ANNALS OF MATHEMATICS
Vol. 45, No. 2, April, 1944

THE SINGULARITIES OF A SMOOTH n-MANIFOLD IN $(2n-1)$-SPACE*

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(Received August 19, 1943)

1. Introduction

We showed in the preceding paper that any smooth n-manifold M^n may be imbedded in $2n$-space E^{2n}. Our primary purpose here is to show that it may be immersed in E^{2n-1}, provided that $n \geqq 2$. Then near any point of M, the mapping f into E^{2n-1} is one-one, but there may be self-intersections (which may be required to lie along curves). Equally important perhaps is the combinatorial study of singularities (points where the mapping is not regular). Along with true manifolds we study also manifolds with boundary. By a *partial manifold*, we mean a manifold with or without boundary. In simple cases, the boundary ∂M of the partial manifold M will be a manifold. (∂M means the point set boundary; it need not coincide with the boundary of the chain M if M is non-orientable.) Since the question of how general ∂M may be allowed to be (we insist at any rate that it be a complex) is a rather difficult one, which we expect to study further in another paper, we will use the term somewhat loosely here. Any special assumptions on ∂M which may be needed will be made at the time.

It is a highly difficult problem to see if the imbedding and immersion theorems of the preceding paper and the present one can be improved upon. Practically the only knowledge we have of this is found in the author's Michigan lecture, [3]. The most important result there for the present problem is the existence of a closed M^4 which cannot be imbedded in E^7. (We have not studied the possibility of immersing it in E^6.) This M^4 is non-orientable; it seems possible that any orientable, or any open or partial, M^4 may be imbedded in E^7, and immersed in E^6. Possibly also any M^3 may be imbedded in E^5!

We touch briefly on the case $n = 1$. Here M is a circle, or a closed, open, or half-open arc. Locally, the mapping f into the line E^1 is expressible as a differentiable real-valued function $x' = f(x)$; the singularities of f are the points where $df/dx = 0$. If the mapping is "semi-regular," the only singularities are maxima and minima of f. It is obvious that a slight alteration of any f will give a mapping g in which this holds. The combinatorial conditions as stated in this paper apply only to the case $n \geqq 2$, but analogues could easily be given for the case $n = 1$.

Suppose now that $n \geqq 2$. If we take a general smooth mapping f of M^2 into E^3, the singularities may be quite wild. But again, a slight alteration of f will reduce them to a single type; see §3 and Fig. 1. The new mapping is semi-regular; these are the mappings which concern us here.

The combinatorial part of the paper consists essentially in counting the

* Presented to the American Mathematical Society, Sept. 9, 1942.

algebraic number $\mathfrak{S}_f(M)$ of singular points (mod 2 if n is even) by means of the mapping f in the boundary of M. (As an immediate corollary, $\mathfrak{S}_f(M) = 0$ or $\equiv 0$ mod 2 for all closed manifolds, orientable or non-orientable.) The manner of counting may be seen from Fig. 1. If we follow around the boundary ∂M, it cuts through M at one point, p^*; hence $\mathfrak{S}_f(M) \equiv 1$ (mod 2). Or, again, let us cut off a strip around the edge of $f(M)$, and spread it out in the form of a circle; we will find that it has a single twist. If we cut this strip into two strips, the twist will show up in the linking of the two new strips; combinatorially, in the fact that (each being considered as a curve) their looping coefficient is ± 1. These two facts are expressible in the form:

$$\mathfrak{S}_f(M) = KI(fM^*, f\partial M) = LC(f\partial M^*, f\partial M), \tag{1.1}$$

M^* being M with a narrow neighborhood of ∂M removed. (1.1) holds in fact for any chain A, using A^*, which is A with a neighborhood of all $(n - 1)$-cells removed. The fundamental theorem states that this is the algebraic number of singular points of M, taken mod 2 if n is even. The proof in the case that M is non-orientable and n is odd is difficult to handle; the intersection theory in Part III is needed to unravel the situation.

The reader may wonder why we are willing to lose preciseness in the results by reducing mod 2 whenever n is even. The answer is, the formulas are only correct after reducing mod 2 (and in fact, this is all we need in the proof of the immersion theorem). This is well illustrated in Fig. 2. There are no singularities; yet $f(\partial M)$ cuts through $f(M)$ twice, each time in the same sense, so that $KI(fM^*, f\partial M) = \pm 2$. In the proof of the immersion theorem, we cut out pieces of $f(M)$ and alter them; it may be necessary to insert twisted pieces such as in Fig. 2 to gain the desired end in case n is even.

Fundamental definitions are as in the preceding paper (including its §4). We note also the following (see also §15 and elsewhere). A vector is *tangent* to M at $p \in M$ if it points into M or along ∂M (if $p \in \partial M$) at p. It is *independent* of M if it is not in the tangent plane to M at p. A vector field $f(p)$ is *independent* of M if each $v(p)$ is independent of M at p. We may use a complex K in place of M; then v is *independent* of K at p if it is independent of each cell σ of K with $p \in \bar{\sigma}$, etc.

2. Outline of the paper

A typical singularity is presented in (3.3), and a mapping of a sphere or plane with just two of them is given in §4. Though there is only one kind of singular point under a semi-regular mapping, we may differentiate between positive and negative ones in case n is odd; see the definition in §5. If a single cell is mapped by f so that it has just one singularity, as illustrated in Fig. 1, the relation to intersections is fairly simple, as noted above. For a partial manifold, or more generally, a complex, the relation is worked out in §7. When we express M and ∂M as chains, and sum, both pairs $\partial\sigma_i^n$, σ_j^n and $\partial\sigma_j^n$, σ_i^n will appear if $i \neq j$. By the commutation rule for Kronecker indices, such terms will cancel out for

n odd, giving the exact value of $\mathfrak{L}_f(M)$ in (8.2); for n even, we get this result only mod 2. If f is deformed, $f(\partial M)$ may cut through itself; yet if n is odd, this does not affect $\mathfrak{L}_f(M)$, as noted in Theorem 4.

To prove the immersion theorem, we need some detailed results on the type of looping coefficients which we mentioned above in cutting a strip into two strips. Lemmas are given which state that certain alterations of f are possible which map ∂M into a given position and let M have given directions at points of ∂M. Next we give a mapping f of an n-cube M_0 (n even) without singularities and with $\mathfrak{L}_f(M_0) = \pm 2$; compare Fig. 2. To prove the immersion theorem in case M is closed and n is odd, we use $\mathfrak{L}_f(M) = 0$ to show that the singular points may be paired, p_i and p_i', the two in a pair being of opposite type. If A is an arc from p_i to p_i', a neighborhood M_1 of A then has the property that $\mathfrak{L}_f(M_1) = 0$. We may therefore alter f in M_1 to remove these two singularities. The other cases do not require much further treatment. Two theorems are then given which discuss the position of ∂M under an immersion of a partial manifold.

Suppose M is non-orientable. Let M_1 be a chain formed by adding together the n-cells of M. Then $\partial M_1 = A + 2B$, where A is the sum of the $(n - 1)$-cells of ∂M, and B is a sum of cells interior to M. Thus, if M^2 is a Möbius strip, ∂M_1^2 is the boundary curve plus twice an arc cutting across the strip. Suppose n is odd. It was proved in Part I that (1.1) counts the singularities, provided that M_1, M_1^* and ∂M_1 are used. But we do not wish to use any interior cells of M; it is necessary to show that these always cancel out. It is clear that in this case we cannot use $KI(fM^*, f\,\partial\, M)$, since by an alteration of f we might move $f\,\partial\, M$ across itself, which would alter the Kronecker index. Moreover, $LC(f\,\partial\, M^*, f\,\partial\, M)$ is not defined, since ∂M^* and ∂M cannot be made into cycles. But if we choose that n-chain $\rho^+ f\,\partial\, M$ formed by deforming $f\,\partial\, M$ in the y_{2n-1}-direction to infinity, we may study $KI(f\,\partial\, M^*, \rho^+ f\,\partial\, M)$. This leads finally to the required result. The definition of $\mathfrak{L}_f(M)$ required, in (20.5), is more complicated than before; its necessity is shown by an example in §21.

In an appendix we take up some topics which are less fundamental in the paper.

I. SINGULARITIES AND INTERSECTIONS

3. The general type of singularity

DEFINITION. The mapping f of the n-manifold M^n (without boundary) into E^{2n-1} is *semi-regular* if it is of class C^{12} (so that we may apply Lemma 2) and is proper, and for each $p \in M$, either f is regular at p or the following holds: With a suitable coordinate system about p,

$$\left.\frac{\partial f}{\partial x_1}\right|_p = 0, \tag{3.1}$$

and the $2n - 1$ vectors

$$\left.\frac{\partial^2 f}{\partial x_1^2}\right|_p, \left.\frac{\partial f}{\partial x_2}\right|_p, \cdots, \left.\frac{\partial f}{\partial x_n}\right|_p, \left.\frac{\partial^2 f}{\partial x_1 \partial x_2}\right|_p, \cdots, \left.\frac{\partial^2 f}{\partial x_1 \partial x_n}\right|_p \tag{3.2}$$

are independent. This condition holds then in any coordinate system for which (3.1) holds; see [4]. The x_1-direction is uniquely determined except as to sense. If M is a partial manifold, we assume also that f is one-one in a neighborhood of the boundary. (See the appendix, Lemma 25.)

Definition. The semi-regular mapping f is *completely semi-regular* if: (a) For any double point $f(p) = f(q)$ (p or q may be in ∂M) the two tangent planes to $f(M)$ there have only a line in common. (b) $f(\partial M)$ does not contain the image of any singular point. (c) If $n \geqq 3$, there are no triple points $f(p) = f(q) = f(r)$; if $n = 2$, there is no such triple point with $p \in \partial M$, and there are no quadruple points. The self-intersections are then along smooth curves; see §22.

Lemma 1. *Arbitrarily close to any f there is a completely semi-regular f'; we may make f' be one-one in a neighborhood of ∂M, and may make f' of class C^∞.*

This is proved without the "completely," for manifolds, in [4]. For partial manifolds, we first imbed a neighborhood of the boundary (using the methods in [1]), then extend the mapping over the interior of M, and apply the proof mentioned to the interior. It is now easy to make the mapping completely semi-regular (see [1], especially §9, (D)).

Lemma 2. *Let f be semi-regular. Then for any singular point p there exist (curvilinear) coordinate systems $(x_1, \cdots, x_n)$ about p and $(y_1, \cdots, y_{2n-1})$ about $f(p)$ such that f is given near p by*

$$y_1 = x_1^2, \qquad \left.\begin{aligned} y_i &= x_i, \\ y_{n+i-1} &= x_1 x_i, \end{aligned}\right\} \quad (i = 2, \cdots, n). \tag{3.3}$$

This also is proved in [4]. If f is of class C^{4r+8}, of class C^∞, or analytic, the new coordinate systems will be of class C^r, of class C^∞, or analytic respectively.

In case $n = 2$, the mapping is

$$x = u^2, \qquad y = v, \qquad z = uv; \tag{3.4}$$

eliminating u and v gives $z = \pm y\sqrt{x}$. For each y, the cross-section is a parabola; as y passes through 0, the parabola degenerates to a half-ray, and opens out again (with sense reversed); see Figure 1. The only self-intersection is at $v = 0$, mapping into the positive x-axis; in the general case, at $x_2 = \cdots = x_n = 0$, mapping into the positive y_1-axis.

4. A mapping of a sphere or plane with just two singularities

The examples we give here not only are interesting as illustrating mappings of whole manifolds with definite singularities, but are useful in the proof of the fundamental Theorem 6, in the case of an open manifold.

Let S_0^n be the n-sphere $x_1^2 + \cdots + x_{n+1}^2 = 1$ in E^{n+1}. We define a smooth mapping f of E^{n+1}, and hence of S_0^n, into E^{2n-1} by the equations

$$y_1 = x_{n+1}, \qquad \left.\begin{aligned} y_i &= x_i, \\ y_{n+i-1} &= x_1 x_i, \end{aligned}\right\} \quad (i = 2, \cdots, n). \tag{4.1}$$

For $n = 2$, transposing terms gives

$$f(x, y, z) = (x, xy, z).$$

The effect of f is to turn the part $x < 0$ of the sphere inside out. More explicitly, f squeezes the cross-sections S_x^1 for each x so that for $x = 0$, the circle S_x^1 turns into a line segment, and for $x < 0$, into an ellipse with sense reversed. There are obviously two singularities, at $(0, 0, \pm 1)$.

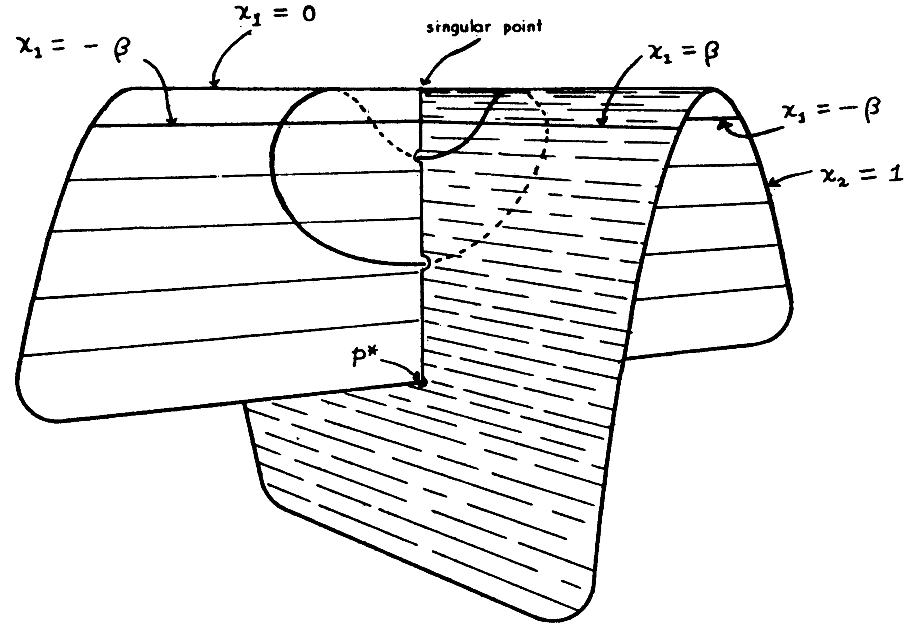

FIG. 1

The matrix of first partial derivatives in (4.1), transposed, is

$$\begin{Vmatrix} 0 & 0 & \cdots & 0 & x_2 & \cdots & x_n \\ 0 & 1 & \cdots & 0 & x_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & x_1 \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{Vmatrix}.$$

To find the singularities, take any $p = (x_1, \cdots, x_{n+1}) \in S_0^n$, and any vector $v = (v_1, \cdots, v_{n+1})$ tangent to S_0^n at p; then v is orthogonal to $p - O$ (O = origin), so that $\sum v_i x_i = 0$. The vector v is mapped by f into a vector we shall call $\nabla f(v, p)$ (which may be considered as the derivative of f along v; see the preceding paper §4); it is

$$\nabla f(v, p) = \sum v_i \frac{\partial f}{\partial x_i}$$
$$= (v_{n+1}, v_2, \cdots, v_n, v_1 x_2 + v_2 x_1, \cdots, v_1 x_n + v_n x_1).$$

Suppose this vanishes, with $v \neq 0$. Then $v_2 = \cdots = v_n = v_{n+1} = 0$, hence $v_1 \neq 0$, and since $v_1 x_i + v_i x_1 = v_1 x_i = 0$ $(i > 1)$, we have $x_2 = \cdots = x_n = 0$. Also $\sum v_i x_i = v_1 x_1 = 0$, and hence $x_1 = 0$, and $x_{n+1} = \pm 1$. Thus the only singular points are

$$p_1 = (0, \cdots, 0, 1), \qquad p_2 = (0, \cdots, 0, -1).$$

Near each p_k we may determine x_{n+1} in terms of $x_1, \cdots, x_n$, and thus write

$$f(x_1, \cdots, x_n, x_{n+1}) = F(x_1, \cdots, x_n) \quad \text{in} \quad S_0^n;$$

we find $\partial F/\partial x_1 |_{p_k} = 0$. Computing $\partial F/\partial x_i$ and $\partial^2 F/\partial x_1 \partial x_i$ at p_1 and at p_2 shows at once that these singularities are of the required type.

From the mapping f in (4.1) we obtain a mapping ϕ of E^n into E^{2n-1} as follows. First, interchange x_1 and x_{n+1}:

$$f_1(x_1, \cdots, x_n, x_{n+1}) = (x_1, \cdots, x_n, x_{n+1}x_2, \cdots, x_{n+1}x_n).$$

Near the point $p_0 = (0, 1, 0, \cdots, 0)$, this is very close to the identity mapping of S_0^n into $S_0^n \subset E^{n+1} \subset E^{2n-1}$; a slight deformation of f_1 into f_2 will bring it to the identity in a neighborhood U of p_0 in S_0^n. By stretching $U - p_0$ into the part of E^n outside some $(n - 1)$-sphere, f_2 transforms into the required mapping.

A mapping f of E^n into E^{2n-1} with two singularities may also be defined as follows:

$$u = (1 + x_1^2) \cdots (1 + x_n^2),$$

$$y_1 = x_1 - \frac{2x_1}{u}, \qquad y_i = x_i \qquad (i = 2, \cdots . n), \tag{4.2}$$

$$y_{n+1} = \frac{1}{u}, \qquad y_{n+i} = \frac{x_1 x_i}{u} \qquad (i = 2, \cdots, n - 1).$$

Note that far from the origin, f is very near the identity. Hence a slight alteration of f will make it the identity outside some sphere. (Compare the proof of Lemma 11.) Comparing with the preceding paper, §2, we see easily that f is regular except at the points $p_\pm = (0, \cdots, 0, \pm 1)$; at these points, $\partial f/\partial x_1 = 0$. At p_+ for example, the vectors (3.2) form a diagonal determinant, whose elements d_i are $d_1 = 2$, $d_i = 1$ $(i = 2, \cdots, n)$, $d_{n+1} = -1$, $d_{n+i} = \frac{1}{2}$ $(i = 2, \cdots, n - 1)$; hence f is semi-regular.

5. The orientation of singular points

We shall discuss the following problem. Given a singular point p and a neighborhood U of p, are there any orientation properties of E or of U determined by the set of points $f(U)$? Let A be the arc of self-intersection through p (i.e. part of the x_1-axis, in the coordinate system of Lemma 2). It turns out that *for n odd, an orientation of E is determined, while for n even, an orientation of E is determined by one of A near p.* We shall show in fact that the following definitions are permissible.

DEFINITIONS. We use the above notations. If n is odd, the singular point is *positive* or *negative* according as the vectors (3.2) determine the negative or positive orientation of E. If n is even, the *positive side* of M at p is the direction along A such that, if the x_1-axis points in that direction, then the vectors (3.2) determine the negative orientation of E. Note that *M need not be oriented or even orientable.* The reason for the choice will appear in Lemma 6.

LEMMA 3. *The above definitions are independent of the coordinate systems employed.*

Take two systems $\{x_i\}$ and $\{x_i'\}$, with $\partial f/\partial x_1 |_p = \partial f/\partial x_1' |_p = 0$. We may rotate the x_i'-axes ($i > 1$), obtaining $\{x_i''\}$, so that

$$\left.\frac{\partial f}{\partial x_i''}\right|_p = \alpha_i \left.\frac{\partial f}{\partial x_i}\right|_p, \qquad \alpha_i > 0 \text{ for } i > 2.$$

The definitions with the $\{x_i''\}$ are the same as with the $\{x_i'\}$.

If $n \geqq 2$ and $\alpha_2 < 0$, let us replace x_2'' by $-x_2''$. This does not affect the orientation of A, and since both $\partial f/\partial x_2'' |_p$ and $\partial^2 f/\partial x_1'' \partial x_2'' |_p$ are reversed in direction, the vectors (3.2) with the new $\{x_i''\}$ determine the same orientation of E as with the old $\{x_i''\}$.

Now if $\alpha_1 < 0$, replace x_1'' by $-x_1''$. Suppose first that n is odd. Then the $n - 1$ vectors $\partial^2 f/\partial x_1'' \partial x_2'' |_p, \cdots, \partial^2 f/\partial x_1'' \partial x_n'' |_p$ are reversed, but no others of (3.2) are changed. Thus the same orientation of E is determined. Suppose next that n is even. Then the orientation of E is reversed; but the new x_i''-axis now points in the other direction along A.

