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An accelerated charge is also absorbing power

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Abstract

An accelerated classical point charge radiates at the Larmor power rate $2e^2a^2/3$, leading to the expectation of an associated radiation reaction force. The famous Abraham-Lorentz-Dirac proposal is plagued with difficulties. Here we note a simple, amazing, and apparently overlooked fact: an accelerated charge is also always absorbing power at exactly the Larmor rate. Implications for radiation reaction and the particle motion are considered. Our analysis supports Kijowski's recent proposal.

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Particles and fields are central concepts in modern physics. The idea of particles interacting through fields is a key paradigm. Its roots are classical, primarily in electrodynamics where Faraday's field idea replaced Newton's "action at a distance". Classical point charges interact locally with the electromagnetic field via the Lorentz force law $\dot{\mathbf{p}} = \mathbf{F} = e(\mathbf{E}+\mathbf{v}\times\mathbf{B})$. This relation, essentially the genesis of the field idea, both *defines* the electromagnetic field and *predicts* the particle motion—with the exception of *self interaction* effects.

From Maxwell's field equations it was found that an accelerating classical point charge radiates power at the Larmor rate. Since the particle apparently looses energy, one expects an associated radiation reaction (damping, friction) force. For over a century many famous physicists struggled with this idea and various associated models for classical particles. The best known reaction force proposal was developed, based on an extended charge model, by Lorentz. Later Dirac gave a derivation of the relativistic version for point particles using the conservation laws. However, the equation of motion, with this radiation reaction force included, has been plagued with numerous difficulties. Here we note a very relevant, simple, amazing fact, which has apparently been overlooked for over 100 years: an accelerated point charge is also *always absorbing power* at *exactly* the Larmor rate. Implications for radiation reaction and the particle motion are considered. Our analysis supports Kijowski's "renormalized electrodynamics".

Maxwell's equations [1–3] (we take $c = 1$), $\partial_\mu F^{\nu\mu} = 4\pi J^\nu$ with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, in the Lorenz gauge $\partial_\mu A^\mu = 0$, lead to the wave equation $\partial_\mu \partial^\mu A^\nu = -4\pi J^\nu$ for the potentials. A solution can be written in terms of the potentials obtained via the retarded or advanced Green function along with an associated solution to the homogeneous equation: $A^\mu = A^\mu_{\rm in} + A^\mu_{\rm ret} = A^\mu_{\rm out} + A^\mu_{\rm adv}$, whence $F^{\mu\nu} = F^{\mu\nu}_{\rm in} + F^{\mu\nu}_{\rm ret} = F^{\mu\nu}_{\rm out} + F^{\mu\nu}_{\rm adv}$.

A point charge e moving along the spacetime path $q^\alpha(\tau)$ generates a current, $J^\mu(x) = e\int_{-\infty}^{+\infty} v^\mu \delta^4(x-q(\tau))d\tau$, where $v^\mu := dq^\mu/d\tau$. The resultant (Liénard-Wiechert) potentials are $A^\alpha = e[v^\alpha/R]$, where $R := |R^\nu v_\nu|$, with $R^\nu := x^\nu - q^\nu(\tau)$. Here and elsewhere the brackets $[\ \ ]$ indicate that the enclosed quantities are to be evaluated at the retarded/advanced time determined implicitly by $R^\mu R_\mu = 0$. The associated fields,

$$F^{\mu\nu}_{\substack{\rm ret\\ \rm adv}} = \pm\left[\frac{e}{R}\frac{d}{d\tau}\left(\frac{v^\mu R^\nu - v^\nu R^\mu}{R}\right)\right], \tag{1}$$

include two types of terms. One, proportional to v^μ, is of the Coulomb $\sim e/R^2$ form and is bound to the charge, moving along with it. The other, proportional to the acceleration $a^\mu := dv^\mu/d\tau$, is long range, having $1/R$ fall off; it is interpreted as radiation which escapes from the charge. In the charge's instantaneous rest frame, in vector notation, $\mathbf{B} = [\hat{\mathbf{R}} \times \mathbf{E}]$, where

$$\mathbf{E} = \mathbf{E}_{\rm cou} + \mathbf{E}_{\rm acl} = e\left[\frac{\hat{\mathbf{R}}}{R^2}\right] + e\left[\frac{\hat{\mathbf{R}} \times (\hat{\mathbf{R}} \times \dot{\mathbf{v}})}{R}\right]. \tag{2}$$

