Aberration of light and relativity of simultaneity

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ABSTRACT

Using an elementary Huygens' construction and the postulates of special relativity, we develop a novel approach to the problem of relativistic aberration of light. We consider a simple two-dimensional model of a plane-wave emitter and derive the relations that give a complete description of the aberration phenomenon. We further emphasize an important point that the standard textbook formula for relativistic aberration describes the aberration of the path taken by a single wavefront, but does not describe the aberration of the whole light beam. We then show that a light ray and a light beam are two different concepts in the case when the source is in uniform rectilinear motion. The approach in this paper clearly shows that relativistic aberration of light is a direct consequence of the relativity of simultaneity.

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I. INTRODUCTION

Relativistic aberration of light is a physical effect defined as a change of the direction of a light ray when observed from different inertial reference frames. It is usually, and sometimes incorrectly, discussed in the context of stellar aberrations [1]. The traditional derivation of the relativistic aberration formula involves a plane-polarized electromagnetic wave and a requirement that its phase is an invariant quantity under a Lorentz transformation [2, 3]. The formula also can be obtained by a direct application of the addition law for relativistic velocities [4]. Both of these methods are obviously based on the standard Lorentz-transformation procedure, which makes the effect of relativistic aberration to look like it is merely a consequence of the Lorentz transformations.

The purpose of the present paper is to provide an additional insight into the problem of relativistic aberration of light by using an elementary Huygens' construction and the postulates of special relativity, and, thus, by employing only the notion of the wave nature of light and without referring to its electromagnetic character. The benefit is three-fold. First, this paper clearly shows that relativistic aberration of light is a direct consequence of the relativity of simultaneity. Second, it points out and further explains the fact that a light ray and a light beam are generally different entities when the light source is moving at a constant velocity. Third, it provides a novel method of analysis to be used by the physics teacher to discuss the problem of relativistic aberration of light and related topics of relativistic optics at a level suitable for beginning undergraduates.

II. ABERRATION AND SIMULTANEITY

We will investigate the case of a plane-polarized light beam emanating from a uniformly moving light source. As a simple two-dimensional model of a plane-wave emitter we use a linear array of identical coherent elementary sources that begin to radiate spherical wavelets simultaneously with respect to the reference frame where the array is at rest. We take the distance between each two adjacent elementary sources to approach zero, which enables us to consider this linear array as an ideal line source. Figure 1 depicts a special case of a horizontal line source. A simple Huygens' construction would show that this idealized line source will emit plane wavefronts perpendicularly to its axis. However, the situation will be different when the line source is moving at a constant velocity v to the right (see Fig. 2). Let A and B denote the elementary sources that lie on the rear and the front of the line source, respectively, and $l = \overline{AB}$ the length of the moving line source. Due to the motion of the line source, the elementary sources along the line source will not flash simultaneously, but continuously, starting with the flash of the source from the rear and ending with the flash of the source from the front. As a consequence of this simultaneity loss, there will exist a certain non-zero time interval between the beginning of the emission of any two elementary sources along the line source, and this time interval will depend on the distance between the sources. Thus, if the elementary sources A and B started to emit wavelets simultaneously in the reference frame where the line source is at rest, then, in the reference frame where the line source is moving at a constant velocity v to the right (Fig. 2), the elementary source at the front will start radiating wavelets Δt_{AB} seconds after the flash of the elementary source at the rear. The time interval Δt_{AB} between the emission of the elementary sources from the rear and the front of the line source is given with

$$\Delta t_{AB} = \frac{lv}{c^2 - v^2}, \tag{1}$$

where c is the speed of light in vacuum. The derivation of Eq. (1) follows directly from the postulates of special relativity if we make a simple analogy with Einstein's thought experiment in which the opposite ends of the train are hit by strokes of lightning by simply substituting the strokes with two identical elementary sources, one at each end of the train [5]. During this time interval Δt_{AB} taken from the beginning of the radiation of the source at A, the source at B has moved a distance $\overline{BB'} = v\Delta t_{AB}$ to the right. At the end of this time interval Δt_{AB}, the source at the front (now, at the point B') will begin to radiate, and the elementary wavefront radiated from A a time Δt_{AB} earlier will be a sphere with radius $\overline{AC} = c\Delta t_{AB}$. The wavefront $\overline{CB'}$ emitted by the line source is an envelope of the spherical wavelets radiated by the point sources along $\overline{AB'}$. We have taken into account the constant light speed postulate, which implies that the elementary wavefront radiated by a moving elementary source will represent a sphere expanding in all directions at a constant speed c.