With all the $\alpha_i > 0$, we may deform the $\{x_i''\}$ system into the $\{x_i\}$ system. The vectors (3.2) remain always independent, so the same orientation of E and of A are determined, completing the proof.

LEMMA 4. *If f is given by* (3.3), *and the coordinate systems determine the positive orientations of M and of E, then for n odd, the singular point is negative, while for n even, the positive direction in M at p is along the negative x_1-axis.*

It is sufficient to show that the matrix formed from the vectors (3.2) has a positive determinant. Differentiating (3.3), we see that the determinant is diagonal, with one 2 and the rest 1's on the diagonal; hence the determinant is $2 > 0$.

6. The intersection $\mathfrak{L}_f(M)$ of M with ∂M under f

We shall count the number of times that $f(\partial M)$ cuts through $f(M)$ in E^{2n-1}. The definition given here suffices in the orientable case; an interpretation in terms of the manner in which M attaches to ∂M will be studied in §9. The latter discussion will apply also to non-orientable partial manifolds such that ∂M can be made into a cycle; see §14.

If A^r and B^s are singular chains in an oriented E^{r+s}, such that $A \cap \partial B = \partial A \cap B = 0$, then their Kronecker index $KI(A, B)$ is defined. In particular, let σ^r and σ^s be oriented cells with just one common interior point p, their tangent planes at p having only p in common. Let $u_1, \cdots, u_r$ be independent vectors tangent to σ^r at p, determining the positive orientation of σ^r; choose $v_1, \cdots, v_s$ similarly for σ^s. Then the intersection is positive or negative ($KI = 1$ or -1) according as $u_1, \cdots, u_r, v_1, \cdots, v_s$ determine the positive or negative orientation of E^{r+s}.

Definition. Let f be a semi-regular mapping of the orientable partial manifold M^n into E^{2n-1}. Choose an orientation of M. With an infinite subdivision of $M - \partial M$, we obtain an infinite singular chain $M^{\odot}$. The boundary of M is oriented, and becomes a chain ∂M. We define

$$\mathfrak{L}_f(M) = KI(f\partial M, fM^{\odot}) = KI(fM^{\odot}, f\partial M), \tag{6.1}$$

if this is finite. If ∂M is compact, it will be finite, since f is proper.

We now give the definition without the help of the infinite chain $M^{\odot}$.

Lemma 5. *Let U be a neighborhood of ∂M in which f is one-one. Let M^* be a singular chain such that*

$$M - M^* \subset U. \tag{6.2}$$

Then (LC = looping coefficient)

$$\mathfrak{L}_f(M) = KI(fM^*, f\,\partial\, M) = LC(f\,\partial\, M^*, f\,\partial\, M). \tag{6.3}$$

For we can write $M^{\odot} = M^* + M'$ where $M' \subset U$; since f is one-one in U, $KI(fM', f\,\partial\, M) = 0$.

Lemma 6. *Let the situation be as in Lemma 2. Let σ be an oriented n-cell, lying in the coordinate system, and obtained from the sphere $\sum x_i^2 = \beta^2$ plus interior by cutting off the part with $x_1 > \alpha$, where $0 < \alpha < \beta$. Then for n odd, $\mathfrak{L}(\sigma) = 1$ or -1 according as the singular point is positive or negative, while for n even, $\mathfrak{L}(\sigma) = 1$ or -1 according as the x_1-axis extends in the positive or negative direction in M.*

Remarks. It is easily seen that for n odd, $\mathfrak{L}(\sigma_1)$ is the same as $\mathfrak{L}(\sigma)$ if σ_1 is obtained from the sphere by cutting off the part $x_1 < -\alpha$; this follows also directly from Theorem 4. In the proof of Theorem 6 for n even, we need only the obvious fact that $\mathfrak{L}(\sigma) = \pm 1$.

Let $p = (-\alpha, 0, \cdots, 0)$, $q = (\alpha, 0, \cdots, 0)$. The only intersection of $f(\partial M)$ with $f(M^{\odot})$ is $f(p) = f(q)$. It is clear from Lemma 3 that we may suppose that the coordinate systems determine the positive orientations of M (or σ) and of E. Now the result of Lemma 4 holds, so that it is sufficient to show that $\mathfrak{L}(\sigma) = -1$.

Let $e_1, \cdots, e_n$ be the unit vectors in E^n; these determine the positive orientation of σ, while at q, the vectors $e_2, \cdots, e_n$ determine the positive orientation of $\partial\sigma$. We must show that the vectors

$$\left.\frac{\partial f}{\partial x_1}\right|_p, \cdots, \left.\frac{\partial f}{\partial x_n}\right|_p, \left.\frac{\partial f}{\partial x_2}\right|_q, \cdots, \left.\frac{\partial f}{\partial x_n}\right|_q \tag{6.4}$$

determine the negative orientation of E^{2n-1}. The two sets of vectors give the matrices

$$\left\| \begin{matrix} -2\alpha & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & -\alpha & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & -\alpha \end{matrix} \right\|, \qquad \left\| \begin{matrix} 0 & 1 & \cdots & 0 & \alpha & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & \alpha \end{matrix} \right\|.$$

Putting the second below the first forms a determinant D which we must prove negative. Subtracting the i^{th} row from the $(n + i - 1)^{\text{th}}$ row $(i = 2, \cdots, n)$ gives a determinant with zeros below the diagonal, whose value is

$$D = (-2\alpha)(2\alpha)^{n-1} = -(2\alpha)^n < 0,$$

as required.

7. The self-intersection of an n-complex mapped into E^{2n-1}

We shall consider mappings of a finite n-complex K into E^{2n-1} *which are one-one in* U for some neighborhood U of K^{n-1}. Of course K might be a subcomplex of a complex of higher dimension. We note that any mapping may be approximated to by one of the required type, even if the cells of K are replaced by more general bounded smooth manifolds; see §16. The considerations will be considerably generalized in Part III.

Let K' be a subdivision of K such that any cell of K' with a vertex in K^{n-1} lies in U. For each oriented σ_i^n of K, let σ_i^* be the sum of the similarly oriented n-cells of K' in σ_i^n which do not touch $\partial\sigma_i^n$. For any chain $A^n = \sum a_i\sigma_i^n$ set $A^* = \sum a_i\sigma_i^*$. The coefficients a_i are integers.

Definition. Generalizing the definition in §6, we set

$$\mathfrak{L}_f(A^n) = KI(fA^*, f\partial A^n) = LC(f\partial A^*, f\partial A^n). \tag{7.1}$$

Note that, if $M = \sum \sigma_i^n$, then $M^* = \sum \sigma_i^*$, which is not the M^* previously used; but the two definitions of $\mathfrak{L}_f(M)$ agree, as is apparent from (8.1) below.

Lemma 7. *Under the above conditions, we have the point set relations*

$$f(\partial\sigma_i^n) \cap f(\partial\sigma_i^*) = 0, \tag{7.2}$$

$$f(\sigma_i^n - \sigma_i^*) \cap f(\partial\sigma_j^*) = 0 \quad for \quad i \neq j. \tag{7.3}$$

For $\partial\sigma_i^n$ and $\partial\sigma_i^*$, also $\sigma_i^n - \sigma_i^*$ and $\partial\sigma_j^*$, are disjoint point sets in U.

Theorem 1. *Let the mapping f of K into E^{2n-1} be one-one in $K^n \cap U$, U a neighborhood of K^{n-1} in K. Then for any n-chain $\sum a_i\sigma_i^n$,*

$$\mathfrak{L}_f(\textstyle\sum a_i\sigma_i^n) = \sum a_i^2\mathfrak{L}_f(\sigma_i^n) \qquad \textit{for } n \textit{ odd}, \tag{7.4}$$

$$\mathfrak{L}_f(\textstyle\sum a_i\sigma_i^n) \equiv \sum a_i\mathfrak{L}_f(\sigma_i^n) \pmod 2 \qquad \textit{for } n \textit{ even}. \tag{7.5}$$

Remark. We could allow double points $f(p) = f(q)$ with both p and q in

K^{n-1}, for this would not destroy the relations (7.2) and (7.3). But $\mathfrak{L}_f$ would not then be invariant under slight deformations.

We may suppose $K = K^n$ in the proof. Set $\tau_i^n = f(\sigma_i^n)$, $\tau_i^* = f(\sigma_i^*)$. First, by (7.2) and (7.3), $KI(\tau_i^*, \partial\tau_j^n)$ has meaning for all i and j. Hence

$$\mathfrak{L}_f(\sum_i a_i\sigma_i^n) = KI(\sum_i a_i\tau_i^*, \partial \sum_j a_j\tau_j^n) = \sum_{i,j} a_i a_j KI(\tau_i^*, \partial\tau_j^n)$$
$$= \sum_i a_i^2 KI(\tau_i^*, \partial\tau_i^n) + \sum_{i<j} a_i a_j[KI(\tau_i^*, \partial\tau_j^n) + KI(\tau_j^*, \partial\tau_i^n)].$$

Now by (7.3), $\partial\tau_j^*$ does not intersect $\tau_i^n - \tau_i^*$ if $i \neq j$, and $\partial\tau_j^n - \partial\tau_j^*$ bounds a chain $\tau_j^n - \tau_j^*$ which does not intersect the boundary of τ_i^*. Hence, by elementary properties of the Kronecker index,

$$KI(\tau_j^*, \partial\tau_i^n) = (-1)^n KI(\partial\tau_j^*, \tau_i^n) = (-1)^n KI(\tau_i^n, \partial\tau_j^*)$$
$$= (-1)^n KI(\tau_i^*, \partial\tau_j^*) = (-1)^n KI(\tau_i^*, \partial\tau_j^n).$$

Consequently

$$\mathfrak{L}_f(\sum_i a_i\sigma_i^n) = \sum_i a_i^2 \mathfrak{L}_f(\sigma_i^n) + [1 + (-1)^n] \sum_{i<j} a_i a_j KI(\tau_i^*, \partial\tau_j^n), \tag{7.6}$$

from which the theorem follows.

8. Relation between singularities and self-intersections in M

We are now ready to prove the fundamental combinatorial theorem in the orientable case. It will be the primary object of Part III to prove the theorem in the non-orientable case; if n is odd, we need a new definition, (20.5), of $\mathfrak{L}_f(M)$, and a slight further restriction on f.

THEOREM 2. *Let f be a semi-regular mapping of the compact partial manifold M^n into E^{n-1}. Then for n odd, $\mathfrak{L}_f(M)$ is the algebraic number of singular points, while for n even, it is congruent to this number* mod 2.

REMARK. We could find an exact expression for $\mathfrak{L}_f(M)$ when n is even with the help of the classification of self-intersections in §22.

First replace f by a completely semi-regular mapping (Lemma 1), which we again call f. Let $p_1, \cdots, p_\mu$ be the singular points. About each p_i choose a cell σ_i^n as in Lemma 6. We may let these be cells of a subdivision of M (which need not be simplicial) into smooth cells.[1] Moreover, by first deforming the cells of the subdivision slightly so that $(n-1)$-cells touch the curves of self-intersection in interior isolated points only, and then deforming slightly again, we may obtain a subdivision K such that f is one-one over K^{n-1}. By Lemma 25, it is one-one in a neighborhood of K^{n-1}. Also, since $M - \sum_{i=1}^{\mu} \sigma_i^n$ contains no singular points, we may suppose that f is one-one in each σ_i^n, $i > \mu$.

Define K' and the σ_i^* as in §7, and M^* as in §6. Now

$$f(\partial M) \cap f[\sum(\sigma_i^n - \sigma_i^*) \cap M^*] = 0,$$

[1] Rather than prove this fact, we could easily construct a subdivision containing cells $\sigma_i'^n$ approximately like the cells σ_i^n, and note that $\mathfrak{L}(\sigma_i'^n) = \mathfrak{L}(\sigma_i^n)$.

and hence

$$KI(fM^*, f\partial M) = KI(\textstyle\sum \sigma_i^*, f\partial M). \tag{8.1}$$

Applying theorem 1 gives

$$\begin{aligned} \mathfrak{L}_f(M) &= \sum_{i=1}^{\mu} \mathfrak{L}_f(\sigma_i^n) && \text{for } n \text{ odd,} \\ \mathfrak{L}_f(M) &\equiv \sum_{i=1}^{\mu} \mathfrak{L}_f(\sigma_i^n) \pmod 2 && \text{for } n \text{ even,} \end{aligned} \tag{8.2}$$

since $\mathfrak{L}(\sigma_i^n) = 0$ for $i > \mu$, as f is one-one there. The theorem now follows from Lemma 6.

THEOREM 3. *Let f be a semi-regular mapping of the closed manifold M^n into E^{2n-1} Then for n odd, the algebraic number of singular points vanishes, while for n even, it vanishes* mod 2.

This is a corollary of the last theorem.

EXAMPLE. For $n = 1$, the theorem says that a (semi-regular) real-valued function defined on a circle has the same number of maxima as minima.

We give finally an invariance theorem whose proof requires Lemma 9 below.

THEOREM 4. *Let f_t be a deformation of the compact partial manifold M such that f_0 and f_1 are semi-regular, and for some neighborhood U of ∂M, each f_t is regular in U. Then if n is odd, f_0 and f_1 have the same algebraic number of singular points, while for n even, they have the same number* mod 2.

The hypothesis on f_t shows that each $\mathfrak{L}_{f_t}(M)$ may be defined as in §9. By Lemma 9, it is constant for n odd, and is constant (mod 2) for n even. The theorem now follows from Theorem 2.

9. Looping coefficients of vector fields in manifolds in space

DEFINITIONS. Let K^r be a finite smooth complex in E^{2r+1}, and let $v(p)$ be a continuous vector field in K^r, independent of K^r. Then there is an $\epsilon_0 > 0$ with the following property. For every ϵ, $0 < \epsilon < \epsilon_0$, if

$$\phi_{v,\epsilon}(K) = \text{all } p + \epsilon v(p), \qquad p \in K, \tag{9.1}$$

then $K \cap \phi_{v,\epsilon}(K) = 0$. Now for any chain A^r in K, $\phi_{v,\epsilon}A$ is a chain in $\phi_{v,\epsilon}(K)$. For cycles A^r, we define

$$\mathfrak{L}(A, v) = LC(\phi_{v,\epsilon}A, A). \tag{9.2}$$

Because of (9.1) it is clear that the result is independent of the choice of ϵ. The definition will be extended in §14.

The following lemma gives the relation to the previous $\mathfrak{L}$.

LEMMA 8. *Let M^n be a partial orientable manifold, and let f be a semi-regular mapping of M^n into E^{2n-1}. Let $v(p)$ be defined in $f(\partial M)$, and independent of $f(\partial M)$, and point into $f(M)$ at each p. Then*

$$\mathfrak{L}_f(M) = \mathfrak{L}(f\partial M, v). \tag{9.3}$$

To prove the lemma, we note that for ϵ sufficiently small, the set of all $\phi_{v,\epsilon'}(\partial M)$ for $0 \leqq \epsilon' \leqq \epsilon$ projects in a one-one manner into a subset M' of M, and M' contains all of M within some neighborhood of ∂M. Setting $M^* = M - M'$ defines a chain M^* to which Lemma 5 applies. Clearly

$$\mathfrak{L}_f(M) = LC(f\partial M^*, f\partial M) = LC(\phi_{v,\epsilon}\partial M, f\partial M) = \mathfrak{L}(f\partial M, v).$$

Let f be an imbedding of the r-manifold M^r in E^{2r+1}, and let $v(p)$ be defined so that $v(p)$ is independent of $f(M)$ at $f(p)$. If deformations f_t and v_t are given so that each f_t is an imbedding and each v_t is independent of $f_t(M)$, clearly $\mathfrak{L}(fM, v)$ is constant. The next lemma (used in the proof of Theorem 4) generalizes this.

Lemma 9. *Let f_t be a regular deformation of M^r in E^{2r+1} such that f_0 and f_1 are imbeddings. Let v_t be a continuous vector field independent of $f_t(M)$ for each t. Then*

$$\mathfrak{L}(f_1M, v_1) = \mathfrak{L}(f_0M, v_0) \qquad \textit{for } r \textit{ even},$$

$$\mathfrak{L}(f_1M, v_1) \equiv \mathfrak{L}(f_0M, v_0) \pmod 2 \qquad \textit{for } r \textit{ odd}.$$

Remark. The lemma clearly holds if M^r is a cycle in a complex K^n, f_t being a regular deformation in each closed $\bar{\sigma}_i^n$. The crossings (see below) may be taken interior to n-cells $\sigma_i^n = U_i$ of K.

It is easily seen by the methods in [1] that a slight alteration of f_t for $0 < t < 1$ will give a new f_t with the following property. If, for a certain t_1, f_{t_1} is not one-one, say $q_0 = f_{t_1}(p_1) = f_{t_1}(p_2)$, then this is the only double point, and the portions U_1 and U_2 of M near p_1 and p_2 are crossing each other as t moves through t_1. That is, if $u_{i1}, \cdots, u_{ir}$ are independent vectors tangent to $f_{t_1}(U_i)$ at q $(i = 1, 2)$, and

$$u' = \left.\frac{\partial f_t(p_1)}{\partial t}\right|_{t=t_1} - \left.\frac{\partial f_t(p_2)}{\partial t}\right|_{t=t_1},$$

then these $2r + 1$ vectors are independent.

Take t' and t'' very close to t_1, with $t' < t_1 < t''$. Since $\mathfrak{L}$ is constant over intervals containing no such t_1, it is sufficient to prove that the relations hold with $f_{t'}$ and $f_{t''}$ replacing f_0 and f_1. We may clearly accomplish the deformation f_t in two steps: first, push $f_{t'}(U_1)$ in the direction of u' so that it crosses $f_{t'}(U_2)$ (pushing $f_{t'}(p_1)$ a distance $(t'' - t')u'$); second, deform the result into $f_{t''}(M)$. We may replace v_t by $v_{t'} = v$ in all this if t' and t'' are close enough together. We now need merely prove the relations for the mappings before and after the first deformation, which we call g_t, using $0 \leqq t \leqq 1$.

Set

$$M_t = g_t(M), \qquad N_t = \phi_{v,\epsilon}(M_t) \qquad (0 \leqq t \leqq 1),$$

with a small $\epsilon > 0$. Except in U_1, these are independent of t. They define singular chains A and B such that (with M oriented)

$$M_1 = M_0 + \partial A, \qquad N_1 = N_0 + \partial B.$$

We find

$$\begin{aligned}\Delta &= \mathfrak{L}(M_1, v) - \mathfrak{L}(M_0, v) = LC(N_1, M_1) - LC(N_0, M_0)\\ &= LC(\partial B, M_1) + LC(N_0, M_1 - M_0)\\ &= LC(\partial B, M_1) + (-1)^{r+1}LC(\partial A, N_0)\\ &= KI(B, M_1) + (-1)^{r+1}KI(A, N_0).\end{aligned}$$

Since $u_{11}, \cdots, u_{1r}, u'$ are independent, we may suppose $A \cap B = 0$. Also g_0 is one-one in $M - U_2$. Hence we may clearly suppose

$$KI(B, g_1(M - U_2)) = KI(A, \phi_{v,\epsilon}g_0(M - U_2)) = 0.$$

Therefore

$$\Delta = KI(B, g_1U_2) + (-1)^{r+1}KI(A, \phi_{v,\epsilon}g_0U_2) = [1 + (-1)^{r+1}]KI(A, g_0U_2),$$

which proves the lemma.

II. The immersion theorem

10. Some deformations related to certain vector fields

We first show when one vector field in $M^r \subset E^{2r+1}$ may be deformed into another one. In Lemma 11 we show how the boundary of a partial manifold may be moved over to a desired position, and in Lemma 12, we show how the boundary may be twisted to point in given directions.

LEMMA 10. *Let $M^r \subset E^{2r+1}$ $(r \geqq 1)$ be a connected closed orientable manifold, and let v_0 and v_1 be continuous vector fields in M, each independent of M. Then there is a deformation $v_t (0 \leqq t \leqq 1)$ of v_0 into v_1 so that each v_t is independent of M, if and only if $\mathfrak{L}(M, v_0) = \mathfrak{L}(M, v_1)$.*

REMARK. If the normal bundle of M^r in E^{2r+1} is simple,[2] the proof is easy to give.

