Inertia: A Purely Relativistic Phenomenon

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The inertial properties of ordinary matter cannot be ascribed solely to the inertial mass appearing in Newton's second law of motion; the contribution made by space-time must also be considered. Herein special relativity is used to show that inertia is ultimately relativistic in origin. Two observers moving relatively in Minkowski space-time are considered. The moving observer accelerates tangentially along a circular path of constant radius, passing by a stationary observer with greater velocity upon each revolution. The stationary observer uses the dilation of time arising between the two observers to derive an expression for the inertial resistance of the moving observer. The form of the resulting expression implies that inertia is chiefly a local relativistic phenomenon. The case in which an arbitrary force acts on an observer undergoing uniform, relativistic translation in Minkowski space-time is then considered. The time dilation approach leads directly to the well-known relativistic form of Newton's second law of motion, derived on the basis of special relativity.

1. Introduction

Previous analyses [1,2] argued that the inertial properties of ordinary matter cannot be ascribed solely to the inertial mass m appearing in Newton's second law of motion,

$$\mathbf{F} = \frac{d}{dt}\left(m\mathbf{v}\right) \tag{1}$$

The role played by space-time must also be considered. Special and general relativity show that inertia is purely relativistic in origin [1,2]. In particular, [1] considers the case of an observer accelerating uniformly under the influence of an external force in flat, Minkowski space-time. As such an observer moves, time intervals in the co-moving reference frame (CMRF) of the observer become increasingly dilated relative to time experienced by observers in Minkowski space-time. The changing time dilation was shown to cause a second force on the observer, given by

$$\mathbf{F} = -E_0 \nabla (dt / d\tau) \tag{2}$$

where E_0 is the rest mass-energy of the observer, and $dt/d\tau$ expresses the distortion of time [3] in the CMRF of the observer relative to Minkowski space-time. Assuming no other forces were present, the second force was identified as the inertial resistance of the observer. The form of the inertial resistance, given herein by Eq. (2), implied that all forms of energy exhibit inertial properties, arising as a direct manifestation of space-time inhomogeneity within accelerating systems [1-5].

In the present analysis, this line of reasoning is continued. The next Section provides a further demonstration that inertia arises out of space-time. As in previous analyses [1,2], two observers who move relatively in Minkowski space-time determine an expression for the force of inertia by exchanging light signals. Unlike previous analyses, however, the moving observer considered herein does not undergo linear acceleration, but rather accelerates tangentially along a circular path of constant radius. Upon each revolution the moving observer passes by the stationary observer with greater velocity. The stationary observer determines the change in time dilation arising between the two observers by measuring the frequency of a light source carried by the moving observer. Using the change in time dilation to derive an expression for the inertial resistance of the moving observer, the stationary observer finds an expression identical to Eq. (2).

In Section 3, the time-dilation approach from Section 2 is used to derive the well-known relativistic form of Newton's second law of motion [6-7]. First an observer initially undergoing uniform, relativistic translation in Minkowski space-time is considered. Then an arbitrary external force is imagined to act on the moving observer, changing the observer's velocity. As the velocity changes, so too does the dilation of time in the CMRF of the moving observer relative to Minkowski space-time. An expression for the changing time dilation is derived and then used to determine an expression for the force acting on the observer. This leads to expressions for the longitudinal and transverse components of the force that are identical to those derived on the basis of special relativity [8].

2. Inertia of an Observer Accelerating Tangentially Along a Circular Path

Consider two observers, residing in Minkowski space-time, who conduct an experiment to show that inertia has its origin in space-time. As depicted in Fig. 1, an S' observer accelerates tangentially along a circular-path with radius vector $\mathbf{r}$, while a second observer remains stationary in an inertial system S. While accelerating, the moving observer carries a light source of proper frequency ν_0. The stationary observer measures the frequency of light as the moving observer passes by; that is, when the relative velocity is entirely transverse, as shown in Fig. 1. As the moving observer acquires greater velocity, the stationary observer finds that the frequency of light becomes increasingly Doppler-shifted according to

$$\nu = \nu_0 \sqrt{1 - v^2 / c^2} \tag{3}$$

where v is the moving observer's instantaneous tangential velocity, and c is the velocity of light. Upon observing the Doppler-shift, the stationary observer measures an initial frequency ν_i;

and then at some later time during the experiment, takes a final measurement of the frequency ν_f.

