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ON THE ORIGINS OF SPACE-TIME AND INERTIA

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Summary

Mach's principle is incorporated into classical General Relativity by writing Einstein's equations as explicitly covariant integral equations involving retarded bi-tensor Green's functions. These Green's functions must be found for the Robertson–Walker metrics in order to discover which, if any, of those universes are Machian.

In this theory all empty universes are non-Machian and the cosmical constant is zero.

It is argued that in any truly Machian theory space-time itself must be caused by the matter and in this theory it may be considered as propagating over itself out from the matter.

Our Mach's principle may also be expressed as a boundary condition on the past singularities of the universe in a way that corresponds to the idea that all space-time and therefore all inertial and gravitational influences come from matter.

1. *Philosophical introduction.* Since Bishop Berkeley (1) criticized Newton's formulation of dynamics the absolute or relative nature of space and space-time has been a subject of controversy. Does space-time exist merely as a consequence of the matter and energy contained in it or are the laws and facts of astronomy at variance with this philosophical concept?

The connection between space-time and local physics is beautifully expressed in Einstein's general theory of relativity. In Eddington's interpretation (2) matter is a secondary phenomenon to the underlying reality of space-time. In such an interpretation space-time is 'flat', $R^{\mu\nu\sigma\tau}=0$, 'empty', $R^{\mu\nu}=0$ or bent $R^{\mu\nu}\neq 0$ at each point. 'Flat' and 'empty' are merely provocative names; still more provocatively we say that the matter *is* 'bent' space.

The aim of this paper is to show that General Relativity under certain boundary conditions completes the philosophically satisfying cycle of being interpretable in two ways. Those who choose to may consider space-time as absolute, but those who like Berkeley reject this idea may consider space-time as being physically caused by the matter in it.

We show that the whole metric tensor, $g_{\alpha\beta}$, at any point may be considered as a tensor potential due to matter within its past light cone. This result is foreshadowed in Davidson's approximate theory (3).

The origin of inertia. Mach (4), Einstein, Weyl (5), Bondi (6), Sciama (7) Wheeler (8) and others have given penetrating discussions of the origin of inertia The problem here again centres around Berkeley's criticism of Newton's work in particular his conclusion that rotation is absolute. It is extended to includ the question of whether all accelerations are relative or absolute. The concep that the inertial reference frames at all points of space-time are determined b some causal law as some average of the positions and motions of the matter of the observable universe, is what we shall call Mach's principle. It is equivalent to Einstein's more appealing dogma (6). There is no inertia of matter against space but only inertia of matter against matter.

The aim of this section is to show that any theory that attributes the inertia of a body to the influence of distant matter must attribute the very space-time in which the body is situated to the same cause.

The observation that the velocity of light measured by any apparatus is independent of uniform motion of both the source and the observer, shows that the light velocity and the associated light-cones are fundamental invariant structures of space-time. In order that no observer should see particles arrive at their destinations before they set out, it is necessary that no particle travel faster than light. In order that a specially large shove shall not send a fast particle through this light barrier inertia must increase without limit as the velocity of light is approached. Thus light cones are both the limiting surfaces onto which the paths of very fast particles tend and invariant structures of the space-time. When referred to accelerated or rotating axes these light cones appear twisted as do the particle paths. From a Machian viewpoint the matter current constituted by the universe itself accelerating or rotating with respect to our axes causes a field which acts on particles to twist their paths. An inconsistent but most illuminating theory of this effect was given by Sciama (7). For consistency the light cones must be twisted in precisely the same way as the limiting particle paths so we *must* attribute this twisting to the same cause. Also from a Berkleian viewpoint a test particle alone in space-time is inconsistent, for the light cones define 'unaccelerated' axes which are unaccelerated with respect to no matter.

This intimate relationship between the space-time and the inertial axes of dynamics implies that we cannot claim inertia to be caused by distant matter without claiming in the same breath that the local space-time frame is also so caused.

2. *Method.* Two key problems have foiled past attempts at a general relativistic theory of Mach's principle.

(1) To show how much inertia is due to each source, inertia must be expressed as a sum over its sources. In general relativity these are represented by the stress–energy–momentum tensor $T_{\mu\nu}$ so we expect integrals of this tensor over space-time. However, if we sum values of $T_{\mu\nu}$ at different points we are summing contributions that transform differently under coordinate transformation so the result is not a tensor anywhere.

This problem has been solved by Synge (9) and by De Witt & Brehme's use of bi-tensors (10). The basic idea here is to consider geodesic tracks from some chosen point $\mathbf{x}$ at which we want the sum tensor. The tensors $T_{\mu\nu}(\mathbf{x}')$ are transported parallelly to $\mathbf{x}$ along these tracks thus becoming tensors at $\mathbf{x}$. Finally they are added to form the sum (or integral) tensor which is a tensor at $\mathbf{x}$. Integrals of tensors depend on the tensor field throughout the volume of integration and on the point at which the integral is to be a tensor. Using this technique De Witt & Brehme (10, 11) and independently Lichnerowitz (12) have developed the theory of vector and tensor Green's functions (sometimes called propagators) for linear differential equations in curved spaces.

(2) The solutions of the non-linear general relativistic equations are non-linearly dependent on the sources $T_{\mu\nu}$. How then can we hope to find expressions which are merely sums over the sources of inertia? Surely in any non-linear theory the fact that body A is affecting body B will influence the effect that C has on B and in the resulting mess it will be impossible to say which bit of the inertia sum comes from what body.

An ingenious way out of this mélée was chosen by Hoyle & Narlikar in the development of their ideas on inertia (13). They find the effect of each mass in the presence of all the others by assuming that inertial effects add linearly when they are considered as propagated over the space-time geometry as it actually is. In Einstein's theory and in the theory of Hoyle & Narlikar this geometry is curved by the matter in it. The propagator of the inertial influence is dependent on the geometry and is thus implicitly dependent on the sources of inertia. The inertia integral though explicitly a linear sum over the sources is implicitly non-linear, being dependent on the sources a second time via the propagator for the geometry. Such a theory is linear over the self-consistent geometry. It is the first aim of this paper to show that conventional general relativity is such a theory.

3. *Basic idea.* Notationally it is unfortunate that geometry, gravity and Green's function all begin with G. We shall use G for the constant of gravity, $g_{\mu\nu}$ for the metric tensor but $G_{\mu\nu}$, $G_{\mu\nu\sigma\tau}$ etc. we will reserve for Green's functions. This leaves us short of a symbol for the Einstein tensor. Since geometry curvature is involved C is natural but Weyl has already used it for his conformal tensor so we use $K_{\mu\nu}$ for the Einstein tensor. Einstein's equations are

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = -8\pi G c^{-2} T_{\mu\nu}. \qquad (1)$$

We are writing

$$K_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R \qquad (2)$$

so Einstein's equations are

$$K_{\mu\nu} = -8\pi G c^{-2} T_{\mu\nu}. \qquad (3)$$

Our first aim is to produce an explicitly linear theory of gravity which is implicitly non-linear and coincides with Einstein's theory. For pedagogical reasons I shall here produce such a theory out of a hat and shall give the details of the construction of this theory in the next section.

