GUNTHER'S PROOF OF NASH'S ISOMETRIC EMBEDDING THEOREM

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1. Preface

Around 1987 a German mathematician named Matthias Gunther found a new way of obtaining the existence of isometric embeddings of a Riemannian manifold. His proof appeared in [1, 2]. His approach avoids the so-called Nash-Moser iteration scheme and, therefore, the need to prove smooth tame or Moser-type estimates for the inverse of the linearized operator. This simplifies the proof of Nash's isometric embedding theorem [3] considerably.

This is an informal expository note describing his proof. It was originally written, because when I first learned Gunther's proof, it had not appeared either in preprint or published form, and I felt that everyone should know about it. Moreover, since he is at Leipzig, which at the time was part of East Germany, very few mathematicians in the U.S. knew about him or his proof.

Since many still seem to be unaware of Gunther's proof, even after he gave a talk at the International Congress of Mathematicians at Kyoto in 1990 and published his proof in the proceedings [2], I have updated this note and continue to distribute it. I do, however, encourage you to seek out Gunther's own presentations of his proof.

2. Introduction

Let M be a smooth n–dimensional manifold. Given an embedding $u : M \to \mathbf{R}^N$, the standard inner product on $\mathbf{R}^N$ induces a Riemannian metric on M. We shall denote this metric by $du \cdot du$. In particular, given a Riemannian metric g on M, we say that the embedding u is *isometric*, if

$$du \cdot du = g$$

Let $N \geq \frac{1}{2}n(n+1)$. A C^2 immersion $u : M \to \mathbf{R}^N$ is *free* if for every $x \in M$,

$$\partial_i u(x), \partial_i \partial_j u(x),\ 1 \leq i, j \leq n,$$

span a $\min(N, n + \frac{1}{2}n(n+1))$–dimensional linear subspace of $\mathbf{R}^N$.

The only place where Gunther's proof differs from earlier proofs of existence lies in showing that given a smooth, free embedding $u_0 : M \to \mathbf{R}^N$ and a smooth Riemannian metric g sufficiently close (in a sense to be made precise later) to $du_0 \cdot du_0$, there exists a smooth embedding $u : M \to \mathbf{R}^N$ close to u_0 such that

$$du \cdot du = g. \tag{1}$$

Although it is not necessary, we shall simplify the exposition by assuming the existence of "global" co–ordinates on M. If M is compact, this is obtained by embedding M smoothly into a torus of larger dimension and extending smoothly the embedding u_0 and the metric g to the torus so that g remains close to $du_0 \cdot du_0$. Otherwise, if all we are trying to prove is a local existence theorem, we can assume that M is diffeomorphic to an open set in $\mathbf{R}^n$. In the discussion below, $x^1, \ldots, x^n$ are assumed to be global co–ordinates on M. (If M does not have global co–ordinates, then all the calculations below should be done using a fixed smooth background metric $\hat{g}$, instead of the flat metric implied by the global co–ordinates, and its Levi–Civita connection. Extra terms involving the curvature of $\hat{g}$ and the covariant derivative of curvature appear, but they are all of lower order and do not affect the proof at all.)

Let $v = u - u_0$ and $h = g - du_0 \cdot du_0$. For convenience we shall denote

$$u_i = \frac{\partial u_0}{\partial x^i},\ u_{ij} = \frac{\partial^2 u_0}{\partial x^i \partial x^j}.$$

Then (1) is equivalent to

$$u_i \cdot \partial_j v + u_j \cdot \partial_i v + \partial_i v \cdot \partial_j v = h_{ij},\ 1 \le i, j \le n. \tag{2}$$

Applying the standard "integration by parts" trick, (2) can be rewritten as

$$\partial_i (u_j \cdot v) + \partial_j (u_i \cdot v) - 2u_{ij} \cdot v + \partial_i v \cdot \partial_j v = h_{ij}. \tag{3}$$

This can be written abstractly in the following form:

$$L_0 v + Q(v, v) = h,$$

where L_0 is a linear operator and Q is bilinear. Nash's trick, when $N \ge \frac{1}{2}n(n+1) + n$, was to observe that the linear differential operator L_0 could be inverted by a zeroth order differential operator M_0. More recently, M. Gromov and Bryant-Griffiths-Yang have found cases where $N < \frac{1}{2}n(n+1)+n$ and L_0 admits a right inverse M_0 which "loses" a fixed number of derivatives. In all cases there is a loss in regularity, so that standard implicit function theorems or contraction map arguments do not seem to apply. Instead, the so–called Nash–Moser iteration scheme must be used.

