Space-time exchange invariance: Special relativity as a symmetry principle

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Special relativity is reformulated as a symmetry property of space-time: space-time exchange invariance. The additional hypothesis of spatial homogeneity is then sufficient to derive the Lorentz transformation without reference to the traditional form of the Principle of Special Relativity. The kinematical version of the latter is shown to be a consequence of the Lorentz transformation. As a dynamical application, the laws of electrodynamics and magnetodynamics are derived from those of electrostatics and magnetostatics respectively. The four-vector nature of the electromagnetic potential plays a crucial role in the last two derivations.
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I. INTRODUCTION

Two postulates were essential for Einstein's original axiomatic derivation[1] of the Lorentz transformation (LT): (i) the Special Relativity Principle and (ii) the hypothesis of the constancy of the velocity of light in all inertial frames (Einstein's second postulate). The Special Relativity Principle, which states that

''**The laws of physics are the same in all inertial frames**,'' had long been known to be respected by Newton's laws of mechanics at the time Einstein's paper was written. Galileo had already stated the principle in 1588 in his ''Dialogues Concerning Two New Sciences.'' The title of Einstein's paper.[1] ''On the Electrodynamics of Moving Bodies,'' and the special role of light in his second postulate seem to link special relativity closely to classical electrodynamics. Indeed, the LT was discovered as the transformation that demonstrates that Maxwell's equations may be written in the same way in any inertial frame, and so manifestly respect the Special Relativity Principle. The same close connection between special relativity and classical electrodynamics is retained in virtually all textbook treatments of the subject, obscuring the essentially geometrical and kinematical nature of special relativistic effects. The latter actually transcend the dynamics of any particular physical system. It was realized, shortly after the space-time geometrical nature of the LT was pointed out by Minkowski,[2] that the domain of applicability of the LT extends beyond the classical electrodynamics considered by Einstein, and that, in fact, Einstein's second postulate is not necessary for its derivation.[3,4] There is now a vast literature devoted to derivations of the LT that do not require the second postulate.[5]

In a recent paper by the present author,[6] the question of the minimum number of postulates, in addition to the Special Relativity Principle, necessary to derive the LT was addressed. The aim of the present paper is somewhat different. The Special Relativity Principle itself is restated in a simple mathematical form which, as will be shown below, has both kinematical and dynamical applications. The new statement is a symmetry condition relating space and time, which, it is conjectured, is respected by the mathematical equations that describe all physical laws.[7] The symmetry condition is first used, together with the postulate of the homogeneity of space, to derive the LT. It is then shown that the Kinematical Special Relativity Principle (KSRP) is a necessary *consequence* of the LT. The KSRP, which describes the reciprocal nature of similar space-time measurements made in two different inertial frames,[8] states that

''**Reciprocal space-time measurements of similar measuring rods and clocks at rest in two different inertial frames *S*, *S*′ by observers at rest in *S*′, *S* respectively, yield identical results**.''

There is no reference here to any physical law. Only space-time events that may constitute the raw material of any observation of a physical process are considered. In the previous literature the KSRP (or some equivalent condition applied to a gedanken experiment[9]) has been used as a necessary postulate to derive the LT.

The symmetry condition that restates the Special Relativity Principle is

(I) ''**The equations describing the laws of physics are invariant with respect to the exchange of space and time coordinates, or, more generally, to the exchange of the spatial and temporal components of four vectors**.''

A corollary is

(II) ''**Predictions of physical theories do not depend on the metric sign convention (space-like or time-like) used to define four-vector scalar products**.''

A proof of this corollary is presented in Sec. IV.

As will become clear during the following discussion, the operation of space-time exchange (*STE*) reveals an invariance property of pairs of physical equations, which are found to map into each other under *STE*. The examples of this discussed below are: the Lorentz transformation equations of space and time, the Maxwell equations describing electrostatics (Gauss' law) and electrodynamics (Ampère's law), and those describing magnetostatics (Gauss' law) and magnetodynamics (the Faraday–Lenz law). It will be demonstrated that each of these three pairs of equations map into each other under *STE*, and so are invariants of the *STE* operator. In the case of the LT equations, imposing *STE* symmetry is sufficient to derive them from a general form of the space transformation equation that respects the classical limit.

The expression ''The equations describing the laws of physics'' in (I) should then be understood as including *both equations* of each *STE* invariant pair. For example, the Gauss equation of electrostatics, considered as an independent physical law, clearly does not respect (I).