We define the Einstein operator $E_{\mu\nu}{}^{\sigma\tau}$ which acts on symmetric tensor potentials $\phi_{\sigma\tau}$:

$$E_{\mu\nu}{}^{\sigma\tau} \phi_{\sigma\tau} \equiv [\tfrac{1}{2} g^{(\sigma}_{(\mu} g^{\tau)}_{\nu)} \nabla^2 + g^{(\sigma}_{(\mu} R^{\tau)}_{\nu)} - \tfrac{1}{2} g^{\sigma\tau} R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R^{\sigma\tau} + \tfrac{1}{4} R g_{\mu\nu} g^{\sigma\tau}] \phi_{\sigma\tau}. \qquad (4)$$

Round brackets about indices denote symmetrization, see equation (42). It will be seen that this covariant differential operator is symmetric in the indices $\sigma\tau$ and in $\mu\nu$ and that $E_{\mu\nu\sigma\tau}$ is symmetric for the pairwise exchange of $\sigma\tau$ and $\mu\nu$. Also $\sqrt{-g} E_{\mu\nu\sigma\tau}$ is self-adjoint and the operator reduces to $\tfrac{1}{2}\nabla^2$ acting on $\phi_{\mu\nu}$ whenever the space is empty. It does not contain the Riemann Christoffel tensor explicitly. It is simple to verify that the Einstein operator acting on $-g_{\sigma\tau}$ yields the Einstein tensor that is

$$E_{\mu\nu}{}^{\sigma\tau}(-g_{\sigma\tau}) = K_{\mu\nu}. \qquad (5)$$

Einstein's equations may therefore be written

$$E_{\mu\nu}{}^{\sigma\tau}(-g_{\sigma\tau}) = -8\pi Gc^{-2}T_{\mu\nu}. \tag{6}$$

We have now demonstrated that $-g_{\sigma\tau}$ is a solution for $\phi_{\sigma\tau}$ of the differential equation

$$E_{\mu\nu}{}^{\sigma\tau}\phi_{\sigma\tau} = -8\pi Gc^{-2}T_{\mu\nu}. \tag{7}$$

Following the extensive discussions of De Witt and Lichnerowitz we shall assume that the Einstein linear operator defined in our Riemannian space possesses a unique retarded Green's function or propagator $\mathscr{G}_{\sigma\tau\chi'\psi'}(\mathbf{x}, \mathbf{x}')$. This will be a second order tensor density of weight $\frac{1}{2}$ at each of the points $\mathbf{x}$ and $\mathbf{x}'$ and will satisfy

$$E_{\mu\nu}{}^{\sigma\tau}\mathscr{G}_{\sigma\tau\chi'\psi'} = -8\pi\bar{g}_{\mu\chi'}\bar{g}_{\nu\psi'}\delta^4(\mathbf{x}, \mathbf{x}') \tag{8}$$

and the retarded condition that $\mathscr{G}_{\sigma\tau\chi'\psi'}$ is zero whenever the point $\mathbf{x}$ lies outside the future light cone of the point $\mathbf{x}'$. In equation (8) $\delta^4(\mathbf{x}, \mathbf{x}')$ is the four dimensional Dirac δ function for the points $\mathbf{x}$ and $\mathbf{x}'$ which is a tensor density of weight $\frac{1}{2}$ at each of $\mathbf{x}$ and $\mathbf{x}'$ and $\bar{g}_{\mu\chi'}$ are the bivectors of parallel geodesic transport between $\mathbf{x}$ and $\mathbf{x}'$ defined by De Witt & Brehme (**10**). $\bar{g}^{\mu}{}_{\chi'}$ may be loosely thought of as a substitution operator which converts and index χ' at $\mathbf{x}'$ to an index μ at $\mathbf{x}$. The retarded solution of equation (7) with $T_{\mu\nu}$ regarded as the source of the potential $\phi_{\sigma\tau}$ is thus

$$\phi_{\sigma\tau}(\mathbf{x}) = Gc^{-2}\int \mathscr{G}_{\sigma\tau\chi'\psi'}^{(\mathbf{x},\mathbf{x}')}\, T^{\chi'\psi'}(\mathbf{x}')\, d^4x'. \tag{9}$$

The integration is taken over all space but owing to the properties of $\mathscr{G}$ it may be reduced to integration over the past light cone of the point $\mathbf{x}$ and its interior.

To verify solution (9) apply the Einstein operator. On the right hand side it operates only on the Green's function since it alone is dependent on $\mathbf{x}$. Thus

$$E_{\mu\nu}{}^{\sigma\tau}\phi_{\sigma\tau} = Gc^{-2}\int E_{\mu\nu}{}^{\sigma\tau}\mathscr{G}_{\sigma\tau\chi'\psi'}T^{\chi'\psi'}\, d^4x'. \tag{10}$$

But this reduces by virtue of the definition of the Green's function to

$$E_{\mu\nu}{}^{\sigma\tau}\phi_{\sigma\tau} = -8\pi Gc^{-2}\int \bar{g}_{\sigma\chi'}\bar{g}_{\tau\psi'}T^{\chi'\psi'}\delta^4(\mathbf{x}, \mathbf{x}')\, d^4x'. \tag{11}$$

But $\bar{g}_{\sigma\chi'} = g_{\sigma\chi'}$ when $\mathbf{x} = \mathbf{x}'$ so by the property of the δ function

$$E_{\mu\nu}{}^{\sigma\tau}\phi_{\sigma\tau} = -8\pi Gc^{-2}T_{\mu\nu} \tag{12}$$

which is the equation (7) that we set out to solve. Expression (9) is the only retarded solution of this equation which does not involve extra free waves so if we want a solution which is physically caused by the sources $T_{\mu\nu}$ it must be identified with $\phi_{\sigma\tau}$; but our Machian philosophy requires that $g_{\sigma\tau}$ should be physically caused by $T_{\mu\nu}$ and $-g_{\sigma\tau}$ obeys the differential equation; so in any Machian universe

$$-g_{\sigma\tau} = Gc^{-2}\int \mathscr{G}_{\sigma\tau\chi'\psi'}T^{\chi'\psi'}\, d^4x'. \tag{13}$$

This remarkable integral for the metric tensor contains Einstein's equations for they may be recovered merely by applying the Einstein operator and the analysis of equations (10)–(12). However over and above the differential equations it contains a causality boundary condition and explicitly demonstrates how each element of metric arises from its source by means of the retarded propagator. The domain of integration in equations (9) and (13) is over all points belonging to the past of timelike or null geodesics through **x**. One may remark that if we have a universe which satisfies Einstein's equations and which satisfies equation (13) for all that part of space-time that lies in the past of a given complete spacelike section of space-time, then that universe satisfies our Mach condition (13) in the whole of space-time. By pushing our spacelike surface back in time we see that we may regard equation (13) as a condition on the singularities at the start of the universe or for non-singular universes as a condition on the infinite past.