The necessity of the condition in the theorem is clear; we shall prove the sufficiency. We first deform v_0 and v_1 into fields of unit normal vectors. Next, let K be a simplicial complex forming a fine enough subdivision of M so that each cell of K is nearly flat, and so that v_0 is nearly constant in each cell. Now as considerations of dimensionality show at once, we may deform v_1 so that $v_1 = v_0$ in K^{r-1}. We now suppose v_0 and v_1 are of this nature.

Let $\{\sigma_i^r\}$ denote the similarly oriented r-cells of K. For each $p \in M$, let $S(p)$ denote the unit r-sphere about p whose plane is normal to M at p, and let $S'(p) \subset S(p)$ denote the subsphere orthogonal to $v_0(p)$. Let S_0^r denote a fixed r-sphere, and q_0, a fixed point of it. For each σ_i^r we introduce a "coordinate system"

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[2] See [3]. The present lemma belongs properly in the subject considered there. Some of the details omitted in the present proof may be found there. In the present paper we need only the case that M is a sphere. By cutting it into two cells and using the theorem that any sphere-bundle over a cell is simple, the proof could be materially simplified in this case.

into the $S(p)$ as follows. For each $p \in \bar{\sigma}_i^r$ and each $q \in S_0^r$, $\xi_i(p, q)$ is a point of $S(p)$; for each p, it is an orthogonal (distance preserving) mapping, and this mapping is continuous in p; furthermore, $\xi_i(p, q_0) = v_0(p)$. (Since the σ_i^r are nearly flat, it is easy to construct ξ_i first over the part S_0^{r-1} of S_0^r orthogonal to q_0, so as to map into the $S'(p)$; it is then uniquely extendable over S_0^r.) Let

$$\xi_i^{-1}(p, q') = q \quad \text{if} \quad q' = \xi_i(p, q).$$

If we orient M and S_0^r, orient the $S(p)$ so that the orientations of M and $S(p)$ at p determine the positive orientation of E, and choose the ξ_i so that they are rotations (i.e. sense-preserving), then each $\xi_i^{-1}(p, \xi_j(p, q))$ for each p will be a rotation.

Set

$$\psi_i(p) = \xi_i^{-1}(p, v_1(p)). \tag{10.1}$$

Since $v_1(p) = v_0(p)$ in $\partial\sigma_i^r$, $\psi_i(p) = q_0$ there. Hence ψ_i maps σ_i^r into S_0^r so that $\partial\sigma_i^r$ goes into q_0, and thus ψ_i has a degree d_i over σ_i^r. Set

$$X(v_1) = \sum d_i\sigma_i^r. \tag{10.2}$$

Since $\dim(K) = r$, this is a cocycle.

Suppose v_1 is deformed as follows. Take any σ_j^{r-1}; $\psi_i(p) = q_0$ here. As t runs from 0 to 1, let $\psi_{it}(\sigma_j^{r-1})$ sweep over S_0^r with the degree α_j, keeping $\psi_{it}(\partial\sigma_j^{r-1}) = q_0$ and $\psi_{i1}(\sigma_j^{r-1}) = q_0$. (Thus if I is the unit interval $0 \leqq t \leqq 1$, and $\Psi_i(t, p) = \psi_{it}(p)$, Ψ_i maps $I \times \sigma_j^{r-1}$ into S_0^r with the degree α_j.) We may extend[3] ψ_{it} over the rest of M, requiring that it be independent of t except in the cells of $St(\sigma_j^{r-1})$. Set

$$v_t'(p) = \xi_i(p, \psi_{it}(p)), \qquad p \in \bar{\sigma}_i^r, \text{ each } i;$$

then $v_0'(p) = v_1(p)$, and each v_t' is a field of unit normal vectors. Let d_i' denote the degree defined with the help of v_1'. Then clearly for any σ_i^r,

$$d_i' = d_i + [\sigma_j^{r-1}:\sigma_i^r]\alpha_j,$$

and hence

$$X(v_1') = X(v_1) + \alpha_j\delta\sigma_j^{r-1}.$$

Since we may carry out this process for each σ_j^{r-1} in turn, we may alter X by any coboundary.

For a small $\epsilon > 0$, if $\omega_k(\sigma_i^r)$ denotes the cell σ_i^r displaced in the direction of v_k a distance ϵ (this mapping need not be one one), then

$$\mathfrak{L}(M, v_k) = LC(\sum \omega_k\sigma_i^r, M) \qquad (k = 1, 2). \tag{10.3}$$

We shall show that

$$LC(\omega_1\sigma_i^r - \omega_0\sigma_i^r, M) = d_i. \tag{10.4}$$

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[3] See ALEXANDROFF-HOPF, *Topologie I*, Berlin, 1936, p. 501 Hilfsatz Ia.

This quantity is defined, since $v_0 = v_1$ in $\partial\sigma_i^r$. Let us flatten σ_i^r into σ', lying in a space E^r. A slight alteration of v_0 will make it constant in $\bar{\sigma}_i^r$. Let E^{r+1} be a plane through a point $p_0 \in \sigma_i^r$, orthogonal to E^r, and consider S_0^r as the unit sphere in E^{r+1} about p_0. If we project the chain $\omega_1\sigma_i^r - \omega_0\sigma_i^r$ parallel to E^r into E^{r+1}, and then in E^{r+1} away from p_0 into S_0^r, $\omega_0\sigma_i^r$ will go into a point, say q_0, and $\omega_1\sigma_i^r$ will go into a chain $\omega'\sigma_i^r$. Now

$$LC(\omega_1\sigma_i^r - \omega_0\sigma_i^r, E^r) = LC(\omega'\sigma_i^r, E^r).$$

This also equals $LC(\omega'\sigma_i^r, p_0)$, considering this as defined in E^{r+1} (which is oriented like $S(p_0)$). We may suppose the ξ_i chosen so that after the above alterations and projection, $v_1(p) + p$ becomes the point $\psi_i(p)$. Hence

$$LC(\omega'\sigma_i^r, E^r) = d_i.$$

Interpreting the looping coefficients as Kronecker indices shows that the looping coefficients with E^r are the same as with M. Thus (10.4) is proved.

Adding the equations (10.4) gives

$$\begin{aligned}\sum d_i &= LC(\sum \omega_1\sigma_i^r - \sum \omega_0\sigma_i^r, M)\\ &= \mathfrak{L}(M, v_1) - \mathfrak{L}(M, v_0) = 0.\end{aligned}$$

Since M is closed, connected and oriented, there is a one-one correspondence h between the cohomology classes of dimension r of M and the integers, given by

$$h(\sum \alpha_i\sigma_i^r) = M\cdot\sum \alpha_i\sigma_i^r = \sum \alpha_i.$$

It follows that

$$h(X(v_1)) = \sum d_i = 0, \qquad X \backsim 0.$$

Consequently we may deform v_1 into v_1' so that $X(v_1') = 0$, i.e. $d_i' = 0$ for each i. Now by a theorem of Hopf,[4] we may deform ψ_i' into q_0 in each $\bar{\sigma}_i^r$, keeping it at q_0 in $\partial\bar{\sigma}_i^r$. This defines a corresponding deformation of v_1' into v_0 in $\bar{\sigma}_i^r$, keeping it fixed in $\partial\sigma_i^r$. Thus v_1 is deformed into v_0 in M, and the lemma is proved.

LEMMA 11. *Let f_0 and f_1 be imbeddings of the manifold M^r in E^ν. Let $L(p)$ be the segment $f_0(p)f_1(p)$. Let no two of these have common points. For each $p \in M$ let there be a plane $T(p) = T^{\nu-r}(p)$ in E^ν such that*

(a) *$f_0(p)$ and $f_1(p)$ are in $T(p)$,*

(b) *$T(p)$ has only $f_i(p)$ in common with the tangent plane to $f_i(M)$ at $f_i(p)$ ($i = 0, 1$),*

(c) *The function $T(p)$ is smooth (Compare* [1], §24).

Then there is a smooth deformation ϕ_t of E^ν ($0 \leqq t \leqq 1$) such that

(d) *each ϕ_t is an imbedding, and ϕ_0 is the identity,*

(e) *$\phi_1(f_0(p)) = f_1(p)$ ($p \in M$),*

(f) *for a given neighborhood U of the set of all segments $L(p)$, $\phi_t(p) = p$ for $p \in E^\nu - U$ and $0 \leqq t \leqq 1$.*

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[4] See ALEXANDROFF-HOPF, loc. cit., p. 504, Satz III$_n$, or H. WHITNEY, Duke Math. J., vol. 3 (1937), pp. 46–50, Appendix.

REMARKS. Any segment $L(p)$ may reduce to a single point $f_0(p) = f_1(p)$. If M is a partial manifold, and $f_0 = f_1$, together with first partial derivatives, in ∂M, the proof below holds, and each ϕ_t is the identity, together with first partial derivatives, at all points of ∂M. The most important application of the lemma is to the case $M^r = \partial M^n$; a given mapping f_0 of M^n is then altered to f_1, so that f_1 is a given mapping in ∂M. If the mappings are of class C^γ, we may make each ϕ_t of class C^γ.

Take any p_0 in M. By (a), (b) and (c), it is easy to see that for some neighborhood U_0 of p_0 and some $\alpha_0 > 0$, the set of points q in planes $T(p)$ with $p \in U_0$ which are within a distance α_0 of $L(p)$ fills out a neighborhood of $L(p_0)$ in E^ν in a smooth one-one way (compare the proof of [1], Lemma 21). Hence, since the $L(p)$ are distinct, there is a positive continuous function $\alpha(p)$ (or a constant $\alpha > 0$ if M is compact) such that if $R(p)$ is the set of points of $T(p)$ within a distance $\alpha(p)$ of $L(p)$, then the $R(p)$ fill out a neighborhood of $\sum L(p)$ in a smooth one-one-way, and $\sum R(p) \subset U$. We may choose a smooth function $\eta(p) > 0$ such that if $L'(p)$ is the segment $L(p)$ extended in each direction by the amount $\eta(p)$, and $C(p)$ is the cylinder (of dimension $\nu - r$) in $T(p)$ with axis $L'(p)$ and of radius $\eta(p)$, then $C(p) \subset R(p)$ for $p \in M$ (see [1], Lemma 25). It is easy to set up an expression depending smoothly on $\eta(p)$ and the (locally oriented) length of $L(p)$, which defines a smooth deformation of $R(p)$ into itself with the properties that it is constant in $R(p) - C(p)$, carries $f_0(p)$ into $f_1(p)$, and is an imbedding for each t. Letting this define ϕ_t in $\sum R(p)$ and setting $\phi_t(p) = p$ in $E^\nu - \sum R(p)$ proves the lemma.

LEMMA 12. *Let $M^n (n \geqq 2)$ be a partial manifold, let ∂M be a closed manifold, and let f be a mapping of class C^2 of M into E^{2n-1} such that in some neighborhood U of ∂M, f is an imbedding. Let $u(p)$ be a smooth vector field in ∂M, pointing into M at p in ∂M; set*

$$v(p) = \nabla f(u, p), \qquad p \in \partial M.$$

Let $v'(p)$ be a smooth vector field in $f(\partial M)$, independent of $f(\partial M)$, such that

$$\mathfrak{L}(f\partial M, v') = \mathfrak{L}(f\partial M, v).$$

Then there is a smooth mapping f' of M into E such that $f' = f$ in $M - U$, f' is an imbedding in U, $f' = f$ in ∂M, f' is arbitrarily close to f (but not together with first derivatives) in M, and

$$\nabla f'(u, p) = v'(p), \qquad p \in \partial M.$$

REMARKS. The assumption that ∂M is closed could be easily removed. A more accurate statement about the class of f and of f' could be given, but we shall not need it.

Since we need define f' in U only, we may consider U as lying in E^{2n-1}, and let f be the identity; then $v(p) = u(p)$. Set

$$p_t^* = p + tv(p), \qquad 0 \leqq t \leqq 1.$$

This is a smooth mapping of $I \times \partial M$ into E^{2n-1}. (Since f is of class C^2, v and p_t^* are of class C^1 in terms of the original coordinate systems in M.) For some $t_0 > 0$, this is an imbedding for the values $0 \leqq t \leqq t_0$, since u is smooth. For t_0 small enough, we may project p_t^* into U. Say p_t^* projects into p_t. Now the points of U near ∂M are uniquely expressible in the form p_t ($p \in \partial M$, $0 \leqq t \leqq t_0$), and

$$\left.\frac{\partial p_t}{\partial t}\right|_{t=0} = v(p).$$

By Lemma 10, there is a deformation $v_t'(p)$ $(0 \leqq t \leqq 1)$ of $v'(p) = v_0'(p)$ into $v(p) = v_1'(p)$, such that each v_t' is independent of ∂M. We may replace $v_t'(p)$ by a smooth function $v_t^*(p)$ as follows. First set

$$v_t''(p) = v_0'(p) \qquad (t \leqq \tfrac{1}{3}), \qquad v_t''(p) = v_1'(p) \qquad (t \geqq \tfrac{2}{3}),$$

$$v_t''(p) = v_{t'}'(p) \qquad (t' = 3(t - \tfrac{1}{3}), \tfrac{1}{3} \leqq t \leqq \tfrac{2}{3}).$$

Then $v_t''(p)$ is smooth except for $\frac{1}{3} \leqq t \leqq \frac{2}{3}$. Now approximate to $v_t''(p)$ by a smooth vector function $v_t^*(p)$ for $\frac{1}{6} \leqq t \leqq \frac{5}{6}$, the approximation being closer and closer, together with first partial derivatives, as $t \to \frac{1}{6}$ or $t \to \frac{5}{6}$. (See [1], Theorem 2, (a) and (d). We could either make use of Theorem III of the author's paper in Trans. Am. Math. Soc., vol. 36 (1934), pp. 63–89, using first derivatives for $t < \frac{1}{3}$ and $t > \frac{2}{3}$, or note simply that in the approximation in Lemma 6, loc. cit., with $m = 0$, the first partial derivatives have automatically the desired approximation property.) Setting $v_t^*(p) = v_t''(p)$ for $t \leqq \frac{1}{6}$ and $t \geqq \frac{5}{6}$ makes v_t^* smooth for $0 \leqq t \leqq 1$ (see [1], Lemma 10). Moreover, with a close enough approximation, $v_t^*(p)$ is independent of ∂M for $0 \leqq t \leqq 1$.

Set

$$v_t(p) = \frac{\partial p_t}{\partial t}.$$

For certain numbers α and β to be determined later, with $0 < \alpha < \beta < t_0$, set

$$p_t' = p + \int_0^t v_{s/\alpha}^*(p)\, ds \qquad (0 \leqq t \leqq \alpha), \tag{10.5}$$

$$p_t' = p + \int_0^\alpha v_{s/\alpha}^*(p)\, ds + \int_0^{t-\alpha} v_s(p)\, ds \qquad (\alpha \leqq t \leqq \beta). \tag{10.6}$$

Cover ∂M with a finite set of coordinate systems $\{x_i\}$. Let V be the maximum of $|v_t^*(p)|$, $|v_t(p)|$, $|\partial v_t^*(p)/\partial x_i|$, $|\partial v_t(p)/\partial x_i|$. Now

$$\frac{\partial p_t'}{\partial t} = v_{t/\alpha}^*(p) \qquad (t \leqq \alpha), \qquad \frac{\partial p_t'}{\partial t} = v_{t-\alpha}(p) \qquad (t \geqq \alpha);$$

since $v_1^*(p) = v_1'(p) = v(p) = v_0(p)$, the mapping thus defined is smooth. Also, since

$$\frac{\partial p_t'}{\partial x_i} = \frac{\partial p}{\partial x_i} + \int_0^t \frac{\partial v_{s/\alpha}^*(p)}{\partial x_i}\, ds \qquad (t \leqq \alpha),$$

and similarly for $t \geqq \alpha$, we find

$$\left| \frac{\partial p'_t}{\partial x_i} - \frac{\partial p}{\partial x_i} \right| \leqq V\beta \qquad (0 \leqq t \leqq \beta).$$

Since the $\partial p/\partial x_i$ are independent for each p, and $\partial p'_t/\partial t$ is independent of them, by choosing β small enough we may insure that the $\partial p'_t/\partial x_i$ and $\partial p'_t/\partial t$ are independent for each p and t; hence the mapping is regular. Moreover, since

$$p_t = p + \int_0^t v_s(p)\, ds,$$

we have

$$p'_\beta - p_\beta = \int_0^\alpha v^*_{s/\alpha}(p)\, ds - \int_{\beta-\alpha}^\beta v_s(p)\, ds,$$

and

$$| p'_\beta - p_\beta | \leqq 2V\alpha, \qquad \left| \frac{\partial p'_\beta}{\partial x_i} - \frac{\partial p_\beta}{\partial x_i} \right| \leqq 2V\alpha,$$

$$\left| \frac{\partial p'_t}{\partial t} - \frac{\partial p_t}{\partial t} \right|_\beta = | v_{\beta-\alpha}(p) - v_\beta(p) | .$$

Hence, keeping β fixed, we may choose α so small that the mapping p'_t at $t = \beta$ is arbitrarily close to that of p_t at β, together with first derivatives.

Set

$$\phi(t, p) = p'_t - p_t \qquad (t \leqq \beta), \qquad\qquad \phi(t, p) = 0 \qquad (t \geqq t_0).$$

This is a mapping of the part of U outside $\beta < t < t_0$, which we have just seen may be taken arbitrarily small, with first derivatives. Hence, by [5], there is a smooth extension of ϕ through $\beta \leqq t \leqq t_0$, which may be taken arbitrarily small, together with first derivatives. Setting

$$p'_t = p_t + \phi(t, p) \qquad (\beta < t < t_0)$$

completes the definition of f' in U. By making ϕ and its first derivatives small enough, we insure that we have a close approximation to f, and that the new mapping is regular. Since

$$\left. \frac{\partial p'_t}{\partial t} \right|_{t=0} = v^*_0(p) = v'_0(p) = v'(p),$$

we have $\nabla f'(u(p), p) = v'(p)$, completing the proof.

11. A twisted cube

We wish to show how, for n even, an n-cube in E^{2n-1} can be slightly altered in position so that, on one face, there will be a "double twist".

THEOREM 5. *Let M_0 be an n-cube in $E^n \subset E^{2n-1}$, n even and $\geqq 2$, and let N_0 be one of its faces. Then there is an immersion f of M_0 in E^{2n-1} with the following properties:*

(1) *f is arbitrarily near the identity Θ.*

(2) *$f = \Theta$ in ∂M_0.*

(3) *$f = \Theta$, together with first derivatives, in $\partial M_0 - N_0$.*

(4) *$\mathfrak{L}_f(M_0) = 2$ or -2 at will.*

It would be easy to make f of class C^∞.

That it is possible to have $\mathfrak{L}_f(M_0) \neq 0$ can be seen at once as follows. Take the mapping of an $(n - 1)$-cube into E^{2n-2} with just one self-intersection as defined in the preceding paper; translating E^{2n-2} in E^{2n-1} gives a mapping f of M_0 into E^{2n-1} with a line of self-intersections, and with ∂M_0 intersecting itself in two points. Make slight deformations so as to remove the self-intersections of ∂M_0. Since n is even, it is easily seen that pulling ∂M_0 away from itself at one of these points in opposite directions has the opposite effect on $\mathfrak{L}_f(M_0)$; hence we may obtain $\mathfrak{L}_f(M_0) \neq 0$.

We must show how a mapping may be obtained to have also the remaining properties. We shall first describe geometrically the case $n = 2$. Take a long rectangle of paper, carry the right hand end up, towards the left, down (cutting through itself), and to the right again; it will then be approximately in its original position, except for the presence of a somewhat cylinder shaped portion near the middle. This mapping may be defined by

$$(11.1) \qquad y_1 = x_1 - \frac{2x_1}{1 + x_1^2}, \qquad y_2 = \frac{1}{1 + x_1^2}, \qquad y_3 = x_2,$$

the y_1-axis pointing East (to the right), the y_2-axis up, and the y_3-axis South. By pulling the right half fairly taut, and a little to one side, the cylindrical piece is made very narrow, and is pulled to a sharp angle, say to about 12° from the direction from left to right. This renders the two long edges nearly straight again. (If a thin strip is cut off one of the long edges, it is found to have no self-intersections, and may be formed from a straight strip by simply twisting one end.)

