EQUIPOTENTIAL SURFACES IN CLOSE BINARY SYSTEMS: REMARKS ON THE TIME-DEPENDENT POTENTIAL FUNCTION

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Abstract. By use of the mass-point model, the equations of the equipotential surfaces are reviewed. A difference between the time-dependent potential function and zero relative velocity surfaces is put in evidence. A drawback in the time-dependent transformation between (ξ, η, ζ) and (x, y, z) coordinate systems is underlined.

1. Introduction

It is a well-known fact that the equilibrium forms of the component stars of close binary systems are defined as the surfaces over which the effective potential of all forces acting within the system remain constant. Such surfaces are named as *equipotential surfaces*, among them the surfaces of zero relative velocity being of great importance for the study of double star evolution.

Much has been written about the problem of equipotential surfaces in close binary systems. But, to my regret, I have no possibility to mention here the great number of papers written about such an interesting subject. Nevertheless, I have to mention some of the most important conditions in which such a question is approachable. If "... the density concentration of the stars constituting our binary is allowed to approach infinity, their shape can be described in a closed *algebraic* form, which is *exact* for any such configuration *irrespective of the proximity of its components or their mass ratio*. Such a model is generally known in the literature, under the name of its originator, as *Roche Model*" (see Kopal, 1989).

In the last time, the problem of the synchronization between orbital motion and axial rotation of the component stars, in a binary system, becomes more and more important. That is why, I have to mention here that, among others, Kruszewski (1966) has established an equation of equipotential surfaces for the case of the nonsynchronism. The aim of the present paper is to demonstrate that Kruszewski's equation does not represent zero relative velocity surfaces from the restricted problem of the three bodies.

2. Equations of the Problem in a First Rotating Frame

Let us consider a close binary system whose components S_1 and S_2 are mass-points and revolve about their common centre of mass along circular orbits with the constant angular velocity ω_k. In addition, we shall consider a rectangular coordinate system

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(X, Y, Z) and its origin taken as the common centre of mass M, but its X-axis will be directed towards the secondary component S_2. Therefore, this coordinate system is considered in rotation with the binary star, XY-plane coincides with orbital plane, while the Z-axis is perpendicular to it (see Figure 1).

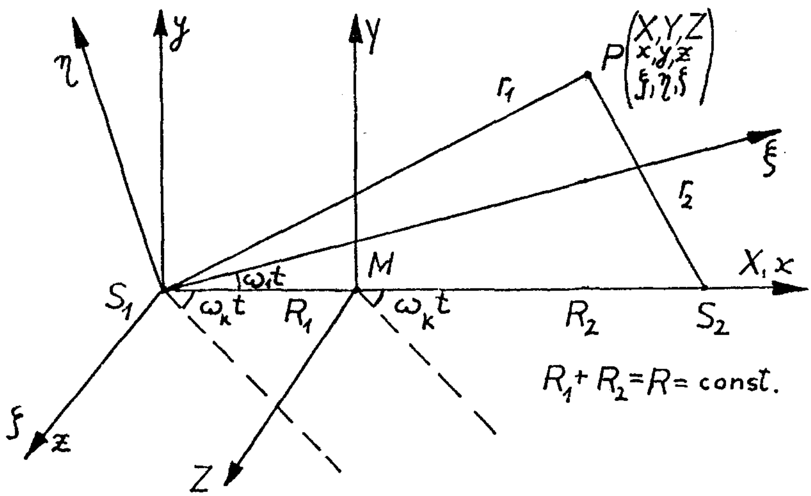

Fig. 1.