The general solution of the linear equation (7) is given by equation (9) together with an additional free wave solution of the homogeneous equation

$$E_{\mu\nu}{}^{\sigma\tau}\phi_{\sigma\tau}=0. \tag{14}$$

The latter could be written as a surface integral of Cauchy data. Mach's principle could then be expressed as the vanishing of this surface integral. However when this past surface is singular as it is in most cosmological models I find it awkward to formulate the criterion in that manner and consider that equation (13) is to be preferred. Equation (13), together with the equation (8) that defines the Green's function, is an integral for the metric in the same way that Poisson's integral

$$\psi(r)=\int G\,\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d^3r', \tag{15}$$

is a solution of Poisson's equation in Newtonian theory.

We have justified the addition of solutions of Einstein's equations provided we first freeze the geometry to be the one we finally end up with and we have shown how this quasi-linearity may be used to express Mach's principle that the whole metric of space-time is caused by the pieces of matter in it. In this respect conventional general relativity has the same properties as the theory of Hoyle & Narlikar. In fact one may consider the space-time as being constructed progressively like a wave. The influence of matter propagates out to make space and it is that space over which the later influences propagate to make the space at a later time. One is tempted to ask 'Is there a Huygen's principle for space?' and then 'What else would it mean if there were?'

4. *Construction of a quasi-linear theory equivalent to Einstein's.* We shall try to construct the simplest possible linear field theory of a second order symmetric tensor field of a zero rest mass particle in a Riemannian space. To do this we shall be guided by three analogies.

(i) Maxwell's electrodynamics;
(ii) Einstein's theory linearized about flat space;
(iii) Einstein's theory linearized about a given Riemannian space.

We shall concentrate our attention on the relationships between the potentials and the fields.

Electrodynamics. The fields are summarized in the field tensor $F_{\mu\nu}$ which possesses the symmetry

$$F_{\mu\nu}=-F_{\nu\mu} \tag{16}$$

and satisfies the Maxwell equation

$$F_{\mu\nu,\sigma}+F_{\sigma\mu,\nu}+F_{\nu\sigma,\mu}=0, \tag{17}$$

where, σ denotes ordinary differentiation with respect to x^σ. These relations are the necessary and sufficient conditions that $F_{\mu\nu}$ can be expressed as the skew derivative of a potential A_μ, i.e.

$$F_{\mu\nu}=A_{\mu,\nu}-A_{\nu,\mu}. \tag{18}$$

General Relativity linearized about flat space. In Einstein's theory linearized about flat space the fields are summarized in the Riemann Christoffel tensor $R_{\kappa\lambda\,\mu\nu}$ which possess the symmetry

$$R_{\kappa\lambda\,\mu\nu}=-R_{\kappa\lambda\,\nu\mu} \tag{19}$$

and satisfies the linearized Bianchi identity

$$R_{\kappa\lambda\,\mu\nu,\sigma}+R_{\kappa\lambda\,\sigma\mu,\nu}+R_{\kappa\lambda\,\nu\sigma,\mu}=0. \tag{20}$$

These relations are by analogy with Maxwell's theory necessary and sufficient conditions for $R_{\kappa\lambda\,\mu\nu}$ to be expressible as the skew derivative of a potential $A_{\kappa\lambda\mu}$, i.e.:

$$R_{\kappa\lambda\mu\nu}=A_{\kappa\lambda\mu,\nu}-A_{\kappa\lambda\nu,\mu}. \tag{21}$$

However in the above we have been unfair on the indices $\kappa\lambda$ which have equal standing with the indices $\mu\nu$ in the Riemann tensor by virtue of the further symmetries

$$R_{\kappa\lambda\,\mu\nu}=-R_{\lambda\kappa\,\mu\nu} \tag{22}$$

$$R_{\kappa\lambda\,\mu\nu}=R_{\mu\nu\,\kappa\lambda}. \tag{23}$$

From these we may deduce that the Riemann tensor may also be expressed as a skew derivative on the other pair of indices.

$$R_{\kappa\lambda\mu\nu}=A_{\kappa\lambda\mu,\nu}-A_{\kappa\lambda\nu,\mu}=A_{\mu\nu\kappa,\lambda}-A_{\mu\nu\lambda,\kappa}. \tag{24}$$

This form suggests that $R_{\kappa\lambda\mu\nu}$ is expressible as a double skew derivative in the sense that $A_{\kappa\lambda\mu}$ is itself the skew derivative of a second order tensor potential $A_{\kappa\mu}$, i.e.:

$$A_{\kappa\lambda\mu}=A_{\kappa\mu,\lambda}-A_{\lambda\mu,\kappa}. \tag{25}$$

In fact this can only be so provided $R_{\kappa\lambda\mu\nu}$ possesses its familiar cyclic symmetry because substitution of equation (25) into equation (21) produces only tensors obeying

$$R_{\kappa\lambda\mu\nu}+R_{\kappa\nu\lambda\mu}+R_{\kappa\mu\nu\lambda}=0. \tag{26}$$

In practice it is well known that tensors satisfying all the symmetries of the Riemann tensor and the linearized Bianchi identities can be expressed in the form

$$R_{\kappa\lambda\mu\nu}=A_{\kappa\lambda\mu,\nu}-A_{\kappa\lambda\nu,\mu}, \tag{27}$$

where

$$A_{\kappa\lambda\mu}=-\tfrac{1}{2}(\delta g_{\kappa\mu,\lambda}-\delta g_{\lambda\mu,\kappa}). \tag{28}$$

Thus the symmetries plus the linearized Bianchi identities are necessary and sufficient for the existence of a second order tensor potential of the Riemann tensor. Furthermore in the linearized theory that potential is $\delta g_{\kappa\mu}$ or alternatively $\eta_{\kappa\mu}+\delta g_{\kappa\mu}$ where $\eta_{\kappa\mu}$ is the Minkowski space metric tensor. In either case the connection between the fields $R_{\kappa\lambda\mu\nu}$ and the potentials $A_{\kappa\mu}$ is

$$-2R_{\kappa\lambda\mu\nu}=(A_{\kappa\mu,\lambda}-A_{\lambda\mu,\kappa})_{,\nu}-(A_{\kappa\nu,\lambda}-A_{\lambda\nu,\kappa})_{,\mu}. \tag{29}$$

It is interesting to point out here that the same theory is obtained by Fierz & Pauli in their discussion of zero rest mass spin two particles (14). In their case the equations are supplemented by the gauge condition $(A_{\kappa\mu}-\frac{1}{2}A\eta_{\kappa\mu})^{,\mu}=0$. This is the normal gauge condition for general relativity linearized about Minkowski space and it is known as the Hilbert–de Donder condition.

