General Relativity and Gravitation, Vol. 28, No. 3, 1996

The Embedding of General Relativity in Five Dimensions

Carlos Romero,[1,2,3] Reza Tavakol[1,4] and Roustam Zalaletdinov[1,5,6]

Received September 15, 1995

We argue that General Relativistic solutions can always be locally embedded in Ricci-flat 5-dimensional spaces. This is a direct consequence of a theorem of Campbell (given here for both a timelike and spacelike extra dimension, together with a special case of this theorem) which guarantees that any n-dimensional Riemannian manifold can be locally embedded in an $(n+1)$-dimensional Ricci-flat Riemannian manifold. This is of great importance in establishing local generality for a proposal recently put forward and developed by Wesson and others, whereby vacuum (4+1)-dimensional field equations give rise to (3+1)-dimensional equations with sources. An important feature of Campbell's procedure is that it automatically guarantees the compatibility of Gauss–Codazzi equations and therefore allows the construction of embeddings to be in principle always possible. We employ this procedure to construct such embeddings in a number of simple cases.

1. INTRODUCTION

There has recently been a fair deal of work on a proposal by Wesson [24]

[1] Astronomy Unit, School of Mathematical Sciences, Queen Mary & Westfield College, Mile End Road, London E1 4NS, UK
[2] Departamento de Física, Universidade Federal da Paraíba, C. Postal 5008 - J. Pessoa -Pb, 58059-970 Brazil
[3] E-mail: car@maths.qmw.ac.uk
[4] E-mail: reza@maths.qmw.ac.uk
[5] E-mail: rmz@maths.qmw.ac.uk
[6] Department of Theoretical Physics, Institute of Nuclear Physics, Uzbek Academy of Sciences, Ulugbek, Tashkent 702132, Uzbekistan, C.I.S.

0001-7701/96/0300-0365$09.50/0

(see also Refs. 25 and 5, and Ref. 19 for the generalisation of this scheme), which gives a prescription for a possible geometrical origin for matter. Briefly the idea is that vacuum (4+1)-dimensional field equations give rise to (3+1)-dimensional equations with sources. This is essentially similar to the Kaluza–Klein [11,14] scheme, where the fifth dimension is utilised as the source of new degrees of freedom to be associated with the electromagnetic field. There are, however, differences between the two schemes. The scheme of Wesson et al does not attempt to geometrise the electromagnetic field and there is no assumption concerning the compactness of the extra dimension. Furthermore, although Wesson initially interpreted the fifth dimension as being associated with the rest mass of particles, the theory itself can be worked out quite independently of this hypothesis.

Now a large proportion of the work in this area is concerned with the study of the properties, consequences and examples of the scheme. They therefore mainly deal with (i) constructing particular examples — some of physical interest — such as the derivation of Friedmann–Robertson–Walker (FRW) models, by choosing appropriate embeddings [10] and (ii) studying the possible observational consequences of the scheme by, for example, looking at the one body problem in this context [23]. In view of the interest that this proposal has attracted, it is also of importance to delve somewhat deeper into its mathematical foundations.

Our main aim in this paper is to ask whether this scheme is general enough to generate all solutions to Einstein's field equations, at least locally.

To answer this question we shall employ an old theorem by Campbell, the outline of the proof of which we give in a modern notation compatible with that employed by Wesson and including spacelike as well as timelike additional dimensions. We also prove a restricted version of this theorem and employ an important property of the Campbell's procedure, namely that the compatibility of Gauss–Codazzi equations are automatically guaranteed, to construct some simple embeddings.

The organisation of the paper is as follows. In Section 2 we give a description of Wesson's procedure. In Section 3 we briefly review some results for embedding GR solutions in flat spaces. Section 4 contains a theorem and a discussion of Campbell's theorem on embedding GR solutions in 5-dimensional Ricci-flat spaces. In Section 5 we give some simple examples of applications of Campbell's theorem and, finally, Section 6 contains our conclusion.

2. WESSON'S PROCEDURE

In this section we briefly describe the mathematical structure of the Wesson's scheme by spelling out its postulates.

