# On Applications of Campbell's Embedding Theorem

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#### Abstract

A little known theorem due to Campbell [1] is employed to establish the local embedding of a wide class of 4–dimensional spacetimes in 5–dimensional Ricci–flat spaces. An embedding for the class of n-dimensional Einstein spaces is also found. The local nature of Campbell's theorem is highlighted by studying the embedding of some lower–dimensional spaces.

## 1 Introduction

There has been considerable interest in recent years in theories of gravity that contain a different number of spatial dimensions from the usual three in general relativity (GR). One physical motivation for considering more than three spatial dimensions arises from the Kaluza–Klein interpretation of the fundamental interactions [2, 3, 4, 5]. Extra spatial dimensions also arise naturally in supergravity [4] and superstring theories [6] and may have played an important role in the evolution of the very early Universe [3, 7].

A new version of 5-dimensional GR has recently been developed by Wesson and others [8, 9, 10]. In this approach, the energy density and pressure of the (3 + 1)dimensional energy-momentum tensor arise directly from the extra components of the (4 + 1)-dimensional Einstein tensor,  ${}^{(5)}G_{ab}$ , where it is assumed that  ${}^{(5)}G_{ab} = 0$ . Thus,

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the physics of (3 + 1)-dimensional cosmologies may be recovered, in principle, from the geometry of (4 + 1)-dimensional, vacuum GR [9]. Direct calculations have verified that the spatially flat, perfect fluid cosmologies may be derived in this way [9, 10].

Lower-dimensional theories of gravity have also been extensively studied in recent years [11]. These theories are interesting because they may provide a solvable framework within which many of the technical and conceptual problems associated with quantum gravitational effects in (3 + 1) dimensions may be addressed. A question that naturally arises in these studies, however, is the extent to which the results and intuitions obtained in lower dimensions may be directly carried over to the (3 + 1)-dimensional environment and vice versa [12, 13]. In general, the precise relationship between these theories and (3 + 1)-dimensional Einstein gravity is not clear. For example, GR does not exhibit a Newtonian limit in (2 + 1) dimensions [14], whereas other theories, such as a modified version of the Brans-Dicke theory, do have such a limit [13].

An investigation into how lower– and higher–dimensional theories of gravity are related to 4–dimensional GR is therefore well motivated from a physical point of view. A potential bridge between gravitational theories of different dimensionality may be found by employing the embedding relationships that exist between spaces and the main purpose of this paper is to investigate such relationships further.

The embedding of manifolds in higher dimensions is also interesting from a purely mathematical point of view. For example, it allows an alternative, invariant classification of known solutions to Einstein's field equations to be made [15]. Furthermore, the embedding method may lead to new solutions. Indeed, the maximal analytic extension of the Schwarzschild solution was independently found in this way [16].

A number of embedding theorems are in existence. It is well known that an analytic Lorentzian space  $V_n(s,t)$ , with s spacelike and t timelike dimensional, where n = s + t, can be locally and isometrically embedded into a higher dimensional, pseudo-euclidean space  $E_N(S,T)$ , where N = S + T,  $n \leq N \leq n(n+1)/2$ , and  $S \geq s$  and  $T \geq t$ are positive integers [18]. The line element of  $E_N$  is given by  $ds^2 = e_A(dx^A)^2$ , where  $A = (0, 1, \ldots, N - 1)$  and  $e_A = \pm 1$ . Thus, no more than ten dimensions are required to embed all 4-dimensional solutions to Einstein's field equations. On the other hand, if the Ricci tensor of  $V_n(s,t)$  is zero, then  $N \geq n+2$  [19]. This implies that no curved, 4-dimensional solutions to the vacuum Einstein equations can be locally embedded in a 5-dimensional flat space; the minimal embedding space is  $E_6$  [20].

There also exists a theorem due to Campbell [1, 23]:

**Theorem:** Any analytic Riemannian space  $V_n(s,t)$  can be locally embedded in a Ricciflat, Riemannian space  $V_{n+1}(\tilde{s}, \tilde{t})$ , where  $\tilde{s} = s$  and  $\tilde{t} = t + 1$ , or  $\tilde{s} = s + 1$  and  $\tilde{t} = t$ .

Campbell's theorem implies that all solutions to the *n*-dimensional Einstein field equations with arbitrary energy-momentum tensor can be embedded, at least locally, in a spacetime that is itself a solution to (n + 1)-dimensional, vacuum GR [25]. (We refer to a local embedding in the usual differential geometric sense, i.e., without any direct reference to the global topology of the embedding or the embedded spaces). This theorem is therefore closely related to Wesson's procedure [8], since it implies that any 4-dimensional cosmology may be locally embedded in a 5-dimensional Ricci-flat space, at least in principle. Very little reference to Campbell's theorem can be found in the literature (see, however, Magaard [23] and Goenner [24]). Recently, Romero, Tavakol and Zalaletdinov [25] outlined the proof of this theorem in a modern notation and discussed its relationship with Wesson's procedure [8]. They also emphasised its constructive nature with the help of some concrete examples.

Here we investigate some further applications of Campbell's theorem. Section 2 summarizes the main points of the theorem. In Sections 3 and 4 we consider the embedding of the general class of 4–dimensional spacetimes that admit a non–twisting null Killing vector [15]. In Section 5 we find an embedding space for the general class of n–dimensional Einstein spaces. We then proceed in Section 6 to discuss some aspects of Campbell's theorem regarding the local and global embeddings of lower–dimensional gravity. We conclude in Section 7.