As the moving observer gains greater velocity, time intervals in the moving observer's CMRF appear increasingly dilated relative to those in stationary system S. The stationary observer can compute the dilation of time by solving Eq. (3) for v^2/c^2 and then substituting the resulting expression into $dt/d\tau = 1\big/\sqrt{1-v^2/c^2}$. Carrying this out leads directly to

$$dt/d\tau = \nu_0/\nu \tag{4}$$

where dt is a time interval in S, and $d\tau$ is an interval of proper time in S' [9]. Thus, the ratio of the proper and observed frequencies gives the stationary observer a direct measure of the time dilation arising between systems S and S'. Using Eq. (4) for frequencies ν_i and ν_f, the stationary observer can then compute two subsequent values for the time dilation:

$$\left(\frac{dt}{d\tau}\right)_i = \frac{\nu_0}{\nu_i},\quad \left(\frac{dt}{d\tau}\right)_f = \frac{\nu_0}{\nu_f} \tag{5a,b}$$

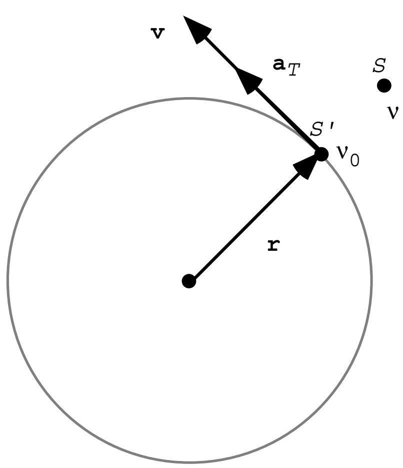

Figure 1. An S' observer undergoes tangential acceleration $\mathbf{a}_T$ along a circular path of constant radius $\mathbf{r}$, while an S observer remains stationary nearby. While accelerating, the S' observer carries a light source of proper frequency ν_0. The S observer measures the frequency of light each time the relative velocity between the two observers is entirely transverse. The S observer finds a frequency ν that is Doppler-shifted due to the velocity of the moving observer.

With values for the time dilation in hand, the stationary observer can compute the force acting on the moving observer by expressing the change in time dilation in the form

$$\left(\frac{dt}{d\tau}\right)_f - \left(\frac{dt}{d\tau}\right)_i = \frac{1}{\sqrt{1-v_f^2/c^2}} - \frac{1}{\sqrt{1-v_i^2/c^2}} \tag{6}$$

where it is tacitly assumed that the magnitude of the tangential acceleration, $\mathbf{a}_T$, remains constant during the experiment. To derive the force by use of Eq. (6), the stationary observer notes that the change in time dilation can be expressed as a line integral of the form

$$\left(\frac{dt}{d\tau}\right)_f - \left(\frac{dt}{d\tau}\right)_i = \int_i^f \nabla\left(\frac{dt}{d\tau}\right)\cdot d\mathbf{l} \tag{7}$$

where $d\mathbf{l}$ is a vector representing an infinitesimal interval of the circular path along which the moving observer travels. Similarly, the right-hand side (RHS) of Eq. (6) can be expressed as an integral of the form

$$\frac{1}{\sqrt{1-v_f^2/c^2}} - \frac{1}{\sqrt{1-v_i^2/c^2}} = \frac{1}{c^2}\int_i^f \frac{\mathbf{v}\cdot d\mathbf{v}}{\left(1-v^2/c^2\right)^{3/2}} \tag{8}$$