For dimensional reasons, the definition of the exchange operation referred to in (I) requires the time coordinate to be multiplied by a universal parameter V with the dimensions of velocity. The new time coordinate with dimension $[L]$,

$$x^0 \equiv Vt, \tag{1.1}$$

may be called the ''causality radius''[10] to distinguish it from the Cartesian spatial coordinate x or the invariant interval s. Since space is three dimensional and time is one dimensional, there is a certain ambiguity in the definition of the exchange operation in (I). Depending on the case under discussion, the space coordinate may be either the magnitude of the spatial vector $x=|\vec{x}|$, or a Cartesian component x^1, x^2, x^3. For any physical problem with a preferred spatial direction (which is the case for the LT), then, by a suitable choice of coordinate system, the identification $x=x^1$, $x^2=x^3=0$ is always possible. The exchange operation in (I) is then simply $x^0 \leftrightarrow x^1$. Formally, the exchange operation is defined by the equations

$$STE x^0 = x^1, \tag{1.2}$$

$$STE x^1 = x^0, \tag{1.3}$$

$$(STE)^2 = 1, \tag{1.4}$$

where *STE* denotes the space time exchange operator. As shown below, for problems where there is no preferred direction, but rather spatial symmetry, it may also be useful to define three exchange operators:

$$x^0 \leftrightarrow x^i, \quad i=1,2,3, \tag{1.5}$$

with associated operations $STE(i)$ analogous to $STE = STE(1)$ in Eqs. (1.2)–(1.4). The operations in Eqs. (1.2)–(1.5) may also be generalized to the case of an arbitrary four-vector with temporal and spatial components A^0 and A^1, respectively.

To clarify the meaning of the *STE* operation, it is of interest to compare it with a different operator acting on space and time coordinates that may be called ''space-time coordinate permutation'' (*STCP*). Consider an equation of the form

$$f(x^0,x^1)=0. \tag{1.6}$$

The *STE* conjugate equation is

$$f(x^1,x^0)=0. \tag{1.7}$$

This equation is different from (1.6) because x^0 and x^1 have different physical meanings. In the *STCP* operation, however, the *values* of the space and time coordinates are interchanged, but no new equation is generated. If $x^0=a$ and $x^1=b$ in Eq. (1.6), then the *STCP* operation applied to the latter yields

$$f(x^0=b,x^1=a)=0. \tag{1.8}$$

This equation is identical in form to (1.6); only its parameters have different values.

The physical meaning of the universal parameter V, and its relation to the velocity of light, c, is discussed in the following section, after the derivation of the LT.

The plan of the paper is as follows. In the following section the LT is derived. In Sec. III, the LT is used to derive the KSRP. The space-time exchange properties of four-vectors and the related symmetries in Minkowski space are discussed in Sec. IV. In Sec. V the space-time exchange symmetries of Maxwell's equations are used to derive electrodynamics (Ampère's law) and magnetodynamics (the Faraday–Lenz law) from the Gauss laws of electrostatics and magnetostatics, respectively. A summary is given in Sec. VI.

II. DERIVATION OF THE LORENTZ TRANSFORMATION

Consider two inertial frames S, S'. S' moves along the common x, x' axis of orthogonal Cartesian coordinate systems in S, S' with velocity v relative to S. The y, y' axes are also parallel. At time $t=t'=0$ the origins of S and S' coincide. In general the transformation equation between the coordinate x in S of a fixed point on the Ox' axis and the coordinate x' of the same point referred to the frame S' is

$$x' = f(x, x^0, \beta), \tag{2.1}$$

where $\beta \equiv v/V$ and V is the universal constant introduced in Eq. (1.1). Differentiating Eq. (2.1) with respect to x^0, for fixed x', gives

$$\left.\frac{dx'}{dx^0}\right|_{x'} = 0 = \left.\frac{dx}{dx^0}\right|_{x'} \frac{\partial f}{\partial x} + \frac{\partial f}{\partial x^0}. \tag{2.2}$$

Since

$$\left.\frac{dx}{dx^0}\right|_{x'} = \frac{1}{V}\left.\frac{dx}{dt}\right|_{x'} = \frac{v}{V} = \beta,$$

the function f must satisfy the partial differential equation:

$$\beta \frac{\partial f}{\partial x} = -\frac{\partial f}{\partial x^0}. \tag{2.3}$$

A sufficient condition for f to be a solution of Eq. (2.3) is that it is a function of $x-\beta x^0$. Assuming also f is a differentiable function, it may be expanded in a Taylor series:

$$x' = \gamma(\beta)(x-\beta x^0) + \sum_{n=2} a_n(\beta)(x-\beta x^0)^n. \tag{2.4}$$

Requiring either spatial homogeneity,[11–13] or that the LT is a unique, single-valued, function of its arguments,[6] requires Eq. (2.4) to be linear, i.e.,

$$a_2(\beta) = a_3(\beta) = \cdots = 0$$

so that

$$x' = \gamma(\beta)(x-\beta x^0). \tag{2.5}$$

Spatial homogeneity implies that Eq. (2.5) is invariant when all spatial coordinates are scaled by any constant factor K. Noting that

$$-\beta = -\frac{1}{V}\left.\frac{dx}{dt}\right|_{x'} = \frac{1}{V}\left.\frac{d(-x)}{dt}\right|_{x'} \tag{2.6}$$

and choosing $K=-1$ gives

$$-x' = \gamma(-\beta)(-x+\beta x^0). \tag{2.7}$$

Hence, Eq. (2.5) is invariant provided that

$$\gamma(-\beta) = \gamma(\beta), \tag{2.8}$$

i.e., $\gamma(\beta)$ is an even function of β.

