In fact, Sewell [54] argues that the restriction of the Minkowski vacuum ω_M to region I, i.e., $\omega_M|_{\mathcal{A}(\mathrm{I})}$ $(=\omega_M|_{\mathrm{I}})$ can be formulated as an algebraic state on $\mathcal{A}_{\mathrm{I}}$ which satisfies the KMS condition at temperature $\beta^{-1} = a/2\pi$ relative to the notion of time translation defined by vector field $\boldsymbol{Z}_{\mathrm{I}} = \partial/\partial \mathrm{t}$ (which then generates the one-parameter group of automorphism $a_{u=\mathrm{t}}$). However, it is necessary to have in mind that the proof that $\omega_M|_{\mathrm{I}}$ is a KMS state does not imply that it is a thermal bath of Rindler particles. The assumption that it is is only a suggestive one. The reason for that statement is that as commented in the main text a detector can indeed be excited when in uniform accelerated motion, but the excitation energy does *not* come from the *pseudo energy* of any hypothetical thermal bath, but from the *real* energy (as inferred from an inertial reference frame) of the source accelerating the device.

References

[1] Arageorgis, A., Earman, J. and Ruetsch, L., Fulling Non-uniqueness and the Unruh Effect: A Primer on Some Aspects of Quantum Field Theory, *Philos. of Sci.*. **70**, 164-202 (2003).

[33] Please, do not confuse $\omega_{\mathrm{I}}|\mathcal{A}(U)$ with $\omega_{\mathrm{I}}|_{\mathcal{A}(U)}$.

[34] The states $\omega_M|\mathcal{A}(U)$ and $\omega_{\mathrm{I}}|\mathcal{A}(U)$ are quasi free Hadamard states, i.e., states for which

[35] Recall that a KMS state is an algebraic state (ζ_u, β) on $\mathcal{A}$ where $\zeta_u : \mathcal{A} \to \mathcal{A}$ one parameter group of automorphisms and $0 \leq \beta < \infty$ such that the condition $\omega(A\zeta_{u\beta}B) = \omega(BA)$. It is a basic result that a state satisfying the KMS condition at t act as a thermal reservoir, in the sense that any finite system coupled to it reaches thermal equilibrium at "temperature" $T = \beta^{-1}$.

[2] Araki, H., Type of the von Neumann Algebra Associated to Free Field, *Prog. Theor. Phys.* **32**, 956-965 (1964).

[3] Bell, J. S., *Speakable and Unspeakable in Quantum Mechanics*, Cambridge Univ. Press, Cambridge, 1987.

[4] Blackadar, B.,*Operator Algebras*, Springer, Heildelberg, 2005. revised text at http://wolfweb.unr.edu/homepage/bruceb/Cycr.pdf.

[5] Boulware, D., Quantum Field Theory in Schwarzschild and Rindler Spaces, *Phys. Rev. D* **11**, 1404- (1975).

[6] Boyer, T. H., Derivation of the Blackbody Radiation Spectrum from the Equivalence Principle in Classical Physics with Classical Electromagnetic Zero-Point Radiation, *Phys. Rev. D* **29**, 1096- (1984).

[7] Bratelli, O, Kishimoto, A.,and Robinson, D. W., Stability Properties and the KMS Condition, *Comm. in Math. Phys.* **61**, 209-328 (1978)

[8] Bratelli, O and Robinson, D. W., *Operator Algebras and Quantum Statistical Mechanics I*, Springer-Verlag, New York,1979.

[9] Bratelli, O and Robinson, D. W., *Operator Algebras and Quantum Statistical Mechanics II*, Springer-Verlag, New York,1996.

[10] Buchholz, D. and Verch, R., Macroscopic Aspects of the Unruh Effect, *Class.Quant.Grav.* **32**, 245004, (2015) [arXiv:1412.5892v4[gr-qc]]

[11] Clifton, R., and Halvorson, H., Are Rindler Quanta Real?, in S. M. Cristensesn (ed.), *Quantum Theory of Gravity*, pp. 66-77, Adam Hilger, Bristol, 1984.

[12] Cozzella,G., Landulfo, A.G.S., Matsas, G. E. A., and Vanzella, D. A. T., *Virtual Observation of the Unruh Effect.* [arXiv:1701.03446v1 [gr-qc]]

[13] Choquet-Bruhat, Y., DeWitt-Morette, C. and Dillard-Bleick, M., *Analysis, Manifolds and Physics* (revised edition), North Holland Publ. Co., Amsterdam, 1982

[14] Colosi, D. and Rätzel, D., The Unruh Effect in General Boundary Quantum Field Theory, *Symm. Integ. Geom: Meth. and Appl.* **9**, 019 (2013). [http://dx.doi.org/10.3842/SIGMA.2013.019]

[15] Crispino, L. C. B., Higuchi, A. and Matsas, G. E. A., The Unruh Effect and its Applications, *Rev. Mod. Phys.* **80**, 786-838 (2008).