From Fig. 2 we see that the wavefront $\overline{CB'}$ will propagate at a speed c at an angle θ from the vertical. From the triangle $AB'C$ the angle θ can be expressed as

$$\sin\theta = \frac{\overline{AC}}{\overline{AB} + \overline{BB'}} = \frac{c\Delta t_{AB}}{l + v\Delta t_{AB}}. \tag{2}$$

We substitute Eq. (1) into Eq. (2) and obtain

$$\sin\theta = \frac{v}{c}, \tag{3}$$

which is the well-known formula that describes the aberration of the transverse light ray in the Michelson-Morley apparatus [6, 7].

Now, consider the situation when the line source is inclined at an angle θ_0 to the horizontal with respect to the reference frame where the line source is at rest (Fig. 3). Evidently, the line source will emit plane wavefronts at an angle θ_0 from the vertical. In the reference frame where the line source is moving at a constant velocity v to the right (see Fig. 4), its inclination angle θ_1 will differ from its inclination angle θ_0 in the stationary case due to the effect of Lorentz contraction along the direction of its motion [7]. It is easy to show that this tilt of the moving line source is given with

$$\tan\theta_1 = \frac{\tan\theta_0}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{4}$$

This relativistic contraction along the velocity vector is a cause for the length $l = \overline{AB}$ of the moving line source to be different from the length $l_0 = \overline{A_0B_0}$ of the stationary line source. Furthermore, the elementary sources along the line source which simultaneously started their radiation in its rest reference frame will not flash simultaneously in the reference frame where the line source is moving. We left to the student as an exercise to show that the time interval Δt_{AB} that elapsed between the flashes of the elementary sources from the rear and the front of the moving line source is given with

$$\Delta t_{AB} = \frac{lv}{c^2 - v^2}\cos\theta_1. \tag{5}$$

We take into account that the elementary source at the front will start radiating spherical wavelets Δt_{AB} seconds later than the elementary source at A. From Fig. 4 we see that during this time interval Δt_{AB}, the line source has covered the distance $\overline{AA'} = \overline{BB'} = v\Delta t_{AB}$ to the right, and the wavelet emitted from A has evolved into a spherical wavefront with radius $\overline{AC} = c\Delta t_{AB}$. The plane wavefront $\overline{B'C}$ emitted by the moving line source is an envelope of the spherical wavelets radiated by the point sources along the line segment $\overline{AB'}$ and it propagates at a speed c at an angle θ from the vertical. From the triangle ACD we have

$$\sin\theta = \frac{\overline{AC}}{\overline{AA'} + \overline{A'D}} = \frac{c\Delta t_{AB}}{v\Delta t_{AB} + \overline{A'D}}. \tag{6}$$

Applying the sine theorem to the triangle $A'B'D$, we get

$$\overline{A'D} = l\,\frac{\sin(\theta-\theta_1)}{\sin\theta}. \tag{7}$$

We substitute Eqs. (5) and (7) into Eq. (6) and simplify the result to obtain

$$\left(1-\frac{v^2}{c^2}\right)\cos\theta\tan\theta_1 = \sin\theta - \frac{v}{c}. \tag{8}$$

By taking the square of Eq. (8) and rearranging the terms, we obtain a quadratic equation in $\sin\theta$

$$\left[1+\left(1-\frac{v^2}{c^2}\right)\tan^2\theta_0\right]\sin^2\theta - \frac{2v}{c}\sin\theta + \frac{v^2}{c^2} - \left(1-\frac{v^2}{c^2}\right)\tan^2\theta_0 = 0, \tag{9}$$

where we also have taken into account Eq. (4). Equation (9) has two solutions in $\sin\theta$,

$$(\sin\theta)_{1,2} = \frac{\frac{v}{c}\cos^2\theta_0 \pm \left(1-\frac{v^2}{c^2}\right)\sin\theta_0}{1-\frac{v^2}{c^2}\sin^2\theta_0}. \tag{10}$$

From the requirement $\theta = \pi/2$ when $\theta_0 = \pi/2$, we find the only solution of Eq. (9) which correctly describes the propagation of the wavefront $\overline{B'C}$

$$\sin\theta = \frac{\sin\theta_0 + \frac{v}{c}}{1+\frac{v}{c}\sin\theta_0}. \tag{11}$$

Equation (11) expresses the law of aberration in its most general form, and it is identical to the formula obtained with the standard methods [2, 3, 4]. Obviously, in the case $\theta_0 = 0$, Eq. (11) reduces to Eq. (3).

