Time in Quantum Geometrodynamics

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Abstract

We revisit the issue of time in quantum geometrodynamics and suggest a quantization procedure on the space of true dynamic variables. This procedure separates the issue of quantization from enforcing the constraints caused by the general covariance symmetries. The resulting theory, unlike the standard approach, takes into account the states that are off shell with respect to the constraints, and thus avoids the problems of time. In this approach, quantum geometrodynamics, general covariance, and the interpretation of time emerge together as parts of the solution of the total problem of geometrodynamic evolution.

1 Introduction.

A proper introduction of time is one of the central issues of any viable programme of gravity quantization. Its resolution is important conceptually as it determines in a profound way the meaning of the quantization procedure or the meaning of the basic structure that contains quantum gravity as a particular case or an approximation. It is also likely that a proper understanding of this issue will provide the clues to answering the questions motivating gravity quantization in the first place, such as avoidance of singularities, the issues of the final and the initial state of the Universe, etc. It might contribute to a

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better understanding of the aspects of modern quantum field theories that depend crucially on the causal structure of spacetime. One can be referred to the recent paper [1] by J. Butterfield and C. J. Isham for an exhaustive exposition of motivational and conceptual problems of quantum gravity, as well as for an overview of modern programmes of gravity quantization.

Quite understandably, the issue of time is formulated differently in different approaches to gravity quantization. In what follows we restrict ourselves to what can be thought of as a slight extension of the canonical gravity quantization programme.

The standard canonical quantum gravity approach [2] is based on the classical dynamic picture of the evolving 3–geometry of a slicing of a spacetime manifold. The slicing is essentially a reference foliation of the spacetime manifold (endowed with a 4–geometry) with respect to which the canonical variables are assigned. It is usually parametrized by the lapse function N and the shift functions N^i . The canonical variables are the 3–metric g_{ik} on a spatial slice Σ of the foliation induced by the spacetime 4–metric, and their canonical conjugate matrix π^{ik} . The latter is related to the extrinsic curvature of Σ when it is considered as embedded in the spacetime.

The customary variational procedure applied to the Hilbert action expressed in terms of the canonical variables produces Hamilton equations describing the time evolution of the canonical variables, with the Hamiltonian given as $N\mathcal{H} + N^i\mathcal{H}_i$ where \mathcal{H} and \mathcal{H}_i are functions of the canonical variables and their spatial derivatives. The procedure is not extended to the derivation of the Hamilton– Jacobi equation in the usual manner as such an equation is rendered to be meaningless with the chosen set of canonical variables (cf. [3]) when the general covariance of the theory is taken into account.

As a way out, the general covariance is introduced in the variational principle from the very onset as the requirement of the action to be invariant with respect to variations of the lapse and shift which leads to the constraint equations (to simplify notations, we omit indices on components of g and π in all equations below)

$$\mathcal{H}(g,\pi;x) = 0,\tag{1}$$

and

$$\mathcal{H}_i(g,\pi;x) = 0,\tag{2}$$

imposed on the canonical variables on each slice. An important feature of general relativity is that its dynamics is fully constrained. It can be shown that if the geometry of spacetime is such that the constraints are satisfied on all the slices of all spatial foliations of spacetime, then the canonical variables necessarily satisfy the Hamilton evolution equations. This feature is often referred to as a key property of general relativity [1] and is interpreted as an argument that the entire theory is coded in the constraints, with the conclusion that the Hamilton equations are redundant and can be ignored in dynamic considerations. Substitution of $\delta S/\delta g$ in the place of p in constraint equations leads to the new set of equations

$$\mathcal{H}\left(g,\frac{\delta S}{\delta g};x\right) = 0,\tag{3}$$

and

$$\mathcal{H}_i\left(g, \frac{\delta S}{\delta g}; x\right) = 0,\tag{4}$$

the first of which is considered to be the Hamilton–Jacobi equation. This assertion is supported by arguments appealing to the variational principle on superspace of 3–geometries (for detailed arguments and the interpretation of other equations cf. [4]).

Dirac's procedure of canonical gravity quantization is based directly on this Hamilton–Jacobi equation and produces the quantum theory that consists of commutation relations imposed on all canonical variables and the Wheeler – DeWitt equation.

The ADM square root quantization procedure is also based entirely on constraints, but in this procedure the set of canonical variables is split in two subsets, embedding variables (four of them altogether; one slicing parameter Ω and three coordinatization parameters α) and true dynamic variables β (two of them) [5], [6], [7]. The set of constraints is solved with respect to the momenta conjugate to the embedding variables. After substituting $\delta S/\delta\Omega$, $\delta S/\delta\alpha$ for p_{Ω} , p_{α} , (where S is the principal Hamilton function) one of the resulting equations (the equation for the momentum conjugate to the slicing parameter) is identified with the Hamilton–Jacobi equation, and its right hand side yields an expression for a new (square root) Hamiltonian. The quantization is based on this equation and produces the quantum theory that consists of the Schrödinger equation and commutation relations imposed on true dynamic variables and their conjugate momenta.

