ADM formalism

From Wikipedia, the free encyclopedia

The **ADM Formalism** developed in 1959 by Richard Arnowitt, Stanley Deser and Charles W. Misner is a Hamiltonian formulation of general relativity. This formulation plays an important role both in quantum gravity and numerical relativity.[2]

Richard Arnowitt, Stanley Deser and Charles Misner at the *ADM-50: A Celebration of Current GR Innovation* conference,[1] in honor of the 50th anniversary of their paper, November 2009.

A comprehensive review of this formalism was published by the same authors in "Gravitation: An introduction to current research" Louis Witten (editor), Wiley NY (1962); chapter 7, pp 227–265. Recently, this has been reprinted in the journal General Relativity and Gravitation [3] The original papers can be found in Physical Review archives.[2][4][5][6][7][8][9][10][11]

Contents

Overview

The formalism supposes that spacetime is foliated into a family of spacelike surfaces Σ_t, labeled by their time coordinate t, and with coordinates on each slice given by x^i . The dynamic variables of this theory are taken to be the metric tensor of three dimensional spatial slices $\gamma_{ij}(t,x^k)$ and their conjugate momenta $\pi^{ij}(t,x^k)$. Using these variables it is possible to define a Hamiltonian, and thereby write the equations of motion for general relativity in the form of Hamilton's equations.

In addition to the twelve variables γ_{ij} and π^{ij}, there are four Lagrange multipliers: the lapse function, N, and components of shift vector field, N_i. These describe how each of the "leaves" Σ_t of the foliation of spacetime are welded together. The equations of motion for these variables can be freely specified; this freedom corresponds to the freedom to specify how to lay out the coordinate system in space and time.

Derivation

Notation

Most references adopt notation in which four dimensional tensors are written in abstract index notation, and that Greek indices are spacetime indices taking values (0, 1, 2, 3) and Latin indices are spatial indices taking values (1, 2, 3). In the derivation here, a superscript (4) is prepended to quantities that typically have both a three-dimensional and a four-dimensional version, such as the metric tensor for three-dimensional slices g_{ij} and the metric tensor for the full four-dimensional spacetime ${}^{(4)}g_{\mu\nu}$.

The text here uses Einstein notation in which summation over repeated indices is assumed.

Two types of derivatives are used: Partial derivatives are denoted either by the operator ∂_i or by subscripts preceded by a comma. Covariant derivatives are denoted either by the operator ∇_i or by subscripts preceded by a semicolon.

The determinant of the metric tensor is represented by g (with no indices). Other tensor symbols written without indices represent the trace of the corresponding tensor such as $\pi = g^{ij}\pi_{ij}$.

Lagrangian Formulation

The starting point for the ADM formulation is the Lagrangian

$$\mathcal{L} = {}^{(4)}R\sqrt{{}^{(4)}g}$$

which is a product of the determinant of the four-dimensional metric tensor for the full spacetime and its Ricci scalar. This is the Lagrangian from the Einstein-Hilbert action.

The desired outcome of the derivation is to define an embedding of three-dimensional spatial slices in the four-dimensional spacetime. The metric of the three-dimensional slices

$$g_{ij} = {}^{(4)}g_{ij}$$

will be the generalized coordinates for a Hamiltonian formulation. The conjugate momenta can then be computed

$$\pi^{ij} = \sqrt{{}^{(4)}g}\left({}^{(4)}\Gamma^0_{pq} - g_{pq}{}^{(4)}\Gamma^0_{rs}g^{rs}\right)g^{ip}g^{jq}$$

using standard techniques and definitions. The symbols ${}^{(4)}\Gamma^0_{ij}$ are Christoffel symbols associated with the metric of the full four-dimensional spacetime. The lapse

$$N = \left(-{}^{(4)}g^{00}\right)^{-1/2}$$

and the shift vector

$$N_i = {}^{(4)}g_{0i}$$

are the remaining elements of the four-metric tensor.

Having identified the quantities for the formulation, the next step is to rewrite the Lagrangian in terms of these variables. The new expression for the Lagrangian

$$\mathcal{L} = -g_{ij}\partial_t\pi^{ij} - NH - N_iP^i - 2\partial_i(\pi^{ij}N_j - \frac{1}{2}\pi N^i + \nabla^i N\sqrt{g})$$

is conveniently written in terms of the two new quantities

$$H = -\sqrt{g}\left[R + g^{-1}\left(\frac{1}{2}\pi^2 - \pi^{ij}\pi_{ij}\right)\right]$$

and

$$P^i = -2\pi^{ij}{}_{;j}$$

which are known as the Hamiltonian constraint and the momentum constraint respectively. Note also that the lapse and the shift appear in the Hamiltonian as Lagrange multipliers.

