Illustrations of the relativistic conservation law for the center of energy

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The relativistic conservation law involving the center of energy is reviewed and illustrated using simple examples from classical electromagnetic theory. It is emphasized that this conservation law is parallel to the conservation laws for energy, linear momentum, and angular momentum, because it arises from the generators of the Poincare group for electromagnetic theory; yet this relativistic law reflecting the continuous flow of energy is not mentioned in text books. The illustrations include situations where external forces are present and are absent. A parallel plate capacitor, a flattened slip-joint solenoid, and two interacting charges are treated.

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I. INTRODUCTION

Classical electrodynamics, like other relativistic Lagrangian field theories, is invariant under the Poincare group involving the operations of spacetime translation, spatial rotation, and proper Lorentz transformation. The associated infinitesimal generators,[1] designated by $\mathbf{P}$, U, $\mathbf{L}$, and $U\vec{\mathcal{X}}$, are associated with conserved quantities. The generator $\mathbf{P}$ of space translations is associated with conservation of linear momentum. The generator U of time translations is associated with conservation of energy. The generator $\mathbf{L}$ of spatial rotations is associated with conservation of angular momentum. The generator $U\vec{\mathcal{X}}$ of proper Lorentz transformations is associated with the uniform motion of the system center of energy.[2] Although the conservation laws of linear momentum, angular momentum, and energy are illustrated by elementary examples in electromagnetism text books,[3] the uniform motion of the center of energy is not.[4] The law expresses the continuous flow of energy in relativistic systems. The invariant motion of the center of energy is known[5] but not widely, and is rarely discussed in the electromagnetism literature.

In this article we review the relativistic law for the invariant motion of the center of energy and then present three simple electromagnetic examples: a parallel plate capacitor, a flattened, slip-joint solenoid, and two interacting point charges. The examples remind us that when calculating the center of energy of an electromagnetic system, relativistic particle equations of motion must be used and all the energy must be considered, including the particle rest energy and kinetic energy, and the distributed energy stored in the electromagnetic field.

II. RELATIVISTIC CONSERVATION LAWS

A. Generators of the Poincare group for electromagnetism

For charged point masses m_i interacting through electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$, the generators of the Poincare group[1] take the forms

$$\mathbf{P}=\sum_i m_i\gamma_i\mathbf{v}_i+\int d^3r\,\frac{1}{4\pi c}\mathbf{E}\times\mathbf{B}\quad\text{(linear momentum)},\tag{1}$$

$$U=\sum_i m_i\gamma_i c^2+\int d^3r\,\frac{1}{8\pi}(E^2+B^2)\quad\text{(energy)},\tag{2}$$

$$\mathbf{L}=\sum_i \mathbf{r}_i\times m_i\gamma_i\mathbf{v}_i+\int d^3r\,\mathbf{r}\times\left(\frac{1}{4\pi c}\mathbf{E}\times\mathbf{B}\right)\quad\text{(angular momentum)},\tag{3}$$

$$U\vec{\mathcal{X}}=\sum_i \mathbf{r}_i m_i\gamma_i c^2+\int d^3r\,\mathbf{r}\,\frac{1}{8\pi}(E^2+B^2)\quad\text{(energy times center of energy)},\tag{4}$$

where $\mathbf{v}_i=d\mathbf{r}_i/dt$ is the time derivative of the particle displacement $\mathbf{r}_i$, and $\gamma_i=(1-v_i^2/c^2)^{-1/2}$, $\mathbf{E}$ and $\mathbf{B}$ represent the electric and magnetic fields, and $\vec{\mathcal{X}}$ is the center of energy of the system. These generators correspond to space translation, time translation, spatial rotation, and proper Lorentz transformation, respectively. In the absence of external forces, the first three quantities are time independent and the fourth has a constant time derivative. The electromagnetic expressions in Eqs. (1)–(3) appear in electromagnetism textbooks, whereas Eq. (4) is usually absent. On account of this omission, we will sketch the derivation of the center of energy expression.

B. Derivation of the center-of-energy law

The center of energy $\vec{\mathcal{X}}$ in Eq. (4) is analogous to the familiar center of (rest) mass $\vec{\mathcal{X}}_{\text{restmass}}$ of nonrelativistic mechanics

$$M\vec{\mathcal{X}}_{\text{rest mass}}=\sum_i m_i\mathbf{r}_i,\quad M=\sum_i m_i,\tag{5}$$

except that all energy contributes. The total energy U in Eq. (2) for a system of charged particles and electromagnetic fields is the sum of the relativistic mechanical energy of each particle $m_i\gamma_i c^2$ and the electromagnetic field energy found by integrating the energy density $u=[1/(8\pi)](E^2+B^2)$ over all space. The center-of-energy expression (4) involves weighting the displacement $\mathbf{r}$ by the amount of the energy located at $\mathbf{r}$. Thus a point mass of energy $m_i\gamma_i c^2$ contributes $\mathbf{r}_i(m_i\gamma_i c^2)$, and the electromagnetic energy ud^3r in a differential volume d^3r contributes $\mathbf{r}(ud^3r)=\mathbf{r}[1/(8\pi)](E^2+B^2)d^3r$. If we sum over the particles and integrate over all the electromagnetic fields in space, we obtain Eq. (4) for the energy times the center of energy $U\vec{\mathcal{X}}$.

The derivation of the law for the invariant motion of the center of energy in electromagnetic theory can be given in a fashion parallel to that given for Poynting's theorem.[6] We consider the integral over all space of $\int d^3r\mathbf{r}(\mathbf{J}\cdot\mathbf{E})$, which represents the volume-integral over the displacement $\mathbf{r}$ weighted by $\mathbf{J}\cdot\mathbf{E}$, the local transfer of power from electromagnetic form to another form due to electric fields on moving charges. Just as for Poynting's theorem, we use Maxwell's equations to rewrite this integral in terms of the electromagnetic fields alone,

$$
\begin{aligned}
\int d^3r\mathbf{r}(\mathbf{J}\cdot\mathbf{E}) &= \int d^3r\mathbf{r}\left[\frac{c}{4\pi}\left(\boldsymbol{\nabla}\times\mathbf{B}-\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}\right)\cdot\mathbf{E}\right] \\
&= \int d^3r\mathbf{r}\left[\frac{-c}{4\pi}\nabla\cdot(\mathbf{E}\times\mathbf{B})+\frac{c}{4\pi}\mathbf{B}\cdot(\nabla\times\mathbf{E}) \right. \\
&\quad \left. -\frac{1}{8\pi}\frac{\partial}{\partial t}E^2\right] \\
&= \int d^3r\mathbf{r}\left[-\boldsymbol{\nabla}\cdot\left(\frac{c}{4\pi}\mathbf{E}\times\mathbf{B}\right) \right. \\
&\quad \left. -\frac{\partial}{\partial t}\left(\frac{1}{8\pi}(E^2+B^2)\right)\right] \\
&= -\int \mathbf{r}\left(\frac{c}{4\pi}\mathbf{E}\times\mathbf{B}\right)\cdot d\mathbf{A} \\
&\quad +\int d^3r\left(\frac{c}{4\pi}\mathbf{E}\times\mathbf{B}\right) \\
&\quad -\frac{d}{dt}\int d^3r\mathbf{r}\left(\frac{1}{8\pi}(E^2+B^2)\right) \\
&= \int d^3r\left(\frac{c}{4\pi}\mathbf{E}\times\mathbf{B}\right) \\
&\quad -\frac{d}{dt}\int d^3r\mathbf{r}\left(\frac{1}{8\pi}(E^2+B^2)\right),
\end{aligned} \tag{6}
$$

where we have used the divergence theorem and have dropped the surface term assuming that the sources of electromagnetic fields are localized.

