Feynman's disk paradox

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frame. The elapsed time shown on this clock between two events at its location is the temporal separation Δt of the events, and their spatial separation $\Delta x = 0$. The elapsed time in the primed frame between the same two events is the temporal separation $\Delta t'$. Application of Eq. (6.2b) to this pair of events yields

$$\Delta t' = \gamma \Delta t. \tag{6.9}$$

It also follows from Eq. (4.6) that

$$\Delta t' = \Delta t (1 - v^2/c^2)^{-1/2}. \tag{6.10}$$

Comparison of Eqs. (6.9) and (6.10) gives the result

$$\gamma = (1 - v^2/c^2)^{-1/2}. \tag{6.11}$$

If we now use the results (6.5), (6.8), and (6.11) in Eqs. (6.1), we obtain the familiar equations of the Lorentz transformation,

$$x' = \frac{x - vt}{(1 - v^2/c^2)^{1/2}}, \quad t' = \frac{t - vx/c^2}{(1 - v^2/c^2)^{1/2}}. \tag{6.12}$$

VII. CONCLUSION

It has been shown above that some of the basic results of special relativistic kinematics can be derived directly from a careful analysis of the relativity of simultaneity. The foundation for this analysis has been the familar thought experiment known as Einstein's train. I do not maintain that the arguments presented here are necessarily the best way to introduce students to the subject. I have found, however, that such arguments provide useful insights, and help in creating some feeling for what appears to most students to be counterintuitive results.

[1] A. Einstein, Ann. Phys. (Leipzig) **17**, 891 (1905); English translation in *The Principle of Relativity* (Dover, New York, 1952), pp. 35–65.
[2] See, for example, A. Einstein, *Relativity* (Holt, New York, 1920), pp. 30–33.
[3] P. G. Bergmann, *Introduction to the Theory of Relativity* (Prentice-Hall, Englewood Cliffs, NJ, 1942), p. 36.
[4] See, for example, pp. 33–36 of the reference cited in Ref. 3.

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A paradox involving the apparent violation of angular momentum conservation is discussed. Electromagnetic induction is used to impart angular momentum to a disk of charges. The paradox is resolved by finding the origin of the angular momentum.

I. INTRODUCTION

A "paradox" concerning electromagnetic induction and the conservation of angular momentum is presented in the *Feynman Lectures*.[1] Its resolution is touched upon in a later chapter,[2] but never treated in detail. Although a recent article in this Journal discussed a similar problem with some of the same features,[3] this author knows of no quantitative treatment of Feynman's paradox. This paper presents such a treatment.

Briefly, the paradox involves a coil of wire, through which a current flows, placed at the center of an insulating disk. Electrostatic charges are embedded symmetrically in the perimeter of the disk. The disk is mounted, on bearings with negligible friction, so that it may rotate freely about its axis. The current in the coil is interrupted, causing a change in the magnetic flux linked by the circle of charge on the perimeter of the disk. According to Faraday's law of electromagnetic induction, the electric field induced at the location of the charges exerts a force on them and causes the disk to rotate. However, the problem may be analyzed from another point of view. While current was flowing in the wire, the disk was at rest. Consequently, after the current is interrupted, the disk should remain at rest in order for angular momentum to be conserved. Further details and a schematic drawing may be found in Ref. 1.

The resolution of this paradox depends upon the realization that static electromagnetic fields have angular momentum which must be considered in computing the total conserved quantity.[4] The gain in angular momentum of the disk compensates for the loss by the field. The result is proved below for a general initial configuration of static charges and steady currents and a final configuration of slowly moving charges, with particular reference to Feynman's disk.

II. PROOF

Let **E** and **B** be the initial electric and magnetic fields of the charges and currents, respectively. The electric field may be expressed in terms of the density of charge ρ by Gauss's law:

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0. \tag{1}$$

The electric field $\mathbf{E}'$ induced by the interruption of the current is given by Faraday's law:

$$\nabla \times \mathbf{E}' = -\frac{\partial \mathbf{B}}{\partial t}. \tag{2}$$

The torque $d\,\mathbf{L}_M/dt$ exerted on the charges is

$$\frac{d\,\mathbf{L}_M}{dt} = \int (\mathbf{r} \times \rho \mathbf{E}') d^3\mathbf{r}, \tag{3}$$

where $\mathbf{L}_M$ is the mechanical angular momentum of the system and $\mathbf{r}$ is the radius vector from an arbitrary origin. Using Eq. (1), this becomes

$$\frac{d\mathbf{L}_M}{dt} = \int (\mathbf{r}\times\mathbf{E}')(\epsilon_0\nabla\cdot\mathbf{E})d^3\mathbf{r}. \tag{4}$$

Consequently, interruption of the current changes the angular momentum by

$$\Delta\,\mathbf{L}_M = \int\int (\mathbf{r}\times\mathbf{E}')\epsilon_0\nabla\cdot\mathbf{E}\,d^3\mathbf{r}\,dt. \tag{5}$$

