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Source: *The American Mathematical Monthly* , Vol. 124, No. 7 (August-September 2017), pp. 621-636

Published by: Taylor & Francis, Ltd. on behalf of the Mathematical Association of America

Stable URL: https://www.jstor.org/stable/10.4169/amer.math.monthly.124.7.621

A Menger Redux: Embedding Metric Spaces Isometrically in Euclidean Space

John C. Bowers and Philip L. Bowers

Abstract. We present geometric proofs of Menger's results on isometrically embedding metric spaces in Euclidean space.

In 1928, Karl Menger [**5**] published the proof of a beautiful characterization of those metric spaces that are isometrically embeddable in the n-dimensional Euclidean space $\mathbb{E}^n$. While a visitor at Harvard University and the Rice Institute in Houston during the 1930–31 academic year, Menger gave courses on metric geometry in which he "considerably shortened and revised [his] original proofs and generalized the formulation." [**6**, p. 721]. The new proofs of the 1930–31 academic year appear in the English language article "New Foundation of Euclidean Geometry" [**6**] published in the *American Journal of Mathematics* in 1931. Leo Liberti and Carlile Lavor in their beautifully written article "Six Mathematical Gems from the History of Distance Geometry" remark on their review of a part of Menger's characterization that "it is remarkable that almost none of the results below offers an intuitive geometrical grasp, such as the proofs of Heron's formula and the Cayley theorem do. As formal mathematics has it, part of the beauty in Menger's work consists in turning the 'visual' geometrical proofs based on intuition into formal symbolic arguments based on sets and relations." [**3**, p. 12] Part of the reason that Menger's results fail to offer an "intuitive geometric grasp" is that Menger offers his results in the general setting of an abstract *congruence system* and *semi-metric spaces*.

We believe that Menger's characterization deserves a wider circulation among the mathematics community. The aim of this article is to explicate Menger's characterization in the category of metric spaces rather than in his original setting of congruence systems and their model semi-metric spaces and in doing so to give a mostly[1] self-contained treatment with straightforward geometric proofs of his characterization. We provide all the background material on the geometry of Euclidean space that is needed to prove Menger's results so that the proofs are accessible to any undergraduate who has mastered the basics of real linear algebra and real inner product spaces.[2] The full characterization as presented in Menger [**6**] may be parsed in three acts. Act 1 reduces the problem of isometrically embedding a metric space X into $\mathbb{E}^n$ to isometrically embedding finite subsets of X into $\mathbb{E}^n$. Act 2 characterizes those metric spaces that fail to embed in $\mathbb{E}^n$ isometrically though each subset with $n+2$ elements does so embed. Acts 1 and 2 together reduce the problem of embedding X isometrically into $\mathbb{E}^n$ to the problem of embedding each subset of X with $n+2$ elements isometrically.

[1] See footnote 3.

[2] *Mastered* being the key word in this sentence. The authors believe that the assignment of this article for careful study could be an appropriate and effective undergraduate project for helping a student master the geometry of Euclidean space using linear algebra and the Euclidean inner product.

http://dx.doi.org/10.4169/amer.math.monthly.124.7.621
MSC: Primary 51K99

Finally, Act 3 gives an algebraic condition on the distances among the points of a set with $n + 2$ elements that guarantees an isometric embedding into $\mathbb{E}^n$.

We give complete geometric proofs of Menger's results of Acts 1 and 2 and the necessity part of Act 3.[3] The approach of this article to Menger's results offers geometric insights not immediately grasped in Menger's original approach. In particular, the construction and characterization of the metric spaces in Act 2 that fail to embed isometrically though each subset with $n + 2$ elements does so embed offer details of the structure of these spaces as well as a geometric understanding that are opaque in Menger's treatment. This geometric approach leads to a characterization of all such spaces with a concrete construction for all of them.

OVERTURE: BACKGROUND ON EUCLIDEAN SPACE. We take as our model of Euclidean n-dimensional space, denoted as $\mathbb{E}^n$, the set of ordered n-tuples $\mathbf{x} = (x_1, \ldots, x_n)$ of real numbers with the usual real vector space operations and the Euclidean inner product $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + \cdots + x_n y_n$. This gives rise to the Euclidean norm $|\mathbf{x}| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ and the Euclidean distance $\mathbf{xy} = |\mathbf{x} - \mathbf{y}|$. In this Overture, we collect the basic geometric facts about Euclidean space that will be used in the proofs of Menger's characterization. The proofs of these basic facts use only elementary linear algebra and are left to the reader. The reader seasoned in the geometry of Euclidean space may skip this Overture safely and head directly to Act 1, referring only to the statements of the lemmas as they are used.

A collection $\{\mathbf{v}_0, \ldots, \mathbf{v}_m\}$ of $m + 1$ points in $\mathbb{E}^n$ is *affinely independent* if the collection $\{\mathbf{v}_1 - \mathbf{v}_0, \ldots, \mathbf{v}_m - \mathbf{v}_0\}$ is linearly independent. Note that this implies that $m \le n$. Affine independence of $\{\mathbf{v}_0, \ldots, \mathbf{v}_m\}$ is equivalent to the condition that, for real coefficients $a_0, \ldots, a_m$, if $\sum_{k=0}^m a_k \mathbf{v}_k = \mathbf{0}$ and $\sum_{k=0}^m a_k = 0$, then $a_0 = \cdots = a_m = 0$. In case $\{\mathbf{v}_0, \ldots, \mathbf{v}_m\}$ is affinely independent, the linear span of the m linearly independent vectors $\{\mathbf{v}_1 - \mathbf{v}_0, \ldots, \mathbf{v}_m - \mathbf{v}_0\}$ is an m-dimensional linear subspace of $\mathbb{E}^n$ that may be expressed as

$$L = \text{span}[\mathbf{v}_1 - \mathbf{v}_0, \ldots, \mathbf{v}_m - \mathbf{v}_0] = \left\{ \sum_{k=0}^m a_k \mathbf{v}_k \colon \sum_{k=0}^m a_k = 0 \right\}.$$

It follows from this that the *affine span* of $\{\mathbf{v}_0, \ldots, \mathbf{v}_m\}$ defined as

$$A = \text{affspan}[\mathbf{v}_0 \ldots, \mathbf{v}_m] = \left\{ \sum_{k=0}^m a_k \mathbf{v}_k \colon \sum_{k=0}^m a_k = 1 \right\}$$

is the *m-dimensional affine plane* obtained by translating the linear subspace L by any element of A so that, in particular, $A = \mathbf{v}_0 + L$. Each element $\mathbf{v}$ of the affine plane A has a unique set of *affine coordinates* $a_0, \ldots, a_m$ with respect to the *affine basis* $V = \{\mathbf{v}_0, \ldots, \mathbf{v}_m\}$ *for* A, with $\mathbf{v} = \sum_{k=0}^m a_k \mathbf{v}_k$ where $\sum_{k=0}^m a_k = 1$. Note in particular that the affine basis V itself is contained in A and that if $\mathbf{x}$ is any element of $\mathbb{E}^n$ not contained in A, then the set $V \cup \{\mathbf{x}\}$ is affinely independent.

A subset C of $\mathbb{E}^n$ is *convex* if, for each pair $\mathbf{x}, \mathbf{y}$ of points in C, the line segment $[\mathbf{x}, \mathbf{y}] = \{t\mathbf{x} + (1 - t)\mathbf{y} : 0 \le t \le 1\}$ is contained in C. The *convex hull* of a set B in $\mathbb{E}^n$ is the intersection of all convex sets containing B and is the smallest convex set containing B.

[3] The sufficiency part of Act 3 is primarily algebraic in nature and depends on a detailed algebraic analysis of certain determinants. As our interest is in explicating the geometry that lies behind Menger's results, we refer the interested reader to Menger's original proof in [**6**] for the sufficiency.

The convex hull of some affinely independent set $V = \{\mathbf{v}_0, \ldots, \mathbf{v}_m\}$ in $\mathbb{E}^n$ is called the *m-simplex* spanned by V, is denoted as $[\mathbf{v}_0, \ldots, \mathbf{v}_m]$, and has the description

$$[\mathbf{v}_0, \ldots, \mathbf{v}_m] = \left\{ \sum_{k=0}^{m} a_k \mathbf{v}_k : \sum_{k=0}^{m} a_k = 1, a_k \geq 0 \text{ for } k = 1, \ldots, m \right\}.$$

Each $\mathbf{v}_k$ is a *vertex* of $[\mathbf{v}_0, \ldots, \mathbf{v}_m]$. The affine span of $V - \{\mathbf{v}_k\}$ has dimension $m - 1$, is called the kth *support plane* of the simplex, and is said to be opposite to the vertex $\mathbf{v}_k$, while the simplex spanned by $V - \{\mathbf{v}_k\}$ is the kth *face* of the simplex. The simplex is *regular* if all the side-lengths $\mathbf{v}_i\mathbf{v}_j$, for $0 \leq i \neq j \leq m$, are the same. Obviously, an m-simplex is contained in the affine span of its vertices. It is not difficult to prove that if $[\mathbf{v}_0, \ldots, \mathbf{v}_n]$ is an n-simplex in $\mathbb{E}^n$ and $\mathbf{u}_k$ is a nonzero vector orthogonal to the kth support plane for $k = 0, \ldots, n$, then the vectors $\mathbf{u}_0, \ldots, \mathbf{u}_n$ span $\mathbb{E}^n$.[4]

The most important property of an affine basis for the proof of Menger's characterization is the following elementary result. Let $\{\mathbf{v}_0, \ldots, \mathbf{v}_m\}$ be an affine basis for the m-dimensional affine plane A in $\mathbb{E}^n$. The result says that an element $\mathbf{a}$ of A is uniquely determined by the $m + 1$ distances $\mathbf{av}_0, \ldots, \mathbf{av}_m$. This implies in particular that the $m + 1$ known distances $\mathbf{av}_k$ for $k = 0, \ldots, m$ uniquely determine the $m + 1$ affine coordinates $a_0, \ldots a_m$ of $\mathbf{a}$ with respect to the affine basis $\{\mathbf{v}_0, \ldots, \mathbf{v}_m\}$.