In such conditions the differential equations of the motion for the infinitesimal body $P(X, Y, Z)$ are (e.g., Tsesevich, 1971)

$$\frac{d^2X}{dt^2} - 2\omega_k \frac{dY}{dt} = \omega_k^2 X - G\frac{m_1}{r_1^3}(X + R_1) - G\frac{m_2}{r_2^3}(X - R_2) = \frac{\partial U}{\partial X},$$

$$\frac{d^2Y}{dt^2} + 2\omega_k \frac{dX}{dt} = \omega_k^2 Y - G\frac{m_1}{r_1^3}Y - G\frac{m_2}{r_2^3}Y = \frac{\partial U}{\partial Y}, \tag{1}$$

$$\frac{d^2Z}{dt^2} = -G\frac{m_1}{r_1^3}Z - G\frac{m_2}{r_2^3}Z = \frac{\partial U}{\partial Z},$$

where m_1 and m_2 are the masses of the two component stars $S_1(R_1, 0, 0)$ *and* $S_2(R_2, 0, 0)$ and G = constant of gravitation. Here r_1 and r_2 denote the distances of the infinitesimal body $P(X, Y, Z)$ from the mass centres of S_1 and S_2; they are defined by

$$r_1^2 = (X + R_1)^2 + Y^2 + Z^2, \qquad r_2^2 = (X - R_2)^2 + Y^2 + Z^2; \tag{2}$$

where $R = R_1 + R_2$ = constant stands for the distance between the centres of the two stars.

In Equations (1) the corresponding potential function U is

$$U = G\frac{m_1}{r_1} + G\frac{m_2}{r_2} + \frac{1}{2}\omega_k^2(X^2 + Y^2), \tag{3}$$

where the first right-hand side term is the gravitational potential of S_1, the second that of S_2. The third term accounts for the centrifugal force due to rotation of the frame of reference (X, Y, Z).

Now, if we multiply Equations (1) by $\mathrm{d}X/\mathrm{d}t$, $\mathrm{d}Y/\mathrm{d}t$, and $\mathrm{d}Z/\mathrm{d}t$, respectively, and adding the resulting equations together, we have

$$\frac{\mathrm{d}^2X}{\mathrm{d}t^2}\frac{\mathrm{d}X}{\mathrm{d}t}+\frac{\mathrm{d}^2Y}{\mathrm{d}t^2}\frac{\mathrm{d}Y}{\mathrm{d}t}+\frac{\mathrm{d}^2Z}{\mathrm{d}t^2}\frac{\mathrm{d}Z}{\mathrm{d}t}=\frac{\partial U}{\partial X}\frac{\mathrm{d}X}{\mathrm{d}t}+\frac{\partial U}{\partial Y}\frac{\mathrm{d}Y}{\mathrm{d}t}+\frac{\partial U}{\partial Z}\frac{\mathrm{d}Z}{\mathrm{d}t}$$

and, after integration with respect to time,

$$\frac{V_1^2}{2}=U-C_1\,;$$

or, for $V_1 = 0$,

$$U - C_1 = 0\,, \tag{4}$$

which represents the surfaces of zero relative velocity expressed in the (X, Y, Z) frame.

Therefore, it is evident that the equipotentials defined by Equation (3) are identical with the surfaces of relative velocity from the restricted problem of three bodies.

3. Hypothesis of the Second Rotating Frame with Keplerian Angular Velocity ω_k

Let us introduce a new rotating frame (x, y, z) such that its origin is now the mass centre of the star S_1, the x-axis points always towards the component S_2, and xy-plane coincides with orbital plane (see Figure 1).

The transformation equations between the system (X, Y, Z) and (x, y, z) are

$$\begin{aligned}&X = x - R_1\,,\quad Y = y\,,\quad Z = z\,,\quad x_1 = 0\,,\quad y_1 = 0\,,\\&z_1 = 0\,,\quad x_2 = R_1 + R_2\,,\quad y_2 = 0\,,\quad z_2 = 0\,,\end{aligned} \tag{5}$$

while for the distances between $P(x, y, z)$ and the two stars S_1 and S_2 we have

