Translation:On bodies that are to be designated as "rigid"

On bodies that are to be designated as "rigid" from the standpoint of the relativity principle;

by G. Herglotz

In his paper "The theory of the rigid electron in the kinematics of the principle of relativity"[1], Born has tried in an obvious way to give a definition of those types of motion of a three-fold extended deformable continuum, that are to be designated as "rigid" from the standpoint of the relativity principle. However, this was actually formulated by him only in one special and easily executed case. In particular the question remained untouched, whether six degrees of freedom can be ascribed to a "rigid" body defined in this way, as it may be wished by us if we want to ascribe the same fundamental meaning to this new "rigid" body in the system of the electromagnetic world-view, as it is ascribed to the ordinary rigid body in the system of the mechanical world-view.

Exactly this question will find its answer in the following lines in so far, as it will be proven that the motion of that "rigid" body is in general — *i.e.* neglecting special, more specified exceptions — unequivocally defined by the arbitrarily specified motion of a single of its points.

Particularly the fact may be mentioned for the purpose of illustration, that when one of its points is fixed, the body of Born can only uniformly rotate around a fixed axis that goes through that point.[2]

Contents

I. Definition of the "rigid" body from the standpoint of the relativity principle.

Following MINKOWSKI's[3] lines of thought, the right angled coordinates *x,y,z* of a material particle, in connection with time *t* when it is located at this point, should be interpreted as the four coordinates of a point of the four-fold extended manifold $R_4(x, y, z, t)$.

Furthermore, a measure-determination should be introduced in this R_4 as well, according to which the square of the distance *ds* of two infinitely adjacent points is (the speed of light should from now on be set equal to 1)

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2 \tag{1}$$

The line elements of real length $\left(ds^2 > 0\right)$ are denoted as space-like, and such of purely imaginary length $\left(ds^2 < 0\right)$ are denoted as time-like. The directions of the line elements of length zero that emerge from one point constitute a real cone — the minimal cone of the relevant point — whose two surfaces are separated by $dt > 0$ and $dt < 0$, and shall be denoted as front-cone and back-cone.

Two directions $(dx : dy : dz : dt)$ and $(dx' : dy' : dz' : dt')$ are normal to each other according to that measure-determination, if

$$dx\, dx' + dy\, dy' + dz\, dz' - dt\, dt' = 0 \tag{2}$$

The elements that are normal to the time-like elements are necessarily space-like, but not *vice versa*.

The group of those ∞^{10} affine transformations (the functional determinant +1) of *x,y,z,t* shall be denoted as motions in R_4, which leave ds^2 unchanged and which don't mutually permute the front- and back-cones. The Lorentz transformations are thus the group of ∞^6 motions, which leave the zero point $x = y = z = t = 0$ fixed — the rotations around the zero point, — and conversely the group of motions emerges from it by addition of the ∞^4 translations.

After these generally known definitions we think of any deformable continuum as in motion in ordinary three-dimensional space $R_3(x, y, z)$ — the coordinates at time t of any material particle (individualized by three parameters ξ, η, ζ) may be:

(3)
$$\begin{cases} x = x(\xi, \eta, \zeta, t), \\ y = y(\xi, \eta, \zeta, t), \\ z = z(\xi, \eta, \zeta, t). \end{cases}$$

To achieve a greater symmetry, some kind of local time τ may be introduced in some way:

(4)
$$\tau = \tau(\xi, \eta, \zeta, t),\ \frac{\partial \tau}{\partial t} > 0$$

by which it can be written in a more uniform way instead of (3):

(5)
$$\begin{cases} x = x(\xi, \eta, \zeta, t), \\ y = y(\xi, \eta, \zeta, t), \\ z = z(\xi, \eta, \zeta, t), \\ t = t(\xi, \eta, \zeta, t). \end{cases}$$

The successive values of *x,y,z,t* for a specified material particle (ξ, η, ζ) now correspond in R_4 to a certain curve $C_{\xi,\eta,\zeta}$ – the world-line of that particle – and its equations in (5) are given for ξ, η, ζ (regarded as fixed) and τ (regarded as variable).

The motion of the entire continuum is thus represented in R_4 by a three-parameter family of curves $C(\xi, \eta, \zeta)$, which is exactly the ∞^3 world-lines of the particles of the continuum.

If we additionally presuppose that no particle of the continuum can travel by the speed of light or faster then the speed of light, then every line element of any curve is time-like. The definition of the "rigid" body that was given by M. Born from the standpoint of the relativity principle, can thus be formulated:

"The continuum is moving as a "rigid body", when in R_4 the world-lines $C(\xi, \eta, \zeta)$ of its points are equidistant curves."[4]

This means, the normal-distance of two infinitely adjacent curves should be constant along themselves, or in other words, the strip that is formed by two infinitely adjacent curves shall everywhere be of equal thickness.

To formulate this condition analytically, we calculate the line element *ds* by (5) in curvilinear coordinates ξ, η, ζ, τ. If we write for uniformities sake:

(6) $$\xi_1 = \xi,\ \xi_2 = \eta,\ \xi_3 = \zeta,\ \xi_4 = \tau,$$

then it shall be

(7) $$ds^2 = \sum_1^4 ij\ A_{ij} d\xi_i d\xi_j$$

Since after general presupposition of subluminal velocities, the elements of the curves $C(\xi, \eta, \zeta)$ are time-like, it is given:

(8) $$A_{44} < 0$$

by introduction of the linear differential form

(9) $$d\nu = A_{14} d\xi_1 + A_{24} d\xi_2 + A_{34} d\xi_3 + A_{44} d\xi_4$$

and the quadratic one that only contains $d\xi, d\eta, d\zeta$:

(10) $$d\sigma^2 = \sum_1^3 ij\ A_{ij} d\xi_i d\xi_j - \frac{1}{A_{44}} (A_{14} d\xi_1 + A_{24} d\xi_2 + A_{34} d\xi_3)^2$$

we can write:

(11) $$ds^2 = d\sigma^2 + \frac{1}{A_{44}} (d\nu)^2$$

If the element ds shall be normal to curve $C_{\xi,\eta,\zeta}$, then it must be:

(12) $$\frac{\partial ds^2}{\partial d\xi_4} = 0,\ i.e.\ d\nu = 0$$

and the normal-distance of curves $C_{\xi,\eta,\zeta}$ and $C_{\xi+d\xi,\eta+d\eta,\zeta+d\zeta}$ are equal to $d\sigma^2$.

