Einstein's mirror and Fermat's principle of least time

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We derive a formula for the reflection of light from a uniformly moving mirror based on Fermat's principle of least time.
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Almost a century ago, Einstein considered the oblique incidence of an electromagnetic wave on a uniformly moving mirror whose velocity is perpendicular to its surface.[1] To derive the equations for the angle of reflection and the wave characteristics of the reflected light, he Lorentz-transformed the equations describing reflection in the reference frame where the mirror was at rest. In recent work,[2] we showed that the problem can be approached in an alternative way by a direct application of the Huygens–Fresnel principle and the constant light speed postulate. The purpose of this paper is to derive the law of reflection from a uniformly moving mirror based on Fermat's principle of least time.

Fermat's principle states the actual path between two points A and B taken by a beam of light is the one that is traversed in the least time.[3] A classic example of the application of Fermat's principle is the derivation of the law of reflection from a stationary mirror (Fig. 1). In this case, the constraint that the light is reflected from a point belonging to the surface of the mirror before it reaches the point B must be taken into account. The light is reflected at a point C whose x coordinate minimizes the time t required for the light to cover the path from A to B at a speed c:

$$t=\frac{\sqrt{d_A^2+x^2}}{c}+\frac{\sqrt{d_B^2+(l-x)^2}}{c}, \tag{1}$$

where d_A and d_B are the shortest distances between the surface of the mirror and A and B, respectively, and l is the distance between the orthogonal projections of A and B on the mirror's surface. If we take the derivative of t with respect to x in Eq. (1) and set the result to zero, we can use simple trigonometry to show that the angles of incidence and reflection are equal.[4]

Now, consider the situation when the mirror is moving at constant velocity v vertically downward (see Fig. 2). We will consider the reflection of a single photon emitted from point A at time t_0. By considering the reflection of a single photon instead of a whole light beam, we may neglect the shift that occurs when the light beam emanating from a stationary source is reflected by a moving mirror.[2] Without loss of generality, we take A and B to lie on the same horizontal line h, and the shortest distance between the fixed horizontal h and the moving mirror at the time of the emission of the photon t_0 to be d_0. Figure 2 shows a sketch of several hypothetical paths of the photon between A and B. Observe that the hypothetical points of reflection off the mirror's surface (C_1, C_2, and C_3) will not lie on the same horizontal due to the motion of the mirror and the finite speed of the photon. For example, by considering the hypothetical path AC_2B, we see that at time t_2 when the photon will hit the mirror at point C_2, the mirror has already crossed the distance $v(t_2-t_0)$ vertically downward from its initial position at time t_0.

According to Fermat's principle, the photon is reflected off the point C for which the path ACB requires the least time.[5] The situation is shown in Fig. 3. Here t_a is the time required for the photon to go from A to C, and t_b is the time from C to B. Observe that point C belongs to the mirror located at a vertical distance vt_a below the position of the mirror at the time when the photon was emitted from A. Thus, the total time required for the photon to travel the path ACB is

$$t=t_a+t_b. \tag{2}$$

By looking at Fig. 3 we see that

$$(ct_a)^2=(d_0+vt_a)^2+x^2, \tag{3}$$

$$(ct_b)^2=(d_0+vt_a)^2+(l-x)^2. \tag{4}$$

To simplify the derivation of the reflection law, we will use a trick. First, we rewrite Eqs. (3) and (4) as

$$t_a=\frac{1}{c}\sqrt{(d_0+vt_a)^2+x^2}, \tag{5}$$

$$t_b=\frac{1}{c}\sqrt{(d_0+vt_a)^2+(l-x)^2}. \tag{6}$$

Then, we substitute Eqs. (5) and (6) into Eq. (2) to obtain

$$t=\frac{1}{c}\sqrt{(d_0+vt_a)^2+x^2}+\frac{1}{c}\sqrt{(d_0+vt_a)^2+(l-x)^2}. \tag{7}$$

The principle of least time requires the value of x (that is, the point of reflection off the moving mirror's surface) to be such that the transit time $t(x)$ of the photon from A to B be a minimum. We take the derivative of Eq. (7) with respect to x, set the result to zero and obtain

$$v\frac{d_0+vt_a}{\sqrt{(d_0+vt_a)^2+x^2}}\frac{dt_a}{dx}+\frac{x}{\sqrt{(d_0+vt_a)^2+x^2}} +v\frac{d_0+vt_a}{\sqrt{(d_0+vt_a)^2+(l-x)^2}}\frac{dt_a}{dx} -\frac{l-x}{\sqrt{(d_0+vt_a)^2+(l-x)^2}}=0. \tag{8}$$

