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ISOMETRIC IMMERSIONS OF RIEMANNIAN SPACES IN EUCLIDEAN SPACES

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Questions of the theory of isometric immersions of Riemannian spaces in Euclidean spaces beginning with the very first results on this topic and also results on immersions of pseudo-Riemannian spaces in pseudo-Euclidean spaces and applications of the theory of immersions in the general theory of relativity are considered.

The paper is devoted to a survey of works on isometric immersions of Riemannian and pseudo-Riemannian spaces in Euclidean and pseudo-Euclidean spaces.

The question of immersions of Riemannian spaces is connected with two distinct approaches to the problem of studying Riemannian manifolds. The first of these consists in investigating an abstractly defined manifold. The second consists in investigating a Riemannian manifold as a submanifold of Euclidean space. The following question arises naturally: Is every n-dimensional Riemannian manifold V^n a submanifold of Euclidean space E^N? In the most general formulation this question was solved positively in the fifties and sixties of our century by the American mathematician Nash [141, 142, 143]. The investigations of Kuiper [122, 123] are closely related to those of Nash. Although the results of Nash and Kuiper are of universal character, they cannot be considered definitive, since they do not give a complete answer to the very important question of the choice of the optimal dimension N of the Euclidean space E^N in which a given V^n or some class of Riemannian spaces is immersed. The corresponding problematics will be formulated in the paper, and a survey of results will be given.

In the paper major coverage is given to papers on immersions of pseudo-Riemannian spaces in pseudo-Euclidean spaces. Interest in this theme is to considerable extent connected with various problems of theoretical physics and theoretical astronomy. Clarifications of the physical character of corresponding results will be given along with a survey of papers on this topic.

We shall not consider in detail the results of immersions of two-dimensional Riemannian metrics, since the surveys [21, 22] are devoted to this problem, while the fundamental papers of Aleksandrov [1] and Pogorelova [18] deal with immersions of two-dimensional metrics of positive curvature. We shall consider these questions and also questions related to immersions in curved spaces only to the extent that they aid in understanding the history of the development of the problem considered.

The bibliography extends to the end of 1976. We shall use the following notation: E^n is n-dimensional Euclidean space, $E^n_{(p,q)}$ is n-dimensional pseudo-Euclidean space with signature (p, q).

1. A Survey of Papers on Isometric Immersions up to 1950

1. Formulation of the Problem. Basic Results. The problem of isometrically embedding of Riemannian space V^n in some Euclidean space E^N was first formulated by Schlaefli [167] in 1873 and, so it seemed to him, not only formulated but also solved. Schlaefli obtained the following equations (we shall henceforth call them the Schlaefli equations):

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$$r_i r_j = g_{ij}, \tag{1}$$

in which $g_{ij} = g_{ij}(x^1, \ldots, x^n)$ are the coordinates of the metric tensor of the Riemannian space V^n; $x^1, \ldots, x^n$, intrinsic coordinates; and $r_i = \partial r / \partial x^i$, where $r = r(x^1, \ldots, x^n)$, is the desired radius vector of the submanifold M^n in E^N on which an inner metric is induced which coincides with the metric of V^n.

Since the number of equations (1) is equal to $s_n = n(n+1)/2$, Schlaefli was convinced that at least locally* V^n can be embedded in E^{s_n}. In other words, the question of the existence of solutions of the system (1) for $N = s_n$ did not arise [in this case the number of equations and the number of unknown functions – the coordinates of the vector $r(x^1, \ldots, x^n)$ – coincide].

For analytic Riemannian metrics V^n the local Schlaefli problem was solved in the work of Janet [114] (1926), Cartan [74] (1927), and Burstin [73] (1931).

To solve the problem of the embedding of analytic Riemannian manifolds V^n in E^{s_n} Cartan made use of the tools of the method of outer forms which he created. It should be noted that so far Cartan's proof has not been simplified nor have the basic ideas of his arguments been clarified.

In the work cited of Janet for the linear element† $ds^2 = g_{11}(dx^1)^2 + g_{ij}dx^i dx^j$ $(i > 1,\ j > 1)$ the Schlaefli equations [see (1)] were brought to the form

$$\begin{gathered} r_{11} r_k = 0, \quad k = 1, 2, \ldots, n, \\ r_{11} r_{lm} = r_{1l} r_{1m} - \frac{1}{2} \frac{\partial^2 g_{lm}}{\partial x^{12}}, \quad l > 1, \quad m > 1 \end{gathered} \tag{2}$$

by differentiations of the simplest algebraic operations. It is clear that if all the vectors r_k, r_{lm} [the number of these vectors is equal to $n(n+1)/2$] are linearly independent, then the system (2) can be solved for the coordinates of the vector r_{11}. As a result, a system of Cauchy–Kovalevskaya type is obtained for which a solution exists. Janet did not prove the possibility of a choice of initial data for which the linear independence of the vectors r_k, r_{lm} is ensured. This plan was realized by Burstin. He constructed an inductive process for the isometric embedding in E^{s_n} of specially chosen submanifolds in V^n of increasing dimensions such that at the n-th step of this process the desired isometric immersion of V^n in E^{s_n} is obtained, while for this immersion all the vectors r_k, r_{lm} are linearly independent. This type of isometric immersion of V^n in E^N subsequently became known as a free immersion.

2. Problem of the Class of a Riemannian Metric. The Problem of Nonimmersibility. In 1886 Schur published the work [169] in which the possibility is established of the local, analytic, isometric immersion of Lobachevskii space H^n in E^N for $N = 2n - 1$. This result is very important in clarifying the problematics of the theory of isometric immersions.

The relation of dimensions (n and $N = 2n - 1$) of the immersed space H^n in the Euclidean space E^N in Schur's result differs sharply from the relation of the dimensions [n and $N = s_n = n(n+1)/2$] in the general result (Schlaefli, Janet, Cartan, and Burstin). It is therefore natural to formulate the following two important problems of the theory of isometric immersions – the problem of the class of the Riemannian metric and the problem of nonimmersibility.

The problem of the class of the Riemannian metric of V^n consists in resolving the question of the minimal dimension N of the Euclidean space E^N in which V^n can be isometrically immersed. The difference $N - n$ is called the class of the Riemannian metric of V^n.

In its original formulation this question pertained to analytic immersions of analytic Riemannian metrics. It became clear only in the fifties that the differentiability conditions in this problem are very basic; it followed from the remarkable result of Nach [141] that if the immersion is only required to be of class C^1, then locally the class of all Riemannian metrics is equal to 1, i.e., locally all Riemannian metrics of dimension n can be immersed as hypersurfaces of class C^1 in E^{n+1}.

We shall subsequently discuss various aspects of this problem of the metric class in surveying other works on the theory of immersions.

The second important problem – the problem of the immersibility of a given Riemannian n-dimensional manifold in Euclidean space E^N of given dimension N – was first formulated by Hilbert in 1900 in his famous Problems [104]. In these Problems Hilbert posed the problem of the existence in E^3 of a complete surface of

*At the time of Schlaefli's work local and global embedding of a manifold were not distinguished.

†Locally the linear element of V^n can always be brought to the indicated form.

constant negative curvature. In 1901 in the work entitled "On surfaces of constant negative curvature" [105, 106] Hilbert proved the impossibility of such a surface in E^3. In other words, a complete two-dimensional Riemannian manifold of constant negative curvature cannot be isometrically immersed in E^3 as a surface of class C^2.*

The problem of nonimmersibility of Riemannian metrics is obviously related to the problem of the metric class – if the class of a given Riemannian metric is known, then the dimension of the Euclidean space in which this metric cannot be immersed is known. However, it is necessary to be more precise here. We have already mentioned the importance of the differentiability conditions on the manifold in E^N on which a given Riemannian metric is induced.

Local and global formulations of the problems also play a basic role in these two problems. Thus, if we consider the question of the immersion of a two-dimensional Riemannian manifold of constant negative curvature (the Lobachevskii plane) in Euclidean space, then here the local and global formulations of the question are distinct – locally the Lobachevskii plane can be immersed in E^3 and therefore the class of its metric is equal to 1; globally it cannot be immersed in E^3 but in E^5 (see Blanusa [74], Rozendorn [23]). Thus, the question of the global metric class of the Lobachevskii plane has so far not been settled.

Before 1950 the problem of the nonimmersibility of a Riemannian metric was investigated in the work of Bianchi [48], Liber [12, 13], and Tompkins [191].