Applying the space-time exchange operations $x \leftrightarrow x^0$, $x' \leftrightarrow (x^0)'$ to Eq. (2.5) gives

$$(x^0)' = \gamma(\beta)(x^0 - \beta x). \tag{2.9}$$

The transformation inverse to (2.9) may, in general, be written as:

$$x^0 = \gamma(\beta')((x^0)' - \beta' x'). \tag{2.10}$$

The same inverse transformation may also be derived by eliminating x between Eqs. (2.5) and (2.9) and rearranging:

$$x^0 = \frac{1}{\gamma(\beta)(1-\beta^2)}((x^0)' + \beta x'). \tag{2.11}$$

Equations (2.10) and (2.11) are consistent provided that

$$\gamma(\beta') = \frac{1}{\gamma(\beta)(1-\beta^2)} \tag{2.12}$$

and

$$\beta' = -\beta. \tag{2.13}$$

Equations (2.8), (2.12), and (2.13) then give[14]

$$\gamma(\beta) = \frac{1}{\sqrt{1-\beta^2}}. \tag{2.14}$$

Equations (2.5) and (2.9) with γ given by (2.14) are the LT equations for space-time points along the common x, x' axis of the frames S, S'. They have been derived here solely from the symmetry condition (I) and the assumption of spatial homogeneity, without any reference to the Principle of Special Relativity.

The physical meaning of the universal parameter V becomes clear when the kinematical consequences of the LT for physical objects are worked out in detail. This is done, for example, in Ref. 6, where it is shown that the velocity of any massive physical object approaches V in any inertial frame in which its energy is much greater than its rest mass. The identification of V with the velocity of light, c, then follows[13,6] if it is assumed that light consists of massless (or almost massless) particles, the light quanta discovered by Einstein in his analysis of the photoelectric effect.[15] That V is the limiting velocity for the applicability of the LT equations is, however, already evident from Eq. (2.14). If $\gamma(\beta)$ is real, then $\beta \leqslant 1$, that is, $v \leqslant V$.

III. DERIVATION OF THE KINEMATICAL SPECIAL RELATIVITY PRINCIPLE

The LT equations (2.5) and (2.9) and their inverses, written in terms of x, x'; t, t', are

$$x' = \gamma(x - vt), \tag{3.1}$$

$$t' = \gamma\left(t - \frac{vx}{V^2}\right), \tag{3.2}$$

$$x = \gamma(x' + vt'), \tag{3.3}$$

$$t = \gamma\left(t' + \frac{vx'}{V^2}\right). \tag{3.4}$$

Consider now observers, at rest in the frames S, S', equipped with identical measuring rods and clocks. The observer in S' places a rod, of length l, along the common x, x' axis. The coordinates in S' of the ends of the rod are x_1', x_2', where $x_2' - x_1' = l$. If the observer in S measures, at time t in his own frame, the ends of the rod to be at x_1, x_2 then, according to Eq. (3.1):

$$x_1' = \gamma(x_1 - vt), \tag{3.5}$$

$$x_2' = \gamma(x_2 - vt). \tag{3.6}$$

Denoting by l_S the apparent length of the rod, as observed from S at time t, Eqs. (3.5) and (3.6) give

$$l_S \equiv x_2 - x_1 = \frac{1}{\gamma}(x_1' - x_2') = \frac{l}{\gamma}. \tag{3.7}$$

Suppose that the observer in S' now makes reciprocal measurements x_1', x_2' of the ends of a similar rod, at rest in S, at time t'. In S the ends of the rod are at the points x_1, x_2, where $l = x_2 - x_1$. Using Eq. (3.3)

$$x_1 = \gamma(x_1' + vt'), \tag{3.8}$$

$$x_2 = \gamma(x_2' + vt') \tag{3.9}$$

and, corresponding to (3.7), there is the relation

$$l_{S'} \equiv x_2' - x_1' = \frac{1}{\gamma}(x_2 - x_1) = \frac{l}{\gamma}. \tag{3.10}$$

Hence, from Eqs. (3.7) and (3.10)

$$l_S = l_{S'} = \frac{1}{\gamma}, \tag{3.11}$$

so that reciprocal length measurements yield identical results.