Postulate I. The fundamental space in which our ordinary 4-dimensional spacetime is locally and isometrically embedded may be described by a 5-dimensional manifold M_5. The line element of this space is given by $ds^2 = g_{ab}dx^a dx^b$ which can always be put, at least locally, in the form

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta + \epsilon\phi^2 d\psi^2, \tag{1}$$

where Greek and Latin indices run from zero to 3 and zero to 4 respectively, $x^a = (x^\alpha, \psi)$ are coordinates, $g_{\alpha\beta} = g_{\alpha\beta}(x^a)$, $\phi = \phi(x^a)$, $\epsilon^2 = 1$[7] and $g_{\alpha\beta}$ is assumed to have signature $(+---)$.

Postulate II. The fundamental 5-dimensional space satisfies the vacuum field equations

$$^{(5)}R_{ab} = 0, \tag{2}$$

where $^{(5)}R_{ab}$ is the Ricci tensor in five dimensions, defined by

$$^{(5)}R_{ab} = \Gamma^c_{ab,c} - \Gamma^c_{ac,b} + \Gamma^c_{ab}\Gamma^d_{cd} - \Gamma^c_{ad}\Gamma^d_{bc}, \tag{3}$$

and where Γ^c_{ab} are the 5-dimensional Christoffel symbols.

Now consider the hypersurface Σ_4 defined by the equation $\psi = \psi_0 =$ constant. This induces the metric $^{(4)}g_{\alpha\beta}(x^\mu)$ in Σ_4 given by

$$^{(4)}g_{\alpha\beta}(x^\mu) = g_{\alpha\beta}(x^\mu, \psi_0). \tag{4}$$

From the above definitions it is clear that by substituting $\psi = \psi_0$ in (2) one can split up the 5-dimensional field equations in the following way:

$$^{(4)}R_{\alpha\beta} = \frac{\phi_{,\alpha;\beta}}{\phi} - \frac{\epsilon}{2\phi^2}\left\{\frac{\phi^*}{\phi}\, g^*_{\alpha\beta} - g^{**}_{\alpha\beta} + {}^{(4)}g^{\lambda\mu} g^*_{\alpha\lambda} g^*_{\beta\mu} - \frac{1}{2}\, {}^{(4)}g^{\mu\nu} g^*_{\mu\nu} g^*_{\alpha\beta}\right\}, \tag{5}$$

$$\epsilon\phi\Box\phi = -\frac{1}{4}\, g^{*\lambda\beta} g^*_{\lambda\beta} - \frac{1}{2}\, {}^{(4)}g^{\lambda\beta} g^{**}_{\lambda\beta} + \frac{1}{2\phi}\, \phi^*\, {}^{(4)}g^{\lambda\beta} g^*_{\lambda\beta}, \tag{6}$$

---

[7] To be compatible with Wesson's and Campbell's notations, ϵ would need to take values -1 and $+1$ respectively. We should, however, note here that if the extra dimension is considered to be timelike, the theory might run into conceptual difficulties concerning the existence of observers travelling along closed timelike curves [9].

$$P^{\beta}_{\alpha;\beta} = 0, \tag{7}$$

where

$$P^{\beta}_{\alpha} \equiv \frac{1}{2\phi}\left({}^{(4)}g^{\beta\sigma} g^{*}_{\sigma\alpha} - \delta^{\beta}_{\alpha}\, {}^{(4)}g^{\mu\nu} g^{*}_{\mu\nu}\right), \tag{8}$$

$\Box\phi = {}^{(4)}g^{\alpha\beta}\phi_{,\alpha;\beta}$ is calculated at $\psi = \psi_0$ and where the symbol (*) denotes the left action of the operator $(\partial/\partial\psi)_{\psi=\psi_0}$.[8] Naturally, using

$${}^{(4)}R = \frac{\epsilon}{4\phi^2}\left[g^{*}_{\mu\nu} g^{*\mu\nu} + {}^{(4)}(g_{\mu\nu} g^{*\mu\nu})^2\right], \tag{9}$$

we can define

$${}^{(4)}T_{\alpha\beta} \equiv {}^{(4)}R_{\alpha\beta} - \tfrac{1}{2}\, {}^{(4)}R\, {}^{(4)}g_{\alpha\beta} \tag{10}$$