Fierz & Pauli (14) write down the Lagrangian or Action density for these field equations and one way to proceed is to adopt their Lagrangian, generalize it to curved space by replacing ordinary derivatives by covariant ones, and so deduce the form of the theory in a general Riemannian space. However such a theory only reproduces the linearized Einstein equations when the curved background Riemannian space is empty and it is impossible to make a quasi-linear theory of Einstein's equations in non-empty space on this basis alone. We shall eventually find such a basis by studying the relationship between the field and the potential in general relativity linearized about a general Riemannian space.

Generalization to curved space-time. It is natural to generalize expression (29) by turning the ordinary derivatives into covariant ones; however this procedure fails on two counts one trivial, one fundamental. Firstly a straight replacement of ordinary by covariant derivatives yields a tensor without the right symmetries because the covariant derivatives do not commute. This objection is trivially circumvented by taking the skew derivatives in either order and averaging the results. (We obtain by this means the same theory as the generalized Fierz–Pauli case alluded to above.) The more fundamental objection is that the resulting field tensor does not satisfy the Bianchi identities identically because the covariant derivatives involved in the statement of the Bianchi identities do not commute with those in our trial expression for $R_{\kappa\lambda\mu\nu}$.

Failure due to non-commutativity leads us to try adding terms that arise from the commutators of derivatives. We therefore consider terms of the form $R^{\epsilon}{}_{\kappa\mu\lambda}A_{\epsilon\nu}$. The combination of such terms with the symmetries of the Riemann tensor is $\Lambda_{\kappa\lambda\mu\nu}$ where

$$\Lambda_{\kappa\lambda\mu\nu}=R^{\epsilon}{}_{\lambda\mu\nu}A_{\kappa\epsilon}-R^{\epsilon}{}_{\kappa\mu\nu}A_{\lambda\epsilon}+R^{\epsilon}{}_{\nu\kappa\lambda}A_{\mu\epsilon}-R^{\epsilon}{}_{\mu\kappa\lambda}A_{\nu\epsilon}. \tag{30}$$

It is natural to add a multiple of Λ to the generalization of expression (29) just proposed and so to try a field tensor of the form

$$R_{\kappa\lambda\mu\nu}=-\tfrac{1}{4}[(A_{\kappa\mu;\lambda}-A_{\lambda\mu;\kappa})_{;\nu}-(A_{\kappa\nu;\lambda}-A_{\lambda\nu;\kappa})_{;\mu}$$
$$+(A_{\kappa\mu;\nu}-A_{\kappa\nu;\mu})_{;\lambda}-(A_{\lambda\mu;\nu}-A_{\lambda\nu;\mu})_{;\kappa}]+a\Lambda_{\kappa\lambda\mu\nu}. \tag{31}$$

Expression (31) still fails to satisfy the Bianchi identities identically, but it does so trivially when $A_{\kappa\mu}\equiv g_{\kappa\mu}$ for then $\Lambda_{\kappa\lambda\mu\nu}=4R_{\kappa\lambda\mu\nu}$ so we merely take $a=\frac{1}{4}$.

Seen from the point of view of general non-linear theories our failure to find a linear covariant relationship between the fields and their tensor potential is easily interpreted. The Bianchi identities are a self-consistency condition on the Riemann tensor of a certain space and knowledge of that space is necessary before we specify the meaning of the covariant derivatives appearing in the Bianchi identities. Our attempt to find a covariant expression for the field tensor $R_{\kappa\lambda\mu\nu}$ in terms of the potentials $A_{\kappa\mu}$ would lead to a solution if the covariant derivatives were with respect to $A_{\kappa\mu}$ regarded as a metric tensor. The Bianchi identities are true of the field tensor if it is the Reimann tensor of the metric in terms of which the Bianchi identities are stated. Seen in this new light we only want to satisfy the Bianchi identities with respect to the self-consistent metric (that is with respect to the potential regarded as a metric tensor). General Relativity linearized about a given metric does precisely that, and the connection between field and potential is again expression (31) with $a=\frac{1}{4}$. We now show this in detail because we are going to base our theory on this connection; but first a word on our basic philosophy.

We want a quasi-linear theory of gravity. To this end we are studying linear tensor theories in curved spaces; pedagogically it is extremely awkward that the only successful theory of this sort is Einstein's theory of gravity linearized about a curved space. In what follows the fields and potentials should be regarded not as being Riemann tensors and metrics but as the natural fields and potentials in terms of which any zero rest mass spin two theory in a curved space would be described. We are trying to solve the problem of Fierz & Pauli in a curved space and we are looking to a particular linear theory as a guide. The fact that this theory is itself a theory of space-time and gravity is unimportant although we use the standard relations for the deduction of the form of the linearized Einstein theory.

General Relativity linearized about curved space. We consider two neighbouring Riemannian spaces with metrics $g_{\mu\nu}$ and $g_{\mu\nu}+\delta g_{\mu\nu}$ on them. We note that $\delta g_{\mu\nu}$ are not physical quantities on account of the freedom of coordinate choice sanctioned by General Relativity. In the present instance this choice has to be made twice, once to choose the coordinates on the unperturbed space and again to choose them on the perturbed one. We have already forgone some freedom by dictating that $\delta g_{\mu\nu}$ shall be small but this only ties down our coordinates to within an arbitrary small transformation. We shall show later that by suitable choice of such transformations we may without loss of generality demand that the $\delta g_{\mu\nu}$ are subject to certain subsidiary or gauge conditions. However, for the present our analysis will be unrestricted by such conditions. Thus we can create a $\delta g_{\mu\nu}$ on a given space by merely changing our mind as to what coordinate system we will use and we can if we want generate waves in the $\delta g_{\mu\nu}$ which travel with the superluminous speed of Eddington's thoughts.

When we compare two neighbouring spaces some such waves will normally be present in the $\delta g_{\mu\nu}$ as well as changes due to the physical difference between the spaces.