Consider now a clock at rest in S' at $x' = 0$. This clock is synchronized with a similar clock in S at $t = t' = 0$, when the spatial coordinate systems in S and S' coincide. Suppose that the observer at rest in S notes the time t recorded by his own clock, when the moving clock records the time τ. At this time, the clock which is moving along the common x, x' axis with velocity v will be situated at $x = vt$. With the definition $\tau_S \equiv t$, and using Eq. (3.2),

$$\tau = \gamma\left(\tau_S - \frac{vx}{V^2}\right) = \gamma\tau_S\left(1 - \frac{v^2}{V^2}\right) = \frac{\tau_S}{\gamma}. \tag{3.12}$$

If the observer at rest in S' makes a reciprocal measurement of the clock at rest in S, which is seen to be at $x' = -vt'$ when it shows the time τ, then according to Eq. (3.4) with $\tau_{S'} \equiv t'$,

$$\tau = \gamma\left(\tau_{S'} + \frac{vx'}{V^2}\right) = \gamma\tau_{S'}\left(1 - \frac{v^2}{V^2}\right) = \frac{\tau_{S'}}{\gamma}. \tag{3.13}$$

Equations (3.12) and (3.13) give

$$\tau_S = \tau_{S'} = \gamma\tau. \tag{3.14}$$

Equations (3.11) and (3.14) prove the Kinematical Special Relativity Principle as stated above. It is a necessary consequence of the LT.

IV. GENERAL SPACE-TIME EXCHANGE SYMMETRY PROPERTIES OF FOUR-VECTORS. SYMMETRIES OF MINKOWSKI SPACE

The LT was derived above for space-time points lying along the common x, x' axis, so that $x = |\vec{x}|$. However, this restriction is not necessary. In the case that $\vec{x} = (x^1, x^2, x^3)$, then x and x' in Eq. (2.5) may be replaced by $x = \vec{x} \cdot \vec{v}/|\vec{v}|$ and $x' = \vec{x}' \cdot \vec{v}/|\vec{v}|$, respectively, where the 1-axis is chosen parallel to $\vec{v}$. The proof proceeds as before with the space-time exchange operation defined as in Eqs. (1.2)–(1.4). The additional transformation equations,

$$y' = y, \tag{4.1}$$

$$z' = z, \tag{4.2}$$

follow from spatial isotropy.[1]

In the above derivation of the LT, application of the *STE* operator generates the LT of time from that of space. It is the pair of equations that is invariant with respect to the *STE* operation. Alternatively, as shown below, by a suitable change of variables, equivalent equations may be defined that are manifestly invariant under the *STE* operation.

The four-vector velocity U and the energy-momentum four-vector P are defined in terms of the space-time four-vector,[2]

$$X \equiv (Vt;x,y,z) = (x^0;x^1,x^2,x^3), \tag{4.3}$$

by the equations

$$U \equiv \frac{dX}{d\tau}, \tag{4.4}$$

$$P \equiv mV, \tag{4.5}$$

where m is the Newtonian mass of the physical object and τ is its proper time, i.e., the time in a reference frame in which the object is at rest. Since τ is a Lorentz invariant quantity, the four-vectors U, P have identical LT properties to X. The properties of U, P under the *STE* operation follow directly from Eqs. (1.2) and (1.3) and the definitions (4.4) and (4.5). Writing the energy-momentum four-vector as

$$P = \left(\frac{E}{V};p,0,0\right) = (p^0,p^1,0,0), \tag{4.6}$$

the *STE* operations: $p^0 \leftrightarrow p^1, (p^0)' \leftrightarrow (p^1)'$ generate the LT equation for energy

$$(p^0)' = \gamma(p^0 - \beta p^1) \tag{4.7}$$

from that of momentum

$$(p^1)' = \gamma(p^1 - \beta p^0) \tag{4.8}$$

or vice versa.

The scalar product of two arbitrary four-vectors C,D,

$$C \cdot D \equiv C^0 D^0 - \vec{C} \cdot \vec{D}, \tag{4.9}$$

can, by choosing the x-axis parallel to $\vec{C}$ or $\vec{D}$, always be written as:

$$C \cdot D = C^0 D^0 - C^1 D^1. \tag{4.10}$$

Defining the *STE* exchange operation for an arbitrary four-vector in a similar way to Eqs. (1.2) and (1.3), then the combined operations $C^0 \leftrightarrow C^1, D^0 \leftrightarrow D^1$ yield

$$C \cdot D \rightarrow C^1 D^1 - C^0 D^0 = -C \cdot D. \tag{4.11}$$

The four-vector product changes sign, and so the combined *STE* operation is equivalent to a change in the sign convention of the metric from space-like to time-like (or vice versa), hence the corollary (II) in Sec. I.