For a system of charged particles interacting through the electromagnetic fields, we differentiate Eq. (4) to obtain

$$
\begin{aligned}
\frac{d(U\vec{\mathcal{X}})}{dt} &= \frac{d}{dt}\left(\sum \mathbf{r}_i m_i\gamma_i c^2+\int d^3r\mathbf{r}\frac{1}{8\pi}(E^2+B^2)\right) \\
&= c^2\sum m_i\gamma_i\mathbf{v}_i+\sum \mathbf{r}_i\frac{d}{dt}(m_i\gamma_i c^2) \\
&\quad +\frac{d}{dt}\int d^3r\mathbf{r}\frac{1}{8\pi}(E^2+B^2) \\
&= c^2\sum m_i\gamma_i\mathbf{v}_i+\int d^3r\mathbf{r}(\mathbf{J}\cdot\mathbf{E}) \\
&\quad +\frac{d}{dt}\int d^3r\mathbf{r}\frac{1}{8\pi}(E^2+B^2) \\
&= c^2\left[\sum m_i\gamma_i\mathbf{v}_i+\int d^3r\left(\frac{1}{4\pi c}\mathbf{E}\times\mathbf{B}\right)\right]=c^2\mathbf{P},
\end{aligned} \tag{7}
$$

where we have used Eqs. (1), (6), and the energy transfer equation for point charges

$$\sum \mathbf{r}_i\frac{d}{dt}(m_i\gamma_i c^2)=\int d^3r\mathbf{r}(\mathbf{J}\cdot\mathbf{E}). \tag{8}$$

The point-charge current density is $\mathbf{J}(\mathbf{r},t)=\Sigma q_i\mathbf{v}_i\delta^3(\mathbf{r}-\mathbf{r}_i(t))$ and $d(m_i\gamma_i c^2)/dt=q_i\mathbf{v}_i\cdot\mathbf{E}(\mathbf{r}_i,t)$. Thus in Eq. (7) we see that the time rate of change of the quantity (energy times the center of energy) is equal to c^2 times the linear momentum of the system. Because the linear momentum and the energy of the system are constant in time, the velocity of the center of energy is constant in time, $d\vec{\mathcal{X}}/dt=\text{const}$.

C. Conservation laws in the presence of external forces on particles

In many cases it is convenient to consider not isolated electromagnetic systems, but rather electromagnetic systems in interaction with external forces $\mathbf{F}_{\text{ext},i}$ acting on the particles of the system. In this case the conservation laws are modified. The sum of the external forces gives the time rate of change of the system linear momentum,

$$\sum_i \mathbf{F}_{\text{ext},i}=\frac{d\mathbf{P}}{dt}. \tag{9}$$

The power delivered by the external forces gives the time rate of change of the system energy

$$\sum_i \mathbf{F}_{\text{ext},i}\cdot\mathbf{v}_i=\frac{dU}{dt}. \tag{10}$$

The sum of the external torques gives the time rate of change of the system angular momentum $\mathbf{L}$ (about the origin)

$$\sum_i \mathbf{r}_i\times\mathbf{F}_{\text{ext},i}=\frac{d\mathbf{L}}{dt}. \tag{11}$$

The law for the change in the energy times the center of energy is unfamiliar.[7] We can obtain the rule by using the modified equation of energy transfer $d(m_i\gamma_i c^2)/dt=[q_i\mathbf{E}(\mathbf{r}_i,t)+\mathbf{F}_{\text{ext},i}]\cdot\mathbf{v}_i$ for the ith particle (multiplied by $\mathbf{r}_i$),

$$\mathbf{r}_i\frac{d}{dt}(m_i\gamma_i c^2)=\mathbf{r}_i(\mathbf{F}_{\text{ext},i}\cdot\mathbf{v}_i)+\mathbf{r}_i(q_i\mathbf{E}\cdot\mathbf{v}_i), \tag{12}$$

and summing over all the particles

$$\sum_i \mathbf{r}_i\frac{d}{dt}(m_i\gamma_i c^2)=\sum_i \mathbf{r}_i(\mathbf{F}_{\text{ext},i}\cdot\mathbf{v}_i)+\int d^3r\mathbf{r}(\mathbf{J}\cdot\mathbf{E}). \tag{13}$$

Now using Eq. (13) for $\int d^3r\mathbf{r}(\mathbf{J}\cdot\mathbf{E})$ in Eq. (6) and noting the first two lines of Eq. (7), we obtain the rule for the center of energy,

$$\sum_i (\mathbf{F}_{\text{ext},i}\cdot\mathbf{v}_i)\mathbf{r}_i=\frac{d(U\vec{\mathcal{X}})}{dt}-c^2\mathbf{P}. \tag{14}$$

Thus the power weighted by the position where the power is delivered equals the time rate of change of the system energy times the center of energy minus c^2 times the system linear momentum.

All of the laws in Eqs. (9)–(11) and (14) can be integrated with respect to time to give integral forms. The integral form for the relativistic center of energy law in Eq. (14) is

$$\sum_i \int_1^2 (d\mathbf{r}_i \cdot \mathbf{F}_{\text{ext},i})\mathbf{r}_i = U_2\vec{\mathcal{X}}_2 - U_1\vec{\mathcal{X}}_1 - c^2\int_1^2 dt\mathbf{P}. \quad (15)$$

In special relativity, the flow of energy has a continuous meaning. Thus the introduction of energy by external forces located at points in space changes the center of energy of the system.

The continuous flow of energy in space for relativistic systems is in contrast with the situation in nonrelativistic mechanics where energy can be suddenly transported from one point in space to another. Thus in nonrelativistic mechanics a long, massless, rigid pole can be used to transport energy instantaneously from one end of the pole to the other. Such poles do not exist in relativistic physics. Rather, in relativistic physics, a system has a well-defined center of energy that moves through space continuously at a speed (in the absence of external forces) not exceeding the speed of light in vacuum c.