It is important to stress that Eq. (11) [and, therefore, Eq. (3)] refers to the aberration of the path taken by a single wavefront, and thus describes the aberration of a light ray. By definition, a light ray is an imaginary line along which the wavefront advances, and it corresponds to the direction of flow of radiant energy [8]. Unlike a light ray, which is a purely mathematical device, a light beam is a physical entity. It refers to the pencil of light (a rod-like volume filled with radiant energy) that emerges from a particular light source. For example, a laser beam corresponds to the cylindrical-shaped volume occupied by the photon stream emerging from a laser. In our case, the light beam could be suitably defined as the volume composed of all the wavefronts emitted by the moving line source. If the line source were stationary, the light ray and the axis along which the light beam evolves would have the same direction, and thus the light ray could be considered to be an unattainable limit on the narrowness of the light beam. However, this is not the case when the source is in uniform rectilinear motion. In the following we will derive a formula for the aberration of a plane-polarized light beam and show that the aberration angles of the light ray and the light beam are generally different.[1]

III. ABERRATION OF THE LIGHT BEAM

The formula for aberration of the light beam can be derived by considering Fig. 5. We have two plane wavefronts subsequently emitted by the moving light source from the points A_1 and A_2 along the line of its motion. The light source is moving at a constant velocity v to the right. Let Δt be the time interval that elapsed between the emission of the wavefronts. The wavefronts will follow two separate paths determined by the aberration angle θ in Eq. (11) at a constant speed c. During the time interval Δt between the emission of the wavefronts, the light source has traveled the path $\overline{A_1 A_2} = v\Delta t$ to the right, and the wavefront emitted at A_1 has crossed the distance $\overline{A_1 A_1'} = c\Delta t$. From Fig. 5 we see that the light beam has made an advancement $\overline{A_2 A_1'}$ at velocity u at an angle δ from the vertical, and, at the same time, has moved the distance $\overline{A_1 A_2}$ as a whole at velocity v to the right. The light source will continue to emit wavefronts from the points along the line of its motion, and at each instant of time all the wavefronts emitted by the moving light source will be arranged along the axis determined by δ. The angle δ is the aberration angle of the light beam, and it determines the axis along which the light beam expands its volume. However, besides the volume of the light beam, which is time-dependent in any case, the location of the axis of the light beam is also time-dependent in the case when the source is moving. The axis of the light beam will move to the right at the same

[1]This fact has already been emphasized in Ref. 7, however, for a particular case of a vertical light beam.

velocity v as the source. Observe that the plane of every propagating wavefront is perpendicular to the wavefront's path determined by the angle θ, but not to the axis along which the light beam expands.

The angle δ that describes the axis of the light beam can be found from the triangle $A_1 A_1' A_2$ by applying the sine theorem

$$\frac{c\Delta t}{\sin(\pi/2+\delta)} = \frac{v\Delta t}{\sin(\theta-\delta)}. \tag{12}$$

By solving Eq. (12) for δ, we get

$$\tan\delta = \frac{\sin\theta - \frac{v}{c}}{\cos\theta}. \tag{13}$$

On the other hand, from Eqs. (4) and (8) we have

$$\tan\theta_0 = \frac{1}{\sqrt{1-v^2/c^2}} \frac{\sin\theta - \frac{v}{c}}{\cos\theta}. \tag{14}$$

Hence, from Eqs. (13) and (14) we obtain

$$\tan\delta = \sqrt{1-\frac{v^2}{c^2}}\tan\theta_0. \tag{15}$$

This is the formula that describes the aberration of the axis along which the light beam expands. It states that the axis of the light beam emanating from a uniformly moving light source will be Lorentz-contracted along the direction of its motion compared with the situation when the source is stationary. We further apply the cosine theorem to the triangle $A_1 A_1' A_2$,

$$(u\Delta t)^2 = (v\Delta t)^2 + (c\Delta t)^2 - 2(v\Delta t)(c\Delta t)\cos(\pi/2-\theta), \tag{16}$$

and use Eq. (11) to obtain the speed u of the expansion of the light beam along the axis determined by δ:

$$u = c\sqrt{1-\frac{v^2}{c^2}}\sqrt{\frac{1-(v/c)\sin\theta_0}{1+(v/c)\sin\theta_0}} . \qquad (17)$$