This splitting (based on the retarded/advanced decomposition) however, although intuitively appealing, is really not physical, for $\nabla\cdot\mathbf{E}_{\rm acl} = (2e/R^2)\mathbf{a}\cdot\hat{\mathbf{R}} \neq 0$ outside of the moving charge. Hence $\mathbf{E}_{\rm acl}$, taken by itself, appears to have conjured out of the vacuum a rather strange source charge density—the associated 4-current is proportional to R^μ and is thus moving at the speed of light [4]. However the "acceleration" fields really do, through the $O(R^{-2})$ part of the Poynting vector, $\mathbf{S}_{\rm acl} = (1/4\pi)\mathbf{E}_{\rm acl}\times\mathbf{B}_{\rm acl} = (1/4\pi)|\mathbf{E}_{\rm acl}|^2\hat{\mathbf{R}} = (e^2a^2/4\pi)R^{-2}\sin^2\theta\,\hat{\mathbf{R}}$, dominate the power radiated out to large distances:

$$P = \oint \mathbf{S}_{\rm acl}\cdot\hat{\mathbf{R}}d\sigma = \frac{2}{3}e^2a^2, \tag{3}$$

the celebrated *Larmor power formula.* Note however, that this calculation is valid *only if* there are no other fields that interfere with the radiation. Real charges are actually immersed in a sea of electromagnetic fields ranging from the 3 K cosmological microwave radiation thorough the output of the sun and stars to radio, television and thermal radiation at ~ 300 K.

In practice one cannot measure the radiation at infinity. Since the radiation rate is proportional to the instantaneous acceleration of the particle, it is inferred that any accelerating charge emits radiation. Rohrlich has rigorously established this as a *local criterion* for radiation [5,3]. Hence our view is that the emission of "radiation" is a local process happening near the charge. An accelerating charge causes a certain type of disturbance of the electromagnetic field in its immediate neighborhood. This disturbance may propagate out to large distances. Instead it may suffer interference from other effects propagating in the field, so that little or no power may actually get out to infinity; but this is a vacuum field propagation issue; it should not be held to the charged particle's account. Note that, because of interference, an outward propagating signal *does not* conserve power in general.

The usual dogma is that the emission of radiation is irreversible and hence there is an associated radiation damping (or friction). If power is emitted then energy conservation considerations lead to a radiation reaction force. The orthodox version is the Abraham-Lorentz-Dirac force [1–3,6–8]. A simple argument (see, e.g., Jackson [2] §17.2) considers the radiated energy

$$\int_{t_1}^{t_2}\frac{2}{3}e^2\dot{\mathbf{v}}\cdot\dot{\mathbf{v}}dt = \frac{2}{3}e^2\left(\dot{\mathbf{v}}\cdot\mathbf{v}\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2}\ddot{\mathbf{v}}\cdot\mathbf{v}dt\right). \tag{4}$$

Under certain conditions the first term on the rhs will vanish (e.g., if the motion is periodic or bounded) for a suitable choice of interval. Then we can identify the radiative reaction force:

$$\mathbf{F}_{\rm rad} = \frac{2}{3}e^2\ddot{\mathbf{v}}. \tag{5}$$

As Jackson says: "It can be considered as an equation which includes in some approximate and time averaged way the reactive effects of the emission of radiation". But many regard this equation (or its relativistic generalization) as fundamental (see e.g., [3–6,9,10]). Wheeler and Feynman [7] observed that: "The existence of this force of radiative reaction is well attested: (a) by the electrical potential required to drive a wireless antenna; (b) by the loss of energy experienced by a charged particle which has been deflected, and therefore accelerated, in its passage near an atomic nucleus; and (c) by the cooling of a glowing body." But each of these processes has its converse, for example a wireless receiver can absorb power and a cool object can absorb heat radiation. Microscopically, the interactions of classical physics were presumed to be time reversible. If a charge could emit radiation it could likewise, under appropriately reversed conditions, absorb radiation. In classical physics the asymmetry between past and future is only a macroscopic statistical effect, certain processes are regarded as being much more probable. And so it should be with electrodynamics. Indeed Wheeler and Feynman [7] attest: "We have to conclude with Einstein[11] [citation in the original] that the irreversibility of the emission process is a phenomenon of statistical mechanics connected with the asymmetry of the initial conditions with respect to time." Our usual physical intuition accepts an accelerated charge emitting power, readily imagining outgoing waves carrying away energy, but regards power absorption as uncommon, imagining it quite unlikely that waves will be prearranged to focus on a charge. We will show that this picture of power absorption is inaccurate.