Substituting Eqs. (7) and (8) into Eq. (6) then gives

$$\int_i^f \nabla\left(\frac{dt}{d\tau}\right)\cdot d\mathbf{l} = \frac{1}{c^2}\int_i^f \frac{\mathbf{v}\cdot d\mathbf{v}}{\left(1-v^2/c^2\right)^{3/2}} \tag{9}$$

The RHS of this expression can be simplified by noticing that

$$\frac{1}{\left(1-v^2/c^2\right)^{3/2}} = \frac{1}{\sqrt{1-v^2/c^2}} + \frac{\left(\mathbf{v}\cdot\mathbf{v}\right)}{c^2\left(1-v^2/c^2\right)^{3/2}} \tag{10}$$

Substituting this relation into Eq. (9), and performing some algebraic manipulation, leads to

$$\int_i^f \nabla\left(\frac{dt}{d\tau}\right)\cdot d\mathbf{l} = \frac{1}{c^2}\int_i^f \mathbf{v}\cdot\left[\frac{d\mathbf{v}}{\sqrt{1-v^2/c^2}} + \frac{\mathbf{v}\left(\mathbf{v}\cdot d\mathbf{v}\right)}{c^2\left(1-v^2/c^2\right)^{3/2}}\right] \tag{11}$$

As a further simplification, the stationary observer notices that

$$d\left(\frac{\mathbf{v}}{\sqrt{1-v^2/c^2}}\right) = \frac{d\mathbf{v}}{\sqrt{1-v^2/c^2}} + \frac{\mathbf{v}\left(\mathbf{v}\cdot d\mathbf{v}\right)}{c^2\left(1-v^2/c^2\right)^{3/2}} \tag{12}$$

Using this, and expressing the moving observer's velocity as $\mathbf{v} = d\mathbf{l}/dt$, Eq. (11) can be recast in the form

$$\int_i^f \nabla\left(\frac{dt}{d\tau}\right)\cdot d\mathbf{l} = \frac{1}{c^2}\int_i^f \frac{d\mathbf{l}}{dt}\cdot d\left(\mathbf{v}\big/\sqrt{1-v^2/c^2}\right) \tag{13}$$

Then, upon writing the RHS in terms of the moving observer's rest mass m, and simplifying a bit, the stationary observer arrives at

$$\int_i^f \nabla\left(\frac{dt}{d\tau}\right)\cdot d\mathbf{l} = \frac{1}{mc^2}\int_i^f \frac{d}{dt}\left(\frac{m\mathbf{v}}{\sqrt{1-v^2/c^2}}\right)\cdot d\mathbf{l} \tag{14}$$

Upon identifying the integrand on the RHS of this expression with the relativistic form of Newton's second law [6-7], the stationary observer writes Eq. (14) as

$$\int_i^f \nabla\left(\frac{dt}{d\tau}\right)\cdot d\mathbf{l} = \frac{1}{E}\int_i^f \mathbf{F}\cdot d\mathbf{l} \tag{15}$$

in which $E = mc^2$ is the moving observer's rest mass-energy, and $\mathbf{F}$ is the external force on the moving observer. While the acceleration takes place, however, a second force acts on the moving observer that has equal magnitude, but opposite direction, to the external force $\mathbf{F}$. Assuming that no other forces are present, the second force is the inertial resistance force of the moving observer, given by $\mathbf{F}_{in} = -\mathbf{F}$. Using this force in Eq. (15), and eliminating the integration on both sides of the expression, the stationary observer then finds that

$$\mathbf{F}_{in} = -E\nabla(dt/d\tau) \tag{16}$$

According to the stationary observer, this is the inertial resistance of the moving observer, arising due to the local inhomogeneity of time within the moving system. From the form of Eq. (16), the stationary observer can deduce that all forms of energy, regardless of embodiment, exhibit inertial properties, which arise as a direct manifestation of the relativistic nature of space-time [1-5].

3. Relativistic Form of Newton's Second Law

The preceding Section showed that inertia has its origin in space-time. In this Section, a time dilation approach is used to derive the well-known relativistic form of Newton's second law of motion [6-7].