The two long edges are now made into straight lines by a slight distortion of 3-space. A contraction in one direction turns the edge (now a rectangle) into a square. The resulting mapping has all the required properties except that there is a twist along two edges instead of along only one. Let us round off the corners. We could now either curl over all the right hand edge (see the proof below), or greatly contract the lower and right hand portions, pulling one twisted part of the edge all the way around the right hand end to a position near the other twisted part (see the figure). The figure shows all these operations except for the straightening of the wavy edge.

We turn now to the general case. By analogy with the above, we shall take the self-intersection defined in the preceding paper, for E^{n-1} mapped into E^{2n-2}, and translate E^{2n-2} in E^{2n-1} but moving it at an angle $\theta = \tan^{-1}(1/10)$ instead of $\pi/2$.

Set

$$u' = (1 + x_1^2) \cdots (1 + x_{n-1}^2),$$

$$(11.2) \qquad \begin{aligned} y_1 &= x_1 - \frac{2x_1}{u'}, & y_i &= x_i & (i &= 2, \cdots, n-1), \\ y_n &= \frac{1}{u'}, & y_{n+i-1} &= \frac{x_1 x_i}{u'} & (i &= 2, \cdots, n-1), \end{aligned}$$

$$y_{2n-1} = x_n + 10x_1 .$$

For x_n fixed, we obtain the mapping referred to. Since that mapping is regular, and x_n appears in y_{2n-1} only, the present mapping is regular. Let us call it f_0. The self-intersections are:

$$(11.3) \qquad f_0(1, 0, \cdots, 0, \alpha - 10) = f_0(-1, 0, \cdots, 0, \alpha + 10).$$

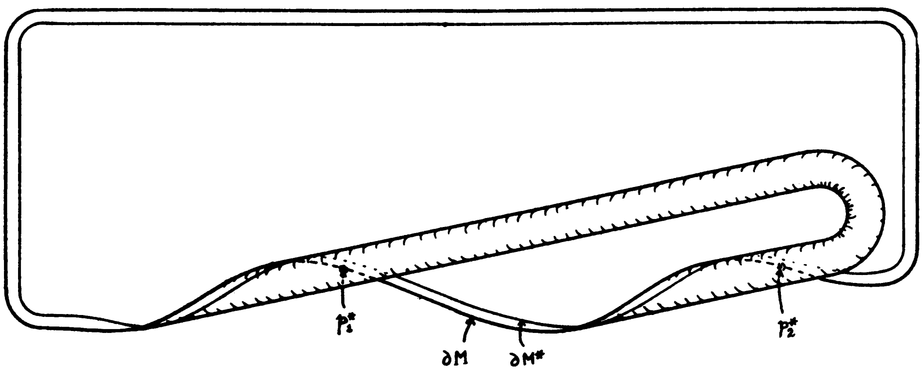

FIG. 2

Let M_1 be the part of E^n defined by $| x_n | \leqq 100$. Consider the affine mapping f' of M_1 into E^{2n-1} defined by omitting the terms in (11.2) containing u':

$$(11.4) \qquad y_i = x_i, \qquad y_{n+i-1} = 0 \qquad (i = 1, \cdots, n-1),$$

$$y_{2n-1} = x_n + 10\, x_1 .$$

Then f_0 is close to f', together with first partial derivatives, except near the x_n-axis. (Taking $n = 2$, the reader is advised to plot the parallelogram $y_1 = x_1$, $y_3 = x_2 + 3x_1$, for $| x_1 | \leqq 3$ and $| x_2 | \leqq 9$.)

Our next object is to replace the mapping f_0 by a mapping f_2 with the same kind of self-intersections, and such that for any $p = (x_1, \cdots, x_n)$, and some a,

$$(11.5) \qquad f_2(p) = f'(p) \quad \text{if} \quad | x_n | = 100 \quad \text{or} \quad \sum_{i=1}^{n-1} x_i^2 \geqq a^2.$$

We shall do this by defining a deformation ϕ_t of E^{2n-1}, and setting

$$f_t(p) = \phi_t(f_0(p)) \qquad (0 \leqq t \leqq 2). \tag{11.6}$$

Let N_1^+ and N_1^- be the parts of ∂M_1 with $x_n = 100$ and $x_n = -100$ respectively. For each $q \in E^{2n-1}$, let $T(q)$ be the n-plane containing q which is parallel to the axes of $y_1, y_n, y_{n+1}, \cdots, y_{2n-2}$. We shall define ϕ_t in two parts. First, ϕ_t $(0 \leqq t \leqq 1)$ will carry each point $q \mp f_0(p)$ near $f'(\partial M_1)$ in the plane $T(q)$ into $f'(p)$, and will be the identity in E^{2n-1} outside a neighborhood of $f'(\partial M_1)$. Then $\phi_1(q)$ is near q if q is far from the y_{2n-1}-axis. Consequently it is easy to define ϕ_t $(1 \leqq t \leqq 2)$ so as to make (11.5) hold.

First, note that

$$\lambda(u) = \lambda(u; a, b) = 1 - 3\left(\frac{u-a}{b-a}\right)^2 + 2\left(\frac{u-a}{b-a}\right)^3 \qquad (a \leqq u \leqq b)$$

has the properties

$$\lambda(a) = 1, \qquad \lambda'(a) = \lambda(b) = \lambda'(b) = 0;$$

hence, if $\lambda = 1$ for $x < a$ and $\lambda = 0$ for $x > b$, λ is smooth. The maximum derivative of $-\lambda$ is at $x = \frac{1}{2}(a + b)$, and has the value $3/[2(b - a)]$.

For each $q = (y_1, \cdots, y_{2n-1})$ in E^{2-1} there is a unique point

$$\begin{aligned} \pi^+(q) &= (x_1, y_2, \cdots, y_{n-1}, 0, \cdots, 0, y_{2n-1}) \qquad \left(x_1 = \frac{y_{2n-1} - 100}{10}\right) \\ &= f'(x_1, y_2, \cdots, y_{n-1}, 100) \end{aligned} \tag{11.7}$$

in $f'(N_1^+) \cap T(q)$. Also,

$$\sigma^+(q) = f_0(x_1, y_2, \cdots, y_{n-1}, 100) \tag{11.8}$$

is in $T(q)$, and is clearly the only point of $f_0(N_1^+)$ in $T(q)$. Set

$$\rho^+(q) = |\sigma^+(q) - q|,$$

and

$$\phi_t(q) = q + t\lambda(\rho^+(q); 1, 9)[\pi^+(q) - \sigma^+(q)], \qquad \rho^+(q) \leqq 9. \tag{11.9}$$

Thus for any $q_0 = f_0(p_0) \in f_0(N_1^+)$ and any q in $T(q_0)$ within a distance 1 of q_0, ϕ_1 moves q by that vector which carries $f_0(p_0)$ into $f'(p_0)$; if q is at a distance 9 from q_0, then $\phi_t(q) = q$.

Set $\omega = y_{2n-1} - 10y_1$, and define the half-spaces

$$E^+: \omega > 0, \qquad E^-: \omega < 0.$$

Since $\omega = 100$ at points of $f'(N_1^+)$, and max $[2x_1/u'] = 1$, $\omega \geqq 99$ at points of $f_0(N_1^+)$. Hence if $\rho^+(q) \leqq 9$, $\omega(q) \geqq 9$, and $q \in E^+$. We may therefore, using N_1^- in place of N_1^+, define $\pi^-(q)$ etc., and define $\phi_t(q)$ for $\rho^-(q) \leqq 9$ so as to have corresponding properties. Setting $\phi_t(q) = q$ in the rest of E^{2n-1} completes the definition of ϕ_t for $t \leqq 1$.

By direct substitution, we find that for any $p = (x_1, \cdots, x_n)$, taking $x_n > 0$,

$$f_1(p) - f_0(p) = \lambda[\rho^+(p); 1, 9]v(p),$$

$$v(p) = \left(\frac{2x_1'}{u''}, 0, \cdots, 0, -\frac{1}{u''}, -\frac{x_1' x_2}{u''}, \cdots, -\frac{x_1' x_{n-1}}{u''}, 0\right),$$

$$x_1' = x_1 + \frac{x_n - 100}{10}, \qquad u'' = (1 + x_1'^2)(1 + x_2^2) \cdots (1 + x_{n-1}^2).$$

Hence, for any $p \in M_1$ with sufficiently large $x_1^2 + \cdots + x_{n-1}^2$, $f_1(p)$ is close to $f_0(p)$ and hence to $f'(p)$, together with first partial derivatives. Consequently, for some a, if $q = f_1(p)$ is at least a distance a from the y_{2n-1}-axis, and hence $y_1^2 + \cdots + y_{n-1}^2$ is large, then $x_1^2 + \cdots + x_{n-1}^2$ is large, and the above statement holds. For such values of q, set

$$(11.10) \qquad w(q) = f'(q) - f_1(q);$$

Then w and its first derivatives are small if q is at least a distance a from the y_{2n-1}-axis; it vanishes in $f'(\partial M_1) = f_1(\partial M_1)$. It follows that w may be extended over E^{2n-1} so that it is small everywhere, together with first partial derivatives, and vanishes in $f'(\partial M_1)$. (This fact may be proved as follows. By a contraction in each $(2n - 1)$-plane $y_{2n-1} - 10y_1 = \alpha$, we may bring the set A_a of points distant at least a from the y_{2n-1}-axis into A_1; then w is carried into w_a say, defined in $[f'(M_1) \cap A_1] \cup f'(\partial M_1)$. By taking a large enough, we may make w_a and its first partial derivatives arbitrarily small. We now apply the theorem of [5]—the fact that A_1 is not bounded is clearly inconsequential,—and reverse the above contraction.) We now set

$$(11.11) \qquad \phi_t(q) = q + (t - 1)w(q) \qquad (1 \leqq t \leqq 2).$$

Then each ϕ_t is an imbedding, and (11.5) holds.

Consider E^n as a subspace of E^{2n-1}. If we define the affine mapping of E^{2n-1}

$$\psi: y_i' = y_i \ (i = 1, \cdots, 2n - 2), \qquad y_{2n-1}' = y_{2n-1} - 10y_1,$$

this carries f_2 into a mapping f_3, where

$$f_3(p) = \psi(f_2(p)) \qquad (p \in M_1)$$

such that, by (11.5),

$$(11.12) \qquad f_3(p) = p \quad \text{if} \quad |x_n| = 10 \quad \text{or} \quad \sum_{i=1}^{n-1} x_i^2 \geqq a^2.$$

Next we shall change f_3 to f_4 so that f_4 = identity, together with first derivatives, at all points of N_1^-. Let M_2 be a partial manifold contained in M_1 and containing all points of M_1 with $x_1^2 + \cdots + x_{n-1}^2 \leqq a^2$. For instance, let M_2 be the set of all points of E^n whose distance from the $(n - 1)$-cell $x_n = 0, x_1^2 + \cdots + x_{n-1}^2 \leqq a^2$, is at most 100. Then $f_3(p) = p$ in ∂M_2. Let $u(p)$ be the inward normal at $p \in \partial M_2$. It is carried into a vector field $v(p)$ in E^{2n-1} by f_3. Let

$v'(p)$ be defined in ∂M_2 and lie in E^{2n-1}, let it equal $v(p)$ in $\partial M_2 - N_1^+ \cup N_1^-$, and let it be the inward normal in $\partial M_2 \cap N_1^-$; define it in $\partial M_2 \cap N_1^+$ so that

$$\text{(11.13)} \qquad \mathfrak{K}(\partial M_2, v') = \mathfrak{K}(\partial M_2, v).$$

Let f_4 be the mapping given by Lemma 12. Then $f_4(p) = p$ in ∂M_2, and setting $f_4(p) = p$ in $M_1 - M_2$ gives the required mapping f_4. We may let $f_4(p) = p$ if $\sum_{i=1}^{n-1} x_i^2 \geqq a^2$.

Now contract E^{2n-1} with a factor 100 in the y_{2n-1}-direction, and with a factor $b \geqq a$ in the other directions. This carries f_4 into a mapping

$$f(p) = \theta(f_4(\theta^{-1}(p))).$$

A certain rectangular parallelopiped M_b' lying in M_1 and containing M_2 is carried into the cube

$$M_0\colon \ |x_i| \leqq 1 \qquad (i = 1, \cdots; n).$$

Since $f_4(p) = p$ in $M_1 - M_a'$ and f_4 leaves y_{2n-1} unchanged, choosing b large enough makes f arbitrarily near the identity Θ in M_0. Clearly $f(p) = p$ in ∂M_0, and $\partial f(p)/\partial x_i = \partial\Theta(p)/\partial x_i$ in $\partial M_0 - N_0$, where $N_0 = \partial M_0 \cap \theta(N_1^+)$. There remains to prove (4) of the theorem.

Since a reflection in E^{2n-1} will cause a change in sign in $\mathfrak{K}$, it is sufficient to show that $\mathfrak{K}_f(M_0) = \pm 2$. Clearly

$$\mathfrak{K}_f(M_0) = \mathfrak{K}_{f_4}(M_b') = \mathfrak{K}_{f_4}(M_2).$$

Since $u(p)$ points into M_2, and f_4 carries $u(p)$ into $v'(p)$ while f_3 carries $u(p)$ into $v(p)$, Lemma 8 and (11.13) give

$$\mathfrak{K}_{f_4}(M_2) = \mathfrak{K}(\partial M_2, v') = \mathfrak{K}(\partial M_2, v) = \mathfrak{K}_{f_3}(M_2).$$

Since the changes from f_0 to f_2 and to f_3 in M_1, and hence in M_2, are obtainable by deformations of E^{2n-1}, which leave Kronecker indices unchanged,

$$\mathfrak{K}_{f_3}(M_2) = \mathfrak{K}_{f_0}(M_2).$$

Hence there remains to prove

$$\text{(11.14)} \qquad \mathfrak{K}_{f_0}(M_2) = KI(f_0 M_2^*, f_0 \partial M_2) = \pm 2.$$

The intersections of $f_0(\partial M_2)$ with $f_0(M_2)$ are:

$$p_1 = (-1, 0, \cdots, 0, 100), \qquad p_1^* = (1, 0, \cdots, 0, 80),$$

$$p_2 = (\ \ 1, 0, \cdots, 0, -100), \qquad p_2 = (-1, 0, \cdots, 0, -80),$$

$$p_i \in \partial M_2, \qquad p_i^* \in M_2^*, \qquad f_0(p_i) = f_0(p_i^*).$$

Let A and B be the parts of M_2 satisfying:

$$A\colon -2 \leqq x_1 \leqq -\tfrac{1}{2}, \qquad B_2\colon \tfrac{1}{2} \leqq x_1 \leqq 2,$$

respectively; we let these be chains, oriented like M_2. Then A and B contain neighborhoods of p_2^* and p_1^* in M_2^* respectively, and ∂A and ∂B contain neighborhoods of p_1 and p_2 in ∂M_2 respectively. Since $f_0(\partial A)$ and $f_0(\partial B)$ do not intersect B and A respectively at other points, and n is even, we have

$$\begin{aligned} KI(f_0M_2^*, f_0\partial M_2) &= KI(f_0B, f_0\partial A) + KI(f_0A, f_0\partial B) \\ &= KI(f_0B, f_0\partial A) + (-1)^n KI(f_0\partial A, f_0B) \\ &= [1 + (-1)^n] KI(f_0B, f_0\partial A) = \pm 2, \end{aligned}$$

completing the proof.

12. Proof of the immersion theorem

We can now prove the following theorem, with the help of Theorem 2; recall that that theorem is proved with the exception of the case M non-orientable, n odd.

THEOREM 6. *Given any n-manifold or partial manifold M ($n \geqq 2$) of class C^ν ($\nu \geqq 1$ finite or infinite), and a continuous mapping f of M into E^{2n-1}, there is an immersion g of M arbitrarily close to f, of class C^ν.*

REMARK. By Lemma 1, we may make the mapping completely semi-regular.

We suppose M is connected; otherwise, we would apply the theorem to each component of M separately. It is sufficient to find an immersion g of class C^1; for a sufficiently close approximation to g by a mapping g' of class C^ν is automatically an immersion also. We may suppose that M is of class C^{16}; if it were not, we could take a manifold or partial manifold M^* of class C^{16}, and an imbedding ϕ of M onto M^*, set $f^*(q) = f(\phi^{-1}(q))$ ($q \in M^*$), find an immersion g^* of M^*, and set $g(p) = g^*(\phi(p))$ ($p \in M$). Finally, by Lemma 1, we may suppose that f is of class C^{16} and semi-regular. We must now divide the proof into four cases.

CASE I. M is closed, and n is odd. By Theorem 3, we may name the singular points $p_1, p_1', \cdots, p_s, p_s'$ in such a manner that p_i and p_i' are of opposite types. If we show how to approximate to f by a function g' in which p_1 and p_1' are no longer singular points, and with no new singular points, a repetition of this process gives a function g approximating to f and without singular points; g is then an immersion.

The method of proof is as follows. Join p_1 to p_1' by an arc A in M. (This may be taken as an arc along which f has a self-intersection.) A neighborhood of A in M may be expressed as the image of a sphere plus interior Q_0^n; f then gives a smooth mapping f^* of Q_0^n into E^{2n-1}, with two singular points q_1 and q_1', of opposite types. If we approximate to f^* by an immersion g^*, which agrees with f^*, together with first partial derivatives, in ∂Q_0^n, this gives the required g' in M.

We shall describe the construction of g^* in the case $n = 2$, in spite of the fact that 2 is not odd. By Lemma 2, we may suppose that f^* near q_1 and q_1' is as shown in Fig. 1. We may choose the mapping of Q_0^n into M so that near $f^*(q_1$

(and similarly, near $f^*(q_1'))f^*(Q_0^n)$ goes up to the top line $x_1 = 0$ in the figure, and down again a short distance, to $x_1 = -\beta$. If we cut off small pieces R and R' of Q_0^n containing q_1 and q_1' respectively, so that the new figure ends at $x_1 = \beta$ near these points, then it is pretty clear that by pushing part of $x_1 = \beta$ into part of $x_1 = -\beta$, we can change f^* to an immersion h', so that $h'(\partial(Q_0^n - R \cup R')) = f^*(\partial Q_0^n)$. Since Q_0^n is easily mapped into $Q_0^n - R \cup R'$, we obtain h with $h(q) = f^*(q)$, $q \in \partial Q_0^n$.

Noting that the part $\partial' R$ of ∂R inside Q_0^n must, under f^*, curl over the top edge in Fig. 1, we see that watching the image of the vector normal to ∂Q_0^n, under f^* and under h', as we go along $\partial R - \partial' R$ and along $\partial' R$, along the latter we obtain one complete twist more than along the former. The same is true near q_1'. As a result, since the singular points are of opposite types, $\mathfrak{L}_h(Q_0^n) = \mathfrak{L}_{f^*}(Q_0^n)$. Therefore we may apply Lemma 12, and obtain from h a mapping g^* which, like h, is an immersion, but agrees with f^*, together with first derivatives, in ∂Q_0^n.

We turn now to the proof for Case I. Turning to §22, (c), we see that given p_1, p_1' may be chosen so that these two points are the ends of arcs A and A', forming a smooth curve in M, and both mapping into a single arc in E^{2n-1}. (We could take, for A, any smooth arc in M which ends like the above arc at p_1 and at p_1'.) Choose coordinate systems about p_1 and p_1' as in Lemma 2; they are of class C^2. With the proper choice of the x_1-direction in each case, the ends of A are given by

$$(12.1) \qquad 0 \leqq x_1 \leqq c, \qquad x_2 = \cdots = x_n = 0.$$

It is easy to define independent smooth vector functions $v_i(p)$ ($i = 2, \cdots, n$) along A so that $v_i(p_1)$ and $\pm v_i(p_1')$ are in the x_i-directions at p_1 and p_1' respectively. By considering M as lying in E^{2n+1}, and projecting the points $p + \sum \alpha_i v_i(p)$ into M, we may define a coordinate system throughout a neighborhood of A. We may extend the system beyond p_1 and p_1'. We have now a neighborhood of A expressed as the imbedding of a cylinder in M, and the ends of the cylinder are mapped approximately into the sets defined by

$$(12.2) \qquad x_1 = -\beta, \qquad x_2^2 + \cdots + x_n^2 \leqq \alpha^2.$$

(Note that x_1 may be replaced by $-x_1$ in Lemma 2.) If we alter the two ends of the cylinder slightly so that they coincide with $x_1 = -\beta$, and then round off the two edges (which are near $x_1 = -\beta$, $x_2^2 + \cdots + x_n^2 = \alpha^2$), we obtain a partial manifold Q_0^n, expressible as a sphere plus interior.

