to detect radioactivity in beryllium with the linear amplifier," Phys. Rev. **44**, 59 (1933); David M. Gans, William D. Harkins, and Henry W. Newton, "Failure to detect radioactivity of beryllium with the Wilson cloud chamber," Phys. Rev. **44**, 310 (1933).
[27]George Gamow, "General stability-problems of atomic nuclei," in *International Conference on Physics*, Vol. 1, *Nuclear Physics*, edited by J. H. Awberry (Cambridge U. P., Cambridge, 1935), pp. 60–71, on p. 62.
[28]Gamow to Bohr, 13 June 1934 (Archive for History of Quantum Physics). Published references to the Pauli–Dirac conjecture include Gamow, Ref. 27, p. 62, and E. Gapon and D. Iwanenko, "Alpha particles in light nuclei," C. R. Acad. Sci. URSS **4**, 276–277 (1934).
[29]G. Gamow, talk on "Negative protons," Cambridge University, 11 June 1934 (Kapitza Club Minute Book, AHQP).
[30]H. A. Bethe, "Masses of light atoms from transmutation data," Phys. Rev. **47**, 633–634 (1935); and in more details in M. Stanley Livingston and H. A. Bethe, "Nuclear dynamics, experimental," Rev. Mod. Phys. **9**, 245–390 (1937) [reprinted in *Basic Bethe. Seminal Articles on Nuclear Physics 1936–37* (AIP, New York, 1986), pp. 331–476].
[31]Gamow, Ref. 27, p. 62.
[32]Reference 27, p. 68.
[33]Gamow to Bohr, 20 January 1935 (AHQP).
[34]G. Gamow, *Structure of Atomic Nuclei and Nuclear Transformation* (Clarendon, Oxford, 1937), pp. 14–18. The first edition of 1931 was entitled *Constitution of Atomic Nuclei and Radioactivity.*
[35]C. D. Anderson, "The positive electron," Phys. Rev. **43**, 491–494 (1933), on p. 494.
[36]Reference 35.
[37]E. J. Williams, "Nature of the high energy particles of penetrating radiation and status of ionization and radiation formulae," Phys. Rev. **45**, 729–730 (1934).
[38]E. J. Williams, "General survey of theory and experiment for high-energy electrons," in *Kernphysik*, edited by E. Bretscher (Springer, Berlin, 1936), pp. 123–141, on p. 123.
[39]H. J. Bhabha, "Negative protons in the cosmic radiation," Nature **139**, 415–416 (1937).
[40]The New York Times, 12 December 1937. Millikan's other discovery claim was of an "X-particle" of mass intermediate between the electron and the proton.
[41]R. A. Millikan, Carnegie Institution Yearbook, No. 36, 1937, p. 364.
[42]See the references in E. Broda, N. Feather, and D. Wilkinson, "A search for negative protons emitted as a result of fission," in *Report of an International Conference on Fundamental Particles and Low Temperatures* (Taylor and Francis, London, 1947), Vol. 1, pp. 114–124.

Some results on the relativistic Doppler effect for accelerated motion

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It is shown how the Doppler-shifted frequencies that result from the uniform acceleration of a source of light, and/or of the receiver, can be derived, and that very simple formulas often apply. For completeness it is shown how a basic equation can be extended to apply in an accelerated frame of reference.

I. INTRODUCTION

Textbooks on special relativity usually consider the Doppler effect for light only for the situation where source and receiver are in uniform relative motion, although Rindler[1] explains how the standard formulas are to be interpreted when the motion is not uniform. A generalization of these formulas (for example, from $[(1-u/c)/(1+u/c)]^{1/2}$ for the frequency ratio for uniform relative motion in one dimension) was recently published by Bachman.[2] A succinct derivation is given below. The frequency ratio for accelerated motion can in principle be worked out in any convenient frame of reference. In practice, a calculation will always be simplest when based in an inertial frame, but as far as I am aware an explicit result has been obtained for one situation only, by Hamilton,[3] in a study of the properties of light in an accelerated frame. The topic is in fact within the scope of an undergraduate course on special relativity.

