Arnowitt-Deser-Misner energy

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The **Arnowitt-Deser-Misner energy** is a universal and useful definition of energy for asymptotically flat solutions of the Einstein equations (with or without matter) having a specified asymptotic falloff ---essentially that of the Schwarzschild metric. This definition was made possible thanks to the canonical formulation of general relativity as a Hamiltonian system as given by Arnowitt, Deser and Misner (1962).

Contents

Introduction

Because of the equivalence principle, there is no generally meaningful local energy-density or stress tensor-density for the gravitational field. This absence was the obstacle to development of the correct global conserved quantities, namely four-momentum and angular momentum, since the corresponding special relativistic invariances under translations and rotations are only realized asymptotically. This problem can be illuminated by an analogy to the more familiar vector gauge fields, and the corresponding contrast there between the abelian—hence uncharged—Maxwell field and the non-abelian Yang-Mills (YM) system which is "charged", namely self-interacting. Definition of color charge in the latter theory (already classically) requires a similar (but less physical) asymptotic "color flatness"; like energy density, color charge density is gauge-dependent. Indeed, a clear picture of the Yang-Mills charge was only given (Abbott and Deser 1982b) after that for energy of cosmological gravity (Abbott and Deser 1982a), where the detailed mechanism of asymptotic Killing vectors was first developed. Once the latter's role was understood, it also became clear how to extend the conserved charge/energy idea to more general systems, such as higher order gravity involving higher orders of curvature (Deser and Tekin 2003, 2007), or to topologically massive models involving Chern-Simons terms (Deser et al. 1982).

Defining the energy

The underlying idea for all systems is a simple one: conserved charge (in the generalized sense to include energy, momentum etc.) is associated with a conserved vector current, as only (contravariant density) vectors j^{μ} can obey ordinary conservation

$$\partial_{\mu} j^{\mu} = 0. \tag{1}$$

in curved backgrounds. Covariantly conserved stress tensor densities $T^{\nu}_{\ \mu}$ must be reduced to vectors, and this is the role of Killing Vector ξ^{μ} . Explicitly, if $\overline{D}_{\nu} T^{\nu}_{\ \mu} = 0$, where $\overline{D}_{\nu}$ is the covariant divergence with respect to some background, for us primarily flat or (Anti)deSitter (A)dS asymptotic spacetimes as we shall see, then

$$\overline{D}_{\mu}(\xi^{\nu} T^{\mu}_{\ \nu}) \equiv \partial_{\mu}(\xi^{\nu} T^{\mu}_{\ \nu}) = 0, \qquad \overline{D}_{\mu}\xi_{\nu} + \overline{D}_{\nu}\xi_{\mu} = 0. \tag{2}$$

The relevance of the Killing vector equation to verifying conservation is obvious for symmetric, covariantly conserved $T^{\mu\nu}$. If j^{μ} falls off sufficiently fast at spatial infinity, then (1) always implies that

$$\partial_0 \int d^3 x j^0 + \oint dS \cdot j = 0 \longrightarrow \partial_0 \int d^3 j^0 \equiv \partial_0 Q = 0. \tag{3}$$

In electrodynamics, the Killing vector is invisible, but in YM where j^{μ} also depends on a color index a, we require a "color-vacuum" asymptotic configuration with a color-space Killing vector ξ_a, $\overline{D}_{\mu} \xi_a = 0$ that "bleaches" $(\overline{D}_{\mu} j^{\mu})_a$ into $\partial_{\mu}(\xi_a j^{\mu}_a)$, where $\overline{D}$ is the usual YM covariant derivative.