The LT equations take a particularly simple form if new variables are defined which have simple transformation properties under the *STE* operation. The variables are

$$x_+ = \frac{x^0 + x^1}{\sqrt{2}}, \tag{4.12}$$

$$x_- = \frac{x^0 - x^1}{\sqrt{2}}. \tag{4.13}$$

x_+, x_- have, respectively, even and odd ''*STE* parity:''

$$STE\, x_+ = x_+, \tag{4.14}$$

$$STE\, x_- = -x_-. \tag{4.15}$$

The manifestly *STE* invariant LT equations expressed in terms of these variables are

$$x'_+ = \alpha x_+, \tag{4.16}$$

$$x'_- = \frac{1}{\alpha} x_-, \tag{4.17}$$

where

$$\alpha = \sqrt{\frac{1-\beta}{1+\beta}}. \tag{4.18}$$

Introducing similar variables for an arbitrary four-vector,

$$C_+ = \frac{C^0 + C^1}{\sqrt{2}}, \tag{4.19}$$

$$C_- = \frac{C^0 - C^1}{\sqrt{2}}, \tag{4.20}$$

the 4-vector scalar product of C and D may be written as

$$C \cdot D = C_+ D_- + C_- D_+. \tag{4.21}$$

In view of the LT equations (4.16) and (4.17), $C \cdot D$ is manifestly Lorentz invariant. The transformations (4.12), (4.13) and (4.19), (4.20) correspond to an anti-clockwise rotation by 45° of the axes of the usual *ct* versus *x* plot. The x_+, x_- axes lie along the light cones of the *x*-*ct* plot (see Fig. 1).

The LT equations (4.16) and (4.17) give a parametric representation of a hyperbola in x_+, x_- space. A point on the latter corresponds to a particular space-time point as viewed in a frame S. The point $x_+ = x_- = 0$ corresponds to the space-time origin of the frame S' moving with velocity βc relative to S. A point at the spatial origin of S' at time $t' = \tau$ will be seen by an observer in S, as β (and hence α) varies, to lie on one of the hyperbolae H_{++}, H_{--} in Fig. 1:

$$x_+ x_- = \frac{c^2 \tau^2}{2} \tag{4.22}$$

with $x_+, x_- > 0$ if $\tau > 0 (H_{++})$ or $x_+, x_- < 0$ if $\tau < 0 (H_{--})$. A point along the x'-axis at a distance s from the origin, at $t' = 0$, lies on the hyperbolae H_{+-}, H_{-+}:

$$x_+ x_- = \frac{-s^2}{2} \tag{4.23}$$

with $x_+ > 0$, $x_- < 0$ if $s > 0 (H_{+-})$ or $x_+ < 0$, $x_- > 0$ if $s < 0 (H_{-+})$. As indicated in Fig. 1 the hyperbolae (4.22) correspond to the past ($\tau < 0$) or the future ($\tau > 0$) of a space-time point at the origin of S or S', whereas (4.23) corresponds to the ''elsewhere'' of the same space-time points, that is, the manifold of all space-time points that are causally

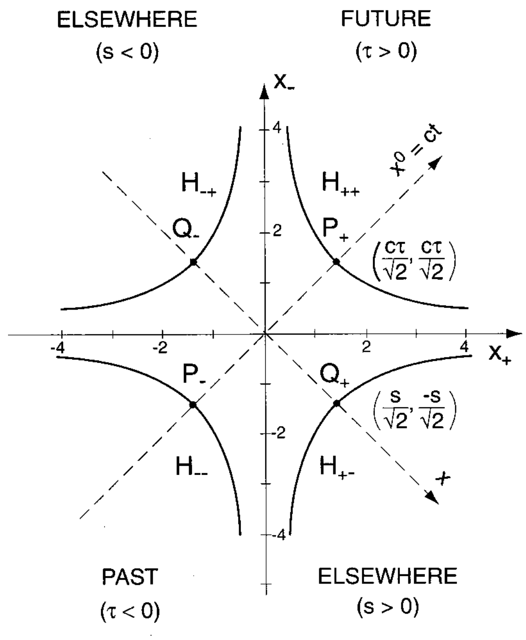

Fig. 1. Space-time points in S' as seen by an observer in S. The hyperbolae H_{++}, H_{--} correspond to points at the origin of S' at time $t'=\tau$. The hyperbolae H_{+-}, H_{-+} correspond to points at $x'=s$ and $t'=0$. See the text for the equations of the hyperbolae and further discussion.

disconnected from them. These are all familiar properties of the Minkowski space *x-ct* plot. One may note, however, the simplicity of the equations (4.16), (4.17), (4.22), and (4.23) containing the ''lightcone'' variables x_+, x_- that have simple transformation properties under the *STE* operation.

Another application of *STE* symmetry may be found in Ref. 16. It is shown there that the apparent distortions of space-time that occur in observations of moving bodies or clocks are related by this symmetry. For example, the Lorentz–Fitzgerald contraction is directly related to time dilatation by the *STE* operations (1.2) and (1.3).

V. DYNAMICAL APPLICATIONS OF SPACE-TIME EXCHANGE SYMMETRY

If a physical quantity is written in a manifestly covariant way, as a function of four-vector products, it will evidently be invariant with respect to *STE* as the exchange operation has the effect only of changing the sign convention for four-vector products from space-like to time-like or vice versa. An example of such a quantity is the invariant amplitude $\mathcal{M}$ for an arbitrary scattering process in quantum field theory. In this case *STE* invariance is equivalent to Corollary II of Sec. I.