It is interesting to note the nonrelativistic limit for the center-of-energy relations in Eqs. (4), (7), and (14). If we divide by a factor of c^2 and allow $c \to \infty$, then all that remains of the energy given in Eq. (2) is the rest-mass contribution $U/c^2 \to \Sigma_i m_i$ with no contribution from the (finite) kinetic energy or electromagnetic energy. Thus in the $c\to\infty$ limit, Eq. (4) becomes the expression for the center of rest mass given in Eq. (5). Also, Eq. (7) reduces to the statement that the total rest mass times the center of rest mass equals the momentum

$$\frac{d}{dt}\left[\left(\sum_i m_i\right)\vec{\mathcal{X}}\right] = \sum_i m_i\mathbf{v}_i = \mathbf{P} \quad (c\to\infty). \quad (16)$$

These results are familiar in Galilean invariant (nonrelativistic) mechanics. On dividing Eq. (14) by c^2 and allowing $c \to\infty$, the left-hand side involving external forces vanishes and the right-hand side involves the same statement in Eq. (16) obtained from the $c\to\infty$ limit of Eq. (7). In nonrelativistic physics, there is a continuous flow of rest mass but not of energy. Thus in nonrelativistic physics there is no separate law regarding the *location* where energy is introduced into the system.

III. ILLUSTRATIONS OF THE CENTER-OF-ENERGY CONSERVATION LAW

A. Quasi-static changes for stationary systems

1. A single point mass

As the simplest possible example of the relativistic conservation laws for stationary systems, we consider a single point mass m at rest at displacement $\mathbf{r}$ in some inertial frame. The conserved quantities associated with Poincare invariance involve the energy $U=mc^2$, linear momentum $\mathbf{P}=0$, angular momentum about the origin $\mathbf{L}=0$, and energy times center of energy $(mc^2)\vec{\mathcal{X}} = mc^2\mathbf{r}$. We now use an external force $\mathbf{F}_{\text{ext}}$ to move the mass from $\mathbf{r}$ to $\mathbf{r}'$ quasi-statically. Because the external force can be chosen arbitrarily small, there is no linear impulse delivered, no net work done, no angular impulse, and no moment of work done. The only conservation law with some nonvanishing terms is the one involving the center of energy. Here the system linear momentum has a nonvanishing time integral so that the integral form of the law in Eq. (15) gives

$$0 = (mc^2\mathbf{r}') - (mc^2\mathbf{r}) - c^2\int_1^2 \mathbf{P}\,dt, \quad (17)$$

which is consistent with the momentum of a particle

$$\mathbf{P} = \frac{m}{\sqrt{1-[(d\mathbf{r}/dt)/c]^2}}\frac{d\mathbf{r}}{dt} \cong m\frac{d\mathbf{r}}{dt} \quad \text{(quasi-static)}. \quad (18)$$

We notice that even though the linear momentum $\mathbf{P}$ can be made as small as desired by taking the external forces sufficiently small, the time integral of the linear momentum gives a nonzero value independent of the magnitude of the external force in the limit $d\mathbf{r}/dt \to 0$. The change in the position of the system center of energy is associated with a flow of momentum as required by special relativity.

In this simplest case where all of the energy is rest-mass energy, we could have divided Eq. (17) by c^2 and have obtained a result valid in nonrelativistic physics where the linear momentum is given by $\mathbf{P}=m\mathbf{v}$. In nonrelativistic physics, the change in rest mass position is continuous and is associated with the flow of linear momentum.

2. Parallel plate capacitor

A parallel plate capacitor provides an illustration of the conservation law for the center of energy when electrostatic energy is involved. The electrostatic energy contributes to the center of energy of the system in relativistic physics, whereas it does not contribute to the center of rest mass which appears in nonrelativistic physics. We consider a capacitor consisting of two parallel conducting plates, each of dimension $L\times L$; the left-hand plate of mass m in the plane with x-coordinate x, and the right-hand plate of mass M in the plane with x-coordinate X. In this section we will discuss quasi-static displacements and assume the masses m and M to be negligible. The plates are centered so that the x axis passes through the center of each plate. Plate m is charged with total charge $+Q$ and plate M with charge $-Q$. It is assumed that the plates form a parallel plate capacitor of small separation $0 < X-x \ll L$ with an electric field given by the electrostatic expression

$$\mathbf{E} = \hat{i}4\pi Q/L^2 \quad (19)$$

between the plates. There is no magnetic field present, and it is assumed that we may neglect the fringing fields outside the plates. The energy $U=[1/(8\pi)]E^2L^2(X-x)$ and the center of energy $\mathcal{X}=(x+X)/2$.

To maintain the capacitor plates at rest, there must be external forces[8] $F_{\text{ext},m} = -Q(E+0)/2 = -2\pi Q^2/L^2 = -F_{\text{ext},M}$ on the left-hand plate at x and on the right-hand plate at X, respectively. The illustration of energy conservation for this situation is easily performed.[9] Thus if the two plates are displaced quasi-statically from x to x' and from X to X', respectively, the work done by the external forces of constraint is found to equal the change in electrostatic energy:

$$F_{\mathrm{ext},m}(x'-x)+F_{\mathrm{ext},M}(X'-X)$$
$$=\frac{2\pi Q^2}{L^2}[-(x'-x)+(X'-X)]$$
$$=\frac{1}{8\pi}\left(\frac{4\pi Q}{L^2}\right)^2 L^2(X'-x')-\frac{1}{8\pi}\left(\frac{4\pi Q}{L^2}\right)^2 L^2(X-x)]. \tag{20}$$

However, in contrast to the work-energy law, the relativistic center-of-energy law in Eq. (15) usually goes unmentioned. During a quasi-static displacement of the plates, there is no magnetic field generated in the region between the plates and therefore no electromagnetic field momentum between the plates. If the plates are displaced quasi-statically from x to x' and from X to X', the left-hand side of Eq. (15) gives

$$\sum_i \int dx_i F_{\mathrm{extx},i}x_i=\int_x^{x'} dx''\left(\frac{-2\pi Q^2}{L^2}\right)x'' + \int_X^{X'} dX''\left(\frac{2\pi Q^2}{L^2}\right)X''$$
$$=\frac{-\pi Q^2(x'^2-x^2)}{L^2}+\frac{\pi Q^2(X'^2-X^2)}{L^2}, \tag{21}$$

and the right-hand side of Eq. (15) gives

$$U_2\mathcal{X}_2-U_1\mathcal{X}_1-c^2\int_1^2 dt P_x=\frac{\pi Q^2(X'^2-x'^2)}{L^2}-\frac{\pi Q^2(X^2-x^2)}{L^2}-0. \tag{22}$$

After some rearrangement, Eqs. (21) and (22) are seen to involve the same quantities on the right-hand sides. Thus moving the capacitor plates illustrates the relativistic center-of-energy law (15) with external forces. The energy introduced by the external forces at the plates provides not only the change in electrostatic energy, but also the continuous motion of the center of electrostatic energy.