Lemma A. *Let* $\mathbf{a}$ *be a point in the affine span* $A \subset \mathbb{E}^n$ *of the affine basis* $\{\mathbf{v}_0, \ldots, \mathbf{v}_m\}$ *for* A. *Let* $\mathbf{b} \in \mathbb{E}^n$. *If* $\mathbf{av}_k = \mathbf{bv}_k$ *for* $k = 0, \ldots, m$, *then* $\mathbf{a} = \mathbf{b}$.

Proof. Let $\mathbf{a}, \mathbf{b} \in \mathbb{E}^n$ with $\mathbf{a} \in A$ and suppose that $\mathbf{av}_k = \mathbf{bv}_k$ for $k = 0, \ldots, m$. If $\mathbf{a} \neq \mathbf{b}$, then the set $M = \{\mathbf{x} \in \mathbb{E}^n : \mathbf{ax} = \mathbf{bx}\}$ of points of $\mathbb{E}^n$ equidistant from both $\mathbf{a}$ and $\mathbf{b}$ is an $(n - 1)$-dimensional affine plane, the *perpendicular bisector* of $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{E}^n$. For $k = 0, \ldots, m$, since $\mathbf{av}_k = \mathbf{bv}_k$, $\mathbf{v}_k$ is a point of M, and this implies that $A \subset M$. Since $\mathbf{a} \in A$, we have that $\mathbf{a} \in M$, which implies that $\mathbf{ba} = \mathbf{aa} = 0$. Therefore, $\mathbf{a} = \mathbf{b}$. ∎

Another straightforward argument using perpendicular bisectors provides a proof of the next result.

Lemma B. *Let* $\{\mathbf{v}_0, \ldots, \mathbf{v}_m\}$ *be an affine basis for the m-dimensional affine plane* A *in* $\mathbb{E}^n$, *where* $m < n$, *and let* B *be an* $(m + 1)$*-dimensional affine plane containing* A. *Let* $\mathbf{x}$ *be a point in* B *that is not in* A, *and let* $\mathbf{x}^*$ *be the isometric reflection in* B *of* $\mathbf{x}$ *through the affine plane* A. *If* $\mathbf{b}$ *is a point of* B *with* $\mathbf{bv}_k = \mathbf{xv}_k$ *for* $k = 0, \ldots, m$, *then either* $\mathbf{b} = \mathbf{x}$ *or* $\mathbf{b} = \mathbf{x}^*$.

We need to know that when two affinely independent point sets in $\mathbb{E}^n$ are congruent, then there is a global self-isometry of $\mathbb{E}^n$ that carries one onto the other. This says that an m-simplex in $\mathbb{E}^n$ is determined up to global isometry by the lengths of its sides, a basic well-known fact in the subdiscipline of rigidity theory for linkages in computational geometry.

Lemma C. *Let* $V = \{\mathbf{v}_0, \ldots, \mathbf{v}_m\}$ *and* $W = \{\mathbf{w}_0, \ldots, \mathbf{w}_m\}$ *be two affinely independent subsets of* $\mathbb{E}^n$ *with* $\mathbf{v}_j\mathbf{v}_k = \mathbf{w}_j\mathbf{w}_k$ *for* $0 \leq j, k \leq m$. *Then there is an isometry* ρ *of* $\mathbb{E}^n$ *with* $\rho(\mathbf{v}_k) = \mathbf{w}_k$ *for* $k = 0, \ldots, m$.

[4] In fact, the collection $\{\mathbf{u}_0, \ldots, \mathbf{u}_n\}$ is affinely independent and any n of the vectors from the list form a vector space basis for $\mathbb{E}^n$.

The proof of this lemma is an easy induction on m using Lemma B and the observation that any isometry between any two affine planes in $\mathbb{E}^n$ extends to an isometry of $\mathbb{E}^n$. We may describe the isometry between A and B, the respective affine spans of the sets V and W of the lemma, as follows. Each point $\mathbf{a}$ of A has a unique set of affine coordinates $a_0, \dots, a_m$ with respect to the affine basis V. Define the function $\lambda : A \to B$ by $\lambda(\mathbf{a}) = \sum_{k=0}^{m} a_k \mathbf{w}_k$. Since both V and W are affine bases for the respective m-dimensional affine planes A and B, λ is a bijection of A onto B, and Lemma A implies that λ is an isometric mapping.

Let $S^{n-1}(\mathbf{c}, r) = \{\mathbf{x} \in \mathbb{E}^n : \mathbf{xc} = r\}$, the $(n-1)$-sphere in $\mathbb{E}^n$ centered at $\mathbf{c}$ of radius $r > 0$. Obviously, the vertices of a 1-simplex $[\mathbf{v}_0, \mathbf{v}_1]$ lie in a unique 0-sphere in $\mathbb{E}^1$, namely $S^0(\mathbf{c}, r) = \{\mathbf{v}_0, \mathbf{v}_1\}$, the sphere of radius $r = |\mathbf{v}_0 - \mathbf{v}_1|/2$ centered at $\mathbf{c} = (\mathbf{v}_0 + \mathbf{v}_1)/2$ in $\mathbb{E}^1$. This is the basis of an easy inductive argument using elementary Euclidean geometry that proves the next lemma.

Lemma D. *Let* $[\mathbf{v}_0, \dots, \mathbf{v}_n]$ *be an* n*-simplex in* $\mathbb{E}^n$*. There is a unique sphere* $S^{n-1}(\mathbf{c}, r)$ *that passes through the* $n+1$ *affinely independent vertices* $\mathbf{v}_0, \dots, \mathbf{v}_n$ *of* $[\mathbf{v}_0, \dots, \mathbf{v}_n]$.

This sphere is the *circumscribed sphere* of the simplex $[\mathbf{v}_0, \dots, \mathbf{v}_n]$ while the *inscribed sphere* is the $(n-1)$-sphere tangent to each face, i.e., which meets each face of the simplex in a single point.

The final geometric facts that we need concern the *inradius* and *circumradius* of a *regular* n-simplex Σ_ε^n of side-length ε. The in- and circumradii are the respective radii of the inscribed and circumscribed spheres, which have a common center since Σ_ε^n is regular. The calculations of these two radii constitute a nice exercise in the use of affine coordinates.

Exercise. Let $\Sigma_{\sqrt{2}}^n = [\mathbf{e}_0, \dots, \mathbf{e}_n]$ be the regular n-simplex in $\mathbb{E}^{n+1}$ with vertices the standard unit coordinate vectors. Then the center of the in- and circumscribed spheres of $\Sigma_{\sqrt{2}}^n$ is $\mathbf{c}_n = \frac{1}{n+1} \sum_{k=0}^{n} \mathbf{e}_k$. It follows that the inradius δ_n and the circumradius ζ_n are given as

$$\delta_n = |\mathbf{c}_n - \mathbf{c}_{n-1}| = \frac{1}{\sqrt{n(n+1)}},$$

$$\zeta_n = |\mathbf{c}_n - \mathbf{e}_n| = \sqrt{\frac{n}{n+1}},$$

and those of Σ_ε^n are obtained from these formulae by multiplying by the scale factor $\varepsilon/\sqrt{2}$.

ACT 1: A REDUCTION TO EMBEDDING FINITELY MANY POINTS ISOMETRICALLY. We are now in a position to state and easily prove the first result of Menger that reduces the question of whether a given metric space embeds isometrically in $\mathbb{E}^n$ to the question of whether each of its finite subsets with at most $n+3$ elements so embeds. Rather than explicitly naming the metric of a metric space, we shall use the convention that the juxtaposition of two points indicates the distance between the points. So, if x and y are points of the metric space X, the expression xy means the distance in X between the two points. This of course is the notation we use in the Overture for the distances between points of $\mathbb{E}^n$. When Y is a subset of X, we always will view Y as a metric subspace, its metric inherited from that of X, and when the metric spaces X and Y are isometric, we will write $X \cong Y$.

Theorem 1. *Let X be a metric space. Then X embeds isometrically in the Euclidean space $\mathbb{E}^n$ if and only if each of its finite subsets with at most $n+3$ elements embeds isometrically in $\mathbb{E}^n$.*

Proof. Suppose each metric subspace of X with at most $n+3$ elements embeds isometrically in $\mathbb{E}^n$. Among all isometric embeddings of subsets of X with at most $n+1$ elements onto affinely independent subsets of $\mathbb{E}^n$, let $Y = \{y_0, \ldots, y_m\} \cong \{\mathbf{y}_0, \ldots, \mathbf{y}_m\} \subset \mathbb{E}^n$, where $m \leq n$ be one with a maximum number of elements. Here of course our notation suggests that $y_k \mapsto \mathbf{y}_k$ for $k = 0, \ldots, m$ under the isometric embedding so that $y_j y_k = \mathbf{y}_j \mathbf{y}_k$ for $0 \leq j, k \leq m$. Let A be the affine span of $\{\mathbf{y}_0, \ldots, \mathbf{y}_m\}$, an m-dimensional affine plane in $\mathbb{E}^n$. We shall show that X embeds isometrically into A.

