ARNOWITT–DESER–MISNER ENERGY AND g_{00}

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For a stationary, asymptotically flat space–time the "Komar energy", associated with the time-like Killing vector and the ADM energy are equal when the latter is evaluated on a Cauchy surface which is asymptotically at rest relative to the Killing vector. The implications of this result on the positivity-of-energy problem in General Relativity are discussed.

In General Relativity there are (at least) two notions of total mass (energy), which can be associated with a stationary spacetime. The first notion (later referred to as KVM), which is a special case of one due to Komar [1], is essentially defined as the integral of the norm of the Killing vector over a 2-sphere lying in some Cauchy surface Σ in the limit as the sphere tends to spatial infinity. The other quantity I consider is the mass according to Arnowitt, Deser and Misner [2]. It is defined as a similar surface integral at infinity, but one which contains only the intrinsic metric of Σ. It is this quantity which is commonly regarded as the true gravitational energy for several reasons, one of them being that it is always conserved (i.e. also in the nonstationary case) and, in fact, coincides in numerical value with the hamiltonian of General Relativity [3].

In this paper the question is asked whether these two quantities agree on certain hypersurfaces Σ in stationary asymptotically flat solutions of Einstein's equations. This is interesting for two (related) reasons:

(1) The KVM is the definition of mass which comes from a consideration of the newtonian limit of General Relativity. To be more precise, the norm of the Killing vector (i.e. the g_{00}-component of the metric in suitable coordinates) is the leading force term in the equations of motion of a slowly moving test particle far away from the source. Hence the KVM can be determined from Kepler orbits which this particle follows.

(2) g_{00} satisfies a Poisson-like equation (essentially $R_0{}^0$ = source term [4]) with a quantity, sometimes called "active gravitational mass density", on the right-hand side, derived from the energy–momentum tensor. This gives rise to rather obvious positivity properties of the KVM, provided the source satisfies certain inequalities. Hence, an answer as to whether KVM equals ADM mass has implications on the much discussed problem of the positivity of the ADM mass [5].

This paper shows that for a stationary, asymptotically flat Einstein space–time (with a source fulfilling a reasonable fall-off condition at spatial infinity), which is spatially diffeomorphic to $\mathbf{R}^3$, KVM and ADM mass are equal, when Σ is chosen to be asymptotically orthogonal to the Killing vector.

A proof of this theorem, though remarkably simple, does not seem to exist in the literature. (Actually, some authors claim that this theorem is false. See, e.g. ref. [6]. They do this on the basis of work of Misner [7] which shows that the ADM and the Komar energy in general disagree.)

However, there are proofs under the additional assumptions of either asymptotic spherical symmetry [8] or validity of linearized theory in the asymptotic regime [9] which have, in general, not been justified mathematically.

I first consider the static case. The space–time manifold is of the form $\{t\} \times \Sigma$. The metric can be written as

$$ds^2 = -V^2\,dt^2 + g_{ij}\,dx^i\,dx^j = -e^{+2U/c^2}\,c^2\,dt^2 + e^{-2U/c^2}\gamma_{ij}\,dx^i dx^j \quad (i = 1,2,3)\,, \tag{1}$$

where $V^2 = -\xi^\mu \xi_\mu$, the norm of the hypersurface-orthogonal Killing vector field, and γ_{ij} are functions of space only. There should exist a coordinate system x^i covering Σ such that space–time is asymptotically flat in the sence that $g_{ij} - \delta_{ij} = O(1/r)$, $\partial_k g_{ij} = O(1/r^2)$, $U = O(1/r)$, $\partial_i U = O(1/r^2)$ with $r = (x_i x^i)^{1/2} \to \infty$. The Einstein field equations give [10]

$$\Delta U = 4\pi G(\rho + p^i{}_i/c^2)\,\mathrm{e}^{-2U/c^2} := 4\pi G \mu \mathrm{e}^{-2U/c^2} , \quad (2)$$

$$-G_{ij} = (2/c^4)(\partial_i U \partial_j U - \tfrac{1}{2}\gamma_{ij}\gamma^{lm}\partial_l U \partial_m U) + (8\pi G/c^4) p_{ij}\, \mathrm{e}^{-2U/c^2} , \quad (3)$$

where Δ is the laplacian relative to γ_{ij}, G_{ij} the Einstein tensor of γ_{ij}, ρ the mass density, p_{ij}/c^2 the stress tensor. If, for example, the source consists of matter in some bounded region of space and a Maxwell field, it is reasonable to assume that $\rho, p_{ij} = O(r^{-4})$. In fact, I shall do with $O(r^{-3-\epsilon})$, $\epsilon > 0$.

The KVM is defined by

$$8\pi GM = \int_{r=\infty} \mathrm{d}S^i \partial_i V^2 . \quad (4)$$

This is well-defined because, by eq. (2),

$$8\pi GM = 8\pi G \int_\Sigma \mathrm{d}^3x \sqrt{\gamma}\, \mu \mathrm{e}^{-2U/c^2} , \quad (5)$$

and the right-hand side of eq. (5) converges.