3. Flattened, slip-joint solenoid

It was pointed out recently[9] that energy calculations for a solenoid can be made analogous to those for a parallel plate capacitor by flattening the solenoid and fitting it with slip joints which allow relative motion of the front and back current sheets while maintaining the continuity of the circulating surface currents. Here we will use this solenoidal configuration to do calculations for a solenoid that are analogous to those given for a capacitor.

The flattened solenoid consists of two large perfectly conducting plates of size $L\times\ell$ with negligible masses m and M located in the planes x and X, respectively, and connected through short perfectly conducting sides parallel to the yz plane which are fitted with slip joints. The slip joints maintain the continuity of the electrical circuit while allowing the plates to move along the x axis, which passes through the centers of the plates. The surface current $\mathbf{K}$ is always perpendicular to the $\hat{k}$ direction and flows around the solenoid, in the $+\hat{j}$ direction in the M plate and in the $-\hat{j}$ direction in the m plate. The surface current $\mathbf{K}$ causes a magnetic field

$$\mathbf{B}=\hat{k}4\pi K/c \tag{23}$$

parallel to the z axis within the flattened solenoid. The magnetic flux Φ through the solenoid is given by the magnitude of $\mathbf{B}$ times the cross-sectional area

$$\Phi=BL(X-x)=4\pi KL(X-x)/c. \tag{24}$$

We assume that the separation between the plates is very small, $0<X-x\ll L,l$, compared to the other dimensions, so that we can neglect the fringing fields.

Here we are interested in the case where external mechanical forces on the left and right current sheets of the flattened solenoid allow these sheets to change position quasi-statically from x to x' and from X to X', respectively. We assume that there is no ohmic resistance in the sheets nor any batteries present, so that the currents of the solenoid flow in such a fashion as to maintain a constant total magnetic flux Φ through the solenoid. The external forces needed to balance the magnetic forces on the current sheets at x and X are given by

$$\mathbf{F}_{\mathrm{ext},x}=\hat{i}\frac{KLl}{c}\frac{(\mathbf{B}+0)}{2}=\hat{i}\frac{1}{8\pi}B^2Ll$$
$$=\hat{i}\frac{1}{8\pi}\frac{\Phi^2 l}{L(X-x)^2}=-\mathbf{F}_{\mathrm{ext},X}, \tag{25}$$

while the energy in the magnetic field is given by

$$U=\frac{1}{8\pi}B^2Ll(X-x)=\frac{1}{8\pi}\frac{\Phi^2 l}{L(X-x)}. \tag{26}$$

Just as in the electrostatic case, it is easy to verify the connection between the external forces and the energy changes for the solenoid. In this case, we use the differential form in Eq. (10) and find

$$\mathbf{F}_{\mathrm{ext},x}\cdot\hat{i}\frac{dx}{dt}+\mathbf{F}_{\mathrm{ext},X}\cdot\hat{i}\frac{dX}{dt}=\frac{1}{8\pi}\frac{\Phi^2 l}{L(X-x)^2}\left(\frac{dx}{dt}-\frac{dX}{dt}\right)$$
$$=\frac{d}{dt}\left(\frac{1}{8\pi}\frac{\Phi^2 l}{L(X-x)}\right)=\frac{dU_{\mathrm{em}}}{dt}, \tag{27}$$

which confirms the energy conservation law.

It also is possible to verify the law in Eq. (14) for the relativistic center of energy. Now because the cross-sectional area of the solenoid is changing, it follows that the magnetic field must be changing, which means that electric fields must be induced. Induced electric fields together with the solenoid magnetic field will lead to electromagnetic field linear momentum and hence to a contribution in Eq. (14) from $c^2\mathbf{P}$. To find the electric field induced when the current sheets are moved apart, we consider a single current sheet observed in a new Lorentz frame. If we consider a current sheet normal to the x axis with a current $\mathbf{K}$ flowing in the $\hat{j}$ direction, then there is a magnetic field $\mathbf{B}=\pm\hat{k}(2\pi/c)K$, the factor of 2π rather than 4π because only a single current sheet is involved. Under a Lorentz transformation to a new inertial frame moving with velocity $v=c\beta$ along the x axis, we find a uniform electric field $\mathbf{E}=\pm\hat{j}\gamma\beta B=\pm\hat{j}\gamma(v/c)(2\pi K/c)$. If we apply this analysis to both plates of the capacitor, we find that there is a net electric field in the region between the moving current sheets,

$$\mathbf{E}=\hat{j}\gamma_m\frac{2\pi K}{c^2}\frac{dx}{dt}+\hat{j}\gamma_M\frac{2\pi K}{c^2}\frac{dX}{dt}. \tag{28}$$

Thus inside the flattened solenoid, there is an electromagnetic linear momentum

$$\begin{aligned}\mathbf{P}&=\frac{1}{4\pi c}\mathbf{E}\times\mathbf{B}Ll(X-x)\\&=\frac{1}{4\pi c}\hat{i}\left(\frac{1}{\sqrt{1-[(dx/dt)/c]^2}}\frac{dx}{dt}\right.\\&\quad\left.+\frac{1}{\sqrt{1-[(dX/dt)/c]^2}}\frac{dX}{dt}\right)\frac{2\pi K}{c^2}\left(\frac{4\pi K}{c}\right)Ll(X-x).\end{aligned} \tag{29}$$

In the quasi-static limit, we drop the terms in $[(dx/dt)/c]^2$ and rewrite the expression for $\mathbf{P}$ in terms of the constant magnetic flux Φ, giving

$$\begin{aligned}c^2\mathbf{P}&=\frac{1}{8\pi}\hat{i}\left(\frac{dx}{dt}+\frac{dX}{dt}\right)\frac{\Phi^2 l}{L(X-x)}\\&=\hat{i}\frac{1}{8\pi}\frac{\Phi^2 l}{L(X-x)^2}\left((X-x)\frac{dx}{dt}+(X-x)\frac{dX}{dt}\right)\\&\quad(\text{quasi-static}).\end{aligned} \tag{30}$$

If the masses m and M of the plates supporting the current sheets are assumed to be negligible, then the position of the center of energy is at the middle of the solenoid volume:

$$U\vec{\mathcal{X}}=\hat{i}\left(\frac{1}{8\pi}B^2Ll(X-x)\right)\frac{(x+X)}{2}=\hat{i}\frac{1}{16\pi}\frac{\Phi^2 l(x+X)}{L(X-x)}. \tag{31}$$

Then the time rate of change of the energy times the center of energy gives

$$\begin{aligned}\frac{d}{dt}(U\vec{\mathcal{X}})&=\frac{d}{dt}\left(\hat{i}\frac{1}{16\pi}\frac{\Phi^2 l(x+X)}{L(X-x)}\right)\\&=\hat{i}\frac{1}{8\pi}\frac{\Phi^2 l}{L(X-x)^2}\left(X\frac{dx}{dt}-x\frac{dX}{dt}\right).\end{aligned} \tag{32}$$

The position-weighted power required on the left-hand side of Eq. (14) is

$$\begin{aligned}(\mathbf{F}_{\text{ext},x}\cdot\mathbf{v})\hat{i}x+\mathbf{F}_{\text{ext},X}\cdot\mathbf{V})\hat{i}X&=\hat{i}\left(\frac{1}{8\pi}\frac{\Phi^2 l}{L(X-x)^2}\frac{dx}{dt}\right)x\\&\quad+\hat{i}\left(\frac{-1}{8\pi}\frac{\Phi^2 l}{L(X-x)^2}\frac{dX}{dt}\right)X.\end{aligned} \tag{33}$$

We now combine Eqs. (30) and (32) and see that the sum of the right-hand sides matches the right-hand side of Eq. (33). Indeed the quasi-static expansion of a solenoid satisfies the relativistic law Eq. (14) for the center of energy.