More interesting results can be obtained from equations where components of four-vectors appear directly. It will now be shown how *STE* invariance may be used to derive Ampère's law and Maxwell's ''displacement current'' from the Gauss law of electrostatics, and the Faraday–Lenz law of magnetic induction from the the Gauss law of magnetostatics (the absence of magnetic charges). Thus electrodynamics and magnetodynamics follow from the laws of electrostatics and magnetostatics, together with space-time exchange symmetry invariance. It will be seen that the four-vector character of the electromagnetic potential plays a crucial role in these derivations.

In the following, Maxwell's equations are written in Heaviside–Lorentz units with $V=c=1$.[17] The four-vector potential $A=(A^0;\vec{A})$ is related to the electromagnetic field tensor $F^{\mu\nu}$ by the equation

$$F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu, \tag{5.1}$$

where

$$\partial^\mu=\left(\frac{\partial}{\partial t};-\vec{\nabla}\right)=(\partial^0;-\vec{\nabla}). \tag{5.2}$$

The electric and magnetic field components, E^k and B^k, respectively, are given, in terms of $F^{\mu\nu}$, by the equations

$$E^k=F^{k0}, \tag{5.3}$$

$$B^k=-\epsilon_{ijk}F^{ij}. \tag{5.4}$$

A time-like metric is used with $C_t=C^0=C_0$, $C_x=C^1=-C_1$, etc., with summation over repeated contravariant (upper) and covariant (lower) indices understood. Repeated Greek indices are summed from 1 to 4 and Roman ones from 1 to 3.

The transformation properties of contravariant and covariant four-vectors under the *STE* operation are now discussed. They are derived from the general condition that four-vector products change sign under the *STE* operation [Eq. (4.11)]. The four-vector product (4.10) is written, in terms of contravariant and covariant four-vectors, as

$$C\cdot D=C^0D_0+C^1D_1. \tag{5.5}$$

Assuming that the contravariant four-vector C^μ transforms according to Eqs. (1.2) and (1.3), i.e.,

$$C^0\leftrightarrow C^1, \tag{5.6}$$

the covariant four-vector D_μ must transform as:

$$D_0\leftrightarrow -D_1 \tag{5.7}$$

in order to respect the transformation property

$$C\cdot D\rightarrow -C\cdot D \tag{5.8}$$

of four-vector products under *STE*.

It remains to discuss the *STE* transformation properties of ∂^μ and the four-vector potential A^μ. In view of the property of $\partial^\mu:\partial^1=-\partial_x=-\partial/\partial x$ [Eq. (5.2)], which is similar to the relation $C_1=-C_x$ for a *covariant* four-vector, it is natural to choose for ∂^μ a *STE* transformation similar to Eq. (5.7):

$$\partial^0\leftrightarrow -\partial^1, \tag{5.9}$$

and hence, in order that $\partial^\mu\partial_\mu$ change sign under *STE*:

$$\partial_0\leftrightarrow\partial_1. \tag{5.10}$$

This is because it is clear that the appearance of a minus sign in the *STE* transformation equation (5.7) is correlated with the minus sign in front of the spatial components of a covariant four-vector, not whether the Lorentz index is an upper or lower one. Thus ∂^μ and ∂_μ transform in an ''anomalous'' manner under *STE* as compared to the convention of Eqs. (5.6) and (5.7). In order that the four-vector product $\partial_\mu A^\mu$ respect the condition (5.8), A^μ and A_μ must then transform under *STE* as:

$$A^0\leftrightarrow -A^1 \tag{5.11}$$

and

$$A_0 \leftrightarrow A_1, \tag{5.12}$$

respectively. That is, they transform in the same way as ∂^μ and ∂_μ, respectively.

Introducing the four-vector electromagnetic current $j^\mu \equiv (\rho;\vec{j})$, Gauss' law of electrostatics may be written as:

$$\vec{\nabla}\cdot\vec{E}=\rho=j^0, \tag{5.13}$$

or, in the manifestly covariant form,

$$(\partial_\mu\partial^\mu)A^0-\partial^0(\partial_\mu A^\mu)=j^0. \tag{5.14}$$

This equation is obtained by writing Eq. (5.13) in covariant notation using Eqs. (5.1) and (5.3) and adding to the left side the identity:

$$\partial_0(\partial^0A^0-\partial^0A^0)=0. \tag{5.15}$$

Applying the space-time exchange operation to Eq. (5.14), with index exchange $0\rightarrow1$ [noting that ∂^0, A^0 transform according to Eqs. (5.9) and (5.11), j^0 according to (5.6), and that the scalar products $\partial_\mu\partial^\mu$ and $\partial_\mu A^\mu$ change sign] yields the equation

$$(\partial_\mu\partial^\mu)A^1-\partial^1(\partial_\mu A^\mu)=j^1. \tag{5.16}$$

The spatial part of the four-vector products on the left side of Eq. (5.16) is

$$\partial_i(\partial^iA^1-\partial^1A^i)=\partial_iF^{i1}=\partial_2B^3-\partial_3B^2=(\vec{\nabla}\times\vec{B})^1, \tag{5.17}$$

where Eqs. (5.1) and (5.4) have been used. The time part of the four-vector products in Eq. (5.16) yields, with Eqs. (5.1) and (5.3),

$$\partial_0(\partial^0A^1-\partial^1A^0)=-\frac{\partial E^1}{\partial t}. \tag{5.18}$$