## 2 The Embedding Theorem of Campbell

We begin by discussing the embedding theorem due to Campbell [1, 23, 25]. Consider the space  $V_n(s,t)$  with metric  ${}^{(n)}g_{\alpha\beta}(x^{\mu})$  and line element<sup>1</sup>

$${}^{(n)}ds^{2} = {}^{(n)}g_{\alpha\beta}(x^{\mu})dx^{\alpha}dx^{\beta}, \qquad (2.1)$$

and let the local embedding of this space in the manifold  $V_{n+1}(\tilde{s}, \tilde{t})$  be given by

$${}^{(n+1)}ds^2 = g_{\alpha\beta}(x^\mu,\psi)dx^\alpha dx^\beta + \epsilon\phi^2(x^\mu,\psi)d\psi^2, \qquad (2.2)$$

where  $\epsilon = \pm 1$  and  $\psi$  is the coordinate that spans the extra dimension. It is assumed that  $g_{\alpha\beta}$ , when restricted to a hypersurface  $\psi = \psi_0$ , results in  ${}^{(n)}g_{\alpha\beta}$ :

$$g_{\alpha\beta}(x^{\mu},\psi_0) = {}^{(n)}g_{\alpha\beta}(x^{\mu}).$$
 (2.3)

According to Campbell's theorem [1], the functional form of the higher-dimensional metric coefficients (2.2) can be determined if functions  $\Omega_{\alpha\beta}(x^{\mu},\psi)$  may be found which satisfy the set of conditions

$$\Omega_{\alpha\beta} = \Omega_{\beta\alpha} \tag{2.4}$$

$$\Omega^{\alpha}{}_{\beta;\alpha} = \Omega_{,\beta} \tag{2.5}$$

$$\Omega_{\alpha\beta}\Omega^{\alpha\beta} - \Omega^2 = -\epsilon^{(n)}R \tag{2.6}$$

on some hypersurface  $\psi = \psi_0$  and if the functions  $g_{\alpha\beta}$  and  $\Omega_{\alpha\beta}$  evolve in accordance with the equations

$$\frac{\partial g_{\alpha\beta}}{\partial \psi} = -2\phi\Omega_{\alpha\beta},\tag{2.7}$$

$$\frac{\partial \Omega^{\alpha}{}_{\beta}}{\partial \psi} = \phi \left( -\epsilon^{(n)} R^{\alpha}{}_{\beta} + \Omega \Omega^{\alpha}{}_{\beta} \right) + \epsilon g^{\alpha \lambda} \phi_{;\lambda\beta}, \qquad (2.8)$$

<sup>&</sup>lt;sup>1</sup>In this paper, Greek indices take values in the range (0, 1, ..., n - 1), Latin indices run from (0, 1, ..., n), semicolons and commas indicate covariant and partial differentiations respectively and space-time metrics have signature (+, -, -, ...).

respectively<sup>2</sup>. In these expressions,  $\Omega^{\alpha}{}_{\beta} \equiv {}^{(n)}g^{\alpha\lambda}\Omega_{\lambda\beta}$ ,  $\Omega \equiv {}^{(n)}g^{\alpha\beta}\Omega_{\alpha\beta}$  and  ${}^{(n)}R \equiv {}^{(n)}R_{\mu\nu}{}^{(n)}g^{\mu\nu}$ .

If Eqs. (2.7) and (2.8) are evaluated on the hypersurface  $\psi = \psi_0$ , it can be shown that Eqs. (2.4)–(2.8) are equivalent to the vacuum, (n + 1)–dimensional GR field equations  ${}^{(n+1)}R_{ab}(x^{\mu},\psi_0) = 0$  [25]. Moreover, it can be proved that Eqs. (2.4)–(2.6) are valid for all  $\psi$  in the neighbourhood of  $\psi_0$  when Eqs. (2.7) and (2.8) are satisfied [1]. This implies that the Ricci tensor of  $V_{n+1}$  vanishes for any  $\psi$  in the neighbourhood of  $\psi_0$ . Consequently, Eq. (2.2) may be viewed as an embedding of the metric (2.1) in a Ricci–flat, (n + 1)– dimensional space.

Applications of Campbell's theorem considered in this paper are based up on the integrability of Eq. (2.8). We consider the embedding of spaces with vanishing Ricci scalar curvature ( $^{(n)}R = 0$ ) and also find an embedding for the class of Einstein spaces ( $^{(n)}R = \text{constant}$ ). In the former case, one solution to Eqs. (2.4)–(2.6) is given by  $\Omega_{\alpha\beta} = 0$ . It should be emphasized, however, that this does not necessarily represent the most general solution possible. Thus, the embedding of spaces with  $^{(n)}R = 0$  may be divided into two subclasses. These correspond to embeddings where all the components of  $\Omega_{\alpha\beta}$  vanish and to those where some (or all) of the components are non-trivial.

As an example of an application of this theorem, we will conclude this Section by discussing the subclass of embeddings where  $\Omega_{\alpha\beta} = 0$ . It follows from Eq. (2.7) that  $\partial g_{\alpha\beta}/\partial \psi = 0$  and  $g_{\alpha\beta}$  is therefore independent of the extra coordinate  $\psi$ . This implies that  $g_{\alpha\beta} = {}^{(n)}g_{\alpha\beta}$  in the neighbourhood of the hypersurface  $\psi = \psi_0$ . The one remaining equation that needs to be solved is Eq. (2.8) which simplifies to

$${}^{(n)}g^{\alpha\lambda}\phi_{;\lambda\beta} = {}^{(n)}R^{\alpha}{}_{\beta}\phi.$$

$$(2.9)$$

We conclude, therefore, that the embedding metric is given by

$${}^{(n+1)}ds^2 = {}^{(n)}g_{\alpha\beta}dx^{\alpha}dx^{\beta} + \epsilon\phi^2 d\psi^2, \qquad (2.10)$$

where  $\phi$  is a solution to Eq. (2.9).

Taking the trace of Eq. (2.9) implies that  $\phi$  must satisfy the massless Klein–Gordon equation,  ${}^{(n)}g^{\alpha\beta}\phi_{;\alpha\beta} = 0$ . If we assume an embedding of this form, therefore, a necessary, but not sufficient, condition on  $\phi$  is that it be an harmonic function of  $x^{\mu}$ . This restriction often provides valuable insight into the generic form that  $\phi$  must take if it is to satisfy the full set of differential equations (2.9). We remark that similar conclusions hold when  $V_n$  is Ricci–flat. Indeed, one solution to Eq. (2.9) in this case is  $\phi = 1$  and this provides a simple proof of theorem III of Romero *et al.* [25].