Consider an S' observer traveling along the x-coordinate axis of inertial system S, as shown in Fig. 2. The observer's initial velocity is $\mathbf{v}_i = \mathrm{v}_{ix}\mathbf{i}$. Then, an arbitrary external force acts on the observer, changing the observer's velocity to $\mathbf{v}_f = \mathrm{v}_{fx}\mathbf{i} + \mathrm{v}_{fy}\mathbf{j} + \mathrm{v}_{fz}\mathbf{k}$. At any point along the observer's world line, while the force is applied, the dilation of time in the moving observer's CMRF relative to system S can be expressed as

$$\frac{dt}{d\tau} = 1\Big/\sqrt{1-\left(\mathrm{v}_{fx}^2 + \mathrm{v}_{fy}^2 + \mathrm{v}_{fz}^2\right)\Big/c^2} \tag{17}$$

where dt is an interval of time in system S, and $d\tau$ is an interval of proper time in the CMRF of the moving observer. Equation (17) can be easily simplified by first writing it in the form

$$\frac{dt}{d\tau} = 1\Big/\sqrt{\left[1 - v_x^2/(c^2 - v_y^2 + v_z^2)\right]\left[1-(v_y^2+v_z^2)/c^2\right]} \tag{18}$$

where, for simplicity, the subscript f has been dropped from v_{fx}, v_{fy}, and v_{fz}. Noting that $(v_y, v_z) << c^2$, and neglecting terms of order greater than v^2/c^2, Eq. (18) can be recast in the approximate form

$$\frac{dt}{d\tau} \cong \left[1+(v_y^2+v_z^2)/2c^2\right]\Big/\sqrt{1-v_x^2/c^2} \tag{19}$$

Eq. (19) is the dilation of time in the CMRF of the moving observer relative to time experienced by observers in inertial system S. To determine the force acting on the moving observer, Eq. (19) can be substituted into $\mathbf{F} = E\nabla(dt/d\tau)$, where E is the moving observer's rest mass-energy. Carrying this out gives

$$\mathbf{F} = \frac{E/2c^2}{\sqrt{1-v_x^2/c^2}}\left[\frac{\partial v_x^2/\partial x}{\left(1-v_x^2/c^2\right)}\hat{\mathbf{i}} + \frac{\partial v_y^2}{\partial y}\hat{\mathbf{j}} + \frac{\partial v_z^2}{\partial z}\hat{\mathbf{k}}\right] \tag{20}$$

in which terms of order greater than v^2/c^2 have again been neglected. To simplify Eq. (20), note that the moving observer's initial and final velocities are related according to $v_f^2 = v_i^2 + 2(\mathbf{a}\cdot\mathbf{x})$, where $\mathbf{a}$ is the acceleration imparted to the moving observer by the external force. For the present case of an observer initially undergoing uniform translation along the x-axis, we have

$$v_x^2 = v_{ix}^2 + 2a_x x,\quad v_y^2 = 2a_y y,\quad v_z^2 = 2a_z z \tag{21a, b, c}$$

where the acceleration (a_x, a_y, a_z) is assumed constant. Substituting Eqs. (21) into Eq. (20), and simplifying somewhat, leads to

$$\mathbf{F} = \frac{m}{\sqrt{1-v_{ix}^2/c^2}}\left[\frac{a_x}{\left(1-v_{ix}^2/c^2\right)}\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}}\right] \tag{22}$$

in which $2a_x x << c^2$ has been assumed. Inspection of Eq. (22) shows that the longitudinal and transverse components of the force are, respectively,

$$\mathbf{F}_{//} = \frac{m\mathbf{a}_{//}}{\left(1-v^2/c^2\right)^{3/2}},\quad \mathbf{F}_{\perp} = \frac{m\mathbf{a}_{\perp}}{\sqrt{1-v^2/c^2}} \tag{23a, b}$$

in which the subscripts have been dropped from v_{ix} for simplicity. These expressions for the force on a moving body are identical to those derived on the basis of the special relativistic form of Newton's second law of motion [9].