which, upon substitution from (5), gives

$$\begin{aligned} {}^{(4)}T_{\alpha\beta} = \frac{\phi_{,\alpha;\beta}}{\phi} - \frac{\epsilon}{2\phi^2}\Bigg\{ & \frac{\phi^*}{\phi} g^{*}_{\alpha\beta} - g^{**}_{\alpha\beta} + {}^{(4)}g^{\lambda\mu} g^{*}_{\alpha\lambda} g^{*}_{\beta\mu} - \frac{1}{2}\, {}^{(4)}g^{\mu\nu} g^{*}_{\mu\nu} g^{*}_{\alpha\beta} \\ & + \frac{1}{4}\, {}^{(4)}g_{\alpha\beta}\left[g^{*}_{\mu\nu} g^{*\mu\nu} + ({}^{(4)}g_{\mu\nu} g^{*\mu\nu})^2\right] \Bigg\}. \end{aligned} \tag{11}$$

Postulate III. The energy-momentum tensor which describes the matter content of the 4-dimensional Universe will be given by eq. (11).

Postulates I–III concern the field equations. To complete the scheme one also needs to describe the motion of free-falling test particles and light rays. The following postulate is put forward by Wesson [10] for this purpose.

Postulate IV. The paths corresponding to the motion of free-falling test particles and light rays are the geodesic lines in the 5-dimensional fundamental (vacuum) space.

This is rather a controversial issue which we shall not discuss here, since it lies somewhat outside the embedding problem we are concerned with in this article.

We should also add that, as was pointed out in [19] with the help of examples, this procedure does not in general lead to physical $T_{\mu\nu}$, with for example a positive energy.

Of related interest is the investigation by Schmidt [20] concerning the generation of minimally coupled scalar fields by higher-dimensional Ricci-flat Kaluza–Klein spaces.

---

[8] Equation (7) corresponds to ${}^{(5)}R_{\alpha 4} = 0$ at $\psi = \psi_0$ and it was put in this special form of a 'conservation law' by Wesson (see Ref. 25).

3. THE EMBEDDING OF GENERAL RELATIVITY SOLUTIONS IN FLAT SPACES

Before discussing the idea of embedding of GR in 5 dimensions in the next section, it is worthwhile to discuss briefly the embedding of solutions of GR in flat spaces.

The origin of such embeddings seems to have had its roots in the idea of relating extra dimensions [1] to elementary particles and also as a way of providing a better geometrical understanding of these solutions when viewed as hypersurfaces of higher-dimensional spaces. In addition, it should be mentioned that embeddings give an invariant classification of solutions of Einstein's equations (Ref. 15, Ch.32).

There are, however, two kinds of embeddings which one can consider: local and global. Throughout this paper we shall be concerned with local embeddings only. Global embeddings of Riemannian spaces are much more complicated and far less studied than local embeddings (see, however, Ref. 3)

Some important theorems of relevance to us here, relating to the local embeddings of n-dimensional Riemannian manifolds in m-dimensional flat spaces ($n \leq m$), are as follows.

Theorem I [7]. Any analytic Riemannian n-dimensional space can be locally and isometrically embedded in some m-dimensional flat space, with $n \leq m \leq n(n+1)/2$.

Theorem II [12]. A non-flat n-dimensional solution of vacuum Einstein equations cannot be embedded in an $(n+1)$-dimensional flat space.

Theorem II was first proved by Kasner in 1921 [12] and it explains why the Schwarzschild solution cannot be embedded in a 5-dimensional flat spacetime, a problem which was also considered some years later by Einstein [6]. Therefore, to (minimally) embed Schwarzschild metric in a flat space one needs at least six dimensions, a result that was again obtained by Kasner [13]. However, as we shall see below, it is possible to embed the Schwarzschild solution in a 5-dimensional Ricci-flat space and this is a special case of a theorem to be proved in the next section.

4. THE EMBEDDING OF GENERAL RELATIVITY IN FIVE DIMENSIONS

As can be seen, Theorem I puts an upper bound on the number of dimensions one needs in order to embed GR locally in a higher-dimensional space, while Theorem II implies that GR vacuum solutions cannot be embedded in 5-dimensional flat space. Nevertheless, local embeddings of GR vacuum solutions in a 5-dimensional Ricci-flat space are always possible and this is guaranteed by the following theorem.

Theorem III. Any analytic n-dimensional Ricci-flat space can be locally embedded in an analytic $(n+1)$-dimensional Ricci-flat space.