Although power absorption situations are actually physically common, power absorption by accelerating charges is rarely mentioned (except in connection with the direct interaction theory [7] where its role is equally as important as emission.). However, the point we wish to make here is not just that a charge could absorb power but that an accelerated charge *must* constantly be absorbing at the same (Larmor) rate as it is emitting.

Including the aforementioned radiation damping force leads to the famous Abraham-Lorentz-Dirac (ALD) equation for a moving point charge [6,1–3], which has been plagued with problems. Briefly, the ALD equation (i) is a 3rd order equation, (ii) seems to violate time reversiblity, (ii) has runaway solutions, (iii) has "unphysical", nonexistent, nonunique and counter intuitive solutions, (iv) violates causality (preacceleration—for an ingenious resolution see [11]). Much research has been devoted to these problems [8], occasionally the controversy has become heated [10]. Dissatisfaction with the ALD has led to many ingenious proposals (see, e.g., [8,12,13]). In our opinion most of the discussions overlook, or at best do not give enough weight to, certain fundamental principles of classical physical interactions:

1. they are time reversible (QED is also time reversible): not only does every emission situation have a corresponding absorption situation, microscopically they are on par and any difference in rate is a consequence of a global asymmetric boundary condition (as Wheeler, Feynman and Einstein believed) [14].

2. they have 2 initial data per degree of freedom: by this reckoning the ALD has 9/2 degrees of freedom.

3. they are local and instantaneous: an influential text states "physics is simple when analyzed locally" [15].

To establish our point about absorption, let us first consider Born's solution [16,17,3] for a uniformly accelerating charge. The path is hyperbolic: $x = 0$, $y = 0$, $z = (\alpha^2 + t^2)^{1/2}$. The field, in cylindrical coordinates, is given by $E_\phi = B_\rho = B_z = 0$, and

$$\begin{aligned} B_\phi &= 8e\alpha^2\rho t/\xi^3, \qquad E_\rho = 8e\alpha^2\rho z/\xi^3, \\ E_z &= -4e\alpha\left(\alpha^2 + t^2 + \rho^2 - z^2\right)/\xi^3, \end{aligned} \tag{6}$$

with $\xi = \left\{4\alpha^2\rho^2 + (\alpha^2 + t^2 - \rho^2 - z^2)^2\right\}^{1/2}$, where $\alpha = a^{-1}$. Note that $\mathbf{E}$ is time symmetric, and $\mathbf{B}$ is time antisymmetric. Note also that the solution has a boost Lorentz symmetry: the field values at a later proper time are just obtained by Lorentz transforming between the respective instantaneous inertial rest frames. Thus all times are essentially equivalent. Without loss of generality we can examine the fields in the frame in which the charge is at rest at the lab time $t = 0$.

It has been noted that $\mathbf{S}$ vanishes at $t = 0$, since $\mathbf{B}$ vanishes at that instant. Pauli [18], in particular, has interpreted this fact to mean that a uniformly accelerated charge does not radiate. However, it has been argued (see, e.g., [3] §5.3) that for radiation we should look along the null cone. With $z = \alpha + \zeta$, calculating on the null cone, $t^2 = r^2 := \rho^2 + \zeta^2$, of the point where the charge is at rest at lab time $t = 0$, we find $E_\rho = e\rho(1 + a\zeta)/r^3$, $E_z = e(\zeta + a\zeta^2 - ar^2)/r^3$, $B_\phi = ea\rho t/r^3$. Hence $\mathbf{S} \cdot \hat{\mathbf{r}} r^2 = (t/4\pi r)e^2 a^2 \sin^2\theta$, which yields, after integration over angles with $t = +r$, the expected Larmor rate, $P = (2/3)e^2a^2/c^3$, for the power radiated along the outgoing null cone (independent of r in this case). *But*, calculating the power *flowing into* the charge along the *incoming* null cone of the same point, $-t = r \geq 0$, gives the negative of the standard Larmor rate! The emitted power is just the absorbed power. Apparently power has come from outside and has simply flowed through the location of the charge to be reemitted.