Define $\rho : X \to A \subset \mathbb{E}^n$ as follows. First, $\rho(y_k) = \mathbf{y}_k$ for $k = 0, \ldots, m$. Now let $x \in X - Y$ and note that the set $Y_x = Y \cup \{x\}$ has $m+2 < n+3$ elements. By hypothesis, Y_x embeds isometrically into $\mathbb{E}^n$, and by Lemma C we may choose an isometric embedding $\rho_x : Y_x \to \mathbb{E}^n$ such that $\rho_x(y_k) = \mathbf{y}_k$ for $k = 0, \ldots, m$. By our choice of the integer m, since Y_x has $m+2$ elements, its image cannot be affinely independent, and this implies that $\rho_x(x)$ is in the affine span of $\{\mathbf{y}_0, \ldots, \mathbf{y}_m\}$, i.e., $\rho_x(x) \in A$. Define $\rho(x) = \rho_x(x)$. Notice that Lemma A implies that $\rho(x)$ is uniquely determined by this prescription, i.e., $\rho(x)$ is independent of the particular initial isometric embedding of Y_x into $\mathbb{E}^n$ and independent of which isometry of $\mathbb{E}^n$ is used, à la Lemma C, to move this initial image of Y_x into position so that $\rho_x(y_k) = \mathbf{y}_k$ for $k = 0, \ldots, m$.

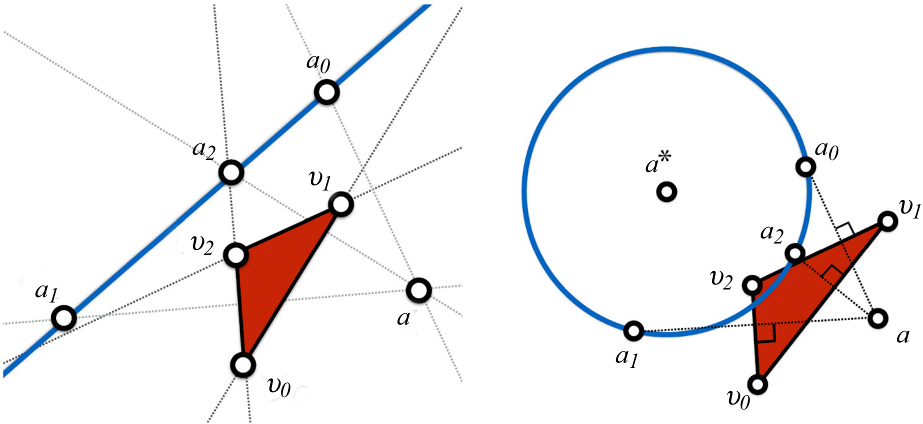
Having defined the function $\rho : X \to A$, we now verify that it is an isometry. We need only verify that the distances xy and $\rho(x)\rho(y)$ agree whenever $x \neq y$ are elements of $X - Y$. Assume then that $x \neq y \in X - Y$. Then the set $Y_{x,y} = Y \cup \{x, y\}$ has $m+3 \leq n+3$ elements and, by hypothesis, there exists an isometric embedding $\lambda : Y_{x,y} \to \mathbb{E}^n$. Let μ be a global isometry of $\mathbb{E}^n$ promised by Lemma C with $\mu(\lambda(y_k)) = \mathbf{y}_k$, for $k = 0, \ldots, m$. As before, by our choice of m, the image of $Y_{x,y}$ under $\mu \circ \lambda$ lies in the affine span A of $\{\mathbf{y}_0, \ldots, \mathbf{y}_m\}$. Two applications of Lemma A imply that the restriction of $\mu \circ \lambda$ to Y_x is equal to the isometric embedding ρ_x and its restriction to Y_y is equal to ρ_y. It follows that $\mathbf{x} \equiv \mu \circ \lambda(x) = \rho_x(x) = \rho(x)$ and, similarly, $\mathbf{y} \equiv \mu \circ \lambda(y) = \rho(y)$. Since $\mu \circ \lambda$ is an isometry, we then have $xy = \mathbf{xy} = \rho(x)\rho(y)$, implying that ρ is an isometric mapping and hence an isometric embedding. ∎

Remark. Note that only the isometric embeddability of finite subsets of X *with at most $n+2$ elements* is used in defining the well-defined mapping ρ of X into $A \subset \mathbb{E}^n$. The only use of the isometric embeddability of $n+3$ points of X appears in the last paragraph of the proof, in proving that the mapping ρ is an isometry. In Act 2, we will use the mapping ρ and the isometries ρ_x constructed in the proof of the theorem in the setting in which every subset of X with $n+2$ elements isometrically embeds in $\mathbb{E}^n$, but X itself fails to embed so.

ACT 2: CHARACTERIZING CERTAIN FINITE SETS THAT FAIL TO EMBED ISOMETRICALLY. In some respects, the results of this section offer the most surprising aspects of Menger's characterization. We begin by asking whether we may improve upon Theorem 1 by reducing the number $n+3$. Specializing to the case $n = 1$ offers a simple but somewhat surprising glimpse into the general results of this section. A metric space that isometrically embeds in the Euclidean line $\mathbb{E}^1$ is said to be *linear*. In this case, the question of the linearity of three points of X is just the question of whether the triangle inequality among an appropriate ordering of the three points is in fact an equality. Indeed, for three pairwise distinct points x, y, and z of the metric space X, we say that y is *between* x and z provided $xz = xy + yz$, which we denote by xyz. Obviously, the triple of points $\{x, y, z\}$ is linear if and only if xyz, or one of its permutations, holds. Theorem 1 says that if every three- and four-point subset of X is linear, then X is linear. Can we do better and say that X is linear whenever each triple of its points is linear? The answer is "no" with a family of similar counterexamples provided by the four-point metric spaces whose elements are the vertices of Euclidean rectangles, with distances the lengths of the shortest edge paths in the rectangles connecting any two. Now the really interesting fact is that *these counterexamples are, up to isometry, the only counterexamples!* What we mean is this. Let X be a metric space in which every triple of its points is linear. Then either X is linear or X consists of precisely four distinct points w, x, y, and z, and wxy, xyz, yzw, zwx, $wx = yz$, and $xy = zw$. In particular, if X has more than four points, then X is linear if and only if any three of its points is linear, improving upon Theorem 1. All of these statements about the $n = 1$ case can be proved by chasing around betweeness relations among four points of X. The aim of this section is to prove that this special case of $n = 1$ is representative of the general case.

First, we address the general question of whether there are any metric spaces with $n + 3$ elements that fail to embed in $\mathbb{E}^n$ isometrically, even though every one of its subsets with at most $n + 2$ elements does so embed. This is answered by the next result, rather surprising to us (especially the uniqueness assertion), which is used later to describe all such examples. To set up the result, let $V = \{\mathbf{v}_0, \ldots, \mathbf{v}_n\} \subset \mathbb{E}^n$ be affinely independent, and therefore an affine basis for $\mathbb{E}^n$, and let $\mathbf{a}$ be any element of $\mathbb{E}^n$. For each $j = 0, \ldots, n$, let A_j be the jth support plane of the n-simplex $[\mathbf{v}_0, \ldots, \mathbf{v}_n]$ opposite $\mathbf{v}_j$ that is affinely spanned by $V_j = V - \{\mathbf{v}_j\}$. Let $\mathbf{a}_j$ be the isometric reflection in $\mathbb{E}^n$ of the point $\mathbf{a}$ through the $(n - 1)$-dimensional support plane A_j. We say that $\mathbf{a}$ is a *polar point* for $[\mathbf{v}_0, \ldots, \mathbf{v}_n]$ if the set $\{\mathbf{a}_0, \ldots, \mathbf{a}_n\}$ is a set of $n + 1$ affinely independent points, which therefore are the vertices of an n-simplex. In this case, Lemma D of the Overture guarantees the existence of the unique circumscribing sphere of the n-simplex $[\mathbf{a}_0, \ldots, \mathbf{a}_n]$, say with center $\mathbf{a}^*$ and radius $r = r(\mathbf{a}, \mathbf{v}_0, \ldots, \mathbf{v}_n)$. The point $\mathbf{a}^*$ is said to be *antipodal* to $\mathbf{a}$, the radius r is the *polar diameter* of the set $\{\mathbf{a}, \mathbf{a}^*\}$, and the ordered pair $(\mathbf{a}, \mathbf{a}^*)$ is said to be *antipodal* with respect to the n-simplex $[\mathbf{v}_0, \ldots, \mathbf{v}_n]$; see Figure 1. It very well may be that $\mathbf{a}$ is *self-antipodal*, meaning that $\mathbf{a} = \mathbf{a}^*$; in fact, $\mathbf{a} = \mathbf{a}^*$ if and only if $\mathbf{a}$ is equidistant from all the support planes $A_0, \ldots, A_n$ of $[\mathbf{v}_0, \ldots, \mathbf{v}_n]$. This includes the incenter of the simplex $[\mathbf{v}_0, \ldots, \mathbf{v}_n]$, the center of its inscribed sphere, as well as, for each $k = 0, \ldots, n$, a point in the intersection of the half-spaces $H_0 \cap \cdots \cap H_{k-1} \cap \widetilde{H}_k \cap H_{k+1} \cap \cdots \cap H_n$, where H_j is the half-space bordered by A_j that contains $\mathbf{v}_j$ and $\widetilde{H}_j$ is the half-space bordered by A_j opposite $\mathbf{v}_j$. Note that if $[\mathbf{v}_0, \ldots, \mathbf{v}_n]$ is a regular simplex, the incenter of $[\mathbf{v}_0, \ldots, \mathbf{v}_n]$ is the only one of these self-antipodal points that is equidistant from the vertices of $[\mathbf{v}_0, \ldots, \mathbf{v}_n]$. We will use this fact in the proof of Theorem 3. A metric space X is an *extension* of the metric space Y if Y embeds isometrically in X.