We have taken into account that t_a is a function of x. By considering the geometry of Fig. 3, Eq. (8) can be rewritten as

$$v\cos\alpha\frac{dt_a}{dx}+\sin\alpha+v\cos\beta\frac{dt_a}{dx}-\sin\beta=0. \tag{9}$$

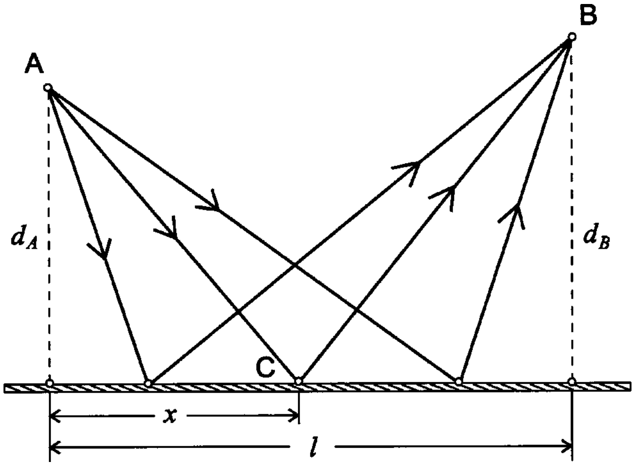

Fig. 1. Several hypothetical ways for a light traveling between A and B to be reflected by a motionless plane mirror.

Here α is the angle of incidence of the photon from the normal n of the mirror's surface, and β is the angle of its reflection.

The value of dt_a/dx can be calculated by taking the derivative of Eq. (3) with respect to x. Thus

$$c^2 t_a \frac{dt_a}{dx} = v(d_0 + vt_a)\frac{dt_a}{dx} + x, \tag{10}$$

or

$$\frac{dt_a}{dx} = \frac{x/ct_a}{c\left(1 - \frac{v}{c}\frac{d_0 + vt_a}{ct_a}\right)}, \tag{11}$$

which, according to Fig. 3, can be rewritten as

$$\frac{dt_a}{dx} = \frac{\sin\alpha}{c\left(1 - \frac{v}{c}\cos\alpha\right)}. \tag{12}$$

We substitute Eq. (12) into Eq. (9) and simplify the result to find the law of reflection of the photon[2]

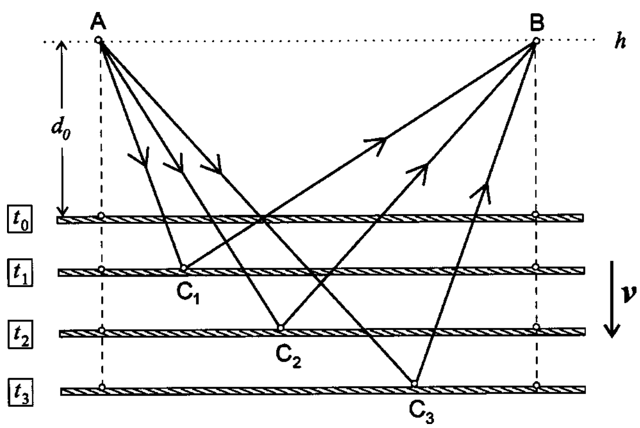

Fig. 2. Several hypothetical paths of the photon emitted from A at time t_0 at different angles from the vertical when the mirror is moving at constant velocity v. Because of the motion of the mirror, the hypothetical points of reflection off the mirror's surface, C_1, C_2, and C_3, at times t_1, t_2 and t_3, will not lie on the same horizontal line.

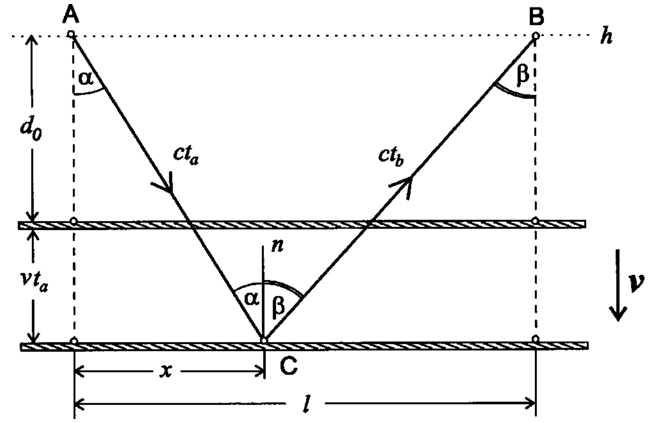

Fig. 3. The least time derivation of the law of reflection from a plane mirror in uniform rectilinear motion.