The Riemann Christoffel tensor is defined by

$$R^{\kappa}{}_{\lambda\mu\nu}=-\Gamma^{\kappa}{}_{\lambda\mu,\nu}+\Gamma^{\kappa}{}_{\lambda\nu,\mu}+\Gamma^{\kappa}{}_{\epsilon\mu}\Gamma^{\epsilon}{}_{\lambda\nu}-\Gamma^{\kappa}{}_{\epsilon\nu}\Gamma^{\epsilon}{}_{\lambda\mu}, \qquad (32)$$

where

$$\Gamma^{\epsilon}{}_{\kappa\lambda}=\tfrac{1}{2}g^{\epsilon\mu}(g_{\kappa\mu,\lambda}+g_{\lambda\mu,\kappa}-g_{\kappa\lambda,\mu}). \qquad (33)$$

From these expressions it follows that

$$\delta R^{\epsilon}{}_{\lambda\mu\nu}=-(\delta\Gamma^{\epsilon}{}_{\lambda\mu})_{;\nu}+(\delta\Gamma^{\epsilon}{}_{\lambda\nu})_{;\mu} \qquad (34)$$

and that

$$\delta\Gamma^{\epsilon}{}_{\lambda\mu}=\tfrac{1}{2}g^{\epsilon\zeta}(\delta g_{\zeta\lambda;\mu}+\delta g_{\zeta\mu;\lambda}-\delta g_{\lambda\mu;\zeta}), \tag{35}$$

where semicolons denote covariant differentiations in the unperturbed space. Further

$$\delta R_{\kappa\lambda\mu\nu}=g_{\kappa\epsilon}\delta R^{\epsilon}{}_{\lambda\mu\nu}+R^{\epsilon}{}_{\lambda\mu\nu}\delta g_{\kappa\epsilon} \tag{36}$$

and so

$$\delta R_{\kappa\lambda\mu\nu}=\begin{Bmatrix}\tfrac{1}{2}[\delta g_{\kappa\lambda;\nu;\mu}+\delta g_{\kappa\nu;\lambda;\mu}-\delta g_{\lambda\nu;\kappa;\mu}-\delta g_{\kappa\lambda;\mu;\nu}-\delta g_{\kappa\mu;\lambda;\nu}+\delta g_{\lambda\mu;\kappa;\nu}] \\ +R^{\epsilon}{}_{\lambda\mu\nu}\delta g_{\kappa\epsilon}.\end{Bmatrix} \tag{37}$$

Making use of the commutation relation for covariant derivatives and the triple symmetry of the Riemann tensor one may re-express this in the form

$$\delta R_{\kappa\lambda\mu\nu}=F_{\kappa\lambda\mu\nu}{}^{\alpha\beta}\delta_{\alpha\beta} \tag{38}$$

where $F_{\kappa\lambda\mu\nu}{}^{\alpha\beta}$ is the covariant differential operator defined by

$$F_{\kappa\lambda\mu\nu}{}^{\alpha\beta}=-\tfrac{1}{4}(D_{\kappa\lambda\mu\nu}{}^{\alpha\beta}+B_{\kappa\lambda\mu\nu}{}^{\alpha\beta}) \tag{39}$$

where $D_{\kappa\lambda\mu\nu}{}^{\alpha\beta}$ is the covariant differential operator of double skew differentiation

$$D_{\kappa\lambda\mu\nu}{}^{\alpha\beta}=2(g^{(\alpha}_{\kappa}g^{\beta)}_{\mu}\nabla_{(\nu}\nabla_{\lambda)}-g^{(\alpha}_{\lambda}g^{\beta)}_{\mu}\nabla_{(\nu}\nabla_{\kappa)}-g^{(\alpha}_{\kappa}g^{\beta)}_{\nu}\nabla_{(\mu}\nabla_{\lambda)}+g^{(\alpha}_{\lambda}g^{\beta)}_{\nu}\nabla_{(\mu}\nabla_{\kappa)}) \tag{40}$$

and

$$B_{\kappa\lambda\mu\nu}{}^{\alpha\beta}=-R^{(\alpha}{}_{\lambda\mu\nu}g^{\beta)}_{\kappa}+R^{(\alpha}{}_{\kappa\mu\nu}g^{\beta)}_{\lambda}-R^{(\alpha}{}_{\nu\kappa\lambda}g^{\beta)}_{\mu}+R^{(\alpha}{}_{\mu\kappa\lambda}g^{\beta)}_{\nu}. \tag{41}$$

∇_{ν} is the operator of covariant differentiation with respect to x^{ν} and round brackets about indices denote symmetrization in the sense

$$Q_{(\alpha\beta)}=\tfrac{1}{2}(Q_{\alpha\beta}+Q_{\beta\alpha}). \tag{42}$$

From their definitions we see that

$$D_{\kappa\lambda\mu\nu}{}^{\alpha\beta}g_{\alpha\beta}=0 \tag{43}$$

and

$$-\tfrac{1}{4}B_{\kappa\lambda\mu\nu}{}^{\alpha\beta}g_{\alpha\beta}=R_{\kappa\lambda\mu\nu}. \tag{44}$$

Thus

$$F_{\kappa\lambda\mu\nu}{}^{\alpha\beta}g_{\alpha\beta}=R_{\kappa\lambda\mu\nu} \tag{45}$$

and since F is a linear operator we have from equation (38)

$$F_{\kappa\lambda\mu\nu}{}^{\alpha\beta}(g_{\alpha\beta}+\delta g_{\alpha\beta})=R_{\kappa\lambda\mu\nu}+\delta R_{\kappa\lambda\mu\nu}. \tag{46}$$

Thus for all metrics $A_{\alpha\beta}$ in the neighbourhood of the metric $g_{\alpha\beta}$ the expression $F_{\kappa\lambda\mu\nu}{}^{\alpha\beta}A_{\alpha\beta}$ gives the Riemann tensor and thus identically satisfies the self-consistent Bianchi identities. By the last phrase we mean that the covariant differentiations in the Bianchi identities are taken with $A_{\alpha\beta}$ as the metric tensor. This property together with the way the same operator naturally arose in equation (31) picks out the operator $F_{\kappa\lambda\mu\nu}{}^{\alpha\beta}$ as the natural operator to apply to the potential to obtain the field.

In the next section we deduce the generalized Hilbert–de Donder covariant condition as the most natural gauge condition to be applied to our potentials $A_{\mu\nu}$.

The slightly new discussion probably has some interest but some may consider the most cogent reasons for adopting this condition are that

(i) It is the natural generalization of that of Fierz & Pauli,
(ii) It is the natural generalization of that used in comparing general relativity with Newtonian theory,
(iii) It greatly simplifies the Einstein operator.

Those not interested in a more detailed discussion of gauge conditions should skip to equation (57) and assume the validity of equation (56).