Obviously, the speed u at which the light beam expands along its axis determined by the aberration angle δ will differ from the speed of light in vacuum. Nevertheless, every wavefront along the light beam will possess a net constant speed c directed at an angle θ from the vertical. This result is due to the motion of the entire light beam at velocity v to the right that we also must take into account.

When the line source is horizontal ($\theta_0 = 0$), Eqs. (11), (15) and (17) reduce to $\sin\theta = v/c$, $\delta = 0$, and $u = c\sqrt{1-v^2/c^2}$, which describe the aberration of the vertical light beam in Einstein's cat experiment discussed in Ref. 7, and also the propagation of the transverse light beam in the Michelson-Morley apparatus. Another interesting example occurs in the case of a vertical line source ($\theta_0 = \pi/2$). In this case, $\sin\theta = 1$, $\delta = \pi/2$, and $u = c - v$, which means that the light beam will expand from the moving light source in the same direction as in the case of a stationary light source, but at a speed ($c - v$) which is smaller than the speed of light in vacuum. The result, however, will not contradict the constant light speed postulate if we also take into account that at the same time the entire volume of the light beam is moving at velocity v to the right. Thus, the motion of the light beam at velocity v in the direction of motion of the light source combined with the expansion of the beam at velocity ($c - v$) in the same direction will give a net constant speed c to every wavefront along the beam.

IV. CONCLUSIONS

Evidently, and unlike the common belief, one should make a clear distinction between a light ray and a light beam when the light source is in uniform rectilinear motion. The usual textbook formula for relativistic aberration refers to aberration of a light ray, but does not give a valid description of aberration of the path taken by the light beam. We have shown that the propagation of the light beam from a source moving at a constant velocity v to the right is a complex phenomenon that consists of the expansion of the light beam at velocity u at an (aberration) angle δ from the vertical, and, at the same time, the motion of the entire light beam at velocity v to the right (see Fig. 6). A revision in the conventional treatments is therefore necessary if the teacher or student is to give a correct description of the propagation of a plane-polarized light beam from a uniformly moving light source.

From the above discussion, we may conclude that relativistic aberration of light is a corollary of relativity of simultaneity. As a subject for future study, we left to the student to explore the situations when the elementary sources are arranged in a more complicated manner. For example, an interesting case occurs when the point sources are arranged in a circle, which represents a simple two-dimensional model of a spherical wave emitter.

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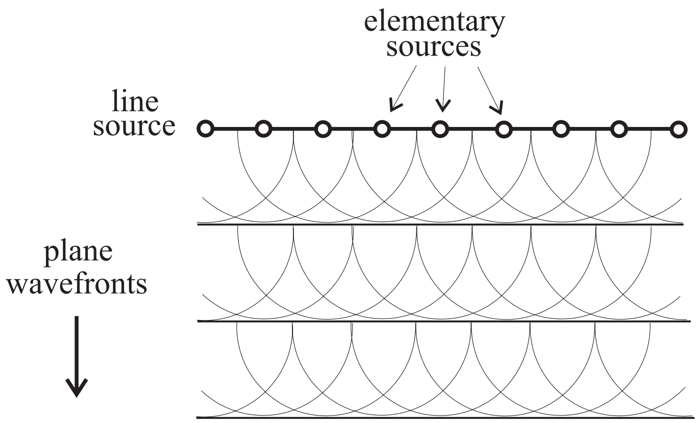

Fig. 1. Two-dimensional model of a plane-wave emitter. The plane wavefronts emitted by the line source will propagate from the line source in the direction which is perpendicular to the axis of the line source. Here, we consider only the wavefronts which propagate vertically downward.