Theorem 2. *Let $V = \{\mathbf{v}_0, \ldots, \mathbf{v}_n\}$ be an affine basis for $\mathbb{E}^n$ and let $\mathbf{a}$ be any element of $\mathbb{E}^n$ other than one of $\mathbf{v}_0, \ldots, \mathbf{v}_n$. Then there is an extension of $V \cup \{\mathbf{a}\}$ to a metric space X with $n + 3$ elements that fails to embed in $\mathbb{E}^n$ isometrically while every one of its subsets with at most $n + 2$ elements does so embed if and only if $\mathbf{a}$ is a polar point for $[\mathbf{v}_0, \ldots, \mathbf{v}_n]$ that does not lie in any of the $n + 1$ support planes A_j, for $j = 0, \ldots, n$; moreover, X is unique up to isometry.*



(a) **a** is not a polar point for T even though it does not lie on any support line of T.

(b) **a** is a polar point for T with $\mathbf{a}^*$ antipodal to **a**, and $\mathbf{a}^*$ is polar for T with $\mathbf{a}^{**} = \mathbf{a}$.

Figure 1. Polar points for a given simplex $T = [\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2]$.

Proof. We make the following preliminary claim. If **a** is a polar point for $[\mathbf{v}_0, \ldots, \mathbf{v}_n]$ with antipodal pair $(\mathbf{a}, \mathbf{a}^*)$, then **a** lies on the support plane A_j, for some $j = 0, \ldots, n$, if and only if $\mathbf{a}^* = \mathbf{v}_j$. Indeed, observe that **a** lies on A_j if and only if $\mathbf{a}_j = \mathbf{a}$. This implies, since the set $\{\mathbf{a}_0, \ldots, \mathbf{a}_n\}$ has exactly $n + 1$ points, that **a** lies on at most one support plane. Assume then that **a** lies on A_j for some $j = 0, \ldots, n$ so that $\mathbf{a}_j = \mathbf{a}$. Then for any index $k \neq j$, $\mathbf{a}_k \neq \mathbf{a}$, and this implies that the support plane A_k is the perpendicular bisector of the two distinct points **a** and $\mathbf{a}_k$. Since $\mathbf{aa}^* = \mathbf{a}_j\mathbf{a}^* = r = \mathbf{a}_k\mathbf{a}^*$, we have $\mathbf{a}^* \in A_k$. Hence,

$$\mathbf{a}^* \in \bigcap\{A_k : k = 0, \ldots, j-1, j+1, \ldots, n\} = \{\mathbf{v}_j\}.$$

For the reverse direction, assume that $\mathbf{a}^* = \mathbf{v}_j$ for some $j = 0, \ldots, n$ and that $\mathbf{a} \notin A_j$. Then $\mathbf{a}^* \in A_k$ for any index $k \neq j$, and this implies that $\mathbf{a}^*\mathbf{a} = \mathbf{a}^*\mathbf{a}_k = r = \mathbf{a}^*\mathbf{a}_j$, Since $\mathbf{a} \notin A_j$, the support plane A_j is the perpendicular bisector of the distinct points **a** and $\mathbf{a}_j$, and since $\mathbf{a}^*\mathbf{a} = r = \mathbf{a}^*\mathbf{a}_j$, we conclude that $\mathbf{a}^* \in A_j$. But then $\mathbf{a}^* \in \cap_{k=0}^{n} A_k = \emptyset$, a contradiction. Therefore, $\mathbf{a} \in A_j$. This proves the preliminary claim.

($\Leftarrow$) Assume that **a** is a polar point for $[\mathbf{v}_0, \ldots, \mathbf{v}_n]$ that does not lie in any of the $n + 1$ support planes A_j, for $j = 0, \ldots, n$. As a point set, let X be any set with precisely $n + 3$ elements and name the distinct elements as $y_0, \ldots, y_n, a, a^*$. Let $Y = \{y_0, \ldots, y_n\}$. Define the function $\rho : X \to \mathbb{E}^n$ by $\rho(y_k) = \mathbf{v}_k$, $\rho(a) = \mathbf{a}$, and $\rho(a^*) = \mathbf{a}^*$. Since **a** does not lie on any support plane A_j, neither **a** nor $\mathbf{a}^*$ is one of the vertices of V. It follows that ρ is bijective between $Y \cup \{a\}$ and $V \cup \{\mathbf{a}\}$ and between $Y \cup \{a^*\}$ and $V \cup \{\mathbf{a}^*\}$. Define distances among the points of X by $xy = \rho(x)\rho(y)$, the Euclidean distance between $\rho(x)$ and $\rho(y)$, provided $\{x, y\} \neq \{a, a^*\}$, and $aa^* = r = r(\mathbf{a}, \mathbf{v}_0, \ldots, \mathbf{v}_n)$. We will show that (i) every subset of X with $n + 2$ elements embeds in $\mathbb{E}^n$, preserving the distances defined among the points of X, and (ii) X does not embed isometrically in $\mathbb{E}^n$. Note that item (i) implies that the distances assigned on X define a metric.

For (i), first note that the restrictions of ρ to both the sets $Y \cup \{a\}$ and to $Y \cup \{a^*\}$ are embeddings into $\mathbb{E}^n$ that preserve the distances defined among the points of X. Consider then subsets with $n + 2$ points that contain both a and a^*. For any $j = 0, \ldots, n$, the restriction of the distance function to the set $X_j = X - \{y_j\}$ is just the pullback to X_j of the Euclidean metric on $V_j \cup \{\mathbf{a}^*, \mathbf{a}_j\}$. To be more specific, let f be the mapping on X_j defined by $f(x) = \rho(x)$ for $x \in Y_j \cup \{a^*\}$ and $f(a) = \mathbf{a}_j$. An application of

Lemma B and the definition of the distances defined among the points of X imply that $xy = f(x)f(y)$ for all $x, y \in X_j$.

We now verify (ii). Suppose that g is an isometric embedding of X into $\mathbb{E}^n$. By composing with an isometry of $\mathbb{E}^n$ if necessary, Lemma C implies that we may assume that $g(y_k) = \mathbf{v}_k$ for $k = 0, \ldots, n$. Two applications of Lemma A imply that $g(a) = \mathbf{a}$ and $g(a^*) = \mathbf{a}^*$. This implies, for every $k = 0, \ldots, n$, that $\mathbf{a}_k\mathbf{a}^* = r = aa^* = g(a)g(a^*) = \mathbf{a}\mathbf{a}^*$ so that $\mathbf{a}^* \in A_k$ for each $k = 0, \ldots, n$ since A_k is the perpendicular bisector of the distinct points $\mathbf{a}_k$ and $\mathbf{a}$. But this is impossible since $\cap_{k=0}^n A_k = \emptyset$.

As $Y \cup \{a\}$ is isometric with $V \cup \{\mathbf{a}\}$, we have shown that X is an extension of $V \cup \{\mathbf{a}\}$ with the desired properties.