[16] Crumeyrolle, A., *Orthogonal and Symplectic Clifford Algebras. Spinor Structures*, Kluwer Acad. Pub., Dordrecht, 1990.

[17] David, F., *A Short Introduction to the Quantum Formalism.* [arXiv;1211-5627v1 {math-ph]]

[18] Davies, P. C. W., Particles Do Not Exist, in Christensen, S.M. (ed.), *Quantum Theory of Gravity*, pp 66-77, Adam Hilger, Bristol, 1984.

[19] Dodson, C. T. J. and Poston, T., *Tensor Geometry* (second edition), Springer Verlag, Berlin, Heidleberg, New York, 1991.

[20] Fedotov, A. M.,Mur, V.D., Narozhny, N. B., Belinski, V.A. and Karnakov, B. M, Quantum Field Aspect of the Unruh Problem, *Phys. Lett. A* **254**, 126-132 (1999).

[21] Frankel, T., *The Geometry of Physics*, Cambridge Univ. Press, Cambridge, 1997.

[22] Fulling, S. A., *Aspects of Quantum Field Theory on Curved Spacetime*, Cambridge Uni. Press, Cambridge, 1989.

[23] Gel'fand, , I. M, and Naimark, M. A., On Embedding of Normed RIngs into the Ring of Operators in Hilbert Space, *Math. Sb* **12**, 197-213 (1943).

[24] Giglio, J. F. T. and Rodrigues, W. A. Jr., Locally Inertial Reference Frames in Lorentzian and Riemann- Cartan Spacetimes, *Ann. der Physik.* **502**, 302-310 (2012).

[25] Ginsburg, V. L. and Frolov, V. P.,Vacuum in Homogeneous Gravitational Field and Excitation of a Uniformly Accelerate Detector, S*ov. Phys Uspeck*i **30**, 1073-1095 (1987).

[26] Grib, A,. A., Mamayev, S. G., and Mostepanenko, V. M., *Vacuum Quantum Effects in Strong Fields*, Friedmann Laboratory Publishing, St. Petersburg, 1994.

[27] Grishchuk, L. P., Zel'Dovich, Ya. B. and Rozhanskill, Equivalence Principle and the Zero-Point Field Fluctuations, *Sov. Phys. JEPT* **65**, 11-14 (1987).

[28] Jackson, J. D., *Classical Electrodynamics*, (third edition), J. Wiley &Sons, New York, 1998.

[29] Kay, B. S., The Double-Wedge Algebra for Quantum Fields on Schwarzschild and Minkowski Spacetimes, *Com. Math. Phys.* **100**, 57-81 (1985).

[30] Kubo, Statistical-Mechanical Theory of Irreversible Processes. I. General Theory and Simple Applications to Magnetic and Conduction Problems", *J. Phys. Soc. of Japan* **12**, 570–586 (1957).

[31] Kay, B. S. and Wald, R. M., Theorems on the Uniquenessand Thermal Properties of Stationary, Nonsingular, Quasi Free States on Spacetimes with a Bifircucate Killing Horizon, *Phys. Reports* **207**, 1709-1714 (1991).

[32] Lyle, S. N., *Uniformly Accelerating Charges. A Treat to the Equivalence Principle*, Fundamental.Theories of Physics. **158**, Springer, Heidelberg, 2008.

[33] Lyle, S. N., *Self-Force and Inertia. Old Light on New Ideas*, Lecture. Notes in Physics. **796,** Springer, Heidelberg, 2010.

[34] Maiorino, J. E. and Rodrigues, W. A. Jr., Maxwell Theory is Still a Source of Surprises, in Dvoeglazov, V. V. (ed.), *Photon: Old Problems in Light of New Ideas,* Nova Sci. Publ., Inc. New York, 2000.

[35] Martin, P. C.; Schwinger, J., Theory of Many-Particle Systems. I, *Phys. Rev.* **115**, 1342–1373 (1959).

[36] Morse,,P. M. and Feshbach, *Methods of Theoretical Physics*, Part II, McGraw-Hill Book Co., Inc., New York,1953.

[37] Motl, L., John Bell Actually Misunderstood Relativity,Too, http://motls.blogspot.com.br/2015/05/john-bell-actually-misunderstood.html

[38] Nakahara, M., *Geometry, Topology and Physics*, Inst. Physics Publ., Bristol and Philadelphia, 1990.

[39] Nöth, M., *The Unruh Effect Without Observer*, M.Sc. Thesis, Ludwig-Maximilians-Universität München, 2016. http://www.mathematik.uni-muenchen.de/~bohmmech/theses/Noeth_Markus_MA.pdf

[40] Ohanian, H. and Ruffini, R., *Gravitation and Spacetime* (second edition), W.W.Norton &Co., New York, 1994.

[41] O'Neill, B., *Semi-Riemannian Geometry With Applications to Relativity*, Academic Press, San Diego, 1983.

[42] Panofski, W. K. H.,and Philiips, M., *Classical Eletricity and Magnetism,* Addison-Wesley Publ. Co., Reading MA, 1969.