In both approaches, the description of time evolution of quantized gravitational fields or systems including such fields becomes extremely troublesome. Any attempt to introduce time that can be used in a way similar to that of time in quantum mechanics or in quantum field theory on a flat background invariably leads to the notorious problems of time [5], some of which are of a conceptual nature while others are technical. Attempts to introduce time in such systems in a universal way from outside, as a reading of a specially designed clock have been unsuccessful and there are all reasons to believe that it is impossible [1], whether the clock is believed to be gravitational (the readings depend only on the variables describing gravitational field) in its nature or it is a matter clock, for as long as it interacts with gravity.

The difficulties of the conceptual nature (the problem of functional evolution, and the multiple choice problem, in Kuchař's terminology) emerge due to the dual nature of time parametrization in general relativity. If spacetime is considered as a manifold it can be coordinatized and sliced in any arbitrary manner. However, this is not sufficient for the description of geometrodynamic evolution. Both slicing and coordinatization need to be tied to the metric on the manifold. The standard way of doing it in classical geometrodynamics involves lapse and shift. These parameters express slicing condition in terms of readings of the clocks of resting test particles, and coordinatization conditions as the metric shift of the coordinate grid on the slice with time. In such a description, the reduction of the dynamic picture to the constraints is based entirely on existence of the unique spacetime metric (although the metric might be not known until the geometrodynamic problem is resolved). While this is not a problem in classical general relativity, there is, in general, no possibility to assign such a unique metric on spacetime in canonical quantum gravity.

With this in mind, keeping the constraints equations as a foundation of geometrodynamics becomes not very meaningful, to say the least. Even in the classical theory, such a procedure results in the treatment of the Hamilton– Jacobi equation, that looks quite artificial and totally different from the standard mechanical considerations. From the point of view of Hamilton dynamics, shift and lapse invariance is just a symmetry of the system and should not be used in deriving dynamic equations.

In fact, the derivation of Hamilton equations does not depend on assumptions of lapse and shift invariance of action (general covariance). It is only in the derivation of the Hamilton–Jacobi equation this invariance becomes involved in a fundamental way and essentially replaces standard dynamic considerations.

As a result, the Hamiltonian of Hamilton equations does not coincide with the Hamiltonian of the Hamilton–Jacobi equation, only the first being related to the Lagrangian in a standard way. Also, the Hamilton–Jacobi equation does not contain the reference to the time evolution at all.

Quantization of the dynamic picture based on the constraints is essentially equivalent to restricting the states of the resulting quantum systems to a "shell" determined by the constraints that are classical in their origin. An attempt to undertake a similar action in quantum mechanics or quantum field theory would be quite disastrous under all but very carefully selected conditions.

One way to resolve this dilemma would be to weaken the requirement of covariance, essentially discarding it in dynamic considerations and recovering it by imposing symmetries on solutions only to the extent and in the sense that is allowed by dynamics. The general covariance in its traditional meaning should be recovered in the classical limit. In a sense, this requirement should determine, at least partially, what constitutes the classical limit in quantum geometrodynamics.

In order to achieve this goal a formulation of geometrodynamic Hamilton– Jacobi theory independent of the symmetry assumptions is needed, in the same spirit as in a standard setting of mechanics. Is there a possibility of writing the Hamilton–Jacobi equation in a way that is closer to that encountered in mechanics? As we have mentioned, the answer is no, if one considers as the object of geometrodynamics the whole 3–metric (or 3–geometry) of the slice [3].

The situation changes dramatically if York's analysis of gravitational degrees of freedom [8] is taken into account and actively utilized. It becomes possible to reformulate classical geometrodynamics in a standard way from the very beginning to the very end and to treat general covariance as a symmetry of gravitational systems. Although the Hamilton–Jacobi equation looks different and makes sense in its traditional form, the resulting description is equivalent to the commonly accepted in classical geometrodynamics. However, quantization based on this new Hamilton–Jacobi equation provides an appropriate interpretation of the conceptual problems of time making them quite natural statements concerning the properties of gravity quantization. It also seems to avoid the technical problems of time, such as the Hilbert space problem, and the spectral analysis problem, as it produces the Schrödinger equation for the state evolution, and the Hamiltonian does not include the square root operation. The procedure has been described previously elsewhere [6], [7], but we do not believe that it is widely known and provide a brief overview of it below.

In this setting, time can be introduced as a slicing parametrization on the spacetime manifold and tied to the metric structure without any contradictions. The metric interpretation of time is coupled with geometrodynamic evolution. The true meaning of time becomes completely determined only after the geometrodynamic evolution problem has been solved. In a sense, quantum geometrodynamic configuration and time emerge together and the meaning of the clock readings is influenced by quantum gravitational system.