Let x_s and x_r be the positions of source and receiver at times t_s and t_r, respectively, in an inertial frame—their motion is, meantime, in one dimension. Successive light flashes are emitted at t_s and $t_s + dt_s$, and received at t_r and $t_r + dt_r$, respectively. The corresponding intervals of proper time are $d\tau_s$ and $d\tau_r$. Then

$$\frac{\nu_r}{\nu_s} = \frac{d\tau_s}{d\tau_r} \tag{1}$$

$$= \frac{dt_s(1-u_s^2/c^2)^{1/2}}{dt_r(1-u_r^2/c^2)^{1/2}}, \tag{2}$$

with $x_r > x_s$, $c(t_r - t_s) = x_r - x_s$, and therefore

$$dt_r(1-u_r/c) = dt_s(1-u_s/c). \tag{3}$$

Combining these, we have the required result:

$$\frac{\nu_r}{\nu_s} = \left(\frac{1+u_s/c}{1-u_s/c}\right)^{1/2}\left(\frac{1-u_r/c}{1+u_r/c}\right)^{1/2}. \tag{4}$$

When u_s and u_r are constant, Eq. (4) reduces to the standard result, that

$$\frac{\nu_r}{\nu_s} = \left(\frac{1-u/c}{1+u/c}\right)^{1/2},$$

involving the relative velocity $u = (u_r - u_s)/(1 - u_r u_s/c^2)$, but as Bachman has emphasized, u_s and u_r must otherwise be taken at different times.

We assume that the source is controlled by an ideal clock, the mechanism of which is unaffected by acceleration, so that ν_s is a constant. It follows from the form of Eq. (1) that ν_r/ν_s is invariant in a transformation to another frame of reference—such as that which we consider in a final section. This point was emphasized by Tolman[4] in his "schematic outline" of "the generalized Doppler effect," from which Eq. (1) can be said to derive. His objective, however, was a wide generalization and he did not include explicitly the condition equivalent to our Eq. (3). To be meaningful, ν_r must eventually be expressed in terms of τ_r.

II. UNIFORM ACCELERATION OF RECEIVER OR SOURCE

The motion specified by

$$\frac{d}{dt}\left(\frac{u}{\sqrt{1-u^2/c^2}}\right)=g \tag{5}$$

is usually referred to as hyperbolic motion, and g, which is invariant in a Lorentz transformation, is sometimes termed the proper acceleration. The acceleration du/dt is equal to g only for $u=0$, but we can imagine a sequence of Lorentz transformations to a sequence of rest frames in each of which the acceleration is momentarily equal to g. In this sense the acceleration is uniform.

With $x=0$ and $u=0$ at $t=0$, the solution of Eq. (5) can be written as

$$x=(c^2/g)[\cosh(g\tau/c)-1], \tag{6}$$

where

$$t=(c/g)\sinh(g\tau/c). \tag{7}$$

It follows that

$$u/c=\tanh(g\tau/c). \tag{8}$$

These are well-known results.[5] Note that we are concerned with the acceleration of a light source or receiver, or of an observer with a clock that records τ, but not as yet of another frame of reference.

Consider first the situation where the source s remains at $x_s=0$ while the receiver r is uniformly accelerated. From Eqs. (4) and (8),

$$\frac{\nu_r}{\nu_s}=\left(\frac{1-u_r/c}{1+u_r/c}\right)^{1/2}=\left(\frac{1-\tanh(g\tau_r/c)}{1+\tanh(g\tau_r/c)}\right)^{1/2}$$
$$=\exp(-g\tau_r/c). \tag{9}$$

This is the result arrived at by Hamilton,[3] following a more difficult route.

However, the essence of the Doppler effect is the relation between τ_s and τ_r. From $c(t_r-t_s)=x_r$, we have, using Eqs. (6) and (7),

$$\sinh\frac{g\tau_r}{c}-\frac{g\tau_s}{c}=\cosh\frac{g\tau_r}{s}-1,$$

from which it follows that

$$1-g\tau_s/c=\exp(-g\tau_r/c). \tag{10}$$

Using Eq. (1), we then obtain Eq. (9). In dealing with hyperbolic motion, where coordinates and time can be directly expressed in terms of proper time, this is a simpler method than the use of Eq. (4). (It can of course also be used when the motion is uniform.)