Let ψ be the imbedding of Q_0^n into M; set

$$f^*(q) = f(\psi(q)) \qquad (q \in Q_0^n),$$
$$q_1 = \psi^{-1}(p_1), \qquad q_1' = \psi^{-1}(p_1').$$

We may clearly suppose that f^* is an imbedding in some neighborhood of ∂Q_0^n. We now consider $(x_1, \cdots, x_n)$ as coordinates in Q_0^n near q_1 and near q_1'; then the sets defined by (12.2) with α replaced by $\alpha' = \alpha/2$ are parts N_0 and N_0' of ∂Q_0^n. Let λ be a real-valued function of class C^2 such that

$$\lambda(t) = 1 \quad \text{if} \quad |t| \leqq \alpha'/2, \qquad \lambda(t) = 0 \quad \text{if} \quad |t| \geqq \alpha'.$$

Let R and R' be the parts of Q_0^n about q_1 and q_1' respectively, defined by

$$(12.3) \qquad -\beta \leqq x_1 \leqq -\beta + 2\beta\lambda[(x_2^2 + \cdots + x_n^2)^{1/2}].$$

Then if $R_0 = \overline{Q_0^n - R \cup R'}$, ∂R_0 is a manifold of class C^1, and f^* immerses R_0 in E^{2n-1}. For each $q \in E^{2n-1}$, let $T_1(q)$ be the n-plane through q parallel to the axes of $y_1, y_{n+1}, y_{n+2}, \cdots, y_{2n-1}$. If $T(p) = T_1(f^*(p))$, $p \in N_0$, these planes cut $f^*(N_0)$ in the manner prescribed in Lemma 11. Moreover, if

$$N_1 = \partial R - N_0, \qquad N_1' = \partial R' - N_0',$$

then each $T(p)$ $(p \in N_0)$ cuts $f^*(N_1)$ in exactly one point $f_1^*(p)$. Pushing N_0 and N_0' onto N_1 and N_1' clearly defines an imbedding θ of Q_0^n onto R_0 such that $f_1^*(p) = f^*(\theta(p))$ agrees with the above f_1^* in N_0 and N_0'. Now f_1^* and f^*, defined in $N_0 - \partial N_0$, satisfy the conditions for f_0 and f_1 in Lemma 11; define ϕ_t by that lemma, and set

$$h_1(p) = \phi_1(f_1^*(p)), \qquad p \in Q_0^n.$$

Carry out a similar deformation of E^{2n-1} about $f_1^*(R')$, forming the mapping h of Q_0^n in E^{2n-1}. Then (see the remarks following the lemma) h is of class C^2, and $h(p) = f^*(p)$ $(p \in \partial Q_0^n)$. Since f^* is an immersion of R_0, f_1^* and h are immersions of Q_0^n. By taking β small enough, we may make h an arbitrarily good approximation to f^* (not of course with first derivatives).

Let $u(p)$ be the inward normal vector at p in ∂Q_0^n. If we replace h by g^*, using Lemma 12, so that

$$g^*(p) = f^*(p), \qquad \nabla g^*(u(p), p) = \nabla f^*(u(p), p) \qquad (p \in \partial Q_0^n),$$

then $g^* = f^*$, together with first partial derivatives, in ∂Q_0^n; hence, if

$$g'(p) = g^*(\psi^{-1}(p)) \quad (p \in \psi(Q_0^n)), \qquad g'(p) = f(p) \quad \text{otherwise},$$

g' will be smooth in M. Since h and hence g^* (see Lemma 12) is an immersion in Q_0^n, g' has no singular points in $\psi(Q_0^n)$, and the proof for Case I will be complete.

To apply Lemma 12, we need merely prove

$$(12.4) \qquad \mathfrak{L}(h\partial Q_0^n, \nabla h(u)) = \mathfrak{L}(h\partial Q_0^n, \nabla f^*(u)).$$

By Theorem 2 and Lemma 8, these numbers are the algebraic number of singular points of h and f^* in Q_0^n. It is 0 for h, since h is regular. It is also 0 for f^*, since f^* has just two singular points, q_1 and q_1', and these are of opposite types.

Case II. M is closed and n is even. By Theorem 3, the number of singular points is even; call them $p_1, p_1', \cdots, p_s, p_s'$. If we proceed as under Case I, the only difficulty is at the last step; the two numbers in (12.4) may differ from each other by any even integer $2k$. Set $M^* = \psi(Q_0^n)$. Choose k points $r_1, \cdots, r_k$ in ∂M^*. For each i, we may express a small piece M_i' of $f(\overline{M - M^*})$ about r_i as the imbedding ϕ_i of an n-cube M_i, one face N_i going into part of $f(\partial M^*)$. If the M_i' are small, and we take $M_i \subset E^{2n-1}$, we may in an obvious fashion extend ϕ_i to be an imbedding of a neighborhood U_i of M_i in E^{2n-1} into E^{2n-1}. Let θ_i be the mapping of M_i into E^{2n-1} given by Theorem 5, with θ_i = identity in ∂M_i, together with first partial derivatives in $\partial M_i - N_i$. Then

$$F(p) = \phi_i \theta_i \phi_i^{-1} f(p) \quad (p \in f^{-1}(M_i')), \quad F(p) = f(p) \quad \text{otherwise},$$

defines a smooth mapping of $M - M^*$, agreeing with f except in the $f^{-1}(M_i')$. Choosing the correct sign in (4) of Theorem 5 in each case, we obtain

$$\text{(12.5)} \qquad \mathfrak{L}(h\partial Q_0^n, \nabla h(u)) = \mathfrak{L}(h\partial Q_0^n, \nabla F^*(u)),$$

where F^* and its first partial derivatives are defined at points of ∂Q_0^n in Q_0^n in terms of F with the help of the imbedding ψ. We may now apply Lemma 12 as before.

Case III. M is open. Choose compact partial manifolds M_i in M by Lemma 20, Appendix. We shall define mappings $f_0 = f, f_1, f_2, \cdots$ with f_i arbitrarily close to f_{i-1}, such that f_i is regular in M_i, and $f_i = f_{i-1}$ in M_{i-1}. Then $g = \lim f_i$ exists and is an immersion.

Suppose f_{i-1} is properly defined. The number of singular points of f_{i-1} in M_i is finite; none are in M_{i-1}. It is sufficient to show how an arbitrarily slight alteration of f_{i-1} will get rid of one of these, say p_1. By (c) of Lemma 20, we may join p_1 to a point p_2 in $M - M_i$ by an arc $A \subset M - M_{i-1}$; we may clearly keep A away from ∂M. Take a small neighborhood U of p_2, and express a neighborhood of $f_{i-1}(U)$ in E^{2n-1} as the image ϕ of part of E^{2n-1}, so that for $E^n \subset E^{2n-1}$, $\phi(E^n)$ contains $f_{i-1}(U)$. Using the last mapping of §4, we may then alter f_{i-1} in U so that it has two singular points, say p_1' and p_1''. If n is odd, then by Theorem 2, these are of opposite types; hence one of these, say p_1', is of opposite type to that of p_1. Applying the proof in Case I, we alter f_{i-1} in a neighborhood of A, together with U, getting rid of the singularities at p_1 and p_1'. If n is even, we apply the proof in Case II, using p_1 and either of p_1', p_1''.

Case IV. M is compact but not closed; then $\partial M \neq 0$. Add a small piece onto M along part of ∂M, and remove a closed n-cell from the new portion, obtaining a new open manifold M', with $M \subset M'$, and extend f through M'. By Case III, we may alter f to give an immersion g of M'; this defines an immersion g of M, and completes the proof of the Theorem.

13. Further immersion theorems

We consider here what may be done with ∂M in an immersion of M.

Theorem 7. *Let f be a smooth mapping of the connected partial manifold M^n into E^{2n-1} $(n \geqq 2)$ which is an imbedding in a neighborhood of ∂M. If M is not compact, there exists an immersion g of M in E^{2n-1} which is arbitrarily close to f*

and equals f in a neighborhood of ∂M. If M is compact and n is odd [is even], g exists if and only if $\mathfrak{L}_f(M)$ *is* $= 0$ [*is* $\equiv 0 \bmod 2$].

Using Lemma 1, we first replace f by a semi-regular mapping, which we again call f. If M is not compact, we may clearly apply the proof in Case III of the last theorem. If M is compact, the proof in Case I or in Case II applies. That the condition is necessary is a consequence of Theorem 2.

THEOREM 8. *Any compact partial manifold M may be immersed in* E^{2n-1} *so that the mapping f is an imbedding in a neighborhood of* ∂M, *and*

$$f(\partial M) \cap f(M - \partial M) = 0.$$

We shall not discuss the case of open manifolds. The theorem being clear if $n = 2$ (all M^2 being known), we assume $n \geqq 3$; also we may suppose M is connected. Let f_0 be the immersion given by Theorem 6; by Lemma 1, we may suppose f_0 is completely semi-regular. By §22, Appendix, the intersections of $f_0(\partial M)$ with $f_0(M - \partial M)$ are on ends of arcs as described in (e_3) and (e_4). Since the number of such arcs is finite, it will be sufficient to show how to get rid of an intersection of either kind.

Consider first an intersection as in (e_4). Let A and B be the arcs of M with $f_0(A) = f_0(B)$, and let U be a neighborhood of A in M. It is easy to define a smooth imbedding g of M in itself which is the identity outside U, and squeezes the whole arc A up into a part of U beyond the end point of A which is in $M - \partial M$, so that

$$g(M) \cap A = 0.$$

(Use a coordinate system about A, as in the proof for Case I of Theorem 6.) If we choose U so small that

$$f_0(U - A) \cap f_0(M - U) = 0,$$

and set $f(p) = f_0(g(p))$, we will clearly have removed the arc of self-intersection without introducing any further intersections.

Now take the case (e_3). Say $f_0(A) = f_0(B)$, all of A being in $M - \partial M$. Since $n \geqq 3$, we may extend A in one direction, forming a smooth arc A' with one end in ∂M, so that

$$f_0(A' - A) \cap f_0(M - A) = 0.$$

We now define g and f as above, with A' in place of A.

III. FURTHER INTERSECTION THEORY

14. Looping coefficients of vector fields in complexes in space

Using the definition of §9, we shall discuss "looping coefficients" of vector fields and pairs of vector fields with pairs of cycles in a smooth (not necessarily finite) complex $K^r \subset E^{2r+1}$. We derive two formulas which are useful, in particular, in studying $\mathfrak{L}_f(M)$ for a partial manifold $M^n, f(M) \subset E^{2n-1}, n = r + 1$.

In the rest of Part III, we study the situation when the above cycles are replaced by chains.

For any r-chain A of K, define the r-chain $\phi_{v,\epsilon}A$ as in §9. *Assuming v independent of K*, we shall take ϵ (or a positive continuous function $\epsilon = \epsilon(p)$ if K is infinite) so small that

$$(14.1) \qquad \phi_{v,\alpha}(K) \cap \phi_{v,\beta}(K) = 0 \quad \text{if} \quad -\epsilon < \alpha < \beta < \epsilon.$$

Generalizing (9.2), set (for finite r-cycles A, B)

$$(14.2) \qquad \mathfrak{L}(A, B, v) = LC(\phi_{v,\epsilon}A, B).$$

If we cut a closed manifold M^n into two parts M_1 and M_2, and $v(p)$ points into $f(M_1)$ at $f(p)$ in $f(\partial M_1)$ (where $f(M) \subset E^{2n-1}$), then the following lemma, with $r = n - 1$, $A = B = f\partial M_1$, relates $\mathfrak{L}_f(M_1)$ to $\mathfrak{L}_f(M_2)$. The lemma will be generalized in (19.10).

LEMMA 13. *Take K and v as above. Then for finite r-cycles A and B,*

$$(14.3) \qquad \mathfrak{L}(A, B, v) = (-1)^{r+1}\mathfrak{L}(B, A, -v).$$

By (14.1), $B = \phi_{v,0}B$ may be deformed into $\phi_{v,-\epsilon}B$ without touching $\phi_{v,\epsilon}A$, $\phi_{v,\epsilon}A$ may be deformed into A without touching $\phi_{v,-\epsilon}B$. Hence

$$LC(\phi_{v,\epsilon}A, B) = (-1)^{r^2+1}LC(B, \phi_{v,\epsilon}A)$$
$$= (-1)^{r+1}LC(\phi_{v,-\epsilon}B, \phi_{v,\epsilon}A) = (-1)^{r+1}LC(\phi_{-v,\epsilon}B, A),$$

which proves the lemma.

REMARK. Let α, β, γ be oriented arcs in E^3 joining the points p and q. Set $A = \alpha - \beta$, $B = \beta - \gamma$. Then it is easy to define v so that

$$\mathfrak{L}(A, B, v) = 1, \qquad \mathfrak{L}(B, A, v) = 0.$$

Suppose there is a small "fin" stretching out from K, in the direction of a vector field $u(p)$ (independent of K). Then $\mathfrak{L}(A, B, v)$, or $\mathfrak{L}_f(M)$ etc., may be determined by studying the intersection of $\phi_{v,\epsilon}A$ etc. with this fin. To show this, note first that by deforming $\phi_{v,\alpha}A$ into $\phi_{v,\beta}A$, we define a chain $\psi_{v,\alpha,\beta}A$ such that

$$(14.4) \qquad \partial\psi_{v,\alpha,\beta}A = \phi_{v,\beta}A - \phi_{v,\alpha}A - \psi_{v,\alpha,\beta}\partial A.$$

Now let $u(p)$ be a continuous vector function in K, independent of K, such that K, u, v are independent in K^{r-1}. That is, for each σ^r, face σ^{r-1}, and $p \in \bar\sigma^{r-1}$, the 2-plane through $u(p)$ and $v(p)$ has only p in common with the tangent plane to $\bar\sigma^r$ at p. We may suppose u, v are unit vector functions. Then we may take any η, $0 < \eta < \epsilon$, and define

$$(14.5) \qquad \mathfrak{L}(A, B, u, v) = KI(\phi_{u,\eta}A, \psi_{v,0,\epsilon}B).$$

Note that if $\partial A = \partial B = 0$, then this is defined without the restriction that K, u and v be independent in K^{r-1}.

This quantity is reducible to the former:

LEMMA 14. *If* $\partial A = \partial B = 0$, *then*

$$\mathfrak{L}(A, B, u, v) = \mathfrak{L}(B, A, v) + (-1)^r \mathfrak{L}(A, B, u). \tag{14.6}$$

In particular,

$$\mathfrak{L}(A, A, v, v) = 2\mathfrak{L}(A, A, v) \quad (r \text{ even}), \qquad = 0 \quad (r \text{ odd}). \tag{14.7}$$

By the method of proof of the last lemma, we find

$$\begin{aligned}\mathfrak{L}(A, B, u, v) &= KI(\psi_{v,0,\epsilon}B, \phi_{u,\eta}A) \\ &= LC(\phi_{v,\epsilon}B - B, \phi_{u,\eta}A) = LC(\phi_{v,\epsilon}B, A) - LC(B, \phi_{u,\eta}A) \\ &= LC(\phi_{v,\epsilon}B, A) + (-1)^r LC(\phi_{u,\eta}A, B),\end{aligned}$$

which gives (14.6).

15. The type of complex we shall use[5]

In the rest of Part III we shall use only complexes that are simplicial, or at least have certain properties of simplicial complexes. In particular:

(a_1) Each closed cell $\bar{\sigma}^n$ may be represented as the (smooth) one-one image $\theta(\bar{\sigma}_0^n)$ of a convex closed cell in E^n.

(a_2) Each closed r-face $\bar{\sigma}^r$ of σ^n is the intersection of $n - r$ $(n - 1)$-faces $\bar{\sigma}_1^{n-1}, \cdots, \bar{\sigma}_{n-r}^{n-1}$ of $\bar{\sigma}^n$.

(a_3) For $p \in \sigma^r$, the tangent planes to the $\bar{\sigma}_i^{n-1}$ there have only the tangent plane to $\bar{\sigma}^r$ in common.

It will be convenient to call the *tangent cone* $\Gamma(\bar{\sigma}^r, p)$ of $\bar{\sigma}^r$ at $p \in \bar{\sigma}^r$ the set of all vectors v tangent to $\bar{\sigma}^r$ at p; i.e. the set of all possible limits $\lim [\phi(t) - p]/t$, $\phi(0) = p$, $\phi(t) \in \bar{\sigma}^r$. The *tangent space* $\bar{\Gamma}(\bar{\sigma}^r, p)$ is the set of all linear combinations of vectors of $\Gamma(\bar{\sigma}^r, p)$. We say a vector of $\bar{\Gamma}$ is *parallel* to $\bar{\sigma}^r$ at p. In terms of this, (a_3) is equivalent to

$$\bar{\Gamma}(\bar{\sigma}^r, p) = \bar{\Gamma}(\bar{\sigma}_1^{n-1}, p) \cap \cdots \cap \bar{\Gamma}(\bar{\sigma}_{n-r}^{n-1}, p) \qquad (p \in \bar{\sigma}^r). \tag{15.1}$$

A cell may be in the form of a cube for example. Note that we may not subdivide a proper face of a cell without subdividing the cell itself, for then (a_3) would be contradicted.

Given $p \in K$, let $\Gamma(K, p)$ denote the set of all tangent cones of closed cells containing p, at p. These form parts of linear spaces which are unrelated except for those corresponding to incident cells. It is not so easy to give meaning to $\bar{\Gamma}(K, p)$.

Suppose f is a smooth mapping of K into E^ν. If f is regular in the part of a closed cell $\bar{\sigma}$ near $p \in \bar{\sigma}$, then

$$\Gamma(f(\bar{\sigma}), f(p)) = \nabla f \Gamma(\bar{\sigma}, p).$$

This is true with K in place of $\bar{\sigma}$; now $\Gamma(f(K), q)$ is formed of cones lying in E^ν.

[5] The author expects to give a more general theory of this subject in a paper on "Complexes of manifolds."

We shall say f is an *immersion* of K if it is regular (in each closed cell) and proper, and is an *imbedding* if, in addition, it is one-one, and further, ∇f is one-one in each $\Gamma(K, p)$. In the latter case, we say K is a complex *in* the second space.

EXAMPLE. No complex may be imbedded onto a pair of tangent circles.

LEMMA 15. *Any complex K^n may be imbedded in E^{2n+1} so that it has no limit set.*

It is not difficult to define an imbedding cell by cell, by the methods in [1]; of course it is easy and standard if K is simplicial. For the last statement (trivial if K is finite), compare [1], p. 665, footnote 32.

We shall let σ_i^r denote the cells of K, and set $\tau_i^r = f(\sigma_i^r)$. If K lies in E^ν, we may let τ_i^r denote its cells, thinking of f as the identity mapping.

LEMMA 16. *Let K lie in E^ν. Then each $\bar{\tau}_i^r$ may be enclosed in a larger cell $\tau_i'^r$, of the same class C^γ as K, such that if τ_j^s is a face of τ_i^r, then $\tau_j'^s$ lies in $\tau_i'^r$.*

REMARK. The lemma extends in an obvious way to the case, if $K = K^n$, that f is an imbedding in a neighborhood of each $\partial\sigma_i^n$.

Of course we take $\tau_i'^0 = \tau_i^0$. Suppose the $\tau_i'^1, \cdots, \tau_i'^{s-1}$ have been constructed. Take any $\bar{\tau}_i^s$, the image $\theta(\bar{\tau}_0^s)$ of a convex cell $\bar{\tau}_0^s$ in E^s. It is easy to define extensions of the boundary cells of τ_0^s, and define θ over these, mapping into the extended faces $\tau_j'^t$ of τ_i^s. Because of (a_3), we may now extend θ over a neighborhood of $\bar{\tau}_i^s$. In a sufficiently small neighborhood, which may be taken as a convex cell, θ is an imbedding.

16. On general position of a complex and vector field in space

Let f be a smooth mapping of a complex K (see §15) into E^ν. We say f is in *general position* if it is proper, and:

(b_1) For each s, f is regular in K^s at all points of $K^{\nu-s}$.