The impossibility of the local immersion of a Riemannian space H^n of constant negative curvature as a hypersurface of Euclidean space was proved in the work cited of Bianchi.

The local nonimmersibility of the space H^n was investigated by Liber in the work cited above. He established that H^n cannot be locally immersed as an analytic surface in E^{2n-2}. Since according to Schur's result H^n can be isometrically immersed in E^{2n-1}, Liber proved that the local class of the metric of H^n under the condition of analytic immersion is equal to $n-1$. We note that the question of the possibility of global immersion of H^n in E^{2n-1} has not been solved.

In 1939 Tompkins (see the work cited above) investigated the question of the global nonimmersibility of multidimensional Riemannian metrics. He proved that a compact, locally Euclidean, n-dimensional Riemannian manifold V^n (e.g., the n-dimensional torus) cannot be regularly (in class C^2) immersed globally in E^{2n-1}. We note that the n-dimensional torus with a flat metric has an immersion in E^{2n}. In the work of Tompkins [191] the connection of the local outer and inner geometry of an immersed manifold is used in the proof of nonimmersibility. For example, in E^3 each regular (of class C^2) developable surface is ruled and if it is complete, then any rectilinear generator of it is complete. This implies the nonimmersibility in the class of surfaces C^2 of the two-dimensional torus with the Euclidean metric in E^3. Tompkins proved that locally any n-dimensional compact manifold with Euclidean metric regularly immersed in Euclidean space of dimension $2n-1$ has a ruled structure in a particular sense, and therefore globally such a manifold cannot be immersed in E^{2n-1}. Tompkins' results were generalized in the work of Chern and Kuiper [76], Otsuki [154] and O'Neil [150]. More will be said about this below.

We note one further result of Tompkins [192]. In this work he constructed an immersion (with self-intersections) of the Klein bottle with Euclidean metric in E^4. Another immersion of the Klein bottle in E^4 (also with self-intersections) was suggested by Ivanov [9].

Above we spoke of the problem of the class of a Riemannian metric, of the natural distinction of the concepts of local and global class of such a metric, and also of the relation of the value of the class of the metric to the differentiability requirements of the immersion. Until 1950 mainly the question of the local metric class under the condition of analyticity of the immersion was investigated. As a rule, it was determined which metrics have a given local class. To this end a system of immersion equations was considered which turned out to be overdetermined, and conditions were found for its compatibility. If these conditions were expressed in terms of the inner metric of the immersed manifold, then as a result the desired characterization of metrics of a given local class was obtained.

The first work on the class problem was that of Schouten and Struik [168] in 1921 in which it was proved that if the Ricci tensor of the space is indentically zero, then the local class of this space is different from one. In other words, either the space is flat and the local class is equal to zero or the space cannot be a hypersurface.

*Kuiper [122] proved the possibility of the isometric immersion of such a manifold in E^3 as a surface of class C^1.

In 1940-43 Rozenson in the work [26-28] obtained a criterion for spaces of local class 1.

Earlier results of Weise [196] and Thomas [190] were used in the work of Rozenson. Results on metrics of class 1 are systematically surveyed in the article of Yanenko [42].

The problem of the local class of Riemannian metrics was discussed in the work of Allendorfer [46] and investigated in detail in the work of Yanenko [43-45]. The results of Yanenko will be discussed in more detail in the next section.

2. Work on the Theory of Immersions after 1950

1. C^1-Isometric Immersions of Nash and Kuiper. A number of fundamental results in the theory of immersions were obtained after 1950.

In 1954 Nash published the work [141] on so-called global C^1-isometric immersions of Riemannian manifolds in Euclidean spaces. The result Nash obtained consists in the following. Suppose that on a closed differentiable manifold M^n there is given a Riemannian metric, and a Riemannian manifold V^n is thus obtained. If M^n can be topologically immersed in Euclidean space E^N ($N \geq n + 2$) as a C^∞-manifold, then V^n can be isometrically immersed in E^N as an n-dimensional surface of class C^1.*

It was noted in the work of Nash that the condition $N \geq n + 2$ can be replaced by the condition $N \geq n + 1$. Kuiper [122] justified precisely the result formulated above for $N \geq n + 1$.

The method proposed by Nash is as follows. Let M^n be topologically immersed in E^N ($N \geq n + 2$) as a C^∞-submanifold. By suitable transformation of this topological immersion in E^N a so-called short immersion of the Riemannian manifold V^n can be obtained, i.e., an n-dimensional submanifold V_1^n in E^N on which the induced Riemannian metric ds_1 is related to the Riemannian metric ds given on V^n at corresponding points and directions by the relation $ds_1/ds \leq a_1 < 1$. By means of the "twisting" operation proposed by Nash the short immersion V_1^n is transformed into a short immersion V_2^n for which the quantity $ds_2/ds \leq a_2 < 1$ satisfies the condition $a_1 < a_2$.

The basic idea of the twisting operation consists in the following. Let $z_1^\alpha = z_1^\alpha(x_i)$ be the parametric equations of V_1^n.† Since $N \geq n + 2$, it is possible to construct on V_1^n two mutually orthogonal vector fields ξ^α and η^α. The embedding V_2^n is defined by the parametric equations

$$z_2^\alpha = z_1^\alpha + \xi^\alpha \frac{\sqrt{a}}{\lambda} \cos \lambda\psi + \eta^\alpha \frac{\sqrt{a}}{\lambda} \sin \lambda\psi,$$

where $a(x_i)$ and $\psi(x_i)$ are functions defined on V_1^n, and λ is an arbitrary constant which is sufficiently large.‡ It is easy to see that the metric tensor of the metric ds_2 induced on V_2^n differs from the metric tensor of the metric ds_1 by the quantity

$$\Delta g_{ij} = a \frac{\partial\psi}{\partial x_i} \frac{\partial\psi}{\partial x_j} + O\left(\frac{1}{\lambda}\right).$$

From this relation it is evident that by choice of the functions a, ψ and the constant λ the required relation $a_1 < a_2 < 1$ can be achieved.

By applying the "twisting" operation to V_2^n, then to V_3^n, etc. a sequence $\{V_k^n\}$ of C^∞-manifolds is constructed in E^N with induced metrics $\{ds_k\}$ which converge to ds, and the V_k^n themselves converge to a C^1-manifold with induced metric which coincides with the given metric.

Surprising corollaries are obtained from the results of Nash and Kuiper just presented. For example, according to Whitney [198] each compact n-dimensional differentiable manifold M^n can be topologically immersed in E^{2n-1} as a C^∞-manifold, and therefore any compact Riemannian manifold V^n can be C^1-isometrically immersed in E^{2n-1}. If we consider any Riemannian manifold homeomorphic to the n-dimensional sphere, then, according to Kuiper's result, this manifold can be C^1-isometrically embedded in E^{n+1}. We note that any Riemannian manifold can be C^1-isometrically immersed in E^{2n-1}.

*Nash established that if the topological immersion of M^n in E^N is an embedding (there are no self-intersections), then V^n can be isometrically embedded in E^N.

†V_1^n can be covered by a system of neighborhoods in each of which there is a system of curvilinear coordinates x_i.

‡This formula explains the term "twisting": for large λ the immersion of V_2^n becomes strongly twisted due to the terms $\cos \lambda\psi$ and $\sin \lambda\psi$.

In the preceding section while discussing the immersion equations obtained by Schlaefli, it was found that the number of equations of this system is equal to $N = s_n = n(n + 1)/2$, and at first glance it therefore seems natural that at least locally n-dimensional Riemannian spaces can be isometrically immersed in E^N for $N = n(n + 1)/2$. The analytic treatment of the results of Nash and Kuiper is thus the more surprising – in the class of C^1-isometric immersions of n-dimensional Riemannian manifolds in E^N for $N \geq n + 2$ (or even for $N \geq n + 1$) the very overdetermined system of immersion equations always has a solution belonging to the class C^1.

A number of substantial refinements to the results of Nash and Kuiper formulated above were obtained in the work [30] of Rokhlin and colleagues.