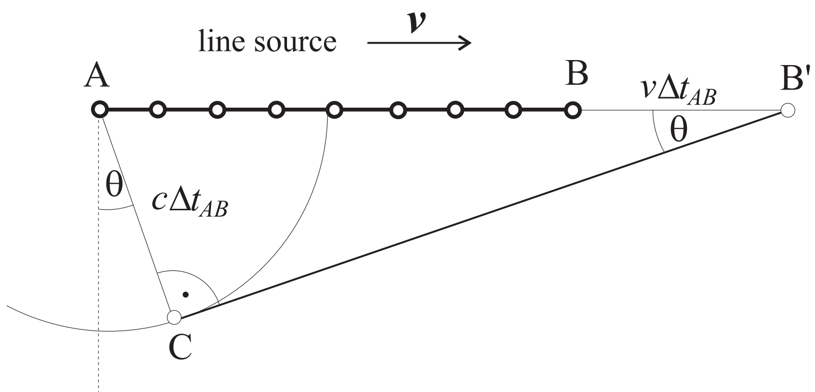

Fig. 2. Huygens' construction of the wavefront emitted when the line source in Fig. 1 is moving at a constant velocity v to the right. The plane wavefront $\overline{CB'}$ will propagate at a speed c at an angle θ from the vertical. Observe that the elementary source at the front will start to emit spherical wavelets at the point B' a time Δt_{AB} later than the elementary source at A.

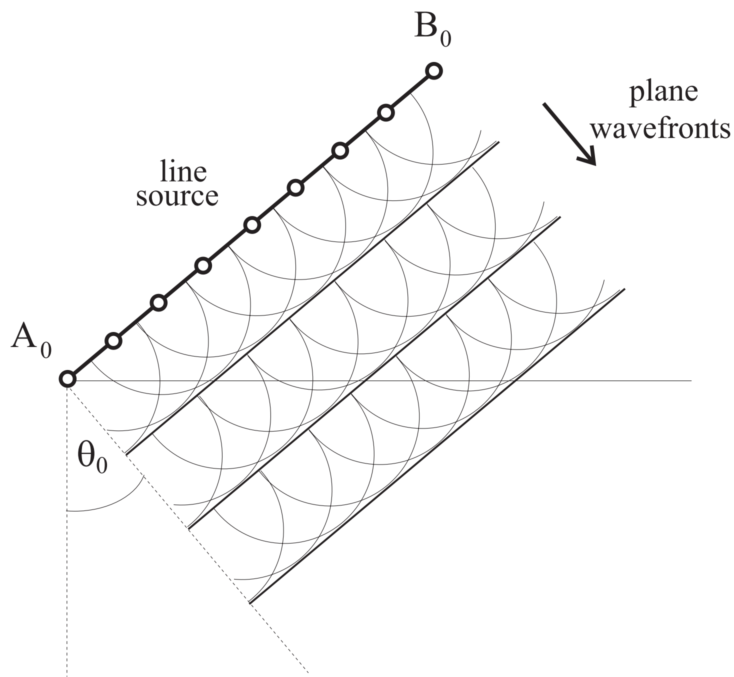

Fig. 3. Huygens' construction of the wavefronts emitted by a stationary line source inclined at an angle θ_0 to the horizontal. The plane wavefronts will propagate at an angle θ_0 from the vertical.