(b_2) For each s, each $p \in K^s$ and each $q \in K^{\nu-s-1}$, if $p \neq q$ then $f(p) \neq f(q)$.

(b_3) If $u_1 \in \Gamma(K^s, p)$, $u_2 \in \Gamma(K^{\nu-s}, p)$, $u_1 \neq u_2$, then $\nabla f(u_1, p) \neq \nabla f(u_2, p)$.

One could combine (b_1) and (b_3) in a more complicated statement. Note that (b_3) uses Γ, not $\bar{\Gamma}$. Any imbedding of K^n into E^ν, $\nu \geqq 2n$, is clearly a mapping in general position. From the above we deduce:

(b_1') f is regular in K^s if $\nu \geqq 2s$.

(b_2') f is one-one in K^s if $\nu \geqq 2s + 1$.

(b_3') ∇f is one-one in $\Gamma(K^s, p)$ if $\nu \geqq 2s$.

Let $(y_1, \cdots, y_\nu)$ be a coordinate system in E^ν, and let π be the projection: $\pi(y_1, \cdots, y_{\nu-1}, y_\nu) = (y_1, \cdots, y_{\nu-1})$ of E^ν into $E^{\nu-1}$

Say f is in *general position with reference to the y_ν-direction* if the above holds, and in addition, πf is in general position in $E^{\nu-1}$. We could obviously generalize this, using a set of independent directions.

Let v be a continuous vector field, defined in a closed subcomplex of K, and with values in E^ν. Then $\nabla\pi v$ has values in $E^{\nu-1}$. We say f and v are in *general position* if f is, and also:

(b_4) For each s, $v(p)$ is independent of any $f(\bar{\sigma}^s)$ at $f(p)$ for all points $p \in K^{\nu-s-1}$.

(b_5) For each s, $v(p)$ is independent of $\nabla f[\bar{\Gamma}(\bar{\tau}_i^s, p) + \bar{\Gamma}(\bar{\tau}_j^{\nu-s-1}, p)]$, where defined.

Letting $\sigma^{\nu-s-1}$ be a face of σ^s shows that (b_4) is a consequence of (b_5). We shall often omit (b_5).

Finally, f and v are in *general position with reference to the y_ν-direction* if the above holds also for πf and $\nabla \pi v$.

LEMMA 17. *Let K be of class C^2, let f map K into E^ν with no limit set, and let $v_1, v_2, \cdots$ be smooth vector functions, each defined in a subcomplex of K and with values in E^ν. Let f and each v_k be in general position. Then by an arbitrarily small rotation of the axes we may make this hold with reference to the y_ν-direction. The lemma holds if we omit* (b_5) *in the hypothesis and conclusion.*

REMARKS. If K is not of class C^2, or the v_k are not smooth, we may use a C^1-homeomorphic K' which is of class C^2, and smooth v_k' approximating to the v_k. The application will be to the case that K_1 is a subdivision of a smooth manifold M^n, $K = K_1^{n-1}$, and the v_k are independent of the $(n - 1)$-cells of K and tangent to the n-cells of K. We could use several independent directions in place of the single y_ν-direction. If $K = K^r$, we could allow the limit set L_f of f to exist, provided it is of zero $(\nu - r - 1)$-extent.

EXAMPLE. Let K be a subdivision of an open arc, and let f map K into E^3 so that it winds like a ball of string, having a 2-sphere as limit set. Then f is proper, but no projection into E^2 is proper.

To prove the lemma, let $S^{\nu-1}$ be the unit sphere in E^ν; its points may be thought of as directions in E^ν. Let R_{s1} be the set of all those directions parallel to $f(K^s)$ at a point of $f(K^{\nu-s-1})$. Let R_{s2} be all those through points $f(p)$ and $f(q)$ with $p \in K^s$, $q \in K^{\nu-s-2}$, $p \neq q$, (which implies $f(p) \neq f(q)$). Let R_{s3} be all directions of vectors $u' = \nabla f(u, p)$, $u = u_1 - u_2$, $u_1 \in \Gamma(K^s, p)$, $u_2 \in \Gamma(K^{\nu-s-1}, p)$, $u_1 \neq u_2$ (which implies $u' \neq 0$). Let R_{s4} be all those defined by $v_k(p)$ plus a vector parallel to some $f(\bar\sigma^s)$ at $p \in K^{\nu-s-2}$. If we are using (b_5), let R_{s5} be all those defined by $v_k(p) + u_1' + u_2'$, $u_i' = \nabla f(u_i)$, $u_1 \in \bar\Gamma(\bar\tau_i^s, p)$, $u_2 \in \bar\Gamma(\bar\tau_j^{\nu-s-2}, p)$.

Since the directions parallel to $f(K^s)$ at a point p form a set of finite $(s - 1)$-extent in $S^{\nu-1}$, and this set varies smoothly with p, R_{s1} is a finite or denumerable sum of sets of finite $(\nu - 2)$-extent.[6] In R_{s2}, since p and q range over sets of finite s- and $(\nu - s - 2)$-extent respectively, R_{s2} is a (denumerable) sum of sets of finite $(\nu - 2)$-extent. In R_{s3}, take any p in any σ^t, and suppose σ^t is a face of σ^s and of $\sigma^{\nu-s-1}$. If $u_1 \in \Gamma(\bar\sigma^s, p)$ and $u_2 \in \Gamma(\bar\sigma^{\nu-s-1}, p)$, these range over sets of finite s- and $(\nu - s - 1)$-extent respectively. But adding a vector of $\Gamma(\bar\sigma^t, p)$ to each leaves their difference unchanged; hence $u_1 - u_2$ ranges over sets of finite $(\nu - 1 - t)$-extent. Letting p vary shows that the directions vary over a sum of sets of finite $(\nu - 2)$-extent. Since the directions defined by $v_k(p) + \nabla f(u, p)$, $u \in \bar\Gamma(\bar\sigma^s, p)$, form a sum of sets of finite s-extent, R_{s4} is as required. In R_{s5}, $v_k(p) + u_1' + u_2'$, p fixed in σ^t, defines directions of finite $(\nu - 2 - t)$-extent; hence R_{s5} is as required.

Consequently by [1], Lemmas 13 and 14, $R = \sum R_{si}$ has no inner points in

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[6] See [1], in particular, Lemma 15. It would be possible to use dimension instead of extent.

$S^{\nu-1}$, so that arbitrarily near the y_ν-direction there is a direction not in R. Rotate axes so as to make this direction the new y_ν-direction. Then since any vector not in this direction projects into a non-zero vector, it is easily seen that f and each v_k are in general position with reference to the new y_ν-direction, completing the proof.

LEMMA 18. *Any smooth complex K^n may be mapped into any E^ν so that it is in general position with reference to a given direction.*

REMARK. This is a generalization of the imbedding and immersion theorems of [1] for manifolds.

By the remark to the last lemma, we may suppose that K is of class C^2. Imbed K^n in E^μ, $\mu = \max [\nu, 2n + 1]$, so that it has no limit set (Lemma 15). It is then in general position. By Lemma 17, applying a small rotation makes it in general position with reference to any chosen direction. If $\mu = \nu$, we let this be the given direction. If $\mu > \nu$, we choose any direction; projecting in this direction into $E^{\mu-1}$ gives a mapping in general position in $E^{\mu-1}$. Repeat the process till we reach E^ν.

17. The fins and corresponding projections

Take a smooth complex $K = K^r \subset E^{2r+1}$, and let K and v be in general position with reference to the y_{2r+1}-direction (using the identity mapping), omitting (b_5). We shall suppose K and v are of class C^2; if this is not so, and we do not wish to change K or v, we could replace the normal planes $T(q)$ below by a smooth function $T(q)$; the properties given will then hold.

Let $\{\tau_i^s\}$ denote the cells of K. Each τ_i^r may be enclosed in a larger cell τ_i' of class C^2 (Lemma 16). Suppose v is defined over $\bar{\tau}_i^r$; extend it to be of class C^2 over τ_i'. With small enough τ_i', we will still have general position. Set

$$\phi_{v,t}(p) = p + tv(p) \qquad (p \in \tau_i'). \tag{17.1}$$

For any point set $R \subset \tau_i'$, let $\Phi_v(R, t)$ denote all points $\phi_{v,t'}(p)$ with $p \in R$ and with $0 \leq t' \leq t$; let $\Phi_v^*(R, t)$ denote the same, with $0 < t' \leq t$. We call these the *fin* and *deleted fin* respectively of τ_i', v and t. Since K and v are of class C^2, t_i' may be chosen so that the *double fin* $\Phi_v(\tau_i', t_i') \cup \Phi_{-v}(\tau_i', t_i')$ is expressed by ϕ as an imbedding of the product of τ_i' and the interval $-t_i' \leq t \leq t_i'$. Moreover, for each q in the double fin, if $T_0(q)$ is the set of all points in the normal plane $T(q)$ to the fin at q which are at a distance $\leq t_i'$ from q, then these fill out a closed neighborhood $W_v(\tau_i', t_i')$ of the interior of the double fin in a one-one way, being again the imbedding of a product. If we set

$$P_v(q') = q \quad \text{if} \quad q' \in T_0(q), \tag{17.2}$$

this is a smooth projection of $W_v(\tau_i', t_i')$ onto the double fin. Finally, let $\Lambda_v(R, t)$ denote all points $q' \in T(q)$, with $q \in \Phi_v^*(p, t)$, $p \in R$, such that $| q' - q | \leq t \, | q - p |$.

Since K is in general position with reference to the y_{2r+1}-direction, omitting (b_5), the (b_k) give

(b_1^*) π is regular in K.
(b_2^*) If $p \in K^{r-1}$, $q \in K$, $p \neq q$, then $\pi p \neq \pi q$.
(b_3^*) Distinct vectors in K at any point map under π into distinct vectors.
(b_4^*) $\nabla \pi v(p)$ is independent of any $\pi \bar{\sigma}^s (s \leqq r)$ at p for $p \in K^{r-1}$.
(b_6^*) There is a neighborhood V^* of K^{r-1} in K such that if $p \in V^*$, $q \in K$, $p \neq q$, then $\pi p \neq \pi q$.

To prove (b_6^*), we use (b_1^*) and (b_3^*) to show that π is one-one in a neighborhood of any point of K^{r-1}, use (b_2^*) to show that π is one-one in K^{r-1}, and apply Lemma 25, §24.

Let $v_1, v_2, \cdots$ be vector functions in K with the same properties as v, such that at most a finite number are defined in any τ_i^r, and for each v_k, some v_l equals $-v_k$. For any R, let $N_t R$ denote all points whose distance from R is $\leqq t$. We shall choose numbers $t_i > 0$ such that the following properties hold, for any τ_i^r and v_k defined in τ_i^r.

(c_1) $\Lambda_{v_k}(\tau_i' \cap N_{t_i}\tau_i^r, t_i) \subset W_{v_k}(\tau_i', t_i')$.
(c_2) $\Phi_{v_k}^*(\tau_i^r, t_i) \cap \Phi_{-v_k}(\tau_i^r, t_i) = 0$.
(c_3) $\pi\Phi_{v_k}^*(\tau_i^r, t_i) \cap \pi\Phi_{-v_k}(\partial\tau_i^r, t_i) = 0$.
(c_4) $\Lambda_{v_k}(\tau_i' \cap N_{t_i}\tau_i^r, t_i) \cap \tau_i^r = 0$.
(c_5) $\pi\Lambda_{v_k}(\tau_i' \cap N_{t_i}\tau_i^r, t_i) \cap \pi\partial\tau_i^r = 0$.
(c_6) $\pi\Lambda_{v_k}(\tau_i' \cap N_{t_i}\partial\tau_i^r, t_i) \cap \pi\tau_i^r = 0$.

It is sufficient to prove each property (c_i) separately. We shall write Λ in place of Λ_{v_k}. We may use a single k, since but a finite number of v_k are defined in τ_i^r. (c_1), (c_2) and (c_4) are clear. The proof of (c_3) is essentially contained in that of (c_5), so we turn to (c_5) and (c_6).

Take $p \in \partial\tau_i^r$. For a small enough t_1, if $q' \in \Lambda(\tau_i' \cap N_{t_1}p, t_1)$, say $q' \in T_0(q)$, $q \in \Phi_{v_k}^*(p')$, $p' \in \tau_i' \cap N_{t_1}p$, then $q - p' = \alpha v_k(p')$, and $|q' - q| \leqq t_1 |q - p'|$, so that the angle between $p'q'$ and $v_k(p)$ is small; also $p' - p$ is approximately a vector $u_1 \in \Gamma(\bar{\tau}_i^r, p)$; furthermore, for any $p'' \in \tau_i^r \cap N_{t_1}p$, $p'' - p$ is approximately a vector $u_2 \in \Gamma(\bar{\tau}_i^r, p)$; therefore, by ($b_4^*$), we may suppose that

$$\nabla\pi[(q' - p') + (p' - p)] \neq \nabla\pi(p'' - p), \qquad \pi q' \neq \pi p''.$$

This gives

$$\pi\Lambda(\tau_i' \cap N_{t_1}p, t_1) \cap \pi(\tau_i^r \cap N_{t_1}p) = 0.$$

Since $\tau_i^r - N_{t_1}p$ is compact, because of (b_2^*) we may clearly take $t_2 \leqq t_1$ so that

$$\pi\Lambda(\tau_i' \cap N_{t_2}p, t_2) \cap \pi(\tau_i^r - N_{t_1}p) = 0.$$

The last two relations give

$$\pi\Lambda(\tau_i' \cap N_{t_2}p, t_2) \cap \pi(\tau_i^r) = 0,$$

and hence (with $t_3 \leqq t_2$)

$$\pi\Lambda(\tau_i' \cap N_{t_3}\partial\tau_i^r, t_3) \cap \pi(\tau_i^r) = 0,$$

proving (c_6). Next, since $(\tau'_i - N_{t_3}\partial\tau^r_i) \cap K^{r-1} = 0$, ($b_2^*$) gives, for some $t_4 \leqq t_3$,

$$\pi\Lambda(\tau'_i - N_{t_3}\partial\tau^r_i, t_4) \cap \pi(\partial\tau^r_i) = 0.$$

Combining this with the last relation gives

$$\pi\Lambda(\tau'_i, t_4) \cap \pi(\partial\tau^r_i) = 0,$$

proving (c_5).

REMARK. With the help of (b_5) we could prove (recalling that K is proper)

$$\pi\Lambda_{v_k}(\tau'_i, t_i) \cap \pi K^{r-1} = 0.$$

18. The numbers $\xi^{\pm}_{i,v}$ and $\zeta^{\pm}_{ij}$

We give here, and with the $\mu^{\pm}_{ij,v}$ of §19, the promised generalization of the looping coefficient of §14. Let K^r and each v_i be in general position in E^{2r+1} with reference to the y_{2r+1}-direction, as in §17. For any (finite) singular chain A in E, let ρ^+A and ρ^-A be the (infinite) singular chains formed by deforming A to infinity in the y_{2r+1}-direction and the negative y_{2r+1}-direction respectively, oriented so that

$$\partial\rho^{\pm}A = -A - \rho^{\pm}\partial A. \tag{18.1}$$

We shall show that the following definitions are permissible.

DEFINITIONS. Choose the t_i so that (c_1) through (c_6) hold. Let $\phi_{v,t}A$ denote the singular chain A, pushed a distance $t \mid v \mid$ in the direction of v, and oriented like A; see (17.1). Set

$$\xi^{\pm}_{i,v_k} = KI(\phi_{v_k,t_i}\tau^r_i, \rho^{\pm}\tau^r_i) \quad (\text{if } v_k \text{ is defined in } \tau^r_i), \tag{18.2}$$

$$\zeta^{\pm}_{ij} = KI(\tau^*_i, \rho^{\pm}\tau^*_j) \quad (i \neq j), \tag{18.3}$$

where τ^*_i is any cell lying in τ^r_i and slightly smaller than τ^r_i, oriented like τ^r_i. Note that $\xi^{\pm}_{i,v_k}$ does not depend on the orientation of τ^r_i.

We shall prove commutation properties of these:

$$\xi^{+}_{i,v_k} = (-1)^{r+1}\xi^{-}_{i,-v_k}, \tag{18.4}$$

$$\zeta^{+}_{ij} = (-1)^{r+1}\zeta^{-}_{ji}. \tag{18.5}$$

In the proofs of these and other relations, the following properties of Kronecker indices are useful:

(d_1) $KI(A, \rho^{\pm}B)$ is defined (for finite A and B) whenever $\dim A + \dim B = 2r$, and

$$A \cap B = 0, \qquad \pi A \cap \pi\partial B = 0, \qquad \pi\partial A \cap \pi B = 0.$$

We must show that

$$\partial A \cap \rho^{\pm}B = 0, \qquad A \cap \partial\rho^{\pm}B \subset (A \cap B) \cup (A \cap \rho^{\pm}\partial B) = 0;$$

these follow from the above relations.

(d_2) If A_λ and B are singular chains, A_λ being continuous in λ ($\lambda_0 \leqq \lambda \leqq \lambda_1$), and each $KI(A_\lambda, B)$ is defined, then $KI(A_{\lambda_0}, B) = KI(A_{\lambda_1}, B)$.

This follows either from continuity, or by making use of chains formed by deforming A_{λ_0} and ∂A_{λ_0}, with standard properties of the index.

(d_3). If $KI(A, B)$ is defined, and C_λ ($\lambda_0 \leqq \lambda \leqq \lambda_1$) satisfies $C_{\lambda_0} = \partial A$, $C_\lambda \cap B = 0$, then this defines a deformation A_λ of A such that $\partial A_\lambda = C_\lambda$ and $KI(A_\lambda, B)$ is defined.

This is clear if we let A_λ equal A plus the "path" of $C_{\lambda'}$, $\lambda_0 \leqq \lambda' \leqq \lambda$.

To prove (18.4), and show incidentally that (18.2) is permissible, choose l so that $-v_k = v_l$. By (d_1), we see that each of

$$KI(\phi_{v_k, t_i}\tau_i^r, \rho^{\pm}\phi_{-v_k, t}\tau_i^r), \qquad KI(\phi_{v_k, t}\tau_i^r, \rho^{\pm}\phi_{-v_k, t_i}\tau_i^r)$$

is defined for $0 \leqq t \leqq t_i$; for the three relations in (d_1) follow from (c_2), (c_3) and (c_5), using both v_k and v_l. Consequently, by (d_2),

$$\xi_{i, v_k}^{+} = KI(\tau_i^r, \rho^{+}\phi_{-v_k, t_i}\tau_i^r) = KI(\rho^{+}\phi_{-v_k, t_i}\tau_i^r, \tau_i^r).$$

In the above proof, if $A = \phi_{-v_k, t_i}\tau_i^r$, we used $\pi A \cap \pi\partial\tau_i^r = \pi\partial A \cap \pi\tau_i^r = 0$. These give

$$KI(\rho^{+}A, \rho^{-}\partial\tau_i^r) = KI(\rho^{+}\partial A, \rho^{-}\tau_i^r) = 0,$$

and therefore, since dim $\rho^{+}A = r + 1$,

$$\begin{aligned}
\xi_{i, v_k}^{+} &= -KI(\rho^{+}A, -\tau_i^r - \rho^{-}\partial\tau_i^r) = -KI(\rho^{+}A, \partial\rho^{-}\tau_i^r) \\
&= (-1)^r KI(\partial\rho^{+}A, \rho^{-}\tau_i^r) = (-1)^{r+1} KI(A + \rho^{+}\partial A, \rho^{-}\tau_i^r) \\
&= (-1)^{r+1} KI(\phi_{-v_k, t_i}\tau_i^r, \rho^{-}\tau_i^r) = (-1)^{r+1}\xi_{i, -v_k}^{-}.
\end{aligned}$$

Next we discuss (18.3) and (18.5). Because of (b_6^*), ζ_{ij}^{+} is defined and independent of the choice of any τ_k^*, so long as $\tau_k^r - \tau_k^* \subset V^*$. Since

$$\pi\tau_i^* \cap \pi\partial\tau_j^* = \pi\partial\tau_i^* \cap \pi\tau_j^* = 0,$$

we find, as in the proof of (18.4),

$$\begin{aligned}
\zeta_{ij}^{+} &= KI(\rho^{+}\tau_j^*, \tau_i^*) = -KI(\rho^{+}\tau_j^*, -\tau_i^* - \rho^{-}\partial\tau_i^*) \\
&= -KI(\rho^{+}\tau_j^*, \partial\rho^{-}\tau_i^*) = (-1)^r KI(\partial\rho^{+}\tau_j^*, \rho^{-}\tau_i^*) \\
&= (-1)^{r+1} KI(\tau_j^* + \rho^{+}\partial\tau_j^*, \rho^{-}\tau_i^*) = (-1)^{r+1}\zeta_{ji}^{-}.
\end{aligned}$$

We shall prove still a lemma regarding the $\xi_{i, v_k}^{\pm}$. Since $\bar\tau_i^r = \theta(\bar\sigma_0^r)$, $\bar\sigma_0^r$ convex in E^r, the parts near $\partial\sigma_0^r$ of radii from an inner point of σ_0^r map under θ into arcs which we may suppose fill out $\tau_i' \cap N_{t_i}\partial\tau_i^r$. With these arcs, we may define a deformation g_λ such that

(e_1) $g_\lambda(p)$ carries p along the arc in $\tau_i' \cap N_{t_i}\partial\tau_i^r$ on which it lies into $\partial\tau_i^r$.