The proof of this theorem is straightforward. Suppose the line element of the n-dimensional space (to be embedded) is given by

$$^{(n)}ds^2 = {}^{(n)}g_{\alpha\beta}(x^\mu)dx^\alpha dx^\beta, \tag{12}$$

where $\alpha, \beta, \ldots = 0, \ldots, n-1$ in this section. Now, let us construct the embedding by defining the line element of the $(n+1)$-dimensional space to be in the form

$$^{(n+1)}ds^2 = {}^{(n)}g_{\alpha\beta}(x^\mu)dx^\alpha dx^\beta + \epsilon d\psi^2. \tag{13}$$

For this special choice of embedding it can be readily shown that the $(n+1)$-dimensional Ricci tensor calculated from (13) is given by

$$^{(n+1)}R_{\alpha\beta} = {}^{(n)}R_{\alpha\beta}, \tag{14}$$

and

$$^{(n+1)}R_{\alpha n} = 0 = {}^{(n+1)}R_{nn}. \tag{15}$$

Thus, n-dimensional Ricci-flat spaces are embedded through (13) in $(n+1)$-dimensional Ricci-flat spaces. It is important to note that this theorem holds irrespective of the signature of the metric of the n-dimensional space and the sign of ϵ. The embedding of Schwarzschild solution obtained by Wesson [22] is then a direct consequence of this theorem. We should add, however, that Theorems I,II and IV (to be stated in the following) are highly non-trivial, whereas the rather trivial Theorem III is included for methodical reasons.

From the point of view of our discussion in this paper, concerning the Wesson's procedure, it would be of importance if Theorem III could be extended to the case of non-vacuum solutions. The question then is whether it is possible to locally embed any arbitrary n-dimensional Riemannian manifold in some $(n+1)$-dimensional Ricci-flat space. Apart from being of mathematical interest, this is of fundamental importance in the context of Wesson's scheme, since an affirmative answer would imply that all solutions of Einstein's equations could in principle be embedded in the 5-dimensional Ricci-flat spaces. Clearly, an equivalent way of formulating the same question is to ask whether any arbitrary $^{(n)}T_{\alpha\beta}$ can be given by means of eq. (11) (see Ref. 17). It turns out that such a theorem exists.

Theorem IV [2]. Any analytic n-dimensional Riemannian space can be locally embedded in a $(n+1)$-dimensional Ricci-flat space.

This theorem was given by Campbell in 1926 with its proof later completed by Magaard [16] (see also Ref. 8), but very little reference to it can be found in the literature. In the light of this and the obscurity of the notation used by Campbell, we shall briefly sketch the main lines of Campbell's proof, in a notation that makes transparent its relationship to Wesson's work and which includes both timelike and spacelike additional dimensions.

Start with a n-dimensional analytic Riemannian manifold with a metric given by

$$ {}^{(n)}ds^2 = {}^{(n)}g_{\alpha\beta}(x^\mu)dx^\alpha dx^\beta, \tag{16} $$

and let the line element of the $(n+1)$-dimensional space be defined by

$$ {}^{(n+1)}ds^2 = g_{\alpha\beta}(x^\mu,\psi)dx^\alpha dx^\beta + \epsilon\phi^2(x^\mu,\psi)d\psi^2, \tag{17} $$

where ψ is the coordinate assigned to the extra dimension and where $g_{\alpha\beta}$ and ϕ are analytic functions of all the $(n+1)$ coordinates. Assume that $g_{\alpha\beta}$, when restricted to a certain hypersurface $\psi=\psi_0$, results in ${}^{(n)}g_{\alpha\beta}$, i.e.

$$ g_{\alpha\beta}(x^\lambda,\psi_0) = {}^{(n)}g_{\alpha\beta}(x^\lambda). \tag{18} $$

Suppose also that the analogues of eqs. (5)–(7) hold for $\psi=\psi_0$. Clearly, this is equivalent to requiring that the $(n+1)$-dimensional Ricci tensor vanishes in the hypersurface $\psi=\psi_0$. Finally define the functions $\Omega_{\alpha\beta}(x^\mu,\psi)$ by the equations

$$ \frac{\partial g_{\alpha\beta}}{\partial\psi} = -2\phi\Omega_{\alpha\beta}\,. \tag{19} $$