4. Two point charges at rest

Our final quasi-static example involves two point charges, one of mass m and charge q at $\mathbf{r}$ and the other of mass M and charge Q at $\mathbf{R}$, both at rest in an inertial frame. Again the analysis involves both electromagnetic field energy and also electromagnetic field momentum as these charges are displaced quasi-statically from $\mathbf{r}$ to $\mathbf{r}'$ and from $\mathbf{R}$ to $\mathbf{R}'$, respectively. In the limit of quasi-static motion, there is no radiation emission on changing the electrostatic configuration, and so the center-of-energy theorem can be verified exactly.

Here again, the only interesting aspects of the conservation laws involve the energy and the center of energy. The external forces needed to move the charges quasi-statically balance the electrostatic forces between the charges

$$\mathbf{F}_{\text{ext},m}=\frac{qQ(\mathbf{R}-\mathbf{r})}{|\mathbf{R}-\mathbf{r}|^3}=-\mathbf{F}_{\text{ext},M}, \tag{34}$$

while the total energy is the rest-mass energy plus the electrostatic energy

$$U=mc^2+Mc^2+\frac{qQ}{|\mathbf{R}-\mathbf{r}|}. \tag{35}$$

The energy conservation law (10) in the quasi-static limit takes the familiar form

$$\begin{aligned}\mathbf{F}_{\text{ext},m}\cdot\mathbf{v}+\mathbf{F}_{\text{ext},M}\cdot\mathbf{V}&=\frac{qQ(\mathbf{R}-\mathbf{r})}{|\mathbf{R}-\mathbf{r}|^3}\cdot(\mathbf{v}-\mathbf{V})\\&=\frac{d}{dt}\left(\frac{qQ}{|\mathbf{R}-\mathbf{r}|}\right)=\frac{dU}{dt},\end{aligned} \tag{36}$$

where $\mathbf{v}=d\mathbf{r}/dt$ and $\mathbf{V}=d\mathbf{R}/dt$, and $U=qQ/|\mathbf{R}-\mathbf{r}|$ is the electrostatic energy associated with the two point charges.

Although this law for energy conservation is familiar, the relativistic law (14) for the center of energy is not. The center of the electromagnetic energy, by symmetry or by direct integration of the interaction energy between the point-charge fields $[1/(8\pi)]\int d^3r\mathbf{r}2\mathbf{E}_m\cdot\mathbf{E}_M$, is located half-way between the two charges so that the energy times the center of energy is given by

$$U\vec{\mathcal{X}}=mc^2\mathbf{r}+Mc^2\mathbf{R}+\frac{qQ}{|\mathbf{R}-\mathbf{r}|}\left(\frac{\mathbf{r}+\mathbf{R}}{2}\right). \tag{37}$$

The evaluation of the position-weighted power on the left-hand side of Eq. (14) involves

$$\begin{aligned}(\mathbf{F}_{\text{ext},m}\cdot\mathbf{v})\mathbf{r}+(\mathbf{F}_{\text{ext},M}\cdot\mathbf{V})\mathbf{R}&=\frac{qQ}{|\mathbf{R}-\mathbf{r}|^3}\{[(\mathbf{R}-\mathbf{r})\cdot\mathbf{v}]\mathbf{r}\\&\quad-[(\mathbf{R}-\mathbf{r})\cdot\mathbf{V}]\mathbf{R}\}.\end{aligned} \tag{38}$$

In the low-velocity limit appropriate for quasi-static changes, the linear momentum of two point charges is given by the sum of the mechanical linear momentum and the linear momentum in the electromagnetic field.[1]

$$\begin{aligned}\mathbf{P}&\cong m\mathbf{v}+m\mathbf{V}+\frac{qQ}{2c^2|\mathbf{R}-\mathbf{r}|}\left(\mathbf{v}+\mathbf{V}\right.\\&\quad\left.+\frac{[(\mathbf{R}-\mathbf{r})\cdot\mathbf{v}](\mathbf{R}-\mathbf{r})+[(\mathbf{R}-\mathbf{r})\cdot\mathbf{V}](\mathbf{R}-\mathbf{r})}{|\mathbf{R}-\mathbf{r}|^2}\right).\end{aligned} \tag{39}$$

For quasi-static displacement, the time rate of change of the energy times the center of energy in Eq. (37) is

$$\frac{d}{dt}(U\vec{\mathcal{X}})=mc^2\mathbf{v}+Mc^2\mathbf{V}+\frac{qQ}{2|\mathbf{R}-\mathbf{r}|}(\mathbf{v}+\mathbf{V})$$
$$-\frac{qQ}{2|\mathbf{R}-\mathbf{r}|^3}[(\mathbf{R}-\mathbf{r})\cdot(\mathbf{V}-\mathbf{v})](\mathbf{r}+\mathbf{R}). \quad (40)$$

Then combining Eqs. (39) and (40), we find

$$\frac{d}{dt}(U\vec{\mathcal{X}})-c^2\mathbf{P}=\frac{qQ}{2|\mathbf{R}-\mathbf{r}|}\times\left(\frac{[(\mathbf{R}-\mathbf{r})\cdot\mathbf{v}](2\mathbf{r})-[(\mathbf{R}-\mathbf{r})\cdot\mathbf{V}](2\mathbf{R})}{|\mathbf{R}-\mathbf{r}|^2}\right). \quad (41)$$

Equation (41) agrees exactly with the position-weighted power expression on the right-hand side of Eq. (38). Hence the relativistic law for the center of energy is illustrated in this case; the Coulomb potential between two point charges fits with the low-velocity limit of electromagnetic theory so as to give continuous motion for the center of energy under quasi-static displacements by external forces.

B. Systems involving acceleration

In the previous examples, we have illustrated how considerations of momentum and electromagnetic energy enter into the relativistic law for the center of energy when treating quasi-static changes of stationary systems. Here we wish to note the role of relativistic energy and momentum for particles. The simplest example seems to be that discussed above in Sec. III A 2 involving a parallel plate capacitor, where now the masses m and M of the plates are no longer treated as negligible and the external forces providing a static configuration are removed. In this case, the parallel plates m and M of the capacitor accelerate toward each other due to electrostatic attraction. We will verify all of the conservation laws for the quantities in Eqs. (1)–(4), and we will note where the distinction between nonrelativistic and relativistic particle mechanics becomes important.