Gauge conditions. For contact with physics it is desirable to have our theory expressed in terms of physical quantities. $\delta g_{\alpha\beta}$ is not such a quantity. We can only obtain a physical tensor $\delta g_{\alpha\beta}$ if we first decide which point in the perturbed space corresponds to any given point in the unperturbed space. Then by giving all corresponding points the same coordinate labels we shall find that $\delta g_{\alpha\beta}$ has physical meaning.

The same coordinates. We consider first how to specify the 'same' point in two neighbouring but physically distinct spaces. We shall use the term 'identical' in the sense used for identical triangles in geometry. When the two spaces are identical but are described in terms of slightly different coordinates, our problem has been solved by the use of *Lie* derivatives. In our case the *Lie* derivatives cannot be made zero because the spaces are not identical. In some sense the *Lie* derivatives must be made as small as possible but to my knowledge the problem has not been solved by the pure mathematicians. I shall take a physicist's short cut as follows.

First set up any coordinate system x^μ on the unperturbed space with metric $g_{\mu\nu}$. Then take any neighbouring coordinates x'^μ on the second space (metric $g'_{\mu\nu}$) and define $\delta g_{\mu\nu}=g_{\mu\nu}-g'_{\mu\nu}$ where $g'_{\mu\nu}$ and $g_{\mu\nu}$ are evaluated at points $\mathbf{x}'$, $\mathbf{x}$ with the same coordinates i.e. $\mathbf{x}'=\mathbf{x}$. It would be both coordinate dependent and stupid to regard such points as 'the same but in the two spaces'. To fix our ideas consider the trivial case when the two spaces are identical. One possible $g'_{\mu\nu}$ is then $g_{\mu\nu}$ and it is our aim to ensure that we use this desirable metric in the dashed space. Had we made the desirable choice we could force the dashed space to fit the other, in the sense that all points with the same coordinates coincide, without any physical deformation (i.e. strain). In these circumstances it is reasonable to regard points with the same coordinates as the 'same' points but in the different spaces. However, with an undesirable choice of coordinates a significant straining of the dashed space will be needed to force points with the same coordinates to fit their counterparts in the unperturbed space. Let us suppose we have some coordinate invariant strain index I which measures the amount of this straining. I will be a functional of $g'_{\mu\nu}$ and $g_{\mu\nu}$. Now we try to reduce the strain index by changing our coordinates on the dashed space. Since the spaces are in fact identical it will be possible to reduce the strain and the strain index to zero. Thus we may discover the desirable coordinates in the dashed space by minimizing the strain index. The advantage of this detailed procedure is that it can be carried out in general when the spaces are not identical. Minimizing the strain index with respect to all choices of coordinates in the dashed space, leads to a coordinate system corresponding to that chosen for the unperturbed space. Furthermore using this system we may regard points with the same coordinates in the two spaces as the 'same' point in those two spaces. This identification is invariant under changes in the chosen coordinates in the unperturbed space due to the invariance of the strain index.

Consider a small element of interval ds joining two neighbouring points P_1, P_2 in the unperturbed space. In any chosen coordinate system on the perturbed space there will be two points P_1', P_2' with same coordinates as P_1 and P_2. The strain involved in forcing P_1' and P_2' into coincidence with P_1, P_2 is the increase in interval per unit interval.

$$\frac{ds-ds'}{ds} = 1 - \left(\frac{g'_{\mu\nu}dx'^{\mu}dx'^{\nu}}{g_{\alpha\beta}dx^{\alpha}dx^{\beta}}\right)^{1/2} = 1 - \left(\frac{(g_{\alpha\beta}-\delta g_{\alpha\beta})dx^{\alpha}dx^{\beta}}{g_{\alpha\beta}dx^{\alpha}dx^{\beta}}\right)^{1/2}$$

$$= \tfrac{1}{2}\delta g_{\alpha\beta}\frac{dx^{\alpha}}{ds}\frac{dx^{\beta}}{ds}. \tag{47}$$

We compare this with the formula for strain in elasticity

$$e_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} \tag{48}$$

and are lead to regard $\tfrac{1}{2}\delta g_{\alpha\beta}$ as the strain tensor involved in forcing the coordinates and spaces to fit. We need an invariant strain index based on this strain tensor. The simplest such invariant is the dilation of the strain integrated over the whole space; that is

$$I = \int \tfrac{1}{2}\delta g_{\alpha\beta}g'^{\alpha\beta}(-g')^{1/2}\,d^4x'. \tag{49}$$

I is actually the change in 4-volume because

$$\tfrac{1}{2}\delta g_{\alpha\beta}g^{\alpha\beta} = \delta(\sqrt{-g})/\sqrt{-g}. \tag{50}$$

In all our applications our two spaces only differ due to a localized disturbance. Far from this disturbance the spaces may be taken to coincide. Under such boundary conditions we shall show that:

(i) $\delta I \equiv 0$ leads to a unique choice of coordinates in the dashed space corresponding to each choice on the original space.

(ii) When the spaces are identical so are the coordinates.

(iii) The covariant relative coordinate condition is the natural generalization of the usual Hilbert–de Donder condition.

(iv) This condition greatly simplifies the mathematical form of the Einstein operator.

(i)–(iv) gives us some confidence that our choice of strain index is the most natural one but we do not claim to have proved it is the only possible or reasonable choice. To find the equations for the minimizing $\delta g_{\alpha\beta}$ we make a small coordinate transformation away from the minimizing coordinates $\mathbf{x}'$ and denote the resulting changes by the symbol δ_1

$$\delta_1 I = \tfrac{1}{2}\int g_{\alpha\beta}\delta_1(g'^{\alpha\beta}\sqrt{-g'})\,d^4x' - \tfrac{1}{2}\int \delta_1(g'_{\alpha\beta}g'^{\alpha\beta}\sqrt{-g'})\,d^4x'.$$

The second integral vanishes owing to the invariance of the 4-volume to coordinate change. Following Schrödinger (15, p. 98) the first integral may be put in the form

$$\delta_1 I = \tfrac{1}{2}\int (g_{\alpha\beta} - \tfrac{1}{2}g'_{\alpha\beta}g'^{\mu\nu}g_{\mu\nu})\delta_1(g'^{\alpha\beta})\sqrt{-g'}\,d^4x'. \tag{51}$$

The condition that $\delta_1 I = 0$ for all $\delta_1(g'^{\alpha\beta})$ that arise from coordinate transformation leads as usual to the equations

$$\nabla_\gamma'[g'^{\gamma\beta}(g_{\alpha\beta} - \tfrac{1}{2}g'_{\alpha\beta}g'^{\mu\nu}g_{\mu\nu})] = 0.$$

But

$$\nabla_\gamma'[g'^{\gamma\beta}(g'_{\alpha\beta} - \tfrac{1}{2}g'_{\alpha\beta}g'^{\mu\nu}g'_{\mu\nu})] = 0$$

and hence on substracting

$$\nabla_\gamma'[g'^{\gamma\beta}(\delta g_{\alpha\beta} - \tfrac{1}{2}g'_{\alpha\beta}g'^{\mu\nu}\delta g_{\mu\nu})] = 0. \tag{52}$$

If we define

$$\delta\gamma_\alpha{}^\beta = g^{\beta\epsilon}\delta g_{\alpha\epsilon} - \tfrac{1}{2}g_\alpha{}^\beta\delta g_{\mu\nu}g^{\mu\nu} \tag{53}$$

then to first order in δ equation (52) reads

$$\nabla_\beta(\delta\gamma_\alpha{}^\beta) = 0. \tag{54}$$

These are the generalized Hilbert–de Donder conditions.