An interesting consequence of this embedding is that it may be repeated indefinitely, at least in principle. That is, the Ricci-flat space with metric (2.10) may itself be embedded in an (n + 2)-dimensional, Ricci-flat space with a line-element given by  ${}^{(n+2)}ds^2 = {}^{(n+1)}ds^2 + \varphi^2 d\theta^2$ , where  $\theta$  represents the extra coordinate and  $\varphi = \varphi(x^{\mu}, \psi, \theta)$ 

$$t \to \psi, \quad h_{\alpha\beta} \to {}^{(3)}g_{\alpha\beta}, \quad K_{\alpha\beta} \to \Omega_{\alpha\beta},$$

where  $h_{\alpha\beta}$  is the metric of the 3-space and  $K_{\alpha\beta}$  is the extrinsic curvature.

<sup>&</sup>lt;sup>2</sup>We note here that with  $\phi = 1$ , n = 3, there is a clear parallel with the language used in the 1 + 3 decomposition employed in the ADM formalism [21] and the initial value formulation of GR [22]. This can be seen through the following identifications:

is an harmonic function satisfying  ${}^{(n+1)}g^{\alpha\lambda}\varphi_{;\lambda\beta} = 0$ . Thus, once a given Lorentzian space  $V_n$  has been embedded in a Ricci–flat space  $V_{n+1}$ , further embeddings in Ricci–flat spaces of progressively higher dimensions can be considered.

Having summarized the steps that need to be taken when applying Campbell's theorem, we proceed in the following Sections to investigate the local embedding of a wide class of Lorentzian spaces with one timelike dimension. We begin by considering the embedding of 4-dimensional spacetimes that admit a non-twisting null Killing vector field  $\mathbf{k}$ , where  $k_{(\mu;\nu)} = 0$ ,  $k_{\mu}k^{\mu} = 0$  and  $k_{[\mu}k_{\nu;\rho]} = 0$ . It can be shown that there are two classes of metrics that admit a Killing vector of this form [15], depending up on whether  $\mathbf{k}$  is covariantly constant,  $k_{\mu;\nu} = 0$ , or whether it satisfies the less severe restriction  $k_{(\mu;\nu)} = 0$ .

# 3 The Embedding of Spacetimes Admitting a Covariantly Constant Null Killing Vector Field

Metrics admitting a covariantly constant, null Killing vector field  $\mathbf{k}$  have the form

$$ds^{2} = dudv + f du^{2} - dx^{2} - dy^{2}, (3.1)$$

where  $k_{\mu} = \partial_{\mu} u$  and f = f(u, x, y) is an arbitrary function that is independent of the coordinate v [15]. The coordinates (x, y) span the spacelike 2-surfaces that are orthogonal to **k** and the surfaces u = constant are null. The Ricci scalar for these spacetimes vanishes for arbitrary f, whilst the Riemann and Ricci tensors are given by  $R_{\mu\nu\rho\sigma} = -2k_{[\mu}\partial_{\nu]}\partial_{[\rho}fk_{\sigma]}$  and  $R_{\mu\nu} = \frac{1}{2}(\partial_T^2 f)k_{\mu}k_{\nu}$ , respectively, where  $\partial_T^2$  is the Laplacian on the transverse 2-surfaces. The only non-zero components of these tensors are

$$R_{uxux} = -\frac{1}{2}f_{,xx}, \quad R_{uxuy} = -\frac{1}{2}f_{,xy}, \quad R_{uyuy} = -\frac{1}{2}f_{,yy}$$
(3.2)

and

$$R_{uu} = \frac{1}{2} \left( f_{,xx} + f_{,yy} \right) \tag{3.3}$$

and it follows from Eq. (3.2) that linear terms in f of the form  $a(u) + b_i(u)x^i$  do not affect the Riemann tensor. They can therefore be transformed away.

The class of spacetimes given by Eq. (3.1) is physically very interesting. They are known as *plane waves* when  $f(u, x^i) = h_{ij}(u)x^ix^j$  for some symmetric function  $h_{ij}(u)$ . These solutions with  $h_{ii} = 0$  were first discussed by Brinkman [31]. A purely gravitational wave is characterized by the condition  $h_{ii}(u) = 0$  and a purely electromagnetic wave corresponds to  $h_{ij}(u) = h(u)\delta_{ij}$ , where  $h(u) \ge 0$  [28]. The amplitudes of the gravitational and electromagnetic waves are given by the trace-free part of  $h_{ij}$  and by  $[\text{Tr}(h_{ij})]^{1/2}$ , respectively. In general, the amplitudes may be arbitrary functions of u.

When f is a solution to the Laplace equation  $\partial_T^2 f = 0$ , the manifold is Ricci-flat. In this case, Eq. (3.1) represents the most general, 4-dimensional solution to the vacuum Einstein field equations with a covariantly constant null vector [15]. The dependence of f on u may be arbitrary and many different solutions can therefore be considered. More general solutions to Laplace's equation, where  $h_{ij}$  also depends on  $x^k$ , are known as plane-fronted waves. Plane-fronted waves are solutions to any gravitational theory whose field equations are given in terms of a second-rank tensor derived from the curvature tensor and its derivatives [30]. This property may be traced to the fact that the curvature is null. Included in this class of theories is string theory [32]. Indeed, plane-fronted waves are *exact* solutions to the classical equations of motion to *all* orders in  $\sigma$ -model perturbation theory [29, 30]. Exact solutions that include non-trivial dilaton and antisymmetric tensor fields can also be found and correspond to the case where  $\partial_T^2 f$  is an arbitrary function of u [30].

