Two novel special relativistic effects: Space dilatation and time contraction

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(Received 1 March 1999; accepted 21 September 1999)

The conventional discussion of the observed distortions of space and time in special relativity (the Lorentz–Fitzgerald contraction and time dilatation) is extended by considering observations, from a stationary frame, of: (i) objects moving with constant velocity and uniformly illuminated during a short time τ_L (their ''luminous proper time'') in their rest frame; these may be called ''transient luminous objects'' and (ii) a moving, extended, array of synchronized ''equivalent clocks'' in a common inertial frame. Application of the Lorentz transformation to (i) shows that such objects, observed from the stationary frame with coarse time resolution in a direction perpendicular to their direction of motion, are seen to be at rest but *longer* in the direction of the relative velocity **v** by a factor $1/\sqrt{1-(v/c)^2}$ (space dilatation) and to (ii) that the moving equivalent clock at any fixed position in the rest frame of the stationary observer is seen to be running *faster* than a similar clock at rest by the factor $1/\sqrt{1-(v/c)^2}$ (time contraction). All four space–time ''effects'' of special relativity are simply classified in terms of the projective geometry of space–time, and the close analogy of these effects to linear spatial perspective is pointed out.

I. INTRODUCTION

In his 1905 paper on special relativity[1] Einstein showed that time dilatation (TD) and the Lorentz–Fitzgerald contraction (LFC), which had previously been introduced in a somewhat *ad hoc* way into classical electrodynamics, are simple consequences of the Lorentz Transformation (LT), that is, of the geometry of space–time.

As an example of the LFC Einstein stated that a sphere moving with velocity v would, ''viewed from the stationary system,'' appear to be contracted by the factor $\sqrt{1-(v/c)^2}$ in its direction of motion where c is the velocity of light in free space. It was only pointed out some 54 years later that if ''viewed'' was interpreted in the conventional sense of ''as seen by the eye, or recorded on a photograph'' then the sphere does not at all appear to be contracted, but is still seen as a sphere with the same dimensions as a stationary one and at the same position.[2–4] It was shown in general[3,4] that transversely viewed moving objects subtending a small solid angle at the observer appear not to be distorted in shape or changed in size, but rather rotated, as compared to a similarly viewed and orientated object at rest. This apparent rotation is a consequence of three distinct physical effects:

(i) the LFC,
(ii) optical aberration,
(iii) different propagation times of photons emitted by different parts of the moving object.

The effect (ii) may be interpreted as the change in direction of photons, emitted by a moving source, due to the LT between the rest frames of the source and the stationary observer. Correcting for (ii) and (iii), the LFC can be deduced as a physical effect, if not directly observed. It was also pointed out by Weinstein[5] that if a single observer is close to a moving object, then, because of the effect of light propagation time delays, it will appear elongated if moving toward the observer and contracted (to an extent greater than the LFC) if moving away. Only an object moving strictly transversely to the line of sight of a close observer shows the LFC.

However, the LFC itself is a physical phenomenon similar in many ways to (iii). The human eye or a photograph taken with a fast shutter record, as a sharp image, the photons incident on it during a short resolution time τ_R. That is, the image corresponds to a projection at an almost fixed time in the frame S of observation. This implies that the photons constituting the observed image are emitted at different times from the different parts, *along the line of sight*, of an extended object. As shown below, the LFC is similarly defined by a fixed time projection in the frame S. The LT then requires that the photons constituting the image of a moving object are also emitted at different times, in the rest frame S' of the object, from the different parts *along its direction of motion*. In the following S will, in general, denote the reference frame of a ''stationary'' observer (space–time coordinates x, y, z, t) while S' refers to the rest frame of an object moving with uniform velocity v in the direction of the positive x axis relative to S (space–time coordinates x', y', z', t').

The purpose of this paper is to point out that the $t=$constant projection of the LFC (see Sec. II) and the $x'=$constant projection of TD (see Sec. III) are not the only physically distinct space–time measurements possible within special relativity. In fact, as will be demonstrated below, there are two others: space dilatation (SD), the $t'=$constant projection and time contraction (TC), the $x=$constant projection. All four ''effects'' are pure consequences of the LT. The additional effects of optical aberration and light propagation delays on the appearance of moving objects and synchronized clocks have been extensively discussed elsewhere.[6]

Although each of the four effects may be simply derived from the projective geometry of the space–time LT, the LFC and TD give rise to more easily observable physical effects, so it is not surprising that they are better known. For example the LFC is essential for the physical interpretation of the Michelson–Morley experiment, and TD is necessary to describe the observed lifetimes of unstable particles decaying in flight. In contrast, the two new effects SD and TC seem to have no similar simple observational consequences. As pointed out below, the most interesting effects are likely to result from SD, which is necessary to describe observations of, for example, a rotating extended object moving with a relativistic transverse velocity. It is easy to conceive a simple experiment involving the observation of two synchronized clocks in space, to test the TC effect. Although it is clearly of interest to work out in more detail such examples, there is no attempt to do so in the present paper, which is devoted to the precise definition of the four possible space–time projections of the LT and a discussion of their interrelations.