($\Rightarrow$) Let X be a metric extension of $V \cup \{\mathbf{a}\}$ with $n + 3$ elements that fails to embed in $\mathbb{E}^n$ isometrically while every one of its subsets with at most $n + 2$ elements does so embed. As above, let $Y = \{y_0, \ldots, y_n\}$, $X = Y \cup \{a, b\}$, and $\rho : Y \cup \{a\} \to V \cup \{\mathbf{a}\}$ be an isometric bijection with $\rho(Y) = V$. By Lemmas A and C and the fact that $Y \cup \{b\}$ admits an isometric embedding in $\mathbb{E}^n$, there is a unique point $\mathbf{b} \in \mathbb{E}^n$ such that the extension of ρ to X via $\rho(b) = \mathbf{b}$ is an isometry on $Y \cup \{b\}$. We claim that $\mathbf{a}$ is a polar point for $[\mathbf{v}_0, \ldots, \mathbf{v}_n]$ and that $\mathbf{b} = \mathbf{a}^*$ is antipodal to $\mathbf{a}$. By hypothesis, ρ is not an isometry, and this implies that $ab \neq \mathbf{ab}$. Lemmas A, B, and C used as in the preceding paragraphs imply that, for each $j = 0, \ldots, n$, the map $g_j : Y_j \cup \{a, b\} \to V_j \cup \{\mathbf{a}_j, \mathbf{b}\}$ defined by $g_j = \rho$ on $Y_j \cup \{b\}$ while $g_j(a) = \mathbf{a}_j$ is an isometry. It follows that $\mathbf{a}_j\mathbf{b} = ab \equiv r$ for $j = 0, \ldots, n$ so that the points $\mathbf{a}_0, \ldots, \mathbf{a}_n$ all lie on the sphere centered at $\mathbf{b}$ of radius r. If the set $\{\mathbf{a}_0, \ldots, \mathbf{a}_n\}$ is affinely independent, then the claim that $\mathbf{a}$ is polar and $\mathbf{a}^* = \mathbf{b}$ is confirmed. Assume then that $\{\mathbf{a}_0, \ldots, \mathbf{a}_n\}$ fails to be affinely independent, and let H be the affine span of $\{\mathbf{a}_0, \ldots, \mathbf{a}_n\}$, an affine plane of dimension at most $n - 1$. Note that the vector $\mathbf{u}_j = \mathbf{a}_j - \mathbf{a}$ is a normal vector to the support plane A_j of the n-simplex $[\mathbf{v}_0, \ldots, \mathbf{v}_n]$, and therefore the vectors $\mathbf{u}_0, \ldots, \mathbf{u}_n$ span $\mathbb{E}^n$. Now if $\mathbf{a}$ were an element of H, then $\{\mathbf{u}_0, \ldots, \mathbf{u}_n\}$ would be contained in the linear subspace $H - \mathbf{a}$ of dimension less than n, contradicting the fact that the vectors $\mathbf{u}_0, \ldots, \mathbf{u}_n$ span $\mathbb{E}^n$. We conclude that $\mathbf{a} \notin H$, and hence, since the points $\mathbf{a}_0, \ldots, \mathbf{a}_n$ all lie on a sphere in H and $\mathbf{a} \notin H$, there is a sphere $S^{n-1}(\mathbf{c}, R)$ in $\mathbb{E}^n$ that contains the point $\mathbf{a}$ as well as the points $\mathbf{a}_0, \ldots, \mathbf{a}_n$. It now follows that, since each support plane A_j is the perpendicular bisector of the pair $\mathbf{a}$ and $\mathbf{a}_j$, the center $\mathbf{c}$ of this sphere is an element of each A_j, but this is impossible as $\cap_{k=0}^n A_k = \emptyset$. We conclude then that $\{\mathbf{a}_0, \ldots, \mathbf{a}_n\}$ is affinely independent, $\mathbf{a}$ is a polar point for $[\mathbf{v}_0, \ldots, \mathbf{v}_n]$, and $\mathbf{b} = \mathbf{a}^*$ is antipodal to $\mathbf{a}$. Now we may observe that $\mathbf{a}$ is not in any support plane A_j, for otherwise the first paragraph of this proof would imply that $\mathbf{b} = \mathbf{a}^* = \mathbf{v}_j$, contradicting that ρ is bijective between $Y \cup \{b\}$ and $V \cup \{\mathbf{b}\}$.

The uniqueness claim follows easily since the proof thus far implies that any metric space extending $V \cup \{\mathbf{a}\}$ as required must have its distances determined by the pair $(\mathbf{a}, \mathbf{a}^*)$ as described in the preceding paragraph. ■

Remark. It is a consequence of the proof of the ($\Rightarrow$) case of Theorem 2 that when $\mathbf{a}$ is a polar point for $[\mathbf{v}_0, \ldots, \mathbf{v}_n]$ that does not lie on any of the $n + 1$ support planes A_j, for $j = 0, \ldots, n$, then $\mathbf{a}^*$ is also a polar point for $[\mathbf{v}_0, \ldots, \mathbf{v}_n]$ that lies on no support plane and $\mathbf{a}^{**} = \mathbf{a}$. In this case, we say that $\mathbf{a}$ and $\mathbf{a}^*$ form a *polar pair* for, and are *antipodal* with respect to, the simplex $[\mathbf{v}_0, \ldots, \mathbf{v}_n]$.

We are now in a position to improve upon Theorem 1.

Theorem 3. *Let X be a metric space for which every subset with at most $n + 2$ elements isometrically embeds in $\mathbb{E}^n$. Then either X isometrically embeds in $\mathbb{E}^n$, or*

X has precisely $n+3$ elements and X contains a subset with $n+1$ elements that embeds isometrically onto an affinely independent subset of $\mathbb{E}^n$.

Proof. We assume that X fails to embed isometrically in $\mathbb{E}^n$ even though all of its subsets with at most $n+2$ elements do so embed. It follows that X has at least $n+3$ elements. Exactly as in the first two paragraphs of the proof of Theorem 1, let $Y = \{y_0, \ldots, y_m\} \cong \{\mathbf{y}_0, \ldots, \mathbf{y}_m\} = V \subset \mathbb{E}^n$, where $m \leq n$ be a subset of X with a maximum number of elements that embeds isometrically onto an affinely independent subset of $\mathbb{E}^n$, let A be the affine span of V, and define the function $\rho : X \to A \subset \mathbb{E}^n$ so that $\rho(y_k) = \mathbf{y}_k$ for $k = 0, \ldots, m$ and $\rho(x) = \rho_x(x)$ for $x \in X - Y$. Though ρ fails to be an isometry, for each $x \in X - Y$, the restriction of ρ to $Y_x = Y \cup \{x\}$, viz., $\rho_x : Y_x \to A$, is an isometry.

We now prove that $m = n$. If not, then $m < n$ and A is an m-dimensional affine plane in $\mathbb{E}^n$ and, as such, is isometric to the m-dimensional Euclidean space $\mathbb{E}^m$. Let Z be any subset of X with precisely $m+3$ elements and note that $m+3 \leq n+2$. By hypothesis, Z embeds isometrically in $\mathbb{E}^n$, and by the definition of m, any such embedding is contained in an affine plane of dimension at most m. It follows that Z embeds in A isometrically and an application of Theorem 1 implies that X embeds in A, and therefore in $\mathbb{E}^n$ isometrically, a contradiction.

Since $m = n$, the set V is an affine basis for $A = \mathbb{E}^n$, and our claim is that there are exactly two elements in $X - Y$. The last paragraph of the proof of Theorem 1 implies that if every subset of X that has $n+3$ elements and contains Y embeds isometrically in $\mathbb{E}^n$, then X itself so embeds. It follows that there are two points a and b of $X - Y$ for which $Y \cup \{a, b\}$ fails to embed isometrically in $\mathbb{E}^n$. Our claim is that $X = Y \cup \{a, b\}$. If not, let c be a point of X not in $Y \cup \{a, b\}$. There are two cases to consider.

Case 1: The sets $Y \cup \{a, c\}$ and $Y \cup \{b, c\}$ do not embed isometrically in $\mathbb{E}^n$. The next two paragraphs will completely determine the metric of the set $Y \cup \{a, b, c\}$ under the hypothesis that the sets $Y \cup \{a, c\}$ and $Y \cup \{b, c\}$ do not embed isometrically in $\mathbb{E}^n$. We then will show that the structure so obtained violates the Pythagorean theorem.

Since $Y \cup \{a, b\}$ is an extension of $V \cup \{\rho(a)\}$ that fails to embed in $\mathbb{E}^n$ isometrically, the proof of Theorem 2 implies that the pair $\{\rho(a), \rho(b)\}$ forms a polar pair for the n-simplex $[\mathbf{y}_0, \ldots, \mathbf{y}_m]$. But this applies to the sets $Y \cup \{a, c\}$ and $Y \cup \{b, c\}$ as well so that $\{\rho(a), \rho(c)\}$ and $\{\rho(b), \rho(c)\}$ also form polar pairs for the same simplex. It follows immediately that $\rho(a) = \rho(b) = \rho(c) \equiv \mathbf{x} \in \mathbb{E}^n$, and therefore $\mathbf{x} = \mathbf{x}^*$.[5] It now follows that a, b, and c satisfy $ab = bc = ca = r$, where r is the polar diameter of the polar pair $\{\mathbf{x}, \mathbf{x}^*\}$, and, for $k = 0, \ldots, n$, $ay_k = by_k = cy_k$.

For any $j = 0, \ldots, n$, let $Y_j = Y - \{y_j\}$, $V_j = \rho(Y_j) = V - \{\mathbf{y}_j\}$, and A_j be the jth support plane for the n-simplex $[\mathbf{y}_0, \ldots, \mathbf{y}_n]$. For $i \neq j \in \{0, \ldots, n\}$, let $Y_{i,j} = Y - \{y_i, y_j\}$, and $A_{i,j} = A_i \cap A_j$, the affine span of $V_{i,j} = V - \{\mathbf{y}_i, \mathbf{y}_j\}$. Since $Y_{i,j} \cup \{a, b, c\}$ has $n+2$ elements, there is an isometric embedding $\tau_{i,j}$ of $Y_{i,j} \cup \{a, b, c\}$ into $\mathbb{E}^n$, and by Lemma C, we may assume that $\tau_{i,j}(y_k) = \mathbf{y}_k$ for $k \neq i, j$. Set $\tau_{i,j}(a) = \mathbf{a}$, $\tau_{i,j}(b) = \mathbf{b}$, and $\tau_{i,j}(c) = \mathbf{c}$ and note that none of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is in the affine plane $A_{i,j}$, for otherwise Lemma A would imply the equality of these three points. It follows that the $(n-2)$-dimensional affine plane $A_{i,j}$ and the 2-dimensional affine plane $B_{i,j}$, the affine span of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, meet at a single point $\mathcal{O}$. Now $A_{i,j}$ may be described as $\{\mathbf{z} \in \mathbb{E}^n : \mathbf{az} = \mathbf{bz} = \mathbf{cz}\}$, and this implies, since $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are the vertices of an equilateral triangle, that the $(n-2)$-dimensional affine plane $A_{i,j}$ and the 2-dimensional affine plane $B_{i,j}$ are orthogonal with the single point of intersection $\mathcal{O}$, which is the

[5] We will show in the next paragraph that the self-antipodal point $\mathbf{x}$ must be the incenter of the n-simplex $[\mathbf{y}_0, \ldots, \mathbf{y}_n]$.