When r remains at $x_r=0$ and s is accelerated, we find in the same way from $c(t_r-t_s)=x_s$ that

$$1+g\tau_r/c=\exp(g\tau_s/c), \tag{11}$$

so that

$$\frac{\nu_r}{\nu_s}=\frac{d\tau_s}{d\tau_r}=\exp\left(\frac{-g\tau_s}{c}\right)=\left(1+\frac{g\tau_r}{c}\right)^{-1}. \tag{12}$$

I have not so far found this simple result in the literature. The singularity at $\tau_r=-c/g$ corresponds to $t_s\to-\infty$; signals emitted over a long time arrive almost simultaneously.

III. SOME MORE GENERAL SITUATIONS

In this section we write $\xi_r=(c^2/g)[\cosh(g\tau_r/c)-1]$, and similarly for s, since the hyperbolic motion will not always have $x=0$ at $t=0$.

When $x_s=0$ and $x_r=x_2+\xi_r$, Eq. (9) applies throughout when the constant x_2 is positive. However, for $x_2<0$, r passes by (through!) s when $x_r=0$, with a discontinuity in the Doppler ratio, from $\exp(-g\tau_r/c)$ for $x_r>0$ to $\exp(g\tau_r/c)$ for $x_r<0$. This is of course a feature of the Doppler effect for motion in one dimension, and we shall avoid it in the future by excluding the possibility of x_r-x_s changing sign.

When $x_r=0$ and $x_s=x_1+\xi_s$ with $x_1>0$, Eq. (12) is modified to

$$\frac{\nu_r}{\nu_s}=\left[1+\frac{g}{c}\left(\tau_r-\frac{x_1}{c}\right)\right]^{-1}. \tag{13}$$

Let s and r start together from the origin with proper accelerations g_1 and g_2, respectively, with $g_2>g_1$ to ensure $x_r>x_s$. We then find in the usual way that

$$\frac{\nu_r}{\nu_s}=\left[\frac{g_1}{g_2}+\left(1-\frac{g_1}{g_2}\right)\exp\frac{g_2\tau_r}{c}\right]^{-1}. \tag{14}$$

The results given in Sec. II can be obtained by putting $g_1=0, g_2=g$ to reproduce Eq. (9), while $g_2\to0$, $g_1=-g$ gives a configuration equivalent to that which led to Eq. (12), and reproduces that equation.

Let $x_s=x_1+\xi_s$ and $x_r=x_2+\xi_r$, with $x_2>x_1$. The proper accelerations are g_1 and g_2, respectively; $g_2>g_1$ certainly ensures $x_r>x_s$, but is more stringent than is necessary. The Doppler ratio is found to be

$$\frac{\nu_r}{\nu_s}=\left[\frac{g_1}{g_2}+g_1\left(\frac{1}{g_1}-\frac{x_1}{c^2}-\frac{1}{g_2}+\frac{x_2}{c^2}\right)\exp\frac{g_2\tau_r}{c}\right]^{-1}. \tag{15}$$

The interesting feature of this result is that ν_r/ν_s becomes time independent if

$$g_1=\frac{g}{1+gx_1/c^2}\quad\text{and}\quad g_2=\frac{g}{1+gx_2/c^2}, \tag{16}$$

when it reduces to

$$\frac{\nu_r}{\nu_s}=\frac{g_2}{g_1}=\frac{1+gx_1/c^2}{1+gx_2/c^2}. \tag{17}$$

Reference to Moller[6] shows that Eqs. (16) are the condition for r and s to be at rest in an accelerated frame of reference. We shall return to this topic in Sec. IV.

The source and the receiver can have initial velocities; a general treatment requires a more general solution of Eq. (5). A reasonably concise treatment is possible for the following example. With $x_s=0$, r is projected from the origin with initial velocity $-v$ and returns there with final velocity v as the result of a proper acceleration g. The Doppler ratio is most readily found by transforming to an inertial frame in which s has constant velocity v, and r performs ordinary hyperbolic motion until it overtakes s. The result is

$$\frac{\nu_r}{\nu_s} = \left(\frac{1 - v/c}{1 + v/c}\right)^{1/2} \exp\frac{g\tau_r}{c}. \tag{18}$$

This applies during the proper time of the round trip, which is $(c/g)\ln[(1 + v/c)/(1 - v/c)]$ for r and $2v/(g\sqrt{1 - v^2/c^2})$ for s. When the roles of s and r are interchanged,

$$\frac{\nu_r}{\nu_s} = \left[\left(\frac{1 + v/c}{1 - v/c}\right)^{1/2} - \frac{g\tau_r}{c}\right]^{-1}. \tag{19}$$