The mathematicians will still require a uniqueness proof. Assume there are two $\delta g_{\alpha\beta}$ called $\delta^1 g_{\alpha\beta}$ and $\delta^2 g_{\alpha\beta}$. Their difference $\delta^2 g_{\alpha\beta} - \delta^1 g_{\alpha\beta}$ will be due to a small coordinate transformation in the dashed space and hence

$$\delta^2 g_{\alpha\beta} - \delta^1 g_{\alpha\beta} = \nabla_\alpha'\xi_\beta + \nabla_\beta'\xi_\alpha$$

where $\mathbf{x}' \to \mathbf{x}' + \xi$ is the small coordinate transformation. Since ξ is already small the dashes on the covariant differentiations may be dropped to first order in δ. Now by hypothesis $\delta^1 g_{\alpha\beta}$ and $\delta^2 g_{\alpha\beta}$ both satisfy the Hilbert–de Donder conditions and both vanish at infinity so their difference must also have this property. Hence ξ must satisfy

$$\nabla_\gamma\{g^{\gamma\beta}[\nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha - \tfrac{1}{2}g_{\alpha\beta}g^{\mu\nu}(\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu)]\} = 0$$

that is

$$\nabla_\gamma\nabla_\alpha\xi^\gamma + \nabla^\gamma\nabla_\gamma\xi_\alpha - \nabla_\alpha(\nabla^\mu\xi_\mu) = 0$$

i.e.

$$\nabla^\gamma\nabla_\gamma\xi_\alpha - R_\alpha{}^\epsilon\xi_\epsilon = 0 \tag{55}$$

which is the vector wave equation in the curved space. When this has a unique solution our boundary conditions that $\boldsymbol{\xi}$ must vanish at infinity will ensure that $\boldsymbol{\xi} \equiv 0$.

The gauge condition

$$\nabla_\beta(A_\alpha{}^\beta - \tfrac{1}{2}g_\alpha{}^\beta A) = 0 \tag{56}$$

where $A = g^{\mu\nu}A_{\mu\nu}$, is satisfied when $A_{\alpha\beta} = \delta g_{\alpha\beta}$. It is also satisfied trivially when $A_{\alpha\beta} = g_{\alpha\beta}$ and so it is satisfied when $A_{\alpha\beta} = g_{\alpha\beta} + \delta g_{\alpha\beta}$. All potentials in our quasi linear theory will be subject to gauge condition (56).

The field equations. Our potentials are related to the fields by

$$F_{\kappa\lambda\mu\nu}{}^{\alpha\beta}A_{\alpha\beta}=R_{\kappa\lambda\mu\nu}. \tag{57}$$

Einstein's equations tell us that the fields are related to their sources by

$$(g^{\kappa\nu}g_\lambda{}^\gamma g_\mu{}^\delta-\tfrac{1}{2}g^{\kappa\nu}g^{\gamma\delta}g_{\lambda\mu})R_{\kappa\lambda\delta\nu}=R_{\lambda\mu}-\tfrac{1}{2}g_{\lambda\mu}R=K_{\lambda\mu}=-8\pi Gc^{-2}T_{\lambda\mu} \tag{58}$$

thus our potentials are related to the sources by

$$K_{\lambda\mu}{}^{\alpha\beta}A_{\alpha\beta}=-8\pi Gc^{-2}T_{\lambda\mu} \tag{59}$$

where the differential operator $K_{\lambda\mu}{}^{\alpha\beta}$ is defined by

$$K_{\lambda\mu}{}^{\alpha\beta}=(g_\lambda{}^\gamma g_\mu{}^\delta-\tfrac{1}{2}g^{\gamma\delta}g_{\lambda\mu})g^{\kappa\nu}F_{\kappa\gamma\delta\nu}{}^{\alpha\beta}. \tag{60}$$

The detailed form of $K_{\lambda\mu}{}^{\alpha\beta}$ is found from equations (39, 40, 41 and 60)

$$K_{\lambda\mu}{}^{\alpha\beta}A_{\alpha\beta}=$$
$$\begin{bmatrix}-\tfrac{1}{2}((g^{\alpha\beta}g_{\lambda\mu}-g^{(\alpha}_{(\lambda}g^{\beta)}_{\mu)})\nabla^2+g^{(\beta}_{(\mu}\nabla^{\alpha)}\nabla_{\lambda)}-g_{\alpha\beta}\nabla_{(\mu}\nabla_{\lambda)}-g_{\lambda\mu}\nabla^{(\alpha}\nabla^{\beta)}+g^{(\alpha}_{(\lambda}\nabla_{\mu)}\nabla^{\beta)}) \\ +\tfrac{1}{2}(R^{(\alpha}{}_{(\lambda\mu)}{}^{\beta)}+R^{(\alpha}_{(\mu}g^{\beta)}_{\lambda)}-R^{\alpha\beta}g_{\lambda\mu})\end{bmatrix}A_{\alpha\beta} \tag{61}$$

$$=\left\{\begin{matrix}\tfrac{1}{2}g^{(\alpha}_{(\lambda}g^{\beta)}_{\mu)}\nabla^2+g^{(\beta}_{(\mu}R^{\alpha)}_{\lambda)}-\tfrac{1}{2}g^{\alpha\beta}R_{\lambda\mu}-\tfrac{1}{2}g^{\lambda\mu}R_{\alpha\beta}+\tfrac{1}{4}g^{\alpha\beta}g_{\lambda\mu}R \\ -[g^{(\alpha}_{(\lambda}\nabla_{\mu)}\nabla^{\beta)}]\end{matrix}\right\}(A_{\alpha\beta}-\tfrac{1}{2}g_{\alpha\beta}A). \tag{62}$$

It appears that both our gauge condition and this operator look simpler when we adopt the new variable

$$\phi_{\alpha\beta}\equiv A_{\alpha\beta}-\tfrac{1}{2}g_{\alpha\beta}A. \tag{63}$$