The $t=$constant projection of the LFC is the space–time measurement appropriate to the ''moving bodies'' of Einstein's original paper and to the photographic recording technique. This medium has no intrinsic time resolution and relies on that provided by a rapidly moving shutter to provide a clear image. The LFC ''works'' as a well-defined physical phenomenon because the ''measuring rod'' or other physical object under observation is assumed to be illuminated during the whole time interval required to make an observation, and so constitutes a continuous source of emitted or reflected photons, such that some are always available in the different space ($\Delta x'$) and time ($\Delta t'$) intervals in S' for every position of the rod corresponding to the time interval $\Delta t=\tau_R$ around the fixed time t in the observer's frame S during which the observation is made. If, however, the physical object of interest has internal motion (rotation, expansion, or contraction) or is only illuminated, in its rest frame S', during a short time interval, the above conditions, which assure that the $t=$constant projection gives a well-defined space–time measurement no longer apply. All such objects, uniformly illuminated for a restricted time τ_L (their ''luminous proper time'') in their rest frames, may be called ''transient luminous objects.'' For such objects it is natural to define a length measurement by taking the $t'=$constant projection in S'. The observation, from the stationary frame S, of such objects is discussed in Sec. II.

In Sec. III time measurements other than the conventional TD of special relativity are considered. The TD phenomenon refers only to a local clock, in the sense that its position in the frame S' is invariant (say at the spatial origin of coordinates $x'=0$). However, the time recorded by any synchronized clock in the same inertial frame is, by definition, identical. Einstein used such an array of ''equivalent clocks'' situated at different positions in the same inertial frame in his original discussion of the relativity of simultaneity.[1] The question addressed in Sec. III is: What will an observer in S see if he looks not only at a given local clock in S', but also at other, synchronized, equivalent clocks at different positions in S', in comparison to a standard clock at rest in his own frame? It is shown that such equivalent clocks may be seen to run slower than, or faster than, the TD prediction for a local clock. In particular they may even appear to *run faster than the standard clock*. This is an example of the time contraction effect mentioned previously.

In Sec. IV the analogy between the Lorentz–Fitzgerald contraction effect and linear perspective in two spatial dimensions is described. Section V points out how all four

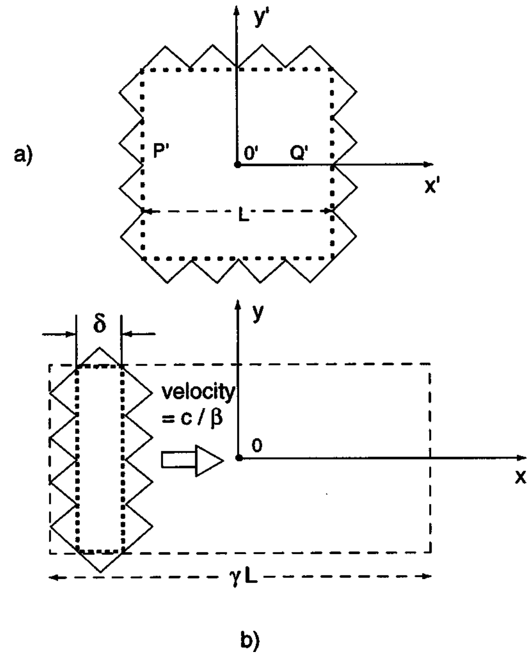

Fig. 1. (a) A square ''transient luminous object'' (indicated by the zig-zag outline) as viewed in the frame S' in which it is at rest. (b) The same object as viewed at a fixed time from the frame S moving with velocity $-\beta c$ parallel to the Ox' axis. It is assumed that the luminous proper time τ_L of the object is small: $\tau_L \ll \beta L/c$. The actual outlines of the objects are shown as short dashed lines. The long dashed rectangle of width γL in (b) shows the outline of the object when viewed with coarse time resolution: $\tau_R \gg \gamma\beta L/c$. The observer in S is assumed to be sufficiently distant from the object that the effects of light propagation delays may be neglected. The object is also assumed to constitute a diffuse photon source so that optical aberration effects are negligible.

space–time ''effects'' (observed distortions of space–time) in special relativity may be described in a unified way in terms of projective geometry, in close analogy with the effect of linear perspective in the perception of space.

II. OBSERVATION OF TRANSIENT LUMINOUS OBJECTS IN MOTION: THE SPACE DILATATION EFFECT

Consider a square planar object centered at the origin of the moving frame S' as shown in Fig. 1(a). The points $P'(x'=-L/2, y'=0)$ and $Q'(x'=L/2, y'=0)$ lie on the vertical edges of the square of side L whose boundary is shown in Fig. 1(a) as the short dashed lines. Suppose now that the square is uniformly illuminated in the time interval $-\tau_L/2<t'<\tau_L/2$ to give the ''transient luminous object'' indicted by the zig-zag lines. The proper time interval τ_L is the ''luminous proper time'' of the object. For example, the surface of the square may be covered with a mosaic of light-emitting diodes that are simultaneously switched on during a time τ_L. The object as seen by an observer, at rest in the stationary system S, viewing the object in a direction perpendicular to the plane $O'x'y'$, is given by the LT connecting space–time points in the frame S' to those in S:

$$x=\gamma(x'+vt'), \tag{2.1}$$

$$t=\gamma\left(t'+\beta\frac{x'}{c}\right), \tag{2.2}$$

Table I. Space–time points on the object at rest in S' (see Fig. 1), at time $t'=0$, as observed in the frames S', S.