Gunther's ingenious trick can be decribed as follows: He finds new *nonlocal* bilinear operators Q_1 and Q_2 such that

$$Q = L_0 Q_1 + Q_2, \tag{4}$$

where Q_1 is zeroth order and Q_2 is of any given negative order, i.e. it is a bilinear smoothing operator. Actually, in the specific situation here, the operator Q_2 will be identically zero. Then the contraction mapping argument can be applied to the equation

$$v = M_0(h - Q_1(v,v)) - Q_2(v,v).$$

The splitting is obtained as follows: Let

$$\Delta = \sum_{i=1}^{n} \partial_i^2.$$

Then $\Delta - 1$ is an invertible elliptic operator on M. Apply it to both sides of (3). Rearranging the terms and then applying $(\Delta-1)^{-1}$ to the resulting equation, we obtain;

$$\partial_i(u_j \cdot v + Q_j(v,v)) + \partial_j(u_i \cdot v + Q_i(v,v)) - 2u_{ij}\cdot v + Q_{ij}(v,v) = h_{ij},$$

where

$$\begin{aligned} Q_i(v,v) &= (\Delta-1)^{-1}(\Delta-1)v\cdot\partial_i v \\ Q_{ij}(v,v) &= (\Delta-1)^{-1}(2\sum_{k=1}^{n}\partial_i\partial_k v\cdot\partial_j\partial_k v + \partial_i v\cdot\partial_j v - 2(\Delta-1)v\cdot\partial_i\partial_j v). \end{aligned}$$

Since u_0 is free, there exists a unique $\mathbf{R}^N$-valued bilinear operator Q_0 such that $u_i\cdot Q_0 = Q_i$ and $u_{ij}\cdot Q_0 = Q_{ij}$. The isometric embedding equation now becomes

$$L_0(v - Q_0(v,v)) = h,$$

where

$$(L_0 v)_{ij} = \partial_i(u_j\cdot v) + \partial_j(u_i\cdot v) - 2u_{ij}\cdot v.$$

Given $h = h_{ij}dx^i dx^j$, define $M_0 h = v$, where for every $x \in M$, $v(x)$ is the unique vector lying in the span of $u_i(x), u_{ij}(x)$, $1 \le i,j \le n$, satisfying the following equations

$$\begin{aligned} u_i\cdot v &= 0 \\ -2u_{ij}\cdot v &= h_{ij} \end{aligned}$$

Clearly, M_0 is a right inverse for L_0. Therefore, to solve (3), it suffices to solve the following:

$$v = M_0 h + Q_0(v,v).$$

Define $\Phi(v) = M_0 h + Q_0(v,v)$. If $\|v\|_{2,\alpha}$, $0 < \alpha < 1$, is sufficiently small, then Φ is a contraction mapping on a neighborhood of $0 \in C^{2,\alpha}(M, \mathbf{R}^N)$. Moreover, the linear operator $I - Q_0(v,\cdot)$ is an elliptic zeroth order operator and therefore if h is $C^{k,\alpha}$, $k \geq 2$, then so is v. In particular, if h is smooth, so is v.

We have therefore obtained the following:

Theorem 1 (Nash, Gunther [3, 1, 2])**.** *Let M be an n-dimensional torus and $u_0 : M \to \mathbf{R}^N$, $N \geq \frac{1}{2}n(n+1)+n$, a smooth, free immersion. Then given $0 < \alpha < 1$, there exists $\epsilon > 0$ (depending on u_0 and α) such that given any $C^{2,\alpha}$ Riemannian metric g, $\|g - du_0 \cdot du_0\|_{2,\alpha} < \epsilon$, there exists a $C^{2,\alpha}$ immersion u close to u_0 such that $du \cdot du = g$. Moreover, if g is $C^{k,\alpha}$, $2 \leq k \leq \infty$, the immersion u is $C^{k,\alpha}$.*

References

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2. ______, *Isometric embeddings of Riemannian manifolds*, Proceedings of the International Congress of Mathematicians (Kyoto, 1990), Mathematical Society of Japan, 1991, pp. 1137–1143.
3. John Nash, *The imbedding problem for Riemannian manifolds*, Annals of Mathematics **63** (1956), 20–63.

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