Equations of Motion

Although the variables in the Lagrangian represent the metric tensor on three-dimensional spaces embedded in the four-dimensional spacetime, it is possible and desirable to use the usual procedures from Lagrangian mechanics to derive "equations of motion" that describe the time evolution of both the metric g_{ij} and its conjugate momentum π^{ij}. The result

$$\partial_t g_{ij} = 2Ng^{-1/2}(\pi_{ij} - \frac{1}{2}\pi g_{ij}) + N_{i;j} + N_{j;i}$$

and

$$\partial_t\pi^{ij} = -N\sqrt{g}(R^{ij} - \frac{1}{2}Rg^{ij}) + \frac{1}{2}Ng^{-1/2}g^{ij}(\pi^{mn}\pi_{mn} - \frac{1}{2}\pi^2) - 2Ng^{-1/2}(\pi^{in}\pi_n{}^j - \frac{1}{2}\pi\pi^{ij})$$
$$-\sqrt{g}(\nabla^i\nabla^j N - g^{ij}\nabla^n\nabla_n N) + \nabla_n(\pi^{ij}N^n) - N^i{}_{;n}\pi^{nj} - N^j{}_{;n}\pi^{ni}$$

is a non-linear set of partial differential equations.

Taking variations with respect to the lapse and shift provide constraint equations

$$H = 0$$

and

$$P^i = 0$$

and the lapse and shift themselves can be freely specified, reflecting the fact that coordinate systems can be freely specified in both space and time.

Application to Quantum Gravity

Using the ADM formulation, it is possible to attempt to construct a quantum theory of gravity, in the same way that one constructs the Schrödinger equation corresponding to a given Hamiltonian in quantum mechanics. That is, replace the canonical momenta $\pi^{ij}(t,x^k)$ and the spatial metric functions by linear functional differential operators

$$\hat{g}_{ij}(t,x^k) \to g_{ij}(t,x^k)$$
$$\hat{\pi}^{ij}(t,x^k) \to -i\frac{\delta}{\delta g_{ij}(t,x^k)}$$

More precisely, the replacing of classical variables by operators is restricted by commutation relations. The hats represents operators in quantum theory. This leads to the Wheeler-deWitt equation.

Application to Numerical Solutions of the Einstein Equations

There are relatively few exact solutions to the Einstein field equations. In order to find other solutions, there is an active field of study known as numerical relativity in which supercomputers are used to find approximate solutions to the equations. In order to construct such solutions numerically, most researchers start with a formulation of the Einstein equations closely related to the ADM formulation. The most common approaches start with an initial value problem based on the ADM formalism.

In Hamiltonian formulations, the basic point is replacement of set of second order equations by another first order set of equations. We may get this second set of equations by Hamiltonian formulation in an easy way. Of course this is very useful for numerical physics, because the reduction of order of differential equations must be done, if we want to prepare equations for a computer.

ADM Energy

ADM energy is a special way to define the energy in general relativity which is only applicable to some special geometries of spacetime that asymptotically approach a well-defined metric tensor at infinity — for example a spacetime that asymptotically approaches Minkowski space. The ADM energy in these cases is defined as a function of the deviation of the metric tensor from its prescribed asymptotic form. In other words, the ADM energy is computed as the strength of the gravitational field at infinity.

The quantity is also called the ADM Hamiltonian, especially if one finds a different formula than the definition above that however leads to the same result.

If the required asymptotic form is time-independent (such as the Minkowski space itself), then it respects the time-translational symmetry. Noether's theorem then implies that the ADM energy is conserved. According to general relativity, the conservation law for the total energy does not hold in more general, time-dependent backgrounds - for example, it is completely violated in physical cosmology. Cosmic inflation in particular is able to produce energy (and mass) from "nothing" because the vacuum energy density is roughly constant, but the volume of the Universe grows exponentially.