We illustrate the emerging meaning of time by considering different examples and introducing times as counted by matter clocks and by gravitational clocks. These clocks are akin to the clocks measuring timelike intervals along the world lines of test particles (matter clock) and the York's extrinsic time parameter. In the end, we discuss the general features of time parametrization and its metric interpretation in quantum geometrodynamics.

2 Geometrodynamic Quantization in General Setting.

According to York's analysis of gravitational degrees of freedom, the set of six parameters describing the slice 3-metric should be split in two subsets, $\{\beta_1, \beta_2\}$ (two functions) and $\{\alpha_1, \alpha_2, \alpha_3, \Omega\}$. The first of these is treated as the set of true gravitational degrees of freedom (the initial values for them can be given freely), while the second is considered to be the set of embedding variables. The α parameters are often referred to as coordinatization parameters, while Ω is called, depending on the context, the slicing parameter, the scale factor, or the many-fingered time parameter. Information relevant to the gravity field is carried by β parameters, while α and Ω essentially describe time. In the original York's analysis the β variables describe the conformal part of a slice 3-geometry, Ω represents the scale factor and the α variables are determined by the choice of coordinatization of 3-slices. The true dynamic variables form what we call a dynamic superspace while the embedding variables are treated as functional parameters.

The idea is to develop geometrodynamics from the very beginning on the dynamic superspace instead of the superspace of 3-metrics or 3-geometries. The variational principle on the dynamic superspace or its phase space (formed by true dynamic variables $\{\beta_1, \beta_2\}$ and their conjugate momenta $\{\pi_{\beta_1}, \pi_{\beta_2}\}$ yields the dynamic equations describing evolution of true dynamic variables. All of these equations depend on lapse and shift and contain embedding variables as functional parameters. These are treated as an external field and is determined by additional equations that do not follow from the variational principle on dynamic superspace. The quantization procedure is performed on the dynamic superspace (only β -s are quantized, i. e. generate commutation relations, while embedding variables form a classical field). The Schrödinger equation is obtained by a quantization procedure from the Hamilton–Jacobi equation on the dynamic superspace and describes the time evolution of the state functional on true dynamic superspace coupled with the external classical field determined by embedding variables. In this paper we introduce such a coupling via a procedure similar to that of Hartree–Fock.

In a more detailed and precise description that follows, we omit indices on variables β and α for the sake of notational simplicity. They can be recovered easily whenever it becomes necessary.

We start from the standard Lagrangian \mathcal{L} (written in terms of the 3-metric, shift and lapse) and the associated action (with appropriate boundary terms, as needed, to remove the terms containing second time derivatives) and we introduce the momenta conjugate to the true dynamic variables

$$\pi_{\beta} = \frac{\partial \mathcal{L}}{\partial \dot{\beta}}.$$
(5)

We then use these π_{β} 's to form the geometrodynamic Hamiltonian \mathcal{H}_{dyn} ,

$$\mathcal{H}_{dyn} = \pi_{\beta}\beta - \mathcal{L}.$$
 (6)

The arguments of the Hamiltonian \mathcal{H}_{dyn} are described by the expression

$$\mathcal{H}_{dyn} = \mathcal{H}_{dyn}(\beta, \pi_{\beta}; \Omega, \alpha). \tag{7}$$

The variables following the semicolon are treated as describing an external field, while the ones preceding the semicolon are the coordinates and momenta of the gravitational true degrees of freedom, i.e. of the true geometrodynamics. The variation of β and π_{β} leads to the equations of geometrodynamics, i.e. to two pairs of Hamilton equations,

$$\dot{\beta} = \frac{\partial \mathcal{H}_{dyn}}{\partial \pi_{\beta}},\tag{8}$$

$$\dot{\pi}_{\beta} = -\frac{\partial \mathcal{H}_{dyn}}{\partial \beta},\tag{9}$$

and, subsequently, to the Hamilton–Jacobi equation

$$\frac{\delta S}{\delta t} = -\mathcal{H}_{DYN}\left(\beta, \frac{\delta S}{\delta\beta}; \Omega, \alpha\right). \tag{10}$$

Here S is a functional of β and, in addition, a function of t,

$$S = S\left[\beta; t\right). \tag{11}$$

and $\frac{\delta}{\delta t}$ is defined by

$$\frac{\partial S}{\partial t} = \int \frac{\delta S}{\delta t} d^3 x. \tag{12}$$

Neither the Hamilton equations (8), (9) nor the Hamilton–Jacobi equation (10) are capable of providing any predictions as their solutions depend on the functional parameters Ω and α which are not yet known. One can complete the system of equations by adding to the Hamilton equations, or to the Hamilton– Jacobi equation, the standard constraint equations of general relativity. They should be satisfied when the solution for β , π_{β} of equations of true geometrodynamics (with appropriate initial data) is substituted in them (we use symbols $[\beta]_s$, $[\pi_{\beta}]_s$ for such a solution)

$$\mathcal{H}^{i}\left([\beta]_{s}, [\pi_{\beta}]_{s}, \Omega, \alpha\right) = 0$$

$$\mathcal{H}\left([\beta]_{s}, [\pi_{\beta}]_{s}, \Omega, \alpha\right) = 0$$
(13)