The parallel plate capacitor example of Sec. III A 2 involves motion along only the x axis. Newton's equations of motion for the plates along the x axis take the form[8]

$$\frac{dp_m}{dt}=Q\frac{(E+0)}{2}=\frac{2\pi Q^2}{L^2}=-\frac{dp_M}{dt}, \quad (42)$$

where the electrostatic force on each plate is due to the average field across the plate or is regarded as due to the electric field due to the other plate. In the approximation of large parallel plates with small separation, there is no magnetic field present even if the plates are moving with finite velocity, so that there is no electromagnetic linear momentum for the system. Therefore the system linear momentum is simply the mechanical momentum of the particles:

$$\mathbf{P}=\hat{i}p_m+\hat{i}p_M. \quad (43)$$

The angular momentum about the origin vanishes,

$$\mathbf{L}=0, \quad (44)$$

because the x axis passes through the center of each plate. The energy of the system includes the mechanical particle energies U_m and U_M and the energy in the electric field $\mathbf{E}=\hat{i}4\pi Q/L^2$ between the plates:

$$U=U_m+U_M+U_{\mathrm{em}}=U_m+U_M+\frac{1}{8\pi}E^2L^2(X-x)$$
$$=U_m+U_M+\frac{2\pi Q^2(X-x)}{L^2}. \quad (45)$$

The energy times the center of energy is

$$U\vec{\mathcal{X}}=\hat{i}\left(U_mx+U_MX+U_{\mathrm{em}}\frac{x+X}{2}\right)$$
$$=\hat{i}\left(U_mx+U_MX+\frac{2\pi Q^2(X-x)}{L^2}\frac{(x+X)}{2}\right). \quad (46)$$

The conservation laws can easily be verified for this system by using the equations of motion. The system linear momentum is constant in time,

$$\frac{d\mathbf{P}}{dt}=\hat{i}\frac{dp_m}{dt}+\hat{i}\frac{dp_M}{dt}=0, \quad (47)$$

as follows from Eq. (42) because the forces on the plates are equal in magnitude and opposite in direction. The system energy is constant in time,

$$\frac{dU}{dt}=\frac{dU_m}{dt}+\frac{dU_M}{dt}+\frac{dU_{\mathrm{em}}}{dt}$$
$$=\left(\frac{dp_m}{dt}-\frac{2\pi Q^2}{L^2}\right)v+\left(\frac{dp_M}{dt}+\frac{2\pi Q^2}{L^2}\right)V=0, \quad (48)$$

as follows from the equations of motion in Eq. (42) when multiplied by $v=dx/dt$ and by $V=dX/dt$ and then added. It is important to note that for both nonrelativistic and relativistic particle energy,

$$\frac{dU_{\mathrm{mech}}}{dt}=\frac{d\mathbf{p}_{\mathrm{mech}}}{dt}\cdot\mathbf{v}. \quad (49)$$

Thus for the nonrelativistic kinetic energy

$$\frac{d}{dt}U_{\text{mech-nonrel}}=\frac{d}{dt}\left(\frac{1}{2}mv^2\right)$$
$$=(m\mathbf{v})\cdot\frac{d\mathbf{v}}{dt}=\frac{d(m\mathbf{v})}{dt}\cdot\mathbf{v}=\frac{d\mathbf{p}_{\mathrm{mech}}}{dt}\cdot\mathbf{v}, \quad (50)$$

while for the relativistic energy

$$\frac{d}{dt}U_{\text{mech-rel}}=\frac{d}{dt}\left(\frac{mc^2}{[1-(v/c)^2]^{1/2}}\right)$$
$$=\frac{m}{[1-(v/c)^2]^{3/2}}\mathbf{v}\cdot\frac{d\mathbf{v}}{dt} \quad (51)$$
$$=\frac{d}{dt}\frac{m\mathbf{v}}{[1-(v/c)^2]^{1/2}}\cdot\mathbf{v}=\frac{d\mathbf{p}_{\mathrm{mech}}}{dt}\cdot\mathbf{v}. \quad (52)$$

The angular momentum is a constant at $\mathbf{L}=0$ for all times. Thus all of the conservation laws treated so far, linear momentum, energy, and angular momentum, have not required the specification of nonrelativistic or relativistic particle mechanics in this electromagnetic system. However, the relativistic law for the center of energy is different; this quantity involves the generator of proper Lorentz transformations and requires a fully relativistic treatment. Thus the time rate of change of the system energy times the center of energy follows from Eqs. (46) and (49) as

$$\frac{d(U\vec{\mathcal{X}})}{dt}=\hat{i}\left(\frac{dp_m}{dt}vx+\frac{dp_M}{dt}VX+U_mv+U_MV+\frac{2\pi Q^2}{L^2}(XV-xv)\right)$$

$$=\hat{i}\left(U_mv+U_MV+\left[\frac{dp_m}{dt}-\frac{2\pi Q^2}{L^2}\right]vx+\left[\frac{dp_M}{dt}+\frac{2\pi Q^2}{L^2}\right]VX\right)$$

$$=\hat{i}(U_mv+U_MV), \tag{53}$$

where the terms in square brackets vanish because of the equations of motion in Eq. (42). We obtain the correct relativistic law (7) provided only that

$$\frac{d}{dt}(U\mathcal{X})=U_mv+U_MV=c^2P. \tag{54}$$

However, as we see from Eq. (43), this relation (54) requires that $U_mv=c^2p_m$ and $U_MV=c^2p_M$. This connection between energy, velocity, and momentum is not true for nonrelativistic particle mechanics. It is true only for the relativistic mechanical energy and momentum, where

$$U_{\text{mech-rel}}=\frac{mc^2}{[1-(v/c)^2]^{1/2}} \text{ and}$$

$$\mathbf{p}_{\text{mech-rel}}=\frac{m\mathbf{v}}{[1-(v/c)^2]^{1/2}}. \tag{55}$$

Thus, provided that we use the exact relativistic expressions for the particle energy and momentum as well as the exact results of electromagnetic theory, the relativistic center-of-energy law (7) is satisfied for a parallel plate capacitor if there are no external forces present and the plates are free to accelerate. We notice that the contributions from both the relativistic mechanical energy and the electromagnetic energy are necessary for the validity of the center-of-energy law.