The motion of s and r need not be on the line joining them. For example, take $x_s = 0$ and let r accelerate in the x direction, starting from x_2, y_2. The basic condition is $c(t_r - t_s) = (x_r^2 + y_2^2)^{1/2}$. Some straightforward algebra gives, with $x_r = x_2 + \xi_r$ as usual,

$$\frac{\nu_r}{\nu_s} = \cosh\frac{g\tau_r}{c} - \frac{x_r \sinh(g\tau_r/c)}{(x_r^2 + y_2^2)^{1/2}}. \tag{20}$$

[With $x_2 < 0$ and $y_2 \to 0$, Eq. (20) reproduces the discontinuity in ν_r/ν_s which was mentioned at the beginning of this section.] When s is also accelerated, for example, along the z axis, we readily obtain a relation between τ_r and τ_s, and from it an equation for ν_r/ν_s. While this is a solution of the problem, ν_r/ν_s cannot be written down as a function of τ_r only, since the equation relating τ_r and τ_s cannot be solved explicitly for τ_s. This is generally true when s is accelerated, unless the situation is one-dimensional.

The method used in this article does not offer much that is new for the discussion of aberration angles. To take an example, suppose that r is moving in the x direction, while s can be moving in any specified direction. The outwardly drawn line from the position of s (at time t_s) to that of r (at time t_r) has length d_{sr} and a direction cosine $h_{sr} = (x_r - x_s)/d_{sr}$ in the inertial frame. This direction cosine is expressible in terms of τ_r and τ_s, and (in principle) in terms of τ_r alone. The corresponding direction cosine for an observer accompanying r, which gives the angle between a continuation of the incoming ray to r and the x direction, is given by the standard formula

$$h'_r = \frac{h_{sr} - u_r/c}{1 - h_{sr}u_r/c},$$

with u_r/c given by Eq. (8).

IV. THE DOPPLER EFFECT IN AN ACCELERATED FRAME OF REFERENCE

Moller[6] has given the equations of transformation between an inertial frame and an accelerated frame, the origin of which makes in the inertial frame the hyperbolic motion described in Sec. II. This is a rigid frame with static properties, and its geometry is Euclidean. Using in this section (xyz) and t for coordinates and (coordinate) time in *this* frame, the metric is given by

$$c^2\,d\tau^2 = c^2\,dt^2(1 + gx/c^2)^2 - (dx^2 + dy^2 + dz^2).$$

The (coordinate) velocity of light is $w = c(1 + gx/c^2)$, and in a one-dimensional situation the time for light to travel from x_s to x_r is therefore $t_r - t_s = \int_s^r dx/w$. Following the steps which were made in an inertial frame, Eqs. (1)–(4), one is led to a simple generalization of Eq. (4), namely,

$$\frac{\nu_r}{\nu_s} = \frac{w_s}{w_r}\left(\frac{1 + u_s/w_s}{1 - u_s/w_s}\right)^{1/2}\left(\frac{1 - u_r/w_r}{1 + u_r/w_r}\right)^{1/2}. \tag{21}$$

There is now a frequency shift for $u_s = u_r = 0$, which an observer in this frame would interpret as an effect of gravity. The equation

$$\frac{\nu_r}{\nu_s} = \frac{w_s}{w_r} = \frac{1 + gx_s/c^2}{1 + gx_r/c^2}, \tag{22}$$

with x_s and x_r constant, is consistent with Eq. (17), since the constant coordinate of s in the accelerated frame is its coordinate in the inertial frame at time zero, and similarly for r.

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[1]W. Rindler, *Special Relativity* (Oliver and Boyd, Edinburgh, 1960), p. 45.

[2]R. A. Bachman, "Generalized relativistic Doppler effect," Am. J. Phys. **54**, 717–719 (1986).

[3]J. Dwayne Hamilton, "The uniformly accelerated reference frame," Am. J. Phys. **46**, 83–89 (1978).

[4]Richard C. Tolman, *Relativity, Thermodynamics and Cosmology* (Oxford U.P., London, 1934), pp. 288–289.

[5]C. Moller, *The Theory of Relativity* (Oxford U.P., London, 1972), pp. 73–74 and 290.

[6]Reference 5, pp. 289–292.