Since R = 0 for all f, we may begin by choosing  $\Omega_{\alpha\beta} = 0$ . An embedding metric is therefore given by Eq. (2.10), where  $\phi = \phi(x^{\mu}, \psi)$  satisfies Eq. (2.9). This equation represents the set of coupled differential equations:

$$f_{,x}\phi_{,v} - \phi_{,ux} = 0 \tag{3.4}$$

$$f_{,y}\phi_{,v} - \phi_{,uy} = 0 \tag{3.5}$$

$$-2f_{,u}\phi_{,v} - f_{,x}\phi_{,x} - f_{,y}\phi_{,y} + 2\phi_{,uu} = (f_{,xx} + f_{,yy})\phi$$
(3.6)

$$\phi_{,uv} = \phi_{,vv} = \phi_{,vx} = \phi_{,vy} = \phi_{,xx} = \phi_{,xy} = \phi_{,yy} = 0.$$
(3.7)

Differentiating Eqs. (3.4) and (3.5) both with respect to x and y implies that

$$f_{,xx}\phi_{,v} = f_{,xy}\phi_{,v} = f_{,yy}\phi_{,v} = 0, \qquad (3.8)$$

where we have employed Eq. (3.7). It follows from Eqs. (3.2) and (3.8) that the Riemann curvature tensor must vanish if  $\phi_{,v} \neq 0$ , thereby implying that the spacetime is flat. Consequently,  $\phi$  must be independent of v when  $\Omega_{\alpha\beta} = 0$ . In this case, Eqs. (3.4)–(3.6) simplify to

$$\phi_{,ux} = 0 \tag{3.9}$$

$$\phi_{,uy} = 0 \tag{3.10}$$

$$-f_{,x}\phi_{,x} - f_{,y}\phi_{,y} + 2\phi_{,uu} = (f_{,xx} + f_{,yy})\phi.$$
(3.11)

The embedding metric is determined once Eqs. (3.9)-(3.11) have been solved subject to the constraints (3.7). We will now consider the vacuum and non-vacuum cases in turn.

#### 3.1 The Embedding of Vacuum Plane–fronted Waves

The right-hand side of Eq. (3.11) vanishes for the vacuum solutions. Differentiating this equation with respect to both x and y then implies that

$$f_{,xx}\phi_{,x} + f_{,xy}\phi_{,y} = 0$$
  
$$f_{,xy}\phi_{,x} + f_{,yy}\phi_{,y} = 0$$
 (3.12)

and combining these two equations implies that

$$f_{,xx}\left(\phi_{,x}^{2}+\phi_{,y}^{2}\right) = 0$$
  
$$f_{,xy}\left(\phi_{,x}^{2}+\phi_{,y}^{2}\right) = 0.$$
 (3.13)

If the embedded spacetime is vacuum and has non-zero curvature, the second derivatives of f with respect to x and y must be non-vanishing. Consequently, Eq. (3.13) can only be

satisfied if  $\phi_{,x}^2 = -\phi_{,y}^2$ . However,  $\phi$  should be a real function if the embedding spacetime is to be physical. Thus,  $\phi$  must be independent of both x and y.

The only non-trivial constraint that remains in Eqs. (3.9)–(3.11), therefore, is that  $\phi_{,uu} = 0$  and this has the general solution  $\phi = a(\psi) + b(\psi)u$ , where a and b are arbitrary functions of the fifth coordinate  $\psi$ . One possible embedding of 4-dimensional, vacuum, plane-fronted waves in a 5-dimensional, Ricci-flat manifold is therefore given by

$$ds^{2} = dudv + f du^{2} - dx^{2} - dy^{2} - (a(\psi) + b(\psi)u)^{2} d\psi^{2}.$$
 (3.14)

A second local embedding of these plane–fronted waves can be found by assuming that

$$\Omega_{\alpha\beta} = \begin{cases} f/(2\psi_0) & \text{if } \alpha = \beta = u \\ 0 & \text{otherwise} \end{cases}$$
(3.15)

on the hypersurface  $\psi = \psi_0$ . In this case, the only non-zero components of  $\Omega^{\alpha}{}_{\beta}$  and  $\Omega^{\alpha\beta}$  are  $\Omega^{v}{}_{u} = 2\Omega_{uu}$  and  $\Omega^{vv} = 4\Omega_{uu}$ , respectively, where indices have been raised with  ${}^{(4)}g^{\alpha\beta}$ . It follows immediately that  $\Omega = \Omega_{\alpha\beta}\Omega^{\alpha\beta} = 0$ , so Eq. (2.6) is satisfied. Moreover, Eq. (2.5) simplifies to  $\Omega^{v}{}_{u;v} = 0$ . Since both  $\Omega^{\alpha}{}_{\beta}$  and  ${}^{(4)}g_{\alpha\beta}$  are independent of v, however, this condition is also satisfied.

Eq. (2.8) is solved by specifying  $\phi = -1$ , since  $\Omega^{\alpha}{}_{\beta}$  is independent of  $\psi$ . Thus, the solution to Eq. (2.7) that satisfies the initial conditions (2.3) on the hypersurface  $\psi = \psi_0$  is given by

$$g_{\alpha\beta} = \begin{cases} (\psi/\psi_0)f & \text{if } \alpha = \beta = u \\ {}^{(4)}g_{\alpha\beta} & \text{otherwise.} \end{cases}$$
(3.16)

It follows, therefore, that when indices are raised with  $g^{\alpha\beta}$ , the only non-zero components of  $\Omega^{\alpha}{}_{\beta}$  and  $\Omega^{\alpha\beta}$  are  $\Omega^{v}{}_{u}$  and  $\Omega^{vv}$ , as before. Thus, Eqs. (2.4) and (2.6) are valid for arbitrary  $\psi$ . The same conclusion holds for Eq. (2.5), since  $g^{\alpha\beta}$  is itself independent of v.

We may conclude, therefore, that the 5–dimensional embedding for this *ansatz* is given by

$$ds^{2} = dudv + \frac{\psi}{\psi_{0}}fdu^{2} - dx^{2} - dy^{2} - d\psi^{2}.$$
(3.17)

It may be verified by direct calculation that this space is Ricci-flat. Its curvature differs from that of Eq. (3.14), however. In particular, we find that  $R_{uxu\psi} = -(1/2\psi_0)f_{,x}$  and  $R_{uyu\psi} = -(1/2\psi_0)f_{,y}$ , whereas these components vanish for the spacetime corresponding to Eq. (3.14).

### 3.2 The Embedding of Electromagnetic Waves and Exact String Backgrounds

We may consider the more general class of spacetimes characterized by

$$f(u, x^k) = h_{ij}(u)x^i x^j + f_T(u, x^k),$$
(3.18)

where  $f_T$  is an arbitrary solution to  $\partial_T^2 f_T = 0$ . These spacetimes are not Ricci-flat if  $\operatorname{Tr}(h_{ij}) \neq 0$ , since  $\partial_T^2 f = 2(h_{11} + h_{22})$ .