[43] Parrott, S., *Relativistic Electrodynamics and Differential Geometry*, Springer-Verlag, New York, 1987

[44] Parrott, S., Radiation from an Uniformily Accelerated Charge and the Equivalence Principle, *Found. Phys.* **32**, 407-440(2002)[arXiv:gr-qc:9303025v8]

[45] Pauli, W., *Theory of Relativity*, Dover edition, Dover Publ. Inc., New York, 1981.

[46] Requardt, M., *On the Reality of Unruh Temperature.* [arXiv:1311.1608v1 [gr-qc]]

[47] Rodrigues, W. A. Jr., and Sharif, M., Equivalence Principle and the Principle of Local Lorentz Invariance, *Found. Phys.* **31**, 1785-1806 (2001).

[48] Rodrigues, W. A. Jr. and Capelas de Oliveira, E., *The Many Faces of Maxwell, Dirac and Einstein Equations. A Clifford Bundle Approach* (second revised and enlarged edition). Lecture Notes in Physics **922**,Springer, Dordrechet, 2016.

[49] Rodrigues, W. A. Jr. and Wainer, S. A.,Equations of Motion and Energy-Momentum 1-Forms for the Coupled Gravitational, Maxwell and Dirac Fields, *Adv, Appl.Clifford Algebras* (2016), DOI: 10-1—7/s0006016-0679-5, .[arXiv: 1601.04878[math-ph]]

[50] Rodrigues, W. A. Jr. and Wainer, S. A., Notes on Conservation Laws, Equations of Motion and Particle Field in Lorentzian and Teleparallel de Sitter Spacetime Structures, *Adv. Math. Phys.* **2016**, 5465263 (2016) [arXiv:1505.02935[math-ph]]

[51] Segal, I, Irreducible Representaions of Operator Algebras, *Bull. Amer. Math. Soc.* **53**, 73-88 (1947).

[52] Sciama, D. W., Candela, P., and Deutch, D., Quantum Fild Theory, Horizons and Thermodynamics, *Adv. in Phys.* **30**, 327-366 (1981)

[53] Sachs, R. K. and Wu, H., *General Relativity for Mathematicians*, Springer-Verlag, New York, Heidelberg,Berlin, 1977.

[54] Swell, G. l., Quantum Fields on Manifolds: PCT and Gravitationally Induced Thermal States, *Ann. Phys.* **141**, 201-224 (1982)

[55] Socolovsy, M., Rindler Space, Unruh Effect and Hawking Temperature, *Ann. Fondation L. de Broglie* **30**, 1-49 (2014).

[56] Susskind, L. and Lindsey, J., *An Introduction to Black Holes, Information and the String Theory Revolution. The Holographic Universe*, World Scientidfic, Singapore 2005.

[57] Takagi, S.,Vacuum Noise and Stress Induced by Uniform Accelerating Hawking-Unruh Effect in Rindler Manifold of Arbitrary Dimension, *Prog. Theor. Phys.* (Suppl.) **88**, 1-142 (1986)

[58] Tolman, R., Rela, tivity, *Thermodynamics and Cosmology,* Dover Publ., Inc., New York, 1987 (first published by Oxford Univ. Press, Oxford,1934).

[59] Turakulov, Z. Ya., Electromagnetic Field of a Charge Moving with Constant Acceleration, *J. Geom. Phys* **14**, 305-308 (1994).

[60] Turakulov, Z. Ya., Infinitesimal Radiation Phenomena, *Turkish. J. Phys.* **19**, 1567-1573 (1995).

[61] Turakulov, Z. Ya..,Geometric Theory of Radiation, in Dowling, J. P. (ed.), *Electron Theory and Quantum Electrodynamics*, NATO ASI SERIES. Series B: Physics vol. 358, pp. 321-325, Plenum Press, New York, 1997.

[62] Unruh, W. H., Notes on Blackhole Evaporation, *Phys. Rev. D* **14**, 870-892 (1976)

[63] Unruh, W. H., and Wald, R. M., What Happens When an Accelerating Observers Detects a Rindler Particle, *Phys. Rev. D* **29**, 1047-1056 (1984).

[64] Verch, R., Local Definitetness, Primitivity, and Quasi Equivalence of Quasi- Free Hadamard-vQunatum States in Curved Spacetime, *Comm. Math. Phys.* **160**, 507-536 (1994).

[65] von Neumann, J. Zur Algebra der Funktional Operationen und Theorie der Normalen Operatoren", *Math. Ann.* **102**, 370–427 (1930).

[66] Wald, R. M., *Quantum Field Theory in Curved Spacetime and Black-Hole Thermodynamics*, The Univ. Chicago Press, Chicago, 1994.

[67] Oziewicz, Z., .Sitarczyk, Cz, Parallel Treatment of Riemannian and Symplectic Clifford Algebras, in Artibano, M., Boudet, R. Helmstetter, J. (eds.), *Clifford Algebras and their Applications in Mathematical Physics*, pp.83-96, Kluwer Acad. Publ., Dordrecht, 1989.