Note that in [5] this solution was derived using only the retarded interaction, so that all fields in the region $t + z > 0$ should have come from the charge. But we have radiation incoming from the past (tracing it back we find that it comes from outside the region $t + z > 0$) converging on the charge. Where did the power come from? We observe that eternal acceleration is not physically realistic. Power flowing into a charge which began accelerating only one second ago could not be traced far back along the incoming null cone (propagation of power along the null cone is not conserved).

For uniform acceleration the power absorption exactly equals the Larmor emission rate, which depends only on the acceleration. Tracing the radiation along the null cone suggests that even for a non-uniformly accelerated particle the rates would still be equal and would depend only on the instantaneous acceleration. To establish this result, we consider the the behavior of the instantaneous fields near a charge undergoing a general acceleration. In addition to a regular part they include singular parts. The electric field, $\mathbf{E}_{\text{sing}} = \mathbf{E}_{-2} + \mathbf{E}_{-1} + \mathbf{E}_0$, includes the unbounded for $\mathbf{r} \to 0$ terms

$$\mathbf{E}_{-2} = \frac{e}{r^2}\hat{\mathbf{r}}, \qquad \mathbf{E}_{-1} = -\frac{e}{2r}(\mathbf{a} + \mathbf{a}\cdot\hat{\mathbf{r}}\hat{\mathbf{r}}), \tag{7}$$

as well as $\mathbf{E}_0 = 3e/8\{(\mathbf{a}\cdot\hat{\mathbf{r}})^2\hat{\mathbf{r}} + 2\mathbf{a}\cdot\hat{\mathbf{r}}\mathbf{a} - a^2\hat{\mathbf{r}}\}$. The latter, like $\mathbf{B}_{\text{sing}} = \mathbf{B}_0 = -(e/2)\dot{\mathbf{a}} \times \hat{\mathbf{r}}$, is bounded but still singular: the limiting value depends on the direction of approach for $\mathbf{r} \to 0$. (It is easily checked that $\mathbf{B}_0$, $\mathbf{E}_0$ and $\mathbf{E}_{-1}$ have vanishing divergence while $\nabla\cdot\mathbf{E}_{-2} = 4\pi\delta^3(\mathbf{r})$.) These expressions can be obtained from the 4-covariant instantaneous expression (which played an important role in Dirac's seminal paper, [6] eq. (60), and Rohrlich's book, [3] eq. (6-68)) or by specializing the more general expression given by Page way back in 1918 [19], eqs. (23,24). (It is amazing that so much of the significance of these expressions was not appreciated until the recent work of Kijowski [20].) These singular *self field* values are in general superposed with some regular solution to the homogeneous equation. The regular parts of the fields at the location of the charge are just given by some constant values. The particular form of $\mathbf{E}_{\text{reg}}(0)$ (and $\mathbf{B}_{\text{reg}}(0)$) is determined by various mathematical/physical choices like the Green function and boundary conditions: in particular using the retarded/advanced Green function along with $\mathbf{E}_{\text{in/out}} = 0$, gives $\mathbf{E}_{\text{reg}}(0) = \mp(2e/3)\dot{\mathbf{a}}$. Thus the only well defined part of the electric field at the location of the particle in its instantaneous rest frame, $\mathbf{E}_{\text{reg}}(0)$, has some definite value which is ultimately determined (via the Maxwell equations) by the fields, particles and boundary conditions elsewhere.

We are now set to establish our general conclusion: that accelerated charges constantly emit and absorb power at the Larmor rate. The key is the singular expansion of the field near the charge. Working in the reference frame in which the charge is instantaneously at rest at $t = 0$, we calculate (to first order is sufficient) the Poynting vector: $\mathbf{S} = (1/4\pi)\mathbf{E}\times\mathbf{B}$, using $\mathbf{E}(t, \mathbf{r}) = \mathbf{E}(0, \mathbf{r}) + \dot{\mathbf{E}}(0, \mathbf{r})t$, along with $\dot{\mathbf{E}}(0, \mathbf{r}) = \nabla \times \mathbf{B}(0, \mathbf{r})$, and similar equations for $\mathbf{B}(t, \mathbf{r})$. The power is determined by the flux integral of $\mathbf{S} \cdot \hat{\mathbf{r}}$. The contribution in the small r limit is just the $O(1/r^2)$ part. To sufficient accuracy,