The other essential ingredient for us is that gauge theories always contain constraints–relations independent of time evolution–because they have fewer dynamical variables than the redundant number needed for a covariant description. These constraint equations (prototypically the Gauss equation $\nabla^2 \phi = -4\pi\rho$) consist, for physical models, of a leading linear term plus nonlinear ones. Thus, in YM, the constraint becomes $\nabla \cdot E^a = -4\pi\rho^a(E, A)$ where ρ^a depends on quadratic and higher powers of the YM fields, as well as possible color matter sources. In Einstein gravity, the same distinction is expressed by the fact that the linearized, "abelian", approximations has the Newtonian constraint $\nabla \cdot \nabla\varphi = -4\pi G T^0_0$, where T^0_0 is the matter energy density, while the full theory has "color" (self-interaction of gravity), with consequent nonlinear constraints. The simplest illustration here is the case of a moment of time-symmetry (essentially one with vanishing conjugate momentum field), where the relevant constraint reads ${}^3R = -4\pi T^0_0$, where ${}^3R$ is the instrinsic 3-slice scalar curvature at this moment. The curvature splits into a sum of linear plus non linear terms, ${}^3R = {}^3R_L + {}^3R_N$, in the expansion of the spatial metric about its flat values. While this expansion may seem ambiguous, the resulting space integrals are entirely unique. The linear term is still the flat Laplacian of a metric component, which leads to an asymptotic expression for energy P_0 as a surface integral at spatial infinity, just like Maxwell charge, as are also the similarly defined spatial momenta P_i, defined from the other three constraints of the model, and for their first moments, the 4-dimensional rotation generators $T_{\mu\nu}$. It can be shown that $P_{\mu} = (P_0, P_i)$ is not only a Lorentz 4-vector, with respect to asymptotic Lorentz transformations, but that it is future timelike for pure gravity and in the presence of physical positive energy (in flat space) matter source. [Null P_{μ} are excluded, since they correspond to plane waves, which violate asymptotic flatness.] Indeed, the positive energy theorems of general relativity (Schoen and Yau 1979; Witten 1981) and supergravity (Deser and Teitelboim 1977) state that P_0 is not only positive, but that vanishing of this single number implies vacuum–everywhere flat spacetime!

The specific expression for P_0 can be written in full asymptotic Killing regalia, but it suffices here to write it in an asymptotically Cartesian coordinate frame at infinity, say putting the system at rest ($P_i = 0$). The relevant "00" constraint is

$$R^0 \equiv -\sqrt{g}[{}^3R + g^{-1}(\frac{1}{2}\pi^2 - \pi^{ij}\pi_{ij})] = \mathrm{I}^0_0.$$

where all metrics are in the intrinsic 3-space and $\pi^{ij} \equiv -\sqrt{g}(K^{ij} - g^{ij}K)$ where K^{ij} is the second fundamental form (essentially the normal of the 3-space into 4-space), and T^0_0 the matter energy density. As we said earlier, ${}^3R$ has a linear term in h_{ij} using the decomposition $g_{ij} = \delta_{ij} + h_{ij}$, and this linear term is

$$R_L \equiv \nabla^2 h_{ii} - \partial^2_{ij} h_{ij} \equiv \nabla^2 h^T,$$

making it similar to $\nabla \cdot E$ in Maxwell. Its asymptotic value determines the total energy at spatial infinity; for the Schwarzschild solution, this gives $P_0 = m$. Note that unlike the standard textbook definition, energy comes from the spatial metric rather than from g_{00}, as explained in (Arnowitt et al. 1962).

The energy E of the gravitational field is just the numerical value of the Hamiltonian for a particular solution of the field equations. In obtaining this numerical value, the form of the Hamiltonian as a function of the canonical variables is irrelevant, and one may make use of the equation, $\mathrm{I}^0{}_0 = \nabla^2 g^T$ to express E as a surface integral. One has then

$$P^0 \equiv E = -\int d^3 r \nabla^2 g^T = -\oint dS_i g^T{}_{,i} = \oint dS_i (g_{ij,j} - g_{jj,i})$$

where dS_i is the two-dimensional surface element at spatial infinity. (It should be emphasized that, while the *energy* and *momentum* are indeed divergences, the integrands in the *Hamiltonian* and in the space translation generators $\mathrm{I}^0{}_\mu[g^{TT}, \pi^{TT}]$ are *not* divergences when expressed as functions of the canonical variables.) Since P^μ is constant, due to the fact that $\mathrm{I}^0{}_\mu$ does not depend on x^μ explicitly, it can be evaluated at any given time. For this purpose, one needs only those initial Cauchy data required to specify the state of the system uniquely. Prior to the imposition of coordinate conditions, these are g_{ij} and π^{ij} (and not, for example, $g_{0\mu}$) When any (asymptotically rectangular) coordinate conditions are imposed, one needs only the two pairs of canonical variables of that frame to specify the state of the system and, of course, to calculate P^μ .

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