incenter of the 2-simplex $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$. This says that $\mathbb{E}^n$ decomposes as an orthogonal direct sum $\mathbb{E}^n = A_{i,j} \perp B_{i,j}$, which implies that V_i along with at least one of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is affinely independent. In turn, this implies that, if the set $Y_i \cup \{a, b, c\}$ were to embed isometrically into $\mathbb{E}^n$, then using Lemma C, $\tau_{i,j}$ could be extended to an isometric embedding of $Y_i \cup \{a, b, c\}$ by $\tau_{i,j}(y_j) = \mathbf{y}_j$. The fact that $ay_j = by_j = cy_j$ then would imply that $\mathbf{y}_j \in A_{i,j}$. This would contradict the fact that the collection V_i, being an affinely independent set of n elements, spans an $(n-1)$-dimensional affine plane and cannot be contained in the $(n-2)$-dimensional affine plane $A_{i,j}$. Thus, $Y_i \cup \{a, b, c\}$, and similarly $Y_j \cup \{a, b, c\}$, cannot embed isometrically in $\mathbb{E}^n$, and therefore both are metric extensions of $V_{i,j} \cup \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ that fail to embed isometrically in $\mathbb{E}^n$. The uniqueness of Theorem 2 now applies to ensure that for any index k not equal to i nor j, $y_i y_k = y_j y_k$, and also $ay_i = ay_j$, $by_i = by_j$, and $cy_i = cy_j$. Because i and j are arbitrary indices, we conclude that all the distances among the $n+1$ points of Y are equal to a common positive value ε, so that the set V is the vertex set of a regular n-simplex Σ^n_ε in $\mathbb{E}^n$, and that all the distances xy for $x \in \{a, b, c\}$ and $y \in Y$ have a common value ζ. Since the incenter of a regular simplex is the only self-antipodal point that is equidistant from the vertices of $[\mathbf{y}_0, \dots, \mathbf{y}_n]$, we conclude that $\mathbf{x}$ is the incenter of the regular simplex Σ^n_ε and hence ζ is the circumradius of Σ^n_ε while the polar diameter $r = ab = bc = ca$ is 2δ, where δ is the inradius of Σ^n_ε.

The preceding paragraphs have uncovered the metric structure of $Y \cup \{a, b, c\}$. Y is isometric to the vertex set of a regular n-simplex Σ^n_ε of side-length ε, $\{a, b, c\}$ is isometric to the vertices of a regular 2-simplex (equilateral triangle) $\Sigma^2_{2\delta}$ of side-length 2δ, where δ is the inradius of Σ^n_ε, and the distance from any point of Y to any of a, b, or c is the distance ζ from the incenter of Σ^n_ε to any of its vertices, the circumradius of Σ^n_ε. Armed with these facts, we are ready to derive a contradiction using the embedding $\tau_{i,j}$ of the immediately preceding paragraph. Specifically, set $i = 0$ and $j = 1$. The set $Y_{0,1} \cup \{a, b, c\} = \{y_2, \dots, y_n, a, b, c\}$ embeds isometrically in $\mathbb{E}^n$ via $\tau_{0,1}$ with image $\{\mathbf{y}_2, \dots, \mathbf{y}_n, \mathbf{a}, \mathbf{b}, \mathbf{c}\}$. Recall that the affine spans $A_{0,1}$ of $\{\mathbf{y}_2, \dots, \mathbf{y}_n\}$ and $B_{0,1}$ of $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ are orthogonal with a single point of intersection $\mathcal{O}$, the incenter of the 2-simplex $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$. From the exercise of the Overture, the circumradius of $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ is $s = (2\delta/\sqrt{2})\zeta_2 = \varepsilon\delta_n\zeta_2$ since $\delta = (\varepsilon/\sqrt{2})\delta_n$. Since the distance from $\mathbf{a}$ to any point of $\{\mathbf{y}_2, \dots, \mathbf{y}_n\}$ is constant, the points $\{\mathbf{y}_2, \dots, \mathbf{y}_n\}$ lie on a sphere in $A_{0,1}$ centered at $\mathcal{O}$. But since the points $\{\mathbf{y}_2, \dots, \mathbf{y}_n\}$ are the vertices of a regular $(n-2)$-simplex of side-length ε, this sphere must be the circumscribing sphere of radius $R = (\varepsilon/\sqrt{2})\zeta_{n-2}$. Consider now the triangle with vertices $\mathbf{a}$, $\mathbf{y}_n$, and $\mathcal{O}$: It has a right angle at $\mathcal{O}$ and side-lengths $\mathcal{O}\mathbf{a} = s$, $\mathcal{O}\mathbf{y}_n = R$, and $\mathbf{a}\mathbf{y}_n = \zeta = (\varepsilon/\sqrt{2})\zeta_n$. An application of the Pythagorean theorem gives $s^2 + R^2 = \zeta^2$, or $2\delta_n^2\zeta_2^2 + \zeta_{n-2}^2 = \zeta_n^2$. The exercise of the Overture then implies that

$$\frac{4}{3}\frac{1}{n(n+1)} + \frac{n-2}{n-1} = \frac{n}{n+1}.$$

Straightforward algebra applies to show that this latter equation has a unique solution, $n = -2$, and this contradicts that the dimension $n \geq 1$ of Euclidean space $\mathbb{E}^n$ satisfies the equation.

Case 2: The set $Y \cup \{b, c\}$ *does embed isometrically in* $\mathbb{E}^n$. In this case, the restriction of ρ to $Y \cup \{b, c\}$ is an isometric embedding. Let $\mathbf{a} = \rho(a)$, $\mathbf{b} = \rho(b)$, and $\mathbf{c} = \rho(c)$. If $Y \cup \{a, c\}$ were to fail to embed isometrically, then the uniqueness of Theorem 2 would imply that $\mathbf{y}_k\mathbf{b} = \mathbf{y}_k\mathbf{c}$ for $k = 0, \dots, n$, which would imply by Lemma A that $\rho(b) = \mathbf{b} = \mathbf{c} = \rho(c)$, contradicting that ρ isometrically embeds $\{b, c\}$. Thus,

ρ isometrically embeds both sets $Y \cup \{a, c\}$ and $Y \cup \{b, c\}$. We will show that this fact—that both these sets with $n+3$ elements do embed isometrically—implies that $Y \cup \{a, b, c\}$ necessarily embeds in $\mathbb{E}^n$ isometrically, a contradiction that finishes the proof.

For the moment, we make two assumptions that are addressed in the next paragraph: (i) the $(n+1)$-element set $\{\mathbf{y}_1 \ldots, \mathbf{y}_n, \mathbf{c}\}$ is affinely independent, and (ii) the n-element set $\{\mathbf{y}_3, \ldots, \mathbf{y}_n, \mathbf{b}, \mathbf{c}\}$ is affinely independent and, hence, spans an $(n-1)$-dimensional affine plane A. By hypothesis, the $(n+1)$-element set $S = \{y_3, \ldots, y_n, a, b, c\}$ embeds in $\mathbb{E}^n$ isometrically, and by Lemma C we may assume the isometric embedding $\mu : S \to \mathbb{E}^n$ satisfies $\mu = \rho$ on $S - \{a\}$ with $\mu(a) = \mathbf{a}'$. Let $\mathbf{a}''$ be the isometric reflection of $\mathbf{a}'$ through A. By hypothesis and Lemma C, the three $(n+2)$-element sets $S_k = \{y_k, y_3, \ldots, y_n, a, b, c\}$, $k = 0, 1, 2$, embed isometrically onto the respective sets $\{\mathbf{y}_k, \mathbf{y}_3, \ldots, \mathbf{y}_n, \alpha_k, \mathbf{b}, \mathbf{c}\}$, $k = 0, 1, 2$, for some $\alpha_k \in \mathbb{E}^n$. By Lemma B, since $\{\mathbf{y}_3, \ldots, \mathbf{y}_n, \mathbf{b}, \mathbf{c}\}$ is affinely independent, each α_k is either $\mathbf{a}'$ or $\mathbf{a}''$. It follows that for at least two values of $k = 0, 1, 2$, the α_k's agree, i.e., without loss of generality, we may assume that $\alpha_1 = \alpha_2 = \mathbf{a}'$, and this implies that $\{y_1, \ldots, y_n, a, b, c\}$ embeds isometrically onto $\{\mathbf{y}_1, \ldots, \mathbf{y}_n, \mathbf{a}', \mathbf{b}, \mathbf{c}\}$. By Lemma A, since ρ restricts to $Y \cup \{a, c\}$ as an isometric embedding and $\{\mathbf{y}_1 \ldots, \mathbf{y}_n, \mathbf{c}\}$ is affinely independent, the distances of a to the points of $\{y_1, \ldots, y_n, c\}$ uniquely determine $\rho(a) = \mathbf{a}$. But we have seen that $\{y_1, \ldots, y_n, a, b, c\}$ is isometric with $\{\mathbf{y}_1, \ldots, \mathbf{y}_n, \mathbf{a}', \mathbf{b}, \mathbf{c}\}$, so the distances of a to the points of $\{y_1, \ldots, y_n, c\}$ uniquely determine the point $\mathbf{a}'$. Thus, $\mathbf{a} = \mathbf{a}'$, and then ρ isometrically embeds $Y \cup \{a, b, c\}$ since then $\mathbf{ab} = \mathbf{a}'\mathbf{b} = ab$. This of course contradicts that $Y \cup \{a, b\}$ fails to embed in $\mathbb{E}^n$ isometrically.