These constraint equations cannot be derived from variational principles on dynamic superspace. Rather, they should be treated as additional symmetries, or the equations for an external field. They do follow from the shift and lapse invariance of the action but their derivation in this new setting depends on the structure of the whole action integral (cf. section 4). As a result, they cannot replace the full set of equations for geometrodynamic evolution. However, the resulting complete system of equations (dynamic equations on conformal superspace and constraint equations) is equivalent to this of the standard geometrodynamics on the superspace of 3–geometries [7].

For the purpose of quantization, we make a transition to the corresponding Schrödinger equation based entirely on dynamics and ignoring the system symmetries

$$i\hbar\frac{\delta\Psi}{\delta t} = \widehat{\mathcal{H}}_{dyn}\left(\beta,\widehat{\pi}_{\beta};\Omega,\alpha\right)\Psi\tag{14}$$

where $\hat{\pi}_{\beta} = \frac{\hbar}{i} \frac{\delta}{\delta\beta}$. The Schrödinger equation (14) implies that commutation relations are imposed only on true dynamic variables and treats embedding variables as external classical fields. The state functional Ψ in this equation is a functional of β and a function of t.

$$\Psi = \Psi\left[\beta, t\right) \tag{15}$$

This Schrödinger equation (with specific initial data) can be solved (cf., for instance the example of the Bianchi 1A cosmological model below). The resulting solution Ψ_s of this Schrödinger equation is not capable of providing any definite predictions as it depends on four functional parameters Ω , α which remain at this stage undetermined. All expectations, such as the expectation values of β

$$<\beta>_{s}=\langle\Psi_{s}|\beta|\Psi_{s}\rangle=\int\Psi_{s}^{*}\beta\Psi_{s}\mathcal{D}\beta$$
(16)

or of $\hat{\pi}_{\beta}$

$$<\pi_{\beta}>_{s}=\langle\Psi_{s}|\hat{\pi}_{\beta}|\Psi_{s}\rangle=\int\Psi_{s}^{*}\hat{\pi}_{\beta}\Psi_{s}\mathcal{D}\beta$$
 (17)

also depend on these functional parameters. To specify these functions we resort to the constraint equations. The treatment of the constraints has nothing to do with quantization of geometrodynamics. It is merely introducing the coupling between already quantized geometrodynamics and the classical field determined by embedding variables. In other words, they take care of the symmetries, which are classical in their nature, to the extent they are capable of doing that.

As in case of classical geometrodynamics, we impose the constraints on the solution of the dynamic equations (Schrödinger equation) with appropriate initial data and in this way, determine the unique values of Ω and α . It is possible that there are several ways to couple the constraints to the quantization of the true dynamic variables, β . As per our previous proposal we impose the four constraints only on the expectation values of the conformal dynamics

$$\mathcal{H}^{i}\left(\langle \beta \rangle_{s}, \langle \pi_{\beta} \rangle_{s}, \Omega, \alpha\right) = 0$$

$$\mathcal{H}\left(\langle \beta \rangle_{s}, \langle \pi_{\beta} \rangle_{s}, \Omega, \alpha\right) = 0$$
(18)

The way evolution occurs can be described as follows. Initial data at $t = t_0$ consist of the initial state functional $\Psi = \Psi_0$ and the initial values (functions) of embedding variables. In addition, lapse and shift are supposed to be given. Equations (16), (17) yield the expectation values (functions) of true dynamic variables and their conjugate momenta. The result are substituted in the constraints (18). After this, the constraints are solved with respect to the time derivatives of embedding variables. A step forward in time (say, with the increment Δt) is performed by integration of obtained expressions to evolve embedding variables and and by integration of the Schrödinger equation (14) to evolve the state functional. This concludes one step forward in time. The next step is performed by repeating the same operations in the same order.

One can be referred to [6], [7] for two particular examples illustrating such geometrodynamic evolution in cases of Bianchi 1A cosmology and Taub cosmology. The first one can and has been solved analytically, while the latter one has been solved numerically.

We provide an abbreviated description of only the first example (Bianchi 1A) as its analytical solution is more useful in presenting the issue of time in quantum geometrodynamics.