The C^1-isometric immersions of Riemannian metrics in Euclidean spaces obtained by means of the method proposed by Nash and perfected by Kuiper have precisely the indicated regularity class C^1 (second derivatives are discontinuous at each point). These embeddings do not have the connection between inner and outer properties which are usual in differential geometry; for this the requirement that the immersions belong to the class C^2 is essential.* There naturally arises the question of for which α in the class of $C^{1,\alpha}$-immersions does this connection "arise" or "disappear." This question was investigated in a cycle of papers of Borisov [3]. He established that for the values $0 \leq \alpha < 1/7$ the corresponding immersions do not have the connection between the outer and inner geometry, while for the values $2/3 < \alpha \leq 1$ this connection is present.

It is also natural to pose the question of nonsmooth isometric immersions. Such immersions of two-dimensional metrics in E^3 were considered by Burago [4, 5].

Shefel' [36-38] posed the question of refining the concept of a regular immersion. According to Shefel', regularity of an immersion depends on the immersed metric belonging to a certain class K and on particular stability properties of the immersion with respect to a group of transformations acting in E^N. For example, if we consider so-called convex† immersions in E^N of two-dimensional metrics of nonnegative curvature defined on the sphere and if we require further that the convexity property be preserved under affine transformations, then it is found that the immersion constructed belongs to $E^3 \subset E^N$.

Shefel' calls such immersions of two-dimensional metrics of nonnegative curvature completely regular. This term is also used in other instances.

2. Nash Theory of Regular Immersions. In 1956 Nash [142] proposed a method of regular isometric immersions of regular Riemannian manifolds in Euclidean spaces. The central feature in Nash's constructions is the generalization to the nonanalytic case of the Cauchy–Kovalevskaya theorem for the immersion equations. He considered the problem of immersion of a metric sufficiently close to a metric of a submanifold S of Euclidean space E^N. For this problem he formed equations for the deformation of the surface necessary to realize the close metric. These equations can be represented as a Cauchy–Kovalevskaya system if the submanifold S in question is a free immersion (the first and second derivatives of the radius vector of S are linearly independent at each point). For this it is necessary that the dimension N be $n(n + 1)/2 + n$ (for manifolds of complicated topological structure the dimension N must be raised).

By means of fine arguments Nash was able to prove the existence of solutions to the system he formed. In other words, the possibility of immersion of metrics sufficiently close to a metric of a submanifold S was proved, and the degree of closeness of the immersed metrics was established.

It is important to emphasize that the arguments of Nash are not related to the specific form of the immersion equations; he obtained a general theorem for the solvability of the Cauchy problem for nonanalytic equations of Cauchy–Kovalevskaya type (under particular additional conditions). The importance of this theorem goes far beyond the framework of the theory of immersions, since it is apparently the first generalization of this type.

Many authors (Schwartz [170], Moser [139, 140], Rokhlin [37]) attempted to give the Nash result a clear formulation. In the opinion of the authors of this survey, this problem has still not been finally resolved, and, in any case, the analytic theorem of J. Nash has not yet been introduced to sufficient extent into the working machinery of mathematics.

*If, for example, in E^3 there is a surface S of class C^1 for which all second derivatives of the functions defining it are discontinuous, then it is not possible to introduce the concept of the usual outer curvature.
†This means that a segment joining any two points of the immersion has no further common points with the immersion.

The possibility of a regular immersion of an arbitrary Riemannian manifold V^n Nash proves by means of his analytic theorem. To this end the line element ds^2 of the manifold V^n is represented as a sum of two line elements $ds_1^2 + ds_2^2$ the first of which is the line element of some freely immersed submanifold. Next, a C^1-isometric immersion of the metric with line element ds_2^2 is constructed, and this immersion is approximated analytically so closely that the error does not exceed the quantity required by the analytic theorem of Nash. Further, the required immersion of the original metric is obtained by correcting the immersion of the first metric.

By means of the method given by Nash it is possible to immerse an arbitrary n-dimensional Riemannian manifold in E^N, where $N = [(3n^2 + 11n)/2](n + 1)$ [for compact manifolds $N = (3n^2 + 11n)/2$].

Embeddings of regular (C^r, $3 \le r < \infty$) Riemannian metrics were considered in the work of Nash indicated; the immersion also had class C^r. In the work [143] Nash showed that in applying his analytic tool to an analytic Riemannian manifold the solution is also analytic. By means of this result, Greene and Jacobowitz [97] proved a theorem on the immersibility of an analytic n-dimensional Riemannian manifold in $E^{(3n^2+11n)/2}$.

Attempts to lower the dimension N were made in the work of Greene [96] and Clarke [78]. Rokhlin and colleagues [30] indicated a geometric method corresponding to the analytic theorem of Nash. It was found that the methods presented in the work of Janet and Burstin can be used also for a global immersion theorem. It was proved in [30] that each Riemannian manifold V^n can be regularly immersed isometrically in E^N, where $N = n(n + 1)/2 + 3n + 5$.

3. Problem of the Class of Riemannian Manifolds. During this period the work of Yanenko [43-45] was the principal work on the problem of the class of Riemannian manifolds.

Yanenko [45] reduced the question of investigating metrics of class 2 to the question of investigating metrics of class 1. As in previous work on the theory of class, the question of flexibility of the immersions obtained is investigated in the work of Yanenko. We also note the work of O'Neil [149] and Takahashi [185] who investigated immersions of class one in nonflat spaces of constant curvature and the work of Sen [173] who classified conformally flat spaces of class 1.

It appears that the problematics of the theory of class were to considerable extent completed in the work of Yanenko. It is likely to be a very tough problem to obtain further effective criteria that a given metric have a given class. After the middle of the fifties only scattered papers were devoted to the problem of the class of proper Riemannian metrics. However, the question of immersion of various classes of metrics (metrics of constant curvature, metrics of negative curvature, etc.) is developing rapidly; we proceed to consider this work in the next sections. The theory of the class of pseudo-Riemannian metrics is also developing; for this see the next section.

4. Immersions of Metrics of Constant Curvature. Papers in which the immersion of special classes of metrics is considered are of considerable interest. The problem of immersions of metrics of constant curvature defined on various manifolds has been investigated the most thoroughly. Local immersions of such metrics are well known, and the basic problem arising here consists in obtaining global immersions having a given topological structure. The main results in this area have been obtained by the Yugoslavian mathematician D. Blanusa.

Generalizing the result of Bieberbach on the immersion of the Lobachevskii plane in Hilbert space, Blanusa constructed a realization in Hilbert space of an infinite Möbius strip with a hyperbolic metric.*

Blanusa [52] also constructed an immersion of n-dimensional Lobachevskii space in Hilbert space. Still another immersion of multidimensional Lobachevskii space in Hilbert space was constructed by Blanusa in [61]. Pursuing these investigations, in [66] Blanusa considered a number of infinite-dimensional spaces which are natural generalizations of finite-dimensional spaces of constant curvature and proved a large number of assertions regarding their mutual immersibility (the presence of such mutual immersibility is a characteristic feature of infinite-dimensional spaces).

In the papers [51, 50, 63] Blanusa turned to the investigation of immersions of an elliptic space. We note that Kuiper showed [125] that the projective plane cannot be topologically immersed in three-dimensional Euclidean space as a convex surface; in any case there thus does not exist a regular realization in three-dimensional Euclidean space of an elliptic plane. Blanusa established that n-dimensional elliptic space has a

*An infinite Möbius strip with hyperbolic metric is obtained by identifying by reflecting the boundaries of the part of the Lobachevskii plane contained between two diverging lines with respect to their common perpendicular.

regular immersion in $E^{n(n+3)/2}$. Here the multidimensional surface, just as the sphere, has geodesics which are sections of the immersion by certain multidimensional planes. It is found that on accounting with this property the indicated dimension of the enveloping space cannot be improved. This result implies the existence of an imbedding of the elliptic plane in E^5.* In [55, 57] Blanusa investigated the question of the immersibility of n-dimensional elliptic space in spherical, hyperbolic, and flat spaces of dimension $n(n+3)/2-1$ (i.e., one less than in the previous estimate). It was found that such immersions actually exist under certain conditions on the parity of the number n and provided that certain inequalities connecting the curvatures of the immersed and enveloping spaces are satisfied. Blanusa returned to this problem in the more recent work [68] where he constructed an immersion of n-dimensional elliptic space in $E^{(n+1)^2}$ and $E^{n(n+3)/2+1}$.

We note that the projective plane with a metric of positive curvature, in particular, the elliptic plane, does not admit a regular imbedding in E^4 (Rokhlin et al. [30]). Moreover, if $n = 2^k$ and $n > 1$, then n-dimensional projective space with a metric of everywhere positive scalar curvature, for example, n-dimensional elliptic space, does not admit isometric imbedding in E^{2n} [30]. We note also the work of Boy [71] of 1903 in which a realization was constructed of the elliptic plane in E^3 as a surface with a singularity.