It might seem that the other examples involving a flattened slip-joint solenoid and two charged particles can be carried over to the case with accelerations when no external forces are present. However, these extensions fail because the electromagnetic behavior is not correctly treated for situations of finite velocity and acceleration. Although the electromagnetic field expressions for a capacitor in the large-plate small-separation approximation do not change at finite velocity and acceleration, the field expressions do change with velocity for the flattened slip-joint solenoid and for point charges. The expressions used in these quasi-static analyses are valid only in the low-velocity limit and can be extended to accelerating systems only in the low-velocity limit. The complications involved are clearly evident for two charged particles. The Darwin Lagrangian[10] correctly describes the interaction of point charges through order v^2/c^2. Even in this order, the particle equations of motion can be exceedingly complex,[11] and beyond this order, the full Maxwell's equations are required to describe the electromagnetic field. These situations involve radiation emission and do not seem to lend themselves to simple examples.

IV. ILLUSTRATING THE CENTER-OF-ENERGY LAW IN OTHER INERTIAL FRAMES

Because the energy times the center of energy is the generator of proper Lorentz transformations, it is natural to wish to see the forms taken by the examples in various inertial frames. The example involving the acceleration of the capacitor plates retains its form under any Lorentz transformation along the x axis. The electric field between the plates remains $\mathbf{E}=\hat{i}4\pi Q/L^2$, and the expressions for the mechanical energy and momentum are unchanged so that the entire analysis is identical for any Lorentz-transformed frame. However, if a Lorentz transformation is made in another direction, then the situation becomes much more complicated. The parallel plate capacitor requires forces of constraint for the stability of its plates. Provided these forces of constraint do no work in a Lorentz-transformed frame, they will not disrupt the conservation laws, just as in our previous calculations. However, in an inertial frame where the forces of constraint do work, there must be a flow of energy, and hence also of momentum, which invalidates any conservation laws that do not take account of these flows.[12] The parallel plates in our examples have finite extent and therefore must have forces of constraint in the y and z directions that prevent the charged plates from flying apart. Thus, our conservation analysis will hold in any inertial frame moving with finite velocity in the x direction because the forces in the y and z directions do no work. The examples involving the flattened slip-joint solenoid and two charged particles at rest with respect to one another also require forces of constraint that must be analyzed carefully.[13]

V. DISCUSSION

Nonrelativistic mechanics is invariant under the group of Galilean transformations. Electrodynamics is invariant under the Poincare group. However, nonrelativistic particle mechanics often is joined with Maxwell's electromagnetic theory to describe physical phenomena. If the elementary examples in Sec. III are assigned as homework, students invariably use nonrelativistic equations of particle motion unless explicitly required to calculate with the relativistic forms. When doing so, students have no trouble with the conservation laws for linear momentum, angular momentum, and energy. Both nonrelativistic particle dynamics and electromagnetism contain these conservation laws, and the examples involve the transfer of these quantities from one system to the other through forces. It is only by considering the invariance of the velocity of the center of energy that we become aware that Poincare invariance enforces strong restrictions on the theory. Nonrelativistic particle equations of motion fail to yield the invariant motion of the center of energy when electromagnetic energy and particle kinetic energy are included.

The three examples of the relativistic conservation laws that we have discussed all involve classical electromagnetism, which is invariant under the Poincare group. The example of the accelerating plates of a parallel plate capacitor illustrates that combinations of nonrelativistic and relativistic physics lead to the conservation laws for linear momentum, angular momentum, and energy, but only fully relativistic systems satisfy the law for the center of energy. Calculation of the center-of-energy motion forces us to notice the distinction between relativistic physics and the alternatives. In relativistic physics, it is not clear that particles can interact through an arbitrary potential function; the $1/r$ Coulomb and Kepler potential appears as part of the relativistic theories of electromagnetism and gravitation. Indeed it seems fascinating that the generator of the $O(4)$ symmetry associated with the Runge-Lenz vector of the nonrelativistic $1/r$ Kepler problem[14] is the nonrelativistic limit of the generator $U\vec{\mathcal{X}}$ for proper Lorentz transformations obtained from the Darwin Lagrangian for the (v^2/c^2)-interaction of two charged particles.[15]

The relativistic conservation laws associated with Poincare invariance require the use of relativistic physics for both the interactions and mechanical energy and momentum.[16] However, both the textbook and research literature contain many examples for which nonrelativistic and relativistic aspects are combined. This arrangement maintains the conservation laws of linear momentum, energy, and angular momentum, but not the relativistic law for the center of energy. For the most part this combination does not lead to significant difficulties in one-step calculations when the particle mechanics is taken as nonrelativistic in the presence of fixed electromagnetic fields and only the particle motion is of interest.[17] However, there are multi-step calculations where the charged particles respond with nonrelativistic motion to electromagnetic fields and in turn the electromagnetic fields arising from the nonrelativistically moving particles are of interest; these calculations lead to questionable conclusions. Thus, for example, the Aharonov-Bohm phase shift involves the (v^2/c^2) interaction of a point charge and a solenoid, but the response of the solenoid to the charged particle's fields often is treated using nonrelativistic physics.[18] One example of the paradoxical and erroneous descriptions that can arise from such a treatment of the charged particle-solenoid interaction is discussed by Coleman and Van Vleck;[1] other discussions have also been given.[19] A second example involves the scattering of random classical radiation by a mechanical scatterer to obtain the equilibrium spectrum corresponding to thermal (blackbody) radiation. It is common to use nonrelativistic mechanical behavior for the scattering charges despite the fact that the electromagnetic fields arising from the nonrelativistically moving particles are of crucial interest in obtaining radiation equilibrium.[20] In some instances,[21] relativistic particle mechanics has been combined with nonrelativistic potential functions in an attempt to discuss classical radiation equilibrium. In all these examples, the relativistic center-of-energy law is violated because the systems do not satisfy Poincare invariance. Yet relativistic transformations are clearly crucial in understanding blackbody radiation because the Planck spectrum can be obtained by Lorentz transformations associated with uniform (proper) acceleration through Lorentz-invariant zero-point radiation.[22] Thus aspects of Lorentz invariance sometimes go unappreciated in both the text book and research literature.

[1]S. Coleman and J. H. Van Vleck, "Origin of 'hidden momentum forces' on magnets," Phys. Rev. **171**, 1370–1375 (1968).

[2]Perhaps in part because there are so few simple examples involving the center of energy, there is a variety of terminology in the literature. Here we have chosen to speak of the "center of energy," following Ref. 1. However, E. F. Taylor and J. A. Wheeler, *Spacetime Physics* (Freeman, San Francisco, 1966), p. 143, discuss the "center of mass" with the understanding that "mass" means "mass-energy" as befits a relativistic theory. L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, New York, 1985), 4th ed., p. 168, discuss the "center of inertia." J. L. Anderson, *Principles of Relativity Physics* (Academic, New York, 1967), pp. 207–208 writes, "We can call **Z** the center of energy of the system of particles, in analogy to the center of mass as defined in Newtonian mechanics. According to Eq. (7-5.7), this point moves like a free particle with a velocity **V**, given by Eq. (7-5.9). If $P^{\mu}P_{\mu}=M^2\geqslant 0$, we can always perform a mapping so that $P^{\mu}=(M,0)$. The corresponding reference frame is called variously, the center of mass, or center of momentum, or center of energy frame. We prefer the latter terminology. In this frame $\mathbf{V}=0$." All of these terms have the same meaning. We have used the term "center of energy" to impress upon the reader that there is a change in point of view from the nonrelativistic "center of (rest) mass" concept where there is no role for electromagnetic energy or even particle kinetic energy and only particle rest masses are involved.

