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The nonreciprocity of relative acceleration in relativity

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We define a particle's acceleration relative to an accelerating observer as that determined in his Møller reference frame. We obtain the general formula for this acceleration in the one-dimensional case and note, inter alia, its nonreciprocity and its extendibility to general relativity. Radar acceleration yields a different result.

1. Introduction

The question as to the correct coordinate-independent definition in relativity of the acceleration of one accelerating observer $\tilde{O}$ relative to another accelerating observer O has recently been raised by one of us [1]. The present paper seeks to supply the answer, at least in the one-dimensional case. We here work exclusively in special relativity, i.e. in flat space–time. However, we claim validity for our main formulae, eqs. (17), (26) and (27) below, even in general relativity. For what general relativity adds to special relativity is curvature, or, roughly speaking, tidal accelerations. So while special-relativistic formulae involving derivatives of the acceleration could be affected, relations involving only velocities and accelerations at a single event go over unchanged to general relativity.

Among other things we find that

(i) relativistic (unlike classical) relative acceleration is not reciprocal, i.e. $\tilde{O}$'s acceleration relative to O is in general not the same numerically as O's acceleration relative to $\tilde{O}$; and

(ii) $\tilde{O}$'s acceleration relative to O can undergo an unexpected sign change when $\tilde{O}$'s velocity relative to O crosses a certain threshold.

As for the nonreciprocity, this was already noted by Rohrlich ([2], especially p. 179), though Rohrlich based his conclusion on coordinate-*dependent* definitions of acceleration. And as for the sign change, a possibly related phenomenon has been noted by Abramowicz and Prasanna [3] in connection with the centrifugal acceleration of circular orbits in Schwarzschild space.

2. The main result

We shall here consider two *collinear* observers O and $\tilde{O}$, thereby restricting the problem to one (spatial) dimension. By collinear we mean that O and $\tilde{O}$ move along a single straight line in some inertial frame, or, in other words, that their world-lines lie in a single two-plane in Minkowski space, at least locally. Moreover we assume that these world-lines intersect, and that it is at the intersection event that O and $\tilde{O}$ are to assign velocities and accelerations to each other. For our purposes O and $\tilde{O}$ must be equipped, at least locally, with a reference frame that allows positions and times to be measured. A "point-observer" cannot in general make such measurements. (Actually, in the one-dimensional case he *can*, namely by radar. We briefly examine this possibility in section 4.)

The natural reference frame for O to choose (and analogously for $\tilde{O}$) is the well-known frame rigidly co-moving with O that has been studied extensively by Møller [4]. We shall call it F. Møller in his equation (8.162) gives the form of the metric of Minkowski space adapted to F. When converted to the opposite signature and to units making the speed of light unity, it reads

$$ds^2 = [1+g(t)x]^2\,dt^2 - dx^2 - dy^2 - dz^2 . \qquad (1)$$

The observer O is at rest at the spatial origin $x=y=z=0$ of F. The metric (1) follows uniquely from the requirements that

(i) the spatial lattice be Euclidean (as measured locally by rulers at rest) and move Born-rigidly, i.e. infinitely small light rods joining the lattice points x, y, z=const suffer no stresses;

(ii) O can have any desired proper acceleration $g(t)$ in the x-direction, t being his proper time;

(iii) all pairs of neighboring coordinate clocks are Einstein synchronous, i.e. light signals exchanged by them at the same time take the same time in either direction. Note that x, y, z measure proper distance in the moving lattice. Thus F is physically determined, and it reduces to the usual inertial frame when $g\equiv 0$. It would indeed seem to be the most natural frame for O to use.