Point	x'	t'	x	t
P'	$-\frac{L}{2}$	0	$-\frac{\gamma L}{2}$	$-\frac{\gamma\beta L}{2c}$
O'	0	0	0	0
Q'	$\frac{L}{2}$	0	$\frac{\gamma L}{2}$	$\frac{\gamma\beta L}{2c}$

where

$$\beta \equiv v/c, \quad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}.$$

It is assumed that the stationary observer is sufficiently distant from the object that the effects of light propagation times are negligible, and that the object is diffusely illuminated so that optical aberration effects may be neglected.[6] In this case any changes in the appearance of the moving object when viewed from the frame S are due solely to the LT. The results of the transformation for $t'=0$ and $x'=-L/2,0,L/2$ are given in Table I. It can be seen that the points P', O', Q are observed at different times in the frame S. This is the well-known effect of the relativity of simultaneity first pointed out in Einstein's classic paper.[1] It can also be seen from Table I that the distance between the positions of P' and Q' as observed in S is γL; that is, the object will appear to be elongated if it is viewed with a time resolution larger than the difference in time, $\gamma\beta L/c$, between the observations in S of the space–time points P' and Q' that are simultaneous in the frame S'. This is the ''space dilatation'' (SD) effect. It will now be discussed in more detail, taking into account the nonzero luminous proper time τ_L of the transient luminous object as well as the resolution time τ_R of the observer, so that the general conditions under which the SD effect occurs are established.

Space–time points of the transient luminous object may be observed at the fixed time t in S provided that

$$x'_{\min} < x' < x'_{\max},$$

where

$$t = \gamma\left(-\frac{\tau_L}{2} + \frac{\beta x'_{\max}}{c}\right) = \gamma\left(\frac{\tau_L}{2} + \frac{\beta x'_{\min}}{c}\right). \tag{2.3}$$

In (2.3) it is assumed that $x'_{\min} > -L/2$, $x'_{\max} < L/2$. The general condition relating τ_L, L, v, and c ensuring the validity of this assumption will be discussed below. Using (2.1) the coordinates in S corresponding to $x'_{\min}$ and $x'_{\max}$ are found to be

$$x_{\max} = \frac{c}{\beta}\left(t + \frac{\tau_L}{2\gamma}\right), \tag{2.4}$$

$$x_{\min} = \frac{c}{\beta}\left(t - \frac{\tau_L}{2\gamma}\right). \tag{2.5}$$

Thus the width δ of the transient luminous object observed at time t in S [indicated by the zig-zag lines in Fig. 1(b); the actual boundary is shown by the short dashed lines] is

$$\delta \equiv x_{\max} - x_{\min} = \frac{c\tau_L}{\beta\gamma}, \tag{2.6}$$

while, as can be seen from (2.4) and (2.5), the observer in S sees a luminous object that moves with velocity c/β, i.e., faster than the velocity of light. In the case of continuous illumination of the object ($\tau_L \to \infty$) the upper and lower limits of the object observed at the fixed time t in S will correspond to the physical boundaries $x'_{\min} = -L/2$, $x'_{\max} = L/2$. Denoting by $t'_{\min}$, $t'_{\max}$ the times in S' corresponding to the observation of these boundaries at time t in S, then, instead of (2.3), the following relation is obtained:

$$t = \gamma\left(t'_{\min} + \frac{\beta L}{2c}\right) = \gamma\left(t'_{\max} - \frac{\beta L}{2c}\right). \tag{2.7}$$

Using (2.1), the boundaries of the object observed in S at time t are then

$$x_{\max} = \frac{L}{2\gamma} + vt, \tag{2.8}$$

$$x_{\min} = -\frac{L}{2\gamma} + vt. \tag{2.9}$$

The width of the object as seen in S is then $x_{\max} - x_{\min} = L/\gamma$, the well-known LFC effect. As can be seen from (2.8) and (2.9) the object is now observed to move in S with velocity v. Thus, in the limit $\tau_L \to \infty$ (continuous illumination of the object) the usual results of special relativity are recovered.

Using (2.1) and (2.2) the upper (U) and lower (L) limits of the space–time region in the stationary frame S swept out by the moving transient luminous object in Fig. 1(b) are

$$x_U = \frac{\gamma}{2}(L + v\tau_L), \tag{2.10}$$

$$x_L = \frac{\gamma}{2}(-L - v\tau_L), \tag{2.11}$$

$$t_U = \frac{\gamma}{2}\left(\tau_L + \frac{\beta L}{c}\right), \tag{2.12}$$

$$t_L = \frac{\gamma}{2}\left(-\tau_L - \frac{\beta L}{c}\right). \tag{2.13}$$

Taking account of the inequality:

$$\frac{\beta L}{c} < \frac{L}{v},$$

it can be seen that if $\tau_L \ll \beta L/c$ the terms containing τ_L in (2.10)–(2.13) may be neglected, so that

$$x_U - x_L \simeq \gamma L, \tag{2.14}$$

$$t_U - t_L \simeq \frac{\gamma\beta L}{c}. \tag{2.15}$$

Thus the space dilatation effect of Table I is recovered in the limit $\tau_L \to 0$. On the other hand, because of the inequality:

$$v\tau_L < \frac{c\tau_L}{\beta},$$

then, if $\tau_L \gg L/v$, the terms containing L in (2.10)–(2.13) may be neglected, leading to the relations:

$$x_U - x_L \simeq \gamma v \tau_L , \tag{2.16}$$

$$t_U - t_L \simeq \gamma \tau_L . \tag{2.17}$$

These are the well-known equations of special relativity describing the motion of a small continously illuminated object as observed in the frame S. The time interval $t_U - t_L$ corresponds to the TD effect and the observed velocity is

$$(x_U - x_L)/(t_U - t_L) = \gamma v \tau_L / \gamma \tau_L = v. \tag{2.18}$$

The conclusions of this section are now summarized. When $\tau_L \ll \beta L/c$, a stationary observer in S with a time resolution $\tau_R \ll \gamma \beta L/c$, viewing the object in the direction transverse to the relative velocity, sees the square object at rest in S' illuminated during the proper time interval τ_L as a narrow rectangular object of width $\delta = c\tau_L/(\beta\gamma)$ moving with velocity c/β and sweeping out during the time $\gamma\beta L/c$ a region of total length γL. If, however, the resolution time τ_R of the observer is much larger than $\gamma\beta L/c$, *the object will appear at rest but elongated by the factor* γ *in the direction of motion*. This is the space dilatation effect. In the contrary case that the luminous proper time τ_L is large ($\tau_L \gg L/v$), the object observed from S moves with velocity v and has an apparent length L/γ due to the well-known LFC effect. Also, in this case, the elapsed times in S and S' are related by the TD effect [Eq. (2.17)].

It should be noted that the ''narrow rectangular object'' referred to above corresponds to the case of uniform illumination of the square object. Actually, because of the relativity of simultaneity, different parts of the square are seen at different times and positions by the stationary observer. If the square were illuminated using different colors: red, yellow, green, blue in four equal bands parallel to the y' axis, in the direction of increasing x', then the moving object in Fig. 1(b) would appear red during the time interval $-\gamma\beta L/2c < t < \gamma\beta L/4c$, yellow during the time $-\gamma\beta L/4c < t < 0$, and so on. The colors will, of course, be seen shifted in frequency according to the relativistic transverse Doppler effet.

If the square is rotated about the y' axis by an angle α, a subtle interplay occurs between the effects of the LT and light propagation time delays. Depending on the values of v and α the rectangular object may be seen, by an observer at rest in the frame S, to move parallel to $\mathbf{v}$ (as in the above-described case $\alpha = 0$), antiparallel to $\mathbf{v}$, or may even be stationary and of length $\gamma L \cos\alpha$. In all cases the total length swept out by the object in the direction of motion is $\gamma L \cos\alpha$. These effects have been described in detail elsewhere.[6]

III. OBSERVATION OF AN ARRAY OF EQUIVALENT MOVING CLOCKS: THE TIME CONTRACTION EFFECT

In this section space–time measurements of an array of synchronized clocks situated in the inertial frame S' will be considered. These clocks may be synchronized by any convenient procedure[7] (see, for example, Ref. 1). For an observer in S' all such clocks are ''equivalent'' in the sense that each of them records, independently of its position, the proper time τ' of the frame S'. For convenience, the array of clocks is assumed to be placed on the wagons of a train which is at rest in S', as shown in Fig. 2(a). The clocks are labeled C_m, $m = \cdots -2, -1, 0, 1, 2, \ldots$ and are situated (with the exception of the ''magic clock'' C_M, see the following) at fixed distances L from each other, along the Ox' axis, which is parallel to the train. It is assumed that the observers in the frames S and S' view the train transversely at a sufficiently large distance that the effects of light propagation time delays may be neglected. It is clear that by considering the limit $L \to 0$ an equivalent clock may be associated with each position on the train and, by extending the ''lattice'' of clocks to three dimensions, to any spatial position in S'.

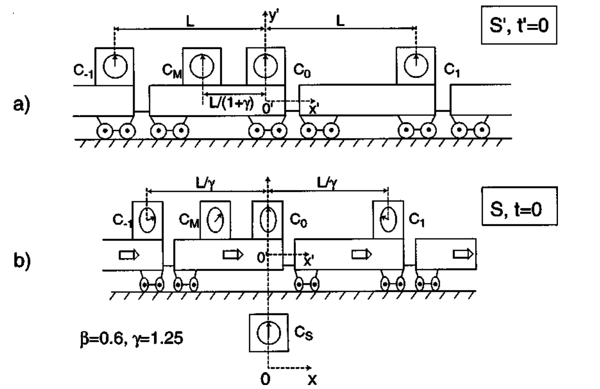

Fig. 2. (a) Positions and times of equivalent clocks on the wagons of a train as seen by a distant observer in the rest frame S' of the train at time $t' = 0$. (b) The positions and times of the same clocks as seen by a distant observer in S at time $t = t' = 0$. The same remarks concerning light propagation time and optical aberration effects as made in the caption of Fig. 1 apply.