We now show that the two assumptions of the preceding paragraph always hold for an appropriate ordering of the vertices of the n-simplex $\sigma = [\mathbf{y}_0, \ldots, \mathbf{y}_n]$. Indeed, since $\cap_{k=0}^n A_k = \emptyset$, there is at least one support plane of σ that does not contain the point $\mathbf{c}$, and without loss of generality we may assume that $\mathbf{c} \notin A_0$. Assumption (i) follows immediately. Now since $\cap_{k=1}^n A_k = \{\mathbf{y}_0\}$ and $\mathbf{c} \neq \mathbf{y}_0$, there is another support plane not containing $\mathbf{c}$, and without loss of generality we may assume that $\mathbf{c} \notin A_1$. Let $\mathbb{E}$ be the affine span of the set $\{\mathbf{y}_2, \ldots, \mathbf{y}_n, \mathbf{c}\}$, an $(n-1)$-dimensional affine plane. If $\mathbf{b} \notin \mathbb{E}$, then assumption (ii) holds since then the larger set $\{\mathbf{y}_2, \ldots, \mathbf{y}_n, \mathbf{b}, \mathbf{c}\}$ is affinely independent. If $\mathbf{b} \in \mathbb{E}$, then the same argument applied above to $\mathbf{c}$ and the n-simplex σ in $\mathbb{E}^n$ applied now to $\mathbf{b}$ and the $(n-1)$-simplex $\tau = [\mathbf{y}_2 \ldots, \mathbf{y}_n, \mathbf{c}]$ in $\mathbb{E}$ implies that there are at least two support planes of τ in the $(n-1)$-dimensional Euclidean space $\mathbb{E}$ that do not contain $\mathbf{b}$. These two affine planes are opposite two of the vertices of τ, and this implies that one of these vertices must be from the list $\mathbf{y}_2, \ldots, \mathbf{y}_n$, so without loss of generality we may assume that $\mathbf{b}$ does not lie on the support plane of τ opposite $\mathbf{y}_2$. Assumption (ii) is now automatic. ■

Theorem 2 offers in its corollary below the claimed improvement upon Theorem 1 as long as X is not a metric space of cardinality $n+3$.

Corollary. *Let X be a metric space whose cardinality is not equal to $n+3$. Then X embeds isometrically in the Euclidean space $\mathbb{E}^n$ if and only if each of its finite subsets with at most $n+2$ elements embeds isometrically in $\mathbb{E}^n$.*

We summarize in the next corollary facts uncovered in this section about all the finite metric spaces with precisely $n+3$ elements that fail to embed isometrically in $\mathbb{E}^n$, even though every subset with at most $n+2$ elements does so embed. This provides a characterization up to isometry of all such metric spaces. Indeed, all such spaces up to isometry are obtained from the vertex set of an n-simplex in $\mathbb{E}^n$ and a polar pair. All distances are the Euclidean ones, except for that between the points of the polar pair, and that distance is the polar diameter of the pair.

Corollary. *Let X be a metric space that fails to embed isometrically in $\mathbb{E}^n$ though all of its subsets with at most $n+2$ elements do so embed. Then X has precisely $n+3$ elements and any subset of X of cardinality $n+1$ isometrically embeds in $\mathbb{E}^n$ as an affine basis. Moreover, for any two points a and b of X, there exists a mapping $\rho : X \to \mathbb{E}^n$ that restricts to isometric embeddings of $X - \{a\}$ and $X - \{b\}$, and then $\rho(a)$ and $\rho(b)$ form a polar pair for the n-simplex σ spanned by the points of $\rho(X - \{a, b\})$. The points $\rho(a)$ and $\rho(b)$ do not lie on any of the $n+1$ support planes of σ. The distance $ab = r$, the polar diameter of the set $\{\rho(a), \rho(a)^*\}$, where $\rho(a)^* = \rho(b)$, while the remaining distances in X are $xy = \rho(x)\rho(y)$, when $\{x, y\} \neq \{a, b\}$. For any $j = 0, \ldots, n$, the polar diameter is $r = \rho(a)_j \rho(a)^* = \rho(a)_j \rho(b)$, where $\rho(a)_j$ is the isometric reflection of $\rho(a)$ through the $(n-1)$-dimensional support plane A_j of the n-simplex σ.*

Now that we have characterized those spaces X with the desired embedding properties into $\mathbb{E}^n$, how common are they? The answer is contained in the next proposition, whose proof is left to the reader.

Proposition. *Let $[\mathbf{v}_0, \ldots, \mathbf{v}_n]$ be an n-simplex in $\mathbb{E}^n$. The set of points $\mathbf{a}$ that are polar to $[\mathbf{v}_0, \ldots, \mathbf{v}_n]$ is a dense open subset of $\mathbb{E}^n$ whose complement has measure zero.*

ACT 3: CAYLEY–MENGER DETERMINANTS AND ISOMETRIC EMBEDDING. Acts 1 and 2 have reduced the question of the isometric embeddability into $\mathbb{E}^n$ of the metric space X with more than $n+3$ elements to the question of the isometric embeddability of each subset of X with $n+2$ elements. Menger in his 1931 paper [**6**] gave a purely algebraic invariant for an $(n+2)$-element metric space that encodes isometric embeddability into $\mathbb{E}^n$. To describe this, we need the classical formula for the volume of a simplex.

Theorem 4. *The n-dimensional volume v_n of an n-simplex $[\mathbf{v}_0, \ldots, \mathbf{v}_n]$ in $\mathbb{E}^n$ is*

$$v_n = v_n([\mathbf{v}_0, \ldots, \mathbf{v}_n]) = \frac{1}{n!}\Big|\det \mathsf{V}\Big|,$$

where, for $k = 0, \ldots, n$, the kth row of the $(n+1) \times (n+1)$ square matrix V consists of the n coordinates of the vertex $\mathbf{v}_k$ augmented by a "1" in the $(n+1)$st position.

The reader may find many references to this in the literature and perhaps one of the shortest proofs is given by P. Stein [**9**]. Since our desire is to give proofs of Menger's results using geometric arguments at the undergraduate level, we give a quick geometric proof, even shorter than Stein's.

Proof. First we derive the n-dimensional volume of a cone C over an $(n-1)$-dimensional base B. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the standard unit basis of $\mathbb{E}^n$ and $A = h\mathbf{e}_n + E^{n-1}$, the $(n-1)$-dimensional affine plane at height $h \geq 0$ above the coordinate hyperplane $E^{n-1} = \text{span}[\mathbf{e}_1, \ldots, \mathbf{e}_{n-1}]$. Let B be a subset of A with $(n-1)$-dimensional volume $v_{n-1} = v_{n-1}(B)$, and let $C = \mathbf{0} * B$, the cone over B with vertex $\mathbf{0}$, the origin of $\mathbb{E}^n$. Now C is just the union of the line segments in $\mathbb{E}^n$ between the origin and the points of B; precisely, $C = \{t\mathbf{b} : 0 \leq t \leq 1, \mathbf{b} \in B\} = \cup_{0 \leq t \leq 1} tB$, where

$tB = \{t\mathbf{b} : \mathbf{b} \in B\}$. For each $0 \le t \le 1$, since all distances are dilated by a factor of t by the dilation $\mathbf{x} \mapsto t\mathbf{x}$, we have $v_{n-1}(tB) = t^{n-1}v_{n-1}(B)$. The "volume by slicing" technology of elementary calculus implies that

$$v_n(C) = \int_0^h v_{n-1}\left(\frac{x}{h}B\right)\,dx = v_{n-1}(B)\int_0^h \frac{x^{n-1}}{h^{n-1}}\,dx = \frac{1}{n}v_{n-1}(B)h.$$

The classical formula for the volume of the n-simplex now follows by induction. Indeed, the basis of the induction, $n = 1$, holds trivially. Assume the formula holds for any $(n-1)$-simplex in $\mathbb{E}^{n-1}$, and let h be the distance from $\mathbf{v}_0$ to the 0th support plane A_0 of $\sigma = [\mathbf{v}_0, \dots, \mathbf{v}_n]$, an n-simplex in $\mathbb{E}^n$. Let $\sigma_0 = [\mathbf{v}_1, \dots, \mathbf{v}_n]$ be the face of σ opposite $\mathbf{v}_0$. First make the simplifying assumption that $A_0 = \mathrm{span}[\mathbf{e}_2, \dots, \mathbf{e}_n]$ and $\mathbf{v}_0 = h\mathbf{e}_1$. Then V, the matrix in the statement of the theorem, has $h, 0, \dots, 0, 1$ as its first row and $h, 0, \dots, 0$ as its first column, and the $n \times n$ minor V_0 of V obtained by striking out the first row and column is the matrix in the statement of the theorem for the simplex σ_0 in the $(n-1)$-dimensional Euclidean space A_0. It follows from this and the inductive hypothesis that $\det \mathsf{V} = h \det \mathsf{V}_0 = (n-1)!hv_{n-1}(\sigma_0)$. Since $\sigma = \mathbf{v}_0 * \sigma_0$, the result of the preceding paragraph implies that $v_n(\sigma) = n^{-1}v_{n-1}(\sigma_0)h = (n!)^{-1}\det \mathsf{V}$, verifying the inductive step.