The condition of rigidity thus reads:

(13) $$\frac{\partial}{\partial \tau} d\sigma^2 = 0$$

in other words, the six coefficients of the quadratic differential form $d\sigma^2$ must be independent from τ.

But also physically, that definition of rigidity can be formulated equally simple.

If the velocity of the particle (ξ, η, ζ) at time t is denoted by s, and its components by u,v,w:

(14) $$s^2 = u^2 + v^2 + w^2$$

then

(15) $$A_{44} = -\left(1 - s^2\right)\left(\frac{\partial t}{\partial \tau}\right)^2$$

(16) $$d\nu = (u\,dx + v\,dy + w\,dz - dt)\frac{\partial t}{\partial \tau}$$

If we put $dt = 0$ and consider all particles of the continuum at the same time t, then

(17) $$ds^2 = dx^2 + dy^2 + dz^2 = d\sigma^2 - \frac{(u\,dx + v\,dy + w\,dz)^2}{1 - s^2}$$

thus

(18) $$d\sigma^2 = dx^2 + dy^2 + dz^2 + \frac{(u\,dx + v\,dy + w\,dz)^2}{1 - s^2}$$

Obviously $d\sigma^2 = \epsilon^2$ is now the equation of an infinitely small ellipsoid of revolution of semi-axes $\epsilon, \epsilon, \epsilon\sqrt{1 - s^2}$, with the particle (ξ, η, ζ) as its center and its velocity direction as its figure-axis. The requirement $(\partial/\partial\tau) d\sigma^2 = 0$ thus means, that the volume elements (which are infinitely small spheres of radius ε in the state of rest) are transformed into an oblate ellipsoid of revolution, with a semi-minor axis $\epsilon\sqrt{1 - s^2}$ in the direction of velocity and a semi-major axis ε normally to it. In other words, the definition of a "rigid" body given by Born can be brought into this extremely suggesting form:

> "*If the velocity within the body is changing in space and time, then the Lorentz-Fitzgerald contraction hypothesis shall be valid for every single volume element.*"[5]

By that, the single volume element possesses a six-fold freedom of motion, because the ellipsoid additionally allows ∞^3 linear homogeneous deformations in it. But is the same true for a body continuously formed by such elements of finite extension throughout? This question shall be dealt with in the following section.

II. Determination of the equidistant families of curves of R_4, that contain an arbitrary given curve.

We imagine that within such a family of curves $C(\xi, \eta, \zeta)$, a certain curve C_0 is somehow chosen, and the parameter values (ξ_0, η_0, ζ_0) may be attributed to it.[6]

Then we use the quadratic differential form $d\sigma^2$ in $d\xi, d\eta, d\zeta$, whose components only depend on ξ, η, ζ, and which, as the square of the length of a space-like line element, has a definite positive character, and we introduce in it the variables *a,b,c* instead of ξ, η, ζ in the following way:

In the three-fold extended manifold of (ξ, η, ζ), we imagine the drawing of the two-parameter dependent family of curves of the extremal (propagating from point (ξ_0, η_0, ζ_0)) of the integral $\int d\sigma$ – the geodesic lines of form $d\sigma^2$ –, and let *b,c* be the values of both parameters for a line that goes through the point (ξ, η, ζ) of that family of curves, and a is the integral $\int d\sigma$ taken along it from (ξ_0, η_0, ζ_0) to (ξ, η, ζ) – the geodesic distance of both points. If (for that geodesic polar coordinate system *a,b,c*) it is given:

$$\left\{\begin{array}{l} \xi = \xi(a, b, c), \\ \eta = \eta(a, b, c), \\ \zeta = \zeta(a, b, c), \end{array}\right. \tag{19}$$

then the differences $\xi - \xi_0,\ \eta - \eta_0,\ \zeta - \zeta_0$ for a = 0 and all *b,c* will vanish, but when divided by a (with convenient choice of *b,c*) then for a = 0 and all *b,c* it will remain finite, and if $d\sigma^2$ is expressed by this variables, it will assume the form:[7]

$$d\sigma^2 = da^2 + \varphi(db, dc) \tag{20}$$

where $\varphi(db, dc)$ also means a definite positive quadratic differential in *db, dc* alone, whose components still contain *a,b,c*, but like the previous of $d\sigma^2$ don't contain τ. And instead of τ another magnitude ϑ of the following kind shall be introduced as well:

We imagine as determined that solution of the differential equation:

(21) $$\frac{d\tau}{da} = -\frac{1}{A_{44}}\left(A_{14}\frac{\partial\xi}{\partial a} + A_{24}\frac{\partial\eta}{\partial a} + A_{34}\frac{\partial\zeta}{\partial a}\right)$$

which assumes the value $\tau = \vartheta$ for a = 0:

(22) $$\tau = \tau(a, b, c, \vartheta)$$

and by this last equation, instead of τ we have introduced the parameter ϑ which becomes identical with τ specifically along the curve C_0.