LEMMA 19. *Let A be any singular chain such that (writing N for N_{t_i} and $\Lambda_v(R)$ for $\Lambda_v(R, t_i)$)*

(18.6) $$A \subset \Lambda_{v_k}(\tau_i' \cap N\tau_i^r), \qquad \partial A \subset \Lambda_{v_k}(\tau_i' \cap N\partial\tau_i^r).$$

Then there is a number a and a chain B such that

(18.7) $$KI(A, \rho^{\pm}\tau_i^r) = a\xi_{i,v_k}^{\pm},$$

(18.8) $$\partial A - a\partial\tau_i^r = \partial B, \qquad B \subset \Lambda_{v_k}(\tau_i' \cap N\partial\tau_i^r) \cup (\tau_i' \cap N\partial\tau_i^r).$$

The last relation determines a uniquely, even if we assume merely $B \subset W_{v_k}(\tau_i' \cap N\partial\tau_i^r, t_i)$.

First, by (c_4), (c_5), (c_6) and (d_1), $KI(A, \rho^{\pm}\tau_i^r)$ is defined. We shall define a deformation h_λ of A ($0 \leqq \lambda \leqq 3$) such that

(18.9) $$h_3A = a\phi_{v_k,t_i}\tau_i^r.$$

Also $KI(h_\lambda A, \rho^{\pm}\tau_i^r)$ is defined for each λ; (18.7) follows from this, together with (18.2) and (d_2).

For any $q \in \Phi_{v_k}^*(\tau_i' \cap N\tau_i^r)$ and $q' \in T_0(q)$, set

$$h_\lambda(q') = (1 - \lambda)q' + \lambda q \qquad (0 \leqq \lambda \leqq 1);$$

this is defined in $\Lambda_{v_k}(\tau_i' \cap N\tau_i^r)$, by ($c_1$), and $h_1(q') = P_{v_k}(q)$, by (17.2). From the definition of Λ_{v_k} we see that $h_\lambda A$ obeys (18.6) so far, simply because A does. Moreover, h_1A is in the deleted fin $\Phi_{v_k}^*(\tau_i' \cap N\tau_i^r)$, and $h_1\partial A \subset \Phi_{v_k}^*(\tau_i' \cap N\partial\tau_i^r)$.

Next, for each $q = \phi_{v_k,t}(p)$ ($p \in \tau_i'$) in the deleted fin, set

$$h_{1+\lambda}(q) = (1 - \lambda)q + \lambda\phi_{v_k,t_i}(p) \qquad (0 \leqq \lambda \leqq 1);$$

this is a deformation of $\Phi_{v_k}^*(\tau_i')$ in itself; applying it to h_1A defines $h_\lambda A$ ($0 \leqq \lambda \leqq 2$) so that (18.6) continues to hold; now

$$h_2A \subset \phi_{v_k,t_i}(\tau_i' \cap N\tau_i^r), \qquad h_2\partial A \subset \phi_{v_k,t_i}(\tau_i' \cap N\partial\tau_i^r).$$

Next, applying ϕ_{v_k,t_i} to the deformation g_λ defines h_λ, $2 \leqq \lambda \leqq 3$; by (e_1), it keeps $\phi_{v_k,t_i}(\tau_i' \cap N\partial\tau_i^r)$ in itself. This defines a deformation of ∂h_2A, and hence of h_2A, so that (18.6) continues to hold; see (d_3). Now

$$h_3A \subset \phi_{v_k,t_i}(\tau_i^r), \qquad h_3\partial A = \partial h_3A \subset \phi_{v_k,t_i}(\partial\tau_i^r).$$

Since ϕ_{v_k,t_i} is one-one in τ_i^r, the only $(r-1)$-cycles in $\phi_{v_k,t_i}(\partial\tau_i^r)$ are multiples of the cycle $\phi_{v_k,t_i}\partial\tau_i^r$; this proves (18.9) and hence (18.7).

The deformation h_λ of ∂A ($0 \leqq \lambda \leqq 3$) defines a chain B_1 such that

$$\partial B_1 = \partial A - a\phi_{v_k,t_i}\partial\tau_i^r;$$

since $\Phi_{v_k}(\partial\tau_i^r)$, properly oriented, is a chain B_2 bounded by $\phi_{v_k,t_i}\partial\tau_i^r - \partial\tau_i^r$, $B = B_1 + aB_2$ satisfies (18.8).

To prove the uniqueness of a in (18.8), suppose it held with a' and B' also; then

$$(a' - a)\partial\tau_i^r = \partial C, \qquad C = B - B' \subset W_{v_k}(\tau_i' \cap N\partial\tau_i^r, t_i).$$

If we contract the fin $\Phi_{v_k}(\tau_i')$ onto τ_i', we may carry C into $C' \subset \tau_i'$, and the above holds with C' in place of C. Since $N_{t_i}\partial\tau_i^r$ does not cover all of τ_i^r, this implies $a' - a = 0$.

19. Application to complexes K^n mapped into E^{2n-1}

In this section we suppose that f, of class C^3, maps K^n in regular position in E^{2n-1} with reference to the y_{2n-1}-direction. We may suppose K imbedded in E^{2n+1}. Extend each cell $\bar{\sigma}_i^s$ to a cell $\sigma_i'^s$ as in Lemma 16; we may extend f over these in turn so that the properties in Lemma 16 and the remark following it hold. Set $\tau_i'^s = f(\sigma_i'^s)$. Taking these cells small enough, we may suppose the properties (b_1), (b_2), (b_3) still hold.

We shall determine open sets U_i and U' in K (which now consists of the $\sigma_i'^s$) such that:

(f_1) $\bar{\sigma}_i^{n-1} \subset U_i$, $K^{n-2} \subset U'$.

(f_2) $U_i \cap U_j \subset U'$ if $i \neq j$.

(f_3) f is an imbedding in $U = \sum U_i$.

(f_4) If $p \in U$, $p' \in U'$, $p \neq p'$, then $\pi f(p) \neq \pi f(p')$.

We shall restrict the U_i and U' further later. That f is locally an imbedding at points of K^{n-1} is an immediate consequence of (b_1) and (b_3), using $s = n$, $\nu - s = n - 1$. The proof of (f_3) and (f_4) is now the same as the proof of (b_6^*). It is easy to choose sets U_i satisfying (f_1) and (f_2) (and thus redefining U) if we make use of the sets $\sigma_j^{n-1} - U'$.

Set $r = n - 1$. For each pair of incident cells σ_i^r and σ_k^n, let $\sigma_{ik}'^n$ be the part of $\sigma_k'^n$ on the side of $\sigma_i'^r$ which includes σ_k^n, and let $u_{ik}(p)$ ($p \in \sigma_i'^r$) be the unit vector tangent to $\sigma_{ik}'^n$ and normal to $\sigma_i'^r$ at $p \in \sigma_i'^r$. Set $v_{ik}(p) = \nabla f(u_{ik}(p), p)$. The following property shows that we may use the results of §§17 and 18.

(f_5) f and each v_{ik}, also f and each $-v_{ik}$, are in general position with reference to the y_{2n-1}-direction, (b_5) being omitted. (Here, f is considered in K^{n-1} only.)

We need merely prove (b_4) for f and πf; this follows at once from the properties of general position; compare (f_3) and (f_4).

Set $V_i = f(U_i)$, $V = f(U)$, $V' = f(U')$.

Next, by further restricting U' and then the U_i, we may obtain:

(f_6) $\tau_{ik}'^n \cap V_i \subset \Lambda_{v_{ik}}(\tau_i'^r \cap N_{t_i}\tau_i^r, t_i)$,

(f_7) $\tau_{ik}'^n \cap V' \subset \Lambda_{v_{ik}}(\tau_i'^r \cap N_{t_i}\partial\tau_i^r, t_i)$.

These are obvious consequences of the definitions of $\tau_{ik}'^n$, v_{ik} and Λ_v.

Now let K' be a subdivision of K so fine that the following holds:

(f_8) Any cell of K' with a vertex in $\bar{\sigma}_i^r$ lies in U_i; any cell with a vertex in K^{r-1} lies in U'.

Each oriented cell σ of K becomes now a chain $Sd\sigma$ of K'. Let $[\sigma_i^{s-1}: \sigma_k^s]$ denote incidence numbers. Define the following chains of K', and use $\tau = f(\sigma)$ as before:

σ_i^{*s} = sum of s-cells of K' in σ_i^s with no vertex in $\partial\sigma_i^s$, each oriented like σ_i^s.

σ_{ik}^* = sum of n-cells of σ_{ik}^n which have a vertex in σ_i^r but none in $\partial\sigma_i^r$, each oriented like $[\sigma_i^r: \sigma_k^n]\sigma_k^n$.

σ_{ik}^{**} = that part of $-\partial\sigma_{ik}^*$ whose cells have no vertex in $\bar{\sigma}_i^r$.

We prove the following relations:

(19.1) $$\partial\tau_{ik}^{*} = \tau_i^{*r} - \tau_{ik}^{**} + A_{ik}, \qquad A_{ik} \subset V'.$$

(19.2) $$\partial\tau_k^{*n} = \sum_i [\sigma_i^r : \sigma_k^n]\tau_{ik}^{**} + B_k, \qquad B_k \subset V'.$$

First, supposing σ_i^r a face of σ_k^n, note that each cell of the chain

$$Sd\tau_k^n - (\tau_k^{*n} + [\sigma_i^r : \sigma_k^n]\tau_{ik}^{*})$$

has a vertex in $\partial\tau_k^n - \tau_i^r$; hence the chain lies in V_i', where $U_i' = \sum_{j\neq i}U_j$, $V_i' = f(U_i')$. Also, clearly

$$\partial Sd\tau_k^n = Sd\partial\tau_k^n = [\sigma_i^r : \sigma_k^n]\tau_i^{*} + C_{ik}, \qquad C_{ik} \subset V_i'.$$

It follows that

$$\partial(\tau_k^{*n} + [\sigma_i^r : \sigma_k^n]\tau_{ik}^{*}) = [\sigma_i^r : \sigma_k^n]\tau_i^{*r} + D_{ik}, \qquad D_{ik} \subset V_i'.$$

This last relation, with (19.1), gives

$$\partial\tau_k^{*n} = [\sigma_i^r : \sigma_k^n]\tau_{ik}^{**} + D_{ik} \pm A_{ik}.$$

Now neither τ_k^{*n} nor τ_{ik}^{**} have any cells with vertices in $\bar{\tau}_i^r$. Hence neither has $D_{ik} \pm A_{ik}$, and it follows that each cell of A_{ik} is a cell of D_{ik}. Therefore $A_{ik} \subset V_i'$. Since τ_{ik}^{*}, τ_i^{*r} and τ_{ik}^{**} are in V_i, each point of A_{ik} is in some $V_i \cap V_j = f(U_i \cap U_j)$ (see (f_3)) with $j \neq i$, and therefore in V', by (f_2). This proves (19.1).

Next, since

$$\sum_{j\neq i} [\sigma_j^r : \sigma_k^n]\tau_{jk}^{**}, \quad D_{ik}, \quad A_{ik} \text{ are in } V_i',$$

the last relation for $\partial\tau_k^{*n}$ gives

$$B_k = \partial\tau_k^{*n} - \sum_l [\sigma_l^r : \sigma_k^n]\tau_{lk}^{**} \subset V_i'.$$

Since B_k is independent of i, this holds for each i. Now if B_k had a cell with a point p in just one V_{i_0} (clearly $B_k \subset \sum V_j$), using i_0 in the last relation would give a contradiction. Hence each p is in at least two, and hence in V'. This proves (19.2).

With the help of the new chains, we shall find new expressions for the numbers in §18:

(19.3) $$\xi_{i,v_{ik}}^{\pm} = KI(\tau_{ik}^{**}, \rho^{\pm}\tau_i^r) \qquad ([\sigma_i^r : \sigma_k^n] \neq 0),$$

(19.4) $$\zeta_{ij}^{\pm} = KI(\tau_{ik}^{**}, \rho^{\pm}\tau_j^r) \qquad ([\sigma_i^r : \sigma_k^n] \neq 0, i \neq j).$$

First, since each cell of σ_{ik}^{**} is a face of a cell of σ_{ik}^{*}, which has a vertex in σ_i^r, (f_8) gives:

(19.5) $$\tau_{ik}^{**} \subset \tau_{ik}'^{n} \cap V_i.$$

Taking the boundary of (19.1) and using (f_8) gives

$$\partial\tau_{ik}^{**} = \partial\tau_i^{*r} + \partial A_{ik} \subset \tau_{ik}^{\prime n} \cap V'. \tag{19.6}$$

These relations, with (f_6) and (f_7), show that we may apply Lemma 19, which gives

$$KI(\tau_{ik}^{**}, \rho^{\pm}\tau_i^r) = a\xi_{i,v_{ik}}^{\pm}.$$

We must show that $a = 1$. By (19.6),

$$\partial\tau_{ik}^{**} - \partial\tau_i^r = \partial[A_{ik} - (\tau_i^r - \tau_i^{*r})].$$

By (f_8) and (19.1), $A_{ik} - (\tau_i^r - \tau_i^{*r}) \subset V'$. Since this chain is also in $\tau_i^{\prime r} \cup \tau_{ik}^{\prime n}$, applying ($f_7$) gives (18.8) with $a = 1$. The uniqueness of a in (18.8) completes the proof of (19.3).

Next, taking $i \neq j$, since

$$\tau_j^r - \tau_j^{*r} \subset V', \qquad \tau_{ik}^{**} \subset V, \qquad (\tau_j^r - \tau_j^{*r}) \cap \tau_{ik}^{**} = 0,$$

(f_4) gives $\pi(\tau_j^r - \tau_j^{*r}) \cap \pi(\tau_{ik}^{**}) = 0$. Hence

$$KI(\tau_{ik}^{**}, \rho^{\pm}\tau_j^r - \rho^{\pm}\tau_j^{*r}) = 0.$$

Similarly, using (19.1), we find the two relations

$$KI(A_{ik}, \rho^{\pm}\tau_j^{*r}) = KI(\tau_{ik}^{*}, \rho^{\pm}\partial\tau_j^{*r}) = 0.$$

By (f_3), $KI(\tau_{ik}^{*}, \tau_j^{*r}) = 0$. This, with the last relations, gives

$$0 = KI(\tau_{ik}^{*}, \partial\rho^{\pm}\tau_j^{*r}) = \pm KI(\tau_i^{*r} - \tau_{ik}^{**}, \rho^{\pm}\tau_j^{*r}).$$

Hence

$$KI(\tau_{ik}^{**}, \rho^{\pm}\tau_j^r) = KI(\tau_{ik}^{**}, \rho^{\pm}\tau_j^{*r}) = KI(\tau_i^{*r}, \rho^{\pm}\tau_j^{*r}),$$

which gives (19.4).

We now give an extension of the results in §14, in particular, of Lemma 13.

Definition. Let K^r and v be in general position in E^{2r+1}, with reference to the y_{2r+1}-direction. With the notations of §18, and a sufficiently small positive continuous $\eta(p)$ in K, set

$$\mu_{ij,v}^{\pm} = KI(\phi_{v,\eta}\tau_i^r, \rho^{\pm}\tau_j^r). \tag{19.7}$$

Save possibly for sign, this is a direct generalization of (14.2). The commutation rule is:

$$\mu_{ij,v}^{+} = (-1)^{r+1}\mu_{ji,-v}^{-}. \tag{19.8}$$

The proof is easily carried out, with the help particularly of (b_5), as follows:

$$\begin{aligned}\mu_{ij,v}^{+} &= KI(\rho^{+}\tau_j^r, \phi_{v,\eta}\tau_i^r) = KI(\rho^{+}\phi_{-v,\eta}\tau_j^r, \tau_i^r)\\ &= -KI(\rho^{+}\phi_{-v,\eta}\tau_j^r, \partial\rho^{-}\tau_i^r) = (-1)^r KI(\partial\rho^{+}\phi_{-v,\eta}\tau_j^r, \rho^{-}\tau_i^r)\\ &= (-1)^{r+1}KI(\phi_{-v,\eta}\tau_j^r, \rho^{-}\tau_i^r) = (-1)^{r+1}\mu_{ji,-v}^{-}.\end{aligned}$$

DEFINITION. With the above notations, set

$$(19.9) \qquad \mathfrak{L}^{\pm}(A, B, v) = KI(\phi_{v,\eta}A, \rho^{\pm}B).$$

The commutation rule, generalizing (14.3), is

$$(19.10) \qquad \mathfrak{L}^{+}(A, B, v) = (-1)^{r+1}\mathfrak{L}^{-}(B, A, -v).$$

To prove this, suppose $A = \sum a_i\tau_i^r$, $B = \sum b_j\tau_j^r$. Then, writing ϕ for $\phi_{v,\eta}$,

$$\begin{aligned}\mathfrak{L}^{+}(A, B, v) &= \sum_{i,j} a_i b_j KI(\phi\tau_i^r, \rho^+\tau_j^r)\\ &= \sum_i a_i b_i \xi_{i,v}^+ + \sum_{i<j} (a_i b_j \mu_{ij,v}^+ + a_j b_i \mu_{ji,v}^+)\\ &= (-1)^{r+1}[\sum_i b_i a_i \xi_{i,-v}^- + \sum_{i<j} (b_i a_j \mu_{ij,-v}^- + b_j a_i \mu_{ji,-v}^-)]\end{aligned}$$

which, by the same proof, equals $(-1)^{r+1}\mathfrak{L}^{-}(B, A, -v)$.

20. Application to partial manifolds M^n mapped into E^{2n-1}

The following theorem contains essentially Theorem 2 if n is odd. Let $\mathfrak{L}_f'(M)$ denote the algebraic number of singular points if n is odd.