The observer in S' will note that each equivalent clock (EC) indicates the same time, as shown in Fig. 2(a). It is now asked how the array of EC will appear to an observer at a fixed position in the frame S when the train is moving with velocity βc parallel to the direction Ox in S [Fig. 2(b)]. It is assumed that the EC C_0 is placed at $x' = 0$ and that it is synchronized with the standard clock C_S, placed at $x = 0$ in

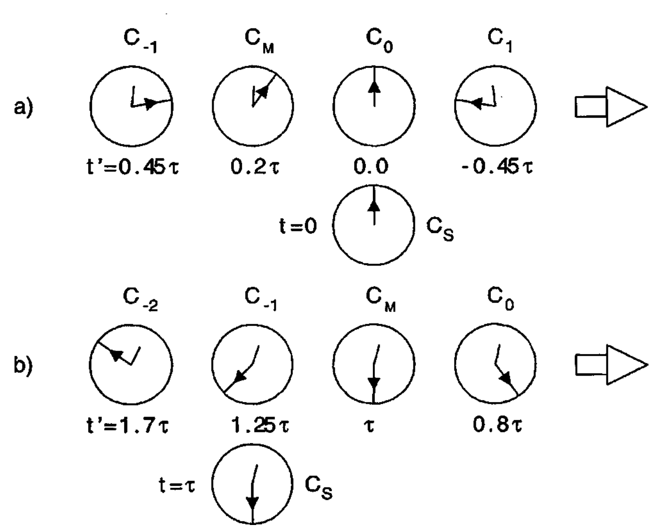

Fig. 3. (a) Times of equivalent clocks on the train (C_{-1}, C_M, C_0, C_1) and the stationary clock C_S, as seen by a distant observer in S at time $t = 0$. (b) The same, at time $t = \tau$. It is assumed that $\beta = 0.6$, $\gamma = 1.25$.

Table II. Times observed in S of equivalent clocks on the moving train in Fig. 2, at the times $t=0$ and $t=\tau$ of the stationary standard clock C_S.

C_S	C_{-2}	C_{-1}	C_M	C_0	C_1	C_2
0	$2\frac{(\gamma^2-1)}{\gamma}\tau$	$\frac{(\gamma^2-1)}{\gamma}\tau$	$\frac{(\gamma-1)}{\gamma}\tau$	0	$-\frac{(\gamma^2-1)}{\gamma}\tau$	$-2\frac{(\gamma^2-1)}{\gamma}\tau$
τ	$\frac{(2\gamma^2-1)}{\gamma}\tau$	$\gamma\tau$	τ	$\frac{\tau}{\gamma}$	$-\frac{(\gamma^2-2)}{\gamma}\tau$	$-\frac{(2\gamma^2-3)}{\gamma}\tau$

S, when $t=t'=0$. All the clocks are similar, that is C_S and each C_m record exactly equal time intervals when they are situated in the same inertial frame.

The appearance of the moving array of EC to an observer in S at $t=0$ is shown in Fig. 2(b), and in more detail in Fig. 3 for both $t=0$ and $t=\tau$. The period τ is the time between the passage of successive EC past C_S. The big hand of C_S in Fig. 3 rotates through 180° during the time τ. Explicit expressions for the observed times are presented in Table II. In Figs. 2(b) and 3 the times indicated by the clocks are shown for $\beta=0.6$. These times are readily calculated using the LT equations (2.1), (2.2). Consider the time indicated by C_1 at $t=0$. The space–time points are

$$S':(L,t'),\quad S:(x,0).$$

Hence, Eqs. (2.1) and (2.2) give

$$x=\gamma(L+vt'), \tag{3.1}$$

$$0=\gamma\left(t'+\frac{\beta L}{c}\right), \tag{3.2}$$

which have the solution $[C_1(t=0)]$:

$$t'=-\frac{\beta L}{c}, \tag{3.3}$$

$$x=\frac{L}{\gamma}. \tag{3.4}$$

As shown in Fig. 2(b), the wagons of the train appear shorter due to the LFC effect [Eq. (3.4)] and also *the wagons at the front end of the train are seen at an earlier proper time than those at the rear end*. Thus a $t=0$ snapshot in S corresponds, not to a fixed t' in S', but one which depends on x': $t'=-\beta x'/c$. This is a consequence of the relativity of simultaneity of space–time events in S and S', as first pointed out by Einstein in Ref. 1. Here it appears in a particularly graphic and striking form. Consider now the time indicated by C_{-1} at $t=\tau$, i.e., when C_{-1} is at the origin of S. The space–time points are

$$S':(-L,t'),\quad S:(0,\tau).$$

Hence, Eqs. (2.1) and (2.2) give

$$0=\gamma(-L+vt'), \tag{3.5}$$

$$\tau=\gamma\left(t'-\frac{\beta L}{c}\right), \tag{3.6}$$

with the solutions $[C_{-1}(t=\tau)]$:

$$t'=\frac{L}{v}, \tag{3.7}$$

$$\tau=\frac{L}{\gamma v}=\frac{t'}{\gamma}, \tag{3.8}$$

so that

$$t'=\gamma\tau. \tag{3.9}$$

The EC at the origin of S at $t=\tau$ shows a later time than C_S, i.e., it is apparently running *faster* than C_S. This is an example of time contraction (TC). *The time contraction effect is exhibited by the EC observed at any fixed position in S*. In fact, if the observer in S can see the EC only when they are near to C_S he (or she) will inevitably conclude that the clocks on the train run fast, not slow as in the classical TD effect (see the following). Suppose that the observer is sitting in a waiting room with the clock C_S and notices the time on the train (the same as C_S) by looking at C_0 as it passes the waiting room window. If he (or she) then compares C_{-1} as it passes the window with C_S it will be seen to be running fast relative to the latter. In order to see the TD effect the observer would (as will now be shown) have to note the time shown by, for example, C_0, at time $t=\tau$ as recorded by C_S in comparison with that shown by the *same clock* C_0 at $t=0$. Using Eq. (3.8), Eq. (3.3) may be written as $[C_1(t=0)]$:

$$t'=-\beta^2\gamma\tau=-\frac{(\gamma^2-1)\tau}{\gamma}. \tag{3.10}$$

This is the formula for the observed time reported in Table II. Now consider C_0 at time $t=\tau$. The space–time points are

$$S':(0,t'),\quad S:(x,\tau).$$

Hence, Eqs. (2.1) and (2.2) give

$$x=\gamma vt', \tag{3.11}$$

$$\tau=\gamma t', \tag{3.12}$$

with the solutions $[C_0(t=\tau)]$:

$$t'=\tau/\gamma, \tag{3.13}$$

$$x=v\tau=L/\gamma. \tag{3.14}$$

So *the EC C_0 at time $t=\tau$ indicates an earlier time, and so is apparently running slower than C_S*. This is the classical time dilatation (TD) effect. It applies to observations of all *local clocks* in S' (i.e., those situated at a fixed value of x') as well as any other EC that has the same value of x'.

As a last example consider the ''magic clock'' C_M shown in Fig. 2(a) at time $t=\tau$. With the space–time points

$$S':(-L/(1+\gamma),t'),\quad S:(x,\tau),$$

Eqs. (2.1) and (2.2) give

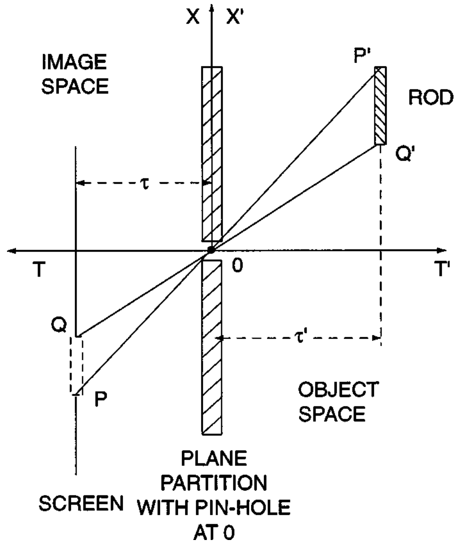

Fig. 4. An example of linear spatial perspective analogous to the Lorentz–Fitzgerald contraction effect.

$$x=\gamma[-L/(1+\gamma)+vt'], \tag{3.15}$$

$$\tau=\gamma\left[t'-\frac{\beta}{c}L/(1+\gamma)\right], \tag{3.16}$$

with the solutions $[C_M(t=\tau)]$:

$$t'=\tau, \tag{3.17}$$

$$x=\gamma v\tau/(1+\gamma), \tag{3.18}$$

where the relation $L=\gamma v\tau$ from Eq. (3.8) has been used. *Thus C_M shows the same time as C_S* at $t=\tau$. Similar moving ''magic clocks'' can be defined that show the same time as C_S at any chosen time t in S. Such a clock is, in general, situated at $x'=-ct(\gamma-1)/\beta\gamma$. All of the other clock times presented in Table II and shown in Figs. 2(b), 3 are calculated in a similar way to the above-mentioned examples by choosing appropriate values of x' and t.

The combined effects of the LT and light propagation delays for light signals moving parallel to the train (corresponding to observations of the array of equivalent clocks by observers on, or close to the train) have been described in detail elsewhere.[6] The observed spatial distortions of the train in this situation were previously considered by Weinstein.[5]

IV. ANALOGY WITH LINEAR PERSPECTIVE IN TWO-DIMENSIONAL SPACE

The analogy between the observed distortions of space–time in special relativity and linear spatial perspective is illustrated in Fig. 4. The ''object space'' on the right is separated from the ''image space'' on the left by a plane partition containing a small aperture (pin hole). Light reflected from the rod PQ in the object space can pass through the pin hole and produce an image on a screen located in the image space. To facilitate the comparison with the Lorentz transformation, the Cartesian axes in the object [image] space are denoted by (X',T') [(X,T)], respectively (see Fig. 4). The T, T' axes are perpendicular to the plane of the partition and pass through the pin hole. The object space is now compared to the rest frame S' of the moving object, with the following correspondences:

$$X'\Leftrightarrow x', \quad T'\Leftrightarrow t',$$

while the image space is compared to the frame S of the stationary observer with the following correspondences:

$$X\Leftrightarrow x, \quad T\Leftrightarrow t.$$

An arbitrary point with coordinates (X',T') on the rod will project into the image space the line:

$$X=-\frac{X'T}{\tau'}, \tag{4.1}$$

which may be compared to the LT equation:

$$x=\frac{1}{\gamma}x'+vt. \tag{4.2}$$

Taking the $T=\tau=$constant projection in (4.1), i.e., setting the screen in the image space parallel to the planar surface at the distance τ from it, gives for the length L_I of the image of the rod:

$$L_I=X_2-X_1=\frac{\tau}{\tau'}(X_1'-X_2')=\frac{\tau}{\tau'}L, \tag{4.3}$$

where the points 1, 2 denote the ends of the rod or of its image. Similarly taking the $t=$constant projection in (4.2) gives, for the apparent length l_I of a rod, parallel to the x axis, of true length l:

$$l_I=x_1-x_2=\frac{1}{\gamma}(x_1'-x_2')=\frac{l}{\gamma} \tag{4.4}$$

corresponding to the LFC effect. The role of the factor $1/\gamma$ in the LT is replaced, in the case of linear perspective, by the ratio τ/τ', which specifies the relative position and orientation of the object and the screen on which it is observed.