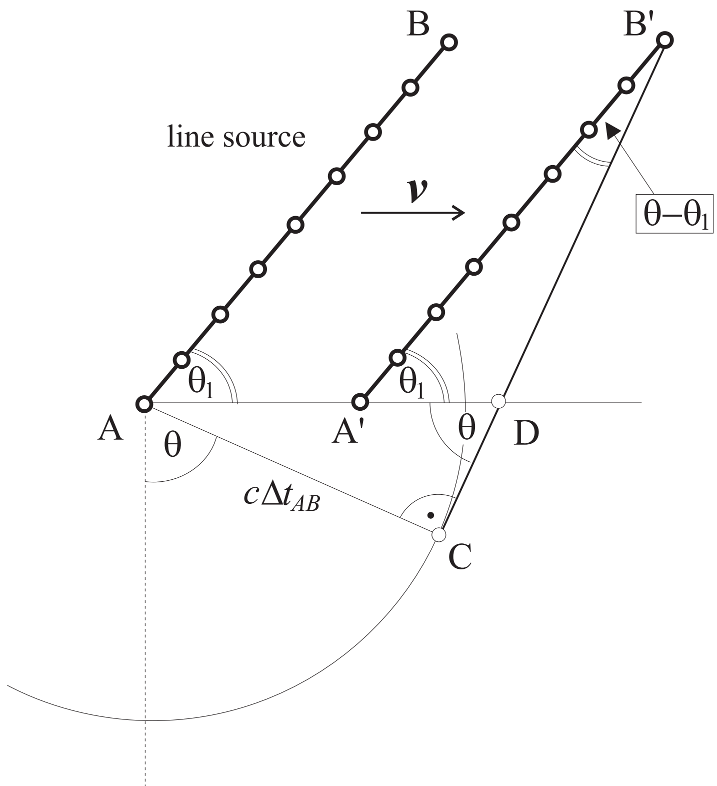

Fig. 4. When the line source in Fig. 3 is in uniform rectilinear motion at velocity v to the right, its inclination angle θ_1 will differ from its inclination angle θ_0 in the stationary case due to the Lorentz contraction along the direction of motion. The elementary source at the front of the line source will start to emit spherical wavelets at the point B' a time Δt_{AB} later than the elementary source at the rear. The plane wavefront emitted by the moving line source is a tangent line of the spherical wavelets radiated by the point sources along the line segment $\overline{AB'}$, and it propagates at velocity c at an angle θ from the vertical.

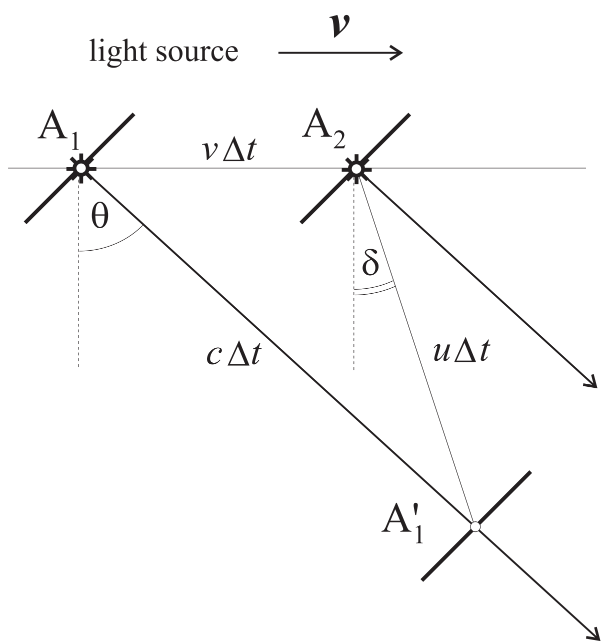

Fig. 5. Two plane wavefronts subsequently emitted by the moving light source at the points A_1 and A_2. Although the wavefronts will move at a speed c at an angle θ from the vertical, the light beam will expand at a speed u along the axis determined by the aberration angle δ. Observe that the entire light beam is also in uniform rectilinear motion at velocity v to the right.

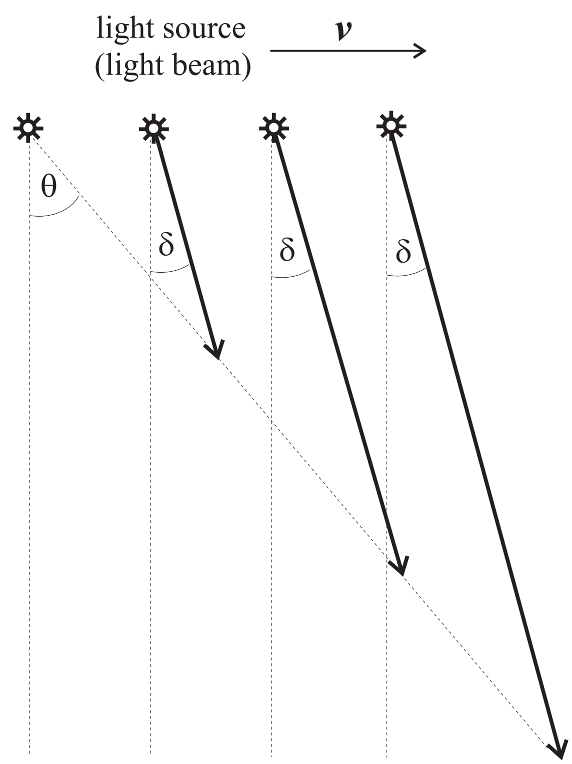

Fig. 6. A schematic diagram describing the time-evolution of a beam of light emanating from a uniformly moving light source.