3 Geometrodynamic Quantization: Bianchi 1A Model.

The Bianchi 1A cosmological model is commonly referred to as the axisymmetric Kasner model [9]. Its metric is determined by two parameters, the scale factor Ω and the anisotropy parameter β

$$ds^{2} = -dt^{2} + e^{-2\Omega} \left(e^{2\beta} dx^{2} + e^{2\beta} dy^{2} + e^{-4\beta} dz^{2} \right).$$
(19)

The choice of this expression for the metric implies that we have chosen $N^i = 0$ and N = 1 values of shift and lapse for this example. As this cosmology is homogeneous the two functions Ω and β are the functions of the time parameter t only. The scalar 4-curvature can be expressed in terms of these two functions to yield the Hilbert action and, after subtracting the boundary term, the cosmological action,

$$I_C = I_H + \frac{3V}{8\pi} \dot{\Omega} e^{-3\Omega} \Big|_{t_0}^{t_f} = \frac{3V}{8\pi} \int_{t_0}^{t_f} \left(\dot{\beta}^2 - \dot{\Omega}^2 \right) e^{-3\Omega} dt,$$
(20)

where $V = \int \int \int dx dy dz$ is the spatial volume element. As it is usually done in case of homogeneous cosmologies, we integrate appropriate quantities over the spatial volumes and work with integrated Lagrangian L, Hamiltonian H and momenta p_{β} rather than with their densities $\mathcal{L}, \mathcal{H}, \pi_{\beta}$.

We treat the scale factor $\Omega(t)$ as the embedding variable and the anisotropy $\beta(t)$ as the dynamic degree of freedom. The momentum conjugate to β is

$$p_{\beta} = \frac{\partial L}{\partial \dot{\beta}} = \frac{3V}{4\pi} e^{-3\Omega} \dot{\beta}.$$
 (21)

The Hamiltonian of the system in our approach can be expressed in terms of the momentum conjugate to β and the Lagrangian.

$$H_{dyn} = p_{\beta}\dot{\beta} - L = \frac{2\pi}{3V} e^{3\Omega} p_{\beta}^2 + \frac{3V}{8\pi} \dot{\Omega}^2 e^{-3\Omega}.$$
 (22)

In the classical theory this Hamiltonian can be used to produce either one pair of Hamilton equations or the equivalent Hamilton–Jacobi equation. In either case, the dynamics picture derived in this way is incomplete. To complete it, we impose the super-Hamiltonian constraint.

$$p_{\beta}^2 = \left(\frac{3V}{4\pi}\right)^2 e^{-6\Omega} \dot{\Omega}^2.$$
(23)

Using the Hamilton–Jacobi equation,

$$\frac{\partial S}{\partial t} = -H_{dyn} \left(\frac{\partial S}{\partial \beta}, \Omega(t), \dot{\Omega}(t) \right), \qquad (24)$$

together with the expression (22) for the Hamiltonian H_{dyn} , we obtain the Schrödinger equation for the axisymmetric Kasner model.

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{2\pi\hbar^2}{3V}e^{3\Omega}\frac{\partial^2\Psi}{\partial\beta^2} + \frac{3V}{8\pi}\dot{\Omega}^2e^{-3\Omega}\Psi.$$
 (25)

The constant \hbar in this equation should be understood as the square of Planck's length scale, rather than the standard Planck constant. The quantum picture based on the Schrödinger equation (25) is incomplete as the scale factor Ω is so far an unknown function of time. To complete the dynamics picture we follow

our prescription and impose, in addition to equation (25), the super-Hamiltonian constraint.

$$\langle p_{\beta} \rangle_{s}^{2} = \left(\frac{4\pi}{3V}\right)^{2} \mathrm{e}^{-6\Omega} \dot{\Omega}^{2}.$$
 (26)

Here $\langle p_{\beta} \rangle_s$ is the expectation value of the momentum $\hat{p}_{\beta} = \frac{\hbar}{i} \frac{\partial}{\partial \beta}$

$$\langle p_{\beta} \rangle_{s} = \langle \Psi_{s} | \hat{p}_{\beta} | \Psi_{s} \rangle = \int_{-\infty}^{\infty} \Psi_{s}^{*}(\beta, t) \hat{p}_{\beta} \Psi_{s}(\beta, t) d\beta$$
 (27)

where Ψ_s is the solution of the Schrödinger equation with specified initial data. The system of equations (25), (26) provide us with a complete quantum dynamic picture of the axisymmetric Kasner model evolution and, when augmented by appropriate initial and boundary conditions, can be solved analytically.

For instance, we can specify the initial data for the Schrödinger equation in the form

$$\Psi(\beta,t)|_{t_0} = \Psi_s(\beta,t_0) = \int_{-\infty}^{\infty} A_k \mathrm{e}^{\frac{i}{\hbar}k\beta} dk$$
(28)

with

$$A_k = C e^{-a(k-k_0)^2}.$$
 (29)

Such a choice introduces the Gaussian wave packet centered initially at the value k_0 of k (we will describe the meaning of k_0 later), and of the initial width determined by the constant a, with C being merely the normalization constant, picked to satisfy the condition $\langle \Psi(\beta, t_0) | \Psi(\beta, t_0) \rangle = 1$.