$$r_1^2 = x^2 + y^2 + z^2\,,\qquad r_2^2 = (x - R)^2 + y^2 + z^2\,. \tag{6}$$

In such conditions the differential equations (1) become

$$\begin{aligned}&\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 2\omega_k\frac{\mathrm{d}y}{\mathrm{d}t} = \omega_k^2(x - R_1) - G\frac{m_1}{r_1^3}x - G\frac{m_2}{r_2^3}(x - R) = \frac{\partial\Psi}{\partial x}\,,\\&\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 2\omega_k\frac{\mathrm{d}x}{\mathrm{d}t} = \omega_k^2 y - G\frac{m_1}{r_1^3}y - G\frac{m_2}{r_2^3}y = \frac{\partial\Psi}{\partial y}\,,\\&\frac{\mathrm{d}^2z}{\mathrm{d}t^2} = -G\frac{m_1}{r_1^3}z - G\frac{m_2}{r_2^3}z = \frac{\partial\Psi}{\partial z}\,;\end{aligned} \tag{7}$$

where the new potential function is defined by

$$\Psi = G \frac{m_1}{r_1} + G \frac{m_2}{r_2} - \omega_k^2 R_1 x + \tfrac{1}{2}\omega_k^2(x^2 + y^2) . \tag{8}$$

From Equations (7), after habitual transformations, we have

$$\tfrac{1}{2}V_2^2 = \Psi - C_2 ;$$

which, for $V_2 = 0$, leads to

$$\Psi - C_2 = 0 . \tag{9}$$

Therefore, Equation (8) defines the equipotential surfaces of zero relative velocity.

4. Hypothesis of the Rotating Frame with an Arbitrary Angular Velocity

Lastly, let us choose a new frame (ξ, η, ζ) with the origin in the same mass centre of the star S_1, but assumed as being in rotation around ζ-axis with an arbitrary angular velocity ω_1 with respect to the (x, y, z) frame (see Figure 1). Therefore, this new system will rotate with an angular velocity $\omega = \omega_1 + \omega_k$ with respect to a rest frame of reference. In particular ω_1 may be taken to be equal to the angular velocity of the axial rotation of S_1 or S_2, but such a case is not important here.

Now, the corresponding transformation equations between (x, y, z) and (ξ, η, ζ) are given by

$$\begin{aligned} x &= \xi \cos \omega_1 t - \eta \sin \omega_1 t , \\ y &= \xi \sin \omega_1 t + \eta \cos \omega_1 t , \\ z &= \zeta ; \end{aligned} \tag{10}$$

and from (6) and (10) it follows that

$$\begin{aligned} r_1^2 &= \xi^2 + \eta^2 + \zeta^2 , \\ r_2^2 &= R^2 + \xi^2 + \eta^2 + \zeta^2 - 2R\xi \cos \omega_1 t + 2R\eta \sin \omega_1 t . \end{aligned} \tag{11}$$

In such conditions the differential equations of the motion are

$$\frac{d^2\xi}{dt^2} - 2\omega \frac{d\eta}{dt} = -\omega_k^2 R_1 \cos \omega_1 t + \omega^2 \xi - G \frac{m_1}{r_1^3} \xi - G \frac{m_2}{r_2^3} (\xi - R \cos \omega_1 t) = \frac{\partial \Omega}{\partial \xi} ,$$

$$\frac{d^2\eta}{dt^2} + 2\omega \frac{d\xi}{dt} = \omega_k^2 R_1 \sin \omega_1 t + \omega^2 \eta - G \frac{m_1}{r_1^3} \eta - G \frac{m_2}{r_2^3} (\eta + R \sin \omega_1 t) = \frac{\partial \Omega}{\partial \eta} ,$$

$$\frac{d^2\zeta}{dt^2} = -G \frac{m_1}{r_1^3} \zeta - G \frac{m_2}{r_2^3} \zeta = \frac{\partial \Omega}{\partial \zeta} ; \tag{12}$$

where $\omega = \omega_1 + \omega_k$ (see Figure 1) and

$$\Omega(\xi, \eta, \zeta, t) = G\,\frac{m_1}{r_1} + G\,\frac{m_2}{r_2} - \omega_k^2 R_1 x + \tfrac{1}{2}\omega^2(\xi^2 + \eta^2) \tag{13a}$$