Now, the general form of  $\phi$  consistent with Eqs. (3.7), (3.9) and (3.10) is given by

$$\phi = a(u) + bx + cy, \tag{3.19}$$

where a(u) is an arbitrary function of u and b and c are arbitrary constants. (We assume for simplicity that  $\phi$  is independent of the fifth coordinate). The embedding spacetime may therefore be determined by finding a solution to Eq. (3.11) that is consistent with Eq. (3.19).

Let us begin with the simpler case where b = c = 0, so that  $\phi$  is a function only of u. Substitution of Eqs. (3.18) and (3.19) into Eq. (3.11) then implies that

$$\frac{d^2a}{du^2} = (h_{11} + h_{22})a. aga{3.20}$$

In this case, the embedding metric is given by

$$ds^{2} = dudv + f du^{2} - dx^{2} - dy^{2} - a^{2}(u)d\psi^{2}$$
(3.21)

and may be expressed in a closed form whenever an exact solution to Eq. (3.20) can be found for a given  $h_{ij}(u)$ . This embedding is general, in the sense that the amplitude  $h_{ij}$  is an arbitrary function of u. The functional form of  $\phi$  is also independent of  $f_T$ , so we may consider arbitrary forms for this latter function, subject to the condition that it satisfies the Laplace equation  $\partial_T^2 f_T = 0$ . Eq. (3.22) corresponds to the embedding of an electromagnetic plane wave of arbitrary amplitude when  $h_{12} = f_T = 0$  and  $h_{11} = h_{22}$ . In this case, a space with vanishing Weyl tensor is embedded in a space with vanishing Ricci tensor of the form

$$ds^{2} = dudv + \left(\frac{1}{2a}\frac{d^{2}a}{du^{2}}\right)\left(x^{2} + y^{2}\right)du^{2} - dx^{2} - dy^{2} - a^{2}(u)d\psi^{2}.$$
 (3.22)

An embedding is also possible if  $(b, c) \neq 0$  and  $f_T = 0$ . In this case, Eq. (3.20) still applies, but the components of  $h_{ij}$  are restricted by the additional constraints

$$(2h_{22} + h_{11})^{1/2} (2h_{11} + h_{22})^{1/2} = \mp h_{12}$$
$$b = \pm \left(\frac{2h_{22} + h_{11}}{2h_{11} + h_{22}}\right)^{1/2} c.$$
(3.23)

We find that the embedding metric is given by

$$ds^{2} = dudv + h_{ij}x^{i}x^{j}du^{2} - dx^{2} - dy^{2} - [a(u) + bx + cy]^{2}d\psi^{2}$$
(3.24)

when Eqs. (3.20) and (3.23) are satisfied.

# 4 The Embedding for Spacetimes Admitting a Non– Constant Null Killing Vector Field

If the null Killing vector is not (covariantly) constant, the metric is given by

$$ds^{2} = 2xdu(dv + mdu) - x^{-1/2}(dx^{2} + dy^{2}), \qquad (4.1)$$

where m = m(u, x, y) is an arbitrary function and is independent of v [15]. The Ricci curvature scalar of these spacetimes vanishes for arbitrary m and the only non-trivial component of the Ricci tensor is given by

$$R_{uu} = x^{1/2} \left( (xm_{,x})_{,x} + xm_{,yy} \right).$$
(4.2)

We will consider the subset of spacetimes that are solutions to vacuum GR. This includes a wide class of spaces, since the dependence of m on u is arbitrary. When m is independent of u and  $R_{uu} = 0$  the spacetimes (4.1) are the stationary van Stockum solutions [15].

Following the discussion of Section 2, we choose  $\Omega_{\alpha\beta} = 0$ , since the Ricci scalar is zero. The (u, u), (u, y), (x, u), (x, v), (x, y) and (y, y) components of Eq. (2.9) are then given by

$$x^{1/2}\phi_{,x} - 2\phi_{,uv} = 0 \tag{4.3}$$

$$\phi_{,vy} = 0 \tag{4.4}$$

$$2xm_{,x}\phi_{,v} - 2x\phi_{,ux} + \phi_{,u} = 0 \tag{4.5}$$

$$2x\phi_{,vx} - \phi_{,v} = 0 \tag{4.6}$$

$$4x\phi_{,xy} + \phi_{,y} = 0 \tag{4.7}$$

$$4x\phi_{,yy} - \phi_{,x} = 0, (4.8)$$

respectively. However, differentiation of Eq. (4.8) with respect to v implies that  $\phi_{,vx} = 0$ , where we have employed Eq. (4.4). Thus, Eq. (4.6) implies that  $\phi$  must be independent of v, but it then follows from Eq. (4.3) that  $\phi$  must also be independent of x. Eq. (4.7) then implies that  $\phi$  must also be independent of y and, finally, Eq. (4.5) implies that  $\phi$  is independent of u. In conclusion, therefore, the only consistent solution is that  $\phi = \phi(\psi)$ when  $\Omega_{\alpha\beta} = 0$ .

An alternative embedding may be found by assuming the ansatz

$$\Omega_{\alpha\beta} = \begin{cases} xm/\psi_0 & \text{if } \alpha = \beta = u \\ 0 & \text{otherwise.} \end{cases}$$
(4.9)

The argument is similar to that followed in the previous Section. Eqs. (2.4)–(2.6) are satisfied on a particular hypersurface  $\psi = \psi_0$ , because  ${}^{(4)}g^{uu} = 0$  and  ${}^{(4)}g^{\alpha\beta}$  and  $\Omega_{\alpha\beta}$  are independent of v. One solution to Eq. (2.8) is given by  $\phi = -1$  and this implies that the integral of Eq. (2.7) is given by

$$g_{\alpha\beta} = \begin{cases} 2xm(\psi/\psi_0) & \text{if } \alpha = \beta = u \\ {}^{(4)}g_{\alpha\beta} & \text{otherwise.} \end{cases}$$
(4.10)

The embedding metric is therefore given by

$$ds^{2} = 2xdudv + 2xm\left(\frac{\psi}{\psi_{0}}\right)du^{2} - x^{-1/2}dx^{2} - x^{-1/2}dy^{2} - d\psi^{2}.$$
(4.11)

It may be verified that Eqs. (2.4)–(2.6) remain valid when indices are raised with  $g^{\alpha\beta}$ , so they are valid for all  $\psi$ .