The solution of the Schrödinger equation with such initial data can be written as

$$\Psi_s(\beta,t) = C\sqrt{\pi} \left(a^2 + \frac{f^2}{\hbar^2}\right)^{-\frac{1}{4}} \exp\left\{-\frac{a}{4\left(a^2 + \frac{f^2}{\hbar^2}\right)} \frac{(\beta - 2k_0 f)^2}{\hbar^2}\right\} e^{iF}, \quad (30)$$

where

$$f = f(t) = \frac{2\pi}{3V} \int_{t_0}^t e^{3\Omega} dt,$$
 (31)

and $F = F(\beta, t)$ is rather involved real valued expression [6] that depends on β , Ω , and $\dot{\Omega}$. However, it has a structure that makes computations of relevant expectation values easy. Such a computation of the expectation $\langle p_{\beta} \rangle_s$ of the momentum $\hat{p}_{\beta} = \frac{\hbar}{i} \frac{\partial}{\partial \beta}$ yields

$$\langle p_{\beta} \rangle_{s} = \langle \Psi_{s} | \widehat{p}_{\beta} | \Psi_{s} \rangle = k_{0}.$$
 (32)

In other words the expectation value of the momentum $\langle p_{\beta} \rangle_s$ does not change with time. It is determined by the k-center of the packet at $t = t_0$. It is clear that this solution of the Schrödinger equation describing the wave packet time evolution cannot provide any definite predictions as it contains as yet undetermined scale factor Ω . To find $\Omega(t)$ we need to substitute this expectation value into the constraint (26) and to solve the resulting equation with respect to Ω . Substitution of $\langle p_{\beta} \rangle_s = k_0$ in (26) yields

$$k_0^2 = \left(\frac{3V}{4\pi}\right)^2 e^{-6\Omega} \dot{\Omega}^2.$$
(33)

Once the solution of this equation is substituted in (30) the geometrodynamic problem (25), (26) for the wave packet (29) is solved completely.

The solution can be used to compute the expectation value for β :

$$\langle \beta \rangle_s = \langle \Psi_s | \beta | \Psi_s \rangle = 2k_0 f(t),$$
(34)

and the variance in β

$$\langle (\beta - \langle \beta \rangle)^2 \rangle_s = \langle \Psi_s | (\beta - \langle \beta \rangle)^2 | \Psi_s \rangle = \frac{\hbar^2 a^2 + f^2}{a}$$
(35)

Thus "the center" of the wave packet evolves as the classical Kasner universe determined by the momentum value equal to k_0 would evolve, while the spread of the packet increases with time. The result is similar to that of the quantum mechanics of a free particle; after all the Bianchi I cosmology is the free-particle analogue of quantum cosmology.

4 Time in Quantum Geometrodynamics.

In sections 2, 3 we have demonstrated that a proper understanding of classical geometrodynamics as the dynamics on the superspace of true dynamic variables amended by the constraints attributed to the universal symmetries of gravitational systems (lapse and shift invariance) opens a way to circumvent the problems of time (for a full discussion, cf. [6], [7]).

The important point is that these problems disappear as soon as the proper object of quantization (true geometrodynamic variables) is chosen. The constraints themselves are of no primary significance in this process. Their presence in the theory reflects the fact that for gravity the true dynamics cannot be decoupled from the evolution of "embedding" variables and that within the classical theory the true dynamic variables picture is limited to a "shell" determined by the constraints. There is absolutely no reason to expect that the last feature will survive after quantization, except, perhaps, for some particular carefully chosen systems. We do not require it and thus avoid the problems of time.

The particular choice of lapse (N = 1) and shift $(N^i = 0)$ in the previous section is quite sufficient to make this point. However, such a choice becomes an obstacle for understanding the subtle differences between our treatment of constraints and the standard one. Also it fixes a particular choice of time and its interpretation, thus precluding the study of alternatives in the important issue of time in quantum geometrodynamics. Essentially, we have become tied to the classical matter clocks at rest as determined by spacelike slices of the Kasner universe. We cannot even switch to another classical clock, such as the one that produces the trace of the extrinsic curvature as its readings, while such a clock might be relevant in resolving the issue of the final singularity for some cosmological models. Such a transition demands releasing of at least lapse Nand making it a function of time N = N(t) (N = N(K)), if we want to assign t = K = Tr(K)).