On the other hand, the ADM energy is given by

$$16\pi E(g) = c^4 \int_{r=\infty} \mathrm{d}S^i (\partial^j g_{ij} - \partial_i g^{jj}) . \quad (6)$$

I claim that $E(g) = Mc^2$. The proof will at the same time show that $E(g)$ converges. Inserting $g_{ij} = \mathrm{e}^{-2U/c^2} \times \gamma_{ij}$ into eq. (6) one obtains

$$E(g) = Mc^2 + E(\gamma) . \quad (7)$$

Now observe that, by eq. (3), $G_{ij}(\gamma)$ goes like $r^{-3-\epsilon}$ while it is expected to go only like r^{-3} in a generic asymptotically flat situation [6]. But one may nicely characterize $E(\gamma)$ by the r^{-3}-part of $G_{ij}(\gamma)$:

Lemma: Let γ_{ij} be asymptotically flat, i.e. $\gamma_{ij} - \delta_{ij} = O(1/r)$, $\partial_k \gamma_{ij} = O(1/r^2)$. Then

$$(16\pi G/c^4) E(\gamma) = \int_{r=\infty} \mathrm{d}S_j\, G_i{}^j(\gamma) x^i , \quad (8)$$

provided $E(\gamma)$ exists.

Remark: By introducing the unit normal and the surface area of the nested family of 2-spheres which is involved here, formula (8) can be given a completely covariant form.

Proof: Consider the identity

$$\gamma\, [G^{ij} + t^{ij}] = \partial_k h^{ijk} . \quad (9)$$

Here t^{ij} is the "3-dimensional Landau–Lifshitz pseudotensor" [11] which is a sum of terms quadratic in $\Gamma^k{}_{lm}(\gamma_{ij})$ and is hence $O(1/r^4)$. $h^{ijk} = -h^{ikj}$ is the superpotential,

$$h^{ijk} = \partial_l [\gamma(\gamma^{ij}\gamma^{kl} - \gamma^{ik}\gamma^{jl})] \to \partial_j \partial_k h^{ijk} = 0 . \quad (10)$$

Partially integrating the identity

$$0 = \int_{\mathbf{R}^3} \mathrm{d}^3x\, (\partial_j \partial_k h_i{}^{jk}) x^i , \quad (11)$$

one obtains, using eq. (10),

$$0 = \int_{r=\infty} \mathrm{d}S^j\, G_{ij} x^i - \int_{\mathbf{R}^3} \mathrm{d}^3x\, \partial_k h^{iik} . \quad (12)$$

After conversion of the last term in eq. (12) into a surface integral, a short calculation gives eq. (8). This proves the lemma. Applying this lemma to γ_{ij}, as determined by eq. (3), implies that $E(\gamma) = 0$. Hence $E(g) = Mc^2$.

Several remarks are in order. When the matter field obeys the strong energy condition [12], one has $\mu \geqslant 0$ and hence $M \geqslant 0$. Therefore $E \geqslant 0$ independently of any weak field assumptions. On the other hand, if only the weak energy condition holds which in our case implies $g \geqslant 0$, then, as shown by Brill and Deser [13] and made rigorous in ref. [6], one knows that $E \geqslant 0$ for weak field excitations. This, then, sets constraints on the extent to which a static solution may violate the strong energy condition when the weak energy condition holds. (Actually, the proof in ref. [6] uses the weak and strong energy condition. But the latter is only needed to establish the existence of a maximal slicing which is trivial in a static manifold. I thank Stefan Beig for discussing this point with me.) For example, $\mu < 0$ is forbidden, provided $\rho \geqslant 0$.

In the stationary case ξ^μ is no longer hypersurface-orthogonal. One may, however, look for a Cauchy surface Σ which, besides from being asymptotically flat intrinsically, is asymptotically orthogonal to ξ^μ. My assumptions hence are that in coordinates where $\Sigma \simeq \mathbf{R}^3$ is given by t = const and ξ^μ by $\partial/\partial t$: $U = (1/c^2)\lg V^2/c^2 = O(1/r)$, $\partial_i U = O(1/r^2)$, $g_{ij} - \delta_{ij} = O(1/r)$, $\partial_k g_{ij} = O(1/r^2)$, $g_{0i} = O(1/r)$, $\partial_j g_{0i} = O(1/r^2)$.

Provided $T_{\mu\nu} = O(r^{-3-\epsilon})$, $8\pi GM = \int_{r=\infty} \mathrm{d}S^i\, \partial_i V^2$ is still well defined [4]. The role played by γ_{ij} in the static case is now taken over by

$$h_{ij} = (c^2/V^2)(g_{ij} - g_{0i}g_{0j}/V^2)\,.$$

Applying the lemma to the stationary field equations (see, e.g., ref. [11], p. 301) give $E(h) = 0$. This, together with the fact that terms containing g_{0i} do not contribute to the energy, gives the desired result $E(g) = Mc^2$.

When the strong energy condition holds, one knows that $\mu \geqslant 0$ (and hence $E \geqslant 0$) if matter is a perfect fluid or if space–time admits a maximal slicing [14].

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