In constructing these immersions of elliptic space Blanusa proceeded from a certain realization of the group of motions of elliptic space as a subgroup of the group of motions of the enveloping space in a way similar to the manner in which the unit sphere may be considered a realization of the orthogonal group in the group of motions of three-dimensional Euclidean space. Subsequently, Kobayashi [120], apparently independently, arrived at a similar idea for the immersion of arbitrary homogeneous spaces in Euclidean spaces. On the basis of this idea he obtained immersions of homogeneous spaces isomorphic to the unitary U(n) and spin Sp(n) groups and also some other more complicated groups of homogeneity in Euclidean spaces.

We note also the recent work of Seidel [171] in which a method is indicated for constructing immersions of elliptic space in Euclidean space if $n + 3$ points of this immersion are known beforehand.

Blanusa also considered the question of the immersion of locally Euclidean spaces in spaces of constant curvature. He considered [54, 56] the question of the immersion of the plane and the flat cylinder in spherical space. This question is of interest, in particular, in connection with the fact that Clifford indicated a two-dimensional torus with Euclidean metric (a so-called flat torus) in the three-dimensional sphere S^3. In [56] Blanusa showed that the two-dimensional cylinder with Euclidean metric (the so-called flat cylinder) can be immersed in the four-dimensional sphere. In [54], by investigating in detail the surface of Clifford, he also constructed an immersion of the flat torus in three-dimensional elliptic space. Further, in [62] Blanusa established that the flat cylinder can be immersed in the three-dimensional sphere, while the Klein bottle with Euclidean metric can be immersed in four-dimensional Euclidean and hyperbolic spaces. We recall that the first immersions of the Klein bottle with a flat metric in four-dimensional Euclidean space were found by Tompkins [192].

In the next cycle of work Blanusa turned to the construction of immersions of n-dimensional Lobachevskii space. In [64] he constructed by very fine analytic methods an immersion of the Lobachevskii plane in E^6 and n-dimensional Lobachevskii space in E^{6n-5}. This was the first regular imbedding of the Lobachevskii plane and space in a finite-dimensional Euclidean space. Modifying somewhat the method of Blanusa, Rozendorn constructed an immersion (with self-intersection) of the Lobachevskii plane in E^5 [23]. In [69] Blanusa constructed an immersion of the hyperbolic plane and of cylinders with a hyperbolic metric in the eight-dimensional sphere, and in [70] he constructed an immersion of n-dimensional Lobachevskii space in the spherical space S^{6n-4}. Finally, in [65] an imbedding of two-dimensional cylinders with a hyperbolic metric in E^7 is constructed.

We mention also the work of Dolbeault-Lemoire [81] who showed that for $n > 2$ there does not exist a regular immersion of E^n in $(n+1)$-dimensional Lobachevskii space.

In [6, 7] Volkov and Vladimirov established that the full Euclidean plane can be immersed in three-dimensional Lobachevskii space only as a horosphere or an equidistant body.

Still another cycle of Blanusa's papers is devoted to immersions of the Möbius strip with various metrics. In connection with the formulation of this problem we recall that according to Kuiper's result [125] there do not exist convex immersions of an infinite Möbius strip in three-dimensional Euclidean space. We note that there do not exist immersions in E^3 of an infinite flat Möbius strip. Blanusa [59] constructed an immersion of an infinite Möbius strip with flat metric in E^5, and in [60] – in four-dimensional Euclidean space and in four-dimensional spherical and hyperbolic spaces. In [67] an immersion of the Möbius strip with hyperbolic metric

*This work was summarized by Blanusa in the survey [58].

in E^{10} was constructed, and in [69] – in E^8 and in the ten-dimensional sphere.

In the work of the Yugoslavian geometer S. Mincic a number of results extending those of Blanusa were obtained. Immersions of the Euclidean space E^{2m} in S^{5m-1} and of E^{2m+1} in (5m + 2)-dimensional elliptic space were constructed. Moreover, an obvious immersion of the flat n-dimensional torus in S^{2n-1} was indicated.

5. Immersions of Metrics of Nonpositive Curvature. In 1952 Chern and Kuiper [76] showed that a closed n-dimensional Riemannian space with curvature in two directions which is nonpositive at all points cannot be C^4-immersed in E^{2n-1}. This result is a natural generalization of the result of Tompkins [191] on the nonimmersibility of the n-dimensional torus with flat metric in E^{2n-1} which was mentioned in the previous section. In this work Chern and Kuiper applied the following important technique. Let F be a closed surface in E^3 lying inside some convex surface Φ. We displace the surface Φ until it is tangent to F. The curvature of the surface F at the point of tangency is not less than the curvature of the surface Φ. This technique was developed in detail by Pogorelov [18] in investigating convex surfaces in three-dimensional Euclidean space. Chern and Kuiper showed that a similar technique of "squeezing" a convex surface can be applied also to surfaces with codimension which is not too large (up to n − 1), which makes it possible to prove the theorem mentioned on the nonimmersibility of metrics of nonpositive curvature. We note that the "squeezing" technique was applied by Sokolov [33] to prove the nonimmersibility of two-dimensional, positive definite metrics of positive curvature in three-dimensional pseudo-Euclidean space (see Sec. 3).

In the same work Chern and Kuiper conjectured that if M is a compact, n-dimensional manifold at each point of which there exists a q-dimensional space along flat elements of which the curvature is nonpositive, then M cannot be embedded in E^{n+q-1}. This was proved by Otsuki in [154, 156, 157].

Many various generalizations of the results of Chern, Kuiper, and Otsuki were subsequently obtained. Kuiper [121] obtained the following result. Let U^2 and U^3 be, respectively, two- and three-dimensional compact manifolds, while all the sectional curvatures of the manifold U^3 are negative. Then the manifold $V^5 = U^3 \times U^2$ cannot be isometrically imbedded in E^8 and S^7. Tachibana [184] extended the results of Chern and Kuiper to immersions of spaces of constant curvature in nonflat spaces of constant curvature. Hartman and Nierenberg [103] proved that if a complete d-dimensional manifold M^d with flat metric has an immersion in E^{d+1}, then M^d is isometric either to E^d or to the cylinder $S^1 \times E^{d-1}$. Further results in this direction were obtained by Hartman [101, 102], Takahashi [186], and Vranceanu [195].

O'Neil [150] proved that if M^n is a compact n-dimensional manifold and M^m is a complete, m-dimensional, simply connected manifold having sectional curvatures connected by the inequality $K(M^n) \leq K(M^m) \leq 0$, then M^n cannot be immersed in M^m for $m < 2n$. O'Neil further proved [152] the following result. Let M^d be a complete d-dimensional Riemannian space with negative sectional curvatures not exceeding the number $c < 0$. If M^d can be immersed in a (d + 1)-dimensional Lobachevskii space of curvature c, then the i-th ($i \geq 2$) Cech cohomology of the space M^d is equal to zero. Stiel [180] proved that if M^d is a d-dimensional compact manifold with nonpositive sectional curvatures, then M^d cannot be imbedded in a (d + k)-dimensional manifold of constant nonpositive curvature for $k < d$. Further results in this direction were obtained by O'Neil and Stiel [153], O'Neil [152], Stiel [178, 179], Maltz [132], Ferus [83], and Nomizu [148].

We consider, finally, the work of Borisenko [2] who proved the following assertions. Let F^l be a compact, l-dimensional surface of class C^3 in a $(2l-1)$-dimensional Riemannian space R^{2l-1}. If the outer sectional curvatures of F^l are negative, then the Euler characteristic of the surface F^l is equal to zero. If F^l is homeomorphic to the sphere and the sectional curvatures of F^l are less than one, then for $l \neq 3$ and $l \neq 7$ it is impossible to imbed F^l in R^{2l-1}. We also note the work of Moore [138] who obtained several results in this area.