This amounts to the choice of metric in the form

$$ds^{2} = -\left[N(t)dt\right]^{2} + e^{-2\Omega(t)} \left[e^{2\beta(t)}dx^{2} + e^{2\beta(t)}dy^{2} + e^{-4\beta(t)}dz^{2}\right].$$
 (36)

instead of (19) (we still retain zero shift $N^i = 0$), which results in the expression for the Lagrangian

$$L = \frac{3v}{8\pi} \frac{1}{N} \left(\dot{\beta}^2 - \dot{\Omega}^2 \right) e^{-3\Omega}$$
(37)

The momentum conjugate to β becomes

$$p_{\beta} = \frac{\partial L}{\partial \dot{\beta}} = \frac{3V}{4\pi} \frac{1}{N} e^{-3\Omega} \dot{\beta}.$$
 (38)

The Hamiltonian of the system can be expressed now in terms of the momentum conjugate to β and the Lagrangian.

$$H_{dyn} = p_{\beta}\dot{\beta} - L = \frac{2\pi}{3V} N e^{3\Omega} p_{\beta}^2 + \frac{3V}{8\pi} \frac{1}{N} \dot{\Omega}^2 e^{-3\Omega}.$$
 (39)

The expression for the action integral

$$I = \int p_{\beta} d\beta - H_{dyn} dt = \int p_{\beta} d\beta - \left[\frac{2\pi}{3V} N e^{3\Omega} p_{\beta}^2 + \frac{3V}{8\pi} \frac{1}{N} \dot{\Omega}^2 e^{-3\Omega}\right] dt \quad (40)$$

differs in its appearance from the standard one. In particular, lapse N is not a mere factor in front of some expression anymore. However, it is easy to see that variation of the action with respect to N produces the constraint

$$p_{\beta}^{2} = \left(\frac{3V}{4\pi}\right)^{2} \frac{1}{N^{2}} e^{-6\Omega} \dot{\Omega}^{2}.$$
 (41)

that coincides with the Hamiltonian constraint of the standard approach when the variables of two different approaches are properly identified.

Just as in previous section, we use the Hamilton–Jacobi equation (24) together with the expression (39) for the Hamiltonian H_{dyn} , to obtain the Schrödinger equation for the axisymmetric Kasner model.

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{2\pi\hbar^2}{3V}N\,\mathrm{e}^{3\Omega}\frac{\partial^2\Psi}{\partial\beta^2} + \frac{3V}{8\pi}\dot{\Omega}^2\frac{1}{N}\,\mathrm{e}^{-3\Omega}\,\Psi.$$
(42)

To complete the dynamic picture, this equation should be amended by the equation

$$< p_{\beta} >_{s}^{2} = \left(\frac{4\pi}{3V}\right)^{2} \frac{1}{N^{2}} e^{-6\Omega} \dot{\Omega}^{2}.$$
 (43)

obtained from the constraint equation (41) via the same procedure as equation (26) of the previous section.

The solution of these equations describing "propagation" of a wave packet of Kasner universes (cf. the previous section) can be written down right away. A simple inspection of equations (42), (43) reveals that they are obtained from the similar equations (25), (26) by replacing everywhere the factor $e^{3\Omega}$ for the factor $N e^{3\Omega}$. The solution of the Schrödinger equation (41) is expressed by (30) with f(t) given by

$$f(t) = \frac{2\pi}{3V} \int_{t_0}^t N \,\mathrm{e}^{3\Omega} dt,$$
(44)

instead of that given by (31). Correction of the equation (33) is just as trivial.

Up until now the lapse function N(t) and, together with it, the meaning of the time parameter has not been specified. A variety of approaches can be used to fix both of them. For a case when shift is fixed (as in our example), one more condition including N(t) should be imposed. One extreme case has been considered in the previous section. The lapse has been fixed and, after this, the meaning of the time parameter has been interpreted. Another possibility is to fix the interpretation of the time parameter, in which case the lapse function will be fixed implicitly and will be determined by an additional condition considered together with the dynamic evolution equations.

As an example we consider the choice of K = TrK as the time parameter. This choice might provide advantages in particular problems (such as avoidance of singularities, etc.) but we are not concerned with this now. In this case equations (42) - (44) take the form

$$i\hbar\frac{\partial\Psi}{\partial K} = -\frac{2\pi\hbar^2}{3V} N e^{3\Omega} \frac{\partial^2\Psi}{\partial\beta^2} + \frac{3V}{8\pi} \left(\frac{d\Omega}{dK}\right)^2 \frac{1}{N} e^{-3\Omega} \Psi$$
(45)

$$\langle p_{\beta} \rangle_{s}^{2} = \left(\frac{4\pi}{3V}\right)^{2} \frac{1}{N^{2}} e^{-6\Omega} \left(\frac{d\Omega}{dK}\right)^{2}.$$
 (46)

and

$$f(K) = \frac{2\pi}{3V} \int_{K_0}^K N(K) e^{3\Omega(K)} dK,$$
(47)

After solving the Schrödinger equation and computing the expectation value $\langle p_{\beta} \rangle_s$, (46) yields

$$k_0^2 = \left(\frac{3V}{4\pi}\right)^2 \frac{1}{N^2} e^{-6\Omega} \left(\frac{d\Omega}{dK}\right)^2.$$
 (48)

which, together with the standard expression (this expression holds for any slicing parametrization, including parametrization by K)

$$K = -\frac{3}{N} \frac{d\Omega}{dK} \tag{49}$$

provides the basis for computing both $\Omega(K)$ and N(K). Indeed, (49) implies

$$\frac{1}{N}\frac{d\Omega}{dK} = -\frac{K}{3} \tag{50}$$

substitution of which in (48) yields the equation for $\Omega(K)$

$$k_0^2 = \frac{V^2}{16\pi^2} K^2 e^{-6\Omega}$$
(51)