To summarize thus far, we have found embedding spaces for the general class of metrics that admit a non-twisting, null Killing vector field. All these spacetimes have vanishing curvature scalar, however. We therefore extend our analysis in the following Section to include the class of spacetimes for which the curvature scalar is covariantly constant.

### 5 The Embedding of Einstein spaces

The class of n-dimensional Einstein spaces is defined by

$${}^{(n)}R^{\alpha}{}_{\beta} = \left(\frac{\kappa}{n}\right){}^{(n)}\delta^{\alpha}{}_{\beta}, \qquad {}^{(n)}R = \kappa, \tag{5.1}$$

where  $\kappa$  is an arbitrary constant [33]. For  $n \geq 3$ , we may define  $\Lambda \equiv (2 - n)\kappa/(2n)$ . This may be viewed as a cosmological constant in the vacuum Einstein field equations  ${}^{(n)}G^{\alpha}{}_{\beta} = \Lambda^{(n)}\delta^{\alpha}{}_{\beta}$ .

We proceed by specifying  $\phi = 1$ . Eq. (2.8) then reduces to

$$\frac{\partial \Omega^{\alpha}{}_{\beta}}{\partial \psi} = \frac{-\epsilon \kappa}{n} \delta^{\alpha}{}_{\beta} + \Omega \Omega^{\alpha}{}_{\beta}.$$
(5.2)

This equation admits the exact solution

$$\Omega^{\alpha}{}_{\beta} = a\psi^{-1}\delta^{\alpha}{}_{\beta} \tag{5.3}$$

on the specific hypersurface

$$\psi = \psi_0 = \pm \left(\frac{an(1+an)}{\epsilon\kappa}\right)^{1/2},\tag{5.4}$$

where a is a constant. When  $\epsilon = -1$  (corresponding to an extra spacelike coordinate), we require for consistency that -1/n < a < 0 if  $\kappa > 0$  and a < -1/n or a > 0 if  $\kappa < 0$ . Conversely, for  $\epsilon = +1$ , we require a > 0 or a < -1/n for  $\kappa > 0$  and -1/n < a < 0 if  $\kappa < 0$ .

We will assume for the moment that the solution (5.3) is valid for arbitrary values of  $\psi$ when  ${}^{(n)}R^{\alpha}{}_{\beta}$  is calculated with  $g_{\alpha\beta}(x^{\mu},\psi)$  rather than with the original metric  ${}^{(n)}g_{\alpha\beta}(x^{\mu})$ . (The validity of this assumption will be verified shortly for a = -1). We may now integrate Eq. (2.7) subject to the initial conditions (2.3). We find that

$$g_{\alpha\beta}(x^{\mu},\psi) = \psi^{-2a} \left(\frac{an(1+an)}{\epsilon\kappa}\right)^{a}{}^{(n)}g_{\alpha\beta}(x^{\mu}).$$
(5.5)

The functions  $\Omega_{\alpha\beta}$  are then determined by this equation and Eq. (5.3). The result is

$$\Omega_{\alpha\beta} = a\psi^{-1-2a} \left(\frac{an(1+an)}{\epsilon\kappa}\right)^a {}^{(n)}g_{\alpha\beta}.$$
(5.6)

However, Eqs. (2.4)—(2.6) must also be satisfied. The functions  $\Omega_{\alpha\beta}$  are symmetric, so Eq. (2.4) is clearly valid. Moreover, they are independent of  $x^{\mu}$ , so Eq. (2.5) is also consistent. On the other hand, Eq. (2.6) is more restrictive because  $\Omega = an\psi^{-1} \neq 0$  in general. Indeed, this equation is only satisfied if a = -1. Thus, the solution (5.3) only applies if  $\epsilon \kappa > 0$ , which implies that  $\epsilon = -1$  ( $\epsilon = +1$ ) for a positive (negative)  $\Lambda$ .

Finally, Eq. (2.8) must be considered when  ${}^{(n)}R^{\alpha}{}_{\beta}$  is calculated with the *n*-dimensional part of the (n+1)-dimensional embedding metric. This will establish the validity of this embedding procedure. For a given metric  $g_{\alpha\beta}$ , we may perform the conformal transformation  $\tilde{g}_{\alpha\beta} = kg_{\alpha\beta}$ , where k is constant. It follows that  $\tilde{R}^{\alpha}{}_{\beta} = k^{-1}R^{\alpha}{}_{\beta}$  and  $\tilde{R} = k^{-1}R$ . Using this property, we are able to calculate  ${}^{(n)}R^{\alpha}{}_{\beta}$  with the metric  $g_{\alpha\beta}(x^{\mu},\psi) = k^{(n)}g_{\alpha\beta}$ , since  $k = \epsilon \kappa \psi^2 / [n(n-1)]$  is a constant. We find that

$${}^{(n)}R^{\alpha}{}_{\beta}(x^{\mu},\psi) = k^{-1}R^{\alpha}{}_{\beta}(x^{\mu}) = \left[\frac{n-1}{\epsilon\psi^2}\right]{}^{(n)}\delta^{\alpha}{}_{\beta}$$
(5.7)

and one can further show by an analogous argument that

$${}^{(n)}R(x^{\mu},\psi) = k^{-1}R = \frac{n(n-1)}{\epsilon\psi^2}.$$
(5.8)

Direct substitution of Eqs. (5.7) and (5.8) then implies that Eqs. (2.6) and (2.8) are valid for arbitrary  $\psi$ . Consequently, the embedding metric is locally Ricci-flat for every value of  $\psi$ , as required, and it is given by

$${}^{(n+1)}ds^2 = \left[\frac{\epsilon\kappa}{n(n-1)}\psi^2\right]{}^{(n)}g_{\alpha\beta}(x^{\mu})dx^{\alpha}dx^{\beta} + \epsilon d\psi^2.$$
(5.9)

The embeddings that we have considered in this paper thus far are local, in the sense that no reference was made to the global topology of either the embedded or embedding space. This is because Campbell's theorem is a local theorem. In the next Section, we shall highlight the local nature of this theorem further by investigating the embeddings of some lower-dimensional spacetimes.