Accordingly, if a particle passing O satisfies the equations $x=x(t)$, $y=z=0$ relative to F, O will assign to it at the moment of coincidence a velocity u and an acceleration a as follows,

$$u=\left(\frac{dx}{dt}\right)_{x=0}, \quad a=\left(\frac{d^2x}{dt^2}\right)_{x=0} . \qquad (2)$$

Now suppose the "particle" in question is our second observer $\tilde{O}$. Let *his* Møller metric be

$$ds^2 = [1+\tilde{g}(\tilde{t})\tilde{x}]^2\,d\tilde{t}^2 - d\tilde{x}^2 - d\tilde{y}^2 - d\tilde{z}^2 . \qquad (3)$$

At the coincidence of O and $\tilde{O}$ we then have, for events on the common x-axis,

$$ds^2 = d\tilde{t}^2 - d\tilde{x}^2 = dt^2 - dx^2 . \qquad (4)$$

Hence at that event a standard Lorentz transformation relates $d\tilde{x}$, $d\tilde{t}$ with dx, dt, which implies that O and $\tilde{O}$ assign equal and opposite velocities

$$u=\left(\frac{dx}{dt}\right)_{d\tilde{x}=0}, \quad \tilde{u}=\left(\frac{d\tilde{x}}{d\tilde{t}}\right)_{dx=0} \qquad (5)$$

to each other.

If $\tilde{O}$ moves according to the equations $x^i=x^i[s]$, $y=z=0$ through F, its four-acceleration components A^i relative to F will be given by

$$A^i = (x^i)'' + \Gamma^i_{jk}(x^j)'(x^k)' , \quad (\ )' = \frac{d}{ds}(\ ) . \qquad (6)$$

It is easily seen that at the coincidence event where $x=0$ and, say, $t=0$, the only nonvanishing Christoffel symbols of the metric (1) are (with x, y, z, t= x^1, x^2, x^3, x^4)

$$\Gamma^1_{44} = \Gamma^4_{14} = g(0) =: g . \qquad (7)$$

Accordingly A^i has only two nonvanishing components at that event, namely

$$A^1 = x'' + gt'^2, \quad A^4 = t'' + 2gt'x' . \qquad (8)$$

Now from the metric (1) we easily find that at $x=0$ (for notation, cf. (2))

$$t' = (1-u^2)^{-1/2} =: \gamma , \qquad (9)$$

$$t'' = \gamma^4 u(a-g) , \qquad (10)$$

$$x' = u\gamma , \qquad (11)$$

$$x'' = \gamma^4(a-u^2 g) . \qquad (12)$$

With the help of these formulae we obtain

$$A^1 = \gamma^4[a+g(1-2u^2)] , \qquad (13)$$

$$A^4 = \gamma^4 u[a+g(1-2u^2)] . \qquad (14)$$

The last two expressions allow us to compute the proper acceleration $\tilde{g}$ of $\tilde{O}$ as the magnitude of A^i; so since

$$\tilde{g}^2 = -g_{ij}A^iA^j = (A^1)^2 - (A^4)^2 , \qquad (15)$$

we find

$$\tilde{g} = \pm\gamma^3[a+g(1-2u^2)] . \qquad (16)$$

It is clear that when $\tilde{O}$'s proper acceleration is in the same sense as that of O, we must choose the positive sign here, and contrariwise, the negative sign. Accordingly we prefer to write (16) as

$$\tilde{g} = \gamma^3[a+g(1-2u^2)] ,$$

i.e.

$$a=\gamma^{-3}[\tilde{g}-g\gamma(2-\gamma^2)]\,, \qquad (17)$$

with the understanding that both g and $\tilde{g}$ can take either sign, and that positive signs correspond to accelerations in the positive x-direction of F.

As noted in the introduction, formula (17) goes over unchanged into general relativity, where local Møller frames are always possible. The condition for the collinearity of the two observers then simply amounts to the coplanarity of their four-velocities and their four-accelerations at the intersection of their world-lines. Formula (17) could be used, for example, by an observer at rest in a static gravitational field, who wishes to determine the proper acceleration $\tilde{g}$ of a particle moving past him in the direction of the field: he knows his own g (now the negative of the field strength) and he measures u and a, whereupon (17) gives $\tilde{g}$.