V. DISCUSSION

The different space–time effects (observed distortions of space or time) in special relativity that have been discussed above are summarized in Table III. These are the well-known LFC and TD effects, space dilatation (SD) introduced in Sec. II, and time contraction (TC) introduced in Sec. III. Each effect is an observed difference Δq $(q=x,x',t,t')$ of two space or time coordinates $(\Delta q=q_1-q_2)$ and corresponds to a constant projection $\tilde{q}=$constant, i.e., $\Delta\tilde{q}=0$ $(\tilde{q}\neq q)$, in another of the four variables x, x', t, t' of the LT. As shown in Table III, the LFC, SD, TC, and TD effects correspond, respectively, to constant t, t', x, and x' projections. After making this projection, the four LT equations give two relations among the remaining three variables. One of these describes the ''space–time distortion'' relating $\Delta t'$ and Δt or $\Delta x'$ and Δx while the other gives the equation shown in the last column (labeled ''Complementary Effect'') in Table III. These equations relate either Δx to Δt (for SD and TD) or $\Delta x'$ to $\Delta t'$ (for LFC and TC). It can be seen from the complementary effect relations that the two space–time points defining the effect (of space–time distortion) are space-like separated for LFC and SD and time-like separated for TC and TD.

Table III. Different observed distortions of space–time in special relativity (see the text).

Name	Observed quantity	Projection	Effect	Complementary effect
Lorentz–Fitzgerald contraction (LFC)	Δx	$\Delta t=0$	$\Delta x=\frac{1}{\gamma}\Delta x'$	$\Delta x'=-\frac{c}{\beta}\Delta t'$
Space dilatation (SD)	Δx	$\Delta t'=0$	$\Delta x=\gamma\Delta x'$	$\Delta x=\frac{c}{\beta}\Delta t$
Time contraction (TC)	$\Delta t'$	$\Delta x=0$	$\Delta t'=\gamma\Delta t$	$\Delta x'=-c\beta\,\Delta t'$
Time dilatation (TD)	$\Delta t'$	$\Delta x'=0$	$\Delta t'=\frac{1}{\gamma}\Delta t$	$\Delta x=c\beta\,\Delta t$

For example, for the LFC when $t_1=t_2=t$, the LT equations for the two space–time points are

$$x_1'=\gamma(x_1-vt), \tag{5.1}$$

$$x_2'=\gamma(x_2-vt), \tag{5.2}$$

$$t_1'=\gamma\left(t-\frac{\beta x_1}{c}\right), \tag{5.3}$$

$$t_2'=\gamma\left(t-\frac{\beta x_2}{c}\right). \tag{5.4}$$

Subtracting (5.1) from (5.2) and (5.3) from (5.4) gives

$$\Delta x'=\gamma\Delta x, \tag{5.5}$$

$$\Delta t'=-\frac{\gamma\beta}{c}\Delta x. \tag{5.6}$$

Equation (5.5) describes the LFC effect, while combining Eqs. (5.5) and (5.6) to eliminate Δx yields the equation for the complementary effect. By taking other projections the other entries of Table III may be calculated in a similar fashion. It is interesting to note that the TD effect can be derived directly from the LFC effect by using the symmetry of the LT equations. Introducing the notation: $s\equiv ct$, the LT may be written as:

$$x'=\gamma(x-\beta s), \tag{5.7}$$

$$s'=\gamma(s-\beta x). \tag{5.8}$$

These equations are invariant[8] under the following transformations:

$$T1:\ x\leftrightarrow s,\quad x'\leftrightarrow s', \tag{5.9}$$

$$T2:\ x\leftrightarrow x',\quad s\leftrightarrow s',\quad \beta\rightarrow-\beta. \tag{5.10}$$

Writing out the LFC entries in the first row of Table III, replacing t, t' by s/c, s'/c; gives

$$\Delta x,\quad \Delta s=0,\quad \Delta x=\frac{\Delta x'}{\gamma},\quad \Delta x'=-\frac{\Delta s'}{\beta}.$$

Applying $T1$ to each entry in this row results in

$$\Delta s,\quad \Delta x=0,\quad \Delta s=\frac{\Delta s'}{\gamma},\quad \Delta s'=-\frac{\Delta x'}{\beta}.$$

Applying $T2$,

$$\Delta s',\quad \Delta x'=0,\quad \Delta s'=\frac{\Delta s}{\gamma},\quad \Delta s=\frac{\Delta x}{\beta}.$$

Replacing Δs, $\Delta s'$ by $c\Delta t$, $c\Delta t'$ yields the last row of Table III which describes the TD effect. Similarly TC can be derived from SD (or vice versa) by successively applying the transformations $T1$, $T2$.