[3]See, for example, D. J. Griffiths, *Introduction to Electrodynamics* (Prentice Hall, Upper Saddle River, NJ, 1999), 3rd ed. Chapter 8 is devoted to conservation laws in electromagnetic theory, including conservation of charge, energy, linear momentum, and angular momentum. There is no mention of the invariant motion of the center of energy. Griffiths' clear organization of the conservation laws made me acutely aware of the absence of the last conservation law of Poincare invariance.

[4]I an not aware of any simple examples of the center-of-energy theorem in electromagnetism textbooks. L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fluids* (Pergamon, Oxford, 1962), 2nd ed, p. 194, give the determination of the "center of inertia" for a collection of interacting point charges as a problem. This problem is repeated on p. 168 of the 4th edition. J. D. Jackson, *Classical Electrodynamics* (J Wiley, New York, 1975), 2nd ed., p. 617, Problem 12.16 asks the reader to derive the uniform motion of the "center of mass" for an arbitrary, localized distribution of source-free electromagnetic fields. The question is repeated as Problem 12.19 in the 3rd edition of 1999.

[5]The basic idea appears in the early work of A. Einstein, "Prinzip von der Erhaltung der Schwerpunktsbewegung und die Trägheit der Energie," Ann. Phys. (Leipzig) **20**, 626–633 (1906). It is also given by E. Bessel-Hagen, "Über die Erhaltungssätze der Elektrodynamik," Math. Ann. **84**, 259–276 (1921). Bessel-Hagen analyzes the conservation laws associated with the conformal group satisfied by Maxwell's equations.

[6]See any standard text on electromagnetic theory, for example, Griffiths in Ref. 3, Sec. 8.1.2.

[7]I am unaware of any previous discussion of the center-of-energy law with external forces.

[8]The forces on each plate can be regarded as due to the average electric field across the plate, or as due to the electric field of the other plate, or as due to the pressure of the electromagnetic field. See, for example, Griffiths in Ref. 3, p. 102, Eq. (2.50), or E. M. Purcell, *Electricity and Magnetism* (McGraw-Hill, New York, 1985), 2nd ed., pp. 30–31.

[9]See, for example, T. H. Boyer, "Electric and magnetic forces and energies for a parallel-plate capacitor and a flattened, slip-joint solenoid," Am. J. Phys. **69**, 1277–1279 (2001).

[10]See, for example, Jackson in Ref. 4, Sec. 12.7.

[11]See, for example, the fields given by L. Page and N. I. Adams, "Action and reaction between moving charges," Am. J. Phys. **13**, 141–147 (1945). These electromagnetic fields follow from the Darwin Lagrangian.

[12]See, for example, the discussion in the introduction of the article by T. H. Boyer, "Example of mass-energy relation: Classical hydrogen atom accelerated or supported in a gravitational field," Am. J. Phys. **66**, 872–876 (1998).

[13]See, for example, T. H. Boyer, "Lorentz-transformation properties for energy and momentum in electromagnetic systems," Am. J. Phys. **53**, 167–171 (1985).

[14]See, for example, H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, MA, 1980), 2nd ed., Sec. 3-9.

[15]J. P. Dahl, "Physical origin of the Runge-Lenz vector," J. Phys. A **30**, 6831–6840 (1997).

[16]F. Rohrlich, in *Classical Charged Particles* (Addison-Wesley, Reading, MA, 1965), p. 210, emphasizes that the combination of nonrelativistic particle mechanics and electromagnetic fields is "inconsistent" in the sense that the combination satisfies neither Galilean invariance nor Lorentz invariance.

[17]See, for example, Ref. 3, Example 5.2, where Griffiths discusses the cycloid motion of a nonrelativistic charged particle in electric and magnetic fields.

[18]See, for example, the calculations by M. Peshkin, I. Talmi, and L. J. Tassie, "The quantum mechanical effects of magnetic fields confined to inaccessible regions," Ann. Phys. (N.Y.) **12**, 426–435 (1961), especially Sec. V.

[19]See, for example, T. H. Boyer, ‘‘Classical electromagnetic interaction of a point charge and a magnetic moment: considerations related to the Aharonov-Bohm Phase shift,’’ Found. Phys. **32**, 1–39 (2002).
[20]See, for example, J. H. Van Vleck, ‘‘The absorption of radiation by multiply periodic orbits, and its relation to the correspondence principle and the Rayleigh-Jeans law. Part II. Calculation of absorption by multiply periodic orbits,’’ Phys. Rev. **24**, 347–365 (1924) and T. H. Boyer, ‘‘Equilibrium of random classical electromagnetic radiation in the presence of a nonrelativistic nonlinear electric dipole oscillator,’’ Phys. Rev. **13**, 2832–2845 (1976).
[21]See R. Blanco, L. Pesquera, and E. Santos, ‘‘Equilibrium between radiation and matter for classical relativistic multiperiodic systems. Derivation of Maxwell-Boltzmann distribution from Rayleigh-Jeans spectrum,’’ Phys. Rev. D **27**, 1254–1287 (1983); ‘‘Equilibrium between radiation and matter for classical relativistic multiperiodic systems. II. Study of radiative equilibrium with Rayleigh-Jeans radiation,’’ *ibid.* **29**, 2240–2254 (1984).
[22]P. C. W. Davies, ‘‘Scalar particle production in Schwarzschild and Rindler metrics,’’ J. Phys. A **8**, 609–616 (1975); W. G. Unruh, ‘‘Notes on black-hole evaporation,’’ Phys. Rev. D **14**, 871–892 (1976); T. H. Boyer, ‘‘Thermal effects of acceleration for a classical dipole oscillator in classical electromagnetic zero-point radiation,’’ *ibid.* **29**, 1089–1095 (1984); D. C. Cole, ‘‘Properties of a classical charged harmonic oscillator accelerated through classical electromagnetic zero-point radiation,’’ *ibid.* **31**, 1972–1981 (1985).

Tablet-Arm Lecture Room Chair. Physics lecture halls used to be filled with ranks of tablet-arm chairs. These very early chairs were designed by Frederick A. P. Barnard of the University of Mississippi shortly before the American Civil War, and were used in the large building that he originally designed as an observatory. The chairs are cast-iron, with leather seat pads, and are more comfortable than one might expect. A few survive in the University Museum. After the war, Barnard became president of Columbia University, and Barnard College is named after him. (Photograph and Notes by Thomas B. Greenslade, Jr., Kenyon College)