The angular momentum of the static fields, before interruption of the current, is[4]

$$\mathbf{L}_F = \epsilon_0\int \mathbf{r}\times(\mathbf{E}\times\mathbf{B})d^3\mathbf{r}. \tag{6}$$

The field $\mathbf{B}$ can be expressed as the time integral of the induced electric field by using Eq. (2):

$$\Delta\,\mathbf{L}_F = -\epsilon_0\int\int \mathbf{r}\times[\mathbf{E}\times(\nabla\times\mathbf{E}')]d^3\mathbf{r}\,dt. \tag{7}$$

In Eqs. (5) and (7), the time dependence of $\mathbf{E}$ has been neglected. This is permissible as long as the motion of the charges is slow, which may be ensured by making them massive. Inclusion of the time variation of $\mathbf{E}$ would necessitate consideration of the radiation field of the accelerating charges, which has also been neglected.

In order for the total angular momentum $\mathbf{L}_M + \mathbf{L}_F$ to be conserved, the two changes must balance:

$$\Delta\,\mathbf{L}_M = -\Delta\,\mathbf{L}_F. \tag{8}$$

If it can be shown that

$$\int (\mathbf{r}\times\mathbf{E}')(\nabla\cdot\mathbf{E})d^3\mathbf{r} = \int \mathbf{r}\times[\mathbf{E}\times(\nabla\times\mathbf{E}')]d^3\mathbf{r}, \tag{9}$$

then Eq. (8) follows *a fortiori*. Since $\mathbf{E}$ is generated by static charges $\nabla\times\mathbf{E} = 0$; likewise $\nabla\cdot\mathbf{E}' = 0$ since $\mathbf{E}'$ is not. Consequently, both sides of Eq. (9) may be symmetrized in $\mathbf{E}$ and $\mathbf{E}'$:

$$\int [(\mathbf{r}\times\mathbf{E}')(\nabla\cdot\mathbf{E}) + (\mathbf{r}\times\mathbf{E})(\nabla\cdot\mathbf{E}')]d^3\mathbf{r}$$
$$= \int \mathbf{r}\times[\mathbf{E}\times(\nabla\times\mathbf{E}') + \mathbf{E}'\times(\nabla\times\mathbf{E})]d^3\mathbf{r}. \tag{10}$$

To show that Eq. (10) holds, only one Cartesian component needs to be considered, e.g., x. Let indices 1, 2, and 3 denote coordinates x, y and z, respectively. The terms in Eq. (10) may be combined and cast in the form

$$\int r_i \frac{\partial}{\partial r_j}(E_k E'_l)\,d^3\mathbf{r}.$$

For terms with $i\neq j$, the integral vanishes if $E_k E'_l$ approaches zero sufficiently fast as r_i approaches infinity. The remaining terms are

$$\int r_3 \frac{\partial}{\partial r_3}(E_2E'_3 + E'_2E_3)d^3\mathbf{r}$$
$$= \int r_2 \frac{\partial}{\partial r_2}(E_2E'_3 + E'_2E_3)d^3\mathbf{r}. \tag{11}$$

As above, if $E_k E'_l$ approaches zero sufficiently fast, an integration by parts reduces Eq. (11) to an identity. These mathematical manipulations are possible if the product $E_i E'_j$ approaches zero at infinity faster that r_k^{-3} for any i, j, and k.[5] This condition is satisfied by the fields in Feynman's disk problem since, for large r, $E \propto 1/r^2$ for static charges, and $E' \propto 1/r^2$ for a field induced by a changing magnetic dipole. Thus it has been shown that the angular momentum stored in the static fields appears as mechanical angular momentum.

Two tacit approximations have been made: the displacement current and the magnetic field generated by the motion of the charges have been neglected. The first effect is simply included by stipulating that $\mathbf{B}$ is the total magnetic field, not just that of the steady currents. As the currents changed, $\mathbf{B}$ would include terms arising from the time dependence of $\mathbf{E}'$. The second effect may be made negligible by making the mass of the disk large, or simply by including the field generated by them in $\mathbf{B}$.

III. DISCUSSION

The Feynman disk paradox is a striking example of the existence of angular momentum of static electromagnetic fields. It demonstrates that motion is not required for a system to possess angular momentum. Of course, currents are charges in motion, but since the charges may be of arbitrary mass and either sign, it is clear that their contribution to the system's angular momentum is not fundamental. Resolution of the paradox necessarily relies upon the transfer of angular momentum from the fields to the charges on the disk.

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[1] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, MA, 1964), Vol. II, p. 17-5.

[2] Reference 1, p. 27-11.

[3] E. Corinaldesi, Am. J. Phys. **48**, 83 (1980).

[4] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, 3rd ed. (Pergamon, Oxford, 1971), p. 79.

[5] This is equivalent to requiring that the flux of angular momentum over a surface tend to zero as the surface approaches infinity.