$$\begin{aligned} r^2\hat{\mathbf{r}} \cdot \mathbf{E} \times \mathbf{B} &\simeq r^2\hat{\mathbf{r}} \cdot (\mathbf{E}_{-2} + \mathbf{E}_{-1} + \mathbf{E}_0 + \nabla \times \mathbf{B}_0 t) \times \{\mathbf{B}_0 - \nabla \times (\mathbf{E}_{-2} + \mathbf{E}_{-1} + \mathbf{E}_0)t\} \\ &\equiv r^2\hat{\mathbf{r}} \cdot (\mathbf{E}_{-1} + \mathbf{E}_0 + \nabla \times \mathbf{B}_0 t) \times \{\mathbf{B}_0 - \nabla \times (\mathbf{E}_{-1} + \mathbf{E}_0)t\}\,. \end{aligned} \tag{8}$$

Dropping terms which vanish in the $t^2 = r^2 \to 0$ limit gives

$$\begin{aligned} r^2\hat{\mathbf{r}} \cdot \mathbf{E} \times \mathbf{B} &\simeq r^2\hat{\mathbf{r}} \cdot \mathbf{E}_{-1} \times \{-\nabla \times \mathbf{E}_{-1}t\} \\ &= r^2\hat{\mathbf{r}} \cdot \left(-e\frac{\mathbf{a}}{2r}\right) \times \left\{e\frac{\mathbf{a} \times \hat{\mathbf{r}}}{r^2}\right\} t \\ &= e^2 \left|\hat{\mathbf{r}} \times \mathbf{a}\right|^2 (t/r) = e^2 a^2 \sin^2\theta(t/r). \end{aligned} \tag{9}$$

Then integration over angles gives $(2/3)e^2a^2(t/r)$, the Larmor power rate multiplied by t/r. This factor is $+1$ for the outgoing future null cone, the usual emission *but*, for $t < 0$, the factor is -1, indicating absorption of power incoming along the past null cone. Doing the calculation in this fashion reveals that the result actually depends only on $\mathbf{E}_{-1}$, the a/r part of the electric field, which is independent of the choice of advanced or retarded potential. Now that the physics is clear, we could do a rigourous covariant calculation. The details need not be given here, it is sufficient to take Rohrlich's establishment of the *local criterion* [5] for the emitted power and time reverse it, replacing retarded by advanced quantities, to find *exactly* the same rate for the absorbed power.

Thus an accelerated charge always emits and absorbs at the Larmor rate locally. There is a continuous stream of power at the Larmor rate through the location of the charge, not a flow from the charge to the field. The charge seems merely to focus the ambient field flow. Such a vision undermines the usual argument for a radiation reaction force based on irreversible emission.

How then should a charge move? Our detailed considerations will be presented elsewhere. Briefly, we expect that a point charge would move according to some suitably adjusted version of the Lorentz force law. The "physics is simple when analyzed locally" philosophy is most easily applied in the instantaneous rest frame of the charge. There the Lorentz force law reduces to the form $\mathbf{F}_{\text{em}} = e\mathbf{E}$. The complication is that $\mathbf{E}$ includes a singular part due to the charge's self field, which does not have a well defined value at the location of the charge. The simplest, most obvious assumption is just to remove these singular terms [21]. Hence we propose that an instantaneously stationary point charged particle interacts with the only well defined part of the electric field at the charge location: $\mathbf{E}_{\text{reg}} := \mathbf{E} - \mathbf{E}_{\text{sing}}$. (Indeed, is there any other option that makes sense?) This *renormalized Lorentz force*

$$\mathbf{F}_{\text{em}} = e\mathbf{E}_{\text{reg}}(0), \tag{10}$$

has, in effect, already been proposed by Kijowski [20]. He took a different approach proposing certain boundary conditions on the field at the location of the particle. He showed that the Maxwell equations then preserved a suitably "renormalized" (by extracting the infinite Coulomb energy) expression for the total energy-momentum of the particle-electromagnetic field system. More recently the initial value problem for this “renormalized electrodynamics” for point charged particles has been considered [22]. It was found that unique solutions exist, there are no runaway solutions.

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