Expressed by a,b,c,τ it will be:

(23) $$\frac{1}{\sqrt{-A_{44}}}\sum_1^4 i\ A_{i4} d\xi_i = B\, db + C\, dc + \Theta\, d\vartheta$$

and da will vanish from the linear differential expression dv.

If we replace in this way the parameter (ξ, η, ζ) [which are constant along any curve] by (a,b,c), and the parameter τ [which is variable along any curve] by ϑ, then it will be

(24) $$ds^2 = da^2 + \varphi(db, dc) - (B\, db + C\, dc + \Theta\, d\vartheta)^2$$

From this form we can immediately conclude[8], that the curves:

b = const., c = const., ϑ = const.

in $R_4(x, y, z, t)$ are extremals of the integral $\int ds$, which means that they are straight lines. Those ∞^3 straight lines $G(b, c, \vartheta)$ thus found, orthogonally intersect the equidistant curves $C(a,b,c)$, because of the missing term with $da\, d\vartheta$ in ds^2. If we especially take from them the ∞^2 straight line that belongs to the same ϑ-value, and imagine the point a = 0 as marked on any of them, then this point is nothing else than the point $\tau = \vartheta$ on the curve C_0 through which they all go, and because they must be perpendicular to curve C_0 at this point, they are exactly the ∞^2 perpendiculars of curve C_0 at point $\tau = \vartheta$. They together form the normal plane of curve C_0 at this point, with which the surface ϑ = const. is consequently identical.

The magnitude a is the length $\int ds$, calculated along the straight line $G_{b,c,\vartheta}$ beginning at the intersection point with curve C_0 – being the distance of the corresponding space point of curve C_0.

The summary of that yields, that the expressions of x,y,z,t by a,b,c,ϑ are necessarily of the form:

$$
\text{(25)} \qquad \begin{cases} x = x_0(\vartheta) + a x_1(b,c,\vartheta), \\ y = y_0(\vartheta) + a y_1(b,c,\vartheta), \\ z = z_0(\vartheta) + a z_1(b,c,\vartheta), \\ t = t_0(\vartheta) + a t_1(b,c,\vartheta). \end{cases}
$$

To simplify the notation, S shall be momentarily denoting a sum that is extended over the four coordinates x,y,z,t, in which, however, the term that is related to the t-coordinate has to be considered as negative.

Since the straight lines $G(b,c,\vartheta)$ are the normals of the curve C_0 (a = 0), it follows

$$
\text{(26)} \qquad \mathrm{S} x_1 \frac{\partial x_0}{\partial \vartheta} = 0
$$

and from that by differentiation to b and c it must be:

$$
\text{(27)} \qquad \mathrm{S} \frac{\partial x_0}{\partial \vartheta} \frac{\partial x_1}{\partial b} = \mathrm{S} \frac{\partial x_0}{\partial \vartheta} \frac{\partial x_1}{\partial c} = 0
$$

If the form ds^2 made by (25) will be identified with the expression (24), then the equations follow:

$$
\varphi(db,dc) - (B\,db + C\,dc)^2 =
$$

$$
\text{(28,α)} \qquad = a^2 \mathrm{S} \left(\frac{\partial x_1}{\partial b} db + \frac{\partial x_1}{\partial c} dc \right)^2 = a^2 \psi(db,dc),
$$

$$
\text{(28,β)} \qquad -B\Theta = a^2 \mathrm{S} \frac{\partial x_1}{\partial b} \frac{\partial x_1}{\partial \vartheta} = a^2 \beta,
$$

$$
\text{(28,γ)} \qquad -C\Theta = a^2 \mathrm{S} \frac{\partial x_1}{\partial c} \frac{\partial x_1}{\partial \vartheta} = a^2 \gamma,
$$

$$
\text{(28,δ)} \qquad -\Theta^2 = \mathrm{S} \left(\frac{\partial x_0}{\partial \vartheta} + a \frac{\partial x_1}{\partial \vartheta} \right)^2,
$$

in which the coefficients of the binary, quadratic differential form $\psi(db,dc)$ and the magnitudes β and γ are evidently independent from a.

Especially we think of ϑ as the "proper time" along the curve C_0, thus

$$\mathrm{S}\left(\frac{\partial x_0}{\partial \vartheta}\right)^2 = -1 \tag{29}$$

which can always be achieved by a convenient choice of τ, with which ϑ coincides along C_0.

From equations (28, δ, β, γ) it follows for a = 0, one after the other:

$$(\Theta)_0 = 1,\ \left(\frac{B}{a^2}\right)_0 = \beta,\ \left(\frac{C}{a^2}\right)_0 = \gamma, \tag{30}$$

and by recognizing, that

$$\left(\frac{B}{a}\right)_0 = \left(\frac{C}{a}\right)_0 = 0 \tag{31}$$

in the same way from (28,α):

$$\left(\frac{1}{a^2}\varphi(db, dc)\right)_0 = \psi(db, dc) \tag{32}$$

and this eventually gives by (28,α):

$$(B\ db + C\ dc)^2 = \varphi(db, dc) - a^2\left(\frac{1}{a^2}\varphi(db, dc)\right)_0 \tag{33}$$

However, as the coefficients of $\varphi(db, dc)$ are free from ϑ, then it follows that the magnitudes B, C and thus β, γ are independent from ϑ.

Now there are two possibilities:

A. *We have* B = 0, C = 0. Then

$$ds^2 = da^2 + \varphi(db, dc) - \Theta^2 d\vartheta^2, \tag{34}$$

from which we can see, that the areas ϑ = const. – *i.e.* the normal planes of curve C_0 – were orthogonally intersected by the curves $C(a,b,c)$. Thus it can be said:

The equidistant curves are the orthogonal-trajectories of a family of planes.