or

$$\Omega(\xi, \eta, \zeta, t) = G\,\frac{m_1}{r_1} + G\,\frac{m_2}{r_2} - \omega_k^2 R_1(\xi \cos\omega_1 t - \eta \sin\omega_1 t) + \tfrac{1}{2}\omega^2(\xi^2 + \eta^2)\,, \tag{13b}$$

where Equation (13) represents Kruszewski's formula for 'equipotential surfaces' expressed in a rotating frame (see Kruszewski, 1966, p. 240, Equation (20)). *This is a time-dependent potential function.*

4.1. The First Remark

If we multiply Equations (12) by $d\xi/dt$, $d\eta/dt$, and $d\zeta/dt$, respectively, and adding the resulting equations together, we have

$$\frac{d^2\xi}{dt^2}\frac{d\xi}{dt} + \frac{d^2\eta}{dt^2}\frac{d\eta}{dt} + \frac{d^2\zeta}{dt^2}\frac{d\zeta}{dt} = \left(\frac{\partial\Omega}{\partial\xi}\frac{d\xi}{dt} + \frac{\partial\Omega}{\partial\eta}\frac{d\eta}{dt} + \frac{\partial\Omega}{\partial\zeta}\frac{d\zeta}{dt} + \frac{\partial\Omega}{\partial t}\right) - \frac{\partial\Omega}{\partial t}\,, \tag{14}$$

where $-\partial\Omega/\partial t$ may be estimated from Equation (13b) and Equations (11). In doing so, we find that

$$-\frac{\partial\Omega}{\partial t} = \left(G\,\frac{m_2}{r_2^3}\,R + \omega_k^2 R_1\right)\omega_1(\xi \sin\omega_1 t + \eta \cos\omega_1 t)\,, \tag{15a}$$

or

$$-\frac{\partial\Omega}{\partial t} = \left(G\,\frac{m_2}{r_2^3}\,R + \omega_k^2 R_1\right)\omega_1 y\,. \tag{15b}$$

Now, after integration with respect to time, for $V = 0$, from (14) we obtain the equation of the surfaces of zero relative velocity which are given by

$$G\,\frac{m_1}{r_1} + G\,\frac{m_2}{r_2} - \omega_k^2 R_1(\xi \cos\omega_1 \mathrm{t} - \eta \sin\omega_1 t) + \tfrac{1}{2}\omega^2(\xi^2 + \eta^2) + \int_{t_0}^{t}\left(-\frac{\partial\Omega}{\partial t}\right) dt = C\,, \tag{16a}$$

or

$$G\frac{m_1}{r_1}+G\frac{m_2}{r_2}-\omega_k^2R_1x+\tfrac{1}{2}\omega^2(\xi^2+\eta^2)+\int_{t_0}^{t}\left(-\frac{\partial\Omega}{\partial t}\right)\mathrm{d}t=C\,. \tag{16b}$$

If we compare Equation (16b) with the time-dependent potential function (see Equation (13a)) it is easy to see that the corresponding difference is determined by the additional term

$$\int_{t_0}^{t}\left(-\frac{\partial\Omega}{\partial t}\right)\mathrm{d}t\,,$$

which was introduced with the purpose that the first term on the right-hand side of Equation (14) be a total differential.

Therefore, *the time-dependent potential function given by Equation* (13) *does not represent zero relative velocity surfaces.*

Of course, for $\omega_1=0$, we have $\partial\Omega/\partial t=0$ and Equation (16) reduces to the conventional potential function (see Equation (8)).

4.2. The second remark

In practice, the potential function is usually expressed in terms of appropriately chosen units. For this purpose, it is convenient to choose the following units: the distance $R=R_1+R_2$ between the mass centres of the two stellar components as unit of length, the sum of the masses (m_1+m_2) of both components as unit of mass and the reciprocal ω_k^{-1} of the Keplerian angular velocity as the unit of time. Therefore, we have $P=2\pi$ for the orbital period and $G=1$.