The results described above rest in final analysis on the fact that imbeddings of manifolds with nonpositive sectional curvatures satisfy a certain condition of saddle-shape type. It is found that immersions of spaces with nonnegative sectional curvatures and not too large codimension satisfy a certain condition of convexity type. Using this condition, Jacobowitz [113] proved that if V^l is a compact, l-dimensional manifold with sectional curvatures not exceeding λ^2, then V^2 cannot be immersed in E^{2l-1} inside a ball of radius λ. With the help of these conditions of convexity type it is also possible to prove various theorems on the convexity of hypersurfaces. There is a broad literature on this question going back to the work of Hadamard [101] in 1897 who proved the convexity of a complete surface of positive curvature in three-dimensional Euclidean space. We mention also the work of Sacksteder [165] who proved that if M^d is a complete, n-dimensional manifold all sectional curvatures of which are nonnegative and at least at one point at least one sectional curvature is positive, then any C^{n+1}-imbedding in E^{n+1} is globally convex.

Another aspect of the problem of the immersion of metrics of nonpositive curvature is related to the question of the existence in E^4 of a compact surface of negative curvature. This question was posed by Chern in the report [75]. An answer to this question was obtained by Rozendorn [25] who constructed an example of such a surface.* In constructing this example essential use is made of an example constructed by Rozendorn in this same work of a surface bounded above of negative curvature in E^3 which is regular everywhere except at a finite number of points. We mention also the work of Rozendorn [24] in which an example is constructed of a bounded, complete surface of nonpositive curvature in E^3.

6. On the Immersion of n-Dimensional Spaces in $E^{n(n+1)/2}$. We have already observed that the "natural" dimension of the enveloping space for which the number of Schlaefli equations coincides with the number of unknown functions in these equations is $s_n = n(n+1)/2$. Rokhlin and others [30] showed that in a certain sense only an everywhere nondense set of n-dimensional metrics can be immersed in $E^{s_n - 1}$. However, at present not a single specific example of a metric which cannot be immersed in E^{s_n-1} is known. There is no question that the construction of such an example would be of considerable interest. It is also not clear† whether all n-dimensional spaces (including nonanalytic) can at least be locally immersed in E^{s_n}. There are only several results on the impossibility of immersing two-dimensional metrics in three-dimensional space. In addition to the classical results of Hilbert and Efimov on the nonimmersibility of complete metrics of negative curvature, we mention the following work.

Poznyak [20] constructed examples of metrics on the sphere and in the disk which have no C^2-immersions globally in E^3. Other examples of this type were constructed by Rokhlin and others [30] and by Greene [93]. Pogorelov [19] constructed an example of a two-dimensional Riemannian metric of class $C^{2,1}$ which does not admit a local immersion of class C^2 in E^3. At present it seems likely that there actually are regular two-dimensional metrics which have no regular local immersions in three-dimensional space. However, this major question can be completely resolved only after constructing an example of an infinitely differentiable metric having no local C^2-immersion in E^3.

7. During the period in question many papers appeared in which the formulation of the local immersion problem was discussed and refined and which also considered its relation to other areas of geometry. These questions are discussed in [15, 16, 92, 109, 111, 119, 134-136, 166, 176, 182].

We also mention the short survey of Friedman of results on the theory of immersions [86] which has played a considerable role in acquainting physicists and mathematicians with this problem.

3. Isometric Immersions of Spaces with Indefinite Metric

1. Immersions in Pseudo-Euclidean Spaces of Large Dimensions. Riemannian spaces with indefinite metric‡ are of interest mainly in connection with applications in the theory of relativity. The problem of isometric immersions of such spaces in pseudo-Euclidean spaces is interesting both from a purely geometric point of view and in connection with certain questions of theoretical physics. Without going into details, we note that new approaches to the problem of the symmetry of elementary particles are connected with the possibility of special isometric immersions of such spaces of physics.

After the fundamental work of Nash [141-143] on the theory of immersions the idea naturally arose of carrying over his techniques to the pseudo-Riemannian case.

The method developed by Nash and improved methods (see [30, 77, 96]) carry over without appreciable changes to the case of immersions of spaces with indefinite metric in pseudo-Euclidean spaces. The corresponding results have been obtained in the papers of Clarke [78], Rokhlin et al. [30], Greene [96], and Sokolov [32]. The best ratio of the dimensions of the immersed pseudo-Riemannian manifold $M^n_{(p,q)}$ and enveloping space $E^m_{(p',q')}$ is obtained in the work noted above of Rokhlin [30]. It is established that $M^n_{(p,q)}$ of class C^∞ can be isometrically immersed in $E^m_{(p',q')}$ if

$$m \geqslant s_n + 3n + 5, \; p' \geqslant n + p, \; q' \geqslant n + q, \; s_n = \frac{n(n+1)}{2}.$$

*The method used by Otsuki [155] to construct such a surface in E^4 is suitable only for the case in which the curvature is nonpositive.

†We recall that in the theorems of Janet, Burstin, and Cartan (see Sec. 1) on local immersion only the case of analytic metrics is considered, and they give no information on immersions of metrics of class C^∞.

‡Riemannian spaces with indefinite metric (a pseudo-Riemannian manifold) are characterized by the fact that at any point the line element ds^2 can be brought to the form $ds^2 = dx^{1^2} + \ldots + dx^{p^2} - dy^{1^2} - \ldots - dy^{q^2}$. The numbers p and q are moreover the same at any point.

Immersion methods related to specific properties of pseudo-Euclidean spaces are proposed in the work of Rokhlin et al. [30] and Sokolov [33]. An interesting fact is that in the general case it is possible to indicate a particular solution of the immersion equations (the Schlaefli equations for the indefinite case) by raising the dimension and special choice of the space $E^m_{(p',q')}$. For example, in the work of Sokolov [33] an immersion of a Riemannian or pseudo-Riemannian metric given in the coordinate ball is sought in a pseudo-Euclidean space $E^{2q}_{(q,q)}$. The main idea of Sokolov is to reduce the nonlinear immersion equations to linear equations by special change of the unknown functions. Namely, in the Schlaefli equations for indefinite metrics and immersions in $E^{2q}_{(q,q)}$

$$\sum_{i=1}^{q} \frac{\partial z_i}{\partial x^k}\frac{\partial z_i}{\partial x^p} - \sum_{i=q+1}^{2q} \frac{\partial z_i}{\partial x^k}\frac{\partial z_i}{\partial x^p} = g_{kp} \tag{4}$$

the following change of the unknown functions is made:

$$z_i = t_i,\ z_{q+i} = h_i + t_i,\ i = 1, \ldots, q.$$

After this change the terms nonlinear in the derivatives $\partial t_i/\partial x^k$ (this group of terms has the form $\sum_{i=1}^{q} \frac{\partial t_i}{\partial x^k}\frac{\partial t_i}{\partial x^p}$) cancel due to the minus in front of the second sum in Eqs. (4). The equations (2) reduce to the following system of equations linear in t_i:

$$\sum_{i=1}^{q} \left(\frac{\partial h_i}{\partial x^k}\frac{\partial t_i}{\partial x^p} - \frac{\partial t_i}{\partial x^k}\frac{\partial h_i}{\partial x^p}\right) = -g_{kp} - \sum_{i=1}^{q} \frac{\partial h_i}{\partial x^k}\frac{\partial h_i}{\partial x^p}. \tag{5}$$

If $q = n(n+1)/2$, then it is not hard to choose functions h_i such that the system (5) has a local solution; moreover, it is possible to write out one such solution, and from its explicit form it follows that it gives an immersion of any ball of the original pseudo-Riemannian space. With the help of certain standard topological techniques (see, e.g., [30]) it is possible to generalize this method to construct immersions of any n-dimensional pseudo-Riemannian spaces in $E^{2s_{2n}}_{(s_{2n},s_{2n})}$.

We note that by means of the methods indicated it is possible to construct C^2-isometric immersions of C^2-pseudo-Riemannian spaces (the analytic theorems of Nash are applicable for smoothness not less than C^3).