With these definitions eqs. (5)–(7) take the form

$$ {}^{(n)}g^{\lambda\mu}(\Omega_{\alpha\beta}\Omega_{\lambda\mu} - 2\Omega_{\alpha\lambda}\Omega_{\beta\mu})\phi + \epsilon\phi_{,\alpha;\beta} - \Omega^*_{\alpha\beta} - \epsilon\phi R_{\alpha\beta} = 0, \tag{20} $$

$$ {}^{(n)}g^{\lambda\beta}(\epsilon\phi_{,\lambda;\beta} - \Omega^*_{\lambda\beta} - \phi\,{}^{(n)}g^{\alpha\rho}\Omega_{\beta\rho}\Omega_{\lambda\alpha}) = 0, \tag{21} $$

and

$$ {}^{(n)}g^{\mu\nu}(\Omega_{\alpha\nu;\mu} - \Omega_{\nu\mu;\alpha}) = 0, \tag{22} $$

where the functions $\Omega_{\alpha\beta}$ and ϕ take values at $\psi=\psi_0$. Now, suppose that one is able to find functions $\Omega_{\alpha\beta}$ which at $\psi=\psi_0$ satisfy the following conditions [2]:

$$\Omega_{\alpha\beta} = \Omega_{\beta\alpha}\,, \tag{23}$$

$$\Omega^{\alpha}_{\beta;\alpha} = \Omega_{\beta}\,, \tag{24}$$

$$\Omega_{\alpha\beta}\Omega^{\alpha\beta} - \Omega^2 = -\epsilon\,{}^{(n)}R, \tag{25}$$

where $\Omega^{\alpha}{}_{\beta} \equiv {}^{(n)}g^{\alpha\lambda}\Omega_{\lambda\beta}$, $\Omega \equiv {}^{(n)}g^{\mu\nu}\Omega_{\mu\nu}$ and ${}^{(n)}R = {}^{(n)}g^{\mu\nu}\,{}^{(n)}R_{\mu\nu}$ is the curvature scalar of the n-dimensional space. Assume further that $g_{\alpha\beta}$ and $\Omega_{\alpha\beta}$ evolve according to (19) and

$$\frac{\partial\Omega^{\alpha}_{\beta}}{\partial\psi} = \phi(-\epsilon\,{}^{(n)}R^{\alpha}_{\beta} + \Omega\Omega^{\alpha}_{\beta}) + \epsilon g^{\alpha\lambda}\phi_{,\lambda;\beta} \tag{26}$$

respectively, with $g_{\alpha\beta}$, in addition to (18), satisfying the initial conditions

$$g^{*}_{\alpha\beta} = -2\phi(x^{\lambda},\psi_0)\Omega_{\alpha\beta}(x^{\lambda},\psi_0), \tag{27}$$

where ${}^{(n)}R^{\alpha}_{\beta}$ is now calculated in terms of $g_{\alpha\beta}(x,\psi)$, which is also used for raising and lowering tensor indices. Then one can prove [2] that (23)–(25) also hold for all ψ in a neighbourhood of ψ_0. It then follows that (19),(24),(25), and (26) would together imply

$${}^{(n+1)}R_{ab} = 0, \tag{28}$$

for any value of ψ in the neighbourhood of ψ_0. In other words, metric (17) represents an embedding of the metric (16) in a Ricci-flat $(n+1)$-dimensional space.

5. SIMPLE APPLICATIONS OF CAMPBELL'S THEOREM

An important feature of Campbell's result is that the procedure employed ensures the compatibility of Gauss–Codazzi equations. As a result, it makes the construction of embeddings always possible, at least in principle. Here we illustrate this scheme with the help of some simple examples.

(a) Suppose we wish to embed the metric

$$ds^2 = dt^2 - t(dx^2 + dy^2 + dz^2), \tag{29}$$

corresponding to a spatially flat FLRW model with a radiation equation of state $p = \rho/3$, in a Ricci-flat space. First, from (29) we calculate ${}^{(4)}R^{\alpha}{}_{\beta}$

and ${}^{(4)}R$ to obtain ${}^{(4)}R^{\alpha}_{\beta} = \mathrm{diag}\,[(3/4t^2), -(1/4t^2), -(1/4t^2), -(1/4t^2)]$ and ${}^{(4)}R = 0$. Thus eq. (25) becomes

$$\Omega_{\alpha\beta}\Omega^{\alpha\beta} - \Omega^2 = 0\,. \tag{30}$$

The simplest choice of $\Omega_{\alpha\beta}$ which would trivially satisfy both (24) and (30) is $\Omega_{\alpha\beta} \equiv 0$. Then, eqs. (19) and (26) read

$$\frac{\partial g_{\alpha\beta}}{\partial\psi} = 0 \tag{31}$$

and

$$\frac{1}{\phi}\, g^{\lambda\alpha}\phi_{,\lambda;\beta} = {}^{(4)}R^{\alpha}_{\beta} \tag{32}$$