See also

- Canonical coordinates
- Canonical gravity
- Hamiltonian mechanics
- Wheeler-deWitt equation

References

1. ^ ADM-50: A Celebration of Current GR Innovation (http://adm-50.physics.tamu.edu)
2. ^ *a* *b* Arnowitt, R.; Deser, S.; Misner, C. (1959). "Dynamical Structure and Definition of Energy in General Relativity". *Physical Review* **116** (5): 1322–1330. Bibcode 1959PhRv..116.1322A (http://adsabs.harvard.edu/abs/1959PhRv..116.1322A) . doi:10.1103/PhysRev.116.1322 (http://dx.doi.org/10.1103%2FPhysRev.116.1322) .
3. ^ Arnowitt, R.; Deser, S.; Misner, C. (2008). "Republication of: The dynamics of general relativity". *General Relativity and Gravitation* **40** (9): 1997–2027. arXiv:gr-qc/0405109 (http://arxiv.org/abs/gr-qc/0405109) . Bibcode 2008GReGr..40.1997A (http://adsabs.harvard.edu/abs/2008GReGr..40.1997A) . doi:10.1007/s10714-008-0661-1 (http://dx.doi.org/10.1007%2Fs10714-008-0661-1) .
4. ^ Arnowitt, R.; Deser, S. (1959). "Quantum Theory of Gravitation: General Formulation and Linearized Theory". *Physical Review* **113** (2): 745–750. Bibcode 1959PhRv..113..745A (http://adsabs.harvard.edu/abs/1959PhRv..113..745A) . doi:10.1103/PhysRev.113.745 (http://dx.doi.org/10.1103%2FPhysRev.113.745) .
5. ^ Arnowitt, R.; Deser, S.; Misner, C. (1960). "Canonical Variables for General Relativity". *Physical Review* **117** (6): 1595–1602. Bibcode 1960PhRv..117.1595A (http://adsabs.harvard.edu/abs/1960PhRv..117.1595A) . doi:10.1103/PhysRev.117.1595 (http://dx.doi.org/10.1103%2FPhysRev.117.1595) .
6. ^ Arnowitt, R.; Deser, S.; Misner, C. (1960). "Finite Self-Energy of Classical Point Particles". *Physical Review Letters* **4** (7): 375–377. Bibcode 1960PhRvL...4..375A (http://adsabs.harvard.edu/abs/1960PhRvL...4..375A) . doi:10.1103/PhysRevLett.4.375 (http://dx.doi.org/10.1103%2FPhysRevLett.4.375) .
7. ^ Arnowitt, R.; Deser, S.; Misner, C. (1960). "Energy and the Criteria for Radiation in General Relativity". *Physical Review* **118** (4): 1100–1104. Bibcode 1960PhRv..118.1100A (http://adsabs.harvard.edu/abs/1960PhRv..118.1100A) . doi:10.1103/PhysRev.118.1100 (http://dx.doi.org/10.1103%2FPhysRev.118.1100) .
8. ^ Arnowitt, R.; Deser, S.; Misner, C. (1960). "Gravitational-Electromagnetic Coupling and the Classical Self-Energy Problem". *Physical Review* **120**: 313–320. Bibcode 1960PhRv..120..313A (http://adsabs.harvard.edu/abs/1960PhRv..120..313A) . doi:10.1103/PhysRev.120.313 (http://dx.doi.org/10.1103%2FPhysRev.120.313) .
9. ^ Arnowitt, R.; Deser, S.; Misner, C. (1960). "Interior Schwarzschild Solutions and Interpretation of Source Terms". *Physical Review* **120**: 321–324. Bibcode 1960PhRv..120..321A (http://adsabs.harvard.edu/abs/1960PhRv..120..321A) . doi:10.1103/PhysRev.120.321 (http://dx.doi.org/10.1103%2FPhysRev.120.321) .
10. ^ Arnowitt, R.; Deser, S.; Misner, C. (1961). "Wave Zone in General Relativity". *Physical Review* **121** (5): 1556–1566. Bibcode 1961PhRv..121.1556A (http://adsabs.harvard.edu/abs/1961PhRv..121.1556A) . doi:10.1103/PhysRev.121.1556 (http://dx.doi.org/10.1103%2FPhysRev.121.1556) .
11. ^ Arnowitt, R.; Deser, S.; Misner, C. (1961). "Coordinate Invariance and Energy Expressions in General Relativity". *Physical Review* **122** (3): 997–1006. Bibcode 1961PhRv..122..997A (http://adsabs.harvard.edu/abs/1961PhRv..122..997A) . doi:10.1103/PhysRev.122.997 (http://dx.doi.org/10.1103%2FPhysRev.122.997) .

- Kiefer, Claus (2007). *Quantum Gravity*. Oxford, New York: Oxford University Press. ISBN 978-0-19-921252-1.