# 6 Local and Global Embedding of Spaces with Lower Dimensions

Clarke [34] has proved that any  $C^{\infty}$ -Riemannian manifold  $V_n$  with  $C^k$ -Riemannian metric  $(k \ge 3)$  of rank r and signature s can be globally  $C^k$ -isometrically embedded in  $E_m(p,q)$ , where

$$m = p + q, \qquad p \ge n - \frac{r+s}{2} + 1$$
 (6.1)

and

$$q \ge \frac{n}{2}(3n+11) \tag{6.2}$$

if  $V_n$  is compact and

$$q \ge \frac{n}{2}(2n^2 + 27) + \frac{5}{2}n^2 + 1 \tag{6.3}$$

if  $V_n$  is non-compact. Clearly, an analogue of this theorem is required, where the embedding space is Ricci, rather than Riemann, flat. Unfortunately, such a theorem does not as yet exist, but the lower bounds (6.2) and (6.3) suggest that more than one extra dimension may generally be needed for global embeddings in Ricci-flat spaces. Nevertheless, some aspects of Campbell's theorem with regard to local and global embeddings can be highlighted by considering lower-dimensional examples.

#### **6.1** Embedding of (1+1)-dimensional Spaces

Campbell's theorem implies that any (1 + 1)-dimensional space can be locally embedded in a 3-dimensional, Ricci-flat space  $M_3$ . However, the Weyl tensor vanishes identically in three dimensions, so the embedding space is necessarily flat, i.e.,  ${}^{(3)}R_{\mu\nu\lambda\rho} \equiv 0$ . Thus, Campbell's theorem is equivalent to Friedman's theorem in this case [18].

This would seem to imply that Campbell's theorem results in a trivial embedding of all (1+1)-dimensional spaces. It is important to emphasize, however, that the *topology* of the embedding space is not specified in this procedure, due to the local nature of the theorem. In principle, therefore, a given space  $V_n$  may be (locally) embedded into more than one higher-dimensional, Ricci-flat space, each of which has a different global topology. This follows since there is usually more than one solution to Eqs. (2.7) and (2.8) consistent with the boundary conditions (2.3). Moreover, the range of the extra coordinate  $\psi$  is not specified in Campbell's approach. It may be either compact or non-compact and this will also affect the topology of the embedding space. Within the context of (1+1) dimensions, this implies that the metric on  $M_3$  may not cover the whole of Minkowski space. It is possible, therefore, that the embedding space may contain singularities and, indeed, it may even exhibit a non-trivial causal structure.

These features may be illustrated by considering different embeddings of the (1 + 1)dimensional Minkowski space  ${}^{(2)}d\eta^2 = dt^2 - dx^2$ , where  $-\infty \leq \{t, x\} \leq +\infty$ . It follows from Section 2 that one class of embedding metrics is given by  ${}^{(3)}ds^2 = {}^{(2)}d\eta^2 - \phi^2 d\psi^2$ , where<sup>3</sup>  $\partial_{\mu}\partial_{\nu}\phi = 0$ . The general solution to these equations is given by  $\phi = a_{\mu}(\psi)x^{\mu}$ , where  $a_{\mu}$  are arbitrary functions.

The whole of  $M_3$  is covered if  $\phi = 1$  and  $-\infty \leq \psi \leq +\infty$ . However, a non-trivial embedding is given by the solution  $\phi = x - t$  when  $\psi$  is a compact coordinate. In this case,

$$^{(3)}ds^2 = dudv - u^2 d\psi^2, \tag{6.4}$$

where  $u \equiv t - x$  and  $v \equiv t + x$  represent null coordinates and  $\psi$  is identified with  $\psi + L$  for some arbitrary constant L. A linear translation on  $\psi$  corresponds to a null boost. Since  $\psi$ is compact, the geometry of a constant x surface, with line–element  $ds^2 = dt^2 - (x-t)^2 d\psi^2$ , is given by  $R \times S^1$ . This surface resembles a lorentzian cone, because there exists a vertex at x = t. Thus, the spacetime (6.4) is geodesically incomplete. It represents a lorentzian orbifold whose vertex moves at the speed of light [35]. In conclusion, therefore, the topology of the embedding space is determined by the specific boundary conditions that are chosen when solving Eq. (2.9), as well as the range of values taken by the extra coordinate.

#### **6.2** Embedding of (2+1)-dimensional Spaces

Further insight may be gained by considering the embedding of 3-dimensional spaces in four dimensions. As an example, we consider the line-element in (2+1) dimensions given by

$$^{(3)}ds^{2} = dt^{2} - d\rho^{2} - \lambda^{2}\rho^{2}d\theta^{2}, \qquad (6.5)$$

where  $\lambda$  is a constant and  $0 \leq \theta \leq 2\pi$ . When  $\lambda = 1 - 4m\bar{G}$ , this represents the spacetime generated by a static point particle of mass m, where  $\bar{G}$  is the gravitational constant

<sup>&</sup>lt;sup>3</sup>In the rest of this Section Greek indices take values in the range (0, 1) only.

in (2 + 1) dimensions [36]. As is well known, this space is flat for  $\rho \neq 0$ . However, it is not globally Ricci-flat. The non-zero components of the Ricci tensor are  ${}^{(3)}R^{\rho}{}_{\rho} =$  ${}^{(3)}R^{\theta}{}_{\theta} = 2\pi\lambda^{-1}(\lambda - 1)\delta^{(2)}(\rho)$ , where  $\delta^{(2)}(\rho)$  is the Dirac delta function at the origin of the 2-surface t = constant, the normalization of which is defined by the condition  $\int_{0}^{2\pi} d\theta \int_{0}^{\infty} d\rho \delta^{(2)}(\rho)\rho = 1$ . In fact, Eq. (6.5) represents a conical spacetime which is flat everywhere except at one point corresponding to the vertex  $\rho = 0$ . (The constant t sections may be viewed as euclidean planes in which a wedge with opening angle  $2\pi(1 - \lambda)$  is cut out and its edges identified).