Clearly (31) together with the initial condition (18) implies that $g_{\alpha\beta} = {}^{(4)}g_{\alpha\beta}$. Thus, we only have to solve (32). Contracting the indices α and β, this equation results in

$$\Box\phi = 0\,. \tag{33}$$

Now since $R^{\alpha}{}_{\beta}$ depends only on the variable t, we assume that $\phi = \phi(t)$ and (33) thus becomes

$$\frac{d(t^{3/2}\dot{\phi})}{dt} = 0, \tag{34}$$

which has the general solution

$$\phi(t) = at^{-1/2} + b, \tag{35}$$

where a and b are arbitrary constants. Taking $a = 1$ and $b = 0$, which ensures that (35) is a solution of (32), this finally gives the 5-dimensional metric

$$ds^2 = dt^2 - t(dx^2 + dy^2 + dz^2) + \epsilon t^{-1} d\psi^2, \tag{36}$$

which is a solution with a shrinking fifth dimension, previously obtained by Wesson [24] and also in a different context by Chodos et al. [4].

(b) As a second example we consider the embedding (in 5-D Ricci-flat space) of de Sitter metric[9]

$$ds^2 = dt^2 - e^{2\sqrt{\Lambda/3}\,t}(dx^2 + dy^2 + dz^2), \tag{37}$$

[9] The embedding of the de Sitter spacetime was already known to de Sitter himself [21].

where Λ is the cosmological constant.

Now taking $\phi = 1$ very much simplifies (26) and (19). Again, calculating ${}^{(4)}R$ and ${}^{(4)}R^{\alpha}_{\beta}$ from (37) we obtain

$$ {}^{(4)}R = -4\Lambda \tag{38} $$

and

$$ {}^{(4)}R^{\alpha}_{\beta} = -\Lambda \delta^{\alpha}_{\beta}\,, \tag{39} $$

which substituted in (26) give

$$ \frac{\partial \Omega^{\alpha}_{\beta}}{\partial \psi} = \epsilon \Lambda \delta^{\alpha}_{\beta} + \Omega \Omega^{\alpha}_{\beta}\,. \tag{40} $$

For the case where the extra dimension is spacelike ($\epsilon = -1$) this equation is satisfied by the functions

$$ \Omega^{\alpha}_{\beta} = -\psi^{-1}\delta^{\alpha}_{\beta}, \tag{41} $$

at $\psi = \psi_0 = \pm\sqrt{3/\Lambda}$. Integrating (19) subject to the initial conditions (18) we obtain

$$ g_{\alpha\beta} = \frac{\Lambda}{3}\,\psi^{2\,(4)}g_{\alpha\beta}\,. \tag{42} $$

It is easily verified that the functions (41) satisfy the conditions (24) and (25). Finally, we write the 5-dimensional metric of the Ricci-flat embedding space as

$$ {}^{(5)}ds^2 = \Lambda\frac{\psi^2}{3}\,dt^2 - \Lambda\frac{\psi^2}{3}\,e^{2\sqrt{\Lambda/3}\,t}(dx^2 + dy^2 + dz^2) - d\psi^2, \tag{43} $$

which induces the metric (37) on the hypersurfaces $\psi = \psi_0 = \pm\sqrt{3/\Lambda}$. We should point out that an equivalent solution was also obtained by Ponce de Leon [18].

(c) As a final example, let us consider the embeddings of vacuum solutions of Einstein's field equations. From eqs. (24)–(26) we conclude that the trivial choice

$$ \Omega_{\alpha\beta} = \phi_{\alpha;\beta} = 0, \tag{44} $$

is a solution. This leads to the embedding

$$ {}^{(5)}ds^2 = {}^{(4)}g_{\alpha\beta}dx^{\alpha}dx^{\beta} - \epsilon\phi^2 d\psi^2, \tag{45} $$

with the function ϕ satisfying (44).