In future we shall regard $\phi_{\alpha\beta}$ as our potential. The following relations are useful for translation from A language to ϕ language.
When

$$A_{\alpha\beta}=g_{\alpha\beta}\text{ then }\phi_{\alpha\beta}=-g_{\alpha\beta} \tag{64}$$

$$A_{\alpha\beta}=\phi_{\alpha\beta}-\tfrac{1}{2}g_{\alpha\beta}\phi \tag{65}$$

$$\phi=g^{\alpha\beta}\phi_{\alpha\beta}=-A. \tag{66}$$

The gauge condition is

$$\nabla^\beta\phi_{\alpha\beta}=0 \tag{67}$$

and as a result of this condition

$$K_{\lambda\mu}{}^{\alpha\beta}A_{\alpha\beta}=E_{\lambda\mu}{}^{\alpha\beta}\phi_{\alpha\beta} \tag{68}$$

where

$$E_{\lambda\mu\alpha\beta}=\tfrac{1}{2}g_{\alpha(\lambda}g_{\mu)\beta}\nabla^2+\tfrac{1}{2}(g_{\beta(\mu}R_{\lambda)\alpha}+g_{\alpha(\mu}R_{\lambda)\beta}-R_{\lambda\mu}g_{\alpha\beta}-R_{\alpha\beta}g_{\lambda\mu})+\tfrac{1}{4}Rg_{\alpha\beta}g_{\lambda\mu}. \tag{69}$$

We have at last reached the Einstein operator of Section 3. Equations (68) and (69) show us that Einstein's equations may be written

$$E_{\lambda\mu}{}^{\alpha\beta}\phi_{\alpha\beta}=-8\pi Gc^{-2}T_{\lambda\mu} \tag{70}$$

it is readily verified that $\phi_{\alpha\beta}=-g_{\alpha\beta}$ satisfies these relations as well as the gauge condition (67). We have therefore derived the quasi-linear theory required in Section 3.

5. *Programme to determine which universes are Machian.* For the Robertson–Walker universes and possibly also for Gödel's universe it seems possible to carry out the following programme:

I, take the metric in some suitable coordinates and write out the form of the Einstein operator (4);

II, solve equation (8) for the Green's function;

III, discover whether equation (13) holds for that universe. It is interesting to note that equation (13) will determine the size and sign of the constant of gravitation if the matter distribution for space-time is known. This follows because the constant of gravitation is involved only on the right hand side.

It is probably no easy matter to carry out this programme and it may turn out that our Mach condition is so restrictive that no universe satisfies it. However even if the mathematics is hard the problem of which universes are Machian is one that will be raised so long as enquiring minds exist and must be of considerable philosophical import.

6. *Comment.* The uniqueness of the theory as presented may readily be called in question. The choice of strain index was supported by strong arguments of expediency and mathematical simplicity, but it is by no means clear nor even likely that another choice of strain index, equally simple after its own fashion, would give the same Mach condition. The whole theory leading to our choice of linear operators is to some extent arbitrary though we have tried to make the simplest and most analogous step at every cross roads. The theory as presented ends with a highly symmetrical self-adjoint Einstein operator with the added bonus that it does not involve the Riemann tensor but only the Ricci tensor. These happy coincidences lead us to believe that we have taken the right path. It is simple to modify and complicate the theory for those who believe that the fundamental law of gravity contains a non-zero dimensionful cosmical constant. The term $\lambda g^{(\sigma}_{\mu} g^{\tau)}_{\nu}$ must be added to the Einstein operator and the rest of the theory is unchanged. However, in my mind such a step is no better than the introduction of an unobservably small mass to the photon with the resulting change in Maxwell's equations. Since the latter change is taken seriously by almost no-one I don't see why the cosmical constant is taken as sensible. Perhaps cosmologists have had too small a subject in the past and have needed more cases to explore which are still simple enough to solve. However were it shown that only if the cosmical constant had some particular value would the universe be Machian, then a theory with that value of the cosmical constant would be attractive. I gather it was this belief that led Einstein to introduce the cosmical term.

Finally, comparison with Wheeler's formulation of Mach's principle in terms of the closure of space-like hypersurfaces, shows that we have chosen what is perhaps an unnecessarily strict interpretation of causality. We have demanded a whole formulation of Mach's principle in terms of Green's functions strictly within and on the light cone. As Wheeler points out a formulation in terms of the closure of space like hypersurfaces is 'ostensibly instantaneous' but can yet be made causal in the sense that all events can be traced to causes within their light cones. Perhaps it would be rational to call our stricter postulate 'Berkeley's principle'.

It seems to me likely that our strictly causal interpretation will imply that the whole of space-time will be in or on the light cone of any of the elements of matter taking part in the initial singularity of the universe, but at present this is only a guess. Penrose's electrodynamic paradox (16) casts interesting light in that direction.

An approach to Mach's principle very different from that considered here has been proposed by Dicke (17).

Two doubts arise, firstly it is not obvious that our Mach condition will be obeyed by any metric, and secondly must those that do so have a $+++-$ signature?

7. *Acknowledgments.* The idea of linearity over the self-consistent space I took from a lecture at Cambridge by Dr J. V. Narlikar. It was a pleasure to learn something of bivectors and Green's functions in curved spaces from Dr W. G. Dixon and R. McLenaghan.

Every aspect of this paper has gained considerably from discussions with Dr D. W. Sciama. It was in these discussions that the form of Mach's principle as an integral for the metric arose (18). He has shared my enthusiasms, pointed out my errors and joined me in advancing theories of this type during the last eighteeen months. He has taught me much of general relativity in the process and has my deepest thanks.

Lastly I would like to thank Drs F. G. Friedlander, S. Hawking and B. Bertotti for talking to me of their later developments of this theme and helping me to learn a new subject, and Professor J. F. Vinen for encouragimg my first groping steps in this direction.

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Note added in proof.—The reader's attention should be drawn to a similar article just published in the Russian Journal for Experimental and Theoretical Physics **51**, 1143, 1966 by B. L. Altshuler. A theory similar to that developed here is presented and it is shown that the nature of the singularity of the Friedmann universe violates the resulting Mach Principle.

Although the operator corresponding to our Einstein operator is of a different form with fewer symmetries, nevertheless it seems likely that the discussion of the singularity will be little changed by this. In fact it may be that the Freidmann universe cannot be Machian in any theory of the type considered here. This would be in line with Penrose's suggestion that perhaps all theories of this type might agree as to which universes were Machian.

Further comparison of the authors work with Altshuler's must await the translation of the latter. It is remarkable that just as the observed low helium abundance in high velocity stars is leading to difficulties in big-bang cosmologies the basis for those cosmologies is being found to be unsatisfactory.