THEOREM 9. *Let M^n be a partial manifold, let K be a subdivision of M (with the properties of §15), and let f be a semi-regular mapping of M into E^{2n-1} so that f maps K in general position. For each σ_i^{n-1} in ∂M, let $v_i(p)$ point into M at $f(p)$, $p \in \bar{\sigma}_i^{n-1}$. Then letting $\sum'$ denote the sum taken over just those cells in ∂M, we have:*

(a) *If n is even,*

$$(20.1) \qquad \sum{}' \xi_{i,v_i}^+ = \sum{}' \xi_{i,v_i}^- = \sum{}' \xi_{i,-v_i}^+ .$$

(b) *If n is odd,*

$$(20.2) \qquad 2\mathfrak{L}_f'(M) = \sum{}' (\xi_{i,v_i}^+ + \xi_{i,v_i}^-) = \sum{}' (\xi_{i,v_i}^+ - \xi_{i,-v_i}^+).$$

We consider first the case that M^n is a single cell σ_k^n. Using the previous notations, setting $\partial_{ik} = [\sigma_i^{n-1}:\sigma_k^n]$ and noting that $\partial_{ik}^2 = 0$ or 1, we find, with the help of (19.2), (19.3) and (19.4),

$$\begin{aligned}\mathfrak{L}_f(\sigma_k^n) &= KI(\tau_k^{*n}, \partial\tau_k^n) = -KI(\tau_k^{*n}, \partial\rho^{\pm}\partial\tau_k^n)\\ &= (-1)^{n-1}KI(\partial\tau_k^{*n}, \rho^{\pm}\partial\tau_k^n) = (-1)^{n-1}\sum_{i,j}\partial_{ik}\partial_{jk}KI(\tau_{ik}^{**}, \rho^{\pm}\tau_j^r)\\ &= (-1)^{n-1}[\sum_i \xi_{i,v_{ik}}^{\pm} + \sum_{i<j}\partial_{ik}\partial_{jk}(\zeta_{ij}^{\pm} + \zeta_{ji}^{\pm})].\end{aligned}$$

Using first + and then −, we find, with the help of (18.5),

$$(-1)^{n-1}\mathfrak{L}_f(\sigma_k^n) = \sum_i \xi_{i,v_{ik}}^+ + \sum_{i<j}\partial_{ik}\partial_{jk}(\zeta_{ij}^+ + \zeta_{ji}^+),$$

$$\mathfrak{L}_f(\sigma_k^n) = (-1)^{n-1}\sum_i \xi_{i,v_{ik}}^- - \sum_{i<j}\partial_{ik}\partial_{jk}(\zeta_{ji}^+ + \zeta_{ij}^+).$$

Adding these and using Theorem 2, gives

$$[1 + (-1)^{n-1}]\mathfrak{L}'_f(\sigma_k^n) = \sum_i [\xi^+_{i,v_{ik}} + (-1)^{n-1}\xi^-_{i,v_{ik}}], \tag{20.3}$$

from which the theorem for this case follows, with the help of (18.4).

By (20.3) and (18.4), we have, in the general case,

$$[1 + (-1)^{n-1}] \sum_k \mathfrak{L}'_f(\sigma_k^n) = \sum_i \sum_k (\xi^+_{i,v_{ik}} - \xi^+_{i,-v_{ik}}),$$

where for each i we sum over all k such that σ_i^{n-1} is a face of σ_k^n. If σ_i^{n-1} is interior to M, then σ_i^{n-1} is a face of two-cells σ_k^n and σ_l^n, and $v_{il} = -v_{ik}$; hence

$$(\xi^+_{i,v_{ik}} - \xi^+_{i,-v_{ik}}) + (\xi^+_{i,v_{il}} - \xi^+_{i,-v_{il}}) = 0.$$

Thus all these values of i drop out. For each i with $\sigma_i^{n-1} \subset \partial M$, there is just one k, and $v_i = v_{ik}$. Hence the theorem follows.

REMARK. Clearly both sides of (20.2) are independent of the subdivision K of M employed. All the terms are independent of the chosen orientations of cells.

DEFINITION. Let M be a partial n-manifold, n odd, with the property that there exists a continuous vector field $u(p)$ defined in ∂M, $u(p)$ being independent of each closed cell of ∂M containing p, and pointing into M. Let K be a subdivision of M and let f be a semi-regular mapping of M into E^{2n-1} such that f maps K in general position with reference to the y_{2n-1}-direction. Set $v(p) = \nabla f(u, p)$. Set

$$A^{n-1} = \sum{}' \sigma_i^{n-1}, \text{ summed over the cells in } \partial M, \tag{20.4}$$

these cells being oriented arbitrarily. Let e be the unit vector in the y_{2n-1}-direction. Then for $\alpha(p) > 0$ sufficiently small in ∂M, and $\eta(p) > 0$ sufficiently much smaller than $\alpha(p)$, define (using the definition of ψ in §14 and recalling that n is odd)

$$\begin{aligned}\mathfrak{L}_f(M) &= \tfrac{1}{2}[KI(\phi_{v,\eta}fA, \psi_{e,0,\alpha}fA) + KI(\phi_{v,\eta}fA, \psi_{-e,0,\alpha}fA)] \\ &= \tfrac{1}{2}[KI(\phi_{v,\eta}fA, \psi_{e,0,\alpha}fA) - KI(\phi_{-v,\eta}fA, \psi_{e,0,\alpha}fA)]\end{aligned} \tag{20.5}$$

REMARK. If the hypothesis of general position in Theorem 9 does not hold, we may apply Lemma 17 to make it hold; that $\mathfrak{L}_f(M)$ is independent of the rotation chosen follows from the proof below, in which general position is assumed.

COMPLETION OF THE PROOF OF THEOREM 2. We must prove $\mathfrak{L}'_f(M) = \mathfrak{L}_f(M)$. Because of (20.2), it is only necessary to show that (for n odd)

$$KI(\phi_{v,\eta}fA, \psi_{e,0,\alpha}fA) = \sum{}' \xi^+_{i,v_i}. \tag{20.6}$$

For each σ_i^{n-1} in ∂M, we may deform $v_i(p)$ into $v(p)$, keeping it tangent to M and independent of $\bar{\sigma}_i^{n-1}$; hence ξ^+_{i,v_i} may be replaced by $\xi^+_{i,v}$. Since both sides of (20.6) are independent of the subdivision employed, and πf is regular in K^{n-1} and hence in ∂M, we may suppose that K is so fine that for each pair σ_i^{n-1},

σ_j^{n-1} of cells of ∂M with a common vertex, πf is an imbedding in $\bar\sigma_i^{n-1} \cup \bar\sigma_j^{n-1}$. Since also πf is regular in M at points of K^{n-2}, it follows at once that for small enough $\alpha(p)$ and $\eta(p)$,

$$KI(\phi_{v,\eta}\tau_i^r, \psi_{e,0,\alpha}\tau_j^r) = 0 \qquad (i \neq j)$$

Exactly as in corresponding proofs in §19, we see that

$$KI(\phi_{v,\eta}\tau_i^r, \psi_{e,0,\alpha}\tau_i^r) = KI(\phi_{v,\eta}\tau_i^r, \rho^+\tau_i^r) = \xi_{i,v}^+ .$$

We find therefore

$$KI(\phi_{v,\eta}fA, \psi_{e,0,\alpha}fA) = \sum_{i,j}{}' KI(\phi_{v,\eta}\tau_i^r, \psi_{e,0,\alpha}\tau_j^r)$$

$$= \sum_i{}' KI(\phi_{v,\eta}\tau_i^r, \psi_{e,0,\alpha}\tau_i^r) = \sum_i{}' \xi_{i,v}^+ ,$$

taken over the cells of ∂M, which completes the proof.

21. The necessity for the type of formula in (20.5) (n odd)

One might expect that a single term on the right in (20.5) would suffice without the factor $\frac{1}{2}$. This is the case if the cells σ_i^r of ∂M can be so oriented that $\sum'\sigma_i^r$ is a cycle, but not in the general case, as we shall show.

To compare the two terms, note that

$$\psi_{-e,0,\alpha} = -\psi_{-e,\alpha,0} = -\psi_{e,-\alpha,0};$$

hence, subtracting one of the terms from the other,

$$\Delta = KI(\phi_{v,\eta}fA, \psi_{e,0,\alpha}fA - \psi_{-e,0,\alpha}fA)$$
$$(21.1) \qquad = KI(\phi_{v,\eta}fA, \psi_{e,-\alpha,\alpha}fA).$$

Let e' be the unit vector in the y_{2n-2}-direction. By deforming $\phi_{v,\eta} C$ into $\phi_{e',\eta}C$, we define θC for all $C \subset \partial M$, with the property

$$\partial\theta C = \phi_{e',\eta}C - \phi_{v,\eta}C - \theta\partial C;$$

we may take this in general position together with the chains ψ considered, with reference to the y_{2n-1}-direction. From the obvious relations (for small $\eta(p)$)

$$\phi_{e',\eta}fA \cap \psi_{e,-\alpha,\alpha}fA = 0,$$

$$\theta fA \cap (\phi_{e,\alpha}fA \cup \phi_{e,-\alpha}fA) = 0,$$

we obtain

$$\Delta = KI(\phi_{v,\eta}fA - \phi_{e',\eta}fA, \psi_{e,-\alpha,\alpha}fA)$$

$$= -KI(\partial\theta fA + \theta\partial fA, \psi_{e,-\alpha,\alpha}fA)$$

$$= -KI(\theta\partial fA, \psi_{e,-\alpha,\alpha}fA) - (-1)^n KI(\theta fA, \partial\psi_{e,-\alpha,\alpha}fA),$$

$$(21.2) \qquad \Delta = -KI(\theta f\partial A, \psi_{e,-\alpha,\alpha}fA) + (-1)^n KI(\theta fA, \psi_{e,-\alpha,\alpha}f\partial A).$$

As a corollary, $\Delta = 0$ if the cells σ_i^r of ∂M can be so oriented that their sum is a cycle.

REMARK. The last fact follows also on applying Lemma 14 to $\mathfrak{L}(fA, fA, v, \pm e)$.

EXAMPLE. We shall show that Δ may be $\neq 0$. Take a cylinder plus interior:

$$x^2 + y^2 \leqq 1, \ -1 \leqq z \leqq 1;$$

identify the two ends, after reflecting one across a diameter: set

$$(x, y, -1) = (x, -y, 1).$$

A "Klein bottle B^2 plus interior" is formed; this is a partial manifold M^3, with $\partial M = B^2$. Let K be a subdivision of M^3. We may imbed B^2 in E^4 in such a fashion that for any continuous vector field $v(p)$ in B^2, independent of B^2 at points of $K^1 \cap B^2$,

$$KI(\phi_{v,\eta}B_0^2, B_0^2) = -4 \text{ in } E^4,$$

B_0^2 being $\sum' \sigma_i^2$, σ_i^2 in B^2, and η being small; see [3], p. 107, Fig. 4, and the top of p. 108. Now take $E^4 \subset E^5$. Then clearly, from the above,

$$\Delta = KI(\phi_{v,\eta}B_0^2, \psi_{e,-\alpha,\alpha}B_0^2) = -4 \text{ in } E^5.$$

If, for instance, we take $v = (e + e')/2^{1/2}$, then (20.5) reads

$$\mathfrak{L}_f(M^3) = \tfrac{1}{2}[-4 + 0] = -2.$$

Mapping M^3 into E^5 so that v points into $f(M)$ gives the result stated.

APPENDIX

22. The self-intersections under a completely semi-regular mapping

Let A be the set of points p of M such that for some $q \neq p$, $f(p) = f(q)$. We shall discuss A and $f(A)$.

Let $p_1, p_2, \cdots$ be the singular points. By Lemma 2 and the definition of completely semi-regularity, there is a neighborhood U_i^* of p_i such that $A \cap U_i^*$ is given by $x_2 = \cdots = x_n = 0$; thus this is an open arc, mapped by f doubly into a half-open arc. We may suppose $\bar{U}_i^* \cap \bar{U}_j^* = 0$ for $i \neq j$.

Choose U_i^{**}, $p_i \in U_i^{**}$, $\bar{U}_i^{**} \subset U_i^*$, and set $M' = M - \sum \bar{U}_i^{**}$. Take p, q in M', $p \neq q$, $f(p) = f(q)$. If $p, q \in M' - \partial M'$, there are neighborhoods U and V of p and q in M' such that $f(U) \cap f(V)$ is a smooth open arc in E, the image of smooth open arcs in M'; if one of p, q is in $\partial M'$, these are half-open arcs. We may cover M' by neighborhoods U_i such that each $f(U_i) \cap f(U_j)$ $(U_i \cap U_j = 0)$ is void or a smooth open or half-open arc. These arcs in M' together with those in $\sum \bar{U}_i^{**}$ fit together to form simple closed curves or arcs; in each direction on each arc, we end either at a singular point or at a point whose image is also the image of a point of ∂M, or we reach no limit point of M (this can occur only if M is open). Moreover, it is not hard to see that there is a grouping of these curves into pairs (the two in a pair need not be distinct), in one of the following ways:

(a) There may be two distinct closed curves, with one closed curve as image.

(b) A single closed curve may be mapped doubly over a closed curve.

(c) If the closed curve contains any singular points, it contains two, and is mapped into an arc.

(d) An arc may have one singular point interior to it; then (d_1) it is open at both ends, or (d_2) one end stops at ∂M and the other stops interior to M.

(e) A pair of arcs, each containing no singular points, may map into an arc. Then (e_1) both are open, or (e_2) each is half open, one ends in ∂M, and the other ends interior to M, or (e_3) one has both ends in ∂M and the other has both ends interior to M, or (e_4) each has one end in ∂M and one end interior to M.

If $n = 2$, we have the same subdivision into cases, but the curves may cut through each other. The case $n = 1$ is trivial. We could further subdivide the cases by taking into account orientation properties.

REMARK. If f is not proper, the set A need not be closed; for example, we may map a strip M^2 around and around in the interior of a torus so that A is dense in M^2. Still A is expressible as a union of curves.

23. On the covering of an open partial manifold by a sequence of compact partial manifolds

Let M be a partial manifold (in particular, a manifold). Since it may be covered by a denumerable number of coordinate systems, and each is compact, M may be covered by a sequence of compact partial manifolds. We wish to show how partial manifolds with certain properties may be chosen. Let a *proper half-open arc* A in a point set R mean the one-one continuous and proper image $\phi(A_0)$ of the half-line $0 \leqq x$. We shall say A runs from $\phi(0)$ to infinity if, for any compact subset B of R, there is an x_0 such that $\phi(x) \in R - B$ for $x \geqq x_0$.

The following lemma is used in the proof of the immersion theorem for open manifolds.

LEMMA 20. *Let M be an open connected partial manifold. Then there is a sequence $M_1, M_2, \cdots$ of compact partial manifolds in M such that*

(a) $M_i \subset M_{i+1} - \partial M_{i+1}$ $(i = 1, 2, \cdots)$,

(b) $M = \sum M_i$,

(c) *any point of $M - M_i$ may be joined to infinity by a half open arc which does not touch M_i.*

REMARKS. It may be shown that each M_i may be taken as connected, and such that ∂M_i is a closed manifold (not necessarily connected); but we do not need these facts here. For the present purpose, by a "partial manifold" we shall mean merely the closure of an open subset of some open manifold.

To start with, let $M_1'', M_2'', \cdots$ be the sets in M covered by a fixed set of coordinate systems, so that $M = \sum M_i''$. Set $M_1' = M_1''$. Since M_1' is compact, we may choose μ_2 so that $M_1' \subset \sum_{k=1}^{\mu_2}(M_k'' - \partial M_k'')$. Set $M_2' = \sum_{k=1}^{\mu_2} M_k''$. In general, choose μ_{i+1} so that $M_i' \subset \sum_{k=1}^{\mu_{i+1}}(M_k'' - \partial M_k'')$, and set $M_{i+1}' = \sum_{k=1}^{\mu_{i+1}} M_k''$. Then $M_i' \subset M_{i+1}' - \partial M_{i+1}'$, and $M = \sum M_i'$.

Let M_i be the set of all points p of M for which the following is not true: For each j there is an arc A joining p to a point $q \in M - M'_j$, such that $A \subset M - M'_i$. We show first that M_i is compact. If not, then there is a sequence p_1, $p_2, \cdots$ of points of M_i with no subsequence which converges in M. Since each M'_j is compact, it contains at most a finite number of the p_k; hence we may suppose the p_k chosen so that $p_j \in M - M'_j$, all j. Since M is connected, there is an arc A_j in M joining p_1 to p_j. Let q_j be the last point of A_j in M'_{i+1} for $j > i$, and let $A(p_j)$ be the arc q_jp_j. Since M'_{i+1} is compact and $\partial M'_{i+1}$ is closed in M'_{i+1}, there is a subsequence $q'_1, q'_2, \cdots$ of $q_1, q_2, \cdots$ converging to a point q of $\partial M'_{i+1}$. For some connected neighborhood U of q, $U \subset M - M'_i$. Choose s so that $q'_k \in U$ for $k \geqq s$. Let p'_k correspond to q'_k. For each $j > s$, $p'_j \in M - M'_j$, and from $A(p'_s)$, $A(p'_j)$, and an arc in U, we find an arc in $M - M'_i$ joining p'_s to p'_j. Hence p'_s is not in M'_i, a contradiction, proving that M_i is compact.

Clearly all boundary points of M_i are in M'_i; hence (a) holds. Since $M'_i \subset M_i$, (b) is true. To prove (c), take $p \in M - M_i$. By definition of the M_j, there is an arc A_i joining p to a point p_{i+1} in $M - M'_{i+1}$, such that $A_i \subset M - M'_i$, there is an arc A_{i+1} joining p_{i+1} to a point p_{i+2} in $M - M'_{i+2}$, such that $A_{i+1} \subset M'_{i+1}$, etc. Since $\sum M'_j = M$, these arcs give a proper half open arc A joining p to infinity, such that $A \subset M - M'_i$. Moreover, $A \subset M - M_i$; for if $q \in A \cap M_i$, then part of A gives an arc in $M - M'_i$ joining q to a point in an arbitrary $M - M_j$, contradicting the definition of M_i. This completes the proof.

24. On proper mappings

We recall the definition from [1]: The *limit set* L_f of a mapping f of a space R into a space R' is the set of points $q \in R'$ such that for some sequence $\{p_k\}$ in R, $f(p_k) \to q$, while $\{p_k\}$ has no limiting point in R. f is *proper* if $f(R) \cap L_f = 0$. Note that if R is compact, the limit set under any mapping is void. If f is proper in R, it is proper in any closed subset of R. If f maps R into a single point, f is proper if and only if R is compact.

REMARK. If f is one–one and continuous, then clearly f is proper if and only if f^{-1} is continuous.

We give first a characterization of proper mappings without use of sequences.

LEMMA 21. *A mapping f of a locally compact separable metric space R into a similar space R' is proper if and only if for each point $p \in R$ (or equally well, each self-compact subset $A \subset R$) there is a neighborhood U of $f(p)$ (or of $f(A)$) in R' and a self-compact subset B of R such that*

$$f(R - B) \cap U = 0. \tag{24.1}$$

We may suppose R is not compact, the lemma being trivial otherwise. Let $R_1, R_2, \cdots$ be self-compact subsets of R with $R_i \subset R_{i+1}$ and $R = \sum R_i$ (see the proof of Lemma 20). Suppose first that the condition in its strong form does not hold; then a self-compact subset A is given; say $A \subset R_k$. Let $\{U_i\}$ be a sequence of neighborhoods of $f(R_k)$ such that $\prod U_i = f(R_k)$. In each $R - R_{k+i}$ choose a point p_i so that $f(p_i) \in U_i$; we may suppose that the p_i are distinct. A subsequence may be chosen so that $f(p_{\lambda_i}) \to q \in f(R_k)$; thus f is not proper.

If, conversely, f is not proper, say $f(p_i) \to q = f(p)$, $\{p_i\}$ having no limit in R. For some subsequence $\{p_{\lambda_i}\}$, $p_{\lambda_i} \in R - R_i$. Since each compact subset of R is in some R_j, the condition does not hold, using p.

We state without proof:

LEMMA 22. *A mapping f of R into R' is proper if and only if antecedents of sets compact in $f(R)$ are sets compact in R.*

LEMMA 23. *A continuous proper mapping of R into R' maps sets closed in R into sets closed in $f(R)$.*

Suppose A is closed, while $f(A)$ is not closed in $f(R)$; then $q_1, q_2, \cdots$ exist in $f(A)$, $q_i \to q$, $q \in f(R) - f(A)$. Choose p_i so that $f(p_i) = q_i$. If the sequence $\{p_i\}$ had a limiting point p, then say $p_{\lambda_i} \to p$; since f is continuous, $q_{\lambda_i} = f(p_{\lambda_i}) \to f(p)$. But $q_{\lambda_i} \to q$; hence $f(p) = q$, and $q \in f(A)$, a contradiction. This shows that f is not proper.

We state without proof:

LEMMA 24. *Let f be continuous and map closed sets into closed sets, and let the antecedents of single points be finite sets of points. Then f is proper.*

The following lemma is needed in one or two places in the present paper.

LEMMA 25. *Let R and R' be locally compact separable metric spaces. Let f, mapping R into R', be continuous, proper, and locally one-one. Let A and B be closed subsets of R such that*

(24.2) *if* $p \in A, \quad q \in B, \quad p \neq q,$ *then* $f(p) \neq f(q).$

Then there are neighborhoods U of A and V of B such that

(24.3) *if* $p \in U, \quad q \in V, \quad p \neq q,$ *then* $f(p) \neq f(q).$

The proof is not very difficult; we expect to give it as an application of much more general ideas in another paper.

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