6. CONCLUSION

Historically, the question of embedding of GR in higher dimensional flat spaces goes back to the work of Kasner in 1921. Recently, however, Wesson and others have developed a scheme in which the embedding space is a 5-dimensional Ricci-flat space. Here by employing Campbell's theorem we have argued that all solutions of GR can be locally embedded in a 5-dimensional Ricci-flat space. This is of great importance for the proposal recently put forward by Wesson, in that it is given a large degree of generality, by showing a *local equivalence* between GR and 5-dimensional vacuum Kaluza–Klein equations. We have also given a less restricted theorem showing that all Ricci-flat n-dimensional Riemannian spaces can be embedded in $(n+1)$-dimensional Ricci-flat spaces. Mathematically, Campbell's theorem implies that if the embedding space is Ricci-flat (rather than flat), then the compatibility of Gauss–Codazzi equations are automatically guaranteed. We use this fact to construct some simple embeddings.

The equivalence referred to here is local. However, based on the usual global results (where the embedding space is Riemann-flat; Ref. 3), one would not expect this equivalence to hold globally. The question then is what is the global status of Wesson's procedure. This together with the application of these ideas to lower dimensional gravity are under consideration and we shall report on them in due course.

ACKNOWLEDGEMENTS

CR was partially supported by CNPq (Brazil), RT was supported by SERC UK Grant number H09454 and RZ was supported by a Royal Society fellowship. CR and RZ would like to thank the School of Mathematical Sciences for hospitality. We also would like to thank the referees for helpful comments.

REFERENCES

1. Applequist, T., Chodos, A., and Freund, P. (1987). *Modern Kaluza–Klein Theories* (Addison-Wesley, New York).
2. Campbell, J. E. (1926). *A Course of Differential Geometry* (Clarendon Press, Oxford).
3. Clarke, C. J. S. (1970). *Proc. Roy. Soc. Lond.* **314A**, 417.
4. Chodos, A., and Detweiler, S. (1980). *Phys. Rev. D***21**, 2167.
5. Coley, A. A. (1994). *Astrophys. J.* **47**, 585.
6. Einstein, A. (1941). *Revista (Universidade Nacional de Tucuman) A***2**, 11.
7. Eisenhart, L. P. (1949). *Riemannian Geometry* (Princeton University Press, Princeton).

8. Goenner, H. (1980). In *General Relativity and Gravitation*, A. Held, ed. (Plenum Press, New York), vol. 1
9. Grøn, Ø. (1988). *Astron. Astrophys.* **193**, 1.
10. Kalligas, D., Wesson, P. S., and Everitt, C. W. F. (1995). *Astrophys. J.* **439**, 548.
11. Kaluza, T. (1921). *Sitzungsber. Preuss. Akad. Wiss. Berlin, Phys. Math.*, **K1** 33, 966.
12. Kasner, E. (1921). *Amer. J. Math.* **43**, 126.
13. Kasner, E. (1921). *Amer. J. Math.* **43**, 130.
14. Klein, O. (1926). *Z. Phys.* **37**, 895.
15. Kramer, D., Stephani, H., MacCallum, M. A. H., and Herlt, E. (1980). *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge).
16. Magaard, L. (1963). "Zur Einbettung Riemannscher Räume in Einstein-Räume und konform-euklidische Räume." PhD thesis, University of Kiel.
17. Mc Manus, D. J. (1994). *J. Math. Phys.* **35**, 4889.
18. Ponce de Leon, J. (1988). *Gen. Rel. Grav.* **20**, 539.
19. Rippl, S., Romero, C., and Tavakol, R. (1995). *Class. Quant. Grav.* **12**, 2411.
20. Schmidt, H.-J. (1990). *Int. J. Mod. Phys.* **A5**, 4661.
21. de Sitter, W. (1918). *Proc. Amsterdam* **20**, 229.
22. Wesson, P. S. (1984). *Gen. Rel. Grav.* **16**, 193.
23. Wesson, P. S. (1992). *Astrophys. J.* **397**, L91.
24. Wesson, P. S. (1992). *Astrophys. J.* **394**, 19.
25. Wesson, P. S., and Ponce de Leon, J. (1992). *J. Math. Phys.* **33**, 3883.