In such conditions it follows that

$$1-\mu=\frac{m_1}{m_1+m_2}=R_2\,,\qquad \mu=\frac{m_2}{m_1+m_2}=R_1\,,\qquad \omega_k+\omega_1=1+f\,; \tag{17}$$

and Equation (13a) becomes

$$\Omega=\frac{1-\mu}{r_1}+\frac{\mu}{r_2}-\mu x+\tfrac{1}{2}(1+f)^2(\xi^2+\eta^2)\,. \tag{18}$$

(see Kruszewski, 1966, p. 241, Equation (23)).

Now, in Kruszewski's mentioned paper (see p. 252, Equation (71)), for $z=\zeta=0$, the potential function Ω is expressed as a function of x and y as

$$\Omega=\frac{1-\mu}{\sqrt{x^2+y^2}}+\frac{\mu}{\sqrt{(x-1)^2+y^2}}-\mu x+\tfrac{1}{2}(1+f)^2(x^2+y^2)\,. \tag{19}$$

As it is easy to see, tacitly, Equations (10) have been used in the form

$$\xi^2 + \eta^2 = x^2 + y^2 \tag{20}$$

in order to perform the transition between the systems (ξ, η, ζ) and (x, y, z).

Of course, it is easy to understand that Equation (20) cannot be used for a transformation between (ξ, η, ζ) and (x, y, z) because the corresponding angular velocity has vanished. In addition we must remember that for the transformation from (x, y, z) to (ξ, η, ζ) we did not use Equation (20), but we have introduced Equations (10) into Equations (7). Therefore, if we have to perform a transformation from (ξ, η, ζ) to (x, y, z), we must have in view the following equations

$$\begin{aligned} \xi &= x \cos \omega_1 t + y \sin \omega_1 \mathrm{t}\,, \\ \eta &= -x \sin \omega_1 t + y \cos \omega_1 t\,, \\ \zeta &= z \end{aligned} \tag{21}$$

and to insert them into Equations (12). Obviously, after we perform all algebraic computations, we come back to Equations (7) with the potential function given by Equation (8).

Therefore, in the rotating system (ξ, η, ζ) the potential function is given by

$$\Omega(\xi, \eta, \zeta, t) = \frac{1-\mu}{r_1} + \frac{\mu}{r_2} - \mu x + \tfrac{1}{2}(1+f)^2(\xi^2 + \eta^2) \tag{18}$$

with

$$\begin{aligned} r_1^2 &= \xi^2 + \eta^2 + \zeta^2\,, \\ r_2^2 &= R^2 + \xi^2 + \eta^2 + \zeta^2 - 2R\xi \cos \omega_1 t + 2R\eta \sin \omega_1 t\,, \\ 1 + f &= \omega_1 + \omega_k\,, \end{aligned}$$

while in (x, y, z) system the potential function is defined by

$$\Psi(x, y, z) = \frac{1-\mu}{r_1} + \frac{\mu}{r_2} - \mu x + \tfrac{1}{2}(x^2 + y^2)) \tag{8a}$$

with

$$r_1^2 = x^2 + y^2 + z^2\,, \qquad r_2^2 + (x-R)^2 + y^2 + z^2\,, \qquad \omega_k = 1\,. \tag{6}$$

Finally, we have considered necessary to underline the above-mentioned mistake in order to prevent a wrong use of the potential function.

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References

Kopal, Z.: 1989, *The Roche Problem*, Kluwer Academic Publishers, Dordrecht, The Netherlands.

Kruszewski, A.: 1966, *Advances in Astronomy and Astrophysics*, Vol. 4, Academic Press, New York, pp. 233–299.

Tsesevich, V. P.: 1971, in V. P. Tsesevich (ed.), *Zatmennye Peremennye Zvezd*, Nauka, Moscow.