2. Immersions of Low Regularity. The method of Nash for constructing C^1-immersions admits generalization to the indefinite case. Rokhlin et al. [30] proved the following result: a compact, pseudo-Riemannian space $M^n_{(p,q)}$ admits a C^1-immersion in $E^m_{(p',q')}$ for

$$p' > p+n,\quad q' > q+n,\quad m > 3n.$$

We note that for the indefinite case the Nash method of C^1-immersions gives somewhat poorer results than for the definite case. This is related to the following circumstance. In constructing immersions in a pseudo-Euclidean space $E^m_{(p',q')}$ it is not sufficient to choose a large dimension m while subjecting the numbers p' and q' only to the obvious necessary inequalities $p' \geq p$, $q' \geq q$. Indeed, there is the following result of Sokolov [33]: Any closed pseudo-Riemannian space $M^n_{(p,q)}$ cannot be C^1-isometrically immersed either in $E^m_{(p,m-p)}$ or $E^m_{(m-q,q)}$ for any arbitrarily large m [if there existed such an immersion, then on this immersion there would be points at which the tangent plane had signature different from (p, q)]. For example, the two-dimensional torus with indefinite metric cannot be C^1-isometrically immersed either in $E^m_{(1,m-1)}$ or in $E^m_{(m-1,1)}$. It is not hard to indicate other restrictions of this type. Thus, for example, for the immersion of an n-dimensional Riemannian space, generally speaking, a pseudo-Euclidean space $E^m_{(p',q')}$ with $p' \geq 2n-1$ is needed. Indeed, if all Riemannian manifolds could be imbedded in some $E^m_{(2n-2,q')}$, then the corresponding differential manifolds on which the pseudo-Riemannian metrics are given could be topologically immersed in E^{2n-2}, which is, in general, impossible [199].

We note also the following result established by Avez [47]. Let $M^4_{(3,1)}$ be a compact, pseudo-Riemannian manifold and suppose that $\det R_{ij} \neq 0$ (R_{ij} is the Ricci tensor). Then $M^4_{(3,1)}$ cannot be isometrically immersed in any five-dimensional pseudo-Euclidean space. The proof uses the connection of the topological structure of the manifold with the curvature.

3. Local Isometric Immersions. Friedman [85] verified that the Janet–Burstin method of proving the existence of a local immersion carries over without appreciable changes to the indefinite case. Namely, each

analytic pseudo-Riemannian space $M^n_{(p;q)}$ has a local isometric immersion as an analytic surface in $E^{\frac{n(n+1)}{2}}_{(p';q')}$, $p' \geq p$, $q' \geq q$.

Vogel [194] and Lense [129] generalized the Janet–Burstin method to the immersions of a space with degenerate metric.† Lense also considered immersions in complex Euclidean spaces.

4. Immersions of Riemannian and Pseudo-Riemannian Metrics in Three-Dimensional Pseudo-Euclidean Space; Some Questions of the Theory of Surfaces in Three-Dimensional Pseudo-Euclidean Space. The systematic study of the question of global immersion in three-dimensional pseudo-Euclidean space and of surfaces in this space has begun only very recently, and many important questions have so far not been investigated. We shall consider a surface in the pseudo-Euclidean space $E^3_{(2;1)}$ with metric $ds^2 = dx^2 + dy^2 - dz^2$. All the results admit reformulation for the other three-dimensional pseudo-Euclidean space $E^3_{(1;2)}$.

A difference of pseudo-Euclidean space from Euclidean space which is important in the theory of immersions is that it contains planes with different metrics (definite, indefinite, and degenerate). Moreover, a different relation between the sign of the curvature and convexity of the space is observed in $E^3_{(2;1)}$ as compared with the space E^3: Convex surfaces in $E^3_{(1;2)}$ have a metric of nonpositive curvature‡ while saddle surfaces have a metric of nonnegative curvature (we recall that in E^3 convex surfaces have a metric of nonnegative curvature, while saddle surfaces have a metric of nonpositive curvature). In order to see this, we consider together with the space $E^3_{(2;1)}$ with the metric $ds^2 = dx^2 + dy^2 - dz^2$ the so-called superposed Euclidean space E^3 with metric $ds^2 = dx^2 + dy^2 + dz^2$. Let K and Δ (K* and Δ*) be, respectively, the discriminant and curvature of the first quadratic form induced on the surface Φ by the metric of the space $E^3_{(2;1)}$ (by the metric of the superposed space E^3). There is then the relation

$$K\Delta^2 + K^*(\Delta^*)^2 = 0. \tag{6}$$

Formula (6) easily implies the connection formulated above between the convexity of the surface and the sign of its curvature.

It is convenient to illustrate this connection of convexity and the sign of the curvature by the example of the sphere in the space $E^3_{(2;1)}$. This sphere is given by the equation

$$|x^2 + y^2 - z^2| = 1.$$

It has three connected components L_+, L_-, L.†† The surfaces L_+, L_- are complete in the sense of the inner definite metric of the surface of constant negative curvature; they constitute an imbedding of the complete Lobachevskii plane in $E^3_{(2;1)}$. The surfaces L_+ and L_- are convex. The surface L is a surface of constant positive curvature and indefinite metric. In the general theory of relativity the metric of the surface L induced by the metric of $E^3_{(2;1)}$ is usually called the two-dimensional de Sitter space–time model. We note that the surface L is a saddle surface.

Convex surfaces with definite metric in $E^3_{(2;1)}$ are just as "natural" a class of surfaces as convex surfaces in Euclidean space. We shall present some results concerning such surfaces. Under specific assumptions of technical character Sokolov [34] proved the following uniqueness theorem for a surface Φ in $E^3_{(2;1)}$ which is complete in the sense of the inner definite metric: If the limit cone of the surface Φ is separated from the isotropic cone, then the surface Φ is uniquely determined by the metric, the orientation, the limit cone, and the limit generator, i.e., by the same elements as a complete convex surface with curvature less than 2π in E^3 (the theorem of Pogorelov [18]). The situation for surfaces with isotropic limit cone is more complicated. To uniquely determine such surfaces, in addition to the elements enumerated, it is necessary to also fix some ruled surface which approximates the surface Φ at infinity more precisely than the limit cone. In another work of Sokolov [35] it is proved that the structure of the limit cone of a surface Φ is closely related to the properties of its metric. Thus, if the curvature of the surface Φ is separated from zero and the limit cone has a smooth directrix, then it necessarily coincides with the isotropic cone. It is proved that if a smooth convex surface in $E^3_{(2;1)}$ has a regular definite metric of strictly negative curvature, then the surface itself is regular.

†A space with degenerate metric is characterized by the fact that at any point the line element ds^2 can be reduced to the form $ds^2 = dx_1^2 + \ldots + dx_p^2$, where p is less than the dimension of the space.

‡The concept of convexity in $E^3_{(2;1)}$ is analogous to that in E^3.

††It is easy to see that in the superposed space E^3 L_+ and L_- are two sheets of the two-sheeted hyperboloid $x^2 + y^2 - z^2 = -1$, while L is the single-sheeted hyperboloid $x^2 + y^2 - z^2 = 1$.

The proof of this theorem is based on the following assertion which is of independent interest: Any sufficiently small neighborhood of a two-dimensional definite metric of strictly negative curvature can be realized in $E^3_{(2;1)}$ as a convex cap. The study of surfaces relative to the superposed space and formula (6) relating the curvatures in the original and superposed spaces are widely used in proof of the assertions formulated above along with other methods. With the help of this formula it is possible to make use in a number of cases of the classical results of the theory of convex surfaces in E^3.

We shall consider the work of Rudyak [31]. The Bonnet theorem that a surface in E^3 is uniquely determined by its first and second quadratic forms is well known. For those surfaces in $E^3_{(2;1)}$ which have definite or indefinite metric it is not hard to prove that they are also uniquely determined by their first and second quadratic forms. However, at those points of a surface Φ in $E^3_{(2;1)}$ at which the tangent plane is also tangent to the isotropic cone a degeneration of the metric, i.e., of the first quadratic form, occurs, while the second form is not defined at all. Rudyak established that under certain conditions on the set of points of degeneracy a surface is uniquely determined by the second quadratic form defined everywhere except at points of degeneracy and by the first quadratic form of the surface. The basis for this proof was an interesting lemma on the maximum principle for surfaces with degenerate metric.

The question of whether it is possible to obtain a generalization of the Bonnet theorem to surfaces in $E^3_{(2;1)}$ with domain of degeneracy of arbitrary type has so far not been solved. One of the approaches can be connected with the investigation of a certain analogue of the second quadratic form, namely, the form $\tilde{\text{II}}$ related to II by the formula

$$\tilde{\text{II}} = \Delta \cdot \text{II},$$

in which $\Delta = \mathscr{E}\mathscr{G} - F^2$ is the determinant of the first quadratic form. The form $\tilde{\text{II}}$ is defined also at points of degeneracy and precisely coincides with the corresponding quantity for the superposed space.

For surfaces in $E^3_{(2;1)}$ it is also possible to prove a number of theorems on nonimmersibility similar to the theorems on the nonimmersibility of metrics of negative curvature in Euclidean space. However, in the present case they can be proved by much simpler methods appropriate to pseudo-Euclidean spaces.