Now, following Campbell's method, the simplest embedding of metric (6.5) for  $\rho \neq 0$  in a (3+1)-dimensional, Ricci-flat spacetime is given by

$$^{(4)}ds^{2} = dt^{2} - d\rho^{2} - \lambda^{2}\rho^{2}d\theta^{2} - d\psi^{2}.$$
(6.6)

If  $\psi$  is a non-compact coordinate  $(-\infty \leq \psi \leq +\infty)$ , Eq. (6.6) is the metric of a static, vacuum cosmic string, where  $\mu \equiv \bar{G}m/G$  is the linear mass density lying on the  $\psi$ -axis and G is the gravitational constant in (3 + 1) dimensions [37, 38]. This spacetime is flat for  $\rho \neq 0$  and the surfaces defined by t = constant and  $\psi = \text{constant}$  have the same topology as a cone. The Ricci components are  ${}^{(4)}R^{\rho}{}_{\rho} = {}^{(4)}R^{\theta}{}_{\theta} = 2\pi\lambda^{-1}(\lambda-1)\delta^{(2)}(\rho)$ .

This embedding (6.6) is not global, however, because the embedding space is only locally Ricci-flat. On the other hand, the space defined by Eq. (6.5) may be globally embedded in (3 + 1)-dimensional Minkowski spacetime. If we define a new coordinate

$$z \equiv (1 - \lambda^2)^{1/2} \rho,$$
 (6.7)

Eq. (6.5) transforms to

$$^{(3)}ds^{2} = dt^{2} - \frac{1}{1 - \lambda^{2}}dz^{2} - \frac{\lambda^{2}z^{2}}{1 - \lambda^{2}}d\theta^{2}.$$
(6.8)

Equation (6.8) represents the induced metric on the  $\xi = 0$  hypersurface of the (3 + 1)-dimensional spacetime

$${}^{(4)}ds^2 = dt^2 - \frac{\lambda^2}{1-\lambda^2}d\xi^2 - \frac{\lambda^2}{1-\lambda^2}(\xi+z)^2d\theta^2 - \frac{1}{1-\lambda^2}dz^2 - \frac{2\lambda^2}{1-\lambda^2}d\xi dz$$
(6.9)

and the coordinate transformation

$$\xi = \frac{\sqrt{1-\lambda^2}}{\lambda}\rho - z \tag{6.10}$$

maps this metric onto Minkowski space.

We conclude, therefore, that the application of Campbell's theorem allows us to embed the spacetime (6.5) into (6.6) locally, but not globally, in the sense that all points on the spacetime (6.5) are included in the embedding. Since the spacetimes represented by Eqs. (6.5) and (6.6) have a conical singularity at the point  $\rho = 0$ , Campbell's theorem will only work for  $\rho \neq 0$ . The reason for this restriction is that Campbell's theorem assumes implicitly that the extra coordinate vector  $\partial/\partial\xi$  is orthogonal to Eq. (6.5), as can be seen from the general expression (2.2). A global embedding of Eq. (6.5) in Minkowski space may be achieved, however, by dropping this restriction. In this case, the embedding spacetime does not inherit the topological defect of the lower-dimensional manifold.

## 7 Discussion and Conclusions

We have employed Campbell's embedding theorem in a number of settings. Firstly, we considered spacetimes for which the Killing vector is covariantly constant. This class includes a number of physically interesting spaces, such as the electromagnetic and gravitational plane waves, as well as the more general plane–fronted waves. Although we found embedding spaces for waves with arbitrary amplitude, these embeddings could in principle be generalized by finding new solutions to Eqs. (2.7) and (2.8). We also considered spacetimes in which the Killing vector is not covariantly constant, including those which are solutions to vacuum GR such as the stationary van Stockum solutions. An embedding for the general class of n-dimensional Einstein spaces was found and we also discussed the local and global embedding of some lower–dimensional spaces.

Campbell's theorem is closely related to Wesson's interpretation of 5–dimensional, vacuum Einstein gravity [8, 9, 10]. In view of this, it would be of interest to consider the embedding of 4–dimensional, cosmological solutions in 5–dimensional, Ricci–flat spaces. For example, inflationary cosmology is thought to be relevant to the physics of the very early Universe [39, 40]. During inflation, the scale factor of the Universe accelerates and this acceleration is driven by the potential energy associated with the self–interactions of a scalar field. Different inflationary solutions correspond to different functional forms for the potential of this field. However, Campbell's theorem implies that all inflationary solutions can also be generated, at least in principle, from 5–dimensional, vacuum Einstein gravity. This implies the existence of a correspondence between inflationary cosmology and Einstein's theory in five dimensions. In principle, such a relationship could be formulated by employing Campbell's theorem.

Although Campbell's theorem relates n-dimensional theories to vacuum (n + 1)dimensional theories, it does not establish a strict equivalence between them. It is therefore important to determine when such theories are *equivalent*. Clearly, this is a more severe restriction than embedability. Two notions of equivalence that could be considered are *dynamical equivalence* and *geodesic equivalence*. Dynamical equivalence would imply that the dynamics of vacuum n-dimensional theories is included in the vacuum (n + 1)dimensional theories. In that case, the embedding would be given by Eq. (2.2) with  $\phi = 1$ . This would then imply that  ${}^{(n+1)}R_{\alpha\beta} = {}^{(n)}R_{\alpha\beta} = 0$  and that  ${}^{(n+1)}R_{\alpha n} = {}^{(n+1)}R_{nn} = 0$  [25].

Alternatively, one may consider geodesic equivalence, in the sense of Mashhoon *et al.* [41]. In this case the (3 + 1) geodesic equation induces a (2 + 1) geodesic equation plus a force term  $F^{\alpha}$ :

$$\frac{d^2x^{\alpha}}{ds^2} + \Gamma^{\alpha}{}_{\beta\gamma}\frac{dx^{\beta}}{ds}\frac{dx^{\gamma}}{ds} = F^{\alpha}.$$
(7.1)

For geodesic equivalence one would therefore require  $F^{\alpha} = 0$ , which is clearly so when

$$\frac{\partial g_{\alpha\beta}}{\partial \psi} = 0. \tag{7.2}$$

It would be interesting to ask whether these equivalences hold in more general settings.

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