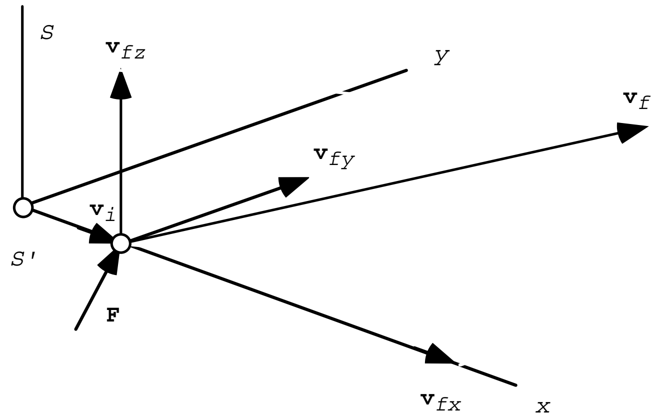

Figure 2. An arbitrary external force acts upon an S' observer moving along the *x*-axis of inertial system S with relativistic velocity $\mathbf{v}_i$. The force changes the S' observer's velocity to $\mathbf{v}_f$, having vector components $(\mathbf{v}_{fx}, \mathbf{v}_{fy}, \mathbf{v}_{fz})$.

4. Discussion

As pointed out in the Introduction, previous analyses used special and general relativity to show that all forms of energy exhibit inertial properties, resulting due to the inhomogeneity of space-time within accelerating systems [1-2]. The present analysis has presented another demonstration of the connection between inertia and relativity. It began with consideration of an observer accelerating tangentially along a circular path of constant radius. It was pointed out that, as such an observer's tangential velocity increases, time in the observer's co-moving reference frame becomes increasingly dilated relative to time experienced by stationary observers in Minkowski space-time. It was then shown that the changing time dilation gives rise to a second force on the moving observer that has equal magnitude, but opposite direction, to the externally applied force. Under the assumption that no other forces are present, the second force must then be the inertial resistance of the moving observer, arising as a direct manifestation of the relativistic nature of space-time [1-2]. The analysis then treated the case in which an arbitrary force is applied to an observer undergoing uniform, relativistic translation in Minkowski space-time. Using the time dilation approach presented herein, the longitudinal and transverse components of the force acting on the moving observer were derived. The form of the resulting expressions was found to be identical to the well-known relativistic form of Newton's second law of motion, derived on the basis of special relativity [6-9].

Notes and References

[1] C.T. Ridgely, "On the Nature of Inertia," Galilean Electrodynamics **11**, 11-15 (2000).

[2] C.T. Ridgely, "On the Origin of Inertia," Galilean Electrodynamics **12**, 17-20 (2001).

[3] The word "distortion" is used in reference to any instance in which local space-time structure deviates from the flat, Minkowski space-time of special relativity.

[4] For an interesting discussion on the inertia of energy, see A. Einstein, "Does the inertia of a body depend upon its energy-content?," in **Einstein, The Principle of Relativity**, pp. 69-71 (Dover, New York, 1952); and H. Weyl, **Space-Time-Matter**, 4th ed., p.202 (Dover, New York, 1952).

[5] For a discussion on the inertia of energy, see Max Born, **Einstein's Theory of Relativity**, p. 286 (Dover, New York, 1965).

[6] See, for example, R. Resnick, **Introduction to Special Relativity**, p. 119 (Wiley, New York, 1968).

[7] See, for example, P.G. Bergmann, **Introduction to the Theory of Relativity**, pp. 103-104 (Dover, New York, 1976).

[8] See, for example, Max Born, Ref. [5], p. 276; and R. Resnick, Ref. [6], p. 125.

[9] See, for example, C. Lanczos, **The Variational Principles of Mechanics**, 4th ed., p. 339 (Dover, New York, 1968); and R. Resnick, Ref. [6], pp. 90-91.