Before proceeding to these theorems, we note that for pseudo-Riemannian spaces and correspondingly for surfaces in pseudo-Euclidean spaces a number of inequivalent definitions of completeness have been proposed, and in the definite case analogous definitions are found equivalent to the usual concept of completeness. Various concepts of completeness are used depending on the content of the problem to be solved (for more details, see the survey of Geroch [93]). In this survey we shall use the concepts of geodesic and outer completeness. A pseudo-Riemannian space is called geodesically complete if on each of its geodesics the geodesic parameter varies from $-\infty$ to $+\infty$ (here, of course, on segments of a geodesic the geodesic parameter may vary within finite limits). The concept of geodesic completeness was introduced by Hopf and Rinow [107]. A surface Φ^n of a pseudo-Euclidean space $E^m_{(p;q)}$ is called outer complete if any limit point of the surface Φ^n belongs to this surface. If Φ is a two-dimensional surface with positive definite metric in $E^3_{(2;1)}$, then the concept of completeness as a Riemannian space (inner completeness) and outer completeness is meaningful for it. It is found that in this case inner completeness implies outer completeness, but not conversely (a counterexample was given by Zel'manov [8]). We recall that for surfaces in E^3 outer completeness implies inner completeness, but not conversely (see Aleksandrov [1]).

It is obvious that a complete space in the sense of an inner definite metric with positive curvature separated from zero cannot be immersed in $E^3_{(2;1)}$. Indeed, such a space is homeomorphic to the sphere and as a closed manifold has not even a C^1 realization in $E^3_{(2;1)}$ (see Sec. 2). probable that a space with definite metric and curvature separated from zero has no outer complete realizations in $E^3_{(2;1)}$ either, but this has not been proved.

For spaces with indefinite metric the following result is known (Sokolov [35]). On an outer complete C^2 surface with indefinite metric in $E^3_{(2;1)}$ the supremum of the Gaussian curvature is nonnegative. In other words, a pseudo-Riemannian space of curvature $K \le -a^2 < 0$ cannot be immersed in $E^3_{(2;1)}$ as an outer complete surface.

The scheme for proving this assertion is as follows. It is shown that if Φ is an outer complete surface with indefinite metric and curvature $K \le -a^2 < 0$, then it constitutes a globally strictly convex, complete surface. It is further found that the total curvature of such a space computed relative to the superposed space must be infinite and in any case exceeds 4π, which cannot be, as is well known.

We remark that it is not clear if there exist in $E^3_{(2;1)}$ geodesically complete surfaces with indefinite metric and curvature $K \le -a^2 < 0$; the existence of such surfaces is improbable.

We note also the work of Maeda and Otsuki [131] and Hou Cheng-Shi [108] who obtained certain generalizations of the results of Chern and Kuiper [76] and Otsuki [154] (see Sec. 2) to the case of immersions in pseudo-Euclidean spaces. Some results on immersions in the space $E^4_{(1;3)}$ were obtained by Mihaileanu [134].

5. There are a number of general results on the problem of the class of pseudo-Riemannian spaces. Some general criteria of class similar to those of Yanenko are obtained in the work of Matsumoto [132]. A number of important results on determining pseudo-Riemannian spaces with immersions of first and second class are obtained by Yakupov [39-41]. In particular, he determined all Einstein spaces with an immersion of first class and found a number of interesting families of Einstein spaces with an immersion of second class. (We recall that an Einstein space is a Riemannian or pseudo-Riemannian space for which the Ricci tensor satisfies the identity $R_{ij} = \varkappa g_{ij}$, $\varkappa = \text{const.}$) Some results on the classification of Einstein spaces with immersions of first and second class are contained also in the work of Kachurina [10] and Lapkovskii [11]. Kachurina proved that any Einstein space with an immersion of first class is either a space of constant curvature or is symmetric in a certain generalized sense.

The work of Rund [164] is devoted to clarifying the outer condition to which the condition $R_{ij} = \varkappa g_{ij}$ corresponds under immersion, i.e., the condition that the space be an Einstein space.

A number of works are devoted to the question of how immersions of lower class are related to particular features of the structure of the Ricci tensor. Certain physical assumptions regarding the properties of space–time with a mathematical formalization which constitutes the pseudo-Riemannian space in question frequently emerge as such conditions. Thus, empty spaces, i.e., having zero Ricci tensor, of class II are considered in the work of Collinson [79]. Lancaster [126] found conformally Euclidean spaces with immersions of first class. Eguchi [82] found criteria that a space of Gödel type be a space of first class. Plebanskii [158] investigated the problem of class for static, spherically symmetric models of space–time.

In the papers cited in this section it is a question of local analytic immersions of analytic metrics.

A number of interesting results on the theory of immersions of spaces with indefinite metric are related to physical investigations in the general theory of relativity. These papers will be discussed in the next section.

4. Applications of the Theory of Immersions in the General Theory of Relativity

Practically since the very creation of the general theory of relativity physicists have been interested in the question of whether it is possible to use a representation of space–time not as an abstract manifold but rather as a surface in some pseudo-Euclidean space to clarify certain difficulties of the theory and also for its further development. In this connection we mention the statements of Einstein on the prospects for using the theory of immersions in the theory of relativity [160]. Although at present these works have not gone beyond the framework of separate, episodic investigations, in this area a rather large volume of material has accumulated which is of both physical and geometric interest.

The following idea unifies all the work on the use of the theory of immersions in the general theory of relativity.* In passing from the special to the general theory of relativity in place of "privileged" Cartesian (inertial) coordinate systems in pseudo-Euclidean space arbitrary curvilinear coordinate systems in curved pseudo-Riemannian space are used. It is found, however, that in many concrete questions it is very difficult to get by without some distinguished Cartesian coordinate system. In principle, it is possible to attempt to combine the curvature of space–time and the use of Cartesian coordinates by using as the latter the Cartesian coordinates of an enveloping space. Naturally, these coordinates are already not independent but are related by the immersion equations. Moreover, it is possible to attempt to use the outer geometric properties of such multidimensional surfaces for the geometrization of various physical quantities (e.g., isotopic spin) which in the general theory of relativity have no geometric interpretation. The organization in Dallas (U.S.A.) in 1965 of a special seminar on the theory of immersions in the general theory of relativity under the supervision of Robinson, Ne'eman, and Friedman [172] bears witness to the popularity of such approaches. We note also the work of Goedecke [94], Goenner [95], and Szekeres [183] inwhich general questions of using the theory of immersions in the general theory of relativity are discussed. In a number of purely mathematical works (e.g., Clarke [78], Greene [96], and Sokolov [32]) it is emphasized that the interest in the theory of immersions was stimulated by physical applications.

*It is here not our task to analyze the problematics of the foundations of the theory of general relativity; therefore, in a number of cases the physical argumentation is somewhat simplified.

The extent to which the theory of immersions is used in these papers is different. For example, in the work of Maldybaeva [14] the theory of immersions remains a convenient tool which is altogether dispensable. More substantial use of the theory of immersions is made in other papers.

We shall first consider the applications of the theory of immersions to the problem of the quantization of the general theory of relativity. Without going into a detailed analysis of all the difficulties encountered in solving quantization problems, we note that there is a rather consistent and extensively developed quantum theory within the framework of the special theory of relativity (so-called relativistic quantum theory). Generalization of this theory with respect to dimension causes no difficulties in principle. However, it is not clear what the generalization of this theory to the curved spaces of the general theory of relativity should be. It is supposed that within the framework of this hypothetical theory space–time itself must be considered in some "quantized," statistical sense. It is possible to imagine several ways of constructing such a theory. One such way is related to the theory of immersions and apparently goes back to the work of Joseph [115]. In this work it is supposed that the classical space–time of the general theory of relativity is really a multidimensional surface in some pseudo-Euclidean space of a higher number of dimensions. Particles adhere to this space by a certain potential U which allows the particles to leave the surface a distance of an order of magnitude not exceeding $\hbar$ (for simplicity we imagine this to be a δ-type potential). Further, the quantization procedure is declared to coincide with the quantization procedure in the enveloping pseudo-Euclidean space accounting with the effect of the potential U which describes the immersion functions. The introduction of so-called second quantization in which the potential U must have a statistical, "quantized" character would mean the quantization of space–time.