It is remarkable that for this particular time parametrization the equation (51) for $\Omega(K)$ is an algebraic equation (generically it should be differential). This equation implies

$$\Omega(K) = \frac{1}{6} \ln\left(\frac{V^2}{16\pi^2} \frac{K^2}{k_0^2}\right)$$
(52)

which, together with (49), yields the expression for N(K)

$$N(K) = -\frac{3}{K} \frac{d\Omega}{dK} = -\frac{1}{K^2}$$
(53)

5 Discussion.

The issue of time in the standard approach to canonical quantum gravity does not seem to have a satisfactory resolution no matter which of the two quantization procedures is used. Both Dirac and ADM quantization do not seem to be able to handle the emerging difficulties of conceptual and technical nature. The difficulties can be traced to identification of the entire 3-geometry of a spacelike slicing as the principal dynamic object. The resulting constrained dynamics consists of proper dynamics that follows from variations of dynamic variables, and of the constraints that enforce the fundamental symmetries of gravitational systems (general covariance) and are obtained by varying lapse and shift. The constraints essentially restrict the evolution of gravitational systems to a "shell". The peculiarity of gravitation is that on this shell all of the dynamics is essentially determined by the constraints, thus rendering the proper dynamic relations obsolete and unnecessary. This does not present any problem in the classical theory. However, quantization of such a classical theory is reduced to quantization of the constraint equations. This essentially amounts to restricting the system states to the shell and neglecting "off shell" contributions, which leads to seemingly intractable problems in describing the time evolution of gravitational systems.

Our suggestion is to separate the true dynamics of gravity fields from enforcing the symmetries. Practically, it is done via putting the dynamic object in the geometrodynamic superspace of true gravitational degrees of freedom and formulating of variational principles on this superspace rather than on the superspace of total 3-geometries, while leaving the embedding variables as free parameters reserved for enforcing of the symmetries on the solutions of proper dynamic equations at a suitable moment. The equations enforcing the symmetries following from the general covariance are not obtained by variations on the geometrodynamic superspace. They come from variations of the action with respect to lapse and shift, as in standard approach. Within the realm of the classical theory, the final outcome of the theory predictions does not change (compared to the standard approach). Only the order of operations is changed. First, we consider all the solutions of the proper dynamic equations (on and off shell), and then force them on the shell via adjusting the embedding parameters. Since the latter enter the solutions for true dynamic variables, the final expressions for these variables, too, depend on the outcome of forcing the system on the shell. In the classical domain, the resulting theory is indistinguishable from the standard one.

The quantum theory, however, becomes quite different. We consider the true geometrodynamics (on the superspace of true dynamic variables or its phase space) as the object of quantization. The resulting equation of the theory is a Schrödinger equation with the Hamiltonian that is not a square root Hamiltonian and thus avoids most of the conceptual as well as technical problems of time. The initial values as well as the solutions of these equations contain embedding variables as functional parameters. They do not know anything about the shell (one can say that they include both on and off shell contributions). By themselves, they cannot either violate or enforce constraints. This should be achieved via adjusting embedding parameters. In addition, the solutions contain lapse and shift functions. These are responsible for the interpretation of time. When imposing constraints, it is important to realize that they might not be enforceable exactly (by simply treating them as additional operator equations obtained by the same procedure as the Schrödinger equation and applied to the state functional) if their operator versions do not commute with the Hamiltonian of the Schrödinger equation and cannot be made to commute via adjusting embedding parameters. It only means that, in this particular problem, off shell contributions cannot be neglected. A weaker version of the constraints should be introduced. The version chosen by us in examples was to impose the constraints on expectations. In this way the covariance requirement is satisfied as much as it can be satisfied. To a reasonable extent, it is exactly satisfied in the sense that predictions on the final hypersurface remain the same if the surface is not changing. The question of the change of the prediction when the surface is changing can be also answered positively by simple count of equations and adjustable parameters. The same is true, under reasonable conditions, about the multiple choice problem of time.

As about interpretation of time (metric interpretation of the slicing parameter) our consideration indicates that, within our approach, it is equivalent to imposing one (or more, if shift is involved) additional relation that involves lapse (or lapse and shift), and will not create any problems if these additional relations do not involve true dynamic variables. We do not believe that even relations involving true dynamic variables can cause considerable difficulties if they are introduced in the way similar to the one described for constraints. In our examples the relations included only embedding variables, thus commuting with the Hamiltonian and not causing any troubles. It should be stressed, however, that all three components of the evolution description for quantum geometrodynamic system — quantum dynamics itself, constraints enforcing the symmetries, and the interpretation of time — emerge together as the solution of the total problem of geometrodynamic evolution.

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