Combining Eqs. (5.16)–(5.18) gives

$$(\vec{\nabla}\times\vec{B})^1-\frac{\partial E^1}{\partial t}=j^1. \tag{5.19}$$

Combining Eq. (5.19) with the two similar equations derived by the index exchanges $0\rightarrow2$, $0\rightarrow3$ in Eq. (5.14) gives

$$(\vec{\nabla}\times\vec{B})-\frac{\partial\vec{E}}{\partial t}=\vec{j}. \tag{5.20}$$

This is Ampère's law, together with Maxwell's displacement current.

The Faraday-Lenz law is now derived by applying the space-time exchange operation to the Gauss law of magnetostatics:

$$\vec{\nabla}\cdot\vec{B}=0. \tag{5.21}$$

Introducing Eqs. (5.4) and (5.1) into Eq. (5.21) gives

$$\partial_1(\partial^3A^2-\partial^2A^3)+\partial_2(\partial^1A^3-\partial^3A^1)+\partial_3(\partial^2A^1-\partial^1A^2)=0. \tag{5.22}$$

Making the exchange $1\rightarrow0$ of space-time indices in Eq. (5.22) and noting that ∂_1 transforms according to Eq. (5.10), whereas ∂^1, A^1 transform as in Eqs. (5.9) and (5.11), respectively, gives

$$\partial_0(\partial^3A^2-\partial^2A^3)+\partial_2(-\partial^0A^3+\partial^3A^0)+\partial_3(-\partial^2A^0-\partial^0A^2)=0. \tag{5.23}$$

Using Eqs. (5.1)–(5.4), Eq. (5.23) may be written as:

$$\frac{\partial B^1}{\partial t}+\partial_2E^3-\partial_3E^2=0, \tag{5.24}$$

or, in three-vector notation,

$$(\vec{\nabla}\times\vec{E})^1=-\frac{\partial B^1}{\partial t}. \tag{5.25}$$

The space-time exchanges $2\rightarrow0$, $3\rightarrow0$ in Eq. (5.22) yield, in a similar manner, the two- and three-components of the Faraday–Lenz law:

$$(\vec{\nabla}\times\vec{E})=-\frac{\partial\vec{B}}{\partial t}. \tag{5.26}$$

Some comments now on the conditions for the validity of the above derivations. It is essential to use the manifestly covariant form of the electrostatic Gauss law, Eq. (5.14), and the manifestly rotationally invariant form, Eq. (5.22), of the magnetostatic Gauss law. For example, the 1-axis may be chosen parallel to the electric field in Eq. (5.13). In this case Eq. (5.14) simplifies to

$$\partial_1(\partial^0A^1-\partial^1A^0)=j^0. \tag{5.27}$$

Applying the space-time exchange operation $0\leftrightarrow1$ to this equation yields only the Maxwell displacement current term in Eq. (5.19). Similarly, choosing the 1-axis parallel to $\vec{B}$ in Eq. (5.21) simplifies Eq. (5.22) to

$$\partial_1(\partial^3A^2-\partial^2A^3)=0. \tag{5.28}$$

The index exchange $1\rightarrow0$ leads then to the equation

$$\frac{\partial B^1}{\partial t}=0. \tag{5.29}$$

instead of the 1-component of the Faraday–Lenz law, as in Eq. (5.24).

The choice of the *STE* transformation properties of contravariant and covariant four-vectors according to Eqs. (5.6) and (5.7) is an arbitrary one. Identical results are obtained if the opposite convention is used. However, ''anomalous'' transformation properties of $\partial^\mu, \partial_\mu$ and A^μ, A_μ, in the sense described above, are essential. This complication results from the upper index on the left side of Eq. (5.2) whereas on the right side the spatial derivative is multiplied by a minus sign. This minus sign changes the *STE* transformation property relative to that, (5.6), of conventional contravariant four-vectors that do not have a minus sign multiplying the spatial components. The upper index on the left side of Eq. (5.2) is a consequence of the Lorentz transformation properties of the four dimensional space-time derivative.[18]

VI. SUMMARY AND DISCUSSION

In this paper the Lorentz transformation for points lying along the common x,x' axis of two inertial frames has been derived from only two postulates: (*i*) the symmetry principle (I), and (*ii*) the homogeneity of space. This is the same number of axioms as used in Ref. 6 where the postulates were the Kinematical Special Relativity Postulate and the uniqueness condition. Since both spatial homogeneity and uniqueness require the LT equations to be linear, the KSRP of Ref. 6 has here, essentially, been replaced by the space-time symmetry condition (I).