The (independent) work of Ne'eman [146] complements the idea of Joseph. In this work it is emphasized that a number of symmetry groups of physical quantities are known which are isomorphic to the orthogonal groups of various Euclidean and pseudo-Euclidean spaces, but these groups have no interpretations in terms of the Poincare group (the group of motions of space–time of the special theory of relativity). These groups are usually interpreted as rotation groups of certain auxiliary spaces. An example of such symmetries is isotopic invariance which consists in the indistinguishability of the proton and neutron for nuclear forces. As a consequence of this, from the point of view of the theory of strong interactions the proton and neutron constitute a single particle – the nucleon – in two states characterized by different values of the so-called isotopic spin. Isotropic spin, just as ordinary spin, is connected with a certain group isomorphic to the group of rotations of some "auxiliary" space. Ne'eman proposed interpreting this and other symmetry groups as subgroups of the group of motions of the orthogonal complement to immersed space–time.

It is significant that the symmetries, as a rule, are violated. For isotopic spin symmetry violation means the following. Although the strong (nuclear) interactions of the proton and neutron are the same, their considerably weaker electromagnetic interactions are distinct; this is treated as a removal of the degeneracy for different values of isotopic spin in the electromagnetic field. From the point of view of the theory of immersions symmetry violation means that these symmetries of the orthogonal complement are local, i.e., they are approximately satisfied only at distances much less than the radii of curvature of the surface in those directions in which rotation occurs. In this connection it is important to emphasize the following. From the universal constants of quantum theory and the theory of relativity it is possible to construct quantities with the dimensions of length and mass which are called, respectively, the Planck length l_{Pl} and the Planck mass m_{Pl}. According to current ideas, it would be natural to consider these characteristic quantities as the characteristic length and mass of the elementary particles. It is actually found, however, that the Planck length is many orders of magnitude less than the characteristic dimensions of the elementary particles, while the Planck mass is many orders of magnitude greater than the characteristic mass of the elementary particles (the differences are so great that as compared with them the differences in masses among the various particles, for example, between the proton and electron, are insignificant). It is presently not clear how this large factor should theoretically be obtained. However, it follows from the results of Nash [142] that the immersion of space–time can be chosen such that in a certain sense the average ratio of the greatest and smallest value of the radii of curvature at a given point be an arbitrarily large prescribed number. It is possible in principle to attempt to relate this additional parameter which arises naturally in the theory of immersions to the large ratio of the Planck mass and the mass of the elementary particles.

The idea of Ne'eman was developed in the work of Ne'eman and Rosen [147]. They attempted to determine precisely the dimension and signature of the enveloping space and to concretely interpret the symmetries known at that time. However, the insufficient development in 1965 both of the theory of immersions and of the theory of symmetries did not allow them to arrive at conclusions which were sufficiently specific and which admitted experimental verification. Moreover, it was proposed to take as the enveloping space that pseudo-Euclidean space which for minimal dimension admits the realization of all (or at least practically all) space−times of the general theory of relativity. Since at that time geometry gave a very excessive estimate for this dimension (~100) as compared with the expected dimensions of the order of $s_4 = 10$ (the estimate currently available $s_4 + 3 \times 4 + 5 = 27$ is also probably very excessive), Rosen [161] attempted to find a lower bound for the desired dimension by investigating immersions of various metrics of the general theory of relativity. He constructed immersions of several dozen classes of metrics. In this connection Rosen [162] and Ianus [110] investigated in more detail metrics which describe isolated spherically symmetric bodies. The question of the immersion of the Schwarzschild and certain similar metrics was further investigated in detail by Fujitani. For space−time models with a magnetic field a number of immersions were found by Navez [145] and Samaranda and Navez [175].

Other results in this direction were obtained by Takeno [187] and Kitamura [117, 118].

It should be mentioned, by the way, that since metrics have been investigated which possess relatively high symmetry, for all metrics considered, immersions have been found in spaces of low dimension (<10). No interesting lower bounds on the desired surface have thus been found. On the other hand, there are a number of results in these papers on spaces of class greater than II.

We shall mention another closely related group of works. In seeking new solutions of the Einstein equations, to distinguish a specific solution one does not, as a rule, resort to the imposition of boundary conditions but rather imposes on the desired space some conditions of symmetry type. The existence of an immersion in some pseudo-Euclidean space of fixed dimension constitutes such conditions in a number of papers. One of the most important solutions of the Einstein equations − the Kasner solution [116] − was found in this way. The work of Kasner was also one of the first to apply the theory of immersions in the general theory of relativity. Other work in the same direction are the papers of Takeno [188] and Stephani [177].

In the papers considered, curved space−time is considered as a certain flat, multidimensional space with given connections (immersion equations). It is found that a similar situation sometimes arises in the theory of mechanics. The problem of constructing a particular class of motions can be interpreted as the question of the structure of suitably curved configuration space (Synge [181]). In order to define connections providing the necessary metric it is sometimes convenient to first construct an immersion of the given metric by means of which the connections are constructed in some standard fashion.

Another area of application of the theory of immersions in the general theory of relativity concerns questions related to complete pseudo-Riemannian spaces. As we have already noted, in the theory of pseudo-Riemannian spaces there is no general concept of completeness, and for each group of problems it is necessary to develop an appropriate concept of completeness.

The question of completeness of solutions in the general theory of relativity has two aspects. First of all, frequently in solving the Einstein equations we obtain a metric which only describes matter and the gravitational field in restricted space−time scales. In other words, the pseudo-Riemannian space M obtained is a proper subset of the desired pseudo-Riemannian space N. It is assumed that the space N itself cannot be represented as a proper subset of any four-dimensional pseudo-Riemannian space (it is said to be nonextendable). Since analytic pseudo-Riemannian spaces are usually considered, the procedure for constructing the space N on the basis of the known space M is called analytic extension of the space M, and N itself is called a maximal analytic extension [90]. At present there exist no sufficiently effective methods of constructing a maximal analytic extension. Fronsdal [87] suggested in place of the analytic extension of the pseudo-Riemannian manifold M to realize an analytic extension of its immersion in pseudo-Euclidean space, which is often much simpler (the concept of the analytic extension of a surface is analogous to that of the analytic extension of an abstract space). In the work of Fronsdal one of the first analytic extensions of the Schwarzschild metric was found. Other examples of extensions of solutions to the Einstein equations by the method of immersions were obtained by Ptazowski [159]. A basic shortcoming of Fronsdal's technique is that the maximal analytic extension of the immersion obtained by means of it is not necessarily a maximal analytic extension as an abstract manifold. For example, a multidimensional surface going out to infinity may admit extension as an abstract manifold. This effect is essentially related to the indefiniteness of the metric of the enveloping space; in Euclidean space it is impossible.

Another aspect of this question is the following. In the general theory of relativity incompleteness (understood in various senses) of nonextendable space−times is related to extremal properties of matter and space−time which make the standard general theory of relativity inapplicable in certain space−time regions. A concept of completeness which is physically and mathematically sufficiently justified has so far not been developed. Without discussing all the papers related to this question, we refer the reader to the work of Geroch

[93]. In the work of Fronsdal just mentioned it was proposed that those metrics of general relativity be considered complete, and hence having no singularity, which have a proper immersion (i.e., an immersion as a surface going out to infinity) in some fixed pseudo-Euclidean space. This idea of singularities was further developed in the work of Dolan [80] and Hajicek [100]. However, using the results of Rokhlin and others [30] it is not hard to show that all pseudo-Riemannian spaces have proper immersions in a pseudo-Euclidean space of sufficiently high dimension. In pseudo-Euclidean spaces of lower dimensions it is not hard to construct examples of outer complete surfaces with metrics which could not justifiably be considered complete from a physical point of view; for example, these metrics might be nonextendable as abstract manifolds. Thus, the concept of completeness proposed by Fronsdal is apparently inadequate to describe the corresponding physical concepts, which, of course, does not detract from its purely geometric interest.

We mention, finally, the work of Tran-hu Phat [193] in which an attempt is made to use the concept of immersions to solve the question of the energy-momentum tensor of the gravitational field and to interpret the concepts of energy and momentum from an outer geometric point of view. In spite of the fact that the theory of immersions actually makes it possible to invariantly fix a particular coordinate system and thus alleviate the solution of the questions indicated, after the work of Isaacson [112] which threw light on this old and complicated question of the foundations of the general theory of relativity, invoking the concepts of the theory of immersions is unnecessary.

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