3. Implications

Observe that when we identify $\tilde{\mathrm{O}}$ with O (i.e. put $x[s]\equiv 0$), we have $u=a=0$ and then (17) yields $\tilde{g}=g$. This verifies that g is indeed the proper acceleration of O. Observe also that in the "classical" case $u\ll 1$, (17) reduces a $a=\tilde{g}-g$, as expected. Note that $g=0$ yields the well-known result $a=\gamma^{-3}\tilde{g}$ when O is inertial. But note above all that in general (17) is not invariant under the substitution $g\mapsto -\tilde{g}$, $\tilde{g}\mapsto -g$. This means that $\tilde{\mathrm{O}}$ ascribes a different acceleration $\tilde{a}$ (numerically) to O than does O to $\tilde{\mathrm{O}}$, i.e. *relative acceleration is nonreciprocal.*

Unlike the classical a, the relativistic a depends on the velocity u of the particle relative to the observer. But it depends on u^2 rather than on u, so that, for given g, $\tilde{g}$ and $|u|$, a is the same for both senses of u. The multiplier $\gamma(2-\gamma^2)$ of g in (17) decreases monotonically from 1 to $-\infty$ as u increases from zero to the speed of light. Thus if we take g to be positive, then in the case $\tilde{g}<g$, where classically a would be negative, a is now negative only for sufficiently small u. For $\gamma(2-\gamma^2)<\tilde{g}/g$, a becomes positive, and ultimately tends to g as u tends to the speed of light.

We finally note that it would be straightforward, in principle, to extend our present analysis to the general case of two coincident observers who need *not* move collinearly; Møller (see ref. [4], eq. (8.154)) has provided the necessary generalization of the metric (1). However, the same extension is not possible for the radar method to be discussed next.

4. Radar velocity and acceleration

Because of the above-mentioned limitation we do not consider the radar method to have the same intrinsic interest as that using the Møller frame. Nevertheless we proceed to obtain a radar formula analogous to (17).

Working in the Møller frame F, we may take the equation of the particle's motion for sufficiently small t to be

$$x=ut+\tfrac{1}{2}at^2+\mathrm{O}(t^3) \qquad (18)$$

with u and a as defined by (2). Now the metric (1) implies that a light signal ($\mathrm{d}s^2=0$) in the x-direction leaving the origin at time t_1 satisfies

$$t-t_1=g^{-1}\ln(1+gx)\,, \qquad (19)$$

while one that returns at time t_2 satisfies

$$t-t_2=-g^{-1}\ln(1+gx)\,. \qquad (20)$$

The intersection event (x, t) of these signals is given by

$$1+gx=\exp[\tfrac{1}{2}g(t_2-t_1)]\,,\quad t=\tfrac{1}{2}(t_2+t_1)\,. \qquad (21)$$

For a to-and-fro radar signal to the particle this event must lie on locus (18). The observer will then assign a radar distance R and radar time T to the particle as follows,

$$R=\tfrac{1}{2}(t_2-t_1)\,,\quad T=\tfrac{1}{2}(t_2+t_1)\,. \qquad (22)$$

With that, eqs. (21) and (18) yield

$$1+g[uT+\tfrac{1}{2}aT^2+\mathrm{O}(T^3)]=\exp(gR) \qquad (23)$$

as the equation of the particle's motion in terms of T and R. Finally O assigns a radar velocity U and a radar acceleration A to the particle as follows,

$$U=\frac{\mathrm{d}R}{\mathrm{d}T},\quad A=\frac{\mathrm{d}^2R}{\mathrm{d}T^2}\,. \qquad (24)$$

To calculate these, we implicitly differentiate (23) twice with respect to T,

$$u+aT+O(T^2)=\exp(gR)U, \tag{25}$$

$$a+O(T)=\exp(gR)(gU^2+A). \tag{26}$$

The values U_O and A_O of U and A at the instant the particle passes O result from these equations on putting $R=T=0$,

$$U_O=u, \quad A_O=a-gu^2. \tag{27}$$

The first of these expressions shows that radar velocity coincides with Møller-frame velocity and is independent of the observer acceleration. The second shows that radar acceleration and Møller-frame acceleration do *not* coincide. When combined with (17), (27) yields the desired radar analogue of (17),

$$A_O=\tilde{g}\gamma^{-3}-g\gamma^{-2}=\gamma^{-3}(\tilde{g}-\gamma g). \tag{28}$$

Thus radar acceleration, too, is nonreciprocal, and now a sign change occurs when $\tilde{g}>g$ and $\gamma=\tilde{g}/g$. As in the case of (17), we claim validity for (28) also in general relativity. In practice, of course, in the one-dimensional case, the radar method is by far the more convenient method for an observer to use. Formulae (27) will then tell the observer what he *would* have found had he constructed his Møller frame.

References

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