Although postulate (I) and the KRSP play equivalent roles in the derivation of the LT, they state in very different ways the physical foundation of special relativity. Postulate (I) is a mathematical statement about the structure of the equations of physics, whereas the KSRP makes, instead, a statement about the relation between space-time measurements performed in two different inertial frames. It is important to note that in neither case do the dynamical laws describing any particular physical phenomenon enter into the derivation of the LT.

Choosing postulate (I) as the fundamental principle of special relativity instead of the Galilean Relativity Principle, as in the traditional approach, has the advantage that a clear distinction is made, from the outset, between classical and relativistic mechanics. Both the former and the latter respect the Galilean Relativity Principle but with different laws. On the other hand, only relativistic equations, such as the LT or Maxwell's equations, respect the symmetry condition (I).

The teaching of, and hence the understanding of, special relativity differs greatly depending on how the parameter V is introduced. In axiomatic derivations of the LT, which do not use Einstein's second postulate, a universal parameter V with the dimensions of velocity necessarily appears at an intermediate stage of the derivation.[19] Its physical meaning, as the absolute upper limit of the observed velocity of *any* physical object, only becomes clear on working out the kinematical consequences of the LT.[6] If Einstein's second postulate is used to introduce the parameter c, as is done in the vast majority of textbook treatments of special relativity, justified by the empirical observation of the constancy of the velocity of light, the actual universality of the theory is not evident. The misleading impression may be given that special relativity is an aspect of classical electrodynamics, the domain of physics in which it was discovered.

Formulating special relativity according to the symmetry principle (I) makes clear the space-time geometrical basis[2] of the theory. The universal velocity parameter V must be introduced at the outset in order even to define the space-time exchange operation. Unlike the Galilean Relativity Principle, the symmetry condition (I) gives a clear test of whether any physical equation is a candidate to describe a universal law of physics. Such an equation must either be invariant under space-time exchange or related by the exchange operation to another equation that also represents a universal law. The invariant amplitudes of quantum field theory are an example of the former case, while the LT equations for space and time correspond to the latter. Maxwell's equations are examples of dynamical laws that satisfy the symmetry condition (I). The laws of electrostatics and magnetostatics (Gauss' law for electric and magnetic charges) are related by the space-time exchange symmetry to the laws of electrodynamics (Ampère's law) and magnetodynamics (the Faraday–Lenz law), respectively. The four-vector character[20] of the electromagnetic potential is essential for these symmetry relations.[21]

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[5]See, for example, Ref. 18 of V. Berzi and V. Gorini, ''Reciprocity Principle and Lorentz Transformations,'' J. Math. Phys. **10**, 1518–1524 (1969). More recent references may be found in Ref. 6, and in J. R. Lucas and P. E. Hodgson, *Space Time and Electromagnetism* (Oxford U.P., Oxford, 1990).

[6]J. H. Field, ''A New Kinematical Derivation of the Lorentz Transformation and the Particle Description of Light,'' Helv. Phys. Acta **70**, 542–564 (1997).

[7]That is, all laws applying to physical systems where the curvature of space-time may be neglected, so that general relativistic effects are unimportant, and may be neglected.

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[10]In J. A. Wheeler and R. P. Feynman, ''Classical Electrodynamics in Terms of Direct Interparticle Action,'' Rev. Mod. Phys. **21**, 425–433 (1949), this quantity is called ''co-time.''

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[14]The positive sign for γ is taken in solving Eq. (2.12). Evidently $\gamma \to 1$ as $\beta \to 0$.

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[17]See, for example, I. J. R. Aitchison and A. J. G. Hey *Gauge Theories in Particle Physics* (Hilger, London, 1982), Appendix C.

[18]See, for example, S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972), p. 36.

[19]See, for example, Eq. (2.36) of Ref. 6.

[20]For a recent discussion of the physical meaning of the three-vector magnetic potential see M. D. Semon and J. R. Taylor, ''Thoughts on the magnetic vector potential,'' Am. J. Phys. **64**, 1361–1369 (1996).

[21]It is often stated in the literature that the potentials $\phi, \vec{A}$ are introduced only for ''reasons of mathematical simplicity'' and ''have no physical meaning.'' See, for example: F. Röhrlich, *Classical Charged Particles* (Addison-Wesley, Reading, MA, 1990), pp. 65–66. Actually, the underlying space-time symmetries of Maxwell's equations can only be expressed by using the four-vector character of A^μ. Also the minimal electromagnetic interaction in the covariant formulation of relativistic quantum mechanics, which is the dynamical basis of quantum electrodynamics, requires the introduction of a quantum field for the photon that has the same four-vector nature as the electromagnetic potential.