To finish the proof, since any n simplex $[\mathbf{v}_0, \dots, \mathbf{v}_n]$ in $\mathbb{E}^n$ may be moved so that $A_0 = \mathrm{span}[\mathbf{e}_2, \dots, \mathbf{e}_n]$ and $\mathbf{v}_0 = h\mathbf{e}_1$ by a rigid motion, all we need do is verify that $\det \mathsf{V}$ is invariant under any translation of the vertices by a vector $\mathbf{a}$ and under the application of any $n \times n$ orthogonal matrix O_n. This is left as a nice exercise in the manipulation of determinants, using that $\det \mathsf{O}_n = 1$, $\det AB = \det A \cdot \det B$, and $\det A = 0$ when one column is a multiple of another. ■

Menger's clever idea is to manipulate the $(n+1) \times (n+1)$ square matrix V to obtain an associated $(n+2) \times (n+2)$ square matrix D whose determinant expresses the squared volume of the n-simplex $[\mathbf{v}_0, \dots, \mathbf{v}_n]$ entirely in terms of $d_{ij} = \mathbf{v}_i\mathbf{v}_j$, $0 \le i, j \le n$, the distances among the vertices of the simplex. Noting that Cayley in his first paper used this same determinant in discussing finite subsets of $\mathbb{E}^3$, Blumenthal [**1**] termed $\det \mathsf{D}$ the *Cayley–Menger determinant*. First consider the two matrices

$$\mathsf{A} = \begin{pmatrix} v_{01} & \dots & v_{0n} & 1 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ v_{n1} & \dots & v_{nn} & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathsf{B} = \begin{pmatrix} v_{01} & \dots & v_{0n} & 0 & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ v_{n1} & \dots & v_{nn} & 0 & 1 \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix},$$

where $\mathbf{v}_k = (v_{k1}, \dots, v_{kn})$ for $k = 0, \dots, n$, and note that $\det \mathsf{A} = \det \mathsf{V} = -\det \mathsf{B}$. The product of A with the transpose of B is

$$\mathsf{AB}^{\mathrm{tr}} = \begin{pmatrix} \langle \mathbf{v}_0, \mathbf{v}_0 \rangle^2 & \dots & \langle \mathbf{v}_0, \mathbf{v}_n \rangle^2 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ \langle \mathbf{v}_n, \mathbf{v}_0 \rangle^2 & \dots & \langle \mathbf{v}_n, \mathbf{v}_n \rangle^2 & 1 \\ 1 & \dots & 1 & 0 \end{pmatrix}$$

and the *Cayley–Menger matrix* is

$$\mathsf{D} = \mathsf{D}(\mathbf{v}_0, \ldots, \mathbf{v}_n) = \begin{pmatrix} d_{00}^2 & \ldots & d_{0n}^2 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ d_{n0}^2 & \ldots & d_{nn}^2 & 1 \\ 1 & \ldots & 1 & 0 \end{pmatrix}.$$

In the Cayley–Menger determinant, $\det \mathsf{D}$, substituting $d_{ij}^2 = \langle \mathbf{v}_i - \mathbf{v}_j, \mathbf{v}_i - \mathbf{v}_j \rangle = \langle \mathbf{v}_i, \mathbf{v}_i \rangle + \langle \mathbf{v}_j, \mathbf{v}_j \rangle - 2\langle \mathbf{v}_i, \mathbf{v}_j \rangle$, then for $i, j = 0, \ldots, n$, multiplying the last row by $\langle \mathbf{v}_i, \mathbf{v}_i \rangle$ and subtracting from the ith row, then multiplying the last column by $\langle \mathbf{v}_j, \mathbf{v}_j \rangle$ and subtracting from the jth column, gives $\det \mathsf{D} = (-2)^n \det \mathsf{AB}^{\mathrm{tr}} = (-1)^{n+1} 2^n (\det \mathsf{V})^2$. Coupling this with Theorem 4 gives the squared-volume of the n-simplex $[\mathbf{v}_0, \ldots, \mathbf{v}_n]$ in $\mathbb{E}^n$ entirely in terms of the distances among its vertices as

$$v_n^2 = v_n([\mathbf{v}_0, \ldots, \mathbf{v}_n])^2 = \frac{(-1)^{n+1}}{2^n (n!)^2} \det \mathsf{D}.$$

Since $v_n^2 \geq 0$, the sign of the determinant $\det \mathsf{D}$ is $(-1)^{n+1}$. Notice that $v_n = 0 = \det \mathsf{D}$ for the points $\mathbf{v}_0, \ldots, \mathbf{v}_n$ in $\mathbb{E}^n$ if and only if the set $\{\mathbf{v}_0, \ldots, \mathbf{v}_n\}$ fails to be affinely independent.

If X is any metric space of cardinality r, the *Cayley–Menger determinant* for X is

$$\det \mathsf{D} = \det \mathsf{D}(X) = \begin{vmatrix} d_{11}^2 & \ldots & d_{1r}^2 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ d_{r1}^2 & \ldots & d_{rr}^2 & 1 \\ 1 & \ldots & 1 & 0 \end{vmatrix},$$

where $x_1, \ldots, x_r$ is some ordering of the points of X and $d_{ij} = x_i x_j$. Note that $\det \mathsf{D}$ is independent of the ordering of the points of X and, obviously, is an isometric invariant of finite metric spaces. This implies the following. If X is a metric space that isometrically embeds in $\mathbb{E}^n$ via the isometry ρ, then for any finite subset $Y \subset X$ with, say, r distinct points, the Cayley–Menger determinant $\det \mathsf{D}(Y)$ either vanishes or has the sign $(-1)^r$. Moreover, the determinant vanishes if and only if $\rho(Y)$ is affinely dependent, which always occurs when $r > n + 1$. This gives a necessary condition for $n + 2$ points of X to embed isometrically in $\mathbb{E}^n$, and Menger showed it to be sufficient.

Theorem 5. *The $(n + 2)$-point metric space X embeds isometrically in $\mathbb{E}^n$ if and only if, for each subset $Y \subset X$, the Cayley–Menger determinant* $\det \mathsf{D}(Y)$ *either vanishes or has the sign $(-1)^{|Y|}$, where $|Y|$ is the cardinality of Y, and* $\det \mathsf{D}(X) = 0$.

Necessity has been demonstrated. The proof of sufficiency is by induction and involves a detailed algebraic analysis. The authors offer no geometric proof of sufficiency and refer the reader to Menger's original argument in [**6**].

We close this act by condensing all three fundamental results of Menger into a characterization of those metric spaces isometrically embeddable in Euclidean space, as he did in [**6**].

Metrical Characterization of Euclidean Sets. *A metric space X embeds isometrically in $\mathbb{E}^n$ if and only if, when X contains more than $n + 3$ points, then*

(i) *for every* $Y \subset X$ *with precisely* $r \leq n+1$ *points, the Cayley–Menger determinant* $\det \mathsf{D}(Y)$ *either vanishes or has the sign* $(-1)^r$, *and*

(ii) *the determinant associated with each* $n+2$ *distinct points of* X *vanishes; and when* X *contains exactly* $n+3$ *points, in addition to these conditions,*

(iii) $\det \mathsf{D}(X) = 0$.

POSTLUDE: DISTANCE AND METRIC GEOMETRY. Menger's 1931 results may be counted as the modern starting point for the discipline of *distance geometry*,[6] a discipline that studies sets based on given distances between pairs of its points. Special interest rests on the relation of general metric spaces to the metric spaces of the classical geometries—spherical and elliptic, Euclidean, and hyperbolic spaces. More recently, distance geometry has found applications to a great variety of fields, including protein structure, molecular structure, wireless sensor networks, multidimensional scaling, and graph rigidity. Liberti et al. [**4**] is a fantastic reference source for these and other applications. Distance geometry has close affinity to and not insignificant overlap with the discipline of *metric geometry*,[7] which concerns the metric relationships that characterize segments and lines and curves and surfaces, and today finds expression in the study of metric curvature and length spaces, with applications to geometric group theory and geometric topology. The classic reference for distance geometry is Leonard Blumenthal's 1953 treatise *Theory and Applications of Distance Geometry* [**1**] and a modern reference for the still-developing discipline of metric geometry is Dmitri Burago, Yuri Burago, and Sergei Ivanov's 2001 treatise *A Course in Metric Geometry* [**2**].

Aside from the original articles [**5**] and [**6**], detailed proofs of Menger's results appear in Chapter IV of Blumenthal [**1**], whose approach is that of Menger's with proofs that are more algebraic and combinatorial than ours. Blumenthal goes much further in that he develops the corresponding isometric embedding results for both spherical and hyperbolic spaces, as well as Hilbert spaces. Of interest also is C. L. Morgan [**7**], where the problem of embedding metric spaces into Euclidean spaces is addressed entirely algebraically. There, Morgan gives determinant conditions, similar in flavor to Menger's use of the Cayley–Menger determinant, to give necessary and sufficient algebraic conditions based on the distances among points for a metric space to admit a Euclidean embedding. This is a little different from Schoenberg's [**8**] earlier algebraic approach for finite metric spaces that gives a condition on the the matrix of squared distances that is necessary and sufficient for embedding into Euclidean space.

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[6] Ancient Greek mathematics made modest contributions to the discipline, but the subject lay dormant until Arthur Cayley's contributions in the 19th century. Menger is the successor to Cayley in the development of the discipline, and his work inspired a generation of geometers to develop the theory to maturity in the middle of the last century.

[7] The early pioneers in this development, no doubt influenced by Menger, were the German-American mathematician Herbert Busemann and the Russian mathematician Aleksandr Aleksandrov, working in the 1940s and 1950s. Mikhail Gromov, along with Eliyahu Rips, revived interest in their work in the 1980s in the early development of *geometric group theory*, one of the most significant developments in geometric topology of the last half century, and one that enjoys intense research activity among geometric topologists today.

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