$$\sin\alpha - \sin\beta = -\frac{v}{c}\sin(\alpha+\beta). \tag{13}$$

According to problem 3 in Ref. 2, we can solve Eq. (13) for the reflected angle β in terms of the incident angle α and the velocity v of the moving mirror

$$\cos\beta = \frac{-2\frac{v}{c} + \left(1 + \frac{v^2}{c^2}\right)\cos\alpha}{1 - 2\frac{v}{c}\cos\alpha + \frac{v^2}{c^2}}. \tag{14}$$

Equation (14) is identical to the result for the angle of reflection obtained by Einstein.[1] Although it might seem that the derivation of Eq. (14) is not connected with special relativity, the result is a direct consequence of the constant light speed postulate. That is, according to the quantum picture of the process of reflection, the photon will be absorbed and re-emitted by the atoms at point C. The implicit assumption in our derivation is that the photon will be re-emitted at the same constant speed c, so that the speed of the photon does not depend on the motion of the source.

Similar considerations can be applied when the moving mirror is inclined at an arbitrary angle to the horizontal. It can be easily demonstrated that the law of reflection for an inclined mirror immediately follows from Eq. (14) if we replace v by the velocity component of the mirror that is normal to its surface.

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[1]A. Einstein, "Zur Elektrodynamik bewegter Körper," Ann. Phys. (Leipzig) **17**, 891–921 (1905); reprinted in *Einstein's Miraculous Year: Five Papers that Changed the Face of Physics*, edited by John Stachel (Princeton University Press, Princeton, 1998).
[2]A. Gjurchinovski, "Reflection of light from a uniformly moving mirror," Am. J. Phys. (to be published).
[3]This formulation of Fermat's principle will be sufficient for our purposes. However, a more precise statement of Fermat's principle is that a light ray going from A to B traverses an optical path length that is stationary with respect to the variations of that path. See, for example, E. Hecht, *Optics*, 2nd ed. (Addison-Wesley, Reading, MA, 1987).
[4]D. Halliday and R. Resnick, *Physics* (Wiley, New York, 1962), pp. 1028–1030.

[5]In terms of modern quantum electrodynamics, the photon explores all the available paths between points *A* and *B* that include reflection from the mirror. It is shown for a stationary mirror that the principle of least time immediately follows as a reasonable approximation of this argument. For a very elegant and elementary discussion of the subject, see R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, MA, 1989), Vol. 1, Chap. 26, and Feynman's original work, ''Space-time approach to non-relativistic quantum mechanics,'' Rev. Mod. Phys. **20**, 367–387 (1948). A very instructive treatment of Feynman's path integral method is given in E. F. Taylor, S. Vokos, J. M. O'Meara, and N. S. Thornber, ''Teaching Feynman's sum-over-paths quantum theory,'' Comput. Phys. **12**, 190–199 (1998). However, unlike the latter work, we stress that when the mirror is in uniform rectilinear motion, the imaginary ''quantum stopwatch'' of the reflected photon will rotate at a different frequency with respect to the same quantum stopwatch of the photon before reflection. This anomaly is due to the frequency shift of the photon which occurs during reflection. Einstein showed that the value of this frequency shift depends on the angle of incidence of the photon and the velocity of the moving mirror [See A. Einstein, ''Zur Elektrodynamik bewegter körper,'' Ann. Phys. (Leipzig) **17**, 891–921 (1905); reprinted in *Einstein's Miraculous Year: Five Papers that Changed the Face of Physics*, edited by John Stachel (Princeton University Press, Princeton, 1998).]

HISTORICAL NOTE ON THE RISE OF KINETIC THEORY

The rise of Kinetic Theory was of a gradual nature, and it is difficult to mention any time at which the theory may be said to have arisen, or any single name to whom honour of its establishment is due. Three stages in its development may be traced. There is first the stage of speculative opinion, unsupported by scientific evidence. Given that a great number of thinkers are speculating as to the structure of matter, it is only in accordance with the laws of probability that some of them should arrive fairly near to the truth. An opinion which turns out ultimately to be near the truth remains, however, of no greater value to the advancement of science than a more erroneous opinion, until scientific reasons can be given for supposing the former to be more accurate than the latter. When this point is reached the theory may be said to have entered upon the second stage of its development; the true and false opinions are still equally in the field, but the former is supplied with weapons for defeating the latter. In the third stage there is general agreement as to the main foundations of the theory and their truth, and labour is devoted no longer to defeating adverse opinion, but to the elaboration of the detail of the theory, and to attempts to extend its boundaries.

Sir James Jeans, *The Dynamical Theory of Gases Fourth Edition* (Dover Publications, INC., 1954), p.11.