Conversely, also the orthogonal-trajectories of any family of planes form an equidistant family of curves, since also with respect to the measure-determination ds^2 in R_4, the theorem remains valid according to which the distance between two moving points is constant when their velocities are always normal to the connecting line.

B. *At least one of the magnitudes B,C is not zero*. Then it follows from (28,*β,γ*), that also *Θ* is independent from *ϑ*, and thus in the expression:

(35) $$ds^2 = da^2 + \varphi(db, dc) - (B\,db + C\,de + \Theta\,d\vartheta)^2$$

all coefficients are free from *ϑ* at all.

In this case, let us consider the one-parameter group of transformations of R_4, in which the point with parameter values (*a,b,c,ϑ*) goes over to those, to which the following parameter values belong:

(36) $$a' = a,\ b' = b,\ c' = c,\ \vartheta' = \vartheta + h$$

According to the things recognized above, it is given for those transformations:

(37) $$ds'^2 = ds^2$$

therefore they are motions in R_4, and since in those motions any single curve *C(a,b,c)* is evidently moved in itself, then the following statement can be made:

The equidistant curves are the trajectories of a one-parametric group of motion.

Conversely, also the trajectories of a one-parameter motion group always form an equidistant family of curves, since the sector that is limited by two infinitely adjacent curves can be moved in itself, thus it must have the same extensions everywhere.[9]

Having noticed this, we think of the curve C_0 as arbitrarily given, and we pose the exercise to find out all equidistant families of curves that belong to that curve.

By (A) we have an unequivocally defined solution in the orthogonal trajectories of the normal planes of C_0 in all cases. Shall there be others besides those, then the related families will necessarily be summarized under (B) and therefore curve C_0 must form a one-parameter group of motions in itself. Conversely, also the trajectories of any one-parameter group of motions of curve C_0 in itself (which don't fix C_0 pointwise) give a solution of the exercise, and by all existing groups of this kind also all other solutions are given.

In order to move C_0 in itself, it is necessary and sufficient, that the three curvatures[10] of that curve – that are invariants of motion and that let the curve remain fixed except its position in space – are constant along the curve, so that the curve is, so to speak, a helix. In addition, if C_0 shall have more than one such group of motions in itself, then there must be motions that let C_0 pointwise remain fixed, and any single one of such motion corresponds to another group of motions of the curve in itself. The fixpoints of any motion in R_4 now forms (here we can neglect the case of a single fixpoint) either a straight line or a plane R_3, and in reverse these formations remain fixed pointwise at ∞^3 or ∞^1 motions.

Depending on whether curve C_0 (with constant curvatures) exists in no space lower than R_3 or R_2 or eventually is a straight line, it has 1 or ∞^1 or ∞^3 one-parameter motion groups in itself, and exactly that is the number of the additional solutions of the exercise, that are given by the trajectories of that group.

If one takes into consideration, that C_0 is the image of a point of a "rigid" body – its world-line – then the answer can be given to the question after the freedom of motion of a "rigid" body:

In Born's kinematics of rigid bodies, the motion of the whole body is generally unequivocally determined by the arbitrarily defined motion of a single point of it.

An exception only takes place, when the world-line of that point in R_4 has constant curvatures, namely in this case – depending on the condition that it doesn't lie in a lower space than (at the most) R_3 or R_2, or eventually it is a straight line – there additionally exist 1 or ∞^1 or ∞^3 possible motions.

However, although by this result the immediately given purpose – the determination of the degrees of freedom of a "rigid" body – is achieved, it is apparently necessary to particularly consider all the possible forms of motion, especially also in the special cases. Although they cannot be used for a general definition of a "rigid" body, they have a special meaning from the standpoint of the relativity principle. It is therefore convenient to incorporate some simple facts of non-Euclidean geometry, which in any case can be used with advantage for questions concerning the theory of relativity – for example for the composition of velocities – as it will be shown at a specific place.

III. The Lorentz transformation and hyperbolic geometry in R_3.

The measure-determination introduced in $R_4(x, y, z, t)$ coincides in the bundle of ∞^3 – for example, the lines that emanate from the origin $O(x = y = z = t = 0)$ – with CAYLEY's metric that is based on the real minimal cone of that point as absolute cone. By projective representation of the line bundle at the points of R_3, it goes over into a real plane of second order, the measure-determination therefore goes over into CAYLEY's metric that is based on that real F_2. The rotations around *O* in R_4 – the Lorentz transformations – correspond to the motions in R_3 which are related to this hyperbolic measure-determination. To give this obvious connection a certain form and take advantage from it for the current purpose, we have to remind some known things of hyperbolic geometry.[11]

To project a line bundle of R_4 through *O* upon R_3 in a simple manner, one only has to set *x,y,z,t* equal to the homogeneous right angled coordinates z_1, z_2, z_3, z_4 in R_3:

(38)
$$z_1 = x,\ z_2 = y,\ z_3 = z,\ z_4 = t,$$

by which the minimal cone in R_3 corresponds to the unit sphere around the origin:

(39)
$$z_1^2 + z_2^2 + z_3^2 - z_4^2 = 0$$

so that it has to serve as absolute plane of the measure-determination that has to be introduced in R_3.