The ''complementary effects'' listed in Table III have the following geometrical interpretations:

LFC $[\Delta x'=-(c/\beta)\Delta t']$. This is the locus of all the points in S' that are observed at the same time $(\Delta t=0)$ in S.

SD $[\Delta x=(c/\beta)\Delta t]$. The locus of the moving object as observed in S [see Fig. 1(b)].

TC $[\Delta x'=-c\beta\Delta t']$. The locus of the position of the local clock in S' observed at a fixed position $(\Delta x=0)$ in S.

TD $(\Delta x=c\beta\Delta t)$. The locus of the position of the moving local clock observed in S.

A remark on the ''observed quantities'' in Table III. For the LFC, SD effects the observed quantity is a length interval in the frame S. The observed space distortion occurs because this length differs from the result of a similar measurement made on the same object in its own rest frame. $\Delta x'$ is not directly measured at the time of observation of the LFC or SD. It is otherwise with the time measurements TD, TC. Here the time intervals indicated *in their own rest frame* by a local moving clock (TD), or different equivalent clocks at the same position in S (TC), are supposed to be directly observed and compared with the time interval Δt registered by an unmoving clock in the observer's rest frame. Thus the effect refers to two simultaneous observations by *the same observer* not to separate observations by *two different observers* as in the case of the LFC and SD.

Einstein's first paper on special relativity[1] showed, for the first time, that the LFC and TD effects could be most simply understood in terms of the geometry of space–time, in contrast to the previous works of Fitzgerald, Larmor, Lorentz, and Poincaré where dynamical and kinematical considerations were always mixed.[9] However it can also be argued that special relativity has a dynamical aspect due to the changes in the electromagnetic field induced by the LT. Indeed, by calculations of the equilibrium positions of an array of point charges in both stationary and uniformly moving frames, Sorensen has shown that the LFC may be derived from dynamical considerations.[10] By considering several different ''electromagnetic clocks'' either at rest or in uniform motion, Jefimenko has demonstrated that the TD effect may also be dynamically derived.[11] Similar considerations, emphasizing the ''dynamical'' rather than the ''kinematical'' aspects of special relativity, have been presented in an article by Bell.[12] Such calculations, based on the properties of electromagnetic fields under the LT, demonstrate the consistency of classical electromagnetism with special relativity, but as pointed out by Bell,[12] in no way supersede the simpler geometrical derivations of the effects. It is not evident to the present author how similar ''dynamical'' derivations of the new SD and TC effects could be performed.

In conclusion, the essential characteristics of the two ''new'' space–time distortions discussed above are summarized.

Space Dilatation (SD): If a luminous object lying along the Ox' axis, at rest in the frame S', is uniformly illuminated for a short time τ_L in this frame it will be observed from a frame S, in uniform motion relative to S' parallel to Ox' at the velocity $-\beta c$, in a direction perpendicular to the relative velocity, as a narrow strip of width $c\tau_L/(\beta\gamma)$, perpendicular to the x axis, moving with the velocity c/β in the same direction as the object. The total distance swept out along the x axis by the strip during the time $\beta L/(c\sqrt{1-\beta^2})$, for which it is visible, is $L/\sqrt{1-\beta^2}$, where L is the length along Ox' of the object as observed in S'. Thus the apparent length of the object when viewed with a time resolution τ_R much larger than $\beta L/(c\sqrt{1-\beta^2})$ is $L/\sqrt{1-\beta^2}$.

Time Contraction (TC): The equivalent clocks in the moving frame S', viewed at the same position in the stationary frame S, apparently run faster by a factor $1/\sqrt{1-\beta^2}$ relative to a clock at rest in S.

ACKNOWLEDGMENTS

I thank G. Barbier and C. Laignel for their valuable help in the preparation of the figures, and an anonymous referee whose pertinent and constructive criticism has allowed me to much improve the presentation of Sec. II.

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[7] If an observer in S' knows the distance D to any of the clocks then the clock is synchronized relative to a local clock at the same position as the observer when it is observed to lag behind the latter by the time D/c when viewed across free space.

[8] Actually the transformation $T2$ yields the inverse of the LT (5.7), and (5.8). The inverse equations may then be solved to recover (5.7) and (5.8).

[9] A detailed discussion of the important differences between Einstein's theory of special relativity, as presented in Ref. 1, and related work of Fitzgerald, Lorentz, and Poincaré is given in Chaps. 7 and 8 of: A. Pais, *Subtle is the Lord, the Science and Life of Albert Einstein* (Oxford University Press, New York, 1982).

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WARTIME APPLICATIONS OF MATHEMATICS

There is one purpose at any rate which the real mathematics may serve in war. When the world is mad, a mathematician may find in mathematics an incomparable anodyne. For mathematics is, of all the arts and sciences, the most austere and the most remote, and a mathematician should be of all men the one who can most easily take refuge where, as Bertrand Russell says, 'one at least of our nobler impulses can best escape from the dreary exile of the actual world'.

G. H. Hardy, *A Mathematician's Apology* (Cambridge University Press, 1969; reprint of 1940 edition), p. 143.