Uniformly accelerated reference frames in special relativity

Edward A. Desloge and R. J. Philpott

Citation: Am. J. Phys. **55**, 252 (1987); doi: 10.1119/1.15197
View online: http://dx.doi.org/10.1119/1.15197
View Table of Contents: http://ajp.aapt.org/resource/1/AJPIAS/v55/i3
Published by the American Association of Physics Teachers

dition inherent in the KS transformation and is closely related to the fact that the two two-dimensional oscillators have the same angular momentum as shown in Eq. (34). Thus, it can be seen that the equivalence between the Coulomb–Kepler problem and a four-dimensional oscillator or a pair of two-dimensional oscillators results, respectively, in the separability of the problem in spherical polar coordinates or parabolic coordinates as shown in Eq. (32) in conjunction with Eqs. (33) and (34).

[1]See, for example, M. Boiteus, Physics **65**, 381 (1973); A. O. Barut, C. K. E. Schneider, and R. Wilson, J. Math. Phys. **20**, 2244 (1979); A. C. Chen, Phys. Rev. A **22**, 333 and 2901E (1980); **23**, 1655 (1981); **25**, 2409 (1982); **26**, 669 (1982); J. Math. Phys. **23**, 412 (1982); M. Kibler and T. Négadi, Lett. Nuovo Cimento **37**, 225 (1983); A. C. Chen and M. Kibler, Phys. Rev. A **31**, 3960 (1985).
[2]R. Rockmore, Am. J. Phys. **43**, 29 (1975).
[3]P. Kustaanheimo and E. Stiefel, J. Reine Angew. Math. **218**, 204 (1965).
[4]The coordinates $u^2 = \xi$, $v^2 = \eta$, and $\phi = \beta + \gamma$ are the squared parabolic coordinates. The coordinates X and Y in Ref. 2 correspond to s_1 and s_3 with $\beta = \gamma = 0$.
[5]It can be shown that $p_\phi = mr^2 \sin^2 \phi\, \dot{\phi} = 4s^2 p_\beta = 4s^2 p_\gamma$.
[6]See, for example, L. I. Schiff, *Quantum Mechanics* (McGraw-Hill, New York, 1958), p. 235; A. C. Chen, Am. J. Phys. **47**, 1073 (1979).
[7]H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, MA, 1950), Chap. 9.

Uniformly accelerated reference frames in special relativity

Edward A. Desloge and R. J. Philpott
Physics Department, Florida State University, Tallahassee, Florida 32306

(Received 5 March 1985; accepted for publication 5 March 1986)

A uniformly accelerated reference frame S is defined as a set of observers who remain at rest with respect to a given observer A who is accelerating at a constant rate with respect to the instantaneously comoving inertial frames. The one-dimensional uniformly accelerated reference frame S is considered. The world lines of A and the other observers making up S are determined. Coordinates useful for describing events in S are carefully defined and the transformation equations between different sets of them are derived. The variation with position in S of the speed and frequency of light waves is determined. The motion of a free-particle in S is determined. Various phenomena in S, ordinarily associated with general relativity, are considered, in particular the asymmetric aging of twins at rest at different positions and the existence of horizons.

I. INTRODUCTION

Most texts on special relativity leave one with the impression that the motion of objects can only be discussed from the point of view of an observer fixed in an inertial frame.[1] Although accelerated motions are discussed,[2] little is said of accelerated observers,[3] and even less of accelerated reference frames.

On the other hand, accelerated reference frames are considered in most general relativity texts, particularly in connection with the principle of equivalence. But there is very little discussion of the properties of such frames and most comments concerning them are vague and occasionally erroneous.[4]

This apparent lack of interest in accelerated reference frames or at least in the details of their properties is quite surprising in view of the fact that the accelerated reference frame is the bridge that Einstein used to go from special relativity to general relativity and also in view of the fact that the properties of such frames are extremely interesting.

In contrast to the above attitude in most textbooks, significant treatments of accelerated frames and related matters can be found in Møller,[5] Rindler,[6] and Misner *et al.*[7] A discussion of Møller's treatment can be found in Sears and Brehme[8] and a survey of the use of accelerated observers and accelerated reference frames in relation to the twin paradox is given by Marder.[9]

A number of instructive treatments have also appeared in this Journal. Lass[10] derives a useful transformation between inertial coordinates and a particular set of coordinates associated with an accelerated frame. Romain[11] discusses and analyzes Lass's transformation. Marsh[12] discusses transformations between inertial coordinates and coordinates associated with an accelerated frame from a more general point of view, and considers both Lass's transformation and Møller's transformation. Rindler[13] shows the relation between Kruskal space in general relativity and the accelerated reference frame. Hamilton[14] discusses in detail some of the observations that would be made in a uniformly accelerated frame. Good[15] considers in conjunction with a film clip some of the observations which would be made by an accelerated observer especially as it relates to the twin paradox.

One or more of the following criticisms can be made of each of the above treatments of accelerated frames: The approach is unnecessarily formal or abstract, key concepts are left undefined, a working knowledge of general relativity is assumed, no attempt is made to give a physical interpretation of the coordinates introduced, the relationship between different sets of coordinates used is not made, and no investigation of the properties of the frame is made.

In this article we consider, within the framework of the special theory of relativity, the nature and properties of uniformly accelerated reference frames. Our objective is (i) to define precisely what is meant by a uniformly accelerated reference frame; (ii) to show the physical meaning of the various coordinates associated with a uniformly accelerated reference frame; and (iii) to demonstrate the unusual properties associated with a uniformly accelerated reference frame; properties normally associated with general relativity, such as the variation in the speed and frequency of light waves with position; the asymmetrical aging of spatially separated twins who remain at rest with respect to one another; and regions into which an observer can enter but not leave.

We have tried to carefully define all quantities introduced, to derive results from as straightforward and elementary view as possible, and to avoid excessive formalism. We assume only a modest knowledge of special relativity theory, and have provided in Sec. III an outline of the results from special relativity which will be needed.

In order to keep the discussion as simple as possible we restrict ourselves to motion along a line in one dimension.

II. DEFINITIONS AND NOTATION

A. Units

To simplify the mathematical expressions which we obtain in this paper, we shall work in a system of units in which $c = g = 1$, where c is the speed of light and g is an arbitrary constant acceleration which we will introduce later. For $g = 9.5\ \mathrm{m/s^2}$, the unit of length is almost exactly one light year and the unit of time is almost exactly 1 yr. The technique for restoring the constants c and g in the resulting expressions can be found in the literature.[16]

B. Standard clocks and standard rods

We assume in what follows that we have available a large number of identically constructed infinitesimally small clocks, which we shall refer to as standard clocks, and which have the property that if any two of them are at some instant at rest with respect to one another at the same point, then at that instant their rates will be equal, regardless of their accelerations.

We also assume that we have available a large number of identically constructed infinitesimal rods which we shall refer to as standard rods, and which have the property that if any two of them are at some instant at rest with respect to one another at the same point, then at that instant their lengths will be equal, regardless of their accelerations.

C. Observers

An observer is considered to be an intelligent point particle,[17] equipped with a standard clock and a standard rod, and having the following properties: (i) negligible interaction with the surroundings, (ii) the ability to move arbitrarily consistent with the usual relativistic limits, and (iii) the ability to communicate with other observers by means of light signals. We assume the universe to be one-dimensional and to be filled with such observers.

D. Reference frames

Each observer defines a world line in space-time, and the world line of each observer can be broken down into a succession of instants in time. If we consider a continuous set of observers whose world lines for simplicity we assume do not cross, then an event can be specified by determining (i) the world line on which the event occurs, or equivalently the observer who is present at the event; and (ii) the point in time, on the above world line, at which the event occurs. Which observer is present at the event determines the "where" of the event, and the point on the observer's world line at which the event occurs determines the "when" of an event.

We define a reference frame to be a continuous set of observers who remain at rest with respect to one another.[18] An observer B will be considered to be at rest with respect to an observer A if it always takes the same amount of time as measured by A for a light signal to make a round trip from A to B and back again. As defined above, a frame of reference is a hypothetical physical entity and not a mathematical entity. A frame of reference is not the same as a set of coordinates. Methods for coordinating a frame of reference will be considered later.

III. INERTIAL FRAMES AND SPECIAL RELATIVITY

A. Introduction

In this section, we have gathered together some results from the special theory of relativity which will be useful to us in later sections. Although the choice and formulation of these results is somewhat unusual, we have made no attempt to provide a detailed justification of them. In most cases the necessary justification can be found in one or another of the many excellent books on special relativity. The purpose of this paper is not to investigate the foundations of special relativity, but to show that uniformly accelerated frames of reference can be handled quite naturally within the framework of special relativity. Implicit in our choice of results is the assumption that the special theory of relativity is properly defined as the theory of flat space-time and is not restricted, as required by some definitions, to observations made with respect to inertial frames of reference.

B. The existence of an inertial frame

We shall assume that there exists at least one frame having the following properties: (i) *if* A and B are arbitrary observers in the frame, L the distance between A and B determined using measuring rods, and $2T$ the time as measured by observer A for a light signal to go from A to B and back to A again, *then* the quantity $c \equiv L/T$ is a constant, that is, it has the same value for every choice of observers A and B in the frame; (ii) if length is defined using measuring rods or light signals, then the resulting geometry is Euclidean.

Any frame having the above properties will be called an inertial frame. The quantity c defined above will be called the speed of light in an inertial frame, and in our system of units has the value 1. We shall later generate, or at least indicate how to generate, with the help of a few additional postulates, a whole set of frames satisfying the above properties, but for our present purposes all we require is the existence of one such frame. We shall assume we have found one and we shall designate it as the inertial frame K.

One important consequence of the above definition is that if an observer in an inertial frame sends out light signals at equal intervals by his clock, they will be received by all other observers in the frame with the same frequency with which they were sent out. It follows that the standard clocks of the observers in an inertial frame can be synchronized.

C. Inertial coordinates

Since a given event will occur at a unique point on the world line of one of the observers making up the inertial frame K, it follows that the frame K can be used to distinguish one event from another. In order to utilize this fact we coordinate the frame K, that is, we assign a number to each observer making up the frame and to each point on his world line. This can be done as follows: (i) Pick one observer O called the reference observer and assign him the position value $x = 0$. (ii) Assign every other observer X a position value x depending on his distance from O measured either by light signals or with measuring rods. (iii) Divide the world line of each observer into equal time intervals using either the standard clock carried by the observer or by means of periodic signals sent out by one particular observer. (iv) Assign a time value to every point on the world line of the observer O by picking a reference point, assigning it the value $t = 0$, and then assigning every other point a value depending on the number of time intervals between it and the time $t = 0$. (v) Assign a time value to one point on the world line of every other observer X by proceeding as follows: Let O reflect a light pulse off X; if the light pulse is sent by O at time t_1 by his clock and the reflected pulse received by O at time t_2 by his clock, then let the time on X's clock at which he receives the pulse be assigned the value $(t_1 + t_2)/2$. (vi) Assign a time value to the remaining points on the world line of each observer either by measuring from the one previously assigned value or by further light signals from the observer O. We will assume that the above procedure, or a similar one, is familiar to the reader, but caution him that when one is dealing with frames other than inertial frames, the alternative procedures permitted in the above steps lead to alternate results.

In what follows we shall obtain all our results by starting with the frame K and we shall refer all results back to this frame for comparison. The letters x and t will be used to designate, respectively, the position and time coordinates with respect to this frame.

D. Basic postulates in the special theory of relativity

Instead of starting from the usual postulates of special relativity, that is, the existence and equivalence of inertial frames and the constancy of the speed of light with respect to these frames, we will base our approach on the assumption that there exists at least one inertial frame and the fact that, with respect to this frame, moving clocks slow down and moving rods contract. More specifically we assume, in addition to the existence of the inertial frame K, the following two postulates A and B:

(A) *If* X is an observer moving with a velocity $\dot{x}$ with respect to the inertial frame K, and $d\tau$ is the time between two neighboring points on the world line of X as measured by X using his standard clock, and dt is the time between the same two events as measured by the observers in K, *then* the times $d\tau$ and dt are related as follows:

$$dt = (1 - \dot{x}^2)^{-1/2}\, d\tau. \qquad (1)$$

(B) *If* X is an observer moving with velocity $\dot{x}$ with respect to the inertial frame K, and $d\sigma$ is the length of the standard rod carried by X, and dx is the distance between the ends of the rod measured at some time t by the observers in K, *then* the distances $d\sigma$ and dx are related as follows:

$$dx = (1 - \dot{x}^2)^{1/2}\, d\sigma. \qquad (2)$$

E. The existence of a set of inertial frames

An observer Y will be at rest with respect to an observer X if the time which elapses *on the clock of X* during the passage of a light signal from X to Y and back again always has the same value. An observer in the inertial frame K can use postulate (A) to determine whether or not Y is at rest with respect to X since postulate A enables the observer in K to determine how much time has elapsed *on the clock of X* between an event in which X sends a light signal to Y and the event in which X receives the reflected signal back again.

If in particular we consider two observers X and Y who at each instant of time t are moving with the same velocity $V(t)$ with respect to the inertial frame K, we will find that if $V(t)$ is a constant then Y will remain at rest with respect to X, but if $V(t)$ is not a constant, then, surprisingly and contrary to what many authors implicitly assume, Y will not be at rest with respect to X.

From the above it follows that a continuous set of observers who are all moving with the same *constant* velocity V with respect to the inertial frame K will constitute a frame of reference, but a continuous set of observers who are all moving with the same *nonconstant* velocity $V(t)$ will not. Furthermore, by exploiting assumptions (A) and (B) it is possible not only to show that the set of all observers moving with a particular constant velocity V with respect to the inertial frame K constitutes a frame of reference, but also that it satisfies the definition of an inertial frame, hence it is an inertial frame.

When inertial frames other than the frame K are used they will be labeled K', K'', ... and we shall assume they have been coordinated in a fashion similar to that employed in the coordination of K. Furthermore, we shall assume that the reference observers and the zero setting on the clocks have been chosen such that the events $(x,t) = (0,0)$, $(x',t') = (0,0)$, ... all correspond to the same event.

F. The Lorentz transformation

Given the inertial frame K and a second inertial frame K' which is moving with a constant velocity V with respect to K, it can be shown from postulates (A) and (B) that the transformation between the coordinates (x,t) in frame K and the coordinates (x',t') in frame K' is given by the Lorentz transformation

$$x' = (x - Vt)/(1 - V^2)^{1/2}, \qquad (3)$$

$$t' = (t - Vx)/(1 - V^2)^{1/2}. \qquad (4)$$

From Eq. (3) it follows that the world lines of the observers making up frame K', that is the lines of constant x', are given by the family of equations $x - Vt =$ constant and from Eq. (4) it follows that the lines of constant t' are given by the family of equations $t - Vx = \text{constant}$. Since a line of constant t' does not coincide with any line of constant t, it follows that spatially separated events which are simultaneous in K' are not simultaneous in K.

IV. UNIFORMLY ACCELERATED OBSERVERS

A. The observer A

Let us consider a particular observer, designated as observer A, whose motion is such that at each instant the set of observers making up the inertial frame with respect to which he is at that instant at rest measure his acceleration to be g, where g is a constant whose value in the units introduced earlier is 1. We shall abbreviate the above statement by saying simply that A is moving with a constant acceleration g with respect to the instantaneously comoving inertial frames.

Let us furthermore choose the reference observer O in frame K and the setting on his clock such that the event corresponding to A being at rest with respect to K has the coordinates $t = 0$ and $x = c^2/g$ or in terms of the units introduced earlier $t = 0$ and $x = 1$.

B. The motion of observer A

Given the observer A we first want to determine his world line $x(t)$ with respect to the inertial frame K. To obtain this we introduce a second arbitrary inertial frame K' that is moving with a velocity V with respect to K. If we knew $x(t)$ for observer A we could find his acceleration with respect to the frame K' by making use of the following equation which can be obtained directly from the Lorentz transformation:

$$\frac{d^2x'}{dt'^2} = \frac{\ddot{x}(1 - V^2)^{3/2}}{(1 - V\dot{x})^3}. \tag{5}$$

Since A is undergoing constant acceleration of unit magnitude with respect to the instantaneously comoving inertial frame it follows that if $\dot{x} = V$ then $d^2x'/dt'^2 = 1$. Putting these values in Eq. (5) we obtain

$$\ddot{x} = (1 - \dot{x}^2)^{3/2}. \tag{6}$$

Equation (6) governs the motion of A with respect to K. Solving for x and noting that when $t = 0$ then $x = 1$ and $\dot{x} = 0$, we obtain

$$x = (1 + t^2)^{1/2}, \tag{7}$$

which is the equation of a hyperbola. As t approaches infinity the velocity of A with respect to K asymptotically approaches the value 1 which in our units is the speed of light. The world line of A as given by Eq. (7) corresponds to the line passing through the point $(x,t) = (1,0)$ in Fig. 1.

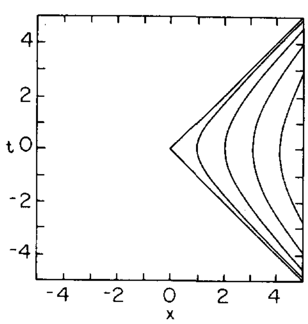

Fig. 1. The world line passing through the point $(x,t) = (1,0)$ is the world line of the uniformly accelerated observer A; see Sec. IV. The remaining world lines are the world lines of observers who remain at rest with respect to A; see Sec. V.

C. Proper time for observer A

If the observer A is moving with a velocity $\dot{x}$ with respect to the inertial frame K, and $d\tau$ is the time between two neighboring points on his world line as measured by him using his standard clock, then the time dt between the same two events as measured by the observers in K is given by Eq. (1). Substituting Eq. (7) in Eq. (1), integrating, and assuming that the clock carried by A is set to zero when A is at rest with respect to K we obtain

$$\tau = \sinh^{-1} t. \tag{8}$$

Equation (8) gives us the relationship which the observer A would find between the reading τ on his clock and the reading t on the clock of the observer in frame K with whom he is at that moment coincident.

V. UNIFORMLY ACCELERATED REFERENCE FRAMES

A. Accelerated reference frame

A reference frame has been defined as a set of observers who remain at rest with respect to one another.[18] We are interested in finding the set of observers who remain at rest with respect to the observer A. One's initial inclination is to assume that any observer who is initially at rest in K at time $t = 0$ and undergoes the same uniform acceleration as A will remain at rest with respect to A. But this is not true, as we shall now show. Let R be an arbitrary observer and $x(t)$ the world line of R. Suppose that at some arbitrary time t_1 the observer A sends a light signal to R which is received at time t_2 and immediately reflected back and received by A at time t_3. Let us further assume that R is located further from the origin of K than A. The position of A at time t_1 is given by $(1 + t_1^2)^{1/2}$, the distance traveled by the light signal between t_1 and t_2 is $t_2 - t_1$, and the position of R at time t_2 is $x(t_2)$. It follows that

$$(1 + t_1^2)^{1/2} + (t_2 - t_1) = x(t_2). \tag{9}$$

Similarly, for the return trip we obtain

$$x(t_2) - (t_3 - t_2) = (1 + t_3^2)^{1/2}. \tag{10}$$

Rearranging Eqs. (9) and (10) we obtain

$$(1 + t_1^2)^{1/2} - t_1 = x(t_2) - t_2, \tag{11}$$

$$(1 + t_3^2)^{1/2} + t_3 = x(t_2) + t_2. \tag{12}$$

Substituting Eq. (8) in Eqs. (11) and (12) and noting that $(1 + \sinh^2 \tau)^{1/2} = \cosh \tau$, we obtain

$$\cosh \tau_1 - \sinh \tau_1 = x(t_2) - t_2, \tag{13}$$

$$\cosh \tau_3 + \sinh \tau_3 = x(t_2) + t_2, \tag{14}$$

or equivalently,

$$e^{-\tau_1} = x(t_2) - t_2, \tag{15}$$

$$e^{\tau_3} = x(t_2) + t_2. \tag{16}$$

Taking the product of Eqs. (15) and (16) we obtain

$$e^{\tau_3 - \tau_1} = x^2(t_2) - t_2^2. \tag{17}$$

If observer R is at rest with respect to A then, irrespective of the time t_2 at which R receives the signal from A, the elapsed time $\tau_3 - \tau_1$ for the round trip as observed by A with his clock should be constant. Setting $t_2 = t$ and $\exp(\tau_3 - \tau_1) = \sigma^2 = \text{constant}$ in Eq. (17) we obtain, for the equation of the world line of R,

$$x = (\sigma^2 + t^2)^{1/2}. \tag{18}$$

The above world line is that of an observer who at time $t = 0$ is at rest at the point $x = \sigma$ and is accelerating with a constant acceleration σ^{-1} with respect to the instantaneously comoving inertial frame. If we had initially assumed that R was located closer to the origin of K than A rather than further away, we would have obtained the same result.

From the above it follows that the set of observers obtained by letting σ range from $0-\infty$ in Eq. (18) constitutes a reference frame, which we will refer to as the uniformly accelerated reference frame S. The world lines in frame K of some of the observers making up frame S are shown in Fig. 1. Note that the observer for whom $\sigma = 0$ is moving with the speed of light at all times other than at $t = 0$.

B. The invariance of the proper distance

Equation (18) describes the world line of an observer R who is one of the observers making up the uniformly accelerated frame S. If we make a Lorentz transformation from the coordinates x,t in the inertial frame K to the coordinates x',t' in another inertial frame K', the equation of the world line of observer R assumes identically the same form. This can be explicitly verified by making the transformation or simply by noting that the quantity $x^2 - t^2$ is an invariant under a Lorentz transformation. It follows that the accelerated frame is always at rest with respect to some member of the set of inertial frames. Furthermore, the distance between any two observers R and R′ measured in the inertial frame with respect to which S is momentarily at rest will always have the same value irrespective of the inertial frame with respect to which it is at rest. We can express this fact by saying that the proper distance between any two observers in S remains constant.

VI. COORDINATE SYSTEMS FOR ACCELERATED REFERENCE FRAMES

A. Coordinate systems

In order to discuss the properties of the accelerated reference frame S we will find it convenient to introduce a number of different coordinate systems.

Let us imagine the reference frame S to be an infinitely long rocket which at $t = 0$ is at rest in frame K with its back end located at $x = 0$ and its front end at $x = +\infty$. Let the rocket be divided into a series of compartments or cabins, each of which is assigned a cabin number σ equal to its x position at $t = 0$, and each of which has mounted on the wall a standard clock whose reading we designate τ, and which is set so that $\tau = 0$ when $t = 0$. The pilot of the rocket, who is identified with the observer we previously designated as observer A, is located in cabin 1; that is the cabin for which $\sigma = 1$. Let each cabin also contain a computer with monitor which links the cabin with the pilot's cabin, and let the computer be programmed in such a way that the monitor displays one constant number r and one varying number θ determined as follows. If R is an arbitrary cabin and A the pilot's cabin, and if a light signal is sent from A at time τ on A's cabin clock to cabin R is reflected back and arrives again at A at time $\tau + \Delta\tau$, then the number r appearing on R's monitor is $+\Delta\tau/2$ if $\sigma > 1$ and $-\Delta\tau/2$ if $\sigma < 1$, and the number θ appearing on R's monitor at the instant the above signal from A arrives at R is $\tau + (\Delta\tau/2)$. Thus the magnitude of the monitor reading r is the distance the pilot A judges R to be from him under the assumption that the speed of light is constant, and the monitor reading θ is the projection of A's clock into cabin R under the same assumption.

Each of the six coordinates introduced above, namely x, t, σ, τ, r, and θ gives us one characteristic of an arbitrary event. Suppose for example an event occurs. Then x is the position of the event as recorded by an observer in K, t is the time of the event as recorded by an observer in K, σ is the number of the cabin in which the event occurs, τ is the time of the event as recorded by the cabin clock in the cabin in which the event occurs, r is the fixed number appearing on the monitor in the cabin in which the event occurs, and θ is the time of the event as recorded by the monitor clock in the cabin in which the event occurs.

It is possible to uniquely identify a given event by any of a variety of pairs of the quantities, x, t, σ, τ, r, and θ. We shall consider only the pairs (x,t) (σ,t), (σ,τ) (σ,θ), and (r,θ). The pairs (x,t), (σ,θ), and (r,θ) are of particular physical significance. The pairs (σ,t) and (σ,τ) are chosen because they are theoretically useful in calculations we will be making.

B. Coordinate transformations

In order to be able to make maximum use of the different coordinate systems introduced above we need to know the equations of transformation between one set and another. This information is obtained below.

Equation (18) gives us $x(\sigma,t)$. If we substitute Eq. (18) in Eq. (1) and integrate we obtain

$$\tau = \sigma \sinh^{-1}(t/\sigma). \tag{19}$$

Combining Eqs. (18) and (19) we obtain

$$x = \sigma \cosh(\tau/\sigma), \tag{20}$$

$$t = \sigma \sinh(\tau/\sigma). \tag{21}$$

Now let us consider the following three events: (i) observer A sends out a light signal at time τ_1, (ii) an observer R who is located in the positive x direction from A receives the signal and returns it to A, and (iii) observer A receives the return signal at time τ_3. The given information uniquely determines all the coordinates of event (ii) in terms of τ_1 and τ_3. To determine these coordinates we note first that Eqs. (15) and (16) can be combined to give

$$x_2 = \tfrac{1}{2}(e^{\tau_3} + e^{-\tau_1}), \tag{22}$$

$$t_2 = \tfrac{1}{2}(e^{\tau_3} - e^{-\tau_1}). \tag{23}$$

If Eqs. (22) and (23) are now substituted into Eqs. (20) and (21) we obtain

$$\sigma_2 = (e^{\tau_3 - \tau_1})^{1/2}, \tag{24}$$

$$\tau_2 = (e^{\tau_3 - \tau_1})^{1/2}[\tfrac{1}{2}(\tau_3 + \tau_1)]. \tag{25}$$

But from the definition of r and θ, we have

$$r_2 = \tfrac{1}{2}(\tau_3 - \tau_1), \tag{26}$$

$$\theta_2 = \tfrac{1}{2}(\tau_3 + \tau_1). \tag{27}$$

Combining Eqs. (24)–(27) we obtain

$$r_2 = \ln \sigma_2, \tag{28}$$

$$\theta_2 = \tau_2/\sigma_2. \tag{29}$$

By proper choice of τ_1 and τ_3, event (ii) can be made arbitrary. Hence

$$r = \ln \sigma, \tag{30}$$

$$\theta = \tau/\sigma. \tag{31}$$

The same result would have been obtained if we had initially assumed that R was in the negative direction with respect to A rather than in the positive direction.

Equations (20), (21), (30), and (31) can be solved to obtain any one of the six unknowns x, t, σ, τ, r, and θ in terms of any one of the pairs of variables (x,t), (σ,t), (σ,τ), (σ,θ), and (r,θ). Hence we can obtain the equations of transformation between any of the above pairs of variables. The results can be summarized in the following set:

$$x = (\sigma^2 + t^2)^{1/2} = \sigma \cosh(\tau/\sigma) = \sigma \cosh \theta = e^r \cosh \theta, \tag{32}$$

$$t = \sigma \sinh(\tau/\sigma) = \sigma \sinh \theta = e^r \sinh \theta, \tag{33}$$

$$\sigma = (x^2 - t^2)^{1/2} = e^r, \tag{34}$$

$$\tau = (x^2 - t^2)^{1/2} \tanh^{-1}(t/x) = \sigma \sinh^{-1}(t/\sigma) = \sigma\theta = e^r\theta, \tag{35}$$

$$r = \tfrac{1}{2} \ln(x^2 - t^2) = \ln \sigma, \tag{36}$$

$$\theta = \tanh^{-1}(t/x) = \sinh^{-1}(t/\sigma) = \tau/\sigma. \tag{37}$$

Note that the transformation from any one member of the set of pairs (x,t), (σ,t), (σ,τ), (σ,θ), and (r,θ) to any other member of the set of pairs can be read off immediately from the set of Eqs. (32)–(37).

The transformation between (x,t) and (r,θ) is the same transformation used by Lass[10]; hence Lass's coordinates are the coordinates which would occur if distances and times were measured by means of light signals sent out by a particular observer. The transformation between (x,t) and (σ,θ) corresponds to Møller's transformation[5] for the case of uniform acceleration; hence Møller's coordinates are the coordinates which would occur if distances were measured with standard rods, and times by means of light signals sent out by a particular observer.

From Eq. (37) it follows that those points in (x,t) space for which θ has some constant value θ_0 define the curve $t = (\tanh \theta_0)x$, which is a straight line through the origin with the slope $|dt/dx| = |\tanh \theta_0| \leqslant 1$. But if (x',t') are the coordinates associated with an inertial frame K' moving with velocity V with respect to K, then those points in (x,t) space for which $t' = 0$ define the curve $t = Vx$, which is also a straight line through the origin, in this case with slope $|dt/dx| = |V| \leqslant 1$. It follows that when the rocket frame S is at rest with respect to one of the inertial frames K' then all of the observers in S will note the same time reading θ on the monitors. The coordinate θ is thus a logical choice for defining simultaneity in S.

C. Local coordinate system

In an inertial frame, distances measured with standard rods and distances measured with light signals give identical results. Furthermore, the clocks of the individual observers making up an inertial frame can be synchronized and will remain synchronized. Neither of the above facts is true for an accelerated frame. However, if in an accelerated frame we restrict ourselves to small distances and short intervals of time, then as we shall show below both of the above conditions are approximately satisfied, and in the limit, as the distances and intervals of time approach zero, the conditions are exactly satisfied.

The distance $\Delta\sigma = \sigma_2 - \sigma_1$ between two points σ_1 and σ_2 in the frame S is the distance that would be obtained if we used a standard measuring rod to measure the distance. The distance $\Delta r = r_2 - r_1$ between two points r_1 and r_2 in the frame S is the distance that would be obtained by the observer A using light signals. The distance from the observer A to an arbitrary point in the frame when measured with standard measuring rods is $\sigma - 1$ and the same distance when measured with light signals is r. If we take the ratio of these two distances we obtain

$$\frac{\text{measuring rod distance}}{\text{light signal distance}} \equiv \alpha = \frac{\sigma - 1}{r} = \frac{e^r - 1}{r} = 1 + \frac{r}{2!} + \frac{r^2}{3!} + \cdots . \tag{38}$$

It follows that as long as $r \ll 1$ the ratio α is approximately one. The condition $r \ll 1$ when converted to dimensional form becomes $r \ll c^2/g$. Hence if g were equal to 9 m/s^2 then as long as $r \ll 10^{16}$ m (which is approximately one light year) the ratio α would be approximately one.

If two events occur at a given point σ in frame S then the time interval $\Delta\tau = \tau_2 - \tau_1$ between the events is the time interval that would be noted by an observer R located at the event and using his own clock; the time interval $\Delta\theta = \theta_2 - \theta_1$ between the two events is the time interval that would be noted by observer A using his own clock and light signals. Consider event (ii) to be an arbitrary event at point σ and event (i) to be the event which occurred at σ when frame S was at rest with respect to K. Then $\tau_1 = 0$, $\tau_2 = \tau$, $\theta_1 = 0$, $\theta_2 = \theta$. Hence

$$\frac{\text{time interval as measured by A}}{\text{time interval as measured by R}} \equiv \beta = \frac{\tau}{\theta} = \sigma = e^r = 1 + r + \frac{r^2}{2!} + \cdots . \tag{39}$$

It follows that as long as $r \ll 1$ the ratio β is approximately one. This is the same result as obtained previously. Note also that

$$\tau - \theta = e^r\theta - \theta = \theta\,(e^r - 1) = \theta(r + r^2/2! + \cdots). \tag{40}$$

Hence the respective clocks of observers R and A, which were initially synchronized, gradually get out of synchronization. In particular, if $g = 9$ m/s^2 then two clocks which are located a distance of 1 m apart will get out of synchronization by about 1 s every 10^{16} s or about 1 s every 3×10^8 yr.

VII. THE PROPERTIES OF A LIGHT WAVE

A. The velocity of light

Let us consider a light pulse which is emitted from $\sigma = 1$ at an arbitrary time. The path of the pulse is given by any one of the following equations:

$$x = A \pm t \tag{41}$$

$$\sigma = A \exp(\pm\theta), \tag{42}$$

$$r = \ln A \pm \theta, \tag{43}$$

where A is a constant whose value depends on the time at which the pulse is emitted and the direction in which it is emitted. It follows that the speed of the pulse is given by

$$\frac{dx}{dt} = \pm 1, \tag{44}$$

$$\frac{d\sigma}{d\theta} = \pm\sigma, \tag{45}$$

$$\frac{dr}{d\theta} = \pm 1. \tag{46}$$

Hence if (x,t) or (r,θ) coordinates are used, the light pulse moves with constant speed. If (σ,θ) coordinates are used, the speed of the light pulse depends on its location, and ranges in value from zero when σ is zero to infinity when σ is infinite. The above differences in the values of the velocity of light arise because the above coordinate systems represent different physical methods of measuring time and distance. The coordinates (r,θ) are the coordinates which would result if an arbitrary observer, observer A in this case, uses the round trip time for a light signal to measure distance, and the time elapsed on his clock to measure time. Obviously the speed of light will in this case by definition come out equal to unity. The coordinates (σ,θ) on the other hand are the coordinates which would result if an arbitrary observer, observer A in this case, uses measuring rods to measure distances, and the time elapsed on his clock to measure time.

B. Constancy of the speed of light

Einstein's second postulate, concerning the constancy of the speed of light, is stated in a variety of ways. Most authors[19] state it essentially as follows: "The speed of light in a vacuum is the same in all inertial frames of reference, regardless of the motion of the light source." Some,[20] however, state it in a stronger form: "The speed of light in a vacuum is the same for all observers, regardless of their motion or the motion of the light source." The former statement is obviously correct, but in light of the analysis in the preceding sections the latter stronger statement must be highly qualified. It can obviously be justified if one assumes that the observer uses reflected light signals to measure the distance to a point P, and then measures the speed of light by determining the time it takes by his clock for a light signal to get to P and back. But in this case the speed of light is the same for all observers by definition.[21] Such a procedure relegates measuring rods to a useless status and gives us no physical information. On the other hand, the strong statement can be alternatively justified if one assumes the observer measures only the local speed of light by placing his clock at one end of his rod and then measuring the time it takes by his clock for a signal to go from the given end of his measuring rod to the other end and back. In the limit, as the length of the standard rod approaches zero the speed of light thus measured will be the same for all observers. With this interpretation of the meaning of the speed of light, the statement that the speed of light is the same for all observers has some physical content. This can be seen by noting that if the length of the measuring rod is increased then the speed of light so measured will change if the observer is one of the observers making up an accelerated frame, but will not change if the observer is one of the observers making up an inertial frame of reference.

C. Variation in the frequency of a light wave with position

Let us suppose the observer A sends out a series of signals which are one unit of time apart according to his proper time. When the observer R located at position σ receives these signals they will arrive one unit of θ time apart, but since $\theta = \tau/\sigma$ they will arrive σ units of time τ apart, where τ is the proper time of the observer R. Hence to the observer R located at σ the frequency of the signals will be

$$\nu = \sigma^{-1}. \tag{47}$$

Thus for an observer for whom $\sigma < 1$ the incoming signal will be blue shifted, and for an observer for whom $\sigma > 1$ the incoming signal will be red shifted. By the same argument, if the observer A is receiving a periodic signal from an observer R for whom $\sigma < 1$ the signal will be red shifted, and if the observer at A is receiving a periodic signal from an observer R for whom $\sigma > 1$ the signal will be blue shifted. It follows that the observer A assumes that clocks located at $\sigma < 1$ are running slow, and those located at $\sigma > 1$ are running fast. In the rocket analogy introduced earlier, if $\sigma < 1$ the cabin clocks will run slower than the monitor clock, and if $\sigma > 1$ the cabin clocks will run faster than the monitor clock.

VIII. FORCE-FREE MOTION OF A PARTICLE

Consider a free-particle which at time $t = 0$ is located at the point $x = 1$ and is moving with a velocity $x = V$, where, by virtue of the limiting speed which such a particle can have, the value of V must lie between -1 and $+1$. These are the initial conditions which would prevail if the observer A projected a particle from himself at time $\theta = 0$.

In terms of the set of coordinates (x,t), the set of coordinates that most naturally would be used by the observers in the inertial frame K, the world line of the particle is given by

$$x = 1 + Vt. \tag{48}$$

Thus in the inertial frame K the particle moves with a constant velocity V away from the point $x = 1$.

In terms of the set of coordinates (σ,θ), one of the sets of coordinates that would be used naturally by the observers in the uniformly accelerated frame S, the world line of the particle is given by

$$\sigma = 1/(\cosh\theta - V\sinh\theta). \tag{49}$$

It follows that in the accelerated frame S the motion appears quite different than in the inertial frame K. World lines for the particle in terms of the set of coordinates (σ,θ) are shown in Fig. 2 for various values of V. If $V = 1$ then $\sigma = e^{\theta}$. If $V = -1$ then $\sigma = e^{-\theta}$. If $0 < V < 1$ then the particle rises to a maximum at $\sigma = (1 - V^2)^{-1/2}$ arriving at time $\theta = \tanh^{-1} V$, and subsequently descends toward $\sigma = 0$ but never arrives. If $-1 \leq V \leq 0$ then the particle simply descends toward $\sigma = 0$ but never arrives.

Of particular interest is the time it takes the particle to go from $\sigma = 1$ to $\sigma = 0$. If the time is measured by the observers in the frame K, the time taken is $(1 - V)^{-1}$. This value is obtained by determining the time at which the curve

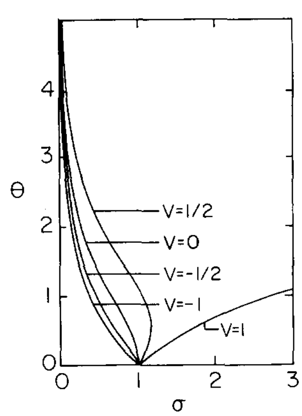

Fig. 2. World lines in (σ,θ) space of free-particles projected from the point $\sigma = 1$ at time $\theta = 0$ with varying initial velocities V.

$x = t$, which is the world line of the observer at $\sigma = 0$, intersects the path of the falling particle as given by Eq. (48). On the other hand, if the time it takes the particle to go from $\sigma = 1$ to $\sigma = 0$ is measured by the observer A using light signals, then the time taken is just the elapsed θ time, which as we have seen above is infinite. Hence the observer A judges that the particle continues to get closer to $\sigma = 0$ but never gets there. Finally, if we consider the projected particle to be an observer, then by his clock the elapsed time will be $(1 + V)^{1/2}/(1 - V)^{1/2}$, which is obtained by integrating the expression $d\tau = (1 - V^2)^{1/2}\, dt$ [see Eq. (1)] between $t = 0$ and $t = (1 - V)^{-1}$.

From the above it follows that if an observer initially located at $\sigma = 1$ in frame S steps out of the frame at $\theta = 0$ and allows himself to fall freely, then he simply steps into frame K and remains at rest in this frame, and will arrive at $\sigma = 0$ after one unit of his time. But from the point of view of the observer A who remains at $\sigma = 1$, the falling observer falls forever and never arrives at $\sigma = 0$.

IX. ASYMMETRICAL AGING OF TWINS

A. Introduction

The fact that two twins A and B who start off together at rest in an inertial frame K, then separate and rejoin one another at some later time may find themselves to have aged differently has been a source of great fascination to both physicists and nonphysicists. According to many textbooks a complete analytical explanation of this phenomenon requires the use of accelerated reference frames, and the handling of accelerated reference frames requires the use of the general theory of relativity. Neither of these statements is true. In the preceding sections we have shown that uniformly accelerated reference frames can be fully treated within the framework of special realtivity. And as emphasized by numerous authors,[22] it is unnecessary to resort to the properties of accelerated frames to handle the asymmetric aging. If one accepts the fact that relative to an inertial frame moving clocks run slow, and that the slowing can be quantitatively determined by Eq. (1), then it is a simple matter to determine the asymmetric aging using only Eq. (1). The introduction of any frame other than frame K is completely superfluous.

Despite the fact that Eq. (1) is all that is needed to determine the asymmetric aging of the twins, the results we have obtained concerning the uniformly accelerated frame can be used to eliminate some of the misconceptions which one encounters in many of the standard treatments, and also to bring out some interesting, generally overlooked facets of the phenomenon.

B. Determination of asymmetric aging with auxiliary frames

The most common approach to the problem of asymmetric aging is to have one twin A remain at rest in a particular inertial frame K and the other twin B to travel away from A and back again by judiciously transferring from one inertial frame to another. Although perfectly correct, the above method generates a certain amount of uncertainty among some because it is necessary for B to undergo periods of acceleration during the transfer from one inertial frame to another. If one assumes that accelerated observers are outside the realm of special relativity then there is indeed a problem. However, with the introduction of uniformly accelerated frames it is possible for twin B to make the round trip in such a way that at every instant he is either in an inertial frame or a uniformly accelerated frame and also in such a way that every transfer between frames occurs when the frames are at rest with respect to one another. The aging of B can then be calculated using the results we have obtained. This procedure does away with the conceptual difficulties encountered when one restricts oneself to inertial frames. However, as mentioned above, the introduction of any frame other than frame K is entirely superfluous if one accepts Eq. (1).

C. Asymmetric aging of twins at rest relative to one another

An interesting result of our earlier analysis is that two twins A and B who are located at different points in the same uniformly accelerated frame will age differently. The following example nicely illustrates this phenomenon and adds a provocative twist.

Consider two twins A and B who at time $t = 0$ are, respectively, located at the points $x = 1$ and $x = 2$ in frame K. Suppose at time $t = 0$ they both board the uniformly accelerated rocket frame S, and then after a time $\Delta\theta = 1$ as noted on the cabin monitors they disembark into the inertial frame K' which at that instant is at rest with respect to the rocket frame. They will then find that the time in frame K' is $t' = 0$ and their respective locations are $x' = 1$ and $x' = 2$. But during the trip the time $\Delta\tau$ which has elapsed on A's clock is 1 while the time $\Delta\tau$ which has elapsed on B's clock is 2. Thus twin A is younger than twin B by an amount of 1. They have aged asymmetrically while remaining at rest relative to one another. Furthermore, the time and place of departure from frame K is the same as the time and place of arrival in frame K'.

D. Asymmetry of the motion

One of the standard arguments against the asymmetric aging of two twins A and B, one of whom, A, stays at rest in an inertial frame K and the other of whom, B, travels away from A and returns, is as follows. The motion of B relative to A is symmetric to the motion of A relative to B, and therefore by the same reasoning that led to the conclusion that B ages less than A one could show that A ages less than B. But it is impossible for both B to be younger than A and also for A to be younger than B. Hence we have the so called twin paradox. The usual resolution of this paradox is to simply point out that the motion is not symmetric since B has undergone accelerations and A has not. Further explanations are usually limited. The results we have obtained provide an interesting confirmation of the asymmetry of the motion.

Suppose twin A is at rest in the inertial frame K at the point $x = 1$, and twin B is at rest in the uniformly accelerated frame S at the point $\sigma = 1$. At time $t = \theta = 0$ the twins are located at the same point and are at rest with respect to one another, but as time progresses they separate. If A determines the distance to B as a function of the time t he obtains $(1 + t^2)^{1/2} - 1$. But if B determines the distance to A as a function of his proper time θ he obtains either $-\ln \operatorname{sech} \theta$ if distance is measured with light signals or $1 - \operatorname{sech} \theta$ if distance is measured with standard rods. In either case the motion of A relative to B and the motion of B relative to A are not symmetric. Hence arguments

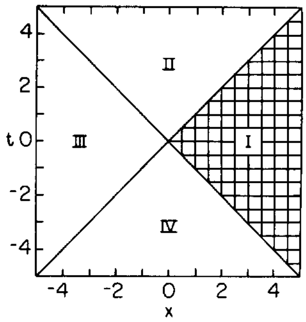

Fig. 3. The shaded region I represents the region in (x,t) space covered by the world lines of the observers making up the uniformly accelerated frame S.

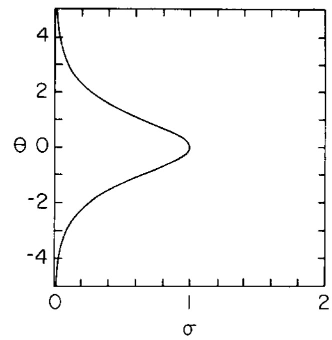

Fig. 5. The world line in (σ,θ) space of the observer who remains at rest at the point $x = 1$ in. the inertial frame K.

against the asymmetric aging of the twins certainly cannot be based on the supposed symmetry of the relative motion. Note that if t and θ are small then $(1 + t^2)^{1/2} - 1 \simeq t^2/2$, $-\ln \operatorname{sech} \theta \simeq -\ln[1 - (\theta^2/2)] \simeq \theta^2/2$, and $1 - \operatorname{sech} \theta \simeq \theta^2/2$. Hence there is an approximate symmetry in the early part of the separation while the relative motion is nonrelativistic.

X. HORIZONS

If the world lines of the observers making up frame S are plotted in x-t space they occupy only the shaded region I shown in Fig. 3. There are a number of consequences of this fact.

First, any event represented by a point lying in the regions II–IV will not be directly observed by the observers making up frame S, that is, it will not occur on the world line of any of the observers making up frame S. In contrast, each event represented by a point in any of the regions I–IV will lie on the world line of one of the observers making up frame K.

Second, if we trace the world line of any observer from its beginning at $t = -\infty$ to its ending at $t = +\infty$, then because the absolute value of the slope dx/dt of such a line cannot exceed 1, or equivalently the absolute value of the slope dt/dx cannot be less than 1, the given line can at most pass through the line $x = -t$ once and the line $x = t$ once. It follows that world lines can leave but not enter region IV and can enter but not leave region II. Thus it is possible for an observer to enter region I at some time $t < 0$ or to leave it at some time $t > 0$, but it is not possible for an observer who is in region I to leave and return again.

So far we have considered the world line of observers entering and leaving region I as they appear in the inertial frame K. But how do they appear in frame S? If we plot in x-t space the world lines of the observers making up frame S and also the lines of constant θ, they appear as shown in Fig. 4. If we now consider the world line $x = 1$, we note that when it enters region I it cuts across the lower half of the line $x = -t$ which corresponds to both $\sigma = 0$ and $\theta = -\infty$, and when it leaves region I it cuts across the upper half of the line $x = +t$ which corresponds to both $\sigma = 0$ and $\theta = +\infty$. Hence the observer whose world line is $x = 1$ enters the frame S at time $\theta = -\infty$ and leaves at time $\theta = +\infty$. It follows that from the viewpoint of an observer in S the observer whose world line is $x = 1$ has been in S from the beginning of time θ and will remain in S until the end of time θ. The world line in σ-θ space of the observer whose world line in x-t space is $x = 1$ is shown in Fig. 5.

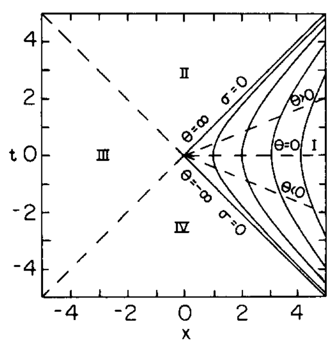

Fig. 4. The dotted lines in region I are lines of constant θ as they appear in (x,t) space. The solid lines in region I are the world lines in (x,t) space of some of the observers making up the uniformly accelerated frame S.

From the above it follows that in the uniformly accelerated reference frame S there is a horizon at $\sigma = 0$. If an observer in S passes through this horizon he will be unable to return again to the frame S. Furthermore, although according to the proper time of the observer who crosses the horizon $\sigma = 0$ the passage from a point in S to a point outside of S took a finite amount of time, to an observer fixed in S the traveling observer continues to get closer to the horizon without ever reaching it.

The above situation is phenomenologically identical to the situation that occurs when an observer passes through the event horizon of a black hole.[23]

XI. CONCLUSION

The preceding discussions demonstrate not only that uniformly accelerated frames have a very natural place in the special theory of relativity, but also that they serve a very useful pedagogical purpose first by clarifying concepts which are sometimes rather carelessly used when they are presented only from the viewpoint of an inertial observer, and second by introducing ideas that provide a natural passage from the special theory of relativity to the general theory of relativity.

[1] See, for example, the comments made in the following excellent books: C. Møller, *The Theory of Relativity* (Clarendon, Oxford, 1952, 1972), 1st ed., p. 49; T. M. Helliwell, *Introduction to Special Relativity* (Allyn and Bacon, Boston, 1966), p. 21; R. D. Sard, *Relativistic Mechanics* (Benjamin, New York, 1970), p. 102; R. Morris, *Time's Arrows* (Simon and Schuster, New York, 1984), p. 172.

[2] However, some authors seem to be wary even of accelerated motions. In one popular book on modern physics the statement is made that "Einstein's special theory of relativity applies only to uniform motion."

[3] If they occur, considerations of accelerated observers are usually found in discussions of the twin paradox.

[4] For example, many treatments of the equivalence principle, especially when discussing the freely falling elevator, leave one with the impression

that if a number of particles at rest at different positions in an inertial frame were simultaneously subjected to the same constant force, then the particles would remain at rest *relative to one another*; hence they would constitute a rigid accelerating frame [see, for example, R. L. Faber, *Differential Geometry and Relativity* (Dekker, New York, 1983), p. 167; R. K. Pathria, *The Theory of Relativity* (Pergamon, New York, 1974), 2nd ed., p. 181; L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, New York, 1975), 4th ed., p. 226]. In fact, they would neither remain at rest with respect to one another nor would their relative positions remain constant with respect to any inertial frame other than the one in which they were initially at rest.
[5]C. Møller, *The Theory of Relativity* (Clarendon, Oxford, 1952, 1972), 1st ed., pp. 253–258; 2nd ed., pp. 289–292.
[6]W. Rindler, *Essential Relativity* (Springer, New York, 1977), 2nd ed., pp. 49–51, 156–164.
[7]C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W. H. Freeman, San Francisco, 1973), Chap. 6.
[8]F. W. Sears and R. W. Brehme, *Introduction to the Theory of Relativity* (Addison-Wesley, Reading, MA, 1968), Chap. 5.
[9]L. Marder, *Time and the Space Traveler* (George Allen and Unwin, London, 1971), pp. 111–123.
[10]H. Lass, Am. J. Phys. **31**, 274 (1963).
[11]J. E. Romain, Am. J. Phys. **32**, 279 (1964).
[12]L. M. Marsh, Am. J. Phys. **33**, 934 (1965).
[13]W. Rindler, Am. J. Phys. **34**, 1174 (1976).
[14]J. D. Hamilton, Am. J. Phys. **46**, 83 (1978).
[15]R. H. Good, Am. J. Phys. **50**, 232 (1982).
[16]E. A. Desloge, Am. J. Phys. **52**, 312 (1984).
[17]See J. L. Synge, *Relativity: The Special Theory* (North-Holland, New York, 1972), 2nd ed., p. 11.
[18]L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, New York, 1975), 4th ed., pp. 229–230.
[19]See, for example, R. L. Faber, *Differential Geometry and Relativity Theory* (Dekker, New York, 1983), p. 109; R. K. Pathria, *The Theory of Relativity* (Pergamon, New York, 1974), 2nd ed., p. 28; W. Pauli, *The Theory of Relativity* (Pergamon, New York, 1958; Dover, New York, 1981), p. 5; W. Rindler, *Introduction to Special Relativity* (Clarendon, Oxford, 1982), p. 9; V. A. Ugarov, *Special Theory of Relativity* (Mir, Moscow, 1979), p. 38; M. Born, *Einstein's Theory of Relativity* (Dover, New York, 1965), p. 232.
[20]See, for example, F. W. Sears and R. W. Brehme, *Introduction to the Theory of Relativity* (Addison-Wesley, Reading, MA, 1968), p. 1; A. P. French, *Special Relativity* (Norton, New York, 1968), p. 72.
[21]See, for example, A. P. French, *Special Relativity* (Norton, New York, 1968), p. 27.
[22]See, for example, W. Rindler, *Essential Relativity* (Springer, New York, 1977), 2nd ed., pp. 45–47.
[23]See, for example, W. Rindler, *Essential Relativity* (Springer, New York, 1977), 2nd ed., pp. 156–165; H. Stephani, *General Relativity* (Cambridge U.P., Cambridge, 1982), pp. 212–218; W. J. Kaufmann, *The Cosmic Frontiers of General Relativity* (Little Brown, Boston, 1977), Chap. 9.

Quantum action-angle-variable analysis of basic systems

Robert A. Leacock and Michael J. Padgett[a)]
Department of Physics and Ames Laboratory, Iowa State University, Ames, Iowa 50011

(Received 30 October 1985; accepted for publication 17 March 1986)

Quantum action-angle variables are used to describe and analyze a number of familiar systems. For a given system, the quantum canonical transformation from the old coordinates, e.g., linear or polar, to the new coordinates, action-angle variables, is found by generalizing the corresponding classical transformation using a method based upon the correspondence principle, the Hermiticity and canonical nature of the old coordinates, and the requirement that the Hamiltonian be independent of the quantum angle variable. The bound-state energy levels and other important system properties follow immediately from the canonical transformation. Harmonic oscillators of various dimensions and the three-dimensional angular momentum system are used as illustrations; these illustrations provide interesting alternatives to the usual quantum treatments.

I. INTRODUCTION, SUMMARY

Quantum mechanics is usually formulated and applied using familiar coordinates, such as the linear coordinates and the spherical polar coordinates, as the fundamental canonically conjugate variables. The fact that quantum mechanics is usually cast in terms of familiar coordinates is unfortunate because other coordinates can offer conceptual and practical advantages. An example from classical mechanics is the theory of action-angle variables.[1] In terms of these variables the system Hamiltonian is free of the angle variables so that the equations of motion can be readily solved. Correspondingly, if one can set up a quantum action-angle-variable theory, the Hamiltonian is again free of the angle variables, and Schrödinger's equation, e.g., is easily solved.

In the present paper a method is formulated whereby quantum action-angle variables can be found for certain systems. The method has as its foundation the classical action-angle-variable theory, and utilizes a form of the correspondence principle, the Hermiticity of the old (original) coordinates, the conjugate nature of the old coordinates, and the fact that the system Hamiltonian must be free of the quantum angle variables. Starting with the clas-