The one- two- and three-dimensional structures of R_4 (which are plane and directed through *O*), are corresponding to the points, straight lines and planes of R_3. By their orientation in relation to the sphere, they visualize the orientation of the corresponding structures to the minimal cone (for example time-like lines = inner points, space-like lines = outer points). Any such structure of R_4 is normal to each other, if the corresponding structures of R_3 are conjugated to each other as regards the polar connectivity at the sphere (for example, two lines perpendicular to each other = two lines of which all are located at the polar plane of the other). The four edges of a polar tetrahedron of the sphere are corresponding in R_4 to four straight lines that pass through *O* and that are mutually normal; if we choose them in a convenient order and take the x', y', z', t'-axis as direction (the corresponding edge in the interior of the sphere, of course, as t'-axis) of a new coordinate system, then those new coordinates x', y', z', t' are connected with the old ones *x,y,z,t* by a Lorentz transformation.

If in addition

(40)
$$p_{ik},\ p_{ik} + p_{ki} = 0, \qquad (i, k = 1, 2, 3, 4)$$

are the components of a vector of second kind (MINKOWSKI), then its two invariants are:

$$
(41) \qquad \begin{cases} D = p_{23}p_{14} + p_{31}p_{24} + p_{12}p_{34}, \\ \Delta = p_{23}^2 + p_{31}^2 + p_{12}^2 + p_{14}^2 + p_{24}^2 + p_{34}^2 \end{cases}
$$

and if one puts, in accordance with the reality relations of p_{ik} (where f_{ik} are to be understood as real magnitudes):

$$
(42) \qquad \begin{cases} p_{23} : p_{31} : p_{12} : ip_{14} : ip_{24} : ip_{34} \\ \qquad = f_{23} : f_{31} : f_{12} : f_{14} : f_{24} : f_{34}, \\ \qquad f_{ik} + f_{ki} = 0, \end{cases}
$$

then the vector of second kind can, regarding the relations of its components, illustrated by the linear complex of R_3:

$$
(43) \qquad \sum_1^4 ik\, f_{ik} \left(z_i z'_k - z'_i z_k\right) = 0
$$

As long as $D \neq 0$, one has a general complex, and then[12] there are two specified real straight lines that are conjugated polars, either with respect to the sphere or with respect to the complex. If one chooses a polar-tetrahedron of the sphere, from which two opposite edges coincide with this straight line, then in the corresponding system x', y', z', t' all components p'_{ik} of the vector of second kind will vanish, except of two whose values can immediately expressed by D and Δ.

However, if $D = 0$ (singular vector according to Minkowski), then this complex becomes a special one, consisting of all straight lines that intersect a specified straight line. This intersects, or is tangent to, or misses the sphere, depending (p_{23}, p_{31}, p_{12} considered as real) on whether $\Delta > 0,\ \Delta = 0,\ \Delta < 0$. The corresponding two-dimensional plane of R_4 can be used for clarification of the vector with respect to the relations of its components. Simultaneously, analogous to the preceding, by its aid the coordinate systems of R_4 can immediately be given in which as much as possible of the vector components will vanish.

Especially any infinitely small Lorentz transformation can be illustrated by a vector of second kind, the points of R_4 are all moving through it perpendicularly to its complex plane. Depending on whether $D \neq 0$ or = 0, then in R_4 only the origin or also the points of the two-dimensional plane (used above) remain fixed. The latter intersects, or is tangent to, or misses the minimal cone depending on whether $\Delta > 0, = 0, < 0$.

If one eventually imagine an arbitrary Lorentz transformation that transforms (x', y', z', t') into (x,y,z,t), then this obviously corresponds to a collineation of R_3 that transforms the unit sphere in itself - exactly a hyperbolic motion of R_3.

Conversely, all of such collineations correspond to two linear homogeneous transformations in *x,y,z,t* with determinant 1 which leave ds^2 unchanged. The identical collineation of z_i particularly corresponds to the two transformations:

$$\left\{\begin{array}{llll} x = \ \ x', & y = \ \ y', & z = \ \ z', & t = \ \ t' \\ x = -x', & y = -y', & z = -z', & t = -t' \end{array}\right. \tag{44}$$

From these, only the first is a Lorentz transformation, since the second one replaces the front- and back-cone of the point *O* against each other. *The Lorentz transformations correspond one-to-one to hyperbolic motions in* R_3.

Now in addition, the sphere will be transformed in itself by any of such a motion, so that the complex parameter (imagined as extended upon it):

$$\mathsf{Z} = \frac{z_1 + iz_2}{z_4 - z_3} = \frac{x+iy}{t-z} \tag{45}$$

is subjected to a linear substitution with generally complex coefficients (the additionally conjugated complex parameter, the conjugated complex substitution), and to all such substitutions corresponds a specified hyperbolic motion in R_3.

If x', y', z', t' are transformed into *x,y,z,t* by a Lorentz transformation, then the magnitudes:

$$\mathsf{Z} = \frac{x+iy}{t-z},\ \mathsf{Z}' = \frac{x'+iy'}{t'-z'} \tag{46}$$

are connected to each other by a linear substitution (with complex coefficients)

$$\mathsf{Z} = \frac{\alpha\mathsf{Z}' + \beta}{\gamma\mathsf{Z}' + \delta} \tag{47}$$

and all such substitutions are corresponding to a certain Lorentz transformation.[13]

If one wants, as it is necessary for the following, to write the one-parameter group of Lorentz transformations, then one only has to take the one-parameter group of linear substitutions in Z, and to form the corresponding Lorentz transformation. The first ones are now (by ϑ we think of the [real] parameter, by λ of an arbitrary real magnitude):

I.	$Z = Z' e^{(1+i\lambda)\vartheta}$	loxodromic	group
II.	$Z = Z' e^{i\vartheta}$	elliptical	″
III.	$Z = Z' e^{\vartheta}$	hyperbolic	″
IV.	$Z = Z' + \vartheta$	parabolic	″

They correspond to the following groups of Lorentz transformation (for the sake of convenience they shall be denoted by the same names):

I. *Loxodromic group*

$$\begin{aligned} x + iy &= (x' + iy')e^{i\lambda\vartheta}, & t - z &= (t' - z')e^{\vartheta}, \\ x - iy &= (x' - iy')e^{-i\lambda\vartheta}, & t + z &= (t' + z')e^{-\vartheta}, \end{aligned}$$

II. *Elliptic group*

$$\begin{aligned} x + iy &= (x' + iy')e^{i\vartheta}, & z &= z', \\ x - iy &= (x' - iy')e^{-i\vartheta}, & t &= t'. \end{aligned}$$

III. *Hyperbolic group*

$$\begin{aligned} x &= x', & t - z &= (t' - z')e^{\vartheta} \\ y &= y', & t + z &= (t' + z')e^{-\vartheta} \end{aligned}$$

IV. *Parabolic group*

$$\begin{aligned} x &= x' + \vartheta(t' - z'), & y &= y' \\ z &= z' + \vartheta x' + \frac{1}{2}\vartheta^2(t' - z'), & t - z &= t' - z'. \end{aligned}$$

One notices at once, that these four groups are only different from each other by the kind of vector of second kind that illustrates the corresponding infinitesimal transformation.

IV. The one-parametric groups of motions in R_4 and the corresponding forms of motion of "rigid" bodies.

If one writes for the sake of symmetry:

$$\begin{cases} x_1 = x, & x_2 = y, \quad x_3 = z, \quad x_4 = it, \\ x'_1 = x', & x'_2 = y', \quad x'_3 = z', \quad x'_4 = it', \end{cases} \tag{48}$$

then any motion in R_4 can be analytically expressed in the form of a linear substitution:

$$x_i = a_i + \sum_1^4 j\, a_{ij} x'_j, \qquad (i = 1, 2, 3, 4). \tag{49}$$

Then $|a_{ij}|$ is an orthogonal determinant of value +1, additionally a_{44} is real positive and the remaining magnitudes a_i, a_{ij} are purely imaginary or real, depending on whether they have the index 4 or not.

If one has by (49) a continuous family of motions that depend on parameter ϑ, that is, a_i and a_{ij} are functions of ϑ, then by differentiation with respect to ϑ while x_i remains constant:

$$\frac{dx'_i}{d\vartheta} + q_i + \sum_1^4 j\, p_{ij} x'_j = 0 \; (i = 1, 2, 3, 4) \tag{50}$$

where it is put:

$$\begin{cases} q_i = \sum_1^4 j\, a_{ij} \dfrac{da_{ij}}{d\vartheta}, & \\ & (i, j = 1, 2, 3, 4) \\ p_{ij} = \sum_1^4 k\, a_{ki} \dfrac{da_{kj}}{d\vartheta}, \qquad p_{ij} + p_{ji} = 0 & \end{cases} \tag{51}$$

therefore also the magnitudes q_i and p_{ij} are purely imaginary or real, depending on whether they have the index 4 or not.

If we interpret (49) as equations of the coordinate transformation from system *S(x,y,z,t)* into a system $S'(x', y', z', t')$ moving against it, then consequently $-q_i$ would be the components of the vector (first kind) of velocity of the origin O' of S' and $-p_{ij}$ would be the components of the vector (second kind) of the angular velocity of S' around O', both times taken by the axis of S'.

If the family of motions forms a group, then (by convenient choice of ϑ) q_i and p_{ij} are independent from ϑ, and conversely the integration of equations (50) always gives a group of motions for arbitrary values of p_i and p_{ij}.

The trajectories of the group – the trajectories of the points fixed at S' according to the interpretation above – which (as we know) form an equidistant family of curves, are illustrated by (49) with constant x_i' and variable ϑ; they of course only apparently depend on four parameters, but actually they depend only on three parameters.

Now, after these remarks, in order to write down the possible one-parameter groups of motion of R_4, and consequently in order to simultaneously write down the corresponding equidistant family of curves, one has to note that when the motions (49) form a group, then the same is true for the rotations:

(52) $$x_i = \sum_1^4 j\, a_{ij} x_j \ (i = 1, 2, 3, 4)$$

and this must be – in general only after performing a suitable Lorentz transformation to x_i and of the same to x_i' – identical with one of the four groups of Lorentz transformations specified in the preceding section; namely with the group I. II. III. IV., depending on whether it applies to the invariant D and Δ of the vector p_{ij}:

(53) $$\begin{cases} \text{I.} \quad D \neq 0, & \text{II.} \quad D = 0, \quad \Delta > 0, \\ \text{II.} \quad D = 0, \quad \Delta < 0, & \text{IV.} \quad D = 0, \quad \Delta = 0. \end{cases}$$

By that, the possible value systems of a_{ij} can immediately be given, though the values associated with a_i in all cases follow from (51):

(54) $$a_i = \sum_1^4 j\, q_j \int a_{ij} d\vartheta \ (i = 1, 2, 3, 4)$$

and specifically we can chose arbitrary constants for q_j, as long as:

$$\sum_1^4 i\left(\frac{dx_i}{d\vartheta}\right)^2 = \sum_1^4 i\left(q_i + \sum_1^4 j\, p_{ij} x'_j\right)^2 < 0 \tag{55}$$

at least for a certain field of values of x'_j. Now, furthermore the values of p_{ij} for the four groups are one after the other:

$$\left.\begin{array}{lll} \text{I.} & p_{21} = -\,p_{12} = \lambda, & p_{34} = -\,p_{43} = i \\ \text{II.} & p_{21} = -\,p_{12} = 1, & \\ \text{III.} & p_{34} = -\,p_{43} = i, & \\ \text{IV.} & q_{31} = -\,p_{13} = 1, & p_{41} = -\,p_{14} = i \end{array}\right\} \text{the remaining } p_{ij} \text{ are always equal to zero}$$

and in connection with (50) they teach us, that by a convenient change of system S' (as well as of the system S) we always can achieve to following more simple value systems of q_i:

$$\begin{array}{lllll} \text{I.} & q_1 = 0 & q_2 = 0, & q_3 = 0, & q_4 = 0, \\ \text{II.} & = 0, & = 0, & = 0, & = \delta i, \\ \text{III.} & = \alpha, & = 0, & = 0, & = 0, \\ \text{IV.} & = 0, & = \beta, & = 0, & = \delta i, \end{array}$$

Now, all this gives the following groups of motion, named after the rotation groups that are contained in them, together with the corresponding equidistant family of curves and motion types of the "rigid" body:

I. *Loxodromic group*

$$\begin{aligned} x + iy &= (x' + iy')e^{i\lambda\vartheta}, & t - z &= (t' - z')e^{\vartheta}, \\ x - iy &= (x' - iy')e^{-i\lambda\vartheta}, & t + z &= (t' + z')e^{-\vartheta}, \end{aligned} \tag{56}$$

If t = 0:

$$x = x_0,\ y = y_0,\ z = z_0,\ \vartheta = \vartheta_0,$$

then for the corresponding motion of a "rigid" body it follows:

(57)
$$\begin{cases} x+iy = (x_0+iy_0)\,e^{i\lambda u}, \qquad u = \lg \dfrac{\sqrt{z_0^2+t^2}-t}{z_0} \\ x-iy = (x_0-iy_0)\,e^{-i\lambda u}, \\ z = \sqrt{z_0^2+t^2}. \end{cases}$$

If one uses cylindrical coordinates $\varrho, \varphi, z, \left(x+iy=\varrho e^{i\varphi}\right)$, then these equations can also be written:

(58)
$$\begin{cases} \varrho = \varrho_0, \\ z = \dfrac{1}{2} u_0 \left(e^{\frac{\varphi-\varphi_0}{\lambda}} + e^{-\frac{\varphi-\varphi_0}{\lambda}}\right), \\ t = \sqrt{z^2-z_0^2}. \end{cases}$$

Thus the points of a "rigid" body are moving upon sphere cylinders around the *Z*-axis along curves, and which, by unwinding of the cylinder on a plane, go over into catenaries with the velocity:

(59)
$$s = \sqrt{\frac{\lambda^2\varrho_0^2+t^2}{z_0^2+t^2}}.$$

The points on the *Z*-axis are moving in this plane by the law denoted by Born as "hyperbolic motion": $z = \sqrt{z_0^2+t^2}$

Their world-lines are lying in R_2, while those of all other points belong to a space not lower than R_4.

II. *Elliptic group*

$$\begin{aligned} x+iy &= (x'+iy')e^{i\vartheta}, \qquad & z &= z', \\ x-iy &= (x'-iy')e^{-i\vartheta}, \qquad & t &= t'+\delta\vartheta. \end{aligned}$$

If t = 0:

$$x=x_0,\; y=y_0,\; z=z_0,\; \vartheta=\vartheta_0,$$

then for the corresponding motion of the "rigid" body it follows:

$$\left\{\begin{aligned} x+iy &= (x_0+iy_0)\,e^{i\frac{t}{\delta}}, \\ x-iy &= (x_0-iy_0)\,e^{-i\frac{t}{\delta}}, \\ z &= z_0. \end{aligned}\right. \tag{60}$$

Thus this body rotates like an ordinary rigid body uniformly around the *Z*-axis.

The world-line of the points on the *Z*-axis are straight lines, the world-lines of all other points belong each to R_3, but not to a space lower than it.

III. *Hyperbolic group.*

$$\begin{aligned} x &= x' + \alpha\vartheta, \qquad & t-z &= (t'-z')e^{\vartheta}, \\ y &= y', & t+z &= (t'+z')e^{-\vartheta}, \end{aligned}$$

If $t = 0$:

$$x = x_0,\ y = y_0,\ z = z_0,\ \vartheta = \vartheta_0,$$

then for the corresponding motion of the "rigid" body it follows:

$$\left\{\begin{aligned} x &= x_0 + \alpha\lg\frac{\sqrt{z_0^2+t^2}-t}{z_0}, \\ y &= y_0, \\ z &= \sqrt{z_0^2+t^2} \end{aligned}\right. \tag{61}$$

Thus its points are moving in planes normal to the *Y*-axis along catenaries:

$$z = \frac{1}{2}z_0\left(e^{\frac{x-x_0}{a}} + e^{-\frac{x-x_0}{a}}\right) \tag{62}$$

with the velocity:

$$s = \sqrt{\frac{a^2+t^2}{z_0^2+t^2}} \tag{63}$$

the world-lines of all points belong each to R_3, but not to a space lower than it.

IV. *Parabolic group.*

$$\begin{aligned} x &= x' + \vartheta\,(t'-z') + \frac{1}{2}\delta\vartheta^2, & y &= y' + \beta\vartheta, \\ z &= z' + \vartheta x' + \frac{1}{2}\vartheta^2\,(t'-z') + \frac{1}{6}\delta\vartheta^3, & t-z &= t'-z' + \delta\vartheta. \end{aligned}$$

Here:

$$\mathsf{S}\left(\frac{dx}{d\vartheta}\right)^2 = (t'-z')^2 + \beta^2 - \delta(2x'+\delta'), \tag{64}$$

thus it is necessarily $\delta \neq 0$.

If we put $\vartheta = \vartheta_0$:

$$x = x_0,\ y = y_0,\ z = z_0,\ t = t_0,$$

then the expressions above remain unchanged due to the group property, as long as one replaces $x', y', z', t', \vartheta$ by $x_0, y_0, z_0, t_0, \vartheta - \vartheta_0$ within them. If one particularly takes:

$$\vartheta_0 = -\frac{1}{\delta}(t'-z'),$$

then $t_0 = z_0$.

Since for the motion of a rigid body only the trajectories of the group are of importance, instead of

$$\vartheta + \frac{1}{\delta}(t'-z'),$$

one can again write ϑ and then formulate the equations for the motion of a "rigid" body as follows:

$$\left\{\begin{aligned} x &= x_0 + \frac{1}{2}\delta\vartheta^2, & y &= y_0 + \beta\vartheta, \\ z &= z_0 + x_0\vartheta + \frac{1}{6}\delta\vartheta^3, & t-z &= \delta\vartheta \end{aligned}\right. \tag{65}$$

Thus all its points are moving along space curves of third order with the velocity:

$$s = \frac{\sqrt{x^2 + 2\delta\,(x - x_0) + \beta^2}}{x + \delta} \tag{66}$$

The world-lines of all points belong each to R_3, but not to a space lower than it.

If one poses for clarities sake the question, which of the formed equidistant families of curves of class (B) simultaneously also belong to class (A), *i.e*, which are orthogonal trajectories of a family of planes, then for the magnitudes q_i and p_{ij} that belong to the corresponding group, it is given:

$$\sum_1^4 i\; q_i dx_i + \sum_1^4 ij\; p_{ij} x_i dx_j = \varphi\; d\psi \tag{67}$$

and for that it is necessary and sufficient, that the 4×5 matrix which emerges from the determinant $|p_{ij}|$ by addition of row q_i, has the rank 1 or 2. The related discussion teaches, that this only occurs for the hyperbolic group II. with α = 0, by which indeed the trajectories are the orthogonal trajectories of the planes $Az + Bt = 0$. The corresponding motion of the "rigid" body reads:

$$x = x_0,\; y = y_0,\; z = \sqrt{z_0^2 + t^2} \tag{68}$$

This most simple translational motion, which was also discussed by Born and denoted as "hyperbolic motion", it thus the only type of motion that simultaneously belong to classes (A) and (B).

Of course, the four types of motion of class (B) formed at this place, can be transformed by an arbitrary Lorentz transformation.

Anyway, due to the given composition it is easy (by a given motion of a point of the "rigid" body) to immediately and explicitly give the possible types of motion of class (B) in addition to the motion of class (A). If, for example, a point of the body is fixed, then its world-line is a straight line, but such one arises as a trajectory only in group II – which immediately gives the fact mentioned at the beginning, that a "rigid" body with a fixed point only can rotate around an axis that goes through it, like an ordinary rigid body.

It may be noticed at the end, that the determination of the always possible motion of a "rigid" body from the motion of one of its points that belong to class (A) – *i.e*, the determination in R_4 of the orthogonal trajectories of the normal planes of that point's world-line – can be traced back to the integration of a Riccatian equation.

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1. M. Born, Ann. d. Phys. **30**. p. 1. 1909.
2. After writing this treatise I became aware of a note by P. Ehrenfest, published in the issue of November 22, 1909 in Physik. Zeitschr., that directly points to this fact by showing in a very simple way, that a body which is once at rest cannot be set into uniform rotation.
3. H. Minkowski, Die Grundgleichungen der elektromagnetischen Vorgänge in bewegten Körpern, Gött. Nachr. 1908; Raum und Zeit, Vortrag, gehalten auf der 80. Naturforscherversammlung zu Köln. Leipzig 1909.
4. By this formulation, the formulas calculated by Born for the case of uniform translation can immediately be written, since the equidistant curves of the (*z,t*)-plane of measure-determination $ds^2 = dz^2 - dt^2$ are of course (analogues to $ds^2 = dz^2 + dt^2$) the orthogonal trajectories of a family of lines, which is exactly the meaning of Born's formulas.
5. The same remark was given by P. Ehrenfest, *l.c.* This is also immediately evident by geometry, if we consider the space-time line that corresponds to a volume element. If its perpendicular cross-section on one location is an infinitely small sphere, then this is because the world-line is equidistant at any place. The cross-section that is perpendicular to the *t*-axis is thus, of course, exactly the preceding ellipsoid.
6. The following (for clarities sake analytically formulated) considerations have a very simple geometrical meaning and are thus transferable on the equidistant family of curves of an arbitrary variational problem.
7. See. G. Darboux, Théorie générale des surfaces 2. livre V. chap. VIII.
8. See. G. Darboux, l.c.
9. In three-dimensional space of ordinary euclidean measure-determination, the equidistant curve systems are either orthogonal trajectories of a family of planes, or coaxial helixes of same pitch.
10. Concerning the differential geometry of curves in higher spaces, see for example G. Landsberg, Crelles Journ. **114**. One has, for achieving the expressions of the

curvatures for the measure-determination used here, only to replace one of the coordinates by *it*.

11. Compare for the complete section especially F. Klein, Nicht-Euklidische Geometrie, Autogr. Vorl., Göttingen 1893, as well as the short introduction concerning projective measure-determination in Fricke-Klein, Autormorphe Funktionen I. Primarily note the imaginative description of hyperbolic motions, which especially make clear the relations of Lorentz transformations.
12. Clebsch-Lindemann, Vorles. über Geometrie 2, 1. p. 343 ff.; see also F. Lindemann, Unendlich kleine Bewegungen und Kraftsysteme bei allgemeiner Maßbestimmung, Diss. Erlangen 1873.
13. For the relevant formulas, see F. Klein